# Liouville Numbers

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**Abstract**

In this work, we define the concept of Liouville numbers as well as the standard construction to obtain Liouville numbers and we prove their most important properties: irrationality and transcendence.

This is historically interesting since Liouville numbers constructed in the standard way where the first numbers that were proven to be transcendental. The proof is very elementary and requires only standard arithmetic and the Mean Value Theorem for polynomials and the boundedness of polynomials on compact intervals.

## Contents



# 1 Liouville Numbers

## 1.1 Preliminary lemmas

**theory** *Liouville-Numbers-Misc*
**imports**
  *Complex-Main*
  *HOL−Computational-Algebra.Polynomial*
**begin**

We will require these inequalities on factorials to show properties of the standard construction later.

**lemma** *fact-ineq*: $n \geq 1 \Longrightarrow fact\ n + k \leq fact\ (n + k)$
**proof** (*induction k*)
  **case** (*Suc k*)
  **from** *Suc* **have** $fact\ n + Suc\ k \leq fact\ (n + k) + 1$ **by** *simp*
  **also from** *Suc* **have** $\ldots \leq fact\ (n + Suc\ k)$ **by** *simp*
  **finally show** *?case* **.**

**qed** *simp-all*

**lemma** *Ints-sum*:
  **assumes** $\bigwedge x.\ x \in A \Longrightarrow f\ x \in \mathbb{Z}$
  **shows** $sum\ f\ A \in \mathbb{Z}$
  **by** (*cases finite A*, *insert assms*, *induction A rule*: *finite-induct*)
    (*auto intro*!: *Ints-add*)

**lemma** *suminf-split-initial-segment'*:
  $summable\ (f :: nat \Rightarrow 'a::real\text{-}normed\text{-}vector) \Longrightarrow$
    $suminf\ f = (\sum n.\ f\ (n + k + 1)) + sum\ f\ \{..k\}$
 **by** (*subst suminf-split-initial-segment*[*of - Suc k*], *assumption*, *subst lessThan-Suc-atMost*)

    *simp-all*

**lemma** *Rats-eq-int-div-int'*: $(\mathbb{Q} :: real\ set) = \{of\text{-}int\ p\ /\ of\text{-}int\ q\ |p\ q.\ q > 0\}$
**proof** *safe*
  **fix** $x :: real$ **assume** $x \in \mathbb{Q}$
  **then obtain** *p q* **where** *pq*: $x = of\text{-}int\ p\ /\ of\text{-}int\ q\ q \neq 0$
    **by** (*subst* (*asm*) *Rats-eq-int-div-int*) *auto*
  **show** $\exists\, p\ q.\ x = real\text{-}of\text{-}int\ p\ /\ real\text{-}of\text{-}int\ q \wedge 0 < q$
  **proof** (*cases* $q > 0$)
    **case** *False*
    **show** *?thesis* **by** (*rule exI*[*of* - $-p$], *rule exI*[*of* - $-q$]) (*insert False pq*, *auto*)
  **qed** (*insert pq*, *force*)
**qed** *auto*

**lemma** *Rats-cases'*:
  **assumes** $(x :: real) \in \mathbb{Q}$
  **obtains** *p q* **where** $q > 0\ x = of\text{-}int\ p\ /\ of\text{-}int\ q$
  **using** *assms* **by** (*subst* (*asm*) *Rats-eq-int-div-int'*) *auto*

The following inequality gives a lower bound for the absolute value of an integer polynomial at a rational point that is not a root.

**lemma** *int-poly-rat-no-root-ge*:
  **fixes** $p :: real\ poly$ **and** $a\ b :: int$
  **assumes** $\bigwedge n.\ coeff\ p\ n \in \mathbb{Z}$
  **assumes** $b > 0\ poly\ p\ (a\ /\ b) \neq 0$
  **defines** $n \equiv degree\ p$
  **shows** $abs\ (poly\ p\ (a\ /\ b)) \geq 1\ /\ of\text{-}int\ b\ \hat{}\ n$
**proof** −
  **let** $?S = (\sum i{\leq}n.\ coeff\ p\ i * of\text{-}int\ a\ \hat{}\ i * (of\text{-}int\ b\ \hat{}\ (n - i)))$
  **from** ‹$b > 0$› **have** *eq*: $?S = of\text{-}int\ b\ \hat{}\ n * poly\ p\ (a\ /\ b)$
   **by** (*simp add*: *poly-altdef power-divide mult-ac n-def sum-distrib-left power-diff*)
  **have** $?S \in \mathbb{Z}$ **by** (*intro Ints-sum Ints-mult assms Ints-power*) *simp-all*
  **moreover from** *assms* **have** $?S \neq 0$ **by** (*subst eq*) *auto*
  **ultimately have** $abs\ ?S \geq 1$ **by** (*elim Ints-cases*) *simp*
  **with** *eq* ‹$b > 0$› **show** *?thesis* **by** (*simp add*: *field-simps abs-mult*)
**qed**

**end**

**theory** *Liouville-Numbers*
**imports**
  *Complex-Main*
  *HOL−Computational-Algebra.Polynomial*
  *Liouville-Numbers-Misc*
**begin**

A Liouville number is a real number that can be approximated well – but not perfectly – by a sequence of rational numbers. "Well", in this context, means that the error of the $n$-th rational in the sequence is bounded by the $n$-th power of its denominator.

Our approach will be the following:

- Liouville numbers cannot be rational.
- Any irrational algebraic number cannot be approximated in the Liouville sense
- Therefore, all Liouville numbers are transcendental.
- The standard construction fulfils all the properties of Liouville numbers.

## 1.2 Definition of Liouville numbers

The following definitions and proofs are largely adapted from those in the Wikipedia article on Liouville numbers. [1]

A Liouville number is a real number that can be approximated well – but not perfectly – by a sequence of rational numbers. The error of the $n$-th term $\frac{p_n}{q_n}$ is at most $q_n^{-n}$, where $p_n \in \mathbb{Z}$ and $q_n \in \mathbb{Z}_{\geq 2}$.

We will say that such a number can be approximated in the Liouville sense.

**locale** *liouville* =
  **fixes** $x$ :: *real* **and** $p$ $q$ :: *nat* ⇒ *int*
  **assumes** *approx-int-pos*: *abs* $(x - p\ n\ /\ q\ n) > 0$
    **and** *denom-gt-1*: $q\ n > 1$
    **and** *approx-int*: *abs* $(x - p\ n\ /\ q\ n) < 1\ /$ *of-int* $(q\ n)$ ^ $n$

First, we show that any Liouville number is irrational.

**lemma** (**in** *liouville*) *irrational*: $x \notin \mathbb{Q}$
**proof**
  **assume** $x \in \mathbb{Q}$

**then obtain** *c d* :: *int* **where** *d*: $x = \textit{of-int}\ c\ /\ \textit{of-int}\ d$ $d > 0$
  **by** (*elim Rats-cases'*) *simp*
**define** *n* **where** $n = \textit{Suc}\ (\textit{nat}\ \lceil \textit{log}\ 2\ d \rceil)$
**note** *q-gt-1* = *denom-gt-1*[*of n*]

**have** *neq*: $c * q\ n \neq d * p\ n$
**proof**
  **assume** $c * q\ n = d * p\ n$
  **hence** $\textit{of-int}\ (c * q\ n) = \textit{of-int}\ (d * p\ n)$ **by** (*simp only*: )
  **with** *approx-int-pos*[*of n*] *d q-gt-1* **show** *False* **by** (*auto simp*: *field-simps*)
**qed**
**hence** *abs-pos*: $0 < \textit{abs}\ (c * q\ n - d * p\ n)$ **by** *simp*

**from** *q-gt-1 d* **have** $\textit{of-int}\ d = 2\ \textit{powr}\ \textit{log}\ 2\ d$ **by** (*subst powr-log-cancel*) *simp-all*
**also from** *d* **have** $\textit{log}\ 2\ (\textit{of-int}\ d) \geq \textit{log}\ 2\ 1$ **by** (*subst log-le-cancel-iff*) *simp-all*
**hence** $2\ \textit{powr}\ \textit{log}\ 2\ d \leq 2\ \textit{powr}\ \textit{of-nat}\ (\textit{nat}\ \lceil \textit{log}\ 2\ d \rceil)$
  **by** (*intro powr-mono*) *simp-all*
**also have** $\ldots = \textit{of-int}\ (2\ \widehat{}\ \textit{nat}\ \lceil \textit{log}\ 2\ d \rceil)$
  **by** (*subst powr-realpow*) *simp-all*
**finally have** $d \leq 2\ \widehat{}\ \textit{nat}\ \lceil \textit{log}\ 2\ (\textit{of-int}\ d) \rceil$
  **by** (*subst* (*asm*) *of-int-le-iff*)
**also have** $\ldots * q\ n \leq q\ n\ \widehat{}\ \textit{Suc}\ (\textit{nat}\ \lceil \textit{log}\ 2\ (\textit{of-int}\ d) \rceil)$
  **by** (*subst power-Suc*, *subst mult.commute*, *intro mult-left-mono power-mono*,
    *insert q-gt-1*) *simp-all*
**also from** *q-gt-1* **have** $\ldots = q\ n\ \widehat{}\ n$ **by** (*simp add*: *n-def*)
**finally have** $1\ /\ \textit{of-int}\ (q\ n\ \widehat{}\ n) \leq 1\ /\ \textit{real-of-int}\ (d * q\ n)$ **using** *q-gt-1 d*
  **by** (*intro divide-left-mono Rings.mult-pos-pos of-int-pos*, *subst of-int-le-iff*) *simp-all*
**also have** $\ldots \leq \textit{of-int}\ (\textit{abs}\ (c * q\ n - d * p\ n))\ /\ \textit{real-of-int}\ (d * q\ n)$ **using** *q-gt-1 d abs-pos*
  **by** (*intro divide-right-mono*) (*linarith*, *simp*)
**also have** $\ldots = \textit{abs}\ (x - \textit{of-int}\ (p\ n)\ /\ \textit{of-int}\ (q\ n))$ **using** *q-gt-1 d*(*2*)
  **by** (*simp-all add*: *divide-simps d*(*1*) *mult-ac*)
**finally show** *False* **using** *approx-int*[*of n*] **by** *simp*
**qed**

Next, any irrational algebraic number cannot be approximated with rational numbers in the Liouville sense.

**lemma** *liouville-irrational-algebraic*:
**fixes** *x* :: *real*
**assumes** *irrationsl*: $x \notin \mathbb{Q}$ **and** *algebraic x*
**obtains** *c* :: *real* **and** *n* :: *nat*
  **where** $c > 0$ **and** $\bigwedge(p::\textit{int})\ (q::\textit{int}).\ q > 0 \Longrightarrow \textit{abs}\ (x - p\ /\ q) > c\ /\ \textit{of-int}\ q\ \widehat{}\ n$
**proof** −
**from** ‹*algebraic x*› **obtain** *p* **where** *p*: $\bigwedge i.\ \textit{coeff}\ p\ i \in \mathbb{Z}$ $p \neq 0$ $\textit{poly}\ p\ x = 0$
  **by** (*elim algebraicE*) *blast*
**define** *n* **where** $n = \textit{degree}\ p$

— The derivative of $p$ is bounded within $\{x - 1..x + 1\}$.
**let** $?f = \lambda t.\ |poly\ (pderiv\ p)\ t|$
**define** $M$ **where** $M = (SUP\ t{\in}\{x{-}1..x{+}1\}.\ ?f\ t)$
**define** *roots* **where** $roots = \{x.\ poly\ p\ x = 0\} - \{x\}$

**define** *A-set* **where** $A\text{-}set = \{1,\ 1/M\} \cup \{abs\ (x' - x)\ |x'.\ x' \in roots\}$
**define** $A'$ **where** $A' = Min\ A\text{-}set$
**define** $A$ **where** $A = A'\ /\ 2$
— We define $A$ to be a positive real number that is less than $1$, $1\ /\ M$ and any distance from $x$ to another root of $p$.

— Properties of $M$, our upper bound for the derivative of $p$:
**have** $\exists\, s{\in}\{x{-}1..x{+}1\}.\ \forall\, y{\in}\{x{-}1..x{+}1\}.\ ?f\ s \geq ?f\ y$
  **by** (*intro continuous-attains-sup*) (*auto intro*!: *continuous-intros*)
**hence** *bdd*: *bdd-above* $(?f\ `\ \{x{-}1..x{+}1\})$ **by** *auto*

**have** *M-pos*: $M > 0$
**proof** −
  **from** $p$ **have** $pderiv\ p \neq 0$ **by** (*auto dest*!: *pderiv-iszero*)
  **hence** *finite* $\{x.\ poly\ (pderiv\ p)\ x = 0\}$ **using** *poly-roots-finite* **by** *blast*
  **moreover have** $\neg finite\ \{x{-}1..x{+}1\}$ **by** (*simp add*: *infinite-Icc*)
  **ultimately have** $\neg finite\ (\{x{-}1..x{+}1\} - \{x.\ poly\ (pderiv\ p)\ x = 0\})$ **by** *simp*
  **hence** $\{x{-}1..x{+}1\} - \{x.\ poly\ (pderiv\ p)\ x = 0\} \neq \{\}$ **by** (*intro notI*) *simp*
  **then obtain** $t$ **where** $t$: $t \in \{x{-}1..x{+}1\}$ **and** $poly\ (pderiv\ p)\ t \neq 0$ **by** *blast*
  **hence** $0 < ?f\ t$ **by** *simp*
  **also from** $t$ **and** *bdd* **have** $\ldots \leq M$ **unfolding** *M-def* **by** (*rule cSUP-upper*)
  **finally show** $M > 0$ **.**
**qed**

**have** *M-sup*: $?f\ t \leq M$ **if** $t \in \{x{-}1..x{+}1\}$ **for** $t$
**proof** −
  **from** *that* **and** *bdd* **show** $?f\ t \leq M$
    **unfolding** *M-def* **by** (*rule cSUP-upper*)
**qed**

— Properties of $A$:
**from** $p$ *poly-roots-finite*[*of p*] **have** *finite A-set*
  **unfolding** *A-set-def roots-def* **by** *simp*
**have** $x \notin roots$ **unfolding** *roots-def* **by** *auto*
**hence** $A' > 0$ **using** *Min-gr-iff*[*OF* ‹*finite A-set*›, *folded A'-def*, *of 0*]
  **by** (*auto simp*: *A-set-def M-pos*)
**hence** *A-pos*: $A > 0$ **by** (*simp add*: *A-def*)

**from** ‹$A' > 0$› **have** $A < A'$ **by** (*simp add*: *A-def*)
**moreover from** *Min-le*[*OF* ‹*finite A-set*›, *folded A'-def*]
  **have** $A' \leq 1$ $A' \leq 1/M$ $\bigwedge x'.\ x' \neq x \Longrightarrow poly\ p\ x' = 0 \Longrightarrow A' \leq abs\ (x' - x)$
  **unfolding** *A-set-def roots-def* **by** *auto*
**ultimately have** *A-less*: $A < 1$ $A < 1/M$ $\bigwedge x'.\ x' \neq x \Longrightarrow poly\ p\ x' = 0 \Longrightarrow A$

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< abs (x′ − x)
    by fastforce+


  — Finally: no rational number can approximate x “well enough”.
  have ∀(a::int) (b :: int). b > 0 ⟶ |x − of-int a / of-int b| > A / of-int b ^ n
  proof (intro allI impI, rule ccontr)
    fix a b :: int
    assume b: b > 0 and ¬(A / of-int b ^ n < |x − of-int a / of-int b|)
    hence ab: abs (x − of-int a / of-int b) ≤ A / of-int b ^ n by simp
    also from A-pos b have A / of-int b ^ n ≤ A / 1
      by (intro divide-left-mono) simp-all
    finally have ab′: abs (x − a / b) ≤ A by simp
    also have ... ≤ 1 using A-less by simp
    finally have ab′′: a / b ∈ {x−1..x+1} by auto

    have no-root: poly p (a / b) ≠ 0
    proof
      assume poly p (a / b) = 0
      moreover from ‹x ∉ ℚ› have x ≠ a / b by auto
      ultimately have A < abs (a / b − x) using A-less(3) by simp
      with ab′ show False by simp
    qed

    have ∃x0∈{x−1..x+1}. poly p (a / b) − poly p x = (a / b − x) ∗ poly (pderiv
p) x0
      using ab′′ by (intro poly-MVT′) (auto simp: min-def max-def)
    with p obtain x0 :: real where x0:
        x0 ∈ {x−1..x+1} poly p (a / b) = (a / b − x) ∗ poly (pderiv p) x0 by auto

    from x0(2) no-root have deriv-pos: poly (pderiv p) x0 ≠ 0 by auto

    from b p no-root have p-ab: abs (poly p (a / b)) ≥ 1 / of-int b ^ n
      unfolding n-def by (intro int-poly-rat-no-root-ge)

    note ab
    also from A-less b have A / of-int b ^ n < (1/M) / of-int b ^ n
      by (intro divide-strict-right-mono) simp-all
    also have ... = (1 / b ^ n) / M by simp
    also from M-pos ab p-ab have ... ≤ abs (poly p (a / b)) / M
      by (intro divide-right-mono) simp-all
    also from deriv-pos M-pos x0(1)
      have ... ≤ abs (poly p (a / b)) / abs (poly (pderiv p) x0)
      by (intro divide-left-mono M-sup) simp-all
    also have |poly p (a / b)| = |a / b − x| ∗ |poly (pderiv p) x0|
      by (subst x0(2)) (simp add: abs-mult)
    with deriv-pos have |poly p (a / b)| / |poly (pderiv p) x0| = |x − a / b|
      by (subst nonzero-divide-eq-eq) simp-all
    finally show False by simp
```

**qed**
**with** *A-pos* **show** *?thesis* **using** *that*[*of A n*] **by** *blast*
**qed**

Since Liouville numbers are irrational, but can be approximated well by rational numbers in the Liouville sense, they must be transcendental.

**lemma** (**in** *liouville*) *transcendental*: ¬*algebraic x*
**proof**
**assume** *algebraic x*
**from** *liouville-irrational-algebraic*[*OF irrational this*]
**obtain** *c n* **where** *cn*:
$c > 0$ $\bigwedge p\ q.\ q > 0 \Longrightarrow c\ /\ \textit{real-of-int}\ q\ \hat{}\ n < |x - \textit{real-of-int}\ p\ /\ \textit{real-of-int}\ q|$
**by** *auto*

**define** *r* **where** $r = \textit{nat}\ \lceil \textit{log}\ 2\ (1\ /\ c)\rceil$
**define** *m* **where** $m = n + r$
**from** *cn*(*1*) **have** $(1/c) = 2\ \textit{powr}\ \textit{log}\ 2\ (1/c)$ **by** (*subst powr-log-cancel*) *simp-all*
**also have** $\textit{log}\ 2\ (1/c) \leq \textit{of-nat}\ (\textit{nat}\ \lceil \textit{log}\ 2\ (1/c)\rceil)$ **by** *linarith*
**hence** $2\ \textit{powr}\ \textit{log}\ 2\ (1/c) \leq 2\ \textit{powr}\ \ldots$ **by** (*intro powr-mono*) *simp-all*
**also have** $\ldots = 2\ \hat{}\ r$ **unfolding** *r-def* **by** (*simp add*: *powr-realpow*)
**finally have** $1\ /\ (2\hat{}r) \leq c$ **using** *cn*(*1*) **by** (*simp add*: *field-simps*)

**have** $\textit{abs}\ (x - p\ m\ /\ q\ m) < 1\ /\ \textit{of-int}\ (q\ m)\ \hat{}\ m$ **by** (*rule approx-int*)
**also have** $\ldots = (1\ /\ \textit{of-int}\ (q\ m)\ \hat{}\ r) * (1\ /\ \textit{real-of-int}\ (q\ m)\ \hat{}\ n)$
**by** (*simp add*: *m-def power-add*)
**also from** *denom-gt-1*[*of m*] **have** $1\ /\ \textit{real-of-int}\ (q\ m)\ \hat{}\ r \leq 1\ /\ 2\ \hat{}\ r$
**by** (*intro divide-left-mono power-mono*) *simp-all*
**also have** $\ldots \leq c$ **by** *fact*
**finally have** $\textit{abs}\ (x - p\ m\ /\ q\ m) < c\ /\ \textit{of-int}\ (q\ m)\ \hat{}\ n$
**using** *denom-gt-1*[*of m*] **by** − (*simp-all add*: *divide-right-mono*)
**with** *cn*(*2*)[*of q m p m*] *denom-gt-1*[*of m*] **show** *False* **by** *simp*
**qed**

## 1.3 Standard construction for Liouville numbers

We now define the standard construction for Liouville numbers.

**definition** *standard-liouville* :: (*nat* ⇒ *int*) ⇒ *int* ⇒ *real* **where**
$\textit{standard-liouville}\ p\ q = (\sum k.\ p\ k\ /\ \textit{of-int}\ q\ \hat{}\ \textit{fact}\ (\textit{Suc}\ k))$

**lemma** *standard-liouville-summable*:
**fixes** *p* :: *nat* ⇒ *int* **and** *q* :: *int*
**assumes** $q > 1$ *range p* ⊆ {*0*..<*q*}
**shows** $\textit{summable}\ (\lambda k.\ p\ k\ /\ \textit{of-int}\ q\ \hat{}\ \textit{fact}\ (\textit{Suc}\ k))$
**proof** (*rule summable-comparison-test'*)
**from** *assms*(*1*) **show** $\textit{summable}\ (\lambda n.\ (1\ /\ q)\ \hat{}\ n)$
**by** (*intro summable-geometric*) *simp-all*
**next**
**fix** *n* :: *nat*

**from** *assms* **have** $p\ n \in \{0..<q\}$ **by** *blast*
**with** *assms*(*1*) **have** *norm* $(p\ n\ /\ \textit{of-int}\ q\ \hat{}\ \textit{fact}\ (\textit{Suc}\ n)) \leq$
$q\ /\ \textit{of-int}\ q\ \hat{}\ (\textit{fact}\ (\textit{Suc}\ n))$ **by** (*auto simp*: *field-simps*)
**also from** *assms*(*1*) **have** $\ldots = 1\ /\ \textit{of-int}\ q\ \hat{}\ (\textit{fact}\ (\textit{Suc}\ n) - 1)$
**by** (*subst power-diff*) (*auto simp del*: *fact-Suc*)
**also have** $\textit{Suc}\ n \leq \textit{fact}\ (\textit{Suc}\ n)$ **by** (*rule fact-ge-self*)
**with** *assms*(*1*) **have** $1\ /\ \textit{real-of-int}\ q\ \hat{}\ (\textit{fact}\ (\textit{Suc}\ n) - 1) \leq 1\ /\ \textit{of-int}\ q\ \hat{}\ n$
**by** (*intro divide-left-mono power-increasing*)
(*auto simp del*: *fact-Suc simp add*: *algebra-simps*)
**finally show** *norm* $(p\ n\ /\ \textit{of-int}\ q\ \hat{}\ \textit{fact}\ (\textit{Suc}\ n)) \leq (1\ /\ q)\ \hat{}\ n$
**by** (*simp add*: *power-divide*)
**qed**

**lemma** *standard-liouville-sums*:
**assumes** $q > 1$ *range* $p \subseteq \{0..<q\}$
**shows** $(\lambda k.\ p\ k\ /\ \textit{of-int}\ q\ \hat{}\ \textit{fact}\ (\textit{Suc}\ k))$ *sums standard-liouville p q*
**using** *standard-liouville-summable*[*OF assms*] **unfolding** *standard-liouville-def*
**by** (*simp add*: *summable-sums*)

Now we prove that the standard construction indeed yields Liouville numbers.

**lemma** *standard-liouville-is-liouville*:
**assumes** $q > 1$ *range* $p \subseteq \{0..<q\}$ *frequently* $(\lambda n.\ p\ n \neq 0)$ *sequentially*
**defines** $b \equiv \lambda n.\ q\ \hat{}\ \textit{fact}\ (\textit{Suc}\ n)$
**defines** $a \equiv \lambda n.\ (\sum k{\leq}n.\ p\ k * q\ \hat{}\ (\textit{fact}\ (\textit{Suc}\ n) - \textit{fact}\ (\textit{Suc}\ k)))$
**shows** *liouville* (*standard-liouville p q*) *a b*
**proof**
**fix** $n :: \textit{nat}$
**from** *assms*(*1*) **have** $1 < q\ \hat{}\ 1$ **by** *simp*
**also from** *assms*(*1*) **have** $\ldots \leq b\ n$ **unfolding** *b-def*
**by** (*intro power-increasing*) (*simp-all del*: *fact-Suc*)
**finally show** $b\ n > 1$ **.**

**note** *summable* = *standard-liouville-summable*[*OF assms*(*1*,*2*)]
**let** $?S = \sum k.\ p\ (k + n + 1)\ /\ \textit{of-int}\ q\ \hat{}\ (\textit{fact}\ (\textit{Suc}\ (k + n + 1)))$
**let** $?C = (q - 1)\ /\ \textit{of-int}\ q\ \hat{}\ (\textit{fact}\ (n{+}2))$

**have** $a\ n\ /\ b\ n = (\sum k{\leq}n.\ p\ k * (\textit{of-int}\ q\ \hat{}\ (\textit{fact}\ (n{+}1) - \textit{fact}\ (k{+}1))\ /\ \textit{of-int}$
$q\ \hat{}\ (\textit{fact}\ (n{+}1))))$
**by** (*simp add*: *a-def b-def sum-divide-distrib of-int-sum*)
**also have** $\ldots = (\sum k{\leq}n.\ p\ k\ /\ \textit{of-int}\ q\ \hat{}\ (\textit{fact}\ (\textit{Suc}\ k)))$
**by** (*intro sum.cong refl*, *subst inverse-divide* [*symmetric*], *subst power-diff* [*symmetric*])
(*insert assms*(*1*), *simp-all add*: *divide-simps fact-mono-nat del*: *fact-Suc*)
**also have** *standard-liouville p q* $- \ldots = ?S$ **unfolding** *standard-liouville-def*
**by** (*subst diff-eq-eq*) (*intro suminf-split-initial-segment′ summable*)
**finally have** *abs* (*standard-liouville p q* $- a\ n\ /\ b\ n$) = *abs* $?S$ **by** (*simp only*: )
**moreover from** *assms* **have** $?S \geq 0$
**by** (*intro suminf-nonneg allI divide-nonneg-pos summable-ignore-initial-segment*

```
summable) force+
  ultimately have eq: abs (standard-liouville p q − a n / b n) = ?S by simp

  also have ?S ≤ (∑k. ?C ∗ (1 / q) ^ k)
  proof (intro suminf-le allI summable-ignore-initial-segment summable)
    from assms show summable (λk. ?C ∗ (1 / q) ^ k)
      by (intro summable-mult summable-geometric) simp-all
  next
    fix k :: nat
    from assms have p (k + n + 1) ≤ q − 1 by force
    with ‹q > 1› have p (k + n + 1) / of-int q ^ fact (Suc (k + n + 1)) ≤
                    (q − 1) / of-int q ^ (fact (Suc (k + n + 1)))
      by (intro divide-right-mono) (simp-all del: fact-Suc)
    also from ‹q > 1› have ... ≤ (q − 1) / of-int q ^ (fact (n+2) + k)
      using fact-ineq[of n+2 k]
      by (intro divide-left-mono power-increasing) (simp-all add: add-ac)
    also have ... = ?C ∗ (1 / q) ^ k
      by (simp add: field-simps power-add del: fact-Suc)
    finally show p (k + n + 1) / of-int q ^ fact (Suc (k + n + 1)) ≤ ... .
  qed
  also from assms have ... = ?C ∗ (∑k. (1 / q) ^ k)
    by (intro suminf-mult summable-geometric) simp-all
  also from assms have (∑k. (1 / q) ^ k) = 1 / (1 − 1 / q)
    by (intro suminf-geometric) simp-all
  also from assms(1) have ?C ∗ ... = of-int q ^ 1 / of-int q ^ fact (n + 2)
    by (simp add: field-simps)
  also from assms(1) have ... ≤ of-int q ^ fact (n + 1) / of-int q ^ fact (n + 2)
    by (intro divide-right-mono power-increasing) (simp-all add: field-simps del:
fact-Suc)
  also from assms(1) have ... = 1 / (of-int q ^ (fact (n + 2) − fact (n + 1)))
    by (subst power-diff) simp-all
  also have fact (n + 2) − fact (n + 1) = (n + 1) ∗ fact (n + 1) by (simp add:
algebra-simps)
  also from assms(1) have 1 / (of-int q ^ ...) < 1 / (real-of-int q ^ (fact (n +
1) ∗ n))
    by (intro divide-strict-left-mono power-increasing mult-right-mono) simp-all
  also have ... = 1 / of-int (b n) ^ n
    by (simp add: power-mult b-def power-divide del: fact-Suc)
  finally show |standard-liouville p q − a n / b n| < 1 / of-int (b n) ^ n .

  from assms(3) obtain k where k: k ≥ n + 1 p k ≠ 0
    by (auto simp: frequently-def eventually-at-top-linorder)
  define k' where k' = k − n − 1
  from assms k have p k ≥ 0 by force
  with k assms have k': p (k' + n + 1) > 0 unfolding k'-def by force
  with assms(1,2) have ?S > 0
    by (intro suminf-pos2[of - k'] summable-ignore-initial-segment summable)
       (force intro!: divide-pos-pos divide-nonneg-pos)+
  with eq show |standard-liouville p q − a n / b n| > 0 by (simp only: )
```

**qed**

We can now show our main result: any standard Liouville number is transcendental.

**theorem** *transcendental-standard-liouville*:
  **assumes** $q > 1$ *range* $p \subseteq \{0..<q\}$ *frequently* $(\lambda k.\ p\ k \neq 0)$ *sequentially*
  **shows** $\neg$*algebraic* (*standard-liouville* $p$ $q$)
**proof** –
  **from** *assms* **interpret**
    *liouville standard-liouville* $p$ $q$
      $\lambda n.\ (\sum k{\le}n.\ p\ k * q\ \hat{}\ (fact\ (Suc\ n) - fact\ (Suc\ k)))$
      $\lambda n.\ q\ \hat{}\ fact\ (Suc\ n)$
    **by** (*rule standard-liouville-is-liouville*)
  **from** *transcendental* **show** *?thesis* **.**
**qed**

In particular: The the standard construction for constant sequences, such as the "classic" Liouville constant $\sum_{n=1}^{\infty} 10^{-n!} = 0.11000100\ldots$, are transcendental.

This shows that Liouville numbers exists and therefore gives a concrete and elementary proof that transcendental numbers exist.

**corollary** *transcendental-standard-standard-liouville*:
  $a \in \{0<..<b\} \Longrightarrow \neg$*algebraic* (*standard-liouville* ($\lambda$-. *int* $a$) (*int* $b$))
  **by** (*intro transcendental-standard-liouville*) *auto*

**corollary** *transcendental-liouville-constant*:
  $\neg$*algebraic* (*standard-liouville* ($\lambda$-. *1*) *10*)
  **by** (*intro transcendental-standard-liouville*) *auto*

**end**

# References


[1] Wikipedia. Liouville number — Wikipedia, the free encyclopedia. https://en.wikipedia.org/w/index.php?title=Liouville_number&oldid=696910651, 2015. [Online; accessed 22-July-2004].