Liouville Numbers

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Abstract

In this work, we define the concept of Liouville numbers as well as the standard construction to obtain Liouville numbers and we prove their most important properties: irrationality and transcendence.

This is historically interesting since Liouville numbers constructed in the standard way where the first numbers that were proven to be transcendental. The proof is very elementary and requires only standard arithmetic and the Mean Value Theorem for polynomials and the boundedness of polynomials on compact intervals.

Contents

1 Liouville Numbers

1.1 Preliminary lemmas

theory *Liouville-Numbers-Misc*
imports
Complex-Main
HOL−Computational-Algebra.Polynomial
begin

We will require these inequalities on factorials to show properties of the standard construction later.

lemma *fact-ineq*: $n \geq 1 \Longrightarrow fact\ n + k \leq fact\ (n + k)$
$\langle proof \rangle$

lemma *Ints-sum*:
assumes $\bigwedge x.\ x \in A \Longrightarrow f\ x \in \mathbb{Z}$
shows $sum\ f\ A \in \mathbb{Z}$

⟨*proof*⟩

lemma *suminf-split-initial-segment'*:
summable (f :: *nat* ⇒ *'a::real-normed-vector*) ⟹
suminf f = ($\sum$ *n*. *f* (*n* + *k* + *1*)) + *sum f* {..*k*}
⟨*proof*⟩

lemma *Rats-eq-int-div-int'*: ($\mathbb{Q}$ :: *real set*) = {*of-int p* / *of-int q* |*p q*. *q* > *0*}
⟨*proof*⟩

lemma *Rats-cases'*:
assumes (*x* :: *real*) ∈ $\mathbb{Q}$
obtains *p q* **where** *q* > *0 x* = *of-int p* / *of-int q*
⟨*proof*⟩

The following inequality gives a lower bound for the absolute value of an integer polynomial at a rational point that is not a root.

lemma *int-poly-rat-no-root-ge*:
fixes *p* :: *real poly* **and** *a b* :: *int*
assumes $\bigwedge$*n*. *coeff p n* ∈ $\mathbb{Z}$
assumes *b* > *0 poly p* (*a* / *b*) ≠ *0*
defines *n* ≡ *degree p*
shows *abs* (*poly p* (*a* / *b*)) ≥ *1* / *of-int b* ^ *n*
⟨*proof*⟩

end

theory *Liouville-Numbers*
imports
Complex-Main
HOL−Computational-Algebra.Polynomial
Liouville-Numbers-Misc
begin

A Liouville number is a real number that can be approximated well – but not perfectly – by a sequence of rational numbers. "Well", in this context, means that the error of the n-th rational in the sequence is bounded by the n-th power of its denominator.

Our approach will be the following:

- Liouville numbers cannot be rational.
- Any irrational algebraic number cannot be approximated in the Liouville sense
- Therefore, all Liouville numbers are transcendental.
- The standard construction fulfils all the properties of Liouville numbers.

1.2 Definition of Liouville numbers

The following definitions and proofs are largely adapted from those in the Wikipedia article on Liouville numbers. [1]

A Liouville number is a real number that can be approximated well – but not perfectly – by a sequence of rational numbers. The error of the n-th term $\frac{p_n}{q_n}$ is at most q_n^{-n}, where $p_n \in \mathbb{Z}$ and $q_n \in \mathbb{Z}_{\geq 2}$.
We will say that such a number can be approximated in the Liouville sense.

locale *liouville* =
  fixes *x* :: *real* **and** *p q* :: *nat* ⇒ *int*
  assumes *approx-int-pos*: *abs* (*x* − *p n* / *q n*) > *0*
    and *denom-gt-1*: *q n* > *1*
    and *approx-int*: *abs* (*x* − *p n* / *q n*) < *1* / *of-int* (*q n*) ^ *n*

First, we show that any Liouville number is irrational.

lemma (**in** *liouville*) *irrational*: *x* ∉ ℚ
⟨*proof*⟩

Next, any irrational algebraic number cannot be approximated with rational numbers in the Liouville sense.

lemma *liouville-irrational-algebraic*:
  fixes *x* :: *real*
  assumes *irrationsl*: *x* ∉ ℚ **and** *algebraic x*
  obtains *c* :: *real* **and** *n* :: *nat*
    where *c* > *0* **and** ⋀(*p*::*int*) (*q*::*int*). *q* > *0* ⟹ *abs* (*x* − *p* / *q*) > *c* / *of-int q* ^ *n*
⟨*proof*⟩

Since Liouville numbers are irrational, but can be approximated well by rational numbers in the Liouville sense, they must be transcendental.

lemma (**in** *liouville*) *transcendental*: ¬*algebraic x*
⟨*proof*⟩

1.3 Standard construction for Liouville numbers

We now define the standard construction for Liouville numbers.

definition *standard-liouville* :: (*nat* ⇒ *int*) ⇒ *int* ⇒ *real* **where**
  standard-liouville p q = (∑*k*. *p k* / *of-int q* ^ *fact* (*Suc k*))

lemma *standard-liouville-summable*:
  fixes *p* :: *nat* ⇒ *int* **and** *q* :: *int*
  assumes *q* > *1* *range p* ⊆ {*0*..<*q*}
  shows *summable* (λ*k*. *p k* / *of-int q* ^ *fact* (*Suc k*))
⟨*proof*⟩

lemma *standard-liouville-sums*:

assumes $q > 1$ *range* $p \subseteq \{0..<q\}$
shows $(\lambda k.\ p\ k\ /\ \textit{of-int}\ q\ \hat{}\ \textit{fact}\ (\textit{Suc}\ k))$ *sums standard-liouville* $p\ q$
⟨*proof*⟩

Now we prove that the standard construction indeed yields Liouville numbers.

lemma *standard-liouville-is-liouville*:
assumes $q > 1$ *range* $p \subseteq \{0..<q\}$ *frequently* $(\lambda n.\ p\ n \neq 0)$ *sequentially*
defines $b \equiv \lambda n.\ q\ \hat{}\ \textit{fact}\ (\textit{Suc}\ n)$
defines $a \equiv \lambda n.\ (\sum k{\le}n.\ p\ k * q\ \hat{}(\textit{fact}\ (\textit{Suc}\ n) - \textit{fact}\ (\textit{Suc}\ k)))$
shows *liouville* (*standard-liouville* $p\ q$) $a\ b$
⟨*proof*⟩

We can now show our main result: any standard Liouville number is transcendental.

theorem *transcendental-standard-liouville*:
assumes $q > 1$ *range* $p \subseteq \{0..<q\}$ *frequently* $(\lambda k.\ p\ k \neq 0)$ *sequentially*
shows ¬*algebraic* (*standard-liouville* $p\ q$)
⟨*proof*⟩

In particular: The the standard construction for constant sequences, such as the "classic" Liouville constant $\sum_{n=1}^{\infty} 10^{-n!} = 0.11000100\ldots$, are transcendental.

This shows that Liouville numbers exists and therefore gives a concrete and elementary proof that transcendental numbers exist.

corollary *transcendental-standard-standard-liouville*:
$a \in \{0<..<b\} \Longrightarrow$ ¬*algebraic* (*standard-liouville* (λ-. *int* a) (*int* b))
⟨*proof*⟩

corollary *transcendental-liouville-constant*:
¬*algebraic* (*standard-liouville* (λ-. *1*) *10*)
⟨*proof*⟩

end

References

[1] Wikipedia. Liouville number — Wikipedia, the free encyclopedia. https://en.wikipedia.org/w/index.php?title=Liouville_number&oldid=696910651, 2015. [Online; accessed 22-July-2004].