Quantifier Elimination for Linear Arithmetic

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Abstract

This article formalizes quantifier elimination procedures for dense linear orders, linear real arithmetic and Presburger arithmetic. In each case both a DNF-based non-elementary algorithm and one or more (doubly) exponential NNF-based algorithms are formalized, including the well-known algorithms by Ferrante and Rackoff and by Cooper. The NNF-based algorithms for dense linear orders are new but based on Ferrante and Rackoff and on an algorithm by Loos and Weispfenning which simulates infinitesimals.

All algorithms are directly executable. In particular, they yield reflective quantifier elimination procedures for HOL itself.

The formalization makes heavy use of locales and is therefore highly modular.

For an exposition of the DNF-based procedures see [5], for the NNF-based procedures see [4].

Contents

1 Logic

theory *Logic*
imports *Main HOL−Library.FuncSet*
begin

We start with a generic formalization of quantified logical formulae using de Bruijn notation. The syntax is parametric in the type of atoms.

declare *Let-def*[*simp*]

datatype (*atoms*: *'a*) *fm* =
TrueF | *FalseF* | *Atom 'a* | *And 'a fm 'a fm* | *Or 'a fm 'a fm* |
Neg 'a fm | *ExQ 'a fm*

notation *map-fm* (‹map_{fm}›)

abbreviation *Imp* **where** *Imp* φ_1 φ_2 ≡ *Or* (*Neg* φ_1) φ_2
abbreviation *AllQ* **where** *AllQ* φ ≡ *Neg*(*ExQ*(*Neg* φ))

definition *neg* **where**
neg φ = (*if* φ=*TrueF then FalseF else if* φ=*FalseF then TrueF else Neg* φ)

definition *and* :: *'a fm* ⇒ *'a fm* ⇒ *'a fm* **where**
and φ_1 φ_2 =
(*if* φ_1=*TrueF then* φ_2 *else if* φ_2=*TrueF then* φ_1 *else*
if φ_1=*FalseF* ∨ φ_2=*FalseF then FalseF else And* φ_1 φ_2)

definition *or* :: *'a fm* ⇒ *'a fm* ⇒ *'a fm* **where**
or φ_1 φ_2 =
(*if* φ_1=*FalseF then* φ_2 *else if* φ_2=*FalseF then* φ_1 *else*

if φ_1=*TrueF* $\vee$ φ_2=*TrueF then TrueF else Or* φ_1 φ_2)

definition *list-conj* :: *'a fm list* $\Rightarrow$ *'a fm* **where**
list-conj fs = *foldr and fs TrueF*

definition *list-disj* :: *'a fm list* $\Rightarrow$ *'a fm* **where**
list-disj fs = *foldr or fs FalseF*

abbreviation *Disj is f* $\equiv$ *list-disj* (*map f is*)

lemmas *atoms-map-fm*[*simp*] = *fm.set-map*

fun *amap-fm* :: (*'a* $\Rightarrow$ *'b fm*) $\Rightarrow$ *'a fm* $\Rightarrow$ *'b fm* ($\langle amap_{fm} \rangle$) **where**
$amap_{fm}$ *h TrueF* = *TrueF* |
$amap_{fm}$ *h FalseF* = *FalseF* |
$amap_{fm}$ *h* (*Atom a*) = *h a* |
$amap_{fm}$ *h* (*And* φ_1 φ_2) = *and* ($amap_{fm}$ *h* φ_1) ($amap_{fm}$ *h* φ_2) |
$amap_{fm}$ *h* (*Or* φ_1 φ_2) = *or* ($amap_{fm}$ *h* φ_1) ($amap_{fm}$ *h* φ_2) |
$amap_{fm}$ *h* (*Neg* φ) = *neg* ($amap_{fm}$ *h* φ)

lemma *amap-fm-list-disj*:
$amap_{fm}$ *h* (*list-disj fs*) = *list-disj* (*map* ($amap_{fm}$ *h*) *fs*)
by(*induct fs*) (*auto simp*:*list-disj-def or-def*)

fun *qfree* :: *'a fm* $\Rightarrow$ *bool* **where**
qfree(*ExQ f*) = *False* |
qfree(*And* φ_1 φ_2) = (*qfree* φ_1 $\wedge$ *qfree* φ_2) |
qfree(*Or* φ_1 φ_2) = (*qfree* φ_1 $\wedge$ *qfree* φ_2) |
qfree(*Neg* φ) = (*qfree* φ) |
qfree φ = *True*

lemma *qfree-and*[*simp*]: $[\![$ *qfree* φ_1; *qfree* φ_2 $]\!]$ $\Longrightarrow$ *qfree*(*and* φ_1 φ_2)
by(*simp add*:*and-def*)

lemma *qfree-or*[*simp*]: $[\![$ *qfree* φ_1; *qfree* φ_2 $]\!]$ $\Longrightarrow$ *qfree*(*or* φ_1 φ_2)
by(*simp add*:*or-def*)

lemma *qfree-neg*[*simp*]: *qfree*(*neg* φ) = *qfree* φ
by(*simp add*:*neg-def*)

lemma *qfree-foldr-Or*[*simp*]:
qfree(*foldr Or fs* φ) = (*qfree* φ $\wedge$ ($\forall \varphi \in$ *set fs*. *qfree* φ))
by (*induct fs*) *auto*

lemma *qfree-list-conj*[*simp*]:
assumes $\forall \varphi \in$ *set fs*. *qfree* φ **shows** *qfree*(*list-conj fs*)
proof −
{ fix *fs* φ
have $[\![\forall \varphi \in$ *set fs*. *qfree* φ; *qfree* φ $]\!]$ $\Longrightarrow$ *qfree*(*foldr and fs* φ)

by (*induct fs*) *auto*
} thus *?thesis* **using** *assms* **by** (*fastforce simp add*:*list-conj-def*)
qed

lemma *qfree-list-disj*[*simp*]:
assumes $\forall \varphi \in$ *set fs. qfree* φ **shows** *qfree*(*list-disj fs*)
proof −
{ fix *fs* φ
have $[\![\forall \varphi \in$ *set fs. qfree* φ; *qfree* $\varphi]\!] \Longrightarrow$ *qfree*(*foldr or fs* φ)
by (*induct fs*) *auto*
} thus *?thesis* **using** *assms* **by** (*fastforce simp add*:*list-disj-def*)
qed

lemma *qfree-map-fm*: *qfree* (map_{fm} *f* φ) = *qfree* φ
by (*induct* φ) *simp-all*

lemma *atoms-list-disjE*:
$a \in$ *atoms*(*list-disj fs*) $\Longrightarrow$ $a \in (\bigcup \varphi \in$ *set fs. atoms* $\varphi)$
apply(*induct fs*)
apply (*simp add*:*list-disj-def*)
apply (*auto simp add*:*list-disj-def Logic.or-def split*:*if-split-asm*)
done

lemma *atoms-list-conjE*:
$a \in$ *atoms*(*list-conj fs*) $\Longrightarrow$ $a \in (\bigcup \varphi \in$ *set fs. atoms* $\varphi)$
apply(*induct fs*)
apply (*simp add*:*list-conj-def*)
apply (*auto simp add*:*list-conj-def Logic.and-def split*:*if-split-asm*)
done

fun *dnf* :: *'a fm* ⇒ *'a list list* **where**
dnf TrueF = [[]] |
dnf FalseF = [] |
dnf (*Atom* φ) = [[φ]] |
dnf (*And* φ_1 φ_2) = [*d1* @ *d2*. *d1* ← *dnf* φ_1, *d2* ← *dnf* φ_2] |
dnf (*Or* φ_1 φ_2) = *dnf* φ_1 @ *dnf* φ_2

fun *nqfree* :: *'a fm* ⇒ *bool* **where**
nqfree(*Atom a*) = *True* |
nqfree TrueF = *True* |
nqfree FalseF = *True* |
nqfree (*And* φ_1 φ_2) = (*nqfree* φ_1 ∧ *nqfree* φ_2) |
nqfree (*Or* φ_1 φ_2) = (*nqfree* φ_1 ∧ *nqfree* φ_2) |
nqfree φ = *False*

lemma *nqfree-qfree*[*simp*]: *nqfree* $\varphi \Longrightarrow$ *qfree* φ
by (*induct* φ) *simp-all*

lemma *nqfree-map-fm*: *nqfree* (*map*$_{fm}$ *f* φ) = *nqfree* φ
by (*induct* φ) *simp-all*

fun *interpret* :: ('*a* ⇒ '*b list* ⇒ *bool*) ⇒ '*a fm* ⇒ '*b list* ⇒ *bool* **where**
interpret h TrueF xs = *True* |
interpret h FalseF xs = *False* |
interpret h (*Atom a*) *xs* = *h a xs* |
interpret h (*And* φ_1 φ_2) *xs* = (*interpret h* φ_1 *xs* ∧ *interpret h* φ_2 *xs*) |
interpret h (*Or* φ_1 φ_2) *xs* = (*interpret h* φ_1 *xs* ∨ *interpret h* φ_2 *xs*) |
interpret h (*Neg* φ) *xs* = (¬ *interpret h* φ *xs*) |
interpret h (*ExQ* φ) *xs* = (∃ *x*. *interpret h* φ (*x*#*xs*))

1.1 Atoms

The locale ATOM of atoms provides a minimal framework for the generic formulation of theory-independent algorithms, in particular quantifier elimination.

locale *ATOM* =
fixes *aneg* :: '*a* ⇒ '*a fm*
fixes *anormal* :: '*a* ⇒ *bool*
assumes *nqfree-aneg*: *nqfree*(*aneg a*)
assumes *anormal-aneg*: *anormal a* ⟹ ∀ *b*∈*atoms*(*aneg a*). *anormal b*

fixes I_a :: '*a* ⇒ '*b list* ⇒ *bool*
assumes I_a-*aneg*: *interpret* I_a (*aneg a*) *xs* = (¬ I_a *a xs*)

fixes *depends*$_0$ :: '*a* ⇒ *bool*
and *decr* :: '*a* ⇒ '*a*
assumes *not-dep-decr*: ¬*depends*$_0$ *a* ⟹ I_a *a* (*x*#*xs*) = I_a (*decr a*) *xs*
assumes *anormal-decr*: ¬ *depends*$_0$ *a* ⟹ *anormal a* ⟹ *anormal*(*decr a*)

begin

fun *atoms*$_0$ :: '*a fm* ⇒ '*a list* **where**
atoms$_0$ *TrueF* = [] |
atoms$_0$ *FalseF* = [] |
atoms$_0$ (*Atom a*) = (*if depends*$_0$ *a then* [*a*] *else* []) |
atoms$_0$ (*And* φ_1 φ_2) = *atoms*$_0$ φ_1 @ *atoms*$_0$ φ_2 |
atoms$_0$ (*Or* φ_1 φ_2) = *atoms*$_0$ φ_1 @ *atoms*$_0$ φ_2 |
atoms$_0$ (*Neg* φ) = *atoms*$_0$ φ

abbreviation *I* **where** *I* ≡ *interpret* I_a

fun *nnf* :: '*a fm* ⇒ '*a fm* **where**
nnf (*And* φ_1 φ_2) = *And* (*nnf* φ_1) (*nnf* φ_2) |
nnf (*Or* φ_1 φ_2) = *Or* (*nnf* φ_1) (*nnf* φ_2) |
nnf (*Neg TrueF*) = *FalseF* |
nnf (*Neg FalseF*) = *TrueF* |

$nnf\ (Neg\ (Neg\ \varphi)) = (nnf\ \varphi)\ |$
$nnf\ (Neg\ (And\ \varphi_1\ \varphi_2)) = (Or\ (nnf\ (Neg\ \varphi_1))\ (nnf\ (Neg\ \varphi_2)))\ |$
$nnf\ (Neg\ (Or\ \varphi_1\ \varphi_2)) = (And\ (nnf\ (Neg\ \varphi_1))\ (nnf\ (Neg\ \varphi_2)))\ |$
$nnf\ (Neg\ (Atom\ a)) = aneg\ a\ |$
$nnf\ \varphi = \varphi$

lemma *nqfree-nnf*: $qfree\ \varphi \Longrightarrow nqfree(nnf\ \varphi)$
by(*induct* φ *rule*:*nnf.induct*)
(*simp-all add*: *nqfree-aneg and-def or-def*)

lemma *qfree-nnf*[*simp*]: $qfree(nnf\ \varphi) = qfree\ \varphi$
by (*induct* φ *rule*:*nnf.induct*)(*simp-all add*: *nqfree-aneg*)

lemma *I-neg*[*simp*]: $I\ (neg\ \varphi)\ xs = I\ (Neg\ \varphi)\ xs$
by(*simp add*:*neg-def*)

lemma *I-and*[*simp*]: $I\ (and\ \varphi_1\ \varphi_2)\ xs = I\ (And\ \varphi_1\ \varphi_2)\ xs$
by(*simp add*:*and-def*)

lemma *I-list-conj*[*simp*]:
$I\ (list\text{-}conj\ fs)\ xs = (\forall \varphi \in set\ fs.\ I\ \varphi\ xs)$
proof −
{ fix *fs* φ
have $I\ (foldr\ and\ fs\ \varphi)\ xs = (I\ \varphi\ xs \wedge (\forall \varphi \in set\ fs.\ I\ \varphi\ xs))$
by (*induct fs*) *auto*
} thus *?thesis* **by**(*simp add*:*list-conj-def*)
qed

lemma *I-or*[*simp*]: $I\ (or\ \varphi_1\ \varphi_2)\ xs = I\ (Or\ \varphi_1\ \varphi_2)\ xs$
by(*simp add*:*or-def*)

lemma *I-list-disj*[*simp*]:
$I\ (list\text{-}disj\ fs)\ xs = (\exists \varphi \in set\ fs.\ I\ \varphi\ xs)$
proof −
{ fix *fs* φ
have $I\ (foldr\ or\ fs\ \varphi)\ xs = (I\ \varphi\ xs \vee (\exists \varphi \in set\ fs.\ I\ \varphi\ xs))$
by (*induct fs*) *auto*
} thus *?thesis* **by**(*simp add*:*list-disj-def*)
qed

lemma *I-nnf*: $I\ (nnf\ \varphi)\ xs = I\ \varphi\ xs$
by(*induct rule*:*nnf.induct*)(*simp-all add*: I_a*-aneg*)

lemma *I-dnf*:
$nqfree\ \varphi \Longrightarrow (\exists as{\in}set\ (dnf\ \varphi).\ \forall a{\in}set\ as.\ I_a\ a\ xs) = I\ \varphi\ xs$
by (*induct* φ) (*fastforce*)+

definition $normal\ \varphi = (\forall a \in atoms\ \varphi.\ anormal\ a)$

lemma *normal-simps*[*simp*]:
normal TrueF
normal FalseF
normal (*Atom a*) $\longleftrightarrow$ *anormal a*
normal (*And* φ_1 φ_2) $\longleftrightarrow$ *normal* φ_1 $\wedge$ *normal* φ_2
normal (*Or* φ_1 φ_2) $\longleftrightarrow$ *normal* φ_1 $\wedge$ *normal* φ_2
normal (*Neg* φ) $\longleftrightarrow$ *normal* φ
normal (*ExQ* φ) $\longleftrightarrow$ *normal* φ
by(*auto simp*:*normal-def*)

lemma *normal-aneg*[*simp*]: *anormal a* $\Longrightarrow$ *normal* (*aneg a*)
by (*simp add*:*anormal-aneg normal-def*)

lemma *normal-and*[*simp*]:
normal φ_1 $\Longrightarrow$ *normal* φ_2 $\Longrightarrow$ *normal* (*and* φ_1 φ_2)
by (*simp add*:*Logic.and-def*)

lemma *normal-or*[*simp*]:
normal φ_1 $\Longrightarrow$ *normal* φ_2 $\Longrightarrow$ *normal* (*or* φ_1 φ_2)
by (*simp add*:*Logic.or-def*)

lemma *normal-list-disj*[*simp*]:
$\forall \varphi \in$ *set fs*. *normal* φ $\Longrightarrow$ *normal* (*list-disj fs*)
apply(*induct fs*)
apply (*simp add*:*list-disj-def*)
apply (*simp add*:*list-disj-def*)
done

lemma *normal-nnf*: *normal* φ $\Longrightarrow$ *normal*(*nnf* φ)
by(*induct* φ *rule*:*nnf.induct*) *simp-all*

lemma *normal-map-fm*:
$\forall$ *a*. *anormal*(*f a*) = *anormal*(*a*) $\Longrightarrow$ *normal* (map_{fm} *f* φ) = *normal* φ
by(*induct* φ) *auto*

lemma *anormal-nnf*:
qfree φ $\Longrightarrow$ *normal* φ $\Longrightarrow$ $\forall$ *a*$\in$*atoms*(*nnf* φ). *anormal a*
apply(*induct* φ *rule*:*nnf.induct*)
apply(*unfold normal-def*)
apply(*simp-all*)
apply (*blast dest*:*anormal-aneg*)+
done

lemma *atoms-dnf*: *nqfree* φ $\Longrightarrow$ *as* $\in$ *set*(*dnf* φ) $\Longrightarrow$ *a* $\in$ *set as* $\Longrightarrow$ *a* $\in$ *atoms* φ
by(*induct* φ *arbitrary*: *as a rule*:*nqfree.induct*)(*auto*)

lemma *anormal-dnf-nnf*:

as ∈ *set*(*dnf*(*nnf* φ)) ⟹ *qfree* φ ⟹ *normal* φ ⟹ *a* ∈ *set as* ⟹ *anormal a*
apply(*induct* φ *arbitrary*: *a as rule*:*nnf*.*induct*)
apply(*simp-all add*: *normal-def*)
apply *clarify*
apply (*metis UnE set-append*)
apply *metis*
apply *metis*
apply *fastforce*
apply (*metis anormal-aneg atoms-dnf nqfree-aneg*)
done

end

end

2 Quantifier elimination

theory *QE*
imports *Logic*
begin

The generic, i.e. theory-independent part of quantifier elimination. Both DNF and an NNF-based procedures are defined and proved correct.

notation (*input*) *Collect* (‹|-|›)

2.1 No Equality

context *ATOM*
begin

2.1.1 DNF-based

Taking care of atoms independent of variable 0:

definition
qelim qe as =
(*let qf* = *qe* [*a*←*as*. *depends*$_0$ *a*];
indep = [*Atom*(*decr a*). *a*←*as*, ¬ *depends*$_0$ *a*]
in and qf (*list-conj indep*))

abbreviation *is-dnf-qe* :: ('*a list* ⇒ '*a fm*) ⇒ '*a list* ⇒ *bool* **where**
is-dnf-qe qe as ≡ ∀ *xs*. *I*(*qe as*) *xs* = (∃ *x*.∀ *a*∈*set as*. *I*$_a$ *a* (*x*#*xs*))

Note that the exported abbreviation will have as a first parameter the type $'b$ of values *xs* ranges over.

lemma *I-qelim*:
assumes *qe*: ⋀*as*. (∀ *a* ∈ *set as*. *depends*$_0$ *a*) ⟹ *is-dnf-qe qe as*
shows *is-dnf-qe* (*qelim qe*) *as* (**is** ∀ *xs*. *?P xs*)
proof

fix *xs*
let *?as0* = *filter* $depends_0$ *as*
let *?as1* = *filter* (*Not* ∘ $depends_0$) *as*
have *I* (*qelim qe as*) *xs* =
(*I* (*qe ?as0*) *xs* ∧ (∀ *a*∈*set*(*map decr ?as1*). I_a *a xs*))
(**is** - = (- ∧ *?B*)) **by**(*force simp add*:*qelim-def*)
also have ... = ((∃ *x*. ∀ *a* ∈ *set ?as0*. I_a *a* (*x*#*xs*)) ∧ *?B*)
by(*simp add*:*qe not-dep-decr*)
also have ... = (∃ *x*. (∀ *a* ∈ *set ?as0*. I_a *a* (*x*#*xs*)) ∧ *?B*) **by** *blast*
also have *?B* = (∀ *a* ∈ *set ?as1*. I_a (*decr a*) *xs*) **by** *simp*
also have (∃ *x*. (∀ *a* ∈ *set ?as0*. I_a *a* (*x*#*xs*)) ∧ ...) =
(∃ *x*. (∀ *a* ∈ *set ?as0*. I_a *a* (*x*#*xs*)) ∧
(∀ *a* ∈ *set ?as1*. I_a *a* (*x*#*xs*)))
by(*simp add*: *not-dep-decr*)
also have ... = (∃ *x*. ∀ *a* ∈ *set*(*?as0* @ *?as1*). I_a *a* (*x*#*xs*))
by (*simp add*:*ball-Un*)
also have ... = (∃ *x*. ∀ *a* ∈ *set*(*as*). I_a *a* (*x*#*xs*))
by *simp blast*
finally show *?P xs* **.**
qed

The generic DNF-based quantifier elimination procedure:

fun *lift-dnf-qe* :: ('*a list* ⇒ '*a fm*) ⇒ '*a fm* ⇒ '*a fm* **where**
lift-dnf-qe qe (*And* φ_1 φ_2) = *and* (*lift-dnf-qe qe* φ_1) (*lift-dnf-qe qe* φ_2) |
lift-dnf-qe qe (*Or* φ_1 φ_2) = *or* (*lift-dnf-qe qe* φ_1) (*lift-dnf-qe qe* φ_2) |
lift-dnf-qe qe (*Neg* φ) = *neg*(*lift-dnf-qe qe* φ) |
lift-dnf-qe qe (*ExQ* φ) = *Disj* (*dnf*(*nnf*(*lift-dnf-qe qe* φ))) (*qelim qe*) |
lift-dnf-qe qe φ = φ

lemma *qfree-lift-dnf-qe*: (⋀*as*. (∀ *a*∈*set as*. $depends_0$ *a*) ⟹ *qfree*(*qe as*))
⟹ *qfree*(*lift-dnf-qe qe* φ)
by (*induct* φ) (*simp-all add*:*qelim-def*)

lemma *qfree-lift-dnf-qe2*: *qe* ∈ *lists* |$depends_0$| → |*qfree*|
⟹ *qfree*(*lift-dnf-qe qe* φ)
using *in-lists-conv-set*[**where** *?'a* = '*a*]
by (*simp add*:*Pi-def qfree-lift-dnf-qe*)

lemma *lem*: ∀ *P A*. (∃ *x*∈*A*. ∃ *y*. *P x y*) = (∃ *y*. ∃ *x*∈*A*. *P x y*) **by** *blast*

lemma *I-lift-dnf-qe*:
assumes ⋀*as*. (∀ *a* ∈ *set as*. $depends_0$ *a*) ⟹ *qfree*(*qe as*)
and ⋀*as*. (∀ *a* ∈ *set as*. $depends_0$ *a*) ⟹ *is-dnf-qe qe as*
shows *I* (*lift-dnf-qe qe* φ) *xs* = *I* φ *xs*
proof(*induct* φ *arbitrary*:*xs*)
case *ExQ* **thus** *?case*
by (*simp add*: *assms I-qelim lem I-dnf nqfree-nnf qfree-lift-dnf-qe*
I-nnf)
qed *simp-all*

lemma *I-lift-dnf-qe2*:
assumes *qe* ∈ *lists* |*depends*$_0$| → |*qfree*|
and ∀ *as* ∈ *lists* |*depends*$_0$|. *is-dnf-qe qe as*
shows *I* (*lift-dnf-qe qe* φ) *xs* = *I* φ *xs*
using *assms in-lists-conv-set*[**where** *?'a* = *'a*]
by(*simp add*:*Pi-def I-lift-dnf-qe*)

Quantifier elimination with invariant (needed for Presburger):

lemma *I-qelim-anormal*:
assumes *qe*: ⋀*xs as*. ∀ *a* ∈ *set as*. *depends*$_0$ *a* ∧ *anormal a* ⟹ *is-dnf-qe qe as*
and *nm*: ∀ *a* ∈ *set as*. *anormal a*
shows *I* (*qelim qe as*) *xs* = (∃ *x*. ∀ *a*∈*set as*. I_a *a* (*x*#*xs*))
proof −
  let *?as0* = *filter depends*$_0$ *as*
  let *?as1* = *filter* (*Not* ∘ *depends*$_0$) *as*
  have *I* (*qelim qe as*) *xs* =
      (*I* (*qe ?as0*) *xs* ∧ (∀ *a*∈*set*(*map decr ?as1*). I_a *a xs*))
    (**is** - = (- ∧ *?B*)) **by**(*force simp add*:*qelim-def*)
  also have ... = ((∃ *x*. ∀ *a* ∈ *set ?as0*. I_a *a* (*x*#*xs*)) ∧ *?B*)
    by(*simp add*:*qe nm not-dep-decr*)
  also have ... = (∃ *x*. (∀ *a* ∈ *set ?as0*. I_a *a* (*x*#*xs*)) ∧ *?B*) **by** *blast*
  also have *?B* = (∀ *a* ∈ *set ?as1*. I_a (*decr a*) *xs*) **by** *simp*
  also have (∃ *x*. (∀ *a* ∈ *set ?as0*. I_a *a* (*x*#*xs*)) ∧ ...) =
          (∃ *x*. (∀ *a* ∈ *set ?as0*. I_a *a* (*x*#*xs*)) ∧
              (∀ *a* ∈ *set ?as1*. I_a *a* (*x*#*xs*)))
    by(*simp add*: *not-dep-decr*)
  also have ... = (∃ *x*. ∀ *a* ∈ *set*(*?as0* @ *?as1*). I_a *a* (*x*#*xs*))
    by (*simp add*:*ball-Un*)
  also have ... = (∃ *x*. ∀ *a* ∈ *set*(*as*). I_a *a* (*x*#*xs*))
    by *simp blast*
  finally show *?thesis* **.**
qed

context notes [[*simp-depth-limit* = *5*]]
begin

lemma *anormal-atoms-qelim*:
  (⋀*as*. ∀ *a* ∈ *set as*. *depends*$_0$ *a* ∧ *anormal a* ⟹ *normal*(*qe as*)) ⟹
   ∀ *a* ∈ *set as*. *anormal a* ⟹ *a* ∈ *atoms*(*qelim qe as*) ⟹ *anormal a*
apply(*auto simp add*:*qelim-def and-def normal-def split*:*if-split-asm*)
apply(*auto simp add*:*anormal-decr dest*!: *atoms-list-conjE*)
 apply(*erule-tac x* = *filter depends*$_0$ *as* **in** *meta-allE*)
 apply(*simp*)
apply(*erule-tac x* = *filter depends*$_0$ *as* **in** *meta-allE*)
apply(*simp*)
done

lemma *normal-lift-dnf-qe*:

assumes $\bigwedge as.\ \forall\, a \in set\ as.\ depends_0\ a \Longrightarrow qfree(qe\ as)$
and $\bigwedge as.\ \forall\, a \in set\ as.\ depends_0\ a \wedge anormal\ a \Longrightarrow normal(qe\ as)$
shows $normal\ \varphi \Longrightarrow normal(lift\text{-}dnf\text{-}qe\ qe\ \varphi)$
proof(*simp add*:*normal-def*, *induct* φ)
case *ExQ* **thus** *?case*
apply (*auto dest*!: *atoms-list-disjE*)
apply(*rule anormal-atoms-qelim*)
prefer *3* **apply** *assumption*
apply(*simp add*:*assms*)
apply (*simp add*:*normal-def qfree-lift-dnf-qe anormal-dnf-nnf assms*)
done
qed (*simp-all add*:*and-def or-def neg-def Ball-def*)

end

context notes [[*simp-depth-limit* = *9*]]
begin

lemma *I-lift-dnf-qe-anormal*:
assumes $\bigwedge as.\ \forall\, a \in set\ as.\ depends_0\ a \Longrightarrow qfree(qe\ as)$
and $\bigwedge as.\ \forall\, a \in set\ as.\ depends_0\ a \wedge anormal\ a \Longrightarrow normal(qe\ as)$
and $\bigwedge xs\ as.\ \forall\, a \in set\ as.\ depends_0\ a \wedge anormal\ a \Longrightarrow is\text{-}dnf\text{-}qe\ qe\ as$
shows $normal\ f \Longrightarrow I\ (lift\text{-}dnf\text{-}qe\ qe\ f)\ xs = I\ f\ xs$
proof(*induct f arbitrary*:*xs*)
case *ExQ* **thus** *?case* **using** *normal-lift-dnf-qe*[*of qe*]
by (*simp add*: *assms*[*simplified normal-def*] *anormal-dnf-nnf I-qelim-anormal lem I-dnf nqfree-nnf qfree-lift-dnf-qe I-nnf normal-def*)
qed (*simp-all add*:*normal-def*)

end

lemma *I-lift-dnf-qe-anormal2*:
assumes $qe \in lists\ |depends_0| \to |qfree|$
and $qe \in lists\ (\ |depends_0| \cap |anormal|\ ) \to |normal|$
and $\forall\, as \in lists(\ |depends_0| \cap |anormal|\ ).\ is\text{-}dnf\text{-}qe\ qe\ as$
shows $normal\ f \Longrightarrow I\ (lift\text{-}dnf\text{-}qe\ qe\ f)\ xs = I\ f\ xs$
using *assms in-lists-conv-set*[**where** *?'a* = *'a*]
by(*simp add*:*Pi-def I-lift-dnf-qe-anormal Int-def*)

2.1.2 NNF-based

fun *lift-nnf-qe* :: (*'a fm* ⇒ *'a fm*) ⇒ *'a fm* ⇒ *'a fm* **where**
lift-nnf-qe qe (*And* φ_1 φ_2) = *and* (*lift-nnf-qe qe* φ_1) (*lift-nnf-qe qe* φ_2) |
lift-nnf-qe qe (*Or* φ_1 φ_2) = *or* (*lift-nnf-qe qe* φ_1) (*lift-nnf-qe qe* φ_2) |
lift-nnf-qe qe (*Neg* φ) = *neg*(*lift-nnf-qe qe* φ) |
lift-nnf-qe qe (*ExQ* φ) = *qe*(*nnf*(*lift-nnf-qe qe* φ)) |
lift-nnf-qe qe φ = φ

lemma *qfree-lift-nnf-qe*: $(\bigwedge \varphi.\ nqfree\ \varphi \Longrightarrow qfree(qe\ \varphi))$

$\Longrightarrow$ *qfree*(*lift-nnf-qe qe* φ)
by (*induct* φ) (*simp-all add*:*nqfree-nnf*)

lemma *qfree-lift-nnf-qe2*:
$qe \in |nqfree| \to |qfree| \Longrightarrow$ *qfree*(*lift-nnf-qe qe* φ)
by(*simp add*:*Pi-def qfree-lift-nnf-qe*)

lemma *I-lift-nnf-qe*:
assumes $\bigwedge\varphi$. *nqfree* $\varphi \Longrightarrow$ *qfree*(*qe* φ)
and $\bigwedge xs\ \varphi$. *nqfree* $\varphi \Longrightarrow I\ (qe\ \varphi)\ xs = (\exists\, x.\ I\ \varphi\ (x\#xs))$
shows $I\ (\textit{lift-nnf-qe}\ qe\ \varphi)\ xs = I\ \varphi\ xs$
proof(*induct* φ *arbitrary*:*xs*)
case *ExQ* **thus** *?case*
by (*simp add*: *assms nqfree-nnf qfree-lift-nnf-qe I-nnf*)
qed *simp-all*

lemma *I-lift-nnf-qe2*:
assumes $qe \in |nqfree| \to |qfree|$
and $\forall\, \varphi \in |nqfree|.\ \forall\, xs.\ I\ (qe\ \varphi)\ xs = (\exists\, x.\ I\ \varphi\ (x\#xs))$
shows $I\ (\textit{lift-nnf-qe}\ qe\ \varphi)\ xs = I\ \varphi\ xs$
using *assms* **by**(*simp add*:*Pi-def I-lift-nnf-qe*)

lemma *normal-lift-nnf-qe*:
assumes $\bigwedge\varphi$. *nqfree* $\varphi \Longrightarrow$ *qfree*(*qe* φ)
and $\bigwedge\varphi$. *nqfree* $\varphi \Longrightarrow$ *normal* $\varphi \Longrightarrow$ *normal*(*qe* φ)
shows *normal* $\varphi \Longrightarrow$ *normal*(*lift-nnf-qe qe* φ)
by (*induct* φ)
(*simp-all add*: *assms Logic.neg-def normal-nnf*
nqfree-nnf qfree-lift-nnf-qe)

lemma *I-lift-nnf-qe-normal*:
assumes $\bigwedge\varphi$. *nqfree* $\varphi \Longrightarrow$ *qfree*(*qe* φ)
and $\bigwedge\varphi$. *nqfree* $\varphi \Longrightarrow$ *normal* $\varphi \Longrightarrow$ *normal*(*qe* φ)
and $\bigwedge xs\ \varphi$. *normal* $\varphi \Longrightarrow$ *nqfree* $\varphi \Longrightarrow I\ (qe\ \varphi)\ xs = (\exists\, x.\ I\ \varphi\ (x\#xs))$
shows *normal* $\varphi \Longrightarrow I\ (\textit{lift-nnf-qe}\ qe\ \varphi)\ xs = I\ \varphi\ xs$
proof(*induct* φ *arbitrary*:*xs*)
case *ExQ* **thus** *?case*
by (*simp add*: *assms nqfree-nnf qfree-lift-nnf-qe I-nnf*
normal-lift-nnf-qe normal-nnf)
qed *auto*

lemma *I-lift-nnf-qe-normal2*:
assumes $qe \in |nqfree| \to |qfree|$
and $qe \in |nqfree| \cap |normal| \to |normal|$
and $\forall\, \varphi \in |normal| \cap |nqfree|.\ \forall\, xs.\ I\ (qe\ \varphi)\ xs = (\exists\, x.\ I\ \varphi\ (x\#xs))$
shows *normal* $\varphi \Longrightarrow I\ (\textit{lift-nnf-qe}\ qe\ \varphi)\ xs = I\ \varphi\ xs$
using *assms* **by**(*simp add*:*Pi-def I-lift-nnf-qe-normal Int-def*)

end

2.2 With equality

DNF-based quantifier elimination can accommodate equality atoms in a generic fashion.

locale *ATOM-EQ* = *ATOM* +
fixes $solvable_0$ :: $'a \Rightarrow bool$
and *trivial* :: $'a \Rightarrow bool$
and $subst_0$ :: $'a \Rightarrow 'a \Rightarrow 'a$
assumes $subst_0$:
$[\![solvable_0\ eq;\ \neg trivial\ eq;\ I_a\ eq\ (x\#xs);\ depends_0\ a]\!]$
$\Longrightarrow I_a\ (subst_0\ eq\ a)\ xs = I_a\ a\ (x\#xs)$
and *trivial*: $trivial\ eq \Longrightarrow I_a\ eq\ xs$
and *solvable*: $solvable_0\ eq \Longrightarrow \exists\, x.\ I_a\ eq\ (x\#xs)$
and *is-triv-self-subst*: $solvable_0\ eq \Longrightarrow trivial\ (subst_0\ eq\ eq)$

begin

definition *lift-eq-qe* :: $('a\ list \Rightarrow 'a\ fm) \Rightarrow 'a\ list \Rightarrow 'a\ fm$ **where**
lift-eq-qe qe as =
(*let as* = [*a*←*as*. ¬ *trivial a*]
in case [*a*←*as*. $solvable_0\ a$] *of*
[] ⇒ *qe as*
| *eq* # *eqs* ⇒
(*let ineqs* = [*a*←*as*. ¬ $solvable_0\ a$]
in list-conj (*map* (*Atom* ∘ ($subst_0\ eq$)) (*eqs* @ *ineqs*))))

theorem *I-lift-eq-qe*:
assumes *dep*: ∀ *a*∈*set as*. $depends_0\ a$
assumes *qe*: ⋀*as*. (∀ *a* ∈ *set as*. $depends_0\ a$ ∧ ¬ $solvable_0\ a$) ⟹
I (*qe as*) *xs* = (∃ *x*. ∀ *a* ∈ *set as*. $I_a\ a\ (x\#xs)$)
shows *I* (*lift-eq-qe qe as*) *xs* = (∃ *x*. ∀ *a* ∈ *set as*. $I_a\ a\ (x\#xs)$)
(**is** *?L* = *?R*)
proof −
let *?as* = [*a*←*as*. ¬ *trivial a*]
show *?thesis*
proof (*cases* [*a*←*?as*. $solvable_0\ a$])
case *Nil*
hence ∀ *a*∈*set as*. ¬ *trivial a* ⟶ ¬ $solvable_0\ a$
by(*auto simp*: *filter-empty-conv*)
thus *?L* = *?R*
by(*simp add*:*lift-eq-qe-def dep qe cong*:*conj-cong*) (*metis trivial*)
next
case (*Cons eq* -)
then have *eq* ∈ *set as* $solvable_0\ eq$ ¬ *trivial eq*
by (*auto simp*: *filter-eq-Cons-iff*)
then obtain *e* **where** $I_a\ eq\ (e\#xs)$ **by**(*metis solvable*)
have ∀ *a* ∈ *set as*. $I_a\ a\ (e\ \#\ xs) = I_a\ (subst_0\ eq\ a)\ xs$
by(*simp add*: $subst_0$[*OF* ‹$solvable_0\ eq$› ‹¬ *trivial eq*› ‹$I_a\ eq\ (e\#xs)$›] *dep*)
thus *?thesis* **using** *Cons dep*

apply(*simp add*: *lift-eq-qe-def*,
clarsimp simp: *filter-eq-Cons-iff ball-Un*)
apply(*rule iffI*)
apply(*fastforce intro*!:*exI*[*of* - *e*] *simp*: *trivial is-triv-self-subst*)
apply (*metis subst*$_0$)
done
qed
qed

definition *lift-dnfeq-qe* = *lift-dnf-qe* ∘ *lift-eq-qe*

lemma *qfree-lift-eq-qe*:
(⋀*as*. ∀ *a*∈*set as*. *depends*$_0$ *a* ⟹ *qfree* (*qe as*)) ⟹
∀ *a*∈*set as*. *depends*$_0$ *a* ⟹ *qfree*(*lift-eq-qe qe as*)
by(*simp add*:*lift-eq-qe-def ball-Un split*:*list.split*)

lemma *qfree-lift-dnfeq-qe*: (⋀*as*. (∀ *a*∈*set as*. *depends*$_0$ *a*) ⟹ *qfree*(*qe as*))
⟹ *qfree*(*lift-dnfeq-qe qe* φ)
by(*simp add*: *lift-dnfeq-qe-def qfree-lift-dnf-qe qfree-lift-eq-qe*)

lemma *I-lift-dnfeq-qe*:
(⋀*as*. (∀ *a* ∈ *set as*. *depends*$_0$ *a*) ⟹ *qfree*(*qe as*)) ⟹
(⋀*as*. (∀ *a* ∈ *set as*. *depends*$_0$ *a* ∧ ¬ *solvable*$_0$ *a*) ⟹ *is-dnf-qe qe as*) ⟹
I (*lift-dnfeq-qe qe* φ) *xs* = *I* φ *xs*
by(*simp add*:*lift-dnfeq-qe-def I-lift-dnf-qe qfree-lift-eq-qe I-lift-eq-qe*)

lemma *I-lift-dnfeq-qe2*:
qe ∈ *lists* |*depends*$_0$| → |*qfree*| ⟹
(∀ *as* ∈ *lists*(|*depends*$_0$| ∩ − |*solvable*$_0$|). *is-dnf-qe qe as*) ⟹
I (*lift-dnfeq-qe qe* φ) *xs* = *I* φ *xs*
using *in-lists-conv-set*[**where** *?'a* = *'a*]
by(*simp add*:*Pi-def I-lift-dnfeq-qe Int-def Compl-eq*)

end

end

3 DLO

theory *DLO*
imports *QE Complex-Main*
begin

3.1 Basics

class *dlo* = *linorder* +
assumes *dense*: *x* < *z* ⟹ ∃ *y*. *x* < *y* ∧ *y* < *z*
and *no-ub*: ∃ *u*. *x* < *u* **and** *no-lb*: ∃ *l*. *l* < *x*

instance *real* :: *dlo*
proof
  fix *r s* :: *real*
  let *?v* = (*r* + *s*) / *2*
  assume *r* < *s*
  hence *r* < *?v* ∧ *?v* < *s* **by** *simp*
  thus ∃ *v*. *r* < *v* ∧ *v* < *s* **..**
next
  fix *r* :: *real*
  have *r* < *r* + *1* **by** *arith*
  thus ∃ *s*. *r* < *s* **..**
next
  fix *r* :: *real*
  have *r* − *1* < *r* **by** *arith*
  thus ∃ *s*. *s* < *r* **..**
qed

datatype *atom* = *Less nat nat* | *Eq nat nat*

fun *is-Less* :: *atom* ⇒ *bool* **where**
is-Less (*Less i j*) = *True* |
is-Less f = *False*

abbreviation *is-Eq* ≡ *Not* ∘ *is-Less*

lemma *is-Less-iff*: *is-Less a* = (∃ *i j*. *a* = *Less i j*)
by(*cases a*) *auto*
lemma *is-Eq-iff*: (∀ *i j*. *a* ≠ *Less i j*) = (∃ *i j*. *a* = *Eq i j*)
by(*cases a*) *auto*
lemma *not-is-Eq-iff*: (∀ *i j*. *a* ≠ *Eq i j*) = (∃ *i j*. *a* = *Less i j*)
by(*cases a*) *auto*

fun neg_{dlo} :: *atom* ⇒ *atom fm* **where**
neg_{dlo} (*Less i j*) = *Or* (*Atom*(*Less j i*)) (*Atom*(*Eq i j*)) |
neg_{dlo} (*Eq i j*) = *Or* (*Atom*(*Less i j*)) (*Atom*(*Less j i*))

fun I_{dlo} :: *atom* ⇒ *'a*::*dlo list* ⇒ *bool* **where**
I_{dlo} (*Eq i j*) *xs* = (*xs*!*i* = *xs*!*j*) |
I_{dlo} (*Less i j*) *xs* = (*xs*!*i* < *xs*!*j*)

fun $depends_{dlo}$ :: *atom* ⇒ *bool* **where**
$depends_{dlo}$(*Eq i j*) = (*i*=*0* | *j*=*0*) |
$depends_{dlo}$(*Less i j*) = (*i*=*0* | *j*=*0*)

fun $decr_{dlo}$ :: *atom* ⇒ *atom* **where**
$decr_{dlo}$ (*Less i j*) = *Less* (*i* − *1*) (*j* − *1*) |
$decr_{dlo}$ (*Eq i j*) = *Eq* (*i* − *1*) (*j* − *1*)

definition [*code del*]: *nnf* = *ATOM.nnf* neg_{dlo}
definition [*code del*]: *qelim* = *ATOM.qelim* $depends_{dlo}$ $decr_{dlo}$
definition [*code del*]: *lift-dnf-qe* = *ATOM.lift-dnf-qe* neg_{dlo} $depends_{dlo}$ $decr_{dlo}$
definition [*code del*]: *lift-nnf-qe* = *ATOM.lift-nnf-qe* neg_{dlo}

hide-const *nnf qelim lift-dnf-qe lift-nnf-qe*

lemmas *DLO-code-lemmas* = *nnf-def qelim-def lift-dnf-qe-def lift-nnf-qe-def*

interpretation *DLO*:
ATOM neg_{dlo} (λ*a*. *True*) I_{dlo} $depends_{dlo}$ $decr_{dlo}$
apply(*unfold-locales*)
apply(*case-tac a*)
apply *simp-all*
apply(*case-tac a*)
apply (*simp-all add*:*linorder-class.not-less-iff-gr-or-eq*
linorder-not-less linorder-neq-iff)
apply(*case-tac a*)
apply(*simp-all add*:*nth-Cons'*)
done

lemmas [*folded DLO-code-lemmas*, *code*] =
DLO.nnf.simps DLO.qelim-def DLO.lift-dnf-qe.simps DLO.lift-dnf-qe.simps

setup ‹*Sign.revert-abbrev* @{*const-abbrev DLO.I*}›

definition *lbounds* **where** *lbounds as* = [*i*. *Less* (*Suc i*) *0* ← *as*]
definition *ubounds* **where** *ubounds as* = [*i*. *Less 0* (*Suc i*) ← *as*]
definition *ebounds* **where**
ebounds as = [*i*. *Eq* (*Suc i*) *0* ← *as*] @ [*i*. *Eq 0* (*Suc i*) ← *as*]

lemma *set-lbounds*: *set*(*lbounds as*) = {*i*. *Less* (*Suc i*) *0* ∈ *set as*}
by(*auto simp*: *lbounds-def split*:*nat.splits atom.splits*)
lemma *set-ubounds*: *set*(*ubounds as*) = {*i*. *Less 0* (*Suc i*) ∈ *set as*}
by(*auto simp*: *ubounds-def split*:*nat.splits atom.splits*)
lemma *set-ebounds*:
set(*ebounds as*) = {*k*. *Eq* (*Suc k*) *0* ∈ *set as* ∨ *Eq 0* (*Suc k*) ∈ *set as*}
by(*auto simp*: *ebounds-def split*: *atom.splits nat.splits*)

abbreviation *LB f xs* ≡ {*xs*!*i*|*i*. *Less* (*Suc i*) *0* ∈ *set*($DLO.atoms_0$ *f*)}
abbreviation *UB f xs* ≡ {*xs*!*i*|*i*. *Less 0* (*Suc i*) ∈ *set*($DLO.atoms_0$ *f*)}
definition *EQ f xs* = {*xs*!*k*|*k*.
Eq (*Suc k*) *0* ∈ *set*($DLO.atoms_0$ *f*) ∨ *Eq 0* (*Suc k*) ∈ *set*($DLO.atoms_0$ *f*)}

lemma *EQ-And*[*simp*]: *EQ* (*And f g*) *xs* = (*EQ f xs* ∪ *EQ g xs*)
by(*auto simp*:*EQ-def*)

lemma *EQ-Or*[*simp*]: *EQ* (*Or f g*) *xs* = (*EQ f xs* ∪ *EQ g xs*)
by(*auto simp*:*EQ-def*)

lemma *EQ-conv-set-ebounds*:
x ∈ *EQ f xs* = (∃ *e*∈*set*(*ebounds*(*DLO.atoms*$_0$ *f*)). *x* = *xs*!*e*)
by(*auto simp*: *EQ-def set-ebounds*)

fun *isubst* **where** *isubst k 0* = *k* | *isubst k* (*Suc i*) = *i*

fun *asubst* :: *nat* ⇒ *atom* ⇒ *atom* **where**
asubst k (*Less i j*) = *Less* (*isubst k i*) (*isubst k j*)|
asubst k (*Eq i j*) = *Eq* (*isubst k i*) (*isubst k j*)

abbreviation *subst* φ *k* ≡ *map*$_{fm}$ (*asubst k*) φ

lemma *I-subst*:
qfree f ⟹ *DLO.I* (*subst f k*) *xs* = *DLO.I f* (*xs*!*k* # *xs*)
apply(*induct f*)
apply(*simp-all*)
apply(*rename-tac a*)
apply(*case-tac a*)
apply(*simp-all add*:*nth.simps split*:*nat.splits*)
done

fun *amin-inf* :: *atom* ⇒ *atom fm* **where**
amin-inf (*Less* - *0*) = *FalseF* |
amin-inf (*Less 0* -) = *TrueF* |
amin-inf (*Less* (*Suc i*) (*Suc j*)) = *Atom*(*Less i j*) |
amin-inf (*Eq 0 0*) = *TrueF* |
amin-inf (*Eq 0* -) = *FalseF* |
amin-inf (*Eq* - *0*) = *FalseF* |
amin-inf (*Eq* (*Suc i*) (*Suc j*)) = *Atom*(*Eq i j*)

abbreviation *min-inf* :: *atom fm* ⇒ *atom fm* (‹*inf*$_-$›) **where**
inf$_-$ ≡ *amap*$_{fm}$ *amin-inf*

fun *aplus-inf* :: *atom* ⇒ *atom fm* **where**
aplus-inf (*Less 0* -) = *FalseF* |
aplus-inf (*Less* - *0*) = *TrueF* |
aplus-inf (*Less* (*Suc i*) (*Suc j*)) = *Atom*(*Less i j*) |
aplus-inf (*Eq 0 0*) = *TrueF* |
aplus-inf (*Eq 0* -) = *FalseF* |
aplus-inf (*Eq* - *0*) = *FalseF* |
aplus-inf (*Eq* (*Suc i*) (*Suc j*)) = *Atom*(*Eq i j*)

abbreviation *plus-inf* :: *atom fm* ⇒ *atom fm* (‹*inf*$_+$›) **where**
inf$_+$ ≡ *amap*$_{fm}$ *aplus-inf*

lemma *min-inf*:
nqfree f $\Longrightarrow \exists x.\ \forall y{\leq}x.$ *DLO.I* (*inf*$_-$ *f*) *xs* = *DLO.I f* (*y* # *xs*)
(**is** - $\Longrightarrow \exists x.$ *?P f x*)
proof(*induct f*)
case (*Atom a*)
show *?case*
proof (*cases a rule*: *amin-inf.cases*)
case *1* **thus** *?thesis* **by**(*auto simp add*:*nth-Cons' linorder-not-less*)
next
case *2* **thus** *?thesis*
by (*simp*) (*metis no-lb linorder-not-less order-less-le-trans*)
next
case *5* **thus** *?thesis*
by(*simp add*:*nth-Cons'*) (*metis no-lb linorder-not-less*)
next
case *6* **thus** *?thesis* **by** *simp* (*metis no-lb linorder-not-less*)
qed *simp-all*
next
case (*And f1 f2*)
then obtain *x1 x2* **where** *?P f1 x1 ?P f2 x2* **by** *fastforce+*
hence *?P* (*And f1 f2*) (*min x1 x2*) **by**(*force simp*:*and-def*)
thus *?case* **..**
next
case (*Or f1 f2*)
then obtain *x1 x2* **where** *?P f1 x1 ?P f2 x2* **by** *fastforce+*
hence *?P* (*Or f1 f2*) (*min x1 x2*) **by**(*force simp*:*or-def*)
thus *?case* **..**
qed *simp-all*

lemma *plus-inf*:
nqfree f $\Longrightarrow \exists x.\forall y{\geq}x.$ *DLO.I* (*inf*$_+$ *f*) *xs* = *DLO.I f* (*y* # *xs*)
(**is** - $\Longrightarrow \exists x.$ *?P f x*)
proof (*induct f*)
have *dlo-bound*: $\bigwedge z{::}'a.\ \exists x.\ \forall y{\geq}x.\ y > z$
proof −
fix *z*
from *no-ub* **obtain** *w* :: *'a* **where** *w* > *z* **..**
then have $\forall y{\geq}w.\ y > z$ **by** *auto*
then show *?thesis z* **..**
qed
case (*Atom a*)
show *?case*
proof (*cases a rule*: *aplus-inf.cases*)
case *1* **thus** *?thesis*
by (*simp add*: *nth-Cons'*) (*metis linorder-not-less*)
next
case *2* **thus** *?thesis* **by** (*auto intro*: *dlo-bound*)
next

```
    case 5 thus ?thesis
      by simp (metis dlo-bound less-imp-neq)
  next
    case 6 thus ?thesis
      by simp (metis dlo-bound less-imp-neq)
  qed simp-all
next
  case (And f1 f2)
  then obtain x1 x2 where ?P f1 x1 ?P f2 x2 by fastforce+
  hence ?P (And f1 f2) (max x1 x2) by(force simp:and-def)
  thus ?case ..
next
  case (Or f1 f2)
  then obtain x1 x2 where ?P f1 x1 ?P f2 x2 by fastforce+
  hence ?P (Or f1 f2) (max x1 x2) by(force simp:or-def)
  thus ?case ..
qed simp-all

context notes [[simp-depth-limit=2]]
begin

lemma LBex:
 ⟦ nqfree f; DLO.I f (x#xs); ¬DLO.I (inf_ f) xs; x ∉ EQ f xs ⟧
  ⟹ ∃ l∈ LB f xs. l < x
proof(induct f)
  case (Atom a) thus ?case
    by (cases a rule: amin-inf.cases)
       (simp-all add: nth.simps EQ-def split: nat.splits)
qed auto


lemma UBex:
 ⟦ nqfree f; DLO.I f (x#xs); ¬DLO.I (inf_+ f) xs; x ∉ EQ f xs ⟧
  ⟹ ∃ u ∈ UB f xs. x < u
proof(induct f)
  case (Atom a) thus ?case
    by (cases a rule: aplus-inf.cases)
       (simp-all add: nth.simps EQ-def split: nat.splits)
qed auto

end

lemma finite-LB: finite(LB f xs)
proof −
  have LB f xs = (λk. xs!k) ‘ set(lbounds(DLO.atoms_0 f))
    by (auto simp:set-lbounds image-def)
  thus ?thesis by simp
qed
```

lemma *finite-UB*: *finite*(*UB f xs*)
proof −
have *UB f xs* = (λ*k*. *xs*!*k*) ‘ *set*(*ubounds*(*DLO.atoms*$_0$ *f*))
by (*auto simp*:*set-ubounds image-def*)
thus *?thesis* **by** *simp*
qed

lemma *qfree-amin-inf*: *qfree* (*amin-inf a*)
by(*cases a rule*:*amin-inf.cases*) *simp-all*

lemma *qfree-min-inf*: *nqfree* φ ⟹ *qfree*(*inf*$_-$ φ)
by(*induct* φ)(*simp-all add*:*qfree-amin-inf*)

lemma *qfree-aplus-inf*: *qfree* (*aplus-inf a*)
by(*cases a rule*:*aplus-inf.cases*) *simp-all*

lemma *qfree-plus-inf*: *nqfree* φ ⟹ *qfree*(*inf*$_+$ φ)
by(*induct* φ)(*simp-all add*:*qfree-aplus-inf*)

end

theory *QEdlo*
imports *DLO*
begin

3.2 DNF-based quantifier elimination

definition *qe-dlo*$_1$ :: *atom list* ⇒ *atom fm* **where**
qe-dlo$_1$ *as* =
(*if Less 0 0* ∈ *set as then FalseF else*
let lbs = [*i*. *Less* (*Suc i*) *0* ← *as*]; *ubs* = [*j*. *Less 0* (*Suc j*) ← *as*];
pairs = [*Atom*(*Less i j*). *i* ← *lbs*, *j* ← *ubs*]
in list-conj pairs)

theorem *I-qe-dlo*$_1$:
assumes *less*: ∀ *a* ∈ *set as*. *is-Less a* **and** *dep*: ∀ *a* ∈ *set as*. *depends*$_{dlo}$ *a*
shows *DLO.I* (*qe-dlo*$_1$ *as*) *xs* = (∃ *x*. ∀ *a* ∈ *set as*. *I*$_{dlo}$ *a* (*x*#*xs*))
(**is** *?L* = *?R*)
proof
let *?lbs* = [*i*. *Less* (*Suc i*) *0* ← *as*]
let *?ubs* = [*j*. *Less 0* (*Suc j*) ← *as*]
let *?Ls* = *set ?lbs* **let** *?Us* = *set ?ubs*
let *?lb* = *Max* (⋃*x*∈*?Ls*. {*xs*!*x*})
let *?ub* = *Min* (⋃*x*∈*?Us*. {*xs*!*x*})
have *2*: *Less 0 0* ∉ *set as* ⟹ ∀ *a* ∈ *set as*.
(∃ *i* ∈ *?Ls*. *a* = *Less* (*Suc i*) *0*) ∨ (∃ *i* ∈ *?Us*. *a* = *Less 0* (*Suc i*))
proof
fix *a* **assume** *Less 0 0* ∉ *set as a* ∈ *set as*

```
    then obtain i j where [simp]: a = Less i j
      using less by (force simp:is-Less-iff)
    with dep obtain k where i = 0 ∧ j = Suc k ∨ i = Suc k ∧ j = 0
      using ‹Less 0 0 ∉ set as› ‹a ∈ set as›
      by auto (metis Nat.nat.nchotomy depends_dlo.simps(2))
    moreover hence i=0 ∧ k ∈ ?Us ∨ j=0 ∧ k ∈ ?Ls
      using ‹a ∈ set as› by force
    ultimately show (∃ i∈?Ls. a=Less (Suc i) 0) ∨ (∃ i∈?Us. a=Less 0 (Suc i))
      by force
  qed
  assume qe1: ?L
  hence 0: Less 0 0 ∉ set as by (auto simp:qe-dlo_1-def)
  with qe1 have 1: ∀ x∈?Ls. ∀ y∈?Us. xs ! x < xs ! y
    by (fastforce simp:qe-dlo_1-def)
  have finite: finite ?Ls finite ?Us by (rule finite-set)+
  { fix i x
    assume Less i 0 ∈ set as | Less 0 i ∈ set as
    moreover hence i ≠ 0 using 0 by iprover
    ultimately have (x#xs) ! i = xs!(i − 1) by (simp add: nth-Cons')
  } note this[simp]
  { assume nonempty: ?Ls ≠ {} ∧ ?Us ≠ {}
    hence Max (⋃ x∈?Ls. {xs!x}) < Min (⋃ x∈?Us. {xs!x})
      using 1 finite by auto
    then obtain m where ?lb < m ∧ m < ?ub using dense by blast
    hence ∀ i∈?Ls. xs!i < m and ∀ j∈?Us. m < xs!j
      using nonempty finite by auto
    hence ∀ a ∈ set as. I_dlo a (m # xs) using 2[OF 0] by(auto simp:less)
    hence ?R .. }
  moreover
  { assume asm: ?Ls ≠ {} ∧ ?Us = {}
    then obtain m where ?lb < m using no-ub by blast
    hence ∀ a∈ set as. I_dlo a (m # xs) using 2[OF 0] asm finite by auto
    hence ?R .. }
  moreover
  { assume asm: ?Ls = {} ∧ ?Us ≠ {}
    then obtain m where m < ?ub using no-lb by blast
    hence ∀ a∈ set as. I_dlo a (m # xs) using 2[OF 0] asm finite by auto
    hence ?R .. }
  moreover
  { assume ?Ls = {} ∧ ?Us = {}
    hence ?R using 2[OF 0] by (auto simp add:less)
  }
  ultimately show ?R by blast
next
  assume ?R
  then obtain x where 1: ∀ a∈ set as. I_dlo a (x # xs) ..
  hence 0: Less 0 0 ∉ set as by auto
  { fix i j
    assume asm: Less i 0 ∈ set as Less 0 j ∈ set as
```

hence $(x\#xs)!i < x\ x < (x\#xs)!j$ **using** *1* **by** *auto+*
hence $(x\#xs)!i < (x\#xs)!j$ **by**(*rule order-less-trans*)
moreover have $\neg(i = 0 \mid j = 0)$ **using** *0 asm* **by** *blast*
ultimately have $xs\ !\ (i - 1) < xs\ !\ (j - 1)$ **by** (*simp add*: *nth-Cons'*)
}
thus *?L* **using** *0 less*
by (*fastforce simp*: $qe\text{-}dlo_1$*-def is-Less-iff split*:*atom.splits nat.splits*)
qed

lemma $I\text{-}qe\text{-}dlo_1$*-pretty*:
$\forall\, a \in set\ as.\ is\text{-}Less\ a \wedge depends_{dlo}\ a \Longrightarrow DLO.is\text{-}dnf\text{-}qe\ -\ qe\text{-}dlo_1\ as$
by(*metis* $I\text{-}qe\text{-}dlo_1$)

definition *subst* :: *nat* ⇒ *nat* ⇒ *nat* ⇒ *nat* **where**
subst i j k = (*if k=0 then if i=0 then j else i else k*) − *1*
fun $subst_0$ :: *atom* ⇒ *atom* ⇒ *atom* **where**
$subst_0$ (*Eq i j*) *a* = (*case a of*
Less m n ⇒ *Less* (*subst i j m*) (*subst i j n*)
| *Eq m n* ⇒ *Eq* (*subst i j m*) (*subst i j n*))

lemma $subst_0$*-pretty*:
$subst_0$ (*Eq i j*) (*Less m n*) = *Less* (*subst i j m*) (*subst i j n*)
$subst_0$ (*Eq i j*) (*Eq m n*) = *Eq* (*subst i j m*) (*subst i j n*)
by *auto*

interpretation DLO_e:
$ATOM\text{-}EQ\ neg_{dlo}\ (\lambda a.\ True)\ I_{dlo}\ depends_{dlo}\ decr_{dlo}$
($\lambda Eq\ i\ j \Rightarrow i{=}0 \vee j{=}0 \mid a \Rightarrow False$)
($\lambda Eq\ i\ j \Rightarrow i{=}j \mid a \Rightarrow False$) $subst_0$
apply(*unfold-locales*)
apply(*fastforce simp*:*subst-def nth-Cons' split*:*atom.splits if-split-asm*)
apply(*simp add*:*subst-def split*:*atom.splits*)
apply(*fastforce simp*:*subst-def nth-Cons' split*:*atom.splits*)
apply(*fastforce simp add*:*subst-def split*:*atom.splits*)
done

setup ‹*Sign.revert-abbrev* @{*const-abbrev* DLO_e*.lift-dnfeq-qe*}›

definition *qe-dlo* = DLO_e*.lift-dnfeq-qe* $qe\text{-}dlo_1$
lemma $qfree\text{-}qe\text{-}dlo_1$: *qfree* ($qe\text{-}dlo_1$ *as*)
by(*auto simp*:$qe\text{-}dlo_1$*-def intro*!:*qfree-list-conj*)

theorem *I-qe-dlo*: *DLO.I* (*qe-dlo* φ) *xs* = *DLO.I* φ *xs*
unfolding *qe-dlo-def*
by(*fastforce intro*!: $I\text{-}qe\text{-}dlo_1$ $qfree\text{-}qe\text{-}dlo_1$ DLO_e*.I-lift-dnfeq-qe*
simp: *is-Less-iff not-is-Eq-iff split*:*atom.splits cong*:*conj-cong*)

theorem *qfree-qe-dlo*: *qfree* (*qe-dlo* φ)
by(*simp add*:*qe-dlo-def* DLO_e*.qfree-lift-dnfeq-qe* $qfree\text{-}qe\text{-}dlo_1$)

end

theory *QEdlo-ex* **imports** *QEdlo*
begin

definition *interpret* :: *atom fm* ⇒ *'a*::*dlo list* ⇒ *bool* **where**
interpret = *Logic.interpret* I_{dlo}

lemma *interpret-Atoms*:
interpret (*Atom* (*Eq i j*)) *xs* = (*xs*!*i* = *xs*!*j*)
interpret (*Atom* (*Less i j*)) *xs* = (*xs*!*i* < *xs*!*j*)
by(*simp-all add*:*interpret-def*)

lemma *interpret-others*:
interpret (*Neg*(*ExQ* (*Neg f*))) *xs* = (∀ *x*. *interpret f* (*x*#*xs*))
interpret (*Or* (*Neg f1*) *f2*) *xs* = (*interpret f1 xs* ⟶ *interpret f2 xs*)
by(*simp-all add*:*interpret-def*)

lemmas *reify-eqs* =
Logic.interpret.simps(*1*,*2*,*4*−*7*)[*of* I_{dlo}, *folded interpret-def*]
interpret-others interpret-Atoms

method-setup *dlo-reify* = ‹
Scan.succeed
(*fn ctxt* =>
Method.SIMPLE-METHOD' (*Reification.tac ctxt* @{*thms reify-eqs*} *NONE*
THEN' simp-tac (*put-simpset HOL-basic-ss ctxt addsimps* [@{*thminterpret-def*}])))
› *dlo reification*

declare I_{dlo}.*simps*(*1*)[*code*]
declare *Logic.interpret.simps*[*code del*]
declare *Logic.interpret.simps*(*1*−*2*)[*code*]

3.3 Examples

lemma ∀ *x*::*real*. ¬ *x* < *x*
apply *dlo-reify*
apply (*subst I-qe-dlo* [*symmetric*])
by *eval*

lemma ∀ *x y*::*real*. ∃ *z*. *x* < *y* ⟶ *x* < *z* ∧ *z* < *y*
apply *dlo-reify*
apply (*subst I-qe-dlo* [*symmetric*])
by *eval*

lemma $\exists\ x{::}real.\ a{+}b < x \wedge x < c{*}d$
apply *dlo-reify*
apply (*subst I-qe-dlo* [*symmetric*])
apply *normalization*
oops

lemma $\forall x{::}real.\ \neg\ x < x$
apply *dlo-reify*
apply (*subst I-qe-dlo* [*symmetric*])
by *eval*

lemma $\forall x\ y{::}real.\ \exists z.\ x < y \longrightarrow x < z \wedge z < y$
apply *dlo-reify*
apply (*subst I-qe-dlo* [*symmetric*])
by *eval*

lemma $\neg(\exists x\ y\ z.\ \forall u{::}real.\ x < x \vee \neg\ x{<}u \vee x{<}y \wedge y{<}z \wedge \neg\ x{<}z)$
apply *dlo-reify*
apply (*subst I-qe-dlo* [*symmetric*])
by *eval*

lemma *qe-dlo*(*AllQ* (*Imp* (*Atom*(*Less 0 1*)) (*Atom*(*Less 1 0*)))) = *FalseF*
by *eval*

lemma *qe-dlo*(*AllQ*(*AllQ* (*Imp* (*Atom*(*Less 0 1*)) (*Atom*(*Less 0 1*))))) = *TrueF*
by *eval*

lemma
qe-dlo(*AllQ*(*ExQ*(*AllQ* (*And* (*Atom*(*Less 2 1*)) (*Atom*(*Less 1 0*)))))) = *FalseF*
by *eval*

lemma *qe-dlo*(*AllQ*(*ExQ*(*ExQ* (*And* (*Atom*(*Less 1 2*)) (*Atom*(*Less 2 0*)))))) = *TrueF*
by *eval*

lemma
qe-dlo(*AllQ*(*AllQ*(*ExQ* (*And* (*Atom*(*Less 1 0*)) (*Atom*(*Less 0 2*)))))) = *FalseF*
by *eval*

lemma *qe-dlo*(*AllQ*(*AllQ*(*ExQ* (*Imp* (*Atom*(*Less 1 2*)) (*And* (*Atom*(*Less 1 0*)) (*Atom*(*Less 0 2*))))))) = *TrueF*
by *eval*

value *qe-dlo*(*AllQ* (*Imp* (*Atom*(*Less 0 1*)) (*Atom*(*Less 0 2*))))

end

theory *QEdlo-fr*
imports *DLO*
begin

3.4 Interior Point Method

This section formalizes a new quantifier elimination procedure based on the idea of Ferrante and Rackoff [2] (see also §4.3) of taking a point between each lower and upper bound as a test point. For dense linear orders it is not obvious how to realize this because we cannot name any intermediate point directly.

fun $asubst_2$ :: $nat \Rightarrow nat \Rightarrow atom \Rightarrow atom\ fm$ **where**
$asubst_2\ l\ u\ (Less\ 0\ 0) = FalseF$ |
$asubst_2\ l\ u\ (Less\ 0\ (Suc\ j)) = Or\ (Atom(Less\ u\ j))\ (Atom(Eq\ u\ j))$ |
$asubst_2\ l\ u\ (Less\ (Suc\ i)\ 0) = Or\ (Atom(Less\ i\ l))\ (Atom(Eq\ i\ l))$ |
$asubst_2\ l\ u\ (Less\ (Suc\ i)\ (Suc\ j)) = Atom(Less\ i\ j)$ |
$asubst_2\ l\ u\ (Eq\ 0\ 0) = TrueF$ |
$asubst_2\ l\ u\ (Eq\ 0\ \text{-}) = FalseF$ |
$asubst_2\ l\ u\ (Eq\ \text{-}\ 0) = FalseF$ |
$asubst_2\ l\ u\ (Eq\ (Suc\ i)\ (Suc\ j)) = Atom(Eq\ i\ j)$

abbreviation $subst_2\ l\ u \equiv amap_{fm}\ (asubst_2\ l\ u)$

lemma *I-subst*$_2$*1*:
$nqfree\ f \Longrightarrow xs!l < xs!u \Longrightarrow DLO.I\ (subst_2\ l\ u\ f)\ xs$
$\Longrightarrow xs!l < x \Longrightarrow x < xs!u \Longrightarrow DLO.I\ f\ (x\#xs)$
proof(*induct f arbitrary*: *x*)
case (*Atom a*) **thus** *?case*
by (*cases* (*l*,*u*,*a*) *rule*: $asubst_2$.*cases*) *auto*
qed *auto*

definition
$nolub\ f\ xs\ l\ x\ u \longleftrightarrow (\forall\, y{\in}\{l{<}..{<}x\}.\ y \notin LB\ f\ xs) \wedge (\forall\, y{\in}\{x{<}..{<}u\}.\ y \notin UB\ f\ xs)$

lemma *nolub-And*[*simp*]:
$nolub\ (And\ f\ g)\ xs\ l\ x\ u = (nolub\ f\ xs\ l\ x\ u \wedge nolub\ g\ xs\ l\ x\ u)$
by(*auto simp*:*nolub-def*)

lemma *nolub-Or*[*simp*]:
$nolub\ (Or\ f\ g)\ xs\ l\ x\ u = (nolub\ f\ xs\ l\ x\ u \wedge nolub\ g\ xs\ l\ x\ u)$
by(*auto simp*:*nolub-def*)

context notes [[*simp-depth-limit*=*3*]]
begin

lemma *innermost-intvl*:
$[\![\ nqfree\ f;\ nolub\ f\ xs\ l\ x\ u;\ l < x;\ x < u;\ x \notin EQ\ f\ xs;$

```
    DLO.I f (x#xs); l < y; y < u⟧
    ⟹ DLO.I f (y#xs)
proof(induct f)
  case (Atom a)
  show ?case
  proof (cases a)
    case (Less i j)
    then show ?thesis using Atom
      unfolding nolub-def
      by (clarsimp simp: nth.simps Ball-def split:if-split-asm nat.splits)
         (metis not-le-imp-less order-antisym order-less-trans)+
  next
    case [simp]: (Eq i j)
    show ?thesis
    proof (cases i)
      case [simp]: 0
      show ?thesis
      proof (cases j)
        case 0 thus ?thesis using Atom by simp
      next
        case Suc thus ?thesis using Atom by(simp add:EQ-def)
      qed
    next
      case [simp]: Suc
      show ?thesis
      proof (cases j)
        case 0 thus ?thesis using Atom by(simp add:EQ-def)
      next
        case Suc thus ?thesis using Atom by simp
      qed
    qed
  qed
next
  case (And f1 f2) thus ?case by (fastforce)
next
  case (Or f1 f2) thus ?case by (fastforce)
qed simp+

lemma I-subst_2 2:
 nqfree f ⟹ xs!l < x ∧ x < xs!u ⟹ nolub f xs (xs!l) x (xs!u)
 ⟹ ∀ x∈{xs!l <..< xs!u}. DLO.I f (x#xs) ∧ x ∉ EQ f xs
 ⟹ DLO.I (subst_2 l u f) xs
proof (induct f)
  case (Atom a) show ?case
    apply (cases (l,u,a) rule: asubst_2.cases)
    apply(insert Atom, auto simp: EQ-def nolub-def split:if-split-asm)
    done
next
  case Or thus ?case by (simp add: Ball-def)(metis innermost-intvl)
```

qed *auto*

end

definition
$qe\text{-}interior_1\ \varphi =$
(*let* $as = DLO.atoms_0\ \varphi$; $lbs = lbounds\ as$; $ubs = ubounds\ as$; $ebs = ebounds\ as$;
$intrs = [And\ (Atom(Less\ l\ u))\ (subst_2\ l\ u\ \varphi).\ l{\leftarrow}lbs,\ u{\leftarrow}ubs]$
in list-disj $(inf_-\ \varphi\ \#\ inf_+\ \varphi\ \#\ intrs\ @\ map\ (subst\ \varphi)\ ebs)$)

lemma *dense-interval*:
assumes *finite L finite U* $l \in L\ u \in U\ l < x\ x < u\ P(x::'a::dlo)$
and *dense*: $\bigwedge y\ l\ u.\ [\![\ \forall\, y{\in}\{l{<}..{<}x\}.\ y \notin L;\ \forall\, y{\in}\{x{<}..{<}u\}.\ y \notin U;$
$l{<}x; x{<}u;\ l{<}y; y{<}u\ ]\!] \Longrightarrow P\ y$
shows $\exists\, l{\in}L.\exists\, u{\in}U.\ l{<}x \wedge x{<}u \wedge (\forall\, y{\in}\{l{<}..{<}x\}.\ y{\notin}L) \wedge (\forall\, y{\in}\{x{<}..{<}u\}.\ y{\notin}U)$
$\wedge\ (\forall\, y.\ l{<}y \wedge y{<}u \longrightarrow P\ y)$
proof −
let *?L* $= \{l{:}L.\ l < x\}$ **let** *?U* $= \{u{:}U.\ x < u\}$
let *?ll* $= Max\ ?L$ **let** *?uu* $= Min\ ?U$
have $?L \neq \{\}$ **using** ‹$l \in L$› ‹$l{<}x$› **by** (*blast intro*:*order-less-imp-le*)
moreover have $?U \neq \{\}$ **using** ‹$u{:}U$› ‹$x{<}u$› **by** (*blast intro*:*order-less-imp-le*)
ultimately have $\forall\, y.\ ?ll{<}y \wedge y{<}x \longrightarrow y \notin L\ \forall\, y.\ x{<}y \wedge y{<}?uu \longrightarrow y \notin U$
using ‹*finite L*› ‹*finite U*› **by** *force*+
moreover have $?ll \in L$
proof
show $?ll \in ?L$ **using** ‹*finite L*› *Max-in*[*OF* - ‹$?L \neq \{\}$›] **by** *simp*
show $?L \subseteq L$ **by** *blast*
qed
moreover have $?uu \in U$
proof
show $?uu \in ?U$ **using** ‹*finite U*› *Min-in*[*OF* - ‹$?U \neq \{\}$›] **by** *simp*
show $?U \subseteq U$ **by** *blast*
qed
moreover have $?ll < x$ **using** ‹*finite L*› ‹$?L \neq \{\}$› **by** *simp*
moreover have $x < ?uu$ **using** ‹*finite U*› ‹$?U \neq \{\}$› **by** *simp*
moreover have $?ll < ?uu$ **using** ‹$?ll{<}x$› ‹$x{<}?uu$› **by** *simp*
ultimately show *?thesis* **using** ‹$l < x$› ‹$x < u$› ‹$?L \neq \{\}$› ‹$?U \neq \{\}$›
by(*blast intro*!:*dense greaterThanLessThan-iff*[*THEN iffD1*])
qed

theorem *I-interior1*:
assumes *nqfree* φ **shows** $DLO.I\ (qe\text{-}interior_1\ \varphi)\ xs = (\exists\, x.\ DLO.I\ \varphi\ (x\#xs))$
(**is** $?QE = ?EX$)
proof
assume *?QE*
{ assume $DLO.I\ (inf_-\ \varphi)\ xs$
hence *?EX* **using** ‹*?QE*› *min-inf*[*of* φ *xs*] ‹*nqfree* φ›
by(*auto simp add*:$qe\text{-}interior_1$*-def amap-fm-list-disj*)

```
  } moreover
  { assume  DLO.I (inf_+ φ) xs
    hence ?EX using ‹?QE› plus-inf[of φ xs] ‹nqfree φ›
      by(auto simp add:qe-interior_1-def amap-fm-list-disj)
  } moreover
  { assume ¬DLO.I (inf_− φ) xs ∧ ¬DLO.I (inf_+ φ) xs ∧
           (∀ x∈EQ φ xs. ¬DLO.I φ (x#xs))
    with ‹?QE› ‹nqfree φ› obtain l u
      where DLO.I (subst_2 l u φ) xs and xs!l < xs!u
    by(fastforce simp: qe-interior_1-def set-lbounds set-ubounds I-subst EQ-conv-set-ebounds)
    moreover then obtain x where xs!l < x ∧ x < xs!u by(metis dense)
    ultimately have DLO.I φ (x # xs)
      using ‹nqfree φ› I-subst_2 1[OF ‹nqfree φ› ‹xs!l < xs!u›] by simp
    hence ?EX .. }
  ultimately show ?EX by blast
next
  let ?as = DLO.atoms_0 φ let ?E = set(ebounds ?as)
  assume ?EX
  then obtain x where x: DLO.I φ (x#xs) ..
  { assume DLO.I (inf_− φ) xs ∨ DLO.I (inf_+ φ) xs
    hence ?QE using ‹nqfree φ› by(auto simp:qe-interior_1-def)
  } moreover
  { assume ∃ k ∈ ?E. DLO.I (subst φ k) xs
    hence ?QE by(force simp:qe-interior_1-def) } moreover
  { assume ¬ DLO.I (inf_− φ) xs and ¬ DLO.I (inf_+ φ) xs
    and ∀ k ∈ ?E. ¬ DLO.I (subst φ k) xs
    hence noE: ∀ e ∈ EQ φ xs. ¬ DLO.I φ (e#xs)
      using ‹nqfree φ› by (force simp:set-ebounds EQ-def I-subst)
    hence x ∉ EQ φ xs using x by fastforce
    obtain l where l ∈ LB φ xs l < x
      using LBex[OF ‹nqfree φ› x ‹¬ DLO.I(inf_− φ) xs› ‹x ∉ EQ φ xs›] ..
    obtain u where u ∈ UB φ xs x < u
      using UBex[OF ‹nqfree φ› x ‹¬ DLO.I(inf_+ φ) xs› ‹x ∉ EQ φ xs›] ..
    have ∃ l∈LB φ xs. ∃ u∈UB φ xs. l<x ∧ x<u ∧ nolub φ xs l x u ∧ (∀ y. l < y
∧ y < u ⟶ DLO.I φ (y#xs))
     using dense-interval[where P = λx. DLO.I φ (x#xs), OF finite-LB finite-UB
‹l:LB φ xs› ‹u:UB φ xs› ‹l<x› ‹x<u› x] x innermost-intvl[OF ‹nqfree φ› - - - ‹x
∉ EQ φ xs›]
      by (simp add:nolub-def)
    then obtain m n where
      Less (Suc m) 0 ∈ set ?as Less 0 (Suc n) ∈ set ?as
      xs!m < x ∧ x < xs!n
      nolub φ xs (xs!m) x (xs!n)
      ∀ y. xs!m < y ∧ y < xs!n ⟶ DLO.I φ (y#xs)
      by blast
    moreover
    hence DLO.I (subst_2 m n φ) xs using noE
      by(force intro!: I-subst_2 2[OF ‹nqfree φ›])
    ultimately have ?QE
```

by(*fastforce simp add*:$qe\text{-}interior_1\text{-}def$ *bex-Un set-lbounds set-ubounds*)
} ultimately show *?QE* **by** *blast*
qed

lemma $qfree\text{-}asubst_2$: *qfree* ($asubst_2$ *l u a*)
by(*cases* (*l*,*u*,*a*) *rule*:$asubst_2.cases$) *simp-all*

lemma $qfree\text{-}subst_2$: *nqfree* $\varphi \Longrightarrow$ *qfree* ($subst_2$ *l u* φ)
by(*induct* φ) (*simp-all add*:$qfree\text{-}asubst_2$)

lemma *qfree-interior1*: *nqfree* $\varphi \Longrightarrow$ *qfree*($qe\text{-}interior_1$ φ)
apply(*simp add*:$qe\text{-}interior_1\text{-}def$)
apply(*rule qfree-list-disj*)
apply (*auto simp*:*qfree-min-inf qfree-plus-inf* $qfree\text{-}subst_2$ *qfree-map-fm*)
done

definition *qe-interior* = $DLO.lift\text{-}nnf\text{-}qe$ $qe\text{-}interior_1$

lemma *qfree-qe-interior*: *qfree*(*qe-interior* φ)
by(*simp add*: *qe-interior-def DLO.qfree-lift-nnf-qe qfree-interior1*)

lemma *I-qe-interior*: *DLO.I* (*qe-interior* φ) *xs* = *DLO.I* φ *xs*
by(*simp add*:*qe-interior-def DLO.I-lift-nnf-qe qfree-interior1 I-interior1*)

end

theory *QEdlo-inf*
imports *DLO*
begin

3.5 Quantifier elimination with infinitesimals

This section presents a new quantifier elimination procedure for dense linear orders based on (the simulation of) infinitesimals. It is a fairly straightforward adaptation of the analogous algorithm by Loos and Weispfenning for linear arithmetic described in §4.4.

fun *asubst-peps* :: *nat* ⇒ *atom* ⇒ *atom fm* (‹$asubst_+$›) **where**
asubst-peps k (*Less 0 0*) = *FalseF* |
asubst-peps k (*Less 0* (*Suc j*)) = *Atom*(*Less k j*) |
asubst-peps k (*Less* (*Suc i*) *0*) = (*if i*=*k then TrueF*
else Or (*Atom*(*Less i k*)) (*Atom*(*Eq i k*))) |
asubst-peps k (*Less* (*Suc i*) (*Suc j*)) = *Atom*(*Less i j*) |
asubst-peps k (*Eq 0 0*) = *TrueF* |
asubst-peps k (*Eq 0 -*) = *FalseF* |
asubst-peps k (*Eq - 0*) = *FalseF* |
asubst-peps k (*Eq* (*Suc i*) (*Suc j*)) = *Atom*(*Eq i j*)

abbreviation *subst-peps* :: *atom fm* ⇒ *nat* ⇒ *atom fm* (‹*subst*$_+$›) **where**
subst$_+$ φ *k* ≡ *amap*$_{fm}$ (*asubst*$_+$ *k*) φ

definition *nolb* φ *xs l x* = (∀ *y*∈{*l*<..<*x*}. *y* ∉ *LB* φ *xs*)

lemma *nolb-And*[*simp*]:
nolb (*And* φ_1 φ_2) *xs l x* = (*nolb* φ_1 *xs l x* ∧ *nolb* φ_2 *xs l x*)
apply(*clarsimp simp*:*nolb-def*)
apply *blast*
done

lemma *nolb-Or*[*simp*]:
nolb (*Or* φ_1 φ_2) *xs l x* = (*nolb* φ_1 *xs l x* ∧ *nolb* φ_2 *xs l x*)
apply(*clarsimp simp*:*nolb-def*)
apply *blast*
done

context notes [[*simp-depth-limit*=*3*]]
begin

lemma *innermost-intvl*:
⟦ *nqfree* φ; *nolb* φ *xs l x*; *l* < *x*; *x* ∉ *EQ* φ *xs*; *DLO.I* φ (*x*#*xs*); *l* < *y*; *y* ≤ *x*⟧
⟹ *DLO.I* φ (*y*#*xs*)
proof(*induct* φ)
case (*Atom a*)
show *?case*
proof (*cases a*)
case (*Less i j*)
then show *?thesis* **using** *Atom*
unfolding *nolb-def*
by (*clarsimp simp*: *nth.simps Ball-def split*:*if-split-asm nat.splits*)
(*metis not-le-imp-less order-antisym order-less-trans*)+
next
case (*Eq i j*) **thus** *?thesis* **using** *Atom*
apply(*clarsimp simp*:*EQ-def nolb-def nth-Cons'*)
apply(*case-tac i*=*0* ∧ *j*=*0*) **apply** *simp*
apply(*case-tac i*≠*0* ∧ *j*≠*0*) **apply** *simp*
apply(*case-tac i*=*0* ∧ *j*≠*0*) **apply** (*fastforce split*:*if-split-asm*)
apply(*case-tac i*≠*0* ∧ *j*=*0*) **apply** (*fastforce split*:*if-split-asm*)
apply *arith*
done
qed
next
case *And* **thus** *?case* **by** (*fastforce*)
next
case *Or* **thus** *?case* **by** (*fastforce*)
qed *simp*+

lemma *I-subst-peps2*:
nqfree $\varphi \Longrightarrow xs!l < x \Longrightarrow nolb\ \varphi\ xs\ (xs!l)\ x \Longrightarrow x \notin EQ\ \varphi\ xs$
$\Longrightarrow \forall\, y \in \{xs!l <..\ x\}.\ DLO.I\ \varphi\ (y\#xs)$
$\Longrightarrow DLO.I\ (subst_+\ \varphi\ l)\ xs$
proof(*induct* φ)
case *FalseF* **thus** *?case*
by *simp* (*metis antisym-conv1 linorder-neq-iff*)
next
case (*Atom a*)
show *?case*
proof(*cases* (*l*,*a*) *rule*:*asubst-peps.cases*)
case *3* **thus** *?thesis* **using** *Atom*
by (*auto simp*: *nolb-def EQ-def Ball-def*)
(*metis One-nat-def antisym-conv1 not-less-iff-gr-or-eq*)
qed (*insert Atom*, *auto simp*: *nolb-def EQ-def Ball-def*)
next
case *Or* **thus** *?case* **by**(*simp add*: *Ball-def*)(*metis order-refl innermost-intvl*)
qed *simp-all*

end

lemma *dense-interval*:
assumes *finite L* $l \in L\ l < x\ P(x::'a::dlo)$
and *dense*: $\bigwedge y\ l.\ [\![\ \forall\, y{\in}\{l{<}..{<}x\}.\ y \notin L;\ l{<}x;\ l{<}y;\ y{\leq}x\ ]\!] \Longrightarrow P\ y$
shows $\exists\, l{\in}L.\ l{<}x \wedge (\forall\, y{\in}\{l{<}..{<}x\}.\ y \notin L) \wedge (\forall\, y.\ l{<}y \wedge y{\leq}x \longrightarrow P\ y)$
proof −
let *?L* = $\{l{\in}L.\ l < x\}$
let *?ll* = *Max ?L*
have *?L* $\neq \{\}$ **using** ‹$l \in L$› ‹$l{<}x$› **by** (*blast intro*:*order-less-imp-le*)
hence $\forall\, y.\ ?ll{<}y \wedge y{<}x \longrightarrow y \notin L$ **using** ‹*finite L*› **by** *force*
moreover have *?ll* $\in L$
proof
show *?ll* $\in$ *?L* **using** ‹*finite L*› *Max-in*[*OF* - ‹*?L* $\neq \{\}$›] **by** *simp*
show *?L* $\subseteq L$ **by** *blast*
qed
moreover have *?ll* $< x$ **using** ‹*finite L*› ‹*?L* $\neq \{\}$› **by** *simp*
ultimately show *?thesis* **using** ‹$l < x$› ‹*?L* $\neq \{\}$›
by(*blast intro*!:*dense greaterThanLessThan-iff*[*THEN iffD1*])
qed

lemma *I-subst-peps*:
nqfree $\varphi \Longrightarrow DLO.I\ (subst_+\ \varphi\ l)\ xs \longrightarrow$
$(\exists\, leps{>}xs!l.\ \forall\, x.\ xs!l < x \wedge x \leq leps \longrightarrow DLO.I\ \varphi\ (x\#xs))$
proof(*induct* φ)
case *TrueF* **thus** *?case* **by** *simp* (*metis no-ub*)
next
case (*Atom a*)
show *?case*

```
  proof (cases (l,a) rule: asubst-peps.cases)
    case 2 thus ?thesis using Atom
      apply(auto)
      apply(drule dense)
      apply(metis One-nat-def xt1(7))
      done
  next
    case 3 thus ?thesis using Atom
      apply(auto)
        apply (metis no-ub)
       apply (metis no-ub less-trans)
      apply (metis no-ub)
      done
  next
    case 4 thus ?thesis using Atom by(auto)(metis no-ub)
  next
    case 5 thus ?thesis using Atom by(auto)(metis no-ub)
  next
    case 8 thus ?thesis using Atom by(auto)(metis no-ub)
  qed (insert Atom, auto)
next
  case And thus ?case
    apply clarsimp
    apply(rule-tac x=min leps lepsa in exI)
    apply simp
    done
next
  case Or thus ?case by force
qed simp-all


definition
qe-eps1(φ) =
(let as = DLO.atoms0 φ; lbs = lbounds as; ebs = ebounds as
 in list-disj (inf_ φ # map (subst+ φ) lbs @ map (subst φ) ebs))

theorem I-qe-eps1:
assumes nqfree φ shows DLO.I (qe-eps1 φ) xs = (∃ x. DLO.I φ (x#xs))
  (is ?QE = ?EX)
proof
  let ?as = DLO.atoms0 φ let ?ebs = ebounds ?as
  assume ?QE
  { assume DLO.I (inf_ φ) xs
    hence ?EX using ‹?QE› min-inf[of φ xs] ‹nqfree φ›
      by(auto simp add:qe-eps1-def amap-fm-list-disj)
  } moreover
  { assume ∀ i ∈ set ?ebs. ¬DLO.I φ (xs!i # xs)
          ¬ DLO.I (inf_ φ) xs
    with ‹?QE› ‹nqfree φ› obtain l where DLO.I (subst+ φ l) xs
```

```
    by(fastforce simp: I-subst qe-eps1-def set-ebounds set-lbounds)
   then obtain leps where DLO.I φ (leps#xs)
    using I-subst-peps[OF ‹nqfree φ›] by fastforce
   hence ?EX .. }
  ultimately show ?EX by blast
next
  let ?as = DLO.atoms0 φ let ?ebs = ebounds ?as
  assume ?EX
  then obtain x where x: DLO.I φ (x#xs) ..
  { assume DLO.I (inf_ φ) xs
   hence ?QE using ‹nqfree φ› by(auto simp:qe-eps1-def)
  } moreover
  { assume ∃k ∈ set ?ebs. DLO.I (subst φ k) xs
   hence ?QE by(auto simp:qe-eps1-def) } moreover
  { assume ¬ DLO.I (inf_ φ) xs
   and ∀k ∈ set ?ebs. ¬ DLO.I (subst φ k) xs
   hence noE: ∀e ∈ EQ φ xs. ¬ DLO.I φ (e#xs) using ‹nqfree φ›
    by (auto simp:set-ebounds EQ-def I-subst nth-Cons' split:if-split-asm)
   hence x ∉ EQ φ xs using x by fastforce
   obtain l where l ∈ LB φ xs l < x
    using LBex[OF ‹nqfree φ› x ‹¬ DLO.I(inf_ φ) xs› ‹x ∉ EQ φ xs›] ..
   have ∃l∈LB φ xs. l<x ∧ nolb φ xs l x ∧
         (∀y. l < y ∧ y ≤ x ⟶ DLO.I φ (y#xs))
    using dense-interval[where P = λx. DLO.I φ (x#xs), OF finite-LB ‹l∈LB
φ xs› ‹l<x› x] x innermost-intvl[OF ‹nqfree φ› - - ‹x ∉ EQ φ xs›]
    by (simp add:nolb-def)
   then obtain m
    where *: Less (Suc m) 0 ∈ set ?as ∧ xs!m < x ∧ nolb φ xs (xs!m) x
         ∧ (∀y. xs!m < y ∧ y ≤ x ⟶ DLO.I φ (y#xs))
    by blast
   then have DLO.I (subst+ φ m) xs
    using noE by(auto intro!: I-subst-peps2[OF ‹nqfree φ›])
   with * have ?QE
    by(simp add:qe-eps1-def bex-Un set-lbounds set-ebounds) metis
  } ultimately show ?QE by blast
qed

lemma qfree-asubst-peps: qfree (asubst+ k a)
by(cases (k,a) rule:asubst-peps.cases) simp-all

lemma qfree-subst-peps: nqfree φ ⟹ qfree (subst+ φ k)
by(induct φ) (simp-all add:qfree-asubst-peps)

lemma qfree-qe-eps1: nqfree φ ⟹ qfree(qe-eps1 φ)
apply(simp add:qe-eps1-def)
apply(rule qfree-list-disj)
apply (auto simp:qfree-min-inf qfree-subst-peps qfree-map-fm)
done
```

definition *qe-eps* = *DLO.lift-nnf-qe qe-eps*$_1$

lemma *qfree-qe-eps*: *qfree*(*qe-eps* φ)
by(*simp add*: *qe-eps-def DLO.qfree-lift-nnf-qe qfree-qe-eps*$_1$)

lemma *I-qe-eps*: *DLO.I* (*qe-eps* φ) *xs* = *DLO.I* φ *xs*
by(*simp add*:*qe-eps-def DLO.I-lift-nnf-qe qfree-qe-eps*$_1$ *I-qe-eps1*)

end

4 Linear real arithmetic

theory *LinArith*
imports *QE HOL−Library.ListVector Complex-Main*
begin

declare *iprod-assoc*[*simp*]

4.1 Basics

4.1.1 Syntax and Semantics

datatype *atom* = *Less real real list* | *Eq real real list*

fun *is-Less* :: *atom* ⇒ *bool* **where**
is-Less (*Less r rs*) = *True* |
is-Less f = *False*

abbreviation *is-Eq* ≡ *Not* ∘ *is-Less*

lemma *is-Less-iff*: *is-Less f* = (∃ *r rs*. *f* = *Less r rs*)
by(*induct f*) *auto*

lemma *is-Eq-iff*: (∀ *i j*. *a* ≠ *Less i j*) = (∃ *i j*. *a* = *Eq i j*)
by(*cases a*) *auto*

fun *neg*$_R$ :: *atom* ⇒ *atom fm* **where**
neg$_R$ (*Less r t*) = *Or* (*Atom*(*Less* (−*r*) (−*t*))) (*Atom*(*Eq r t*)) |
neg$_R$ (*Eq r t*) = *Or* (*Atom*(*Less r t*)) (*Atom*(*Less* (−*r*) (−*t*)))

fun *hd-coeff* :: *atom* ⇒ *real* **where**
hd-coeff (*Less r cs*) = (*case cs of* [] ⇒ *0* | *c*#- ⇒ *c*) |
hd-coeff (*Eq r cs*) = (*case cs of* [] ⇒ *0* | *c*#- ⇒ *c*)

definition *depends*$_R$ *a* = (*hd-coeff a* ≠ *0*)

fun *decr*$_R$ :: *atom* ⇒ *atom* **where**
decr$_R$ (*Less r rs*) = *Less r* (*tl rs*) |
decr$_R$ (*Eq r rs*) = *Eq r* (*tl rs*)

fun I_R :: *atom* ⇒ *real list* ⇒ *bool* **where**
I_R (*Less r cs*) *xs* = (*r* < ⟨*cs*,*xs*⟩) |
I_R (*Eq r cs*) *xs* = (*r* = ⟨*cs*,*xs*⟩)

definition $atoms_0$ = *ATOM*.$atoms_0$ $depends_R$

interpretation *R*: *ATOM* neg_R (λ*a*. *True*) I_R $depends_R$ $decr_R$
rewrites *ATOM*.$atoms_0$ $depends_R$ = $atoms_0$
proof *goal-cases*
case *1*
thus *?case*
apply(*unfold-locales*)
apply(*case-tac a*)
apply *simp-all*
apply(*case-tac a*)
apply *simp-all*
apply *arith*
apply *arith*
apply(*case-tac a*)
apply(*simp add*:$depends_R$-*def split*:*list.splits*)
done
next
case *2*
thus *?case* **by**(*simp add*:$atoms_0$-*def*)
qed

setup ‹*Sign.revert-abbrev* @{*const-abbrev R.I*}›
setup ‹*Sign.revert-abbrev* @{*const-abbrev R.lift-nnf-qe*}›

4.1.2 Shared constructions

fun *combine* :: (*real* ∗ *real list*) ⇒ (*real* ∗ *real list*) ⇒ *atom* **where**
combine (r_1,cs_1) (r_2,cs_2) = *Less* (r_1−r_2) (cs_2 − cs_1)

definition *lbounds as* = [(*r*/*c*, (−*1*/*c*) ∗$_s$ *cs*). *Less r* (*c*#*cs*) ← *as*, *c*>*0*]
definition *ubounds as* = [(*r*/*c*, (−*1*/*c*) ∗$_s$ *cs*). *Less r* (*c*#*cs*) ← *as*, *c*<*0*]
definition *ebounds as* = [(*r*/*c*, (−*1*/*c*) ∗$_s$ *cs*). *Eq r* (*c*#*cs*) ← *as*, *c*≠*0*]

lemma *set-lbounds*:
set(*lbounds as*) = {(*r*/*c*, (−*1*/*c*) ∗$_s$ *cs*)|*r c cs*. *Less r* (*c*#*cs*) ∈ *set as* ∧ *c*>*0*}
by(*force simp*: *lbounds-def split*:*list.splits atom.splits if-splits*)
lemma *set-ubounds*:
set(*ubounds as*) = {(*r*/*c*, (−*1*/*c*) ∗$_s$ *cs*)|*r c cs*. *Less r* (*c*#*cs*) ∈ *set as* ∧ *c*<*0*}
by(*force simp*: *ubounds-def split*:*list.splits atom.splits if-splits*)
lemma *set-ebounds*:

set(*ebounds as*) = {(*r*/*c*, (−*1*/*c*) ∗$_s$ *cs*)|*r c cs*. *Eq r* (*c*#*cs*) ∈ *set as* ∧ *c*≠*0*}
by(*force simp*: *ebounds-def split*:*list.splits atom.splits if-splits*)

abbreviation *EQ* **where**
EQ f xs ≡ {(*r* − ⟨*cs*,*xs*⟩)/*c*|*r c cs*. *Eq r* (*c*#*cs*) ∈ *set*(*R.atoms*$_0$ *f*) ∧ *c*≠*0*}
abbreviation *LB* **where**
LB f xs ≡ {(*r* − ⟨*cs*,*xs*⟩)/*c*|*r c cs*. *Less r* (*c*#*cs*) ∈ *set*(*R.atoms*$_0$ *f*) ∧ *c*>*0*}
abbreviation *UB* **where**
UB f xs ≡ {(*r* − ⟨*cs*,*xs*⟩)/*c*|*r c cs*. *Less r* (*c*#*cs*) ∈ *set*(*R.atoms*$_0$ *f*) ∧ *c*<*0*}

fun *asubst* :: *real* ∗ *real list* ⇒ *atom* ⇒ *atom* **where**
asubst (*r*,*cs*) (*Less s* (*d*#*ds*)) = *Less* (*s* − *d*∗*r*) (*d* ∗$_s$ *cs* + *ds*) |
asubst (*r*,*cs*) (*Eq s* (*d*#*ds*)) = *Eq* (*s* − *d*∗*r*) (*d* ∗$_s$ *cs* + *ds*) |
asubst (*r*,*cs*) (*Less s* []) = *Less s* [] |
asubst (*r*,*cs*) (*Eq s* []) = *Eq s* []

abbreviation *subst* *φ rcs* ≡ *map*$_{fm}$ (*asubst rcs*) *φ*

definition *eval* :: *real* ∗ *real list* ⇒ *real list* ⇒ *real* **where**
eval rcs xs = *fst rcs* + ⟨*snd rcs*,*xs*⟩

lemma *I-asubst*:
I$_R$ (*asubst t a*) *xs* = *I*$_R$ *a* (*eval t xs* # *xs*)
proof(*cases a*)
case (*Less r cs*)
thus *?thesis* **by**(*cases t*, *cases cs*,
simp-all add:*eval-def distrib-left iprod-left-add-distrib*)
arith
next
case (*Eq r cs*)
thus *?thesis*
by(*cases t*, *cases cs*, *simp-all add*:*eval-def distrib-left iprod-left-add-distrib*)
arith
qed

lemma *I-subst*:
qfree φ ⟹ *R.I* (*subst φ t*) *xs* = *R.I φ* (*eval t xs* # *xs*)
by(*induct φ*)(*simp-all add*:*I-asubst*)
lemma *I-subst-pretty*:
qfree φ ⟹ *R.I* (*subst φ* (*r*,*cs*)) *xs* = *R.I φ* ((*r* + ⟨*cs*,*xs*⟩) # *xs*)
by(*simp add*:*I-subst eval-def*)

fun *min-inf* :: *atom fm* ⇒ *atom fm* (‹*inf*$_-$›) **where**
inf$_-$ (*And φ*$_1$ *φ*$_2$) = *and* (*inf*$_-$ *φ*$_1$) (*inf*$_-$ *φ*$_2$) |
inf$_-$ (*Or φ*$_1$ *φ*$_2$) = *or* (*inf*$_-$ *φ*$_1$) (*inf*$_-$ *φ*$_2$) |
inf$_-$ (*Atom*(*Less r* (*c*#*cs*))) =

```
  (if c<0 then TrueF else if c>0 then FalseF else Atom(Less r cs)) |
inf_ (Atom(Eq r (c#cs))) = (if c=0 then Atom(Eq r cs) else FalseF) |
inf_ φ = φ

fun plus-inf :: atom fm ⇒ atom fm (‹inf+›) where
inf+ (And φ1 φ2) = and (inf+ φ1) (inf+ φ2) |
inf+ (Or φ1 φ2) = or (inf+ φ1) (inf+ φ2) |
inf+ (Atom(Less r (c#cs))) =
  (if c>0 then TrueF else if c<0 then FalseF else Atom(Less r cs)) |
inf+ (Atom(Eq r (c#cs))) = (if c=0 then Atom(Eq r cs) else FalseF) |
inf+ φ = φ

lemma qfree-min-inf: qfree φ ⟹ qfree(inf_ φ)
by(induct φ rule:min-inf.induct) simp-all

lemma qfree-plus-inf: qfree φ ⟹ qfree(inf+ φ)
by(induct φ rule:plus-inf.induct) simp-all

lemma min-inf:
  nqfree f ⟹ ∃x. ∀y≤x. R.I (inf_ f) xs = R.I f (y # xs)
  (is - ⟹ ∃x. ?P f x)
proof(induct f)
  case (Atom a)
  show ?case
  proof (cases a)
    case (Less r cs)
    show ?thesis
    proof(cases cs)
      case Nil thus ?thesis using Less by simp
    next
      case (Cons c cs)
      { assume c=0 hence ?thesis using Less Cons by simp }
      moreover
      { assume c<0
        hence ?P (Atom a) ((r − ⟨cs,xs⟩ + 1)/c) using Less Cons
          by(auto simp add: field-simps)
        hence ?thesis .. }
      moreover
      { assume c>0
        hence ?P (Atom a) ((r − ⟨cs,xs⟩ − 1)/c) using Less Cons
          by(auto simp add: field-simps)
        hence ?thesis .. }
      ultimately show ?thesis by force
    qed
  next
    case (Eq r cs)
    show ?thesis
    proof(cases cs)
      case Nil thus ?thesis using Eq by simp
```

```
    next
      case (Cons c cs)
      { assume c=0 hence ?thesis using Eq Cons by simp }
      moreover
      { assume c<0
        hence ?P (Atom a) ((r − ⟨cs,xs⟩ + 1)/c) using Eq Cons
          by(auto simp add: field-simps)
        hence ?thesis .. }
      moreover
      { assume c>0
        hence ?P (Atom a) ((r − ⟨cs,xs⟩ − 1)/c) using Eq Cons
          by(auto simp add: field-simps)
        hence ?thesis .. }
      ultimately show ?thesis by force
    qed
  qed
next
  case (And f1 f2)
  then obtain x1 x2 where ?P f1 x1 ?P f2 x2 by fastforce+
  hence ?P (And f1 f2) (min x1 x2) by(force simp:and-def)
  thus ?case ..
next
  case (Or f1 f2)
  then obtain x1 x2 where ?P f1 x1 ?P f2 x2 by fastforce+
  hence ?P (Or f1 f2) (min x1 x2) by(force simp:or-def)
  thus ?case ..
qed simp-all

lemma plus-inf:
  nqfree f ⟹ ∃ x. ∀ y≥x. R.I (inf+ f) xs = R.I f (y # xs)
  (is - ⟹ ∃ x. ?P f x)
proof(induct f)
  case (Atom a)
  show ?case
  proof (cases a)
    case (Less r cs)
    show ?thesis
    proof(cases cs)
      case Nil thus ?thesis using Less by simp
    next
      case (Cons c cs)
      { assume c=0 hence ?thesis using Less Cons by simp }
      moreover
      { assume c<0
        hence ?P (Atom a) ((r − ⟨cs,xs⟩ − 1)/c) using Less Cons
          by(auto simp add: field-simps)
        hence ?thesis .. }
      moreover
      { assume c>0
```

```
      hence ?P (Atom a) ((r − ⟨cs,xs⟩ + 1)/c) using Less Cons
        by(auto simp add: field-simps)
      hence ?thesis .. }
    ultimately show ?thesis by force
  qed
 next
  case (Eq r cs)
  show ?thesis
  proof(cases cs)
    case Nil thus ?thesis using Eq by simp
  next
    case (Cons c cs)
    { assume c=0 hence ?thesis using Eq Cons by simp }
    moreover
    { assume c<0
      hence ?P (Atom a) ((r − ⟨cs,xs⟩ − 1)/c) using Eq Cons
        by(auto simp add: field-simps)
      hence ?thesis .. }
    moreover
    { assume c>0
      hence ?P (Atom a) ((r − ⟨cs,xs⟩ + 1)/c) using Eq Cons
        by(auto simp add: field-simps)
      hence ?thesis .. }
    ultimately show ?thesis by force
  qed
 qed
next
 case (And f1 f2)
 then obtain x1 x2 where ?P f1 x1 ?P f2 x2 by fastforce+
 hence ?P (And f1 f2) (max x1 x2) by(force simp:and-def)
 thus ?case ..
next
 case (Or f1 f2)
 then obtain x1 x2 where ?P f1 x1 ?P f2 x2 by fastforce+
 hence ?P (Or f1 f2) (max x1 x2) by(force simp:or-def)
 thus ?case ..
qed simp-all

context notes [[simp-depth-limit = 4]]
begin

lemma LBex:
 ⟦ nqfree f; R.I f (x#xs); ¬R.I (inf_ f) xs; x ∉ EQ f xs ⟧
  ⟹ ∃ l∈ LB f xs. l < x
apply(induct f)
apply simp
apply simp
apply(rename-tac a)
apply(case-tac a)
```

apply(*auto simp add*: $depends_R$*-def field-simps split*:*if-splits list.splits*)
apply *fastforce+*
done

lemma *UBex*:
⟦ *nqfree f*; *R.I f* (*x*#*xs*); ¬*R.I* (*inf*$_+$ *f*) *xs*; *x* ∉ *EQ f xs* ⟧
⟹ ∃ *u*∈ *UB f xs*. *x* < *u*
apply(*induct f*)
apply *simp*
apply *simp*
apply(*rename-tac a*)
apply(*case-tac a*)
apply(*auto simp add*: $depends_R$*-def field-simps split*:*if-splits list.splits*)
apply *fastforce+*
done

end

lemma *finite-LB*: *finite*(*LB f xs*)
proof −
have *LB f xs* = (λ(*r*,*cs*). *r* + ⟨*cs*,*xs*⟩) ‘ *set*(*lbounds*($R.atoms_0$ *f*))
by (*force simp*:*set-lbounds image-def field-simps*)
thus *?thesis* **by** *simp*
qed

lemma *finite-UB*: *finite*(*UB f xs*)
proof −
have *UB f xs* = (λ(*r*,*cs*). *r* + ⟨*cs*,*xs*⟩) ‘ *set*(*ubounds*($R.atoms_0$ *f*))
by (*force simp*:*set-ubounds image-def field-simps*)
thus *?thesis* **by** *simp*
qed

end

theory *QElin*
imports *LinArith*
begin

4.2 Fourier

definition $qe\text{-}FM_1$ :: *atom list* ⇒ *atom fm* **where**
$qe\text{-}FM_1$ *as* = *list-conj* [*Atom*(*combine p q*). *p*←*lbounds as*, *q*←*ubounds as*]

theorem $I\text{-}qe\text{-}FM_1$:
assumes *less*: ∀ *a* ∈ *set as*. *is-Less a* **and** *dep*: ∀ *a* ∈ *set as*. $depends_R$ *a*
shows *R.I* ($qe\text{-}FM_1$ *as*) *xs* = (∃ *x*. ∀ *a* ∈ *set as*. I_R *a* (*x*#*xs*)) (**is** *?L* = *?R*)
proof

let *?Ls* = *set*(*lbounds as*) **let** *?Us* = *set*(*ubounds as*)
let *?lbs* = *UN* (*r*,*cs*):*?Ls*. {*r* + ⟨*cs*,*xs*⟩}
let *?ubs* = *UN* (*r*,*cs*):*?Us*. {*r* + ⟨*cs*,*xs*⟩}
have *fins*: *finite ?lbs* ∧ *finite ?ubs* **by** *auto*
have *2*: ∀ *f*∈ *set as*. ∃ *r c cs*. *f* = *Less r* (*c*#*cs*) ∧
(*c*>*0* ∧ (*r*/*c*,(−*1*/*c*)∗$_s$ *cs*) ∈ *?Ls* ∨ *c*<*0* ∧ (*r*/*c*,(−*1*/*c*)∗$_s$ *cs*) ∈ *?Us*)
using *dep less*
by(*fastforce simp*:*set-lbounds set-ubounds is-Less-iff depends*$_R$*-def*
split:*list.splits*)
assume *?L*
have *1*: ∀ *x*∈*?lbs*. ∀ *y*∈*?ubs*. *x* < *y*
proof (*rule ballI*)+
fix *x y* **assume** *x*∈*?lbs y*∈*?ubs*
then obtain *r cs*
where (*r*,*cs*) ∈ *?Ls* ∧ *x* = *r* + ⟨*cs*,*xs*⟩ **by** *fastforce*
moreover from ‹*y*∈*?ubs*› **obtain** *s ds*
where (*s*,*ds*) ∈ *?Us* ∧ *y* = *s* + ⟨*ds*,*xs*⟩ **by** *fastforce*
ultimately show *x*<*y* **using** ‹*?L*›
by(*fastforce simp*:*qe-FM*$_1$*-def algebra-simps iprod-left-diff-distrib*)
qed
{ assume *nonempty*: *?lbs* ≠ {} ∧ *?ubs* ≠ {}
hence *Max ?lbs* < *Min ?ubs* **using** *fins 1*
by(*blast intro*: *Max-less-iff*[*THEN iffD2*] *Min-gr-iff*[*THEN iffD2*])
then obtain *m* **where** *Max ?lbs* < *m* ∧ *m* < *Min ?ubs*
using *dense*[**where** *'a* = *real*] **by** *blast*
hence ∀ *a* ∈ *set as*. *I*$_R$ *a* (*m*#*xs*) **using** *2 nonempty*
apply (*auto simp*: *Ball-def*)
apply (*auto simp*: *Bex-def*)
apply (*fastforce simp*: *field-simps*)
done
hence *?R* **.. }**
moreover
{ assume *asm*: *?lbs* ≠ {} ∧ *?ubs* = {}
have ∀ *a* ∈ *set as*. *I*$_R$ *a* ((*Max ?lbs* + *1*) # *xs*)
proof
fix *a* **assume** *a* ∈ *set as*
then obtain *r c cs*
where *a* = *Less r* (*c*#*cs*) *c*>*0* (*r*/*c*,(−*1*/*c*)∗$_s$ *cs*) ∈ *?Ls*
using *asm 2*
by (*fastforce simp*: *field-simps*)
moreover hence (*r* − ⟨*cs*,*xs*⟩)/*c* ≤ *Max ?lbs*
using *asm fins*
by(*auto intro*!: *Max-ge-iff*[*THEN iffD2*])
(*force simp add*:*field-simps*)
ultimately show *I*$_R$ *a* ((*Max ?lbs* + *1*) # *xs*) **by** (*simp add*: *field-simps*)
qed
hence *?R* **.. }**
moreover
{ assume *asm*: *?lbs* = {} ∧ *?ubs* ≠ {}

have $\forall\, a \in$ *set as*. I_R *a* ((*Min ?ubs* − *1*) # *xs*)
proof
fix *a* **assume** *a* ∈ *set as*
then obtain *r c cs*
where *a* = *Less r* (*c*#*cs*) *c*<*0* (*r*/*c*,(−*1*/*c*)∗$_s$ *cs*) ∈ *?Us*
using *asm 2* **by** *fastforce*
moreover hence *Min ?ubs* ≤ (*r* − ⟨*cs*,*xs*⟩)/*c*
using *asm fins*
by(*auto intro*!: *Min-le-iff*[*THEN iffD2*])
(*force simp add*:*field-simps*)
ultimately show I_R *a* ((*Min ?ubs* − *1*) # *xs*) **by** (*simp add*: *field-simps*)
qed
hence *?R* **.. }**
moreover
{ assume *?lbs* = {} ∧ *?ubs* = {}
hence *?R* **using** *2 less* **by** *auto* (*rule*, *fast*)
}
ultimately show *?R* **by** *blast*
next
let *?Ls* = *set*(*lbounds as*) **let** *?Us* = *set*(*ubounds as*)
assume *?R*
then obtain *x* **where** *1*: ∀ *a*∈ *set as*. I_R *a* (*x*#*xs*) **..**
{ fix *r c cs s d ds*
assume *Less r* (*c*#*cs*) ∈ *set as* *0* < *c Less s* (*d*#*ds*) ∈ *set as d* < *0*
hence *r* < *c*∗*x* + ⟨*cs*,*xs*⟩ *s* < *d*∗*x* + ⟨*ds*,*xs*⟩ *c* > *0 d* < *0*
using *1* **by** *auto*
hence (*r* − ⟨*cs*,*xs*⟩)/*c* < *x x* < (*s* − ⟨*ds*,*xs*⟩)/*d* **by**(*simp add*:*field-simps*)+
hence (*r* − ⟨*cs*,*xs*⟩)/*c* < (*s* − ⟨*ds*,*xs*⟩)/*d* **by** *arith*
}
thus *?L* **by** (*auto simp*: *qe-FM*$_1$*-def iprod-left-diff-distrib less field-simps set-lbounds set-ubounds*)
qed

corollary *I-qe-FM*$_1$*-pretty*:
∀ *a* ∈ *set as*. *is-Less a* ∧ *depends*$_R$ *a* ⟹ *R.is-dnf-qe qe-FM*$_1$ *as*
by(*metis I-qe-FM*$_1$)

fun *subst*$_0$ :: *atom* ⇒ *atom* ⇒ *atom* **where**
subst$_0$ (*Eq r* (*c*#*cs*)) *a* = (*case a of*
Less s (*d*#*ds*) ⇒ *Less* (*s* − (*r*∗*d*)/*c*) (*ds* − (*d*/*c*) ∗$_s$ *cs*)
| *Eq s* (*d*#*ds*) ⇒ *Eq* (*s* − (*r*∗*d*)/*c*) (*ds* − (*d*/*c*) ∗$_s$ *cs*))

lemma *subst*$_0$*-pretty*:
subst$_0$ (*Eq r* (*c*#*cs*)) (*Less s* (*d*#*ds*)) = *Less* (*s* − (*r*∗*d*)/*c*) (*ds* − (*d*/*c*) ∗$_s$ *cs*)
subst$_0$ (*Eq r* (*c*#*cs*)) (*Eq s* (*d*#*ds*)) = *Eq* (*s* − (*r*∗*d*)/*c*) (*ds* − (*d*/*c*) ∗$_s$ *cs*)
by *auto*

lemma *I-subst*$_0$: *depends*$_R$ *a* ⟹ *c* ≠ *0* ⟹

I_R ($subst_0$ (Eq r (c#cs)) a) xs = I_R a ((r − ⟨cs,xs⟩)/c # xs)
apply(*cases a*)
by (*auto simp add*: $depends_R$-*def iprod-left-diff-distrib algebra-simps diff-divide-distrib split*:*list.splits*)

interpretation R_e:
ATOM-EQ neg_R (λ*a*. *True*) I_R $depends_R$ $decr_R$
(λ*Eq* - (*c*#-) ⇒ *c* ≠ *0* | - ⇒ *False*)
(λ*Eq r cs* ⇒ *r*=*0* ∧ (∀ *c*∈ *set cs*. *c*=*0*) | - ⇒ *False*) $subst_0$
apply(*unfold-locales*)
apply(*simp del*:$subst_0$.*simps add*:*I-subst*$_0$ *split*:*atom.splits list.splits*)
apply(*simp add*: *iprod0-if-coeffs0 split*:*atom.splits*)
apply(*simp split*:*atom.splits list.splits*)
apply(*rename-tac r ds c cs*)
apply(*rule-tac x*= (*r* − ⟨*cs*,*xs*⟩)/*c* **in** *exI*)
apply (*simp add*: *algebra-simps diff-divide-distrib*)
apply(*simp add*: *self-list-diff set-replicate-conv-if*
split:*atom.splits list.splits*)
done

definition *qe-FM* = R_e.*lift-dnfeq-qe qe-FM*$_1$

lemma *qfree-qe-FM*$_1$: *qfree* (*qe-FM*$_1$ *as*)
by(*auto simp*:*qe-FM*$_1$*-def intro*!:*qfree-list-conj*)

corollary *I-qe-FM*: *R.I* (*qe-FM* φ) *xs* = *R.I* φ *xs*
unfolding *qe-FM-def*
apply(*rule* R_e.*I-lift-dnfeq-qe*)
apply(*rule qfree-qe-FM*$_1$)
apply(*rule allI*)
apply(*rule I-qe-FM*$_1$)
prefer *2* **apply** *blast*
apply(*clarify*)
apply(*drule-tac x*=*a* **in** *bspec*) **apply** *simp*
apply(*simp add*: $depends_R$-*def split*:*atom.splits list.splits*)
done

theorem *qfree-qe-FM*: *qfree* (*qe-FM f*)
by(*simp add*:*qe-FM-def* R_e.*qfree-lift-dnfeq-qe qfree-qe-FM*$_1$)

4.2.1 Tests

lemmas *qesimps* = *qe-FM-def* R_e.*lift-dnfeq-qe-def* R_e.*lift-eq-qe-def R.qelim-def qe-FM*$_1$*-def lbounds-def ubounds-def list-conj-def list-disj-def and-def or-def* $depends_R$-*def*

lemma *qe-FM*(*TrueF*) = *TrueF*
by(*simp add*:*qesimps*)

lemma
qe-FM(*ExQ* (*And* (*Atom*(*Less 0* [*1*])) (*Atom*(*Less 0* [−*1*])))) = *Atom*(*Less 0* [])
by(*simp add*:*qesimps*)

lemma
qe-FM(*ExQ* (*And* (*Atom*(*Less 0* [*1*])) (*Atom*(*Less* (− *1*) [−*1*])))) = *Atom*(*Less* (− *1*) [])
by(*simp add*:*qesimps*)

end

theory *QElin-opt*
imports *QElin*
begin

4.2.2 An optimization

Atoms are simplified asap.

definition
asimp a = (*case a of*
Less r cs ⇒ (*if* ∀ *c*∈ *set cs. c* = *0*
then if r<*0 then TrueF else FalseF*
else Atom a) |
Eq r cs ⇒ (*if* ∀ *c*∈ *set cs. c* = *0*
then if r=*0 then TrueF else FalseF else Atom a*))

lemma *asimp-pretty*:
asimp (*Less r cs*) =
(*if* ∀ *c*∈ *set cs. c* = *0*
then if r<*0 then TrueF else FalseF*
else Atom(*Less r cs*))
asimp (*Eq r cs*) =
(*if* ∀ *c*∈ *set cs. c* = *0*
then if r=*0 then TrueF else FalseF*
else Atom(*Eq r cs*))
by(*auto simp*:*asimp-def*)

definition *qe-FMo*$_1$ :: *atom list* ⇒ *atom fm* **where**
qe-FMo$_1$ *as* = *list-conj* [*asimp*(*combine p q*). *p*←*lbounds as*, *q*←*ubounds as*]

lemma *I-asimp*: *R.I* (*asimp a*) *xs* = I_R *a xs*
by(*simp add*:*asimp-def iprod0-if-coeffs0 split*:*atom.split*)

lemma *I-qe-FMo*$_1$: *R.I* (*qe-FMo*$_1$ *as*) *xs* = *R.I* (*qe-FM*$_1$ *as*) *xs*
by(*simp add*:*qe-FM*$_1$*-def qe-FMo*$_1$*-def I-asimp*)

definition *qe-FMo* = R_e.*lift-dnfeq-qe qe-FMo*$_1$

lemma *qfree-qe-FMo*$_1$: *qfree* (*qe-FMo*$_1$ *as*)
by(*auto simp*:*qe-FM*$_1$*-def qe-FMo*$_1$*-def asimp-def intro*!:*qfree-list-conj*
split:*atom.split*)

corollary *I-qe-FMo*: *R.I* (*qe-FMo* φ) *xs* = *R.I* φ *xs*
unfolding *qe-FMo-def*
apply(*rule* R_e*.I-lift-dnfeq-qe*)
apply(*rule qfree-qe-FMo*$_1$)
apply(*rule allI*)
apply(*subst I-qe-FMo*$_1$)
apply(*rule I-qe-FM*$_1$)
prefer *2* **apply** *blast*
apply(*clarify*)
apply(*drule-tac x*=*a* **in** *bspec*) **apply** *simp*
apply(*simp add*: *depends*$_R$*-def split*:*atom.splits list.splits*)
done

theorem *qfree-qe-FMo*: *qfree* (*qe-FMo f*)
by(*simp add*:*qe-FMo-def* R_e*.qfree-lift-dnfeq-qe qfree-qe-FMo*$_1$)

end

theory *FRE*
imports *LinArith*
begin

4.3 Ferrante-Rackoff

This section formalizes a slight variant of Ferrante and Rackoff's algorithm [2]. We consider equalities separately, which improves performance.

fun *between* :: *real* ∗ *real list* ⇒ *real* ∗ *real list* ⇒ *real* ∗ *real list*
where *between* (*r*,*cs*) (*s*,*ds*) = ((*r*+*s*)/*2*, (*1*/*2*) ∗$_s$ (*cs*+*ds*))

definition *FR*$_1$ :: *atom fm* ⇒ *atom fm* **where**
FR$_1$ φ =
(*let as* = *R.atoms*$_0$ φ; *lbs* = *lbounds as*; *ubs* = *ubounds as*; *ebs* = *ebounds as*;
intrs = [*subst* φ (*between l u*) . *l* ← *lbs*, *u* ← *ubs*]
in list-disj (*inf*$_-$ φ # *inf*$_+$ φ # *intrs* @ *map* (*subst* φ) *ebs*))

lemma *dense-interval*:
assumes *finite L finite U l* ∈ *L u* ∈ *U l* < *x x* < *u P*(*x*::*real*)
and *dense*: ⋀*y l u*. ⟦ ∀ *y*∈{*l*<..<*x*}. *y* ∉ *L*; ∀ *y*∈{*x*<..<*u*}. *y* ∉ *U*;
l<*x*;*x*<*u*; *l*<*y*;*y*<*u* ⟧ ⟹ *P y*
shows ∃ *l*∈*L*.∃ *u*∈*U*. *l*<*u* ∧ (∀ *y*. *l*<*y* ∧ *y*<*u* ⟶ *P y*)
proof −
let *?L* = {*l*:*L*. *l* < *x*} **let** *?U* = {*u*:*U*. *x* < *u*}

```
  let ?ll = Max ?L let ?uu = Min ?U
  have ?L ≠ {} using ‹l ∈ L› ‹l<x› by (blast intro:order-less-imp-le)
  moreover have ?U ≠ {} using ‹u:U› ‹x<u› by (blast intro:order-less-imp-le)
  ultimately have ∀ y. ?ll<y ∧ y<x ⟶ y ∉ L ∀ y. x<y ∧ y<?uu ⟶ y ∉ U
    using ‹finite L› ‹finite U› by force+
  moreover have ?ll ∈ L
  proof
    show ?ll ∈ ?L using ‹finite L› Max-in[OF - ‹?L ≠ {}›] by simp
    show ?L ⊆ L by blast
  qed
  moreover have ?uu ∈ U
  proof
    show ?uu ∈ ?U using ‹finite U› Min-in[OF - ‹?U ≠ {}›] by simp
    show ?U ⊆ U by blast
  qed
  moreover have ?ll < x using ‹finite L› ‹?L ≠ {}› by simp
  moreover have x < ?uu using ‹finite U› ‹?U ≠ {}› by simp
  moreover have ?ll < ?uu using ‹?ll<x› ‹x<?uu› by arith
  ultimately show ?thesis using ‹l < x› ‹x < u› ‹?L ≠ {}› ‹?U ≠ {}›
    by(blast intro!:dense greaterThanLessThan-iff[THEN iffD1])
qed

lemma dense:
  ⟦ nqfree f; ∀ y∈{l<..<x}. y ∉ LB f xs; ∀ y∈{x<..<u}. y ∉ UB f xs;
     l < x; x < u; x ∉ EQ f xs;  R.I f (x#xs); l < y; y < u⟧
   ⟹ R.I f (y#xs)
proof(induct f)
  case (Atom a)
  show ?case
  proof (cases a)
    case (Less r cs)
    show ?thesis
    proof (cases cs)
      case Nil thus ?thesis using Atom Less by (simp add:depends_R-def)
    next
      case (Cons c cs)
      hence r < c*x + ⟨cs,xs⟩ using Atom Less by simp
      { assume c=0 hence ?thesis using Atom Less Cons by simp }
      moreover
      { assume c<0
        hence x < (r - ⟨cs,xs⟩)/c (is - < ?u) using ‹r < c*x + ⟨cs,xs⟩›
          by (simp add: field-simps)
        have ?thesis
        proof (rule ccontr)
          assume ¬ R.I (Atom a) (y#xs)
          hence ?u ≤ y using Atom Less Cons ‹c<0›
            by (auto simp add: field-simps)
          hence ?u < u using ‹y<u› by simp
          with ‹x<?u› show False using Atom Less Cons ‹c<0›
```

```
        by(auto simp:depends_R-def)
      qed } moreover
    { assume c>0
      hence x > (r − ⟨cs,xs⟩)/c (is - > ?l) using ‹r < c∗x + ⟨cs,xs⟩›
        by (simp add: field-simps)
      have ?thesis
      proof (rule ccontr)
        assume ¬ R.I (Atom a) (y#xs)
        hence ?l ≥ y using Atom Less Cons ‹c>0›
          by (auto simp add: field-simps)
        hence ?l > l using ‹y>l› by simp
        with ‹?l<x› show False using Atom Less Cons ‹c>0›
          by (auto simp:depends_R-def)
      qed }
    ultimately show ?thesis by force
  qed
 next
  case (Eq r cs)
  show ?thesis
  proof (cases cs)
    case Nil thus ?thesis using Atom Eq by (simp add:depends_R-def)
  next
    case (Cons c cs)
    hence r = c∗x + ⟨cs,xs⟩ using Atom Eq by simp
    { assume c=0 hence ?thesis using Atom Eq Cons by simp }
    moreover
    { assume c≠0
      hence ?thesis using ‹r = c∗x + ⟨cs,xs⟩› Atom Eq Cons ‹l<y› ‹y<u›
        by (auto simp: depends_R-def split: if-splits) }
    ultimately show ?thesis by force
  qed
 qed
next
 case (And f1 f2) thus ?case
  by auto (metis (no-types, opaque-lifting))+
next
 case (Or f1 f2) thus ?case
  by auto (metis (no-types, opaque-lifting))+
qed fastforce+

theorem I-FR_1:
assumes nqfree φ shows R.I (FR_1 φ) xs = (∃ x. R.I φ (x#xs))
 (is ?FR = ?EX)
proof
 assume ?FR
 { assume R.I (inf_- φ) xs
  hence ?EX using ‹?FR› min-inf[OF ‹nqfree φ›, where xs=xs]
    by(auto simp add:FR_1-def)
 } moreover
```

```
  { assume R.I (inf_+ φ) xs
    hence ?EX using ‹?FR› plus-inf[OF ‹nqfree φ›, where xs=xs]
      by(auto simp add:FR_1-def)
  } moreover
  { assume ∃x ∈ EQ φ xs. R.I φ (x#xs)
    hence ?EX using ‹?FR› by(auto simp add:FR_1-def)
  } moreover
  { assume ¬R.I (inf_− φ) xs ∧ ¬R.I (inf_+ φ) xs ∧
           (∀x∈EQ φ xs. ¬R.I φ (x#xs))
    with ‹?FR› obtain r cs s ds
      where R.I (subst φ (between (r,cs) (s,ds))) xs
      by(auto simp: FR_1-def eval-def
        diff-divide-distrib set-ebounds I-subst ‹nqfree φ›) blast
    hence R.I φ (eval (between (r,cs) (s,ds)) xs # xs)
      by(simp add:I-subst ‹nqfree φ›)
    hence ?EX .. }
  ultimately show ?EX by blast
next
  assume ?EX
  then obtain x where x: R.I φ (x#xs) ..
  { assume R.I (inf_− φ) xs ∨ R.I (inf_+ φ) xs
    hence ?FR by(auto simp:FR_1-def)
  } moreover
  { assume x ∈ EQ φ xs
    then obtain r cs
      where (r,cs) ∈ set(ebounds(R.atoms_0 φ)) ∧ x = r + ⟨cs,xs⟩
      by(force simp:set-ebounds field-simps)
    moreover hence R.I (subst φ (r,cs)) xs using x
      by(auto simp: I-subst ‹nqfree φ› eval-def)
    ultimately have ?FR by(force simp:FR_1-def) } moreover
  { assume ¬ R.I (inf_− φ) xs and ¬ R.I (inf_+ φ) xs and x ∉ EQ φ xs
    obtain l where l ∈ LB φ xs l < x
      using LBex[OF ‹nqfree φ› x ‹¬ R.I (inf_− φ) xs› ‹x ∉ EQ φ xs›] ..
    obtain u where u ∈ UB φ xs x < u
      using UBex[OF ‹nqfree φ› x ‹¬ R.I (inf_+ φ) xs› ‹x ∉ EQ φ xs›] ..
   have ∃l∈LB φ xs. ∃u∈UB φ xs. l<u ∧ (∀y. l < y ∧ y < u ⟶ R.I φ (y#xs))
      using dense-interval[where P = λx. R.I φ (x#xs), OF finite-LB finite-UB
‹l:LB φ xs› ‹u:UB φ xs› ‹l<x› ‹x<u› x] x dense[OF ‹nqfree φ› - - - - ‹x ∉ EQ φ
xs›] by simp
    then obtain r c cs s d ds
      where Less r (c # cs) ∈ set (R.atoms_0 φ) Less s (d # ds) ∈ set (R.atoms_0
φ)
          ⋀y. (r − ⟨cs,xs⟩) / c < y ⟹ y < (s − ⟨ds,xs⟩) / d ⟹ R.I φ (y # xs)
        and ∗: c > 0 d < 0 (r − ⟨cs,xs⟩) / c < (s − ⟨ds,xs⟩) / d
      by blast
    moreover
      have (r − ⟨cs,xs⟩) / c < eval (between (r / c, (−1 / c) ∗_s cs) (s / d, (−1 /
d) ∗_s ds)) xs (is ?P)
        and eval (between (r / c, (−1 / c) ∗_s cs) (s / d, (−1 / d) ∗_s ds)) xs < (s
```

```
– ⟨ds,xs⟩) / d (is ?Q)
    proof –
      from * have [simp]: c * (c * (d * (d * 4))) > 0
        by (simp add: algebra-split-simps)
      from * have c * s + d * ⟨cs,xs⟩ < d * r + c * ⟨ds,xs⟩
        by (simp add: field-simps)
      with * have (2 * c * c * d) * (d * r + c * ⟨ds,xs⟩)
        < (2 * c * c * d) * (c * s + d * ⟨cs,xs⟩)
        and (2 * c * d * d) * (c * s + d * ⟨cs,xs⟩)
        < (2 * c * d * d) * (d * r + c * ⟨ds,xs⟩) by simp-all
    with * show ?P and ?Q by (auto simp add: field-simps eval-def iprod-left-add-distrib)
    qed
    ultimately have ?FR
     by (fastforce simp: FR_1-def bex-Un set-lbounds set-ubounds set-ebounds I-subst
‹nqfree φ›)
  } ultimately show ?FR by blast
qed


definition FR = R.lift-nnf-qe FR_1


lemma qfree-FR_1: nqfree φ ⟹ qfree (FR_1 φ)
apply(simp add:FR_1-def)
apply(rule qfree-list-disj)
apply(auto simp:qfree-min-inf qfree-plus-inf set-ubounds set-lbounds set-ebounds
image-def qfree-map-fm)
done

theorem I-FR: R.I (FR φ) xs = R.I φ xs
by(simp add:I-FR_1 FR-def R.I-lift-nnf-qe qfree-FR_1)

theorem qfree-FR: qfree (FR φ)
by(simp add:FR-def R.qfree-lift-nnf-qe qfree-FR_1)

end


theory QElin-inf
imports LinArith
begin
```

4.4 Quantifier elimination with infinitesimals

This section formalizes Loos and Weispfenning's quantifier elimination procedure based on (the simulation of) infinitesimals [3].

```
fun asubst-peps :: real * real list ⇒ atom ⇒ atom fm (‹asubst_+›) where
asubst-peps (r,cs) (Less s (d#ds)) =
  (if d=0 then Atom(Less s ds) else
```

```
   let u = s − d∗r; v = d ∗s cs + ds; less = Atom(Less u v)
   in if d<0 then less else Or less (Atom(Eq u v))) |
asubst-peps rcs (Eq r (d#ds)) = (if d=0 then Atom(Eq r ds) else FalseF) |
asubst-peps rcs a = Atom a

abbreviation subst-peps :: atom fm ⇒ real ∗ real list ⇒ atom fm (‹subst+›)
where subst+ φ rcs ≡ amapfm (asubst+ rcs) φ

definition nolb f xs l x = (∀ y∈{l<..<x}. y ∉ LB f xs)

lemma nolb-And[simp]:
  nolb (And f g) xs l x = (nolb f xs l x ∧ nolb g xs l x)
apply(clarsimp simp:nolb-def)
apply blast
done

lemma nolb-Or[simp]:
  nolb (Or f g) xs l x = (nolb f xs l x ∧ nolb g xs l x)
apply(clarsimp simp:nolb-def)
apply blast
done

context notes [[simp-depth-limit=4]]
begin

lemma innermost-intvl:
  ⟦ nqfree f; nolb f xs l x; l < x; x ∉ EQ f xs; R.I f (x#xs); l < y; y ≤ x⟧
  ⟹ R.I f (y#xs)
proof(induct f)
  case (Atom a)
  show ?case
  proof (cases a)
    case [simp]: (Less r cs)
    show ?thesis
    proof (cases cs)
      case Nil thus ?thesis using Atom by (simp add:dependsR-def)
    next
      case [simp]: (Cons c cs)
      hence r < c∗x + ⟨cs,xs⟩ using Atom by simp
      { assume c=0 hence ?thesis using Atom by simp }
      moreover
      { assume c<0
        hence x < (r − ⟨cs,xs⟩)/c (is - < ?u) using ‹r < c∗x + ⟨cs,xs⟩›
          by (simp add: field-simps)
        have ?thesis
        proof (rule ccontr)
          assume ¬ R.I (Atom a) (y#xs)
          hence ?u ≤ y using Atom ‹c<0›
            by (auto simp add: field-simps)
```

```
        with ‹x<?u› show False using Atom ‹c<0›
          by(auto simp:depends_R-def)
      qed } moreover
    { assume c>0
      hence x > (r − ⟨cs,xs⟩)/c (is - > ?l) using ‹r < c∗x + ⟨cs,xs⟩›
        by (simp add: field-simps)
      then have ?l < y using Atom ‹c>0›
          by (auto simp:depends_R-def Ball-def nolb-def)
            (metis linorder-not-le antisym order-less-trans)
      hence ?thesis using ‹c>0› by (simp add: field-simps)
    } ultimately show ?thesis by force
  qed
 next
  case [simp]: (Eq r cs)
  show ?thesis
  proof (cases cs)
    case Nil thus ?thesis using Atom by (simp add:depends_R-def)
  next
    case [simp]: (Cons c cs)
    hence r = c∗x + ⟨cs,xs⟩ using Atom by simp
    { assume c=0 hence ?thesis using Atom by simp }
    moreover
    { assume c≠0
      hence ?thesis using ‹r = c∗x + ⟨cs,xs⟩› Atom
        by (auto simp: depends_R-def split: if-splits) }
    ultimately show ?thesis by force
  qed
 qed
next
 case (And f1 f2) thus ?case by (fastforce)
next
 case (Or f1 f2) thus ?case by (fastforce)
qed simp+

definition EQ2 = EQ

lemma EQ2-Or[simp]: EQ2 (Or f g) xs = (EQ2 f xs ∪ EQ2 g xs)
by(auto simp:EQ2-def)

lemma EQ2-And[simp]: EQ2 (And f g) xs = (EQ2 f xs ∪ EQ2 g xs)
by(auto simp:EQ2-def)

lemma innermost-intvl2:
 ⟦ nqfree f; nolb f xs l x; l < x; x ∉ EQ2 f xs; R.I f (x#xs); l < y; y ≤ x⟧
 ⟹ R.I f (y#xs)
unfolding EQ2-def by(blast intro:innermost-intvl)

lemma I-subst-peps2:
 nqfree f ⟹ r+⟨cs,xs⟩ < x ⟹ nolb f xs (r+⟨cs,xs⟩) x
```

$\Longrightarrow \forall\, y \in \{r{+}\langle cs{,}xs\rangle <.. x\}.\ R.I\ f\ (y\#xs) \wedge y \notin EQ2\ f\ xs$
$\Longrightarrow R.I\ (subst_+\ f\ (r{,}cs))\ xs$
proof(*induct f*)
case *FalseF* **thus** *?case*
by *simp* (*metis antisym-conv1 linorder-neq-iff*)
next
case (*Atom a*)
show *?case*
proof(*cases* ((*r*,*cs*),*a*) *rule*:*asubst-peps.cases*)
case (*1 r cs s d ds*)
{ assume *d=0* **hence** *?thesis* **using** *Atom 1* **by** *auto* **}**
moreover
{ assume *d<0*
have $s < d{*}x + \langle ds{,}xs\rangle$ **using** *Atom 1* **by** *simp*
moreover have $d{*}x < d{*}(r + \langle cs{,}xs\rangle)$ **using** ‹*d<0*› *Atom 1*
by (*simp add*: *mult-strict-left-mono-neg*)
ultimately have $s < d * (r + \langle cs{,}xs\rangle) + \langle ds{,}xs\rangle$ **by**(*simp add*:*algebra-simps*)
hence *?thesis* **using** *1*
by (*auto simp add*: *iprod-left-add-distrib algebra-simps*)
} moreover
{ let *?L* $= (s - \langle ds{,}xs\rangle)\ /\ d$ **let** *?U* $= r + \langle cs{,}xs\rangle$
assume *d>0*
hence $?U < x$ **and** $\forall\, y.\ ?U < y \wedge y < x \longrightarrow y \neq ?L$
and $\forall\, y.\ ?U < y \wedge y \leq x \longrightarrow ?L < y$ **using** *Atom 1*
by(*simp-all add*:*nolb-def depends*$_R$*-def Ball-def field-simps*)
hence $?L < ?U \vee ?L = ?U$
by (*metis linorder-neqE-linordered-idom order-refl*)
hence *?thesis* **using** *Atom 1* ‹*d>0*›
by (*simp add*: *iprod-left-add-distrib field-simps*)
} ultimately show *?thesis* **by** *force*
next
case *2* **thus** *?thesis* **using** *Atom*
by (*fastforce simp*: *nolb-def EQ2-def depends*$_R$*-def field-simps split*: *if-split-asm*)
qed (*insert Atom*, *auto*)
next
case *Or* **thus** *?case* **by**(*simp add*:*Ball-def*)(*metis order-refl innermost-intvl2*)
qed *simp-all*

end

lemma *I-subst-peps*:
nqfree f $\Longrightarrow R.I\ (subst_+\ f\ (r{,}cs))\ xs \Longrightarrow$
$(\exists\, leps{>}r{+}\langle cs{,}xs\rangle.\ \forall\, x.\ r{+}\langle cs{,}xs\rangle < x \wedge x \leq leps \longrightarrow R.I\ f\ (x\#xs))$
proof(*induct f*)
case *TrueF* **thus** *?case* **by** *simp* (*metis less-add-one*)
next
case (*Atom a*)
show *?case*
proof (*cases* ((*r*,*cs*),*a*) *rule*: *asubst-peps.cases*)

```
    case (1 r cs s d ds)
    { assume d=0 hence ?thesis using Atom 1 by auto (metis less-add-one) }
    moreover
    { assume d<0
      with Atom 1 have r + ⟨cs,xs⟩ < (s − ⟨ds,xs⟩)/d (is ?a < ?b)
        by(simp add:field-simps iprod-left-add-distrib)
      then obtain x where ?a < x x < ?b by(metis dense)
      hence ∀ y. ?a < y ∧ y ≤ x ⟶ s < d*y + ⟨ds,xs⟩
        using ‹d<0› by (simp add:field-simps)
    (metis add-le-cancel-right mult-le-cancel-left order-antisym linear mult.commute
xt1(8))
      hence ?thesis using 1 ‹?a<x› by auto
    } moreover
    { let ?a = s − d * r let ?b = ⟨d *s cs + ds,xs⟩
      assume d>0
      with Atom 1 have ?a < ?b ∨ ?a = ?b by auto
      hence ?thesis
      proof
        assume ?a = ?b
        thus ?thesis using ‹d>0› Atom 1
          by(simp add:field-simps iprod-left-add-distrib)
         (metis add-0-left add-less-cancel-right distrib-left mult.commute mult-strict-left-mono)
      next
        assume ?a < ?b
        { fix x assume r+⟨cs,xs⟩ < x ∧ x ≤ r+⟨cs,xs⟩ + 1
          hence d*(r + ⟨cs,xs⟩) < d*x
            using ‹d>0› by(metis mult-strict-left-mono)
          hence s < d*x + ⟨ds,xs⟩ using ‹d>0› ‹?a < ?b›
            by (simp add:algebra-simps iprod-left-add-distrib)
        }
        thus ?thesis using 1 ‹d>0›
          by(force simp: iprod-left-add-distrib)
      qed
    } ultimately show ?thesis by (metis less-linear)
  qed (insert Atom, auto split:if-split-asm intro: less-add-one)
next
  case And thus ?case
    apply clarsimp
    apply(rule-tac x=min leps lepsa in exI)
    apply simp
    done
next
  case Or thus ?case by force
qed simp-all

lemma dense-interval:
assumes finite L l ∈ L l < x P(x::real)
and dense: ⋀y l. ⟦ ∀ y∈{l<..<x}. y ∉ L; l<x; l<y; y≤x ⟧ ⟹ P y
shows ∃ l∈L. l<x ∧ (∀ y∈{l<..<x}. y ∉ L) ∧ (∀ y. l<y ∧ y≤x ⟶ P y)
```

proof −
let *?L* = {*l*∈*L*. *l* < *x*}
let *?ll* = *Max ?L*
have *?L* ≠ {} **using** ‹*l* ∈ *L*› ‹*l*<*x*› **by** (*blast intro*:*order-less-imp-le*)
hence ∀ *y*. *?ll*<*y* ∧ *y*<*x* ⟶ *y* ∉ *L* **using** ‹*finite L*› **by** *force*
moreover have *?ll* ∈ *L*
proof
show *?ll* ∈ *?L* **using** ‹*finite L*› *Max-in*[*OF* - ‹*?L* ≠ {}›] **by** *simp*
show *?L* ⊆ *L* **by** *blast*
qed
moreover have *?ll* < *x* **using** ‹*finite L*› ‹*?L* ≠ {}› **by** *simp*
ultimately show *?thesis* **using** ‹*l* < *x*› ‹*?L* ≠ {}›
by(*blast intro*!:*dense greaterThanLessThan-iff*[*THEN iffD1*])
qed

definition
qe-eps$_1$(*f*) =
(*let as* = *R.atoms*$_0$ *f*; *lbs* = *lbounds as*; *ebs* = *ebounds as*
in list-disj (*inf*$_-$ *f* # *map* (*subst*$_+$ *f*) *lbs* @ *map* (*subst f*) *ebs*))

theorem *I-eps1*:
assumes *nqfree f* **shows** *R.I* (*qe-eps*$_1$ *f*) *xs* = (∃ *x*. *R.I f* (*x*#*xs*))
(**is** *?QE* = *?EX*)
proof
let *?as* = *R.atoms*$_0$ *f* **let** *?ebs* = *ebounds ?as*
assume *?QE*
{ **assume** *R.I* (*inf*$_-$ *f*) *xs*
hence *?EX* **using** ‹*?QE*› *min-inf*[*of f xs*] ‹*nqfree f*›
by(*auto simp add*:*qe-eps*$_1$*-def amap-fm-list-disj*)
} **moreover**
{ **assume** ∀ *x* ∈ *EQ f xs*. ¬*R.I f* (*x*#*xs*)
¬ *R.I* (*inf*$_-$ *f*) *xs*
with ‹*?QE*› ‹*nqfree f*› **obtain** *r cs* **where** *R.I* (*subst*$_+$ *f* (*r*,*cs*)) *xs*
by (*fastforce simp*: *qe-eps*$_1$*-def set-ebounds diff-divide-distrib eval-def I-subst*
‹*nqfree f*›)
then obtain *leps* **where** *R.I f* (*leps*#*xs*)
using *I-subst-peps*[*OF* ‹*nqfree f*›] **by** *fastforce*
hence *?EX* .. }
ultimately show *?EX* **by** *blast*
next
let *?as* = *R.atoms*$_0$ *f* **let** *?ebs* = *ebounds ?as*
assume *?EX*
then obtain *x* **where** *x*: *R.I f* (*x*#*xs*) ..
{ **assume** *R.I* (*inf*$_-$ *f*) *xs*
hence *?QE* **using** ‹*nqfree f*› **by**(*auto simp*:*qe-eps*$_1$*-def*)
} **moreover**
{ **assume** ∃ *rcs* ∈ *set ?ebs*. *R.I* (*subst f rcs*) *xs*
hence *?QE* **by**(*auto simp*:*qe-eps*$_1$*-def*) } **moreover**
{ **assume** ¬ *R.I* (*inf*$_-$ *f*) *xs*

and $\forall\, rcs \in set\ ?ebs.\ \neg\ R.I\ (subst\ f\ rcs)\ xs$
hence *noE*: $\forall\, e \in EQ\ f\ xs.\ \neg\ R.I\ f\ (e\#xs)$ **using** ‹*nqfree f*›
by (*force simp*:*set-ebounds I-subst diff-divide-distrib eval-def split*:*if-split-asm*)
hence $x \notin EQ\ f\ xs$ **using** *x* **by** *fastforce*
obtain *l* **where** $l \in LB\ f\ xs\ l < x$
using *LBex*[*OF* ‹*nqfree f*› *x* ‹$\neg\ R.I(inf_{\_}\ f)\ xs$› ‹$x \notin EQ\ f\ xs$›] **..**
have $\exists\, l{\in}LB\ f\ xs.\ l{<}x \wedge nolb\ f\ xs\ l\ x \wedge$
$(\forall\, y.\ l < y \wedge y \leq x \longrightarrow R.I\ f\ (y\#xs))$
using *dense-interval*[**where** $P = \lambda x.\ R.I\ f\ (x\#xs)$, *OF finite-LB* ‹$l{\in}LB\ f\ xs$›
‹$l{<}x$› *x*] *x innermost-intvl*[*OF* ‹*nqfree f*› - - ‹$x \notin EQ\ f\ xs$›]
by (*simp add*:*nolb-def*)
then obtain *r c cs*
where $*$: $Less\ r\ (c\#cs) \in set(R.atoms_0\ f) \wedge c{>}0 \wedge$
$(r - \langle cs,xs\rangle)/c < x \wedge nolb\ f\ xs\ ((r - \langle cs,xs\rangle)/c)\ x$
$\wedge\ (\forall\, y.\ (r - \langle cs,xs\rangle)/c < y \wedge y \leq x \longrightarrow R.I\ f\ (y\#xs))$
by *blast*
then have $R.I\ (subst_+\ f\ (r/c,\ (-1/c) *_s\ cs))\ xs$ **using** *noE*
by(*auto intro*!: *I-subst-peps2*[*OF* ‹*nqfree f*›]
simp:*EQ2-def diff-divide-distrib algebra-simps*)
with $*$ **have** *?QE*
by(*simp add*:$qe\text{-}eps_1$*-def bex-Un set-lbounds*) *metis*
} ultimately show *?QE* **by** *blast*
qed

lemma *qfree-asubst-peps*: *qfree* ($asubst_+$ *rcs a*)
by(*cases* (*rcs*,*a*) *rule*:*asubst-peps.cases*) *simp-all*

lemma *qfree-subst-peps*: *nqfree* $\varphi \Longrightarrow$ *qfree* ($subst_+\ \varphi$ *rcs*)
by(*induct* φ) (*simp-all add*:*qfree-asubst-peps*)

lemma $qfree\text{-}qe\text{-}eps_1$: *nqfree* $\varphi \Longrightarrow$ *qfree*($qe\text{-}eps_1\ \varphi$)
apply(*simp add*:$qe\text{-}eps_1$*-def*)
apply(*rule qfree-list-disj*)
apply (*auto simp*:*qfree-min-inf qfree-subst-peps qfree-map-fm*)
done

definition $qe\text{-}eps = R.lift\text{-}nnf\text{-}qe\ qe\text{-}eps_1$

lemma *qfree-qe-eps*: *qfree*(*qe-eps* φ)
by(*simp add*: *qe-eps-def R.qfree-lift-nnf-qe* $qfree\text{-}qe\text{-}eps_1$)

lemma *I-qe-eps*: $R.I\ (qe\text{-}eps\ \varphi)\ xs = R.I\ \varphi\ xs$
by(*simp add*:*qe-eps-def R.I-lift-nnf-qe* $qfree\text{-}qe\text{-}eps_1$ *I-eps1*)

end

5 Presburger arithmetic

theory *PresArith*

imports *QE HOL−Library.ListVector*
begin

declare *iprod-assoc*[*simp*]

5.1 Syntax

datatype *atom* =
  Le int int list | *Dvd int int int list* | *NDvd int int int list*

fun *divisor* :: *atom* ⇒ *int* **where**
divisor (*Le i ks*) = *1* |
divisor (*Dvd d i ks*) = *d* |
divisor (*NDvd d i ks*) = *d*

fun neg_Z :: *atom* ⇒ *atom fm* **where**
neg_Z (*Le i ks*) = *Atom*(*Le* (*1*−*i*) (−*ks*)) |
neg_Z (*Dvd d i ks*) = *Atom*(*NDvd d i ks*) |
neg_Z (*NDvd d i ks*) = *Atom*(*Dvd d i ks*)

fun *hd-coeff* :: *atom* ⇒ *int* **where**
hd-coeff (*Le i ks*) = (*case ks of* [] ⇒ *0* | *k*#- ⇒ *k*) |
hd-coeff (*Dvd d i ks*) = (*case ks of* [] ⇒ *0* | *k*#- ⇒ *k*) |
hd-coeff (*NDvd d i ks*) = (*case ks of* [] ⇒ *0* | *k*#- ⇒ *k*)

fun $decr_Z$ :: *atom* ⇒ *atom* **where**
$decr_Z$ (*Le i ks*) = *Le i* (*tl ks*) |
$decr_Z$ (*Dvd d i ks*) = *Dvd d i* (*tl ks*) |
$decr_Z$ (*NDvd d i ks*) = *NDvd d i* (*tl ks*)

fun I_Z :: *atom* ⇒ *int list* ⇒ *bool* **where**
I_Z (*Le i ks*) *xs* = (*i* ≤ ⟨*ks*,*xs*⟩) |
I_Z (*Dvd d i ks*) *xs* = (*d dvd i*+⟨*ks*,*xs*⟩) |
I_Z (*NDvd d i ks*) *xs* = (¬ *d dvd i*+⟨*ks*,*xs*⟩)

definition $atoms_0$ = *ATOM.atoms*$_0$ (λ*a*. *hd-coeff a* ≠ *0*)

interpretation *Z*:
  ATOM neg_Z (λ*a*. *divisor a* ≠ *0*) I_Z (λ*a*. *hd-coeff a* ≠ *0*) $decr_Z$
  rewrites *ATOM.atoms*$_0$ (λ*a*. *hd-coeff a* ≠ *0*) = $atoms_0$
proof *goal-cases*
  case *1*
  thus *?case*
    apply(*unfold-locales*)
    apply(*case-tac a*)
    apply *simp-all*
    apply(*case-tac a*)
    apply *simp-all*

apply(*case-tac a*)
apply (*simp-all*)
apply *arith*
apply(*case-tac a*)
apply(*simp-all add*: *split*: *list.splits*)
apply(*case-tac a*)
apply *simp-all*
done
next
case *2*
thus *?case* **by**(*simp add*:$atoms_0$-*def*)
qed

setup ‹*Sign.revert-abbrev* @{*const-abbrev Z.I*}›
setup ‹*Sign.revert-abbrev* @{*const-abbrev Z.lift-dnf-qe*}›

abbreviation
hd-coeff-is1 a ≡
(*case a of Le - - ⇒ hd-coeff a* ∈ {*1*,−*1*} | *- ⇒ hd-coeff a* = *1*)

fun *asubst* :: *int* ⇒ *int list* ⇒ *atom* ⇒ *atom* **where**
asubst i′ ks′ (*Le i* (*k*#*ks*)) = *Le* (*i* − *k*∗*i′*) (*k* ∗$_s$ *ks′* + *ks*) |
asubst i′ ks′ (*Dvd d i* (*k*#*ks*)) = *Dvd d* (*i* + *k*∗*i′*) (*k* ∗$_s$ *ks′* + *ks*) |
asubst i′ ks′ (*NDvd d i* (*k*#*ks*)) = *NDvd d* (*i* + *k*∗*i′*) (*k* ∗$_s$ *ks′* + *ks*) |
asubst i′ ks′ a = *a*

abbreviation *subst* :: *int* ⇒ *int list* ⇒ *atom fm* ⇒ *atom fm*
where *subst i ks* ≡ map_{fm} (*asubst i ks*)

lemma *IZ-asubst*: I_Z (*asubst i ks a*) *xs* = I_Z *a* ((*i* + ⟨*ks*,*xs*⟩) # *xs*)
apply (*cases a*)
apply (*rename-tac list*)
apply (*case-tac list*)
apply (*simp-all add*:*algebra-simps iprod-left-add-distrib*)
apply (*rename-tac list*)
apply (*case-tac list*)
apply (*simp-all add*:*algebra-simps iprod-left-add-distrib*)
apply (*rename-tac list*)
apply (*case-tac list*)
apply (*simp-all add*:*algebra-simps iprod-left-add-distrib*)
done

lemma *I-subst*:
qfree φ ⟹ *Z.I* φ ((*i* + ⟨*ks*,*xs*⟩) # *xs*) = *Z.I* (*subst i ks* φ) *xs*
by (*induct* φ) (*simp-all add*:*IZ-asubst*)

lemma *divisor-asubst*[*simp*]: *divisor* (*asubst i ks a*) = *divisor a*
by(*induct i ks a rule*:*asubst.induct*) *auto*

definition *lbounds as* = [(*i*,*ks*). *Le i* (*k*#*ks*) ← *as*, *k*>*0*]
definition *ubounds as* = [(*i*,*ks*). *Le i* (*k*#*ks*) ← *as*, *k*<*0*]
lemma *set-lbounds*:
set(*lbounds as*) = {(*i*,*ks*)|*i k ks*. *Le i* (*k*#*ks*) ∈ *set as* ∧ *k*>*0*}
by(*auto simp*: *lbounds-def split*:*list.splits atom.splits if-splits*)
lemma *set-ubounds*:
set(*ubounds as*) = {(*i*,*ks*)|*i k ks*. *Le i* (*k*#*ks*) ∈ *set as* ∧ *k*<*0*}
by(*auto simp*: *ubounds-def split*:*list.splits atom.splits if-splits*)

lemma *lbounds-append*[*simp*]: *lbounds*(*as* @ *bs*) = *lbounds as* @ *lbounds bs*
by(*simp add*:*lbounds-def*)

5.2 LCM and lemmas

fun *zlcms* :: *int list* ⇒ *int* **where**
zlcms [] = *1* |
zlcms (*i*#*is*) = *lcm i* (*zlcms is*)

lemma *dvd-zlcms*: *i* ∈ *set is* ⟹ *i dvd zlcms is*
by(*induct is*) *auto*

lemma *zlcms-pos*: ∀ *i* ∈ *set is*. *i*≠*0* ⟹ *zlcms is* > *0*
by(*induct is*)(*auto simp*:*lcm-pos-int*)

lemma *zlcms0-iff*[*simp*]: (*zlcms is* = *0*) = (*0* ∈ *set is*)
by (*metis mod-by-0 dvd-eq-mod-eq-0 dvd-zlcms zlcms-pos less-le*)

lemma *elem-le-zlcms*: ∀ *i* ∈ *set is*. *i* ≠ *0* ⟹ *i* ∈ *set is* ⟹ *i* ≤ *zlcms is*
by (*metis dvd-zlcms zdvd-imp-le zlcms-pos*)

5.3 Setting coeffiencients to 1 or -1

fun *hd-coeff1* :: *int* ⇒ *atom* ⇒ *atom* **where**
hd-coeff1 m (*Le i* (*k*#*ks*)) =
(*if k*=*0 then Le i* (*k*#*ks*)
else let m′ = *m div* (*abs k*) *in Le* (*m*′∗*i*) (*sgn k* # (*m*′ ∗*s ks*))) |
hd-coeff1 m (*Dvd d i* (*k*#*ks*)) =
(*if k*=*0 then Dvd d i* (*k*#*ks*)
else let m′ = *m div k in Dvd* (*m*′∗*d*) (*m*′∗*i*) (*1* # (*m*′ ∗*s ks*))) |
hd-coeff1 m (*NDvd d i* (*k*#*ks*)) =
(*if k*=*0 then NDvd d i* (*k*#*ks*)
else let m′ = *m div k in NDvd* (*m*′∗*d*) (*m*′∗*i*) (*1* # (*m*′ ∗*s ks*))) |
hd-coeff1 - *a* = *a*

The def of *hd-coeff1* on *Dvd* and *NDvd* is different from the *Le* because it allows the resulting head coefficient to be 1 rather than 1 or -1. We show

that the other version has the same semantics:

lemma ⟦ $k \neq 0$; k *dvd* m ⟧ ⟹
  I_Z (*hd-coeff1* m (*Dvd* d i (k#ks))) (x#e) = (*let* m' = m *div* (*abs* k) *in*
  I_Z (*Dvd* (m'∗d) (m'∗i) (*sgn* k # (m' ∗$_s$ ks))) (x#e))
apply(*auto simp*:*algebra-simps abs-if sgn-if*)
 apply(*simp add*: *zdiv-zminus2-eq-if dvd-eq-mod-eq-0*[*THEN iffD1*] *algebra-simps*)
 apply (*metis diff-conv-add-uminus add.left-commute dvd-minus-iff minus-add-distrib*)
apply(*simp add*: *zdiv-zminus2-eq-if dvd-eq-mod-eq-0*[*THEN iffD1*] *algebra-simps*)
apply (*metis diff-conv-add-uminus add.left-commute dvd-minus-iff minus-add-distrib*)
done

lemma *I-hd-coeff1-mult-a*: **assumes** m>0
shows *hd-coeff a dvd m* | *hd-coeff a* = 0 ⟹ I_Z (*hd-coeff1 m a*) (m∗x#xs) = I_Z a (x#xs)
proof(*induct a*)
  case [*simp*]: (*Le i ks*)
  show *?case*
  proof(*cases ks*)
    case *Nil* **thus** *?thesis* **by** *simp*
  next
    case [*simp*]: (*Cons k ks'*)
    show *?thesis*
    proof *cases*
      assume k=0 **thus** *?thesis* **by** *simp*
    next
      assume k≠0
      with *Le* **have** |k| *dvd m* **by** *simp*
      let *?m'* = m *div* |k|
      have *?m'* > 0 **using** ‹|k| *dvd m*› *pos-imp-zdiv-pos-iff* ‹m>0› ‹k≠0›
        by(*simp add*:*zdvd-imp-le*)
      have *1*: k∗(x∗*?m'*) = *sgn* k ∗ x ∗ m
      proof −
        have k∗(x∗*?m'*) = (*sgn* k ∗ *abs* k) ∗ (x ∗ *?m'*)
          by(*simp only*: *mult-sgn-abs*)
        also have ... = *sgn* k ∗ x ∗ (*abs* k ∗ *?m'*) **by** *simp*
        also have ... = *sgn* k ∗ x ∗ m
          using *dvd-mult-div-cancel*[*OF* ‹|k| *dvd m*›] **by**(*simp add*:*algebra-simps*)
        finally show *?thesis* **.**
      qed
      have I_Z (*hd-coeff1 m* (*Le i ks*)) (m∗x#xs) ⟷
            (i∗*?m'* ≤ *sgn* k ∗ m∗x + *?m'* ∗ ⟨ks',xs⟩)
        using ‹k≠0› **by**(*simp add*: *algebra-simps*)
      also have ... ⟷ *?m'*∗i ≤ *?m'* ∗ (k∗x + ⟨ks',xs⟩) **using** *1*
        by(*simp* (*no-asm-simp*) *add*:*algebra-simps*)
      also have ... ⟷ i ≤ k∗x + ⟨ks',xs⟩ **using** ‹*?m'*>0›
        by *simp*
      finally show *?thesis* **by**(*simp*)
    qed

```
  qed
next
  case [simp]: (Dvd d i ks)
  show ?case
  proof(cases ks)
    case Nil thus ?thesis by simp
  next
    case [simp]: (Cons k ks')
    show ?thesis
    proof cases
      assume k=0 thus ?thesis by simp
    next
      assume k≠0
      with Dvd have k dvd m by simp
      let ?m' = m div k
      have ?m' ≠ 0 using ‹k dvd m› zdiv-eq-0-iff ‹m>0› ‹k≠0›
        by(simp add:linorder-not-less zdvd-imp-le)
      have 1: k*(x*?m') = x * m
      proof −
        have k*(x*?m') = x*(k*?m') by(simp add:algebra-simps)
        also have ... = x*m using dvd-mult-div-cancel[OF ‹k dvd m›]
          by(simp add:algebra-simps)
        finally show ?thesis .
      qed
      have I_Z (hd-coeff1 m (Dvd d i ks)) (m*x#xs) ⟷
            (?m'*d dvd ?m'*i + m*x + ?m' * ⟨ks',xs⟩)
        using ‹k≠0› by(simp add: algebra-simps)
      also have ... ⟷ ?m'*d dvd ?m' * (i + k*x + ⟨ks',xs⟩) using 1
        by(simp (no-asm-simp) add:algebra-simps)
      also have ... ⟷ d dvd i + k*x + ⟨ks',xs⟩ using ‹?m'≠0› by(simp)
      finally show ?thesis by(simp add:algebra-simps)
    qed
  qed
next
  case [simp]: (NDvd d i ks)
  show ?case
  proof(cases ks)
    case Nil thus ?thesis by simp
  next
    case [simp]: (Cons k ks')
    show ?thesis
    proof cases
      assume k=0 thus ?thesis by simp
    next
      assume k≠0
      with NDvd have k dvd m by simp
      let ?m' = m div k
      have ?m' ≠ 0 using ‹k dvd m› zdiv-eq-0-iff ‹m>0› ‹k≠0›
        by(simp add:linorder-not-less zdvd-imp-le)
```

```
    have 1: k*(x*?m') = x * m
    proof −
      have k*(x*?m') = x*(k*?m') by(simp add:algebra-simps)
      also have ... = x*m using dvd-mult-div-cancel[OF ‹k dvd m›]
        by(simp add:algebra-simps)
      finally show ?thesis .
    qed
    have I_Z (hd-coeff1 m (NDvd d i ks)) (m*x#xs) ⟷
        ¬(?m'*d dvd ?m'*i + m*x + ?m' * ⟨ks',xs⟩)
      using ‹k≠0› by(simp add: algebra-simps)
    also have ... ⟷ ¬ ?m'*d dvd ?m' * (i + k*x + ⟨ks',xs⟩) using 1
      by(simp (no-asm-simp) add:algebra-simps)
    also have ... ⟷ ¬ d dvd i + k*x + ⟨ks',xs⟩ using ‹?m'≠0› by(simp)
    finally show ?thesis by(simp add:algebra-simps)
   qed
  qed
qed


lemma I-hd-coeff1-mult: assumes m>0
shows qfree φ ⟹ ∀ a ∈ set(Z.atoms₀ φ). hd-coeff a dvd m ⟹
 Z.I (map_fm (hd-coeff1 m) φ) (m*x#xs) = Z.I φ (x#xs)
proof(induct φ)
  case (Atom a)
  thus ?case using I-hd-coeff1-mult-a[OF ‹m>0›] by auto
qed simp-all

end


theory QEpres
imports PresArith
begin
```

5.4 DNF-based quantifier elimination

```
definition
hd-coeffs1 as =
 (let m = zlcms(map hd-coeff as)
  in Dvd m 0 [1] # map (hd-coeff1 m) as)

lemma I-hd-coeffs1:
assumes 0: ∀ a∈set as. hd-coeff a ≠ 0 shows
  (∃ x. ∀ a ∈ set(hd-coeffs1 as). I_Z a (x#xs)) =
   (∃ x. ∀ a ∈ set as. I_Z a (x#xs)) (is ?B = ?A)
proof −
  let ?m = zlcms(map hd-coeff as)
  have ?m>0 using 0 by(simp add:zlcms-pos)
  have ?A = (∃ x. ∀ a ∈ set as. I_Z (hd-coeff1 ?m a) (?m*x#xs))
```

```
  by (simp add:I-hd-coeff1-mult-a[OF ‹?m>0›] dvd-zlcms 0)
 also have ... = (∃ x. ?m dvd x+0 ∧ (∀ a ∈ set as. I_Z (hd-coeff1 ?m a) (x#xs)))
  by(rule unity-coeff-ex[THEN meta-eq-to-obj-eq])
 finally show ?thesis by(simp add:hd-coeffs1-def)
qed


abbreviation is-dvd a ≡ case a of Le - - ⇒ False | - ⇒ True

definition
qe-pres_1 as =
 (let ds = filter is-dvd as; (d::int) = zlcms(map divisor ds); ls = lbounds as
  in if ls = []
     then Disj [0..d − 1] (λn. list-conj(map (Atom ∘ asubst n []) ds))
     else
     Disj ls (λ(li,lks).
       Disj [0..d − 1] (λn.
         list-conj(map (Atom ∘ asubst (li + n) (−lks)) as))))
```

Note the optimization in the case $ls = []$: only the divisibility atoms are tested, not the inequalities. This complicates the proof.

```
lemma I-cyclic:
assumes is-dvd a and hd-coeff a = 1 and i mod divisor a = j mod divisor a
shows I_Z a (i#e) = I_Z a (j#e)
proof (cases a)
 case (Dvd d l ks)
 with ‹hd-coeff a = 1› obtain ks' where [simp]: ks = 1#ks'
  by(simp split:list.splits)
 have (l + (i + ⟨ks',e⟩)) mod d = (l + (j + ⟨ks',e⟩)) mod d (is ?l=?r)
 proof −
  have ?l = (l mod d + (i + ⟨ks',e⟩) mod d) mod d
   by (simp add: mod-add-eq)
  also have (i + ⟨ks',e⟩) mod d = (i mod d + ⟨ks',e⟩ mod d) mod d
   by (simp add: mod-add-eq)
  also have i mod d = j mod d
   using ‹i mod divisor a = j mod divisor a› Dvd by simp
  also have (j mod d + ⟨ks',e⟩ mod d) mod d = (j + ⟨ks',e⟩) mod d
   by(rule mod-add-eq)
  also have (l mod d + (j + ⟨ks',e⟩) mod d) mod d = ?r
   by(rule mod-add-eq)
  finally show ?thesis .
 qed
 thus ?thesis using Dvd by (simp add:dvd-eq-mod-eq-0)
next
 case (NDvd d l ks)
 with ‹hd-coeff a = 1› obtain ks' where [simp]: ks = 1#ks'
  by(simp split:list.splits)
 have (l + (i + ⟨ks',e⟩)) mod d = (l + (j + ⟨ks',e⟩)) mod d (is ?l=?r)
 proof −
```

```
    have ?l = (l mod d + (i + ⟨ks',e⟩) mod d) mod d
      by (simp add: mod-add-eq)
    also have (i + ⟨ks',e⟩) mod d = (i mod d + ⟨ks',e⟩ mod d) mod d
      by (simp add: mod-add-eq)
    also have i mod d = j mod d
      using ‹i mod divisor a = j mod divisor a› NDvd by simp
    also have (j mod d + ⟨ks',e⟩ mod d) mod d = (j + ⟨ks',e⟩) mod d
      by(rule mod-add-eq)
    also have (l mod d + (j + ⟨ks',e⟩) mod d) mod d = ?r
      by(rule mod-add-eq)
    finally show ?thesis .
  qed
  thus ?thesis using NDvd by (simp add:dvd-eq-mod-eq-0)
next
  case Le thus ?thesis using ‹is-dvd a› by simp
qed

lemma I-qe-pres_1:
assumes norm: ∀ a ∈ set as. divisor a ≠ 0
and hd: ∀ a ∈ set as. hd-coeff-is1 a
shows Z.I (qe-pres_1 as) xs = (∃ x. ∀ a∈ set as. I_Z a (x#xs))
proof −
  let ?lbs = lbounds as
  let ?ds = filter is-dvd as
  let ?lcm = zlcms(map divisor ?ds)
  let ?Ds = {a ∈ set as. is-dvd a}
  let ?Us = {a ∈ set as. case a of Le - (k#-) ⇒ k < 0 | - ⇒ False}
  let ?Ls = {a ∈ set as. case a of Le - (k#-) ⇒ k > 0 | - ⇒ False}
  have as: set as = ?Ds ∪ ?Ls ∪ ?Us (is - = ?S)
  proof −
    { fix x assume x ∈ set as
      hence x ∈ ?S using hd by (cases x rule: atom.exhaust)(auto split:list.splits)
}
    moreover
    { fix x assume x ∈ ?S
      hence x ∈ set as by auto }
    ultimately show ?thesis by blast
  qed
  have 1: ∀ a ∈ ?Ds. hd-coeff a = 1 using hd by(fastforce split:atom.splits)
  show ?thesis  (is ?QE = (∃ x. ?P x))
  proof
    assume ?QE
    { assume ?lbs = []
      with ‹?QE› obtain n where n < ?lcm and
        A: ∀ a ∈ ?Ds. I_Z a (n#xs) using 1
        by(auto simp:IZ-asubst qe-pres_1-def)
      have ?Ls = {} using ‹?lbs = []› set-lbounds[of as]
        by (auto simp add:filter-empty-conv split:atom.split list.split)
      have ∃ x. ?P x
```

proof *cases*
assume *?Us = {}*
with ‹*?Ls = {}*› **have** *set as = ?Ds* **using** *as* **by**(*simp* (*no-asm-use*))*blast*
hence *?P n* **using** *A* **by** *auto*
thus *?thesis* **..**
next
assume *?Us ≠ {}*
let *?M = {⟨tl ks, xs⟩ − i|ks i. Le i ks ∈ ?Us}* **let** *?m = Min ?M*
have *finite ?M*
proof −
have *finite* ((*λLe i ks ⇒ ⟨tl ks, xs⟩ − i*) ‘
{a∈set as. ∃ i k ks. k<0 ∧ a = Le i (k#ks)})
(**is** *finite ?B*)
by *simp*
also have *?B = ?M* **using** *hd*
by(*fastforce simp*:*image-def neq-Nil-conv split*:*atom.splits list.splits*)
finally show *?thesis* **by** *auto*
qed
have *?M ≠ {}*
proof −
from ‹*?Us ≠ {}*› **obtain** *i k ks* **where** *Le i (k#ks) ∈ ?Us ∧ k<0*
by (*fastforce split*:*atom.splits list.splits*)
thus *?thesis* **by** *auto*
qed
let *?k = (n − ?m) div ?lcm + 1* **let** *?x = n − ?k ∗ ?lcm*
have *∀ a ∈ ?Ds. I_Z a (?x # xs)*
proof (*intro allI ballI*)
fix *a* **assume** *a ∈ ?Ds*
let *?d = divisor a*
have *2*: *?d dvd ?lcm* **using** ‹*a ∈ ?Ds*› **by**(*simp add*:*dvd-zlcms*)
have *?x mod ?d = n mod ?d* (**is** *?l = ?r*)
proof −
have *?l = (?r − ((?k ∗ ?lcm) mod ?d)) mod ?d*
by (*simp add*: *mod-diff-eq*)
also have *(?k ∗ ?lcm) mod ?d = 0*
by(*simp add*: *dvd-eq-mod-eq-0*[*symmetric*] *dvd-mult*[*OF 2*])
finally show *?thesis* **by** *simp*
qed
thus *I_Z a (?x#xs)* **using** *A I-cyclic*[*of a n ?x*] ‹*a ∈ ?Ds*› *1* **by** *auto*
qed
moreover
have *∀ a∈ ?Us. I_Z a (?x#xs)*
proof
fix *a* **assume** *a ∈ ?Us*
then obtain *l ks* **where** [*simp*]: *a = Le l (−1#ks)* **using** *hd*
by(*fastforce split*:*atom.splits list.splits*)
have *?m ≤ ⟨ks,xs⟩ − l*
using *Min-le-iff*[*OF* ‹*finite ?M*› ‹*?M ≠ {}*›] ‹*a ∈ ?Us*› **by** *fastforce*
moreover have *(n − ?m) mod ?lcm < ?lcm*

```
        by(simp add: pos-mod-bound[OF zlcms-pos] norm)
      ultimately show I_Z a (?x#xs)
        by(simp add:minus-mod-eq-mult-div [symmetric] algebra-simps)
    qed
    moreover
    have set as = ?Ds ∪ ?Us using as ‹?Ls = {}›
      by (simp (no-asm-use)) blast
    ultimately have ?P(?x) by auto
    thus ?thesis ..
  qed }
 moreover
 { assume ?lbs ≠ []
  with ‹?QE› obtain il ksl m
    where ∀ a∈set as. I_Z (asubst (il + m) ksl a) xs
    by(auto simp:qe-pres_1-def)
  hence ?P(il + m + ⟨ksl,xs⟩) by(simp add:IZ-asubst)
  hence ∃ x. ?P x .. }
 ultimately show ∃ x. ?P x by blast
next
 assume ∃ x. ?P x then obtain x where x: ?P x ..
 show ?QE
 proof cases
  assume ?lbs = []
  moreover
  have ∃ x. 0 ≤ x ∧ x < ?lcm ∧ (∀ a ∈ ?Ds. I_Z a (x # xs))
    (is ∃ x. ?P x)
  proof
    { fix a assume a ∈ ?Ds
      hence I_Z a ((x mod ?lcm) # xs) = I_Z a (x # xs) using 1
        by (fastforce del:iffI intro: I-cyclic
            simp: mod-mod-cancel dvd-zlcms) }
    thus ?P(x mod ?lcm) using x norm by(simp add: zlcms-pos)
  qed
  ultimately show ?thesis by (auto simp:qe-pres_1-def IZ-asubst)
 next
  assume ?lbs ≠ []
  let ?L = {i - ⟨ks,xs⟩ |ks i. (i,ks) ∈ set(lbounds as)}
  let ?lm = Max ?L
  let ?n = (x - ?lm) mod ?lcm
  have finite ?L
  proof -
    have finite((λ(i,ks). i-⟨ks,xs⟩) ` set(lbounds as) ) (is finite ?B)
      by simp
    also have ?B = ?L by auto
    finally show ?thesis by auto
  qed
  moreover have ?L ≠ {} using ‹?lbs ≠ []›
    by(fastforce simp:neq-Nil-conv)
  ultimately have ?lm ∈ ?L by(rule Max-in)
```

then obtain *li lks* **where** (*li*,*lks*)∈ *set ?lbs* **and** *lm*: *?lm* = *li*−⟨*lks*,*xs*⟩
by *blast*
moreover
have *n*: *0* ≤ *?n* ∧ *?n* < *?lcm* **using** *norm* **by**(*simp add*:*zlcms-pos*)
moreover
{ fix *a* **assume** *a* ∈ *set as*
with *x* **have** I_Z *a* (*x* # *xs*) **by** *blast*
have I_Z *a* ((*li* + *?n* − ⟨*lks*,*xs*⟩) # *xs*)
proof −
{ assume *a* ∈ *?Ls*
then obtain *i ks* **where** [*simp*]: *a* = *Le i* (*1*#*ks*) **using** *hd*
by(*fastforce split*:*atom.splits list.splits*)
from ‹*a* ∈ *?Ls*› **have** *i*−⟨*ks*,*xs*⟩ ∈ *?L* **by**(*fastforce simp*:*set-lbounds*)
hence *i*−⟨*ks*,*xs*⟩ ≤ *li* − ⟨*lks*,*xs*⟩
using *lm*[*symmetric*] ‹*finite ?L*› ‹*?L* ≠ {}› **by** *auto*
hence *?thesis* **using** *n* **by** *simp* **}**
moreover
{ assume *a* ∈ *?Us*
then obtain *i ks* **where** [*simp*]: *a* = *Le i* (−*1*#*ks*) **using** *hd*
by(*fastforce split*:*atom.splits list.splits*)
have *Le li* (*1*#*lks*) ∈ *set as* **using** ‹(*li*,*lks*) ∈ *set ?lbs*› *hd*
by(*auto simp*:*set-lbounds*)
hence *li* − ⟨*lks*,*xs*⟩ ≤ *x* **using** *x* **by** *auto*
hence (*x* − *?lm*) *mod ?lcm* ≤ *x* − *?lm*
using *lm* **by**(*simp add*: *zmod-le-nonneg-dividend*)
hence *?thesis* **using** ‹I_Z *a* (*x* # *xs*)› *lm* **by** *auto* **}**
moreover
{ assume *a* ∈ *?Ds*
have *?thesis*
proof(*rule I-cyclic*[*THEN iffD2*, *OF - - -* ‹I_Z *a* (*x* # *xs*)›])
show *is-dvd a* **using** ‹*a* ∈ *?Ds*› **by** *simp*
show *hd-coeff a* = *1* **using** ‹*a* ∈ *?Ds*› *hd*
by(*fastforce split*:*atom.splits list.splits*)
have *li* + (*x*−*?lm*) *mod ?lcm* − ⟨*lks*,*xs*⟩ = *?lm* + (*x*−*?lm*) *mod ?lcm*
using *lm* **by** *arith*
hence (*li* + (*x*−*?lm*) *mod ?lcm* − ⟨*lks*,*xs*⟩) *mod divisor a* =
(*?lm* + (*x*−*?lm*) *mod ?lcm*) *mod divisor a* **by** (*simp only*:)
also have ... =
(*?lm mod divisor a* + (*x*−*?lm*) *mod ?lcm mod divisor a*) *mod divisor a*
by (*simp add*: *mod-add-eq*)
also have
... = (*?lm mod divisor a* + (*x*−*?lm*) *mod divisor a*) *mod divisor a*
using ‹*is-dvd a*› ‹*a*∈ *set as*›
by(*simp add*: *mod-mod-cancel dvd-zlcms*)
also have ... = (*?lm* + (*x*−*?lm*)) *mod divisor a*
by(*rule mod-add-eq*)
also have ... = *x mod divisor a* **by** *simp*
finally
show (*li* + *?n* − ⟨*lks*,*xs*⟩) *mod divisor a* = *x mod divisor a*

```
            using norm by(auto simp: zlcms-pos)
          qed }
        ultimately show ?thesis using ‹a ∈ set as› as by blast
      qed
    }
    ultimately show ?thesis using ‹?lbs ≠ []›
      by (simp (no-asm-simp) add:qe-pres$_1$-def IZ-asubst split-def)
        (force simp del:int-nat-eq)
  qed
 qed
qed

lemma divisors-hd-coeffs1:
assumes div0: ∀ a∈set as. divisor a ≠ 0 and hd0: ∀ a∈set as. hd-coeff a ≠ 0
and a: a∈set (hd-coeffs1 as) shows divisor a ≠ 0
proof −
 let ?m = zlcms(map hd-coeff as)
 from a have a = Dvd ?m 0 [1] ∨ (∃ b ∈ set as. a = hd-coeff1 ?m b)
  (is ?A ∨ ?B)
  by(auto simp:hd-coeffs1-def)
 thus ?thesis
 proof
  assume ?A thus ?thesis using hd0 by(auto)
 next
  assume ?B
  then obtain b where b ∈ set as and [simp]: a = hd-coeff1 ?m b ..
  hence b: hd-coeff b ≠ 0 divisor b ≠ 0 using div0 hd0 by auto
  show ?thesis
  proof (cases b)
   case (Le i ks) thus ?thesis using b by(auto split:list.splits)
  next
   case [simp]: (Dvd d i ks)
   then obtain k ks′ where [simp]: ks = k#ks′ using b
    by(auto split:list.splits)
   have k: k ∈ set(map hd-coeff as) using ‹b ∈ set as› by force
   have zlcms (map hd-coeff as) div k ≠ 0
    using b hd0 dvd-zlcms[OF k]
    by(auto simp add:dvd-def)
   thus ?thesis using b by (simp)
  next
   case [simp]: (NDvd d i ks)
   then obtain k ks′ where [simp]: ks = k#ks′ using b
    by(auto split:list.splits)
   have k: k ∈ set(map hd-coeff as) using ‹b ∈ set as› by force
   have zlcms (map hd-coeff as) div k ≠ 0
    using b hd0 dvd-zlcms[OF k]
    by(auto simp add:dvd-def)
   thus ?thesis using b by (simp)
  qed
```

```
  qed
qed

lemma hd-coeff-is1-hd-coeffs1:
assumes hd0: ∀ a∈set as. hd-coeff a ≠ 0
and a: a∈set (hd-coeffs1 as) shows hd-coeff-is1 a
proof −
  let ?m = zlcms(map hd-coeff as)
  from a have a = Dvd ?m 0 [1] ∨ (∃ b ∈ set as. a = hd-coeff1 ?m b)
    (is ?A ∨ ?B)
    by(auto simp:hd-coeffs1-def)
  thus ?thesis
  proof
    assume ?A thus ?thesis using hd0 by simp
  next
    assume ?B
    then obtain b where b ∈ set as and [simp]: a = hd-coeff1 ?m b ..
    hence b: hd-coeff b ≠ 0 using hd0 by auto
    show ?thesis using b
      by (cases b) (auto simp: sgn-if split:list.splits)
  qed
qed


lemma I-qe-pres_1-o:
 ⟦ ∀ a ∈ set as. divisor a ≠ 0; ∀ a∈set as. hd-coeff a ≠ 0 ⟧ ⟹
  Z.I ((qe-pres_1 ∘ hd-coeffs1) as) e = (∃ x. ∀ a∈ set as. I_Z a (x#e))
apply(simp)
apply(subst I-qe-pres_1)
  apply(simp add: divisors-hd-coeffs1)
 apply(simp add: hd-coeff-is1-hd-coeffs1)
using I-hd-coeffs1 apply(simp)
done

definition qe-pres = Z.lift-dnf-qe (qe-pres_1 ∘ hd-coeffs1)

lemma qfree-qe-pres-o: qfree ((qe-pres_1 ∘ hd-coeffs1) as)
by(auto simp:qe-pres_1-def intro!:qfree-list-disj)


lemma normal-qe-pres_1-o:
  ∀ a ∈ set as. hd-coeff a ≠ 0 ∧ divisor a ≠ 0 ⟹
   Z.normal ((qe-pres_1 ∘ hd-coeffs1) as)
  supply image-cong-simp [cong del]
apply(auto simp:qe-pres_1-def Z.normal-def
   dest!:atoms-list-disjE atoms-list-conjE)

apply(simp add: hd-coeffs1-def)
 apply(erule disjE) apply fastforce
```

apply (*clarsimp*)
apply(*case-tac xa*)
 apply(*rename-tac list*) **apply**(*case-tac list*) **apply** *fastforce* **apply** (*simp split*:*if-split-asm*)
 apply(*rename-tac list*) **apply**(*case-tac list*) **apply** *fastforce*
 apply (*simp split*:*if-split-asm*) **apply** *fastforce*
 apply(*erule disjE*) **prefer** *2* **apply** *fastforce*
 apply(*simp add*:*zdiv-eq-0-iff*)
 apply(*subgoal-tac a* ∈ *set*(*map hd-coeff as*))
  prefer *2* **apply** *force*
 apply(*subgoal-tac* ∀ *i*∈ *set*(*map hd-coeff as*). *i* ≠ *0*)
  prefer *2* **apply** *simp*
 apply (*metis elem-le-zlcms linorder-not-le zlcms-pos*)
apply(*rename-tac list*) **apply**(*case-tac list*) **apply** *fastforce*
apply (*simp split*:*if-split-asm*) **apply** *fastforce*
apply(*simp add*:*zdiv-eq-0-iff*)
apply(*subgoal-tac* ∀ *i*∈ *set*(*map hd-coeff as*). *i* ≠ *0*)
 prefer *2* **apply** *simp*
apply(*subgoal-tac a* ∈ *set*(*map hd-coeff as*))
 prefer *2* **apply** *force*
apply(*erule disjE*)
 apply (*metis elem-le-zlcms linorder-not-le*)
apply(*erule disjE*)
 apply (*metis linorder-not-le zlcms-pos*)
apply *fastforce*

apply(*simp add*: *hd-coeffs1-def*)
 apply(*erule disjE*) **apply** *fastforce*
apply (*clarsimp*)
apply(*case-tac xa*)
 apply(*rename-tac list*) **apply**(*case-tac list*) **apply** *fastforce* **apply** (*simp split*:*if-split-asm*)
 apply(*rename-tac list*) **apply**(*case-tac list*) **apply** *fastforce*
 apply (*simp split*:*if-split-asm*) **apply** *fastforce*
 apply(*erule disjE*) **prefer** *2* **apply** *fastforce*
 apply(*simp add*:*zdiv-eq-0-iff*)
 apply(*subgoal-tac a* ∈ *set*(*map hd-coeff as*))
  prefer *2* **apply** *force*
 apply(*subgoal-tac* ∀ *i*∈ *set*(*map hd-coeff as*). *i* ≠ *0*)
  prefer *2* **apply** *simp*
 apply (*metis elem-le-zlcms linorder-not-le zlcms-pos*)
apply(*rename-tac list*) **apply**(*case-tac list*) **apply** *fastforce*
apply (*simp split*:*if-split-asm*) **apply** *fastforce*
apply(*simp add*:*zdiv-eq-0-iff*)
apply(*subgoal-tac* ∀ *i*∈ *set*(*map hd-coeff as*). *i* ≠ *0*)
 prefer *2* **apply** *simp*
apply(*subgoal-tac a* ∈ *set*(*map hd-coeff as*))
 prefer *2* **apply** *force*
apply(*erule disjE*)
 apply (*metis elem-le-zlcms linorder-not-le*)
apply(*erule disjE*)

apply (*metis linorder-not-le zlcms-pos*)
apply *fastforce*
done

theorem *I-pres-qe*: *Z.normal* $\varphi \Longrightarrow$ *Z.I* (*qe-pres* φ) *xs* = *Z.I* φ *xs*
by(*simp add*:*qe-pres-def Z.I-lift-dnf-qe-anormal I-qe-pres*$_1$*-o qfree-qe-pres-o normal-qe-pres*$_1$*-o del*:*o-apply*)

theorem *qfree-pres-qe*: *qfree* (*qe-pres f*)
by(*simp add*:*qe-pres-def Z.qfree-lift-dnf-qe qfree-qe-pres-o del*:*o-apply*)

end

theory *Cooper*
imports *PresArith*
begin

5.5 Cooper

This section formalizes Cooper's algorithm [1].

lemma *set-atoms0-iff*:
qfree $\varphi \Longrightarrow a \in set(Z.atoms_0\ \varphi) \longleftrightarrow a \in atoms\ \varphi \wedge hd\text{-}coeff\ a \neq 0$
by(*induct* φ) (*auto split*:*if-split-asm*)

definition
hd-coeffs1 φ =
(*let m* = *zlcms*(*map hd-coeff* (*Z.atoms*$_0$ φ))
in And (*Atom*(*Dvd m 0* [*1*])) (*map*$_{fm}$ (*hd-coeff1 m*) φ))

lemma *I-hd-coeffs1*:
assumes *qfree* φ
shows ($\exists x.$ *Z.I* (*hd-coeffs1* φ) (*x*#*xs*)) = ($\exists x.$ *Z.I* φ (*x*#*xs*)) (**is** *?L* = *?R*)
proof −
let *?l* = *zlcms*(*map hd-coeff* (*Z.atoms*$_0$ φ))
have *?l*>*0* **by**(*simp add*: *zlcms-pos set-atoms0-iff*[*OF* ‹*qfree* φ›])
have *?L* = ($\exists x.$ *?l dvd x*+*0* $\wedge$ *Z.I* (*map*$_{fm}$ (*hd-coeff1 ?l*) φ) (*x*#*xs*))
by(*simp add*:*hd-coeffs1-def*)
also have ... = ($\exists x.$ *Z.I* (*map*$_{fm}$ (*hd-coeff1 ?l*) φ) (*?l*∗*x*#*xs*))
by(*rule unity-coeff-ex*[*THEN meta-eq-to-obj-eq*,*symmetric*])
also have ... = *?R*
by(*simp add*: *I-hd-coeff1-mult*[*OF* ‹*?l*>*0*› ‹*qfree* φ›] *dvd-zlcms*)
finally show *?thesis* **.**
qed

fun *min-inf* :: *atom fm* ⇒ *atom fm* (‹*inf*$_-$›) **where**
inf$_-$ (*And* φ_1 φ_2) = *and* (*inf*$_-$ φ_1) (*inf*$_-$ φ_2) |
inf$_-$ (*Or* φ_1 φ_2) = *or* (*inf*$_-$ φ_1) (*inf*$_-$ φ_2) |

```
inf_ (Atom(Le i (k#ks))) =
  (if k<0 then TrueF else if k>0 then FalseF else Atom(Le i (0#ks))) |
inf_ φ = φ


definition
qe-cooper_1 φ =
 (let as = Z.atoms_0 φ; d = zlcms(map divisor as); ls = lbounds as
  in or (Disj [0..d − 1] (λn. subst n [] (inf_ φ)))
        (Disj ls (λ(i,ks).
           Disj [0..d − 1] (λn. subst (i + n) (−ks) φ))))


lemma min-inf:
  nqfree f ⟹ ∀ a∈set(Z.atoms_0 f). hd-coeff-is1 a
   ⟹ ∃ x.∀ y<x. Z.I (inf_ f) (y # xs) = Z.I f (y # xs)
  (is - ⟹ - ⟹ ∃ x. ?P f x)
proof(induct f rule: min-inf.induct)
  case (3 i k ks)
  { assume k=0 hence ?case using 3 by simp }
  moreover
  { assume k= −1
    hence ?P (Atom(Le i (k#ks))) (−i + ⟨ks,xs⟩ − 1) using 3 by auto
    hence ?case .. }
  moreover
  { assume k=1
    hence ?P (Atom(Le i (k#ks))) (i − ⟨ks,xs⟩ − 1) using 3 by auto
    hence ?case .. }
  ultimately show ?case using 3 by auto
next
  case (1 f1 f2)
  then obtain x1 x2 where ?P f1 x1 ?P f2 x2 by fastforce+
  hence ?P (And f1 f2) (min x1 x2) by simp
  thus ?case ..
next
  case (2 f1 f2)
  then obtain x1 x2 where ?P f1 x1 ?P f2 x2 by fastforce+
  hence ?P (Or f1 f2) (min x1 x2) by simp
  thus ?case ..
qed simp-all


lemma min-inf-repeats:
  nqfree φ ⟹ ∀ a∈set(Z.atoms_0 φ). divisor a dvd d ⟹
  Z.I (inf_ φ) ((x − k∗d)#xs) = Z.I (inf_ φ) (x#xs)
proof(induct φ rule:min-inf.induct)
  case (4-4 da i ks)
  show ?case
  proof (cases ks)
```

```
    case Nil thus ?thesis by simp
  next
    case (Cons j js)
    show ?thesis
    proof cases
      assume j=0 thus ?thesis using Cons by simp
    next
      assume j≠0
      hence da dvd d using Cons 4-4 by simp
      hence da dvd i + (j * x - j * (k * d) + ⟨js,xs⟩) ⟷
            da dvd i + (j * x + ⟨js,xs⟩)
      proof -
        have da dvd i + (j * x - j * (k * d) + ⟨js,xs⟩) ⟷
             da dvd (i + j*x + ⟨js,xs⟩) - (j*k)*d
          by(simp add: algebra-simps)
        also have ... ⟷ da dvd i + j*x + ⟨js,xs⟩ using ‹da dvd d›
          by (metis dvd-diff zdvd-zdiffD dvd-mult mult.commute)
        also have ... ⟷ da dvd i + (j * x + ⟨js,xs⟩)
          by(simp add: algebra-simps)
        finally show ?thesis .
      qed
      then show ?thesis using Cons by (simp add:ring-distribs)
    qed
  qed
next
  case (4-5 da i ks)
  show ?case
  proof (cases ks)
    case Nil thus ?thesis by simp
  next
    case (Cons j js)
    show ?thesis
    proof cases
      assume j=0 thus ?thesis using Cons by simp
    next
      assume j≠0
      hence da dvd d using Cons 4-5 by simp
      hence da dvd i + (j * x - j * (k * d) + ⟨js,xs⟩) ⟷
            da dvd i + (j * x + ⟨js,xs⟩)
      proof -
        have da dvd i + (j * x - j * (k * d) + ⟨js,xs⟩) ⟷
             da dvd (i + j*x + ⟨js,xs⟩) - (j*k)*d
          by(simp add: algebra-simps)
        also have ... ⟷ da dvd i + j*x + ⟨js,xs⟩ using ‹da dvd d›
          by (metis dvd-diff zdvd-zdiffD dvd-mult mult.commute)
        also have ... ⟷ da dvd i + (j * x + ⟨js,xs⟩)
          by(simp add: algebra-simps)
        finally show ?thesis .
      qed
```

```
    then show ?thesis using Cons by (simp add:ring-distribs)
   qed
  qed
qed simp-all


lemma atoms-subset: qfree f ⟹ set(Z.atoms₀(f::atom fm)) ≤ atoms f
by (induct f) auto


lemma β:
  ⟦ nqfree φ;  ∀ a∈set(Z.atoms₀ φ). hd-coeff-is1 a;
     ∀ a∈set(Z.atoms₀ φ). divisor a dvd d; d > 0;
     ¬(∃ j∈{0 .. d − 1}. ∃ (i,ks) ∈ set(lbounds(Z.atoms₀ φ)).
         x = i − ⟨ks,xs⟩ + j); Z.I φ (x#xs) ⟧
  ⟹ Z.I φ ((x−d)#xs)
proof(induct φ)
  case (Atom a)
  show ?case
  proof (cases a)
    case (Le i js)
    show ?thesis
    proof (cases js)
      case Nil thus ?thesis using Le Atom by simp
    next
      case (Cons k ks) thus ?thesis using Le Atom
        by (auto simp:lbounds-def Ball-def split:if-split-asm) arith
    qed
  next
    case (Dvd m i js)
    show ?thesis
    proof (cases js)
      case Nil thus ?thesis using Dvd Atom by simp
    next
      case (Cons k ks)
      show ?thesis
      proof cases
        assume k=0 thus ?thesis using Cons Dvd Atom by simp
      next
        assume k≠0
        hence m dvd d using Cons Dvd Atom by auto
        have m dvd i + (x + ⟨ks,xs⟩) ⟹ m dvd i + (x − d + ⟨ks,xs⟩)
          (is ?L ⟹ -)
        proof −
          assume ?L
          hence m dvd i + (x + ⟨ks,xs⟩) − d
            by (metis ‹m dvd d› dvd-diff)
          thus ?thesis by(simp add:algebra-simps)
        qed
```

```
      thus ?thesis using Atom Dvd Cons by(auto split:if-split-asm)
    qed
  qed
 next
  case (NDvd m i js)
  show ?thesis
  proof (cases js)
    case Nil thus ?thesis using NDvd Atom by simp
  next
    case (Cons k ks)
    show ?thesis
    proof cases
      assume k=0 thus ?thesis using Cons NDvd Atom by simp
    next
      assume k≠0
      hence m dvd d using Cons NDvd Atom by auto
      have m dvd i + (x - d + ⟨ks,xs⟩) ⟹ m dvd i + (x + ⟨ks,xs⟩)
        (is ?L ⟹ -)
      proof -
        assume ?L
        hence m dvd i + (x + ⟨ks,xs⟩) - d by(simp add:algebra-simps)
        thus ?thesis by (metis ‹m dvd d› zdvd-zdiffD)
      qed
      thus ?thesis using Atom NDvd Cons by(auto split:if-split-asm)
    qed
  qed
 qed
qed force+


lemma periodic-finite-ex:
 assumes dpos: (0::int) < d and modd: ∀ x k. P x = P(x - k*d)
 shows (∃ x. P x) = (∃ j∈{0..d - 1}. P j)
 (is ?LHS = ?RHS)
proof
 assume ?LHS
 then obtain x where P: P x ..
 have x mod d = x - (x div d)*d
   by(simp add:mult-div-mod-eq [symmetric] ac-simps eq-diff-eq)
 hence Pmod: P x = P(x mod d) using modd by simp
 have P(x mod d) using dpos P Pmod by simp
 moreover have x mod d ∈ {0..d - 1} using dpos by auto
 ultimately show ?RHS ..
qed auto

lemma cpmi-eq: (0::int) < D ⟹ (∃ z. ∀ x. x < z ⟶ (P x = P1 x))
⟹ ∀ x.¬(∃ j∈{0..D - 1}. ∃ b∈B. P(b+j)) ⟶ P (x) ⟶ P (x - D)
⟹ ∀ x. ∀ k. P1 x = P1(x-k*D)
⟹ (∃ x. P(x)) = ((∃ j∈{0..D - 1}. P1(j)) ∨ (∃ j∈{0..D - 1}. ∃ b∈B. P(b+j)))
```

apply(*rule iffI*)
prefer *2*
apply(*drule minusinfinity*)
apply *assumption+*
apply(*fastforce*)
apply *clarsimp*
apply(*subgoal-tac* $\bigwedge k.\ 0 \leq k \Longrightarrow \forall x.\ P\ x \longrightarrow P\ (x - k*D)$)
apply(*frule-tac* $x = x$ **and** $z=z$ **in** *decr-lemma*)
apply(*subgoal-tac P1*$(x - (|x - z| + 1) * D)$)
prefer *2*
apply(*subgoal-tac* $0 \leq (|x - z| + 1)$)
prefer *2* **apply** *arith*
apply *fastforce*
apply(*drule* (*1*) *periodic-finite-ex*)
apply *blast*
apply(*blast dest*:*decr-mult-lemma*)
done

theorem *cp-thm*:
assumes *nq*: *nqfree* φ
and *u*: $\forall a{\in}set(Z.atoms_0\ \varphi).$ *hd-coeff-is1 a*
and *d*: $\forall a{\in}set(Z.atoms_0\ \varphi).$ *divisor a dvd d*
and *dp*: $d > 0$
shows $(\exists x.\ Z.I\ \varphi\ (x\#xs)) =$
$(\exists j{\in}\{0..d - 1\}.\ Z.I\ (inf_{-}\ \varphi)\ (j\#xs) \vee$
$(\exists (i,ks) \in set(lbounds(Z.atoms_0\ \varphi)).\ Z.I\ \varphi\ ((i - \langle ks,xs\rangle + j)\ \#\ xs)))$
(**is** $(\exists x.\ ?P\ (x)) = (\exists\ j{\in}\ ?D.\ ?M\ j \vee (\exists (i,ks){\in}\ ?B.\ ?P\ (?I\ i\ ks + j))))$
proof−
from *min-inf*[*OF nq u*] **have** *th*: $\exists z.\forall x{<}z.\ ?P\ x = ?M\ x$ **by** *blast*
let $?B' = \{?I\ i\ ks\ |i\ ks.\ (i,ks) \in ?B\}$
have *BB'*: $(\exists j{\in}?D.\ \exists (i,ks){\in}\ ?B.\ ?P\ (?I\ i\ ks + j)) = (\exists j \in ?D.\ \exists b \in ?B'.\ ?P\ (b + j))$ **by** *auto*
hence *th2*: $\forall\ x.\ \neg\ (\exists\ j \in ?D.\ \exists\ b \in ?B'.\ ?P\ ((b + j))) \longrightarrow ?P\ (x) \longrightarrow ?P\ ((x - d))$
using β[*OF nq u d dp, of - xs*] **by**(*simp add*:*Bex-def*) *metis*
from *min-inf-repeats*[*OF nq d*]
have *th3*: $\forall\ x\ k.\ ?M\ x = ?M\ (x{-}k{*}d)$ **by** *simp*
from *cpmi-eq*[*OF dp th th2 th3*] *BB'* **show** *?thesis* **by** *simp blast*
qed

lemma *qfree-min-inf*[*simp*]: *qfree* $\varphi \Longrightarrow$ *qfree* $(inf_{-}\ \varphi)$
by (*induct* φ *rule*:*min-inf.induct*) *simp-all*

lemma $I\text{-}qe\text{-}cooper_1$:
assumes *norm*: $\forall a{\in}atoms\ \varphi.$ *divisor* $a \neq 0$
and *hd*: $\forall a{\in}set(Z.atoms_0\ \varphi).$ *hd-coeff-is1 a* **and** *nqfree* φ

shows $Z.I\ (qe\text{-}cooper_1\ \varphi)\ xs = (\exists\, x.\ Z.I\ \varphi\ (x\#xs))$
proof −
  let *?as* = $Z.atoms_0\ \varphi$
  let *?d* = *zlcms*(*map divisor ?as*)
  have *?d* > *0* **using** *norm atoms-subset*[*of* φ] ‹*nqfree* φ›
    by(*fastforce intro*:*zlcms-pos*)
  have *alld*: $\forall\, a{\in}set(Z.atoms_0\ \varphi).$ *divisor a dvd ?d* **by**(*simp add*:*dvd-zlcms*)
  from *cp-thm*[*OF* ‹*nqfree* φ› *hd alld* ‹*?d*>*0*›]
  show *?thesis* **using** ‹*nqfree* φ›
    by (*simp add*:$qe\text{-}cooper_1$*-def I-subst*[*symmetric*] *split-def algebra-simps*) *blast*
qed

lemma *divisor-hd-coeff1-neq0*:
  qfree $\varphi \Longrightarrow a \in atoms\ \varphi \Longrightarrow$ *divisor a* $\neq$ *0* $\Longrightarrow$
   divisor (*hd-coeff1* (*zlcms* (*map hd-coeff* ($Z.atoms_0\ \varphi$))) *a*) $\neq$ *0*
apply (*case-tac a*)

apply *simp*
apply(*rename-tac list*) **apply**(*case-tac list*) **apply** *simp* **apply**(*simp split*:*if-split-asm*)

apply *simp*
apply(*rename-tac list*)**apply**(*case-tac list*) **apply** *simp*
apply(*clarsimp split*:*if-split-asm*)
apply(*hypsubst-thin*)
apply(*subgoal-tac a* ∈ *set*(*map hd-coeff* ($Z.atoms_0\ \varphi$)))
 apply(*subgoal-tac* $\forall\, i{\in}set(map\ hd\text{-}coeff\ (Z.atoms_0\ \varphi)).\ i \neq 0$)
  apply (*metis dvd-zlcms mult-eq-0-iff dvd-mult-div-cancel zlcms0-iff*)
 apply (*simp add*:*set-atoms0-iff*)
apply(*fastforce simp*:*image-def set-atoms0-iff Bex-def*)

apply *simp*
apply(*rename-tac list*) **apply**(*case-tac list*) **apply** *simp*
apply(*clarsimp split*:*if-split-asm*)
apply(*hypsubst-thin*)
apply(*subgoal-tac a* ∈ *set*(*map hd-coeff* ($Z.atoms_0\ \varphi$)))
 apply(*subgoal-tac* $\forall\, i{\in}set(map\ hd\text{-}coeff\ (Z.atoms_0\ \varphi)).\ i \neq 0$)
  apply (*metis dvd-zlcms mult-eq-0-iff dvd-mult-div-cancel zlcms0-iff*)
 apply (*simp add*:*set-atoms0-iff*)
apply(*fastforce simp*:*image-def set-atoms0-iff Bex-def*)
done

lemma *hd-coeff-is1-hd-coeff1*:
  hd-coeff (*hd-coeff1 m a*) $\neq$ *0* $\longrightarrow$ *hd-coeff-is1* (*hd-coeff1 m a*)
by (*induct a rule*: *hd-coeff1.induct*) (*simp-all add*:*zsgn-def*)

lemma *I-cooper1-hd-coeffs1*: *Z.normal* $\varphi \Longrightarrow$ *nqfree* φ
      $\Longrightarrow Z.I\ (qe\text{-}cooper_1(hd\text{-}coeffs1\ \varphi))\ xs = (\exists\, x.\ Z.I\ \varphi\ (x\ \#\ xs))$
apply(*simp add*:*Z.normal-def*)
apply(*subst* $I\text{-}qe\text{-}cooper_1$)

apply(*clarsimp simp*:*hd-coeffs1-def image-def set-atoms0-iff divisor-hd-coeff1-neq0*)
apply (*clarsimp simp*:*hd-coeffs1-def qfree-map-fm set-atoms0-iff*
hd-coeff-is1-hd-coeff1)
apply(*simp add*:*hd-coeffs1-def nqfree-map-fm*)
apply(*simp add*: *I-hd-coeffs1*)
done

definition *qe-cooper* = *Z.lift-nnf-qe* (*qe-cooper*$_1$ ∘ *hd-coeffs1*)

lemma *qfree-cooper1-hd-coeffs1*: *qfree* $\varphi \Longrightarrow$ *qfree* (*qe-cooper*$_1$ (*hd-coeffs1* φ))
by(*auto simp*:*qe-cooper*$_1$*-def hd-coeffs1-def qfree-map-fm*
intro!: *qfree-or qfree-and qfree-list-disj qfree-min-inf*)

lemma *normal-min-inf*: *Z.normal* $\varphi \Longrightarrow$ *Z.normal*(*inf*$_-$ φ)
by(*induct* φ *rule*:*min-inf.induct*) *simp-all*

lemma *normal-cooper1*: *Z.normal* $\varphi \Longrightarrow$ *Z.normal*(*qe-cooper*$_1$ φ)
by(*simp add*:*qe-cooper*$_1$*-def Logic.or-def Z.normal-map-fm normal-min-inf split-def*)

lemma *normal-hd-coeffs1*: *qfree* $\varphi \Longrightarrow$ *Z.normal* $\varphi \Longrightarrow$ *Z.normal*(*hd-coeffs1* φ)
by(*auto simp*: *hd-coeffs1-def image-def set-atoms0-iff*
divisor-hd-coeff1-neq0 Z.normal-def)

theorem *I-cooper*: *Z.normal* $\varphi \Longrightarrow$ *Z.I* (*qe-cooper* φ) *xs* = *Z.I* φ *xs*
by(*simp add*:*qe-cooper-def Z.I-lift-nnf-qe-normal qfree-cooper1-hd-coeffs1 I-cooper1-hd-coeffs1*
normal-cooper1 normal-hd-coeffs1)

theorem *qfree-cooper*: *qfree* (*qe-cooper* φ)
by(*simp add*:*qe-cooper-def Z.qfree-lift-nnf-qe qfree-cooper1-hd-coeffs1*)

end

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