

# Liouville Numbers

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## Abstract

In this work, we define the concept of Liouville numbers as well as the standard construction to obtain Liouville numbers and we prove their most important properties: irrationality and transcendence.

This is historically interesting since Liouville numbers constructed in the standard way were the first numbers that were proven to be transcendental. The proof is very elementary and requires only standard arithmetic and the Mean Value Theorem for polynomials and the boundedness of polynomials on compact intervals.

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## 1 Liouville Numbers

### 1.1 Preliminary lemmas

```
theory Liouville-Numbers-Misc
imports
  Complex-Main
  HOL-Computational-Algebra.Polynomial
begin
```

We will require these inequalities on factorials to show properties of the standard construction later.

**lemma** *fact-ineq*:  $n \geq 1 \implies \text{fact } n + k \leq \text{fact } (n + k)$

**proof** (*induction k*)

**case** (*Suc k*)

**from** *Suc* **have**  $\text{fact } n + \text{Suc } k \leq \text{fact } (n + k) + 1$  **by** *simp*

**also from** *Suc* **have**  $\dots \leq \text{fact } (n + \text{Suc } k)$  **by** *simp*

**finally show** *?case* .

**qed** *simp-all*

**lemma** *Ints-sum*:

**assumes**  $\bigwedge x. x \in A \implies f x \in \mathbb{Z}$

**shows**  $\text{sum } f A \in \mathbb{Z}$

**by** (*cases finite A, insert assms, induction A rule: finite-induct*)  
(*auto intro!: Ints-add*)

**lemma** *suminf-split-initial-segment'*:

*summable* ( $f :: \text{nat} \Rightarrow 'a::\text{real-normed-vector}$ )  $\implies$

$\text{suminf } f = (\sum n. f (n + k + 1)) + \text{sum } f \{..k\}$

**by** (*subst suminf-split-initial-segment[of - Suc k], assumption, subst lessThan-Suc-atMost*)

*simp-all*

**lemma** *Rats-eq-int-div-int'*:  $(\mathbb{Q} :: \text{real set}) = \{\text{of-int } p / \text{of-int } q \mid p \ q. \ q > 0\}$

**proof** *safe*

**fix**  $x :: \text{real}$  **assume**  $x \in \mathbb{Q}$

**then obtain**  $p \ q$  **where**  $pq: x = \text{of-int } p / \text{of-int } q \ q \neq 0$

**by** (*subst (asm) Rats-eq-int-div-int*) *auto*

**show**  $\exists p \ q. x = \text{real-of-int } p / \text{real-of-int } q \wedge 0 < q$

**proof** (*cases q > 0*)

**case** *False*

**show** *?thesis* **by** (*rule exI[of - -p], rule exI[of - -q]*) (*insert False pq, auto*)

**qed** (*insert pq, force*)

**qed** *auto*

**lemma** *Rats-cases'*:

**assumes**  $(x :: \text{real}) \in \mathbb{Q}$

**obtains**  $p \ q$  **where**  $q > 0 \ x = \text{of-int } p / \text{of-int } q$

**using** *assms* **by** (*subst (asm) Rats-eq-int-div-int'*) *auto*

The following inequality gives a lower bound for the absolute value of an integer polynomial at a rational point that is not a root.

**lemma** *int-poly-rat-no-root-ge*:

**fixes**  $p :: \text{real poly}$  **and**  $a \ b :: \text{int}$

**assumes**  $\bigwedge n. \text{coeff } p \ n \in \mathbb{Z}$

**assumes**  $b > 0 \ \text{poly } p \ (a / b) \neq 0$

**defines**  $n \equiv \text{degree } p$

**shows**  $\text{abs } (\text{poly } p \ (a / b)) \geq 1 / \text{of-int } b \wedge n$

**proof** –

**let**  $?S = (\sum i \leq n. \text{coeff } p \ i * \text{of-int } a \wedge i * (\text{of-int } b \wedge (n - i)))$

**from**  $\langle b > 0 \rangle$  **have**  $\text{eq}: ?S = \text{of-int } b \wedge n * \text{poly } p \ (a / b)$

**by** (*simp add: poly-altdef power-divide mult-ac n-def sum-distrib-left power-diff*)

**have**  $?S \in \mathbb{Z}$  **by** (*intro Ints-sum Ints-mult assms Ints-power*) *simp-all*

**moreover from** *assms* **have**  $?S \neq 0$  **by** (*subst eq*) *auto*

**ultimately have**  $\text{abs } ?S \geq 1$  **by** (*elim Ints-cases*) *simp*

**with**  $\text{eq } \langle b > 0 \rangle$  **show** *?thesis* **by** (*simp add: field-simps abs-mult*)

**qed**

**end**

**theory** *Liouville-Numbers*

**imports**

*Complex-Main*

*HOL-Computational-Algebra.Polynomial*

*Liouville-Numbers-Misc*

**begin**

A Liouville number is a real number that can be approximated well – but not perfectly – by a sequence of rational numbers. “Well“, in this context, means that the error of the  $n$ -th rational in the sequence is bounded by the  $n$ -th power of its denominator.

Our approach will be the following:

- Liouville numbers cannot be rational.
- Any irrational algebraic number cannot be approximated in the Liouville sense
- Therefore, all Liouville numbers are transcendental.
- The standard construction fulfils all the properties of Liouville numbers.

## 1.2 Definition of Liouville numbers

The following definitions and proofs are largely adapted from those in the Wikipedia article on Liouville numbers. [1]

A Liouville number is a real number that can be approximated well – but not perfectly – by a sequence of rational numbers. The error of the  $n$ -th term  $\frac{p_n}{q_n}$  is at most  $q_n^{-n}$ , where  $p_n \in \mathbb{Z}$  and  $q_n \in \mathbb{Z}_{\geq 2}$ .

We will say that such a number can be approximated in the Liouville sense.

**locale** *liouville* =

**fixes**  $x :: \text{real}$  **and**  $p\ q :: \text{nat} \Rightarrow \text{int}$

**assumes** *approx-int-pos*:  $\text{abs } (x - p\ n / q\ n) > 0$

**and** *denom-gt-1*:  $q\ n > 1$

**and** *approx-int*:  $\text{abs } (x - p\ n / q\ n) < 1 / \text{of-int } (q\ n) \wedge n$

First, we show that any Liouville number is irrational.

**lemma** (*in liouville*) *irrational*:  $x \notin \mathbb{Q}$

**proof**

**assume**  $x \in \mathbb{Q}$

```

then obtain  $c d :: \text{int}$  where  $d: x = \text{of-int } c / \text{of-int } d \ d > 0$ 
  by (elim Rats-cases') simp
define  $n$  where  $n = \text{Suc } (\text{nat } \lceil \log 2 \ d \rceil)$ 
note  $q\text{-gt-1} = \text{denom-gt-1}[\text{of } n]$ 

have  $\text{neg}: c * q \ n \neq d * p \ n$ 
proof
  assume  $c * q \ n = d * p \ n$ 
  hence  $\text{of-int } (c * q \ n) = \text{of-int } (d * p \ n)$  by (simp only: )
  with  $\text{approx-int-pos}[\text{of } n] \ d \ q\text{-gt-1}$  show False by (auto simp: field-simps)
qed
hence  $\text{abs-pos}: 0 < \text{abs } (c * q \ n - d * p \ n)$  by simp

from  $q\text{-gt-1} \ d$  have  $\text{of-int } d = 2^{\text{powr } \log 2 \ d}$  by (subst powr-log-cancel) simp-all
also from  $d$  have  $\log 2 \ (\text{of-int } d) \geq \log 2 \ 1$  by (subst log-le-cancel-iff) simp-all
hence  $2^{\text{powr } \log 2 \ d} \leq 2^{\text{powr } \text{of-nat } (\text{nat } \lceil \log 2 \ d \rceil)}$ 
  by (intro powr-mono) simp-all
also have  $\dots = \text{of-int } (2^{\text{nat } \lceil \log 2 \ d \rceil})$ 
  by (subst powr-realpow) simp-all
finally have  $d \leq 2^{\text{nat } \lceil \log 2 \ (\text{of-int } d) \rceil}$ 
  by (subst (asm) of-int-le-iff)
also have  $\dots * q \ n \leq q \ n^{\text{Suc } (\text{nat } \lceil \log 2 \ (\text{of-int } d) \rceil)}$ 
  by (subst power-Suc, subst mult.commute, intro mult-left-mono power-mono,
    insert q-gt-1) simp-all
also from  $q\text{-gt-1}$  have  $\dots = q \ n^{\wedge} n$  by (simp add: n-def)
finally have  $1 / \text{of-int } (q \ n^{\wedge} n) \leq 1 / \text{real-of-int } (d * q \ n)$  using  $q\text{-gt-1} \ d$ 
  by (intro divide-left-mono Rings.mult-pos-pos of-int-pos, subst of-int-le-iff)
simp-all
also have  $\dots \leq \text{of-int } (\text{abs } (c * q \ n - d * p \ n)) / \text{real-of-int } (d * q \ n)$  using
 $q\text{-gt-1} \ d \ \text{abs-pos}$ 
  by (intro divide-right-mono) (linarith, simp)
also have  $\dots = \text{abs } (x - \text{of-int } (p \ n) / \text{of-int } (q \ n))$  using  $q\text{-gt-1} \ d(2)$ 
  by (simp-all add: divide-simps d(1) mult-ac)
finally show False using  $\text{approx-int}[\text{of } n]$  by simp
qed

```

Next, any irrational algebraic number cannot be approximated with rational numbers in the Liouville sense.

**lemma** *liouville-irrational-algebraic*:

```

fixes  $x :: \text{real}$ 
assumes irrationsl:  $x \notin \mathbb{Q}$  and algebraic  $x$ 
obtains  $c :: \text{real}$  and  $n :: \text{nat}$ 
  where  $c > 0$  and  $\bigwedge (p::\text{int}) (q::\text{int}). q > 0 \implies \text{abs } (x - p / q) > c / \text{of-int } q^{\wedge} n$ 
proof -
  from  $\langle \text{algebraic } x \rangle$  guess  $p$  by (elim algebraicE) note  $p = \text{this}$ 
  define  $n$  where  $n = \text{degree } p$ 

```

— The derivative of  $p$  is bounded within  $\{x - 1..x + 1\}$ .

**let**  $?f = \lambda t. |poly (pderiv p) t|$   
**define**  $M$  **where**  $M = (SUP\ t \in \{x-1..x+1\}. ?f\ t)$   
**define**  $roots$  **where**  $roots = \{x. poly\ p\ x = 0\} - \{x\}$

**define**  $A$ -set **where**  $A$ -set =  $\{1, 1/M\} \cup \{abs\ (x' - x) \mid x'. x' \in roots\}$   
**define**  $A'$  **where**  $A' = Min\ A$ -set  
**define**  $A$  **where**  $A = A' / 2$

— We define  $A$  to be a positive real number that is less than  $1::'a$ ,  $1 / M$  and any distance from  $x$  to another root of  $p$ .

— Properties of  $M$ , our upper bound for the derivative of  $p$ :

**have**  $\exists s \in \{x-1..x+1\}. \forall y \in \{x-1..x+1\}. ?f\ s \geq ?f\ y$   
**by** (*intro continuous-attains-sup*) (*auto intro!: continuous-intros*)  
**hence**  $bdd$ :  $bdd$ -above ( $?f\ ' \{x-1..x+1\}$ ) **by** *auto*

**have**  $M$ -pos:  $M > 0$

**proof** —

**from**  $p$  **have**  $pderiv\ p \neq 0$  **by** (*auto dest!: pderiv-iszero*)  
**hence**  $finite\ \{x. poly\ (pderiv\ p)\ x = 0\}$  **using** *poly-roots-finite* **by** *blast*  
**moreover** **have**  $\neg finite\ \{x-1..x+1\}$  **by** (*simp add: infinite-Icc*)  
**ultimately** **have**  $\neg finite\ (\{x-1..x+1\} - \{x. poly\ (pderiv\ p)\ x = 0\})$  **by** *simp*  
**hence**  $\{x-1..x+1\} - \{x. poly\ (pderiv\ p)\ x = 0\} \neq \{\}$  **by** (*intro notI*) *simp*  
**then obtain**  $t$  **where**  $t: t \in \{x-1..x+1\}$  **and**  $poly\ (pderiv\ p)\ t \neq 0$  **by** *blast*  
**hence**  $0 < ?f\ t$  **by** *simp*  
**also from**  $t$  **and**  $bdd$  **have**  $\dots \leq M$  **unfolding**  $M$ -def **by** (*rule cSUP-upper*)  
**finally show**  $M > 0$  .

**qed**

**have**  $M$ -sup:  $?f\ t \leq M$  **if**  $t \in \{x-1..x+1\}$  **for**  $t$

**proof** —

**from** *that* **and**  $bdd$  **show**  $?f\ t \leq M$   
**unfolding**  $M$ -def **by** (*rule cSUP-upper*)

**qed**

— Properties of  $A$ :

**from**  $p$  *poly-roots-finite*[*of p*] **have**  $finite\ A$ -set  
**unfolding**  $A$ -set-def  $roots$ -def **by** *simp*  
**have**  $x \notin roots$  **unfolding**  $roots$ -def **by** *auto*  
**hence**  $A' > 0$  **using** *Min-gr-iff*[*OF*  $\langle finite\ A$ -set $\rangle$ , *folded*  $A'$ -def, *of 0*]  
**by** (*auto simp: A-set-def M-pos*)  
**hence**  $A$ -pos:  $A > 0$  **by** (*simp add: A-def*)

**from**  $\langle A' > 0 \rangle$  **have**  $A < A'$  **by** (*simp add: A-def*)

**moreover from** *Min-le*[*OF*  $\langle finite\ A$ -set $\rangle$ , *folded*  $A'$ -def]

**have**  $A' \leq 1\ A' \leq 1/M \wedge x'. x' \neq x \implies poly\ p\ x' = 0 \implies A' \leq abs\ (x' - x)$

**unfolding**  $A$ -set-def  $roots$ -def **by** *auto*

**ultimately have**  $A$ -less:  $A < 1\ A < 1/M \wedge x'. x' \neq x \implies poly\ p\ x' = 0 \implies A < abs\ (x' - x)$

by *fastforce+*

— Finally: no rational number can approximate  $x$  “well enough”.

**have**  $\forall (a::int) (b::int). b > 0 \longrightarrow |x - \text{of-int } a / \text{of-int } b| > A / \text{of-int } b \wedge n$   
**proof** (*intro allI impI, rule ccontr*)

**fix**  $a\ b :: int$

**assume**  $b: b > 0$  **and**  $\neg(A / \text{of-int } b \wedge n < |x - \text{of-int } a / \text{of-int } b|)$

**hence**  $ab: \text{abs } (x - \text{of-int } a / \text{of-int } b) \leq A / \text{of-int } b \wedge n$  **by** *simp*

**also from**  $A\text{-pos } b$  **have**  $A / \text{of-int } b \wedge n \leq A / 1$

**by** (*intro divide-left-mono simp-all*)

**finally have**  $ab': \text{abs } (x - a / b) \leq A$  **by** *simp*

**also have**  $\dots \leq 1$  **using**  $A\text{-less}$  **by** *simp*

**finally have**  $ab'': a / b \in \{x-1..x+1\}$  **by** *auto*

**have**  $\text{no-root}: \text{poly } p (a / b) \neq 0$

**proof**

**assume**  $\text{poly } p (a / b) = 0$

**moreover from**  $\langle x \notin \mathbb{Q} \rangle$  **have**  $x \neq a / b$  **by** *auto*

**ultimately have**  $A < \text{abs } (a / b - x)$  **using**  $A\text{-less}(3)$  **by** *simp*

**with**  $ab'$  **show** *False* **by** *simp*

**qed**

**have**  $\exists x0 \in \{x-1..x+1\}. \text{poly } p (a / b) - \text{poly } p x = (a / b - x) * \text{poly } (pderiv\ p) x0$

**using**  $ab''$  **by** (*intro poly-MVT'*) (*auto simp: min-def max-def*)

**with**  $p$  **obtain**  $x0 :: real$  **where**  $x0:$

$x0 \in \{x-1..x+1\}$   $\text{poly } p (a / b) = (a / b - x) * \text{poly } (pderiv\ p) x0$  **by** *auto*

**from**  $x0(2)$   $\text{no-root}$  **have**  $\text{deriv-pos}: \text{poly } (pderiv\ p) x0 \neq 0$  **by** *auto*

**from**  $b\ p\ \text{no-root}$  **have**  $p\text{-ab}: \text{abs } (\text{poly } p (a / b)) \geq 1 / \text{of-int } b \wedge n$

**unfolding**  $n\text{-def}$  **by** (*intro int-poly-rat-no-root-ge*)

**note**  $ab$

**also from**  $A\text{-less } b$  **have**  $A / \text{of-int } b \wedge n < (1/M) / \text{of-int } b \wedge n$

**by** (*intro divide-strict-right-mono simp-all*)

**also have**  $\dots = (1 / b \wedge n) / M$  **by** *simp*

**also from**  $M\text{-pos } ab\ p\text{-ab}$  **have**  $\dots \leq \text{abs } (\text{poly } p (a / b)) / M$

**by** (*intro divide-right-mono simp-all*)

**also from**  $\text{deriv-pos } M\text{-pos } x0(1)$

**have**  $\dots \leq \text{abs } (\text{poly } p (a / b)) / \text{abs } (\text{poly } (pderiv\ p) x0)$

**by** (*intro divide-left-mono M-sup simp-all*)

**also have**  $|\text{poly } p (a / b)| = |a / b - x| * |\text{poly } (pderiv\ p) x0|$

**by** (*subst x0(2)*) (*simp add: abs-mult*)

**with**  $\text{deriv-pos}$  **have**  $|\text{poly } p (a / b)| / |\text{poly } (pderiv\ p) x0| = |x - a / b|$

**by** (*subst nonzero-divide-eq-eq simp-all*)

**finally show** *False* **by** *simp*

**qed**

**with**  $A$ -pos show ?thesis **using** that[ $of A n$ ] **by** blast  
qed

Since Liouville numbers are irrational, but can be approximated well by rational numbers in the Liouville sense, they must be transcendental.

**lemma** (in *liouville*) *transcendental*:  $\neg$ algebraic  $x$

**proof**

assume algebraic  $x$

from *liouville-irrational-algebraic*[*OF irrational this*] **guess**  $c n$  . **note**  $cn = this$

**define**  $r$  where  $r = nat \lceil \log 2 (1 / c) \rceil$

**define**  $m$  where  $m = n + r$

from  $cn(1)$  **have**  $(1/c) = 2 \text{ powr } \log 2 (1/c)$  **by** (*subst powr-log-cancel*) *simp-all*

**also have**  $\log 2 (1/c) \leq of\text{-nat } (nat \lceil \log 2 (1/c) \rceil)$  **by** *linarith*

**hence**  $2 \text{ powr } \log 2 (1/c) \leq 2 \text{ powr } \dots$  **by** (*intro powr-mono*) *simp-all*

**also have**  $\dots = 2^{\wedge r}$  **unfolding**  $r$ -def **by** (*simp add: powr-realpow*)

**finally have**  $1 / (2^{\wedge r}) \leq c$  **using**  $cn(1)$  **by** (*simp add: field-simps*)

**have**  $abs (x - p m / q m) < 1 / of\text{-int } (q m)^{\wedge m}$  **by** (*rule approx-int*)

**also have**  $\dots = (1 / of\text{-int } (q m)^{\wedge r}) * (1 / real\text{-of-int } (q m)^{\wedge n})$

**by** (*simp add: m-def power-add*)

**also from** *denom-gt-1*[ $of m$ ] **have**  $1 / real\text{-of-int } (q m)^{\wedge r} \leq 1 / 2^{\wedge r}$

**by** (*intro divide-left-mono power-mono*) *simp-all*

**also have**  $\dots \leq c$  **by** *fact*

**finally have**  $abs (x - p m / q m) < c / of\text{-int } (q m)^{\wedge n}$

**using** *denom-gt-1*[ $of m$ ] **by**  $-$  (*simp-all add: divide-right-mono*)

**with**  $cn(2)$ [ $of q m p m$ ] *denom-gt-1*[ $of m$ ] **show** *False* **by** *simp*

qed

### 1.3 Standard construction for Liouville numbers

We now define the standard construction for Liouville numbers.

**definition** *standard-liouville* ::  $(nat \Rightarrow int) \Rightarrow int \Rightarrow real$  **where**

*standard-liouville*  $p q = (\sum k. p k / of\text{-int } q^{\wedge} fact (Suc k))$

**lemma** *standard-liouville-summable*:

**fixes**  $p :: nat \Rightarrow int$  **and**  $q :: int$

**assumes**  $q > 1$  *range*  $p \subseteq \{0..<q\}$

**shows** *summable*  $(\lambda k. p k / of\text{-int } q^{\wedge} fact (Suc k))$

**proof** (*rule summable-comparison-test*)

from *assms*(1) **show** *summable*  $(\lambda n. (1 / q)^{\wedge} n)$

**by** (*intro summable-geometric*) *simp-all*

**next**

**fix**  $n :: nat$

from *assms* **have**  $p n \in \{0..<q\}$  **by** *blast*

**with** *assms*(1) **have**  $norm (p n / of\text{-int } q^{\wedge} fact (Suc n)) \leq$

$q / of\text{-int } q^{\wedge} (fact (Suc n))$  **by** (*auto simp: field-simps*)

**also from** *assms*(1) **have**  $\dots = 1 / of\text{-int } q^{\wedge} (fact (Suc n) - 1)$

by (subst power-diff) (auto simp del: fact-Suc)  
 also have  $\text{Suc } n \leq \text{fact } (\text{Suc } n)$  by (rule fact-ge-self)  
 with  $\text{assms}(1)$  have  $1 / \text{real-of-int } q \wedge (\text{fact } (\text{Suc } n) - 1) \leq 1 / \text{of-int } q \wedge n$   
 by (intro divide-left-mono power-increasing)  
 (auto simp del: fact-Suc simp add: algebra-simps)  
 finally show  $\text{norm } (p \ n / \text{of-int } q \wedge \text{fact } (\text{Suc } n)) \leq (1 / q) \wedge n$   
 by (simp add: power-divide)  
 qed

**lemma** *standard-liouville-sums*:  
 assumes  $q > 1$  range  $p \subseteq \{0..<q\}$   
 shows  $(\lambda k. p \ k / \text{of-int } q \wedge \text{fact } (\text{Suc } k))$  sums *standard-liouville*  $p \ q$   
 using *standard-liouville-summable*[OF *assms*] **unfolding** *standard-liouville-def*  
 by (simp add: *summable-sums*)

Now we prove that the standard construction indeed yields Liouville numbers.

**lemma** *standard-liouville-is-liouville*:  
 assumes  $q > 1$  range  $p \subseteq \{0..<q\}$  frequently  $(\lambda n. p \ n \neq 0)$  sequentially  
 defines  $b \equiv \lambda n. q \wedge \text{fact } (\text{Suc } n)$   
 defines  $a \equiv \lambda n. (\sum k \leq n. p \ k * q \wedge (\text{fact } (\text{Suc } n) - \text{fact } (\text{Suc } k)))$   
 shows *liouville* (*standard-liouville*  $p \ q$ )  $a \ b$

**proof**

fix  $n :: \text{nat}$   
 from  $\text{assms}(1)$  have  $1 < q \wedge 1$  by *simp*  
 also from  $\text{assms}(1)$  have  $\dots \leq b \ n$  **unfolding** *b-def*  
 by (intro *power-increasing*) (*simp-all* del: *fact-Suc*)  
 finally show  $b \ n > 1$ .  
  
**note** *summable* = *standard-liouville-summable*[OF  $\text{assms}(1,2)$ ]  
**let**  $?S = \sum k. p \ (k + n + 1) / \text{of-int } q \wedge (\text{fact } (\text{Suc } (k + n + 1)))$   
**let**  $?C = (q - 1) / \text{of-int } q \wedge (\text{fact } (n+2))$   
  
**have**  $a \ n / b \ n = (\sum k \leq n. p \ k * (\text{of-int } q \wedge (\text{fact } (n+1) - \text{fact } (k+1)) / \text{of-int } q \wedge (\text{fact } (n+1))))$   
 by (*simp* add: *a-def* *b-def* *sum-divide-distrib* *of-int-sum*)  
**also have**  $\dots = (\sum k \leq n. p \ k / \text{of-int } q \wedge (\text{fact } (\text{Suc } k)))$   
 by (intro *sum.cong* *reft*, *subst* *inverse-divide* [*symmetric*], *subst* *power-diff* [*symmetric*])  
 (*insert*  $\text{assms}(1)$ , *simp-all* add: *divide-simps* *fact-mono-nat* del: *fact-Suc*)  
**also have** *standard-liouville*  $p \ q - \dots = ?S$  **unfolding** *standard-liouville-def*  
 by (*subst* *diff-eq-eq*) (*intro* *suminf-split-initial-segment'* *summable*)  
**finally have**  $\text{abs } (\text{standard-liouville } p \ q - a \ n / b \ n) = \text{abs } ?S$  by (*simp* only: )  
**moreover from**  $\text{assms}$  **have**  $?S \geq 0$   
 by (*intro* *suminf-nonneg* *allI* *divide-nonneg-pos* *summable-ignore-initial-segment* *summable*) *force*+  
**ultimately have**  $\text{eq: } \text{abs } (\text{standard-liouville } p \ q - a \ n / b \ n) = ?S$  by *simp*  
  
**also have**  $?S \leq (\sum k. ?C * (1 / q) \wedge k)$



**proof** (*intro suminf-le allI summable-ignore-initial-segment summable*)  
**from** *assms* **show**  $\text{summable } (\lambda k. ?C * (1 / q) ^ k)$   
**by** (*intro summable-mult summable-geometric*) *simp-all*  
**next**  
**fix**  $k :: \text{nat}$   
**from** *assms* **have**  $p (k + n + 1) \leq q - 1$  **by** *force*  
**with**  $\langle q > 1 \rangle$  **have**  $p (k + n + 1) / \text{of-int } q ^ \text{fact } (Suc (k + n + 1)) \leq$   
 $(q - 1) / \text{of-int } q ^ (\text{fact } (Suc (k + n + 1)))$   
**by** (*intro divide-right-mono*) (*simp-all del: fact-Suc*)  
**also from**  $\langle q > 1 \rangle$  **have**  $\dots \leq (q - 1) / \text{of-int } q ^ (\text{fact } (n+2) + k)$   
**using** *fact-ineq*[*of n+2 k*]  
**by** (*intro divide-left-mono power-increasing*) (*simp-all add: add-ac*)  
**also have**  $\dots = ?C * (1 / q) ^ k$   
**by** (*simp add: field-simps power-add del: fact-Suc*)  
**finally show**  $p (k + n + 1) / \text{of-int } q ^ \text{fact } (Suc (k + n + 1)) \leq \dots$   
**qed**  
**also from** *assms* **have**  $\dots = ?C * (\sum k. (1 / q) ^ k)$   
**by** (*intro suminf-mult summable-geometric*) *simp-all*  
**also from** *assms* **have**  $(\sum k. (1 / q) ^ k) = 1 / (1 - 1 / q)$   
**by** (*intro suminf-geometric*) *simp-all*  
**also from** *assms*(1) **have**  $?C * \dots = \text{of-int } q ^ 1 / \text{of-int } q ^ \text{fact } (n + 2)$   
**by** (*simp add: field-simps*)  
**also from** *assms*(1) **have**  $\dots \leq \text{of-int } q ^ \text{fact } (n + 1) / \text{of-int } q ^ \text{fact } (n + 2)$   
**by** (*intro divide-right-mono power-increasing*) (*simp-all add: field-simps del: fact-Suc*)  
**also from** *assms*(1) **have**  $\dots = 1 / (\text{of-int } q ^ (\text{fact } (n + 2) - \text{fact } (n + 1)))$   
**by** (*subst power-diff*) *simp-all*  
**also have**  $\text{fact } (n + 2) - \text{fact } (n + 1) = (n + 1) * \text{fact } (n + 1)$  **by** (*simp add: algebra-simps*)  
**also from** *assms*(1) **have**  $1 / (\text{of-int } q ^ \dots) < 1 / (\text{real-of-int } q ^ (\text{fact } (n + 1) * n))$   
**by** (*intro divide-strict-left-mono power-increasing mult-right-mono*) *simp-all*  
**also have**  $\dots = 1 / \text{of-int } (b n) ^ n$   
**by** (*simp add: power-mult b-def power-divide del: fact-Suc*)  
**finally show**  $|\text{standard-liouville } p q - a n / b n| < 1 / \text{of-int } (b n) ^ n$ .  
  
**from** *assms*(3) **obtain**  $k$  **where**  $k: k \geq n + 1 \ p \ k \neq 0$   
**by** (*auto simp: frequently-def eventually-at-top-linorder*)  
**define**  $k'$  **where**  $k' = k - n - 1$   
**from** *assms*  $k$  **have**  $p \ k \geq 0$  **by** *force*  
**with**  $k$  *assms* **have**  $k': p (k' + n + 1) > 0$  **unfolding**  $k'$ -*def* **by** *force*  
**with** *assms*(1,2) **have**  $?S > 0$   
**by** (*intro suminf-pos2*[*of - k'*] *summable-ignore-initial-segment summable*)  
(*force intro!: divide-pos-pos divide-nonneg-pos*)  
**with** *eq* **show**  $|\text{standard-liouville } p q - a n / b n| > 0$  **by** (*simp only:*)  
**qed**

We can now show our main result: any standard Liouville number is transcendental.

**theorem** *transcendental-standard-liouville*:  
**assumes**  $q > 1$  range  $p \subseteq \{0..<q\}$  frequently  $(\lambda k. p k \neq 0)$  sequentially  
**shows**  $\neg$ algebraic (standard-liouville  $p$   $q$ )  
**proof** –  
**from** *assms* **interpret**  
*liouville standard-liouville*  $p$   $q$   
 $\lambda n. (\sum k \leq n. p k * q ^ \wedge (\text{fact } (\text{Suc } n) - \text{fact } (\text{Suc } k)))$   
 $\lambda n. q ^ \wedge \text{fact } (\text{Suc } n)$   
**by** (*rule standard-liouville-is-liouville*)  
**from** *transcendental* **show** ?thesis .  
**qed**

In particular: The the standard construction for constant sequences, such as the “classic” Liouville constant  $\sum_{n=1}^{\infty} 10^{-n!} = 0.11000100\dots$ , are transcendental.

This shows that Liouville numbers exists and therefore gives a concrete and elementary proof that transcendental numbers exist.

**corollary** *transcendental-standard-standard-liouville*:  
 $a \in \{0 < .. < b\} \implies \neg$ algebraic (standard-liouville  $(\lambda-. \text{int } a)$   $(\text{int } b)$ )  
**by** (*intro transcendental-standard-liouville*) *auto*

**corollary** *transcendental-liouville-constant*:  
 $\neg$ algebraic (standard-liouville  $(\lambda-. 1)$  10)  
**by** (*intro transcendental-standard-liouville*) *auto*

**end**

## References

- [1] Wikipedia. Liouville number — Wikipedia, the free encyclopedia. [https://en.wikipedia.org/w/index.php?title=Liouville\\_number&oldid=696910651](https://en.wikipedia.org/w/index.php?title=Liouville_number&oldid=696910651), 2015. [Online; accessed 22-July-2004].