The Lévy-Prokhorov Metric

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Abstract

We formalize the Lévy-Prokhorov metric, a metric on finite measures, mainly following the lecture notes by Gaans [4]. This entry includes the following formalization.

- Characterizations of closed sets, open sets, and topology by limit.
- A special case of Alaoglu's theorem.
- Weak convergence and the Portmanteau theorem.
- The Lévy-Prokhorov metric and its completeness and separability.
- The equivalence of the topology of weak convergence and the topology generated by the Lévy-Prokhorov metric.
- Prokhorov's theorem.
- Equality of two σ-algebras on the space of finite measures. One is the Borel algebra of the Lévy-Prokhorov metric and the other is the least σ-algebra that makes $(\lambda\mu.\ \mu(A))$ measurable for all measurable sets A.
- The space of finite measures on a standard Borel space is also a standard Borel space.

Contents

1 Preliminaries

theory *Lemmas-Levy-Prokhorov*
  imports *Standard-Borel-Spaces.StandardBorel*
begin

lemma(**in** *Metric-space*) [*measurable*]:
  shows *mball-sets*: $mball\ x\ e \in sets\ (borel\text{-}of\ mtopology)$
  and *mcball-sets*: $mcball\ x\ e \in sets\ (borel\text{-}of\ mtopology)$
  ⟨*proof*⟩

lemma *Metric-space-eq-MCauchy*:
  assumes $Metric\text{-}space\ M\ d\ \bigwedge x\ y.\ x \in M \Longrightarrow y \in M \Longrightarrow d\ x\ y = d'\ x\ y$
    and $\bigwedge x\ y.\ d'\ x\ y = d'\ y\ x\ \bigwedge x\ y.\ d'\ x\ y \geq 0$
  shows $Metric\text{-}space.MCauchy\ M\ d\ xn \longleftrightarrow Metric\text{-}space.MCauchy\ M\ d'\ xn$
⟨*proof*⟩

lemma *borel-of-compact*: *Hausdorff-space X* $\Longrightarrow$ *compactin X K* $\Longrightarrow$ *K* $\in$ *sets* (*borel-of X*)
⟨*proof*⟩

lemma *prob-algebra-cong*: *sets M* = *sets N* $\Longrightarrow$ *prob-algebra M* = *prob-algebra N*
⟨*proof*⟩

lemma *topology-eq-closedin*: *X* = *Y* $\longleftrightarrow$ ($\forall$ *C*. *closedin X C* $\longleftrightarrow$ *closedin Y C*)
⟨*proof*⟩

Another version of *finite-measure ?M* $\Longrightarrow$ *countable* {*x*. *Sigma-Algebra.measure ?M* {*x*} $\neq$ *0*}

lemma(**in** *finite-measure*) *countable-support-sets*:
assumes *disjoint-family-on Ai D*
shows *countable* {*i*∈*D*. *measure M* (*Ai i*) $\neq$ *0*}
⟨*proof*⟩

1.1 Finite Sum of Measures

definition *sum-measure* :: *'b measure* ⇒ *'a set* ⇒ (*'a* ⇒ *'b measure*) ⇒ *'b measure*
where
sum-measure M I Mi ≡ *measure-of* (*space M*) (*sets M*) (λ*A*. ∑ *i*∈*I*. *emeasure* (*Mi i*) *A*)

lemma *sum-measure-cong*:
assumes *sets M* = *sets M'* ⋀*i*. *i* ∈ *I* $\Longrightarrow$ *N i* = *N' i*
shows *sum-measure M I N* = *sum-measure M' I N'*
⟨*proof*⟩

lemma [*simp*]:
shows *space-sum-measure*: *space* (*sum-measure M I Mi*) = *space M*
and *sets-sum-measure*[*measurable-cong*]: *sets* (*sum-measure M I Mi*) = *sets M*
⟨*proof*⟩

lemma *emeasure-sum-measure*:
assumes [*measurable*]:*A* ∈ *sets M* **and** ⋀*i*. *i* ∈ *I* $\Longrightarrow$ *sets* (*Mi i*) = *sets M*
shows *emeasure* (*sum-measure M I Mi*) *A* = (∑ *i*∈*I*. *Mi i A*)

⟨*proof*⟩

lemma *sum-measure-infinite*: *infinite I* $\Longrightarrow$ *sum-measure M I Mi* = *null-measure M*
⟨*proof*⟩

lemma *nn-integral-sum-measure*:
assumes *f* ∈ *borel-measurable M* **and** [*measurable-cong*]: ⋀*i*. *i* ∈ *I* $\Longrightarrow$ *sets* (*Mi i*) = *sets M*

shows $(\int^+ x.\ f\ x\ \partial sum\text{-}measure\ M\ I\ Mi) = (\sum i{\in}I.\ (\int^+ x.\ f\ x\ \partial(Mi\ i)))$
⟨*proof*⟩

corollary *integrable-sum-measure-iff-ne*:
fixes $f :: 'a \Rightarrow 'b::\{banach,\ second\text{-}countable\text{-}topology\}$
assumes [*measurable-cong*]: $\bigwedge i.\ i \in I \Longrightarrow sets\ (Mi\ i) = sets\ M$ **and** *finite I* **and** $I \neq \{\}$
shows $integrable\ (sum\text{-}measure\ M\ I\ Mi)\ f \longleftrightarrow (\forall i{\in}I.\ integrable\ (Mi\ i)\ f)$
⟨*proof*⟩

corollary *integrable-sum-measure-iff*:
fixes $f :: 'a \Rightarrow 'b::\{banach,\ second\text{-}countable\text{-}topology\}$
assumes [*measurable-cong*]: $\bigwedge i.\ i \in I \Longrightarrow sets\ (Mi\ i) = sets\ M$ **and** *finite I*
and [*measurable*]: $f \in borel\text{-}measurable\ M$
shows $integrable\ (sum\text{-}measure\ M\ I\ Mi)\ f \longleftrightarrow (\forall i{\in}I.\ integrable\ (Mi\ i)\ f)$
⟨*proof*⟩

lemma *integral-sum-measure*:
fixes $f :: 'a \Rightarrow 'b::\{banach,\ second\text{-}countable\text{-}topology\}$
assumes [*measurable-cong*]:$\bigwedge i.\ i \in I \Longrightarrow sets\ (Mi\ i) = sets\ M$ $\bigwedge i.\ i \in I \Longrightarrow integrable\ (Mi\ i)\ f$
shows $(\int x.\ f\ x\ \partial sum\text{-}measure\ M\ I\ Mi) = (\sum i{\in}I.\ (\int x.\ f\ x\ \partial(Mi\ i)))$
⟨*proof*⟩

Lemmas related to scale measure

lemma *integrable-scale-measure*:
fixes $f :: 'a \Rightarrow 'b::\{banach,\ second\text{-}countable\text{-}topology\}$
assumes *integrable M f*
shows $integrable\ (scale\text{-}measure\ (ennreal\ r)\ M)\ f$
⟨*proof*⟩

lemma *integral-scale-measure*:
assumes $r \geq 0$ *integrable M f*
shows $(\int x.\ f\ x\ \partial scale\text{-}measure\ (ennreal\ r)\ M) = r * (\int x.\ f\ x\ \partial M)$
⟨*proof*⟩

lemma
fixes $c :: ereal$
assumes c: $c \neq -\infty$ **and** a: $\bigwedge n.\ 0 \leq a\ n$
shows *liminf-cadd*: $liminf\ (\lambda n.\ c + a\ n) = c + liminf\ a$
and *limsup-cadd*: $limsup\ (\lambda n.\ c + a\ n) = c + limsup\ a$
⟨*proof*⟩

lemma(**in** *Metric-space*) *frontier-measure-zero-balls*:
assumes $sets\ N = sets\ (borel\text{-}of\ mtopology)$ $finite\text{-}measure\ N$ $M \neq \{\}$
and $e > 0$ **and** *separable-space mtopology*
obtains *ai ri* **where**
$(\bigcup i{::}nat.\ mball\ (ai\ i)\ (ri\ i)) = M$ $(\bigcup i{::}nat.\ mcball\ (ai\ i)\ (ri\ i)) = M$
$\bigwedge i.\ ai\ i \in M$ $\bigwedge i.\ ri\ i > 0$ $\bigwedge i.\ ri\ i < e$

$\bigwedge i.$ *measure* N (*mtopology frontier-of* (*mball* (*ai* i) (*ri* i))) $= 0$
$\bigwedge i.$ *measure* N (*mtopology frontier-of* (*mcball* (*ai* i) (*ri* i))) $= 0$
⟨*proof*⟩

lemma *finite-measure-integral-eq-dense*:
assumes *finite*: *finite-measure* N *finite-measure* M
and *sets-N*:*sets* N = *sets* (*borel-of* X) **and** *sets-M*: *sets* M = *sets* (*borel-of* X)
and *dense*:*dense-in* (*mtopology-of* (*cfunspace* X *euclidean-metric*)) F
and *integ-eq*:$\bigwedge f$::- ⇒ *real*. $f \in F \Longrightarrow (\int x.\ f\ x\ \partial N) = (\int x.\ f\ x\ \partial M)$
and f:*continuous-map* X *euclideanreal* f *bounded* (f ‘ *topspace* X)
shows $(\int x.\ f\ x\ \partial N) = (\int x.\ f\ x\ \partial M)$
⟨*proof*⟩

1.2 Sequentially Continuous Maps

definition *seq-continuous-map* :: $'a$ *topology* ⇒ $'b$ *topology* ⇒ ($'a$ ⇒ $'b$) ⇒ *bool* **where**
seq-continuous-map X Y f ≡ (∀ *xn* x. *limitin* X *xn* x *sequentially* ⟶ *limitin* Y ($\lambda n.$ f (*xn* n)) (f x) *sequentially*)

lemma *seq-continuous-map*:
seq-continuous-map X Y f ⟷ (∀ *xn* x. *limitin* X *xn* x *sequentially* ⟶ *limitin* Y ($\lambda n.$ f (*xn* n)) (f x) *sequentially*)
⟨*proof*⟩

lemma *seq-continuous-map-funspace*:
assumes *seq-continuous-map* X Y f
shows $f \in$ *topspace* X → *topspace* Y
⟨*proof*⟩

lemma *seq-continuous-iff-continuous-first-countable*:
assumes *first-countable* X
shows *seq-continuous-map* X Y = *continuous-map* X Y
⟨*proof*⟩

1.3 Sequential Compactness

definition *seq-compactin* :: $'a$ *topology* ⇒ $'a$ *set* ⇒ *bool* **where**
seq-compactin X S
⟷ $S \subseteq$ *topspace* X ∧ (∀ *xn*. (∀ n::*nat*. *xn* $n \in S$) ⟶ (∃ l∈S. ∃ a::*nat* ⇒ *nat*. *strict-mono* a ∧ *limitin* X (*xn* ∘ a) l *sequentially*))

definition *seq-compact-space* X ≡ *seq-compactin* X (*topspace* X)

lemma *seq-compactin-subset-topspace*: *seq-compactin* X S ⟹ $S \subseteq$ *topspace* X
⟨*proof*⟩

lemma *seq-compactin-empty*[*simp*]: *seq-compactin* X {}
⟨*proof*⟩

lemma *seq-compactin-seq-compact*[*simp*]: *seq-compactin euclidean* $S \longleftrightarrow$ *seq-compact* S
⟨*proof*⟩

lemma *image-seq-compactin*:
assumes *seq-compactin* X S *seq-continuous-map* X Y f
shows *seq-compactin* Y $(f$ ‘ $S)$
⟨*proof*⟩

lemma *closed-seq-compactin*:
assumes *seq-compactin* X K $C \subseteq K$ *closedin* X C
shows *seq-compactin* X C
⟨*proof*⟩

corollary *closedin-seq-compact-space*:
seq-compact-space $X \Longrightarrow$ *closedin* X $C \Longrightarrow$ *seq-compactin* X C
⟨*proof*⟩

lemma *seq-compactin-subtopology*: *seq-compactin* (*subtopology* X S) $T \longleftrightarrow$ *seq-compactin* X $T \wedge T \subseteq S$
⟨*proof*⟩

corollary *seq-compact-space-subtopology*: *seq-compactin* X $S \Longrightarrow$ *seq-compact-space* (*subtopology* X S)
⟨*proof*⟩

lemma *seq-compactin-PiED*:
assumes *seq-compactin* (*product-topology* X I) (Pi_E I S)
shows (Pi_E I $S = \{\}$ $\vee$ ($\forall$ $i \in I$. *seq-compactin* (X i) (S i)))
⟨*proof*⟩

lemma *metrizable-seq-compactin-iff-compactin*:
assumes *metrizable-space* X
shows *seq-compactin* X $S \longleftrightarrow$ *compactin* X S
⟨*proof*⟩

corollary *metrizable-seq-compact-space-iff-compact-space*:
shows *metrizable-space* $X \Longrightarrow$ *seq-compact-space* $X \longleftrightarrow$ *compact-space* X
⟨*proof*⟩

1.4 Lemmas for Limsup and Liminf

lemma *real-less-add-ex-less-pair*:
fixes x w v :: *real*
assumes $x < w + v$
shows $\exists\, y\ z.\ x = y + z \wedge y < w \wedge z < v$
⟨*proof*⟩

lemma *ereal-less-add-ex-less-pair*:

fixes $x\ w\ v :: ereal$
assumes $-\infty < w\ -\infty < v\ x < w + v$
shows $\exists\, y\ z.\ x = y + z \wedge y < w \wedge z < v$
$\langle proof \rangle$

lemma *real-add-less*:
fixes $x\ w\ v :: real$
assumes $w + v < x$
shows $\exists\, y\ z.\ x = y + z \wedge w < y \wedge v < z$
$\langle proof \rangle$

lemma *ereal-add-less*:
fixes $x\ w\ v :: ereal$
assumes $w + v < x$
shows $\exists\, y\ z.\ x = y + z \wedge w < y \wedge v < z$
$\langle proof \rangle$

A generalized version of $\neg\ (liminf\ ?u = \infty \wedge liminf\ ?v = -\infty \vee liminf\ ?u = -\infty \wedge liminf\ ?v = \infty) \Longrightarrow liminf\ ?u + liminf\ ?v \leq liminf\ (\lambda n.\ ?u\ n + ?v\ n)$.

lemma *ereal-Liminf-add-mono*:
fixes $u\ v{::}'a \Rightarrow ereal$
assumes $\neg((Liminf\ F\ u = \infty \wedge Liminf\ F\ v = -\infty) \vee (Liminf\ F\ u = -\infty \wedge Liminf\ F\ v = \infty))$
shows $Liminf\ F\ (\lambda n.\ u\ n + v\ n) \geq Liminf\ F\ u + Liminf\ F\ v$
$\langle proof \rangle$

A generalized version of $limsup\ (\lambda n.\ ?u\ n + ?v\ n) \leq limsup\ ?u + limsup\ ?v$.

lemma *ereal-Limsup-add-mono*:
fixes $u\ v{::}'a \Rightarrow ereal$
shows $Limsup\ F\ (\lambda n.\ u\ n + v\ n) \leq Limsup\ F\ u + Limsup\ F\ v$
$\langle proof \rangle$

1.5 A Characterization of Closed Sets by Limit

There is a net which charactrize closed sets in terms of convergence. Since Isabelle/HOL's convergent is defined through filters, we transform the net to a filter. We refer to the lecture notes by Shi [3] for the conversion.

definition *derived-filter* $:: ['i\ set,\ 'i \Rightarrow\ 'i \Rightarrow bool] \Rightarrow\ 'i\ filter$ **where**
$derived\text{-}filter\ I\ op \equiv (\bigsqcap i{\in}I.\ principal\ \{j{\in}I.\ op\ i\ j\})$

lemma *eventually-derived-filter*:
assumes $A \neq \{\}$
and $refl{:}\bigwedge a.\ a \in A \Longrightarrow rel\ a\ a$
and $trans{:}\bigwedge a\ b\ c.\ a \in A \Longrightarrow b \in A \Longrightarrow c \in A \Longrightarrow rel\ a\ b \Longrightarrow rel\ b\ c \Longrightarrow rel\ a\ c$
and $pair\text{-}bounded{:}\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow \exists\, c{\in}A.\ rel\ a\ c \wedge rel\ b\ c$

shows *eventually P* (*derived-filter A rel*) $\longleftrightarrow$ ($\exists\, i{\in}A.\ \forall\, n{\in}A.\ rel\ i\ n \longrightarrow P\ n$)
⟨*proof*⟩

definition *nhdsin-sets* :: *'a topology* ⇒ *'a* ⇒ *'a set filter* **where**
nhdsin-sets X x ≡ *derived-filter* {*U*. *openin X U* ∧ *x* ∈ *U*} (⊇)

lemma *eventually-nhdsin-sets*:
assumes *x* ∈ *topspace X*
shows *eventually P* (*nhdsin-sets X x*) $\longleftrightarrow$ (∃ *U*. *openin X U* ∧ *x* ∈ *U* ∧ (∀ *V*. *openin X V* ⟶ *x* ∈ *V* ⟶ *V* ⊆ *U* ⟶ *P V*))
⟨*proof*⟩

lemma *eventually-nhdsin-setsI*:
assumes *x* ∈ *topspace X* ⋀*U*. *x* ∈ *U* ⟹ *openin X U* ⟹ *P U*
shows *eventually P* (*nhdsin-sets X x*)
⟨*proof*⟩

lemma *nhdsin-sets-bot*[*simp*, *intro*]:
assumes *x* ∈ *topspace X*
shows *nhdsin-sets X x* ≠ ⊥
⟨*proof*⟩

corollary *limitin-nhdsin-sets*: *limitin X xn x* (*nhdsin-sets X x*) $\longleftrightarrow$ *x* ∈ *topspace X* ∧ (∀ *U*. *openin X U* ⟶ *x* ∈ *U* ⟶ (∃ *V*. *openin X V* ∧ *x* ∈ *V* ∧ (∀ *W*. *openin X W* ⟶ *x* ∈ *W* ⟶ *W* ⊆ *V* ⟶ *xn W* ∈ *U*)))
⟨*proof*⟩

lemma *closedin-limitin*:
assumes *T* ⊆ *topspace X* ⋀*xn x*. *x* ∈ *topspace X* ⟹ (⋀*U*. *x* ∈ *U* ⟹ *openin X U* ⟹ *xn U* ≠ *x*) ⟹ (⋀*U*. *x* ∈ *U* ⟹ *openin X U* ⟹ *xn U* ∈ *T*) ⟹ (⋀*U*. *xn U* ∈ *topspace X*) ⟹ *limitin X xn x* (*nhdsin-sets X x*) ⟹ *x* ∈ *T*
shows *closedin X T*
⟨*proof*⟩

corollary *closedin-iff-limitin-eq*:
fixes *X* :: *'a topology*
shows *closedin X C*
$\longleftrightarrow$ *C* ⊆ *topspace X* ∧
(∀ *xi x* (*F* :: *'a set filter*). (∀ *i*. *xi i* ∈ *topspace X*) ⟶ *x* ∈ *topspace X*
⟶ ($\forall_F$ *i in F*. *xi i* ∈ *C*) ⟶ *F* ≠ ⊥ ⟶ *limitin X xi x F* ⟶ *x* ∈ *C*)
⟨*proof*⟩

lemma *closedin-iff-limitin-sequentially*:
assumes *first-countable X*
shows *closedin X S* $\longleftrightarrow$ *S* ⊆ *topspace X* ∧ (∀ σ *l*. *range* σ ⊆ *S* ∧ *limitin X* σ *l sequentially* ⟶ *l* ∈ *S*) (**is** *?lhs=?rhs*)
⟨*proof*⟩

1.6 A Characterization of Topology by Limit

lemma *topology-eq-filter*:
fixes $X :: 'a\ topology$
assumes $topspace\ X = topspace\ Y$
and $\bigwedge(F :: 'a\ set\ filter)\ xi\ x.\ (\bigwedge i.\ xi\ i \in topspace\ X) \Longrightarrow x \in topspace\ X \Longrightarrow limitin\ X\ xi\ x\ F \longleftrightarrow limitin\ Y\ xi\ x\ F$
shows $X = Y$
$\langle proof \rangle$

lemma *topology-eq-limit-sequentially*:
assumes $topspace\ X = topspace\ Y$
and $first\text{-}countable\ X\ first\text{-}countable\ Y$
and $\bigwedge xn\ x.\ (\bigwedge n.\ xn\ i \in topspace\ X) \Longrightarrow x \in topspace\ X \Longrightarrow limitin\ X\ xn\ x\ sequentially \longleftrightarrow limitin\ Y\ xn\ x\ sequentially$
shows $X = Y$
$\langle proof \rangle$

1.7 A Characterization of Open Sets by Limit

corollary *openin-limitin*:
assumes $U \subseteq topspace\ X\ \bigwedge xi\ x.\ x \in topspace\ X \Longrightarrow (\bigwedge i.\ xi\ i \in topspace\ X) \Longrightarrow limitin\ X\ xi\ x\ (nhdsin\text{-}sets\ X\ x) \Longrightarrow x \in U \Longrightarrow \forall_F\ i\ in\ (nhdsin\text{-}sets\ X\ x).\ xi\ i \in U$
shows $openin\ X\ U$
$\langle proof \rangle$

corollary *openin-iff-limitin-eq*:
fixes $X :: 'a\ topology$
shows $openin\ X\ U \longleftrightarrow U \subseteq topspace\ X \wedge (\forall\ xi\ x\ (F :: 'a\ set\ filter).\ (\forall\ i.\ xi\ i \in topspace\ X) \longrightarrow x \in U \longrightarrow limitin\ X\ xi\ x\ F \longrightarrow (\forall_F\ i\ in\ F.\ xi\ i \in U))$
$\langle proof \rangle$

corollary *limitin-openin-sequentially*:
assumes $first\text{-}countable\ X$
shows $openin\ X\ U \longleftrightarrow U \subseteq topspace\ X \wedge (\forall\ xn\ x.\ x \in U \longrightarrow limitin\ X\ xn\ x\ sequentially \longrightarrow (\exists\ N.\ \forall\ n{\geq}N.\ xn\ n \in U))$
$\langle proof \rangle$

lemma *upper-semicontinuous-map-limsup-iff*:
fixes $f :: 'a \Rightarrow ('b :: \{complete\text{-}linorder,linorder\text{-}topology\})$
assumes $first\text{-}countable\ X$
shows $upper\text{-}semicontinuous\text{-}map\ X\ f \longleftrightarrow (\forall\ xn\ x.\ limitin\ X\ xn\ x\ sequentially \longrightarrow limsup\ (\lambda n.\ f\ (xn\ n)) \leq f\ x)$
$\langle proof \rangle$

1.8 Lemmas for Upper/Lower-Semi Continuous Maps

lemma *upper-semicontinuous-map-limsup-real*:
fixes $f :: 'a \Rightarrow real$

assumes *first-countable X*
shows *upper-semicontinuous-map X f* $\longleftrightarrow$ ($\forall$ *xn x. limitin X xn x sequentially* $\longrightarrow$ *limsup* ($\lambda n.$ *f* (*xn n*)) $\leq$ *f x*)
⟨*proof*⟩

lemma *lower-semicontinuous-map-liminf-iff*:
fixes *f* :: *'a* $\Rightarrow$ (*'b* :: {*complete-linorder*,*linorder-topology*})
assumes *first-countable X*
shows *lower-semicontinuous-map X f* $\longleftrightarrow$ ($\forall$ *xn x. limitin X xn x sequentially* $\longrightarrow$ *f x* $\leq$ *liminf* ($\lambda n.$ *f* (*xn n*)))
⟨*proof*⟩

lemma *lower-semicontinuous-map-liminf-real*:
fixes *f* :: *'a* $\Rightarrow$ *real*
assumes *first-countable X*
shows *lower-semicontinuous-map X f* $\longleftrightarrow$ ($\forall$ *xn x. limitin X xn x sequentially* $\longrightarrow$ *f x* $\leq$ *liminf* ($\lambda n.$ *f* (*xn n*)))
⟨*proof*⟩

end

2 Alaoglu's Theorem

theory *Alaoglu-Theorem*
imports *Lemmas-Levy-Prokhorov*
Riesz-Representation.Riesz-Representation
begin

We prove (a special case of) Alaoglu's theorem for the space of continuous functions. We refer to Section 9 of the lecture note by Heil [1].

2.1 Metrizability of the Space of Uniformly Bounded Positive Linear Functionals

lemma *metrizable-functional*:
fixes *X* :: *'a topology* **and** *r* :: *real*
defines *prod-space* $\equiv$ *powertop-real* (*mspace* (*cfunspace X euclidean-metric*))
defines *B* $\equiv$ {φ∈*topspace prod-space*. φ (λx∈*topspace X*. *1*) $\leq$ *r* $\wedge$ *positive-linear-functional-on-CX X* φ}
defines Φ $\equiv$ *subtopology prod-space B*
assumes *compact*: *compact-space X* **and** *met*: *metrizable-space X*
shows *metrizable-space* Φ
⟨*proof*⟩

2.2 Alaoglu's Theorem

According to Alaoglu's theorem, $\{\varphi \in C(X)^* \mid \|\varphi\| \leq r\}$ is compact. We show that $\Phi = \{\varphi \in C(X)^* \mid \|\varphi\| \leq r \wedge \varphi$ is positive$\}$ is compact. Note that

$\|\varphi\| = \varphi(1)$ when $\varphi \in C(X)^*$ is positive.

theorem *Alaoglu-theorem-real-functional*:
fixes X :: *'a topology* **and** r :: *real*
defines *prod-space* $\equiv$ *powertop-real* (*mspace* (*cfunspace* X *euclidean-metric*))
defines $B \equiv \{\varphi \in$*topspace prod-space*. φ ($\lambda x \in$*topspace* X. *1*) $\leq r \wedge$ *positive-linear-functional-on-CX* X $\varphi\}$
assumes *compact*: *compact-space* X **and** *ne*: *topspace* $X \neq \{\}$
shows *compactin prod-space* B
⟨*proof*⟩

theorem *Alaoglu-theorem-real-functional-seq*:
fixes X :: *'a topology* **and** r :: *real*
defines *prod-space* $\equiv$ *powertop-real* (*mspace* (*cfunspace* X *euclidean-metric*))
defines $B \equiv \{\varphi \in$*topspace prod-space*. φ ($\lambda x \in$*topspace* X. *1*) $\leq r \wedge$ *positive-linear-functional-on-CX* X $\varphi\}$
assumes *compact*:*compact-space* X **and** *ne*: *topspace* $X \neq \{\}$ **and** *met*: *metrizable-space* X
shows *seq-compactin prod-space* B
⟨*proof*⟩

end

3 General Weak Convergence

theory *General-Weak-Convergence*
imports *Lemmas-Levy-Prokhorov*
Riesz-Representation.Regular-Measure
begin

We formalize the notion of weak convergence and equivalent conditions. The formalization of weak convergence in HOL-Probability is restricted to probability measures on real numbers. Our formalization is generalized to finite measures on any metric spaces.

3.1 Topology of Weak Convegence

definition *weak-conv-topology* :: *'a topology* $\Rightarrow$ *'a measure topology* **where**
weak-conv-topology $X \equiv$
topology-generated-by
($\bigcup f \in \{f$. *continuous-map* X *euclideanreal* $f \wedge (\exists B. \forall x \in$*topspace* X. $|f\,x| \leq B)\}$.
Collect (*openin* (*pullback-topology* $\{N$. *sets* N = *sets* (*borel-of* X) $\wedge$ *finite-measure* $N\}$
($\lambda N. \int x.\ f\ x\ \partial N$) *euclideanreal*)))

lemma *topspace-weak-conv-topology*[*simp*]:
topspace (*weak-conv-topology* X) = $\{N$. *sets* N = *sets* (*borel-of* X) $\wedge$ *finite-measure* $N\}$
⟨*proof*⟩

lemma *openin-weak-conv-topology-base*:
assumes f:*continuous-map X euclideanreal f* **and** B:$\bigwedge x.\ x \in$ *topspace X* $\Longrightarrow |f\ x| \leq B$
and U:*open U*
shows *openin* (*weak-conv-topology X*) $((\lambda N.\ \int x.\ f\ x\ \partial N)$ −‘ U
$\cap$ {N. *sets N* = *sets* (*borel-of X*) $\wedge$ *finite-measure N*})
⟨*proof*⟩

lemma *continuous-map-weak-conv-topology*:
assumes f:*continuous-map X euclideanreal f* **and** B:$\bigwedge x.\ x \in$ *topspace X* $\Longrightarrow |f\ x| \leq B$
shows *continuous-map* (*weak-conv-topology X*) *euclideanreal* $(\lambda N.\ \int x.\ f\ x\ \partial N)$
⟨*proof*⟩

lemma *weak-conv-topology-minimal*:
assumes *topspace Y* = {N. *sets N* = *sets* (*borel-of X*) $\wedge$ *finite-measure N*}
and $\bigwedge f\ B$. *continuous-map X euclideanreal f*
$\Longrightarrow (\bigwedge x.\ x \in$ *topspace X* $\Longrightarrow |f\ x| \leq B) \Longrightarrow$ *continuous-map Y euclideanreal* $(\lambda N.\ \int x.\ f\ x\ \partial N)$
shows *openin* (*weak-conv-topology X*) $U \Longrightarrow$ *openin Y U*
⟨*proof*⟩

lemma *weak-conv-topology-continuous-map-integral*:
assumes *continuous-map X euclideanreal f* $\bigwedge x.\ x \in$ *topspace X* $\Longrightarrow |f\ x| \leq B$
shows *continuous-map* (*weak-conv-topology X*) *euclideanreal* $(\lambda N.\ \int x.\ f\ x\ \partial N)$
⟨*proof*⟩

3.2 Weak Convergence

abbreviation *weak-conv-on* :: ($'a \Rightarrow 'b$ *measure*) $\Rightarrow 'b$ *measure* $\Rightarrow 'a$ *filter* $\Rightarrow 'b$ *topology* $\Rightarrow$ *bool*
(‹′((-)/ $\Rightarrow_{WC}$ (-)′) (-)/ *on* (-)› [*56*, *55*] *55*) **where**
weak-conv-on Ni N F X $\equiv$ *limitin* (*weak-conv-topology X*) *Ni N F*

abbreviation *weak-conv-on-seq* :: (*nat* $\Rightarrow 'b$ *measure*) $\Rightarrow 'b$ *measure* $\Rightarrow 'b$ *topology* $\Rightarrow$ *bool*
(‹′((-)/ $\Rightarrow_{WC}$ (-)′) *on* (-)› [*56*, *55*] *55*) **where**
weak-conv-on-seq Ni N X $\equiv$ *weak-conv-on Ni N sequentially X*

3.3 Limit in Topology of Weak Convegence

lemma *weak-conv-on-def*:
weak-conv-on Ni N F X $\longleftrightarrow$
$(\forall_F\ i$ *in F*. *sets* (*Ni i*) = *sets* (*borel-of X*) $\wedge$ *finite-measure* (*Ni i*)) $\wedge$ *sets N* = *sets* (*borel-of X*)
$\wedge$ *finite-measure N*
$\wedge$ ($\forall f$. *continuous-map X euclideanreal f* $\longrightarrow$ ($\exists B.\ \forall x{\in}$*topspace X*. $|f\ x| \leq B$)

$\longrightarrow ((\lambda i.\ \int x.\ f\ x\ \partial Ni\ i) \longrightarrow (\int x.\ f\ x\ \partial N))\ F)$
⟨*proof*⟩

lemma *weak-conv-on-def'*:
assumes $\bigwedge i.$ *sets* (*Ni i*) = *sets* (*borel-of X*) **and** $\bigwedge i.$ *finite-measure* (*Ni i*)
and *sets N* = *sets* (*borel-of X*) **and** *finite-measure N*
shows *weak-conv-on Ni N F X*
$\longleftrightarrow$ ($\forall f.$ *continuous-map X euclideanreal f* $\longrightarrow$ ($\exists B.\ \forall x{\in}$*topspace X*. $|f\ x|$ $\le B$)
$\longrightarrow ((\lambda i.\ \int x.\ f\ x\ \partial Ni\ i) \longrightarrow (\int x.\ f\ x\ \partial N))\ F)$
⟨*proof*⟩

lemmas *weak-conv-seq-def* = *weak-conv-on-def*[**where** *F*=*sequentially*]

lemma *weak-conv-on-const*:
($\bigwedge i.$ *Ni i* = *N*) $\Longrightarrow$ *sets N* = *sets* (*borel-of X*)
$\Longrightarrow$ *finite-measure N* $\Longrightarrow$ *weak-conv-on Ni N F X*
⟨*proof*⟩

lemmas *weak-conv-on-seq-const* = *weak-conv-on-const*[**where** *F*=*sequentially*]

context *Metric-space*
begin

abbreviation *mweak-conv* ≡ (λ*Ni N F*. *weak-conv-on Ni N F mtopology*)
abbreviation *mweak-conv-seq* ≡ λ*Ni N*. *mweak-conv Ni N sequentially*

lemmas *mweak-conv-def* = *weak-conv-on-def*[**where** *X*=*mtopology*,*simplified*]
lemmas *mweak-conv-seq-def* = *weak-conv-seq-def*[**where** *X*=*mtopology*,*simplified*]

end

3.4 The Portmanteau Theorem

locale *mweak-conv-fin* = *Metric-space* +
fixes *Ni* :: *'b* ⇒ *'a measure* **and** *N* :: *'a measure* **and** *F*
assumes *sets-Ni*:$\forall_F$ *i in F*. *sets* (*Ni i*) = *sets* (*borel-of mtopology*)
and *sets-N*[*measurable-cong*]: *sets N* = *sets* (*borel-of mtopology*)
and *finite-measure-Ni*: $\forall_F$ *i in F*. *finite-measure* (*Ni i*)
and *finite-measure-N*: *finite-measure N*
begin

interpretation *N*: *finite-measure N*
⟨*proof*⟩

lemma *space-N*: *space N* = *M*
⟨*proof*⟩

lemma *space-Ni*: $\forall_F$ *i in F*. *space* (*Ni i*) = *M*

⟨*proof*⟩

lemma *eventually-Ni*: $\forall_F$ *i in F. space (Ni i) = M* $\wedge$ *sets (Ni i) = sets (borel-of mtopology)* $\wedge$ *finite-measure (Ni i)*
⟨*proof*⟩

lemma *measure-converge-bounded'*:
assumes $((\lambda n.\ \mathit{measure}\ (\mathit{Ni}\ n)\ M) \longrightarrow \mathit{measure}\ N\ M)\ F$
obtains K **where** $\bigwedge A.\ \forall_F\ x\ \mathit{in}\ F.\ \mathit{measure}\ (\mathit{Ni}\ x)\ A \leq K$ $\bigwedge A.\ \mathit{measure}\ N\ A \leq K$
⟨*proof*⟩

lemma
assumes $F \neq \bot$ $\forall_F\ x\ \mathit{in}\ F.\ \mathit{measure}\ (\mathit{Ni}\ x)\ A \leq K$ $\mathit{measure}\ N\ A \leq K$
shows *Liminf-measure-bounded*: $\mathit{Liminf}\ F\ (\lambda i.\ \mathit{measure}\ (\mathit{Ni}\ i)\ A) < \infty$ $0 \leq \mathit{Liminf}\ F\ (\lambda i.\ \mathit{measure}\ (\mathit{Ni}\ i)\ A)$
and *Limsup-measure-bounded*: $\mathit{Limsup}\ F\ (\lambda i.\ \mathit{measure}\ (\mathit{Ni}\ i)\ A) < \infty$ $0 \leq \mathit{Limsup}\ F\ (\lambda i.\ \mathit{measure}\ (\mathit{Ni}\ i)\ A)$
⟨*proof*⟩

lemma *mweak-conv1*:
fixes f:: *'a* $\Rightarrow$ *real*
assumes *mweak-conv Ni N F*
and *uniformly-continuous-map Self euclidean-metric f*
shows $(\exists B.\ \forall x{\in}M.\ |f\ x| \leq B) \Longrightarrow ((\lambda n.\ \mathit{integral}^L\ (\mathit{Ni}\ n)\ f) \longrightarrow \mathit{integral}^L\ N\ f)\ F$
⟨*proof*⟩

lemma *mweak-conv2*:
assumes $\bigwedge f$:: *'a* $\Rightarrow$ *real. uniformly-continuous-map Self euclidean-metric f* $\Longrightarrow$ $(\exists B.\ \forall x{\in}M.\ |f\ x| \leq B)$
$\Longrightarrow ((\lambda n.\ \mathit{integral}^L\ (\mathit{Ni}\ n)\ f) \longrightarrow \mathit{integral}^L\ N\ f)\ F$
and *closedin mtopology A*
shows $\mathit{Limsup}\ F\ (\lambda x.\ \mathit{ereal}\ (\mathit{measure}\ (\mathit{Ni}\ x)\ A)) \leq \mathit{ereal}\ (\mathit{measure}\ N\ A)$
⟨*proof*⟩

lemma *mweak-conv3*:
assumes $\bigwedge A.\ \mathit{closedin}\ \mathit{mtopology}\ A \Longrightarrow \mathit{Limsup}\ F\ (\lambda n.\ \mathit{measure}\ (\mathit{Ni}\ n)\ A) \leq \mathit{measure}\ N\ A$
and $((\lambda n.\ \mathit{measure}\ (\mathit{Ni}\ n)\ M) \longrightarrow \mathit{measure}\ N\ M)\ F$
and *openin mtopology U*
shows $\mathit{measure}\ N\ U \leq \mathit{Liminf}\ F\ (\lambda n.\ \mathit{measure}\ (\mathit{Ni}\ n)\ U)$
⟨*proof*⟩

lemma *mweak-conv3'*:
assumes $\bigwedge U.\ \mathit{openin}\ \mathit{mtopology}\ U \Longrightarrow \mathit{measure}\ N\ U \leq \mathit{Liminf}\ F\ (\lambda n.\ \mathit{measure}\ (\mathit{Ni}\ n)\ U)$
and $((\lambda n.\ \mathit{measure}\ (\mathit{Ni}\ n)\ M) \longrightarrow \mathit{measure}\ N\ M)\ F$
and *closedin mtopology A*

shows *Limsup* F ($\lambda n.$ *measure* (*Ni* n) A) $\leq$ *measure* N A
⟨*proof*⟩

lemma *mweak-conv4*:
assumes $\bigwedge A.$ *closedin mtopology* $A \Longrightarrow$ *Limsup* F ($\lambda n.$ *measure* (*Ni* n) A) $\leq$ *measure* N A
and $\bigwedge U.$ *openin mtopology* $U \Longrightarrow$ *measure* N $U \leq$ *Liminf* F ($\lambda n.$ *measure* (*Ni* n) U)
and [*measurable*]: $A \in$ *sets* (*borel-of mtopology*)
and *measure* N (*mtopology frontier-of* A) $= 0$
shows (($\lambda n.$ *measure* (*Ni* n) A) $\longrightarrow$ *measure* N A) F
⟨*proof*⟩

lemma *mweak-conv5*:
assumes $\bigwedge A.$ $A \in$ *sets* (*borel-of mtopology*) $\Longrightarrow$ *measure* N (*mtopology frontier-of* A) $= 0$
$\Longrightarrow$ (($\lambda n.$ *measure* (*Ni* n) A) $\longrightarrow$ *measure* N A) F
shows *mweak-conv Ni N F*
⟨*proof*⟩

lemma *mweak-conv-eq*: *mweak-conv Ni N F*
$\longleftrightarrow$ ($\forall f$::$'a \Rightarrow$ *real*. *continuous-map mtopology euclidean* $f \longrightarrow$ ($\exists B.$ $\forall x{\in}M.$ $|f\ x|$ $\leq B$)
$\longrightarrow$ (($\lambda n.$ $\int x.\ f\ x\ \partial Ni\ n$) $\longrightarrow$ ($\int x.\ f\ x\ \partial N$)) F)
⟨*proof*⟩

lemma *mweak-conv-eq1*: *mweak-conv Ni N F*
$\longleftrightarrow$ ($\forall f$::$'a \Rightarrow$ *real*. *uniformly-continuous-map Self euclidean-metric* $f \longrightarrow$ ($\exists B.$ $\forall x{\in}M.$ $|f\ x| \leq B$)
$\longrightarrow$ (($\lambda n.$ $\int x.\ f\ x\ \partial Ni\ n$) $\longrightarrow$ ($\int x.\ f\ x\ \partial N$)) F)
⟨*proof*⟩

lemma *mweak-conv-eq2*: *mweak-conv Ni N F*
$\longleftrightarrow$ (($\lambda n.$ *measure* (*Ni* n) M) $\longrightarrow$ *measure* N M) F $\wedge$ ($\forall A.$ *closedin mtopology* A
$\longrightarrow$ *Limsup* F ($\lambda n.$ *measure* (*Ni* n) A) $\leq$ *measure* N A)
⟨*proof*⟩

lemma *mweak-conv-eq3*: *mweak-conv Ni N F*
$\longleftrightarrow$ (($\lambda n.$ *measure* (*Ni* n) M) $\longrightarrow$ *measure* N M) F $\wedge$
($\forall U.$ *openin mtopology* $U \longrightarrow$ *measure* N $U \leq$ *Liminf* F ($\lambda n.$ *measure* (*Ni* n) U))
⟨*proof*⟩

lemma *mweak-conv-eq4*: *mweak-conv Ni N F*
$\longleftrightarrow$ ($\forall A \in$ *sets* (*borel-of mtopology*). *measure* N (*mtopology frontier-of* A) $= 0$
$\longrightarrow$ (($\lambda n.$ *measure* (*Ni* n) A) $\longrightarrow$ *measure* N A) F)
⟨*proof*⟩

corollary *mweak-conv-imp-limit-space*:
assumes *mweak-conv Ni N F*
shows $((\lambda i.\ \mathit{measure}\ (\mathit{Ni}\ i)\ M) \longrightarrow \mathit{measure}\ N\ M)\ F$
$\langle \mathit{proof} \rangle$

end

lemma
assumes *metrizable-space X*
and $\forall_F\ i\ \mathit{in}\ F.\ \mathit{sets}\ (\mathit{Ni}\ i) = \mathit{sets}\ (\mathit{borel\text{-}of}\ X)\ \forall_F\ i\ \mathit{in}\ F.\ \mathit{finite\text{-}measure}\ (\mathit{Ni}\ i)$
and $\mathit{sets}\ N = \mathit{sets}\ (\mathit{borel\text{-}of}\ X)\ \mathit{finite\text{-}measure}\ N$
shows *weak-conv-on-eq1*:
weak-conv-on Ni N F X
$\longleftrightarrow ((\lambda n.\ \mathit{measure}\ (\mathit{Ni}\ n)\ (\mathit{topspace}\ X)) \longrightarrow \mathit{measure}\ N\ (\mathit{topspace}\ X))\ F$
$\wedge (\forall A.\ \mathit{closedin}\ X\ A \longrightarrow \mathit{Limsup}\ F\ (\lambda n.\ \mathit{measure}\ (\mathit{Ni}\ n)\ A) \leq \mathit{measure}\ N$
$A)$ (**is** *?eq1*)
and *weak-conv-on-eq2*:
weak-conv-on Ni N F X
$\longleftrightarrow ((\lambda n.\ \mathit{measure}\ (\mathit{Ni}\ n)\ (\mathit{topspace}\ X)) \longrightarrow \mathit{measure}\ N\ (\mathit{topspace}\ X))\ F$
$\wedge (\forall U.\ \mathit{openin}\ X\ U \longrightarrow \mathit{measure}\ N\ U \leq \mathit{Liminf}\ F\ (\lambda n.\ \mathit{measure}\ (\mathit{Ni}\ n)$
$U))$ (**is** *?eq2*)
and *weak-conv-on-eq3*:
weak-conv-on Ni N F X
$\longleftrightarrow (\forall A \in \mathit{sets}\ (\mathit{borel\text{-}of}\ X).\ \mathit{measure}\ N\ (X\ \mathit{frontier\text{-}of}\ A) = 0$
$\longrightarrow ((\lambda n.\ \mathit{measure}\ (\mathit{Ni}\ n)\ A) \longrightarrow \mathit{measure}\ N\ A)\ F)$ (**is** *?eq3*)
$\langle \mathit{proof} \rangle$

end

4 The Lévy-Prokhorov Metric

theory *Levy-Prokhorov-Distance*
imports *Lemmas-Levy-Prokhorov General-Weak-Convergence*
begin

4.1 The Lévy-Prokhorov Metric

lemma *LPm-ne'*:
assumes *finite-measure M finite-measure N*
shows $\exists e{>}0.\ \forall A\ B\ C\ D.\ \mathit{measure}\ M\ A \leq \mathit{measure}\ N\ (B\ A\ e) + e \wedge \mathit{measure}$
$N\ C \leq \mathit{measure}\ M\ (D\ C\ e) + e$
$\langle \mathit{proof} \rangle$

locale *Levy-Prokhorov* = *Metric-space*
begin

definition $\mathcal{P} \equiv \{N.\ \mathit{sets}\ N = \mathit{sets}\ (\mathit{borel\text{-}of}\ \mathit{mtopology}) \wedge \mathit{finite\text{-}measure}\ N\}$

lemma *inP-D*:

assumes $N \in \mathcal{P}$
shows *finite-measure N sets N* $=$ *sets* (*borel-of mtopology*) *space N* $=$ *M*
⟨*proof*⟩

declare *inP-D*(*2*)[*measurable-cong*]

lemma *inP-I*: *sets N* $=$ *sets* (*borel-of mtopology*) $\Longrightarrow$ *finite-measure N* $\Longrightarrow$ $N \in \mathcal{P}$
⟨*proof*⟩

lemma *inP-iff*: $N \in \mathcal{P} \longleftrightarrow$ *sets N* $=$ *sets* (*borel-of mtopology*) $\wedge$ *finite-measure N*
⟨*proof*⟩

lemma *M-empty-P*:
assumes $M = \{\}$
shows $\mathcal{P} = \{\} \vee \mathcal{P} = \{$*count-space* $\{\}\}$
⟨*proof*⟩

lemma *M-empty-P'*:
assumes $M = \{\}$
shows $\mathcal{P} = \{\} \vee \mathcal{P} = \{$*null-measure* (*borel-of mtopology*)$\}$
⟨*proof*⟩

lemma *inP-mweak-conv-fin-all*:
assumes $\bigwedge i.$ *Ni i* $\in \mathcal{P}$ $N \in \mathcal{P}$
shows *mweak-conv-fin M d Ni N F*
⟨*proof*⟩

lemma *inP-mweak-conv-fin*:
assumes $\forall_F$ *i in F*. *Ni i* $\in \mathcal{P}$ $N \in \mathcal{P}$
shows *mweak-conv-fin M d Ni N F*
⟨*proof*⟩

definition *LPm* :: *'a measure* $\Rightarrow$ *'a measure* $\Rightarrow$ *real* **where**
LPm N L $\equiv$
if $N \in \mathcal{P} \wedge L \in \mathcal{P}$ *then*
($\bigsqcap$ $\{e.\ e > 0 \wedge (\forall A \in$*sets* (*borel-of mtopology*).
measure N A $\leq$ *measure L* ($\bigcup a \in A.$ *mball a e*) $+$ *e* $\wedge$
measure L A $\leq$ *measure N* ($\bigcup a \in A.$ *mball a e*) $+$ *e*)$\}$)
else 0

lemma *bdd-below-Levy-Prokhorov*:
bdd-below $\{e.\ e > 0 \wedge (\forall A \in$*sets* (*borel-of mtopology*).
measure N A $\leq$ *measure L* ($\bigcup a \in A.$ *mball a e*) $+$ *e* $\wedge$
measure L A $\leq$ *measure N* ($\bigcup a \in A.$ *mball a e*) $+$ *e*)$\}$
⟨*proof*⟩

lemma *LPm-ne*:
assumes $N \in \mathcal{P}$ $L \in \mathcal{P}$

shows $\{e.\ e > 0 \wedge (\forall A{\in}sets\ (borel\text{-}of\ mtopology).$
$measure\ N\ A \leq measure\ L\ (\bigcup a{\in}A.\ mball\ a\ e) + e \wedge$
$measure\ L\ A \leq measure\ N\ (\bigcup a{\in}A.\ mball\ a\ e) + e)\}$
$\neq \{\}$
$\langle proof \rangle$

lemma *LPm-imp-le*:
assumes $e > 0$
and $\bigwedge B.\ B \in sets\ (borel\text{-}of\ mtopology) \Longrightarrow measure\ L\ B \leq measure\ N\ (\bigcup a{\in}B.\ mball\ a\ e) + e$
and $\bigwedge B.\ B \in sets\ (borel\text{-}of\ mtopology) \Longrightarrow measure\ N\ B \leq measure\ L\ (\bigcup a{\in}B.\ mball\ a\ e) + e$
shows $LPm\ L\ N \leq e$
$\langle proof \rangle$

lemma *LPm-le-max-measure*: $LPm\ L\ N \leq max\ (measure\ L\ (space\ L))\ (measure\ N\ (space\ N))$
$\langle proof \rangle$

lemma *LPm-less-then*:
assumes $N \in \mathcal{P}$ **and** $L \in \mathcal{P}$
and $LPm\ N\ L < e\ A \in sets\ (borel\text{-}of\ mtopology)$
shows $measure\ N\ A \leq measure\ L\ (\bigcup a{\in}A.\ mball\ a\ e) + e\ measure\ L\ A \leq measure\ N\ (\bigcup a{\in}A.\ mball\ a\ e) + e$
$\langle proof \rangle$

lemma *LPm-nonneg*: $0 \leq LPm\ N\ L$
$\langle proof \rangle$

lemma *LPm-open*: $LPm\ L\ N = (if\ L \in \mathcal{P} \wedge N \in \mathcal{P}\ then$
$(\bigsqcap\ \{e.\ e > 0 \wedge (\forall A{\in}\{U.\ openin\ mtopology\ U\}.$
$measure\ L\ A \leq measure\ N\ (\bigcup a{\in}A.\ mball\ a\ e) + e \wedge$
$measure\ N\ A \leq measure\ L\ (\bigcup a{\in}A.\ mball\ a\ e) + e)\})$
$else\ 0)$
$\langle proof \rangle$

lemma *LPm-closed*: $LPm\ L\ N = (if\ L \in \mathcal{P} \wedge N \in \mathcal{P}\ then$
$(\bigsqcap\ \{e.\ e > 0 \wedge (\forall A{\in}\{U.\ closedin\ mtopology\ U\}.$
$measure\ L\ A \leq measure\ N\ (\bigcup a{\in}A.\ mball\ a\ e) + e \wedge$
$measure\ N\ A \leq measure\ L\ (\bigcup a{\in}A.\ mball\ a\ e) + e)\})$
$else\ 0)$
$\langle proof \rangle$

lemma *LPm-compact*:
assumes *separable-space mtopology mcomplete*

shows *LPm L N* = (*if* $L \in \mathcal{P} \wedge N \in \mathcal{P}$ *then*
($\bigsqcap$ {*e*. $e > 0 \wedge (\forall A \in \{U.$ *compactin mtopology U*}.
measure L A $\leq$ *measure N* ($\bigcup a \in A$. *mball a e*)
+ *e* $\wedge$
measure N A $\leq$ *measure L* ($\bigcup a \in A$. *mball a e*)
+ *e*)})
else 0)
⟨*proof*⟩

sublocale *LPm*: *Metric-space* $\mathcal{P}$ *LPm*
⟨*proof*⟩

4.2 Convervence and Weak Convergence

lemma *converge-imp-mweak-conv*:
assumes *limitin LPm.mtopology Ni N F*
shows *mweak-conv Ni N F*
⟨*proof*⟩

lemma *mweak-conv-imp-converge*:
assumes *separable-space mtopology*
and *mweak-conv Ni N F*
shows *limitin LPm.mtopology Ni N F*
⟨*proof*⟩

corollary *conv-iff-mweak-conv*: *separable-space mtopology* $\Longrightarrow$ *limitin LPm.mtopology Ni N F* $\longleftrightarrow$ *mweak-conv Ni N F*
⟨*proof*⟩

4.3 Separability

lemma *LPm-countable-base*:
assumes *ai*:*mdense* (*range ai*)
shows *LPm.mdense*
((λ(*k*,*bi*). *sum-measure*
(*borel-of mtopology*) {..*k*}
(λ*i*. *scale-measure* (*ennreal* (*bi i*)) (*return* (*borel-of mtopology*)
(*ai i*))))
' (*SIGMA k*:(*UNIV* :: *nat set*). ({..*k*} $\rightarrow_E$ $\mathbb{Q} \cap \{0..\}$))) (**is** *LPm.mdense*
?D)
⟨*proof*⟩

lemma *separable-LPm*:
assumes *separable-space mtopology*
shows *separable-space LPm.mtopology*
⟨*proof*⟩

lemma *closedin-bounded-measures*:
closedin LPm.mtopology {*N*. *sets N* = *sets* (*borel-of mtopology*) $\wedge$ *N* (*space N*)
$\leq$ *ennreal r*}

⟨*proof*⟩

lemma *closedin-subprobs*:
closedin LPm.mtopology $\{N.$ *subprob-space N* $\wedge$ *sets N* $=$ *sets* (*borel-of mtopology*)$\}$
⟨*proof*⟩

lemma *closedin-probs*: *closedin LPm.mtopology* $\{N.$ *prob-space N* $\wedge$ *sets N* $=$ *sets* (*borel-of mtopology*)$\}$
⟨*proof*⟩

4.4 The Lévy-Prokhorov Metric and Topology of Weak Convegence

lemma *weak-conv-topology-le-LPm-topology*:
assumes*openin* (*weak-conv-topology mtopology*) *S*
shows *openin LPm.mtopology S*
⟨*proof*⟩

lemma *LPmtopology-eq-weak-conv-topology*:
assumes *separable-space mtopology*
shows *LPm.mtopology* $=$ *weak-conv-topology mtopology*
⟨*proof*⟩

end

corollary
assumes *metrizable-space X separable-space X*
shows *metrizable-weak-conv-topology*:*metrizable-space* (*weak-conv-topology X*)
and *separable-weak-conv-topology*:*separable-space* (*weak-conv-topology X*)
⟨*proof*⟩

end

5 Prokhorov's Theorem

theory *Prokhorov-Theorem*
imports *Levy-Prokhorov-Distance*
Alaoglu-Theorem
begin

5.1 Prokhorov's Theorem

context *Levy-Prokhorov*
begin

lemma *relatively-compact-imp-tight-LP*:
assumes $\Gamma \subseteq \mathcal{P}$ *separable-space mtopology mcomplete*
and *compactin LPm.mtopology* (*LPm.mtopology closure-of* Γ)

shows *tight-on-set mtopology* Γ
$\langle proof \rangle$

lemma *mcompact-imp-LPmcompact*:
assumes *compact-space mtopology*
shows *compactin LPm.mtopology* $\{N.$ *sets* $N =$ *sets* (*borel-of mtopology*) $\wedge$ N (*space* N) $\leq$ *ennreal* $r\}$
(**is** *compactin* - *?N*)
$\langle proof \rangle$

lemma *tight-imp-relatively-compact-LP*:
assumes $\Gamma \subseteq \{N.$ *sets* $N =$ *sets* (*borel-of mtopology*) $\wedge$ N (*space* N) $\leq$ *ennreal* $r\}$ *separable-space mtopology*
and *tight-on-set mtopology* Γ
shows *compactin LPm.mtopology* (*LPm.mtopology closure-of* Γ)
$\langle proof \rangle$

corollary *Prokhorov-theorem-LP*:
assumes $\Gamma \subseteq \{N.$ *sets* $N =$ *sets* (*borel-of mtopology*) $\wedge$ *emeasure* N (*space* N) $\leq$ *ennreal* $r\}$
and *separable-space mtopology mcomplete*
shows *compactin LPm.mtopology* (*LPm.mtopology closure-of* Γ) $\longleftrightarrow$ *tight-on-set mtopology* Γ
$\langle proof \rangle$

5.2 Completeness of the Lévy-Prokhorov Metric

lemma *mcomplete-tight-on-set*:
assumes $\Gamma \subseteq \mathcal{P}$ *mcomplete*
and $\bigwedge e\, f.\ e > 0 \Longrightarrow f > 0$
$\Longrightarrow \exists\, an\ n.\ an$ ` $\{..n::nat\} \subseteq M \wedge (\forall N{\in}\Gamma.$ *measure* N $(M - (\bigcup i{\leq}n.$ *mball* $(an\ i)\ f)) \leq e)$
shows *tight-on-set mtopology* Γ
$\langle proof \rangle$

lemma *mcomplete-LPmcomplete*:
assumes *mcomplete separable-space mtopology*
shows *LPm.mcomplete*
$\langle proof \rangle$

5.3 Equivalence of Separability, Completeness, and Compactness

lemma *return-inP*[*simp*]:*return* (*borel-of mtopology*) $x \in \mathcal{P}$
$\langle proof \rangle$

lemma *LPm-return-eq*:
assumes $x \in M$ $y \in M$
shows *LPm* (*return* (*borel-of mtopology*) x) (*return* (*borel-of mtopology*) y) $=$

min 1 (*d x y*)
⟨*proof*⟩

corollary *LPm-return-eq-capped-dist*:
assumes $x \in M$ $y \in M$
shows *LPm* (*return* (*borel-of mtopology*) *x*)(*return* (*borel-of mtopology*) *y*) = *capped-dist 1 x y*
⟨*proof*⟩

corollary *MCauchy-iff-MCauchy-return*:
assumes *range xn* $\subseteq M$
shows *MCauchy xn* $\longleftrightarrow$ *LPm.MCauchy* ($\lambda n.$ *return* (*borel-of mtopology*) (*xn n*))
⟨*proof*⟩

lemma *conv-conv-return*:
assumes *limitin mtopology xn x sequentially*
shows *limitin LPm.mtopology* ($\lambda n.$ *return* (*borel-of mtopology*) (*xn n*)) (*return* (*borel-of mtopology*) *x*) *sequentially*
⟨*proof*⟩

lemma *conv-iff-conv-return*:
assumes *range xn* $\subseteq M$ $x \in M$
shows *limitin mtopology xn x sequentially*
$\longleftrightarrow$ *limitin LPm.mtopology* ($\lambda n.$ *return* (*borel-of mtopology*) (*xn n*))
(*return* (*borel-of mtopology*) *x*) *sequentially*
⟨*proof*⟩

lemma *continuous-map-return*: *continuous-map mtopology LPm.mtopology* ($\lambda x.$ *return* (*borel-of mtopology*) *x*)
⟨*proof*⟩

lemma *homeomorphic-map-return*:
homeomorphic-map mtopology
(*subtopology LPm.mtopology* (($\lambda x.$ *return* (*borel-of mtopology*) *x*) ‘ *M*))
($\lambda x.$ *return* (*borel-of mtopology*) *x*)
⟨*proof*⟩

corollary *homeomorphic-space-mtopology-return*:
mtopology homeomorphic-space (*subtopology LPm.mtopology* (($\lambda x.$ *return* (*borel-of mtopology*) *x*) ‘ *M*))
⟨*proof*⟩

lemma *closedin-returnM*: *closedin LPm.mtopology* (($\lambda x.$ *return* (*borel-of mtopology*) *x*) ‘ *M*)
⟨*proof*⟩

corollary *separable-iff-LPm-separable*: *separable-space mtopology* $\longleftrightarrow$ *separable-space LPm.mtopology*

$\langle proof \rangle$

corollary *LPmcomplete-mcomplete*:
assumes *LPm.mcomplete*
shows *mcomplete*
$\langle proof \rangle$

corollary *mcomplete-iff-LPmcomplete*: *separable-space mtopology* $\Longrightarrow$ *mcomplete* $\longleftrightarrow$ *LPm.mcomplete*
$\langle proof \rangle$

lemma *LPmcompact-imp-mcompact*: *compact-space LPm.mtopology* $\Longrightarrow$ *compact-space mtopology*
$\langle proof \rangle$

end

corollary *Polish-space-weak-conv-topology*:
assumes *Polish-space X*
shows *Polish-space* (*weak-conv-topology X*)
$\langle proof \rangle$

5.4 Prokhorov Theorem for Topology of Weak Convergence

lemma *relatively-compact-imp-tight*:
assumes *Polish-space X* $\Gamma \subseteq \{N.$ *sets N* $=$ *sets* (*borel-of X*) $\wedge$ *finite-measure N*$\}$
and *compactin* (*weak-conv-topology X*) (*weak-conv-topology X closure-of* Γ)
shows *tight-on-set X* Γ
$\langle proof \rangle$

lemma *tight-imp-relatively-compact*:
assumes *metrizable-space X separable-space X*
$\Gamma \subseteq \{N.$ *N* (*space N*) $\leq$ *ennreal r* $\wedge$ *sets N* $=$ *sets* (*borel-of X*)$\}$
and *tight-on-set X* Γ
shows *compactin* (*weak-conv-topology X*) (*weak-conv-topology X closure-of* Γ)
$\langle proof \rangle$

lemma *Prokhorov*:
assumes *Polish-space X* $\Gamma \subseteq \{N.$ *N* (*space N*) $\leq$ *ennreal r* $\wedge$ *sets N* $=$ *sets* (*borel-of X*)$\}$
shows *tight-on-set X* $\Gamma \longleftrightarrow$ *compactin* (*weak-conv-topology X*) (*weak-conv-topology X closure-of* Γ)
$\langle proof \rangle$

corollary *tight-on-set-imp-convergent-subsequence*:
fixes *Ni* :: *nat* $\Rightarrow$ *- measure*
assumes *metrizable-space X separable-space X*
and *tight-on-set X* (*range Ni*) $\bigwedge i.$ (*Ni i*) (*space* (*Ni i*)) $\leq$ *ennreal r*

shows $\exists\, a\ N.\ \textit{strict-mono}\ a \wedge \textit{finite-measure}\ N \wedge \textit{sets}\ N = \textit{sets}\ (\textit{borel-of}\ X)$
$\wedge\ N\ (\textit{space}\ N) \leq \textit{ennreal}\ r \wedge \textit{weak-conv-on}\ (Ni \circ a)\ N\ \textit{sequentially}\ X$
⟨*proof*⟩

end

theory *Space-of-Finite-Measures*
imports *Prokhorov-Theorem*
begin

6 Measurable Space of Finite Measures

6.1 Measurable Space of Finite Measures

We define the measurable space of all finite measures in the same way as *subprob-algebra*.

definition *finite-measure-algebra* :: $'a$ *measure* $\Rightarrow$ $'a$ *measure measure* **where**
finite-measure-algebra $K =$
$(\textit{SUP}\ A \in \textit{sets}\ K.\ \textit{vimage-algebra}\ \{M.\ \textit{finite-measure}\ M \wedge \textit{sets}\ M = \textit{sets}\ K\}$ $(\lambda M.\ \textit{emeasure}\ M\ A)\ \textit{borel})$

lemma *space-finite-measure-algebra*:
$\textit{space}\ (\textit{finite-measure-algebra}\ A) = \{M.\ \textit{finite-measure}\ M \wedge \textit{sets}\ M = \textit{sets}\ A\}$
⟨*proof*⟩

lemma *finite-measure-algebra-cong*: $\textit{sets}\ M = \textit{sets}\ N \Longrightarrow \textit{finite-measure-algebra}\ M = \textit{finite-measure-algebra}\ N$
⟨*proof*⟩

lemma *measurable-emeasure-finite-measure-algebra*[*measurable*]:
$a \in \textit{sets}\ A \Longrightarrow (\lambda M.\ \textit{emeasure}\ M\ a) \in \textit{borel-measurable}\ (\textit{finite-measure-algebra}\ A)$
⟨*proof*⟩

lemma *measurable-measure-finite-measure-algebra*[*measurable*]:
$a \in \textit{sets}\ A \Longrightarrow (\lambda M.\ \textit{measure}\ M\ a) \in \textit{borel-measurable}\ (\textit{finite-measure-algebra}\ A)$
⟨*proof*⟩

lemma *finite-measure-measurableD*:
assumes N: $N \in \textit{measurable}\ M\ (\textit{finite-measure-algebra}\ S)$ **and** x: $x \in \textit{space}\ M$
shows $\textit{space}\ (N\ x) = \textit{space}\ S$
and $\textit{sets}\ (N\ x) = \textit{sets}\ S$
and $\textit{measurable}\ (N\ x)\ K = \textit{measurable}\ S\ K$
and $\textit{measurable}\ K\ (N\ x) = \textit{measurable}\ K\ S$
⟨*proof*⟩

⟨*ML*⟩

context
fixes K M N **assumes** K: $K \in$ *measurable* M (*finite-measure-algebra* N)
begin

lemma *finite-measure-space-kernel*: $a \in$ *space* $M \Longrightarrow$ *finite-measure* (K a)
⟨*proof*⟩

lemma *sets-finite-kernel*: $a \in$ *space* $M \Longrightarrow$ *sets* (K a) = *sets* N
⟨*proof*⟩

lemma *measurable-emeasure-finite-kernel*[*measurable*]:
$A \in$ *sets* $N \Longrightarrow$ (λa. *emeasure* (K a) A) $\in$ *borel-measurable* M
⟨*proof*⟩

end

lemma *measurable-finite-measure-algebra*:
($\bigwedge a$. $a \in$ *space* $M \Longrightarrow$ *finite-measure* (K a)) $\Longrightarrow$
($\bigwedge a$. $a \in$ *space* $M \Longrightarrow$ *sets* (K a) = *sets* N) $\Longrightarrow$
($\bigwedge A$. $A \in$ *sets* $N \Longrightarrow$ (λa. *emeasure* (K a) A) $\in$ *borel-measurable* M) $\Longrightarrow$
$K \in$ *measurable* M (*finite-measure-algebra* N)
⟨*proof*⟩

lemma *measurable-finite-markov*:
$K \in$ *measurable* M (*finite-measure-algebra* M) $\longleftrightarrow$
($\forall x \in$*space* M. *finite-measure* (K x) $\wedge$ *sets* (K x) = *sets* M) $\wedge$
($\forall A \in$*sets* M. (λx. *emeasure* (K x) A) $\in$ *measurable* M *borel*)
⟨*proof*⟩

lemma *measurable-finite-measure-algebra-generated*:
assumes *eq*: *sets* N = *sigma-sets* Ω G **and** *Int-stable* G $G \subseteq$ *Pow* Ω
assumes *subsp*: $\bigwedge a$. $a \in$ *space* $M \Longrightarrow$ *finite-measure* (K a)
assumes *sets*: $\bigwedge a$. $a \in$ *space* $M \Longrightarrow$ *sets* (K a) = *sets* N
assumes $\bigwedge A$. $A \in G \Longrightarrow$ (λa. *emeasure* (K a) A) $\in$ *borel-measurable* M
assumes Ω: (λa. *emeasure* (K a) Ω) $\in$ *borel-measurable* M
shows $K \in$ *measurable* M (*finite-measure-algebra* N)
⟨*proof*⟩

lemma *space-finite-measure-algebra-empty*: *space* $N = \{\} \Longrightarrow$ *space* (*finite-measure-algebra* N) = {*null-measure* N}
⟨*proof*⟩

lemma *sets-subprob-algebra-restrict*:
sets (*subprob-algebra* M) = *sets* (*restrict-space* (*finite-measure-algebra* M) {N. *subprob-space* N})
(**is** *sets* *?L* = *sets* *?R*)
⟨*proof*⟩

6.2 Equivalence between Spaces of Finite Measures

Corollary 17.21 [2].

lemma(**in** *Levy-Prokhorov*) *openin-lower-semicontinuous*:
  assumes *openin mtopology U*
  shows *lower-semicontinuous-map LPm.mtopology* (λN. *measure N U*)
  ⟨*proof*⟩

lemma(**in** *Levy-Prokhorov*) *closedin-upper-semicontinuous*:
  assumes *closedin mtopology A*
  shows *upper-semicontinuous-map LPm.mtopology* (λN. *measure N A*)
  ⟨*proof*⟩

context *Levy-Prokhorov*
begin

We show that the measurable space generated from *LPm.mtopology* is equal to *finite-measure-algebra* (*borel-of LPm.mtopology*).

lemma *sets-LPm1*: *sets* (*finite-measure-algebra* (*borel-of mtopology*))
  $\subseteq$ *sets* (*borel-of LPm.mtopology*) (**is** *sets ?Giry* $\subseteq$ *sets ?Levy*)
⟨*proof*⟩

lemma *sets-LPm2*:
  assumes *mcomplete separable-space mtopology*
  shows *sets* (*borel-of LPm.mtopology*) $\subseteq$ *sets* (*finite-measure-algebra* (*borel-of mtopology*))
  (**is** *sets ?Levy* $\subseteq$ *sets ?Giry*)
⟨*proof*⟩

corollary *sets-LPm-eq-sets-finite-measure-algebra*:
  assumes *mcomplete separable-space mtopology*
  shows *sets* (*borel-of LPm.mtopology*) = *sets* (*finite-measure-algebra* (*borel-of mtopology*))
  ⟨*proof*⟩

end

corollary *weak-conv-topology-eq-finite-measure-algebra*:
  assumes *Polish-space X*
  shows *sets* (*borel-of* (*weak-conv-topology X*)) = *sets* (*finite-measure-algebra* (*borel-of X*))
⟨*proof*⟩

corollary *weak-conv-topology-eq-subprob-algebra*:
  assumes *Polish-space X*
  shows *sets* (*borel-of* (*subtopology* (*weak-conv-topology X*) {*N*. *subprob-space N* $\wedge$ *sets N* = *sets* (*borel-of X*)}))
  = *sets* (*subprob-algebra* (*borel-of X*)) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

corollary *weak-conv-topology-eq-prob-algebra*:
assumes *Polish-space* X
shows *sets* (*borel-of* (*subtopology* (*weak-conv-topology* X) $\{N.$ *prob-space* $N \wedge$ *sets* $N =$ *sets* (*borel-of* X)$\}$))
$=$ *sets* (*prob-algebra* (*borel-of* X)) (**is** *?lhs* $=$ *?rhs*)
⟨*proof*⟩

6.3 Standardness

lemma *closedin-weak-conv-topology-r*:
closedin (*weak-conv-topology* X) $\{N.$ *sets* $N =$ *sets* (*borel-of* X) $\wedge$ N (*space* N) $\leq$ *ennreal* $r\}$
⟨*proof*⟩

lemma *closedin-weak-conv-topology-subprob*:
closedin (*weak-conv-topology* X) $\{N.$ *subprob-space* $N \wedge$ *sets* $N =$ *sets* (*borel-of* X)$\}$
⟨*proof*⟩

lemma *closedin-weak-conv-topology-prob*:
closedin (*weak-conv-topology* X) $\{N.$ *prob-space* $N \wedge$ *sets* $N =$ *sets* (*borel-of* X)$\}$
⟨*proof*⟩

corollary
assumes *standard-borel* M
shows *standard-borel-finite-measure-algebra*: *standard-borel* (*finite-measure-algebra* M)
and *standard-borel-ne-finite-measure-algebra*: *standard-borel-ne* (*finite-measure-algebra* M)
and *standard-borel-subprob-algebra*: *standard-borel* (*subprob-algebra* M)
and *standard-borel-prob-algebra*: *standard-borel* (*prob-algebra* M)
⟨*proof*⟩

corollary
assumes *standard-borel-ne* M
shows *standard-borel-ne-subprob-algebra*: *standard-borel-ne* (*subprob-algebra* M)
and *standard-borel-ne-prob-algebra*: *standard-borel-ne* (*prob-algebra* M)
⟨*proof*⟩

end

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