

The Lévy-Prokhorov Metric

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Abstract

We formalize the Lévy-Prokhorov metric, a metric on finite measures, mainly following the lecture notes by Gaans [4]. This entry includes the following formalization.

- Characterizations of closed sets, open sets, and topology by limit.
- A special case of Alaoglu's theorem.
- Weak convergence and the Portmanteau theorem.
- The Lévy-Prokhorov metric and its completeness and separability.
- The equivalence of the topology of weak convergence and the topology generated by the Lévy-Prokhorov metric.
- Prokhorov's theorem.
- Equality of two σ -algebras on the space of finite measures. One is the Borel algebra of the Lévy-Prokhorov metric and the other is the least σ -algebra that makes $(\lambda\mu, \mu(A))$ measurable for all measurable sets A .
- The space of finite measures on a standard Borel space is also a standard Borel space.

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1 Preliminaries

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theory Lemmas-Levy-Prokhorov
  imports Standard-Borel-Spaces.StandardBorel
begin

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lemma(in Metric-space) [measurable]:
  shows mball-sets: mball x e ∈ sets (borel-of mtopology)
  and mcball-sets: mcball x e ∈ sets (borel-of mtopology)
  <proof>

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lemma Metric-space-eq-MCauchy:
  assumes Metric-space M d ∧ x y. x ∈ M ⇒ y ∈ M ⇒ d x y = d' x y
  and ∧ x y. d' x y = d' y x ∧ x y. d' x y ≥ 0
  shows Metric-space.MCauchy M d xn ↔ Metric-space.MCauchy M d' xn
  <proof>

```

lemma *borel-of-compact*: Hausdorff-space $X \implies$ compactin X $K \implies K \in$ sets
(borel-of X)
(proof)

lemma *prob-algebra-cong*: sets $M =$ sets $N \implies$ prob-algebra $M =$ prob-algebra N
(proof)

lemma *topology-eq-closedin*: $X = Y \iff (\forall C. \text{closedin } X C \iff \text{closedin } Y C)$
(proof)

Another version of *finite-measure* ? $M \implies$ countable $\{x. \text{Sigma-Algebra.measure } ?M \{x\} \neq 0\}$

lemma(in *finite-measure*) *countable-support-sets*:
assumes *disjoint-family-on* $A_i D$
shows *countable* $\{i \in D. \text{measure } M (A_i i) \neq 0\}$
(proof)

1.1 Finite Sum of Measures

definition *sum-measure* :: 'b measure \Rightarrow 'a set \Rightarrow ('a \Rightarrow 'b measure) \Rightarrow 'b measure
where
sum-measure $M I M_i \equiv$ *measure-of* (space M) (sets M) $(\lambda A. \sum_{i \in I}. \text{emeasure } (M_i i) A)$

lemma *sum-measure-cong*:
assumes *sets* $M =$ sets $M' \wedge i. i \in I \implies N i = N' i$
shows *sum-measure* $M I N =$ *sum-measure* $M' I N'$
(proof)

lemma [*simp*]:
shows *space-sum-measure*: *space* (*sum-measure* $M I M_i$) = *space* M
and *sets-sum-measure*[*measurable-cong*]: *sets* (*sum-measure* $M I M_i$) = *sets* M
(proof)

lemma *emeasure-sum-measure*:
assumes [*measurable*]: $A \in$ sets M **and** $\wedge i. i \in I \implies$ sets $(M_i i) =$ sets M
shows *emeasure* (*sum-measure* $M I M_i$) $A = (\sum_{i \in I}. M_i i A)$

(proof)

lemma *sum-measure-infinite*: *infinite* $I \implies$ *sum-measure* $M I M_i =$ *null-measure* M
(proof)

lemma *nn-integral-sum-measure*:
assumes $f \in$ *borel-measurable* M **and** [*measurable-cong*]: $\wedge i. i \in I \implies$ sets $(M_i i) =$ sets M

shows $(\int^+ x. f x \partial \text{sum-measure } M I Mi) = (\sum i \in I. (\int^+ x. f x \partial (Mi i)))$
 ⟨proof⟩

corollary *integrable-sum-measure-iff-ne:*

fixes $f :: 'a \Rightarrow 'b :: \{\text{banach, second-countable-topology}\}$
assumes $[\text{measurable-cong}]: \bigwedge i. i \in I \implies \text{sets } (Mi i) = \text{sets } M \text{ and finite } I \text{ and } I \neq \{\}$
shows $\text{integrable } (\text{sum-measure } M I Mi) f \longleftrightarrow (\forall i \in I. \text{integrable } (Mi i) f)$
 ⟨proof⟩

corollary *integrable-sum-measure-iff:*

fixes $f :: 'a \Rightarrow 'b :: \{\text{banach, second-countable-topology}\}$
assumes $[\text{measurable-cong}]: \bigwedge i. i \in I \implies \text{sets } (Mi i) = \text{sets } M \text{ and finite } I$
and $[\text{measurable}]: f \in \text{borel-measurable } M$
shows $\text{integrable } (\text{sum-measure } M I Mi) f \longleftrightarrow (\forall i \in I. \text{integrable } (Mi i) f)$
 ⟨proof⟩

lemma *integral-sum-measure:*

fixes $f :: 'a \Rightarrow 'b :: \{\text{banach, second-countable-topology}\}$
assumes $[\text{measurable-cong}]: \bigwedge i. i \in I \implies \text{sets } (Mi i) = \text{sets } M \bigwedge i. i \in I \implies \text{integrable } (Mi i) f$
shows $(\int x. f x \partial \text{sum-measure } M I Mi) = (\sum i \in I. (\int x. f x \partial (Mi i)))$
 ⟨proof⟩

Lemmas related to scale measure

lemma *integrable-scale-measure:*

fixes $f :: 'a \Rightarrow 'b :: \{\text{banach, second-countable-topology}\}$
assumes $\text{integrable } M f$
shows $\text{integrable } (\text{scale-measure } (\text{ennreal } r) M) f$
 ⟨proof⟩

lemma *integral-scale-measure:*

assumes $r \geq 0 \text{ integrable } M f$
shows $(\int x. f x \partial \text{scale-measure } (\text{ennreal } r) M) = r * (\int x. f x \partial M)$
 ⟨proof⟩

lemma

fixes $c :: \text{ereal}$
assumes $c: c \neq -\infty \text{ and } a: \bigwedge n. 0 \leq a n$
shows $\text{liminf-cadd: } \text{liminf } (\lambda n. c + a n) = c + \text{liminf } a$
and $\text{limsup-cadd: } \text{limsup } (\lambda n. c + a n) = c + \text{limsup } a$
 ⟨proof⟩

lemma(in *Metric-space*) *frontier-measure-zero-balls:*

assumes $\text{sets } N = \text{sets } (\text{borel-of } m\text{topology}) \text{ finite-measure } N M \neq \{\}$
and $e > 0 \text{ and separable-space } m\text{topology}$
obtains $ai \ ri \text{ where}$
 $(\bigcup i::\text{nat. } m\text{ball } (ai i) (ri i)) = M (\bigcup i::\text{nat. } m\text{cball } (ai i) (ri i)) = M$
 $\bigwedge i. ai i \in M \bigwedge i. ri i > 0 \bigwedge i. ri i < e$

$\bigwedge i. \text{measure } N (\text{mtopology frontier-of } (\text{mball } (ai \ i) \ (ri \ i))) = 0$
 $\bigwedge i. \text{measure } N (\text{mtopology frontier-of } (\text{mcball } (ai \ i) \ (ri \ i))) = 0$
 <proof>

lemma *finite-measure-integral-eq-dense*:

assumes *finite*: *finite-measure* N *finite-measure* M
and *sets-N*: *sets* $N = \text{sets } (\text{borel-of } X)$ **and** *sets-M*: *sets* $M = \text{sets } (\text{borel-of } X)$
and *dense*: *dense-in* (*mtopology-of* (*cfunspace* X *euclidean-metric*)) F
and *integ-eq*: $\bigwedge f :: - \Rightarrow \text{real}. f \in F \implies (\int x. f \ x \ \partial N) = (\int x. f \ x \ \partial M)$
and *f*: *continuous-map* X *euclideanreal* *f* *bounded* (*f* ' *topspace* X)
shows $(\int x. f \ x \ \partial N) = (\int x. f \ x \ \partial M)$
 <proof>

1.2 Sequentially Continuous Maps

definition *seq-continuous-map* :: $'a \text{ topology} \Rightarrow 'b \text{ topology} \Rightarrow ('a \Rightarrow 'b) \Rightarrow \text{bool}$
where

seq-continuous-map $X \ Y \ f \equiv (\forall xn \ x. \text{limitin } X \ xn \ x \ \text{sequentially} \longrightarrow \text{limitin } Y$
 $(\lambda n. f \ (xn \ n)) \ (f \ x) \ \text{sequentially})$

lemma *seq-continuous-map*:

seq-continuous-map $X \ Y \ f \longleftrightarrow (\forall xn \ x. \text{limitin } X \ xn \ x \ \text{sequentially} \longrightarrow \text{limitin } Y$
 $(\lambda n. f \ (xn \ n)) \ (f \ x) \ \text{sequentially})$
 <proof>

lemma *seq-continuous-map-funspace*:

assumes *seq-continuous-map* $X \ Y \ f$
shows $f \in \text{topspace } X \rightarrow \text{topspace } Y$
 <proof>

lemma *seq-continuous-iff-continuous-first-countable*:

assumes *first-countable* X
shows *seq-continuous-map* $X \ Y = \text{continuous-map } X \ Y$
 <proof>

1.3 Sequential Compactness

definition *seq-compactin* :: $'a \text{ topology} \Rightarrow 'a \text{ set} \Rightarrow \text{bool}$ **where**

seq-compactin $X \ S$
 $\longleftrightarrow S \subseteq \text{topspace } X \wedge (\forall xn. (\forall n :: \text{nat}. xn \ n \in S) \longrightarrow (\exists l \in S. \exists a :: \text{nat} \Rightarrow \text{nat}.$
 $\text{strict-mono } a \wedge \text{limitin } X \ (xn \ o \ a) \ l \ \text{sequentially}))$

definition *seq-compact-space* $X \equiv \text{seq-compactin } X \ (\text{topspace } X)$

lemma *seq-compactin-subset-topspace*: *seq-compactin* $X \ S \implies S \subseteq \text{topspace } X$

<proof>

lemma *seq-compactin-empty[simp]*: *seq-compactin* $X \ \{\}$

<proof>

lemma *seq-compactin-seq-compact*[simp]: *seq-compactin euclidean S* \longleftrightarrow *seq-compact S*

<proof>

lemma *image-seq-compactin*:

assumes *seq-compactin X S seq-continuous-map X Y f*

shows *seq-compactin Y (f ' S)*

<proof>

lemma *closed-seq-compactin*:

assumes *seq-compactin X K C \subseteq K closedin X C*

shows *seq-compactin X C*

<proof>

corollary *closedin-seq-compact-space*:

seq-compact-space X \implies closedin X C \implies seq-compactin X C

<proof>

lemma *seq-compactin-subtopology*: *seq-compactin (subtopology X S) T* \longleftrightarrow *seq-compactin X T \wedge T \subseteq S*

<proof>

corollary *seq-compact-space-subtopology*: *seq-compactin X S \implies seq-compact-space (subtopology X S)*

<proof>

lemma *seq-compactin-PiED*:

assumes *seq-compactin (product-topology X I) (Pi_E I S)*

shows *(Pi_E I S = {} \vee ($\forall i \in I$. seq-compactin (X i) (S i)))*

<proof>

lemma *metrizable-seq-compactin-iff-compactin*:

assumes *metrizable-space X*

shows *seq-compactin X S \longleftrightarrow compactin X S*

<proof>

corollary *metrizable-seq-compact-space-iff-compact-space*:

shows *metrizable-space X \implies seq-compact-space X \longleftrightarrow compact-space X*

<proof>

1.4 Lemmas for Limsup and Liminf

lemma *real-less-add-ex-less-pair*:

fixes *x w v :: real*

assumes *x < w + v*

shows $\exists y z. x = y + z \wedge y < w \wedge z < v$

<proof>

lemma *ereal-less-add-ex-less-pair*:

fixes $x w v :: ereal$
assumes $-\infty < w - \infty < v x < w + v$
shows $\exists y z. x = y + z \wedge y < w \wedge z < v$
 $\langle proof \rangle$

lemma *real-add-less*:
fixes $x w v :: real$
assumes $w + v < x$
shows $\exists y z. x = y + z \wedge w < y \wedge v < z$
 $\langle proof \rangle$

lemma *ereal-add-less*:
fixes $x w v :: ereal$
assumes $w + v < x$
shows $\exists y z. x = y + z \wedge w < y \wedge v < z$
 $\langle proof \rangle$

A generalized version of $\neg (liminf ?u = \infty \wedge liminf ?v = -\infty \vee liminf ?u = -\infty \wedge liminf ?v = \infty) \implies liminf ?u + liminf ?v \leq liminf (\lambda n. ?u n + ?v n)$.

lemma *ereal-Liminf-add-mono*:
fixes $u v :: 'a \Rightarrow ereal$
assumes $\neg((Liminf F u = \infty \wedge Liminf F v = -\infty) \vee (Liminf F u = -\infty \wedge Liminf F v = \infty))$
shows $Liminf F (\lambda n. u n + v n) \geq Liminf F u + Liminf F v$
 $\langle proof \rangle$

A generalized version of $limsup (\lambda n. ?u n + ?v n) \leq limsup ?u + limsup ?v$.

lemma *ereal-Limsup-add-mono*:
fixes $u v :: 'a \Rightarrow ereal$
shows $Limsup F (\lambda n. u n + v n) \leq Limsup F u + Limsup F v$
 $\langle proof \rangle$

1.5 A Characterization of Closed Sets by Limit

There is a net which characterize closed sets in terms of convergence. Since Isabelle/HOL's convergent is defined through filters, we transform the net to a filter. We refer to the lecture notes by Shi [3] for the conversion.

definition *derived-filter* :: $['i \text{ set}, 'i \Rightarrow 'i \Rightarrow bool] \Rightarrow 'i \text{ filter}$ **where**
 $derived-filter I op \equiv (\prod i \in I. principal \{j \in I. op i j\})$

lemma *eventually-derived-filter*:
assumes $A \neq \{\}$
and *refl*: $\bigwedge a. a \in A \implies rel a a$
and *trans*: $\bigwedge a b c. a \in A \implies b \in A \implies c \in A \implies rel a b \implies rel b c \implies rel a c$
and *pair-bounded*: $\bigwedge a b. a \in A \implies b \in A \implies \exists c \in A. rel a c \wedge rel b c$

shows eventually P (derived-filter A rel) $\longleftrightarrow (\exists i \in A. \forall n \in A. \text{rel } i \ n \longrightarrow P \ n)$
 ⟨proof⟩

definition nhdsin-sets :: 'a topology \Rightarrow 'a \Rightarrow 'a set filter **where**
 nhdsin-sets $X \ x \equiv$ derived-filter $\{U. \text{openin } X \ U \wedge x \in U\}$ (\supseteq)

lemma eventually-nhdsin-sets:

assumes $x \in \text{topspace } X$
shows eventually P (nhdsin-sets $X \ x)$ $\longleftrightarrow (\exists U. \text{openin } X \ U \wedge x \in U \wedge (\forall V. \text{openin } X \ V \longrightarrow x \in V \longrightarrow V \subseteq U \longrightarrow P \ V))$
 ⟨proof⟩

lemma eventually-nhdsin-setsI:

assumes $x \in \text{topspace } X \wedge U. x \in U \Longrightarrow \text{openin } X \ U \Longrightarrow P \ U$
shows eventually P (nhdsin-sets $X \ x)$
 ⟨proof⟩

lemma nhdsin-sets-bot[simp, intro]:

assumes $x \in \text{topspace } X$
shows nhdsin-sets $X \ x \neq \perp$
 ⟨proof⟩

corollary limitin-nhdsin-sets: limitin $X \ x \ n \ x$ (nhdsin-sets $X \ x$) $\longleftrightarrow x \in \text{topspace } X \wedge (\forall U. \text{openin } X \ U \longrightarrow x \in U \longrightarrow (\exists V. \text{openin } X \ V \wedge x \in V \wedge (\forall W. \text{openin } X \ W \longrightarrow x \in W \longrightarrow W \subseteq V \longrightarrow x \ n \ W \in U)))$
 ⟨proof⟩

lemma closedin-limitin:

assumes $T \subseteq \text{topspace } X \wedge x \ n \ x. x \in \text{topspace } X \Longrightarrow (\bigwedge U. x \in U \Longrightarrow \text{openin } X \ U \Longrightarrow x \ n \ U \neq x) \Longrightarrow (\bigwedge U. x \in U \Longrightarrow \text{openin } X \ U \Longrightarrow x \ n \ U \in T) \Longrightarrow (\bigwedge U. x \ n \ U \in \text{topspace } X) \Longrightarrow \text{limitin } X \ x \ n \ x$ (nhdsin-sets $X \ x$) $\Longrightarrow x \in T$
shows closedin $X \ T$
 ⟨proof⟩

corollary closedin-iff-limitin-eq:

fixes $X ::$ 'a topology
shows closedin $X \ C$
 $\longleftrightarrow C \subseteq \text{topspace } X \wedge$
 $(\forall x \ i \ x \ (F ::$ 'a set filter). $(\forall i. x \ i \ i \in \text{topspace } X) \longrightarrow x \in \text{topspace } X$
 $\longrightarrow (\forall_F \ i \ \text{in } F. x \ i \ i \in C) \longrightarrow F \neq \perp \longrightarrow \text{limitin } X \ x \ i \ x \ F \longrightarrow x \in C)$
 ⟨proof⟩

lemma closedin-iff-limitin-sequentially:

assumes first-countable X
shows closedin $X \ S \longleftrightarrow S \subseteq \text{topspace } X \wedge (\forall \sigma \ l. \text{range } \sigma \subseteq S \wedge \text{limitin } X \ \sigma \ l$ sequentially $\longrightarrow l \in S)$ (is ?lhs=?rhs)
 ⟨proof⟩

1.6 A Characterization of Topology by Limit

lemma *topology-eq-filter*:

fixes $X :: 'a \text{ topology}$
assumes $\text{topspace } X = \text{topspace } Y$
and $\bigwedge (F :: 'a \text{ set filter}) \ xi \ x. (\bigwedge i. \ xi \ i \in \text{topspace } X) \implies x \in \text{topspace } X \implies$
 $\text{limitin } X \ xi \ x \ F \longleftrightarrow \text{limitin } Y \ xi \ x \ F$
shows $X = Y$
 $\langle \text{proof} \rangle$

lemma *topology-eq-limit-sequentially*:

assumes $\text{topspace } X = \text{topspace } Y$
and $\text{first-countable } X \ \text{first-countable } Y$
and $\bigwedge xn \ x. (\bigwedge n. \ xn \ i \in \text{topspace } X) \implies x \in \text{topspace } X \implies \text{limitin } X \ xn \ x$
 $\text{sequentially} \longleftrightarrow \text{limitin } Y \ xn \ x \ \text{sequentially}$
shows $X = Y$
 $\langle \text{proof} \rangle$

1.7 A Characterization of Open Sets by Limit

corollary *openin-limitin*:

assumes $U \subseteq \text{topspace } X \ \bigwedge xi \ x. \ x \in \text{topspace } X \implies (\bigwedge i. \ xi \ i \in \text{topspace } X) \implies$
 $\text{limitin } X \ xi \ x \ (\text{nhdsin-sets } X \ x) \implies x \in U \implies \forall_F \ i \ \text{in } (\text{nhdsin-sets } X \ x). \ xi$
 $i \in U$
shows $\text{openin } X \ U$
 $\langle \text{proof} \rangle$

corollary *openin-iff-limitin-eq*:

fixes $X :: 'a \text{ topology}$
shows $\text{openin } X \ U \longleftrightarrow U \subseteq \text{topspace } X \wedge (\forall xi \ x \ (F :: 'a \text{ set filter}). \ (\forall i. \ xi \ i \in \text{topspace } X) \longrightarrow x \in U \longrightarrow \text{limitin } X \ xi \ x \ F \longrightarrow (\forall_F \ i \ \text{in } F. \ xi \ i \in U))$
 $\langle \text{proof} \rangle$

corollary *limitin-openin-sequentially*:

assumes $\text{first-countable } X$
shows $\text{openin } X \ U \longleftrightarrow U \subseteq \text{topspace } X \wedge (\forall xn \ x. \ x \in U \longrightarrow \text{limitin } X \ xn \ x \ \text{sequentially} \longrightarrow (\exists N. \ \forall n \geq N. \ xn \ n \in U))$
 $\langle \text{proof} \rangle$

lemma *upper-semicontinuous-map-limsup-iff*:

fixes $f :: 'a \Rightarrow ('b :: \{\text{complete-linorder, linorder-topology}\})$
assumes $\text{first-countable } X$
shows $\text{upper-semicontinuous-map } X \ f \longleftrightarrow (\forall xn \ x. \ \text{limitin } X \ xn \ x \ \text{sequentially} \longrightarrow \text{limsup } (\lambda n. \ f \ (xn \ n)) \leq f \ x)$
 $\langle \text{proof} \rangle$

1.8 Lemmas for Upper/Lower-Semi Continuous Maps

lemma *upper-semicontinuous-map-limsup-real*:

fixes $f :: 'a \Rightarrow \text{real}$

assumes *first-countable X*
shows *upper-semicontinuous-map X f* $\longleftrightarrow (\forall xn\ x. \text{limitin } X\ xn\ x\ \text{sequentially})$
 $\longrightarrow \text{limsup } (\lambda n. f\ (xn\ n)) \leq f\ x$
 $\langle \text{proof} \rangle$

lemma *lower-semicontinuous-map-liminf-iff:*
fixes $f :: 'a \Rightarrow ('b :: \{ \text{complete-linorder}, \text{linorder-topology} \})$
assumes *first-countable X*
shows *lower-semicontinuous-map X f* $\longleftrightarrow (\forall xn\ x. \text{limitin } X\ xn\ x\ \text{sequentially})$
 $\longrightarrow f\ x \leq \text{liminf } (\lambda n. f\ (xn\ n))$
 $\langle \text{proof} \rangle$

lemma *lower-semicontinuous-map-liminf-real:*
fixes $f :: 'a \Rightarrow \text{real}$
assumes *first-countable X*
shows *lower-semicontinuous-map X f* $\longleftrightarrow (\forall xn\ x. \text{limitin } X\ xn\ x\ \text{sequentially})$
 $\longrightarrow f\ x \leq \text{liminf } (\lambda n. f\ (xn\ n))$
 $\langle \text{proof} \rangle$

end

2 Alaoglu's Theorem

theory *Alaoglu-Theorem*
imports *Lemmas-Levy-Prokhorov*
Riesz-Representation.Riesz-Representation
begin

We prove (a special case of) Alaoglu's theorem for the space of continuous functions. We refer to Section 9 of the lecture note by Heil [1].

2.1 Metrizable of the Space of Uniformly Bounded Positive Linear Functionals

lemma *metrizable-functional:*
fixes $X :: 'a\ \text{topology}$ **and** $r :: \text{real}$
defines $\text{prod-space} \equiv \text{powertop-real } (\text{mspace } (\text{cfunspace } X\ \text{euclidean-metric}))$
defines $B \equiv \{ \varphi \in \text{topspace } \text{prod-space}. \varphi\ (\lambda x \in \text{topspace } X. 1) \leq r \wedge \text{positive-linear-functional-on-} CX\ X\ \varphi \}$
defines $\Phi \equiv \text{subtopology } \text{prod-space } B$
assumes *compact: compact-space X* **and** *met: metrizable-space X*
shows *metrizable-space* Φ
 $\langle \text{proof} \rangle$

2.2 Alaoglu's Theorem

According to Alaoglu's theorem, $\{ \varphi \in C(X)^* \mid \|\varphi\| \leq r \}$ is compact. We show that $\Phi = \{ \varphi \in C(X)^* \mid \|\varphi\| \leq r \wedge \varphi \text{ is positive} \}$ is compact. Note that

$\|\varphi\| = \varphi(1)$ when $\varphi \in C(X)^*$ is positive.

theorem *Alaoglu-theorem-real-functional*:

fixes $X :: 'a \text{ topology}$ **and** $r :: \text{real}$

defines *prod-space* \equiv *powertop-real* (*mSPACE* (*cfunSPACE* X *euclidean-metric*))

defines $B \equiv \{\varphi \in \text{topSPACE } \text{prod-space}. \varphi(\lambda x \in \text{topSPACE } X. 1) \leq r \wedge \text{positive-linear-functional-on-CX } X \varphi\}$

assumes *compact*: *compact-space* X **and** *ne*: *topSPACE* $X \neq \{\}$

shows *compactin prod-space* B

<proof>

theorem *Alaoglu-theorem-real-functional-seq*:

fixes $X :: 'a \text{ topology}$ **and** $r :: \text{real}$

defines *prod-space* \equiv *powertop-real* (*mSPACE* (*cfunSPACE* X *euclidean-metric*))

defines $B \equiv \{\varphi \in \text{topSPACE } \text{prod-space}. \varphi(\lambda x \in \text{topSPACE } X. 1) \leq r \wedge \text{positive-linear-functional-on-CX } X \varphi\}$

assumes *compact*: *compact-space* X **and** *ne*: *topSPACE* $X \neq \{\}$ **and** *met*: *metrizable-space* X

shows *seq-compactin prod-space* B

<proof>

end

3 General Weak Convergence

theory *General-Weak-Convergence*

imports *Lemmas-Levy-Prokhorov*

Riesz-Representation.Regular-Measure

begin

We formalize the notion of weak convergence and equivalent conditions. The formalization of weak convergence in HOL-Probability is restricted to probability measures on real numbers. Our formalization is generalized to finite measures on any metric spaces.

3.1 Topology of Weak Convergence

definition *weak-conv-topology* $:: 'a \text{ topology} \Rightarrow 'a \text{ measure topology}$ **where**

weak-conv-topology $X \equiv$

topology-generated-by

$(\bigcup f \in \{f. \text{continuous-map } X \text{ euclideanreal } f \wedge (\exists B. \forall x \in \text{topSPACE } X. |f x| \leq B)\}.$

Collect (*openin* (*pullback-topology* $\{N. \text{sets } N = \text{sets } (\text{borel-of } X) \wedge \text{finite-measure } N\}$

$(\lambda N. \int x. f x \partial N) \text{ euclideanreal}))$)

lemma *topSPACE-weak-conv-topology[simp]*:

topSPACE (*weak-conv-topology* X) = $\{N. \text{sets } N = \text{sets } (\text{borel-of } X) \wedge \text{finite-measure } N\}$

<proof>

lemma *openin-weak-conv-topology-base*:

assumes f :continuous-map X euclideanreal f **and** B : $\bigwedge x. x \in \text{topspace } X \implies |f x| \leq B$
and U :open U
shows openin (weak-conv-topology X) (($\lambda N. \int x. f x \partial N$) - ' U
 $\cap \{N. \text{sets } N = \text{sets (borel-of } X) \wedge \text{finite-measure } N\}$)
<proof>

lemma *continuous-map-weak-conv-topology*:

assumes f :continuous-map X euclideanreal f **and** B : $\bigwedge x. x \in \text{topspace } X \implies |f x| \leq B$
shows continuous-map (weak-conv-topology X) euclideanreal ($\lambda N. \int x. f x \partial N$)
<proof>

lemma *weak-conv-topology-minimal*:

assumes $\text{topspace } Y = \{N. \text{sets } N = \text{sets (borel-of } X) \wedge \text{finite-measure } N\}$
and $\bigwedge B. \text{continuous-map } X \text{ euclideanreal } f$
 $\implies (\bigwedge x. x \in \text{topspace } X \implies |f x| \leq B) \implies \text{continuous-map } Y$
euclideanreal ($\lambda N. \int x. f x \partial N$)
shows openin (weak-conv-topology X) $U \implies \text{openin } Y U$
<proof>

lemma *weak-conv-topology-continuous-map-integral*:

assumes continuous-map X euclideanreal f $\bigwedge x. x \in \text{topspace } X \implies |f x| \leq B$
shows continuous-map (weak-conv-topology X) euclideanreal ($\lambda N. \int x. f x \partial N$)
<proof>

3.2 Weak Convergence

abbreviation *weak-conv-on* :: ('a \implies 'b measure) \implies 'b measure \implies 'a filter \implies 'b topology \implies bool

('((-)/ \implies_{WC} (-)' (-)/ on (-) [56, 55] 55) **where**
weak-conv-on $Ni N F X \equiv \text{limitin (weak-conv-topology } X) Ni N F$

abbreviation *weak-conv-on-seq* :: (nat \implies 'b measure) \implies 'b measure \implies 'b topology \implies bool

('((-)/ \implies_{WC} (-)' on (-) [56, 55] 55) **where**
weak-conv-on-seq $Ni N X \equiv \text{weak-conv-on } Ni N \text{ sequentially } X$

3.3 Limit in Topology of Weak Convergence

lemma *weak-conv-on-def*:

weak-conv-on $Ni N F X \longleftrightarrow$
 $(\forall F i \text{ in } F. \text{sets (} Ni i) = \text{sets (borel-of } X) \wedge \text{finite-measure (} Ni i)) \wedge \text{sets } N =$
 $\text{sets (borel-of } X)$
 $\wedge \text{finite-measure } N$
 $\wedge (\forall f. \text{continuous-map } X \text{ euclideanreal } f \longrightarrow (\exists B. \forall x \in \text{topspace } X. |f x| \leq$
 $B))$

$\longrightarrow ((\lambda i. \int x. f x \partial N i) \longrightarrow (\int x. f x \partial N)) F$
 <proof>

lemma *weak-conv-on-def'*:

assumes $\bigwedge i. \text{sets } (N i) = \text{sets } (\text{borel-of } X)$ **and** $\bigwedge i. \text{finite-measure } (N i)$
and $\text{sets } N = \text{sets } (\text{borel-of } X)$ **and** $\text{finite-measure } N$

shows *weak-conv-on* $N i N F X$

$\longleftrightarrow (\forall f. \text{continuous-map } X \text{ euclideanreal } f \longrightarrow (\exists B. \forall x \in \text{topspace } X. |f x| \leq B))$

$\longrightarrow ((\lambda i. \int x. f x \partial N i) \longrightarrow (\int x. f x \partial N)) F$

<proof>

lemmas *weak-conv-seq-def* = *weak-conv-on-def* [**where** $F = \text{sequentially}$]

lemma *weak-conv-on-const*:

$(\bigwedge i. N i = N) \implies \text{sets } N = \text{sets } (\text{borel-of } X)$

$\implies \text{finite-measure } N \implies \text{weak-conv-on } N i N F X$

<proof>

lemmas *weak-conv-on-seq-const* = *weak-conv-on-const* [**where** $F = \text{sequentially}$]

context *Metric-space*

begin

abbreviation *mweak-conv* $\equiv (\lambda N i N F. \text{weak-conv-on } N i N F \text{ mtopology})$

abbreviation *mweak-conv-seq* $\equiv \lambda N i N. \text{mweak-conv } N i N \text{ sequentially}$

lemmas *mweak-conv-def* = *weak-conv-on-def* [**where** $X = \text{mtopology, simplified}$]

lemmas *mweak-conv-seq-def* = *weak-conv-seq-def* [**where** $X = \text{mtopology, simplified}$]

end

3.4 The Portmanteau Theorem

locale *mweak-conv-fin* = *Metric-space* +

fixes $N i :: 'b \Rightarrow 'a \text{ measure}$ **and** $N :: 'a \text{ measure}$ **and** F

assumes $\text{sets-}N i: \forall_F i \text{ in } F. \text{sets } (N i) = \text{sets } (\text{borel-of } \text{mtopology})$

and $\text{sets-}N[\text{measurable-cong}]: \text{sets } N = \text{sets } (\text{borel-of } \text{mtopology})$

and $\text{finite-measure-}N i: \forall_F i \text{ in } F. \text{finite-measure } (N i)$

and $\text{finite-measure-}N: \text{finite-measure } N$

begin

interpretation $N: \text{finite-measure } N$

<proof>

lemma *space-N*: $\text{space } N = M$

<proof>

lemma *space-Ni*: $\forall_F i \text{ in } F. \text{space } (N i) = M$

<proof>

lemma *eventually-Ni*: $\forall_F i$ in F . $\text{space } (Ni\ i) = M \wedge \text{sets } (Ni\ i) = \text{sets } (\text{borel-of } m\text{topology}) \wedge \text{finite-measure } (Ni\ i)$

<proof>

lemma *measure-converge-bounded'*:

assumes $((\lambda n. \text{measure } (Ni\ n)\ M) \longrightarrow \text{measure } N\ M)\ F$

obtains K **where** $\bigwedge A. \forall_F x$ in F . $\text{measure } (Ni\ x)\ A \leq K \wedge A. \text{measure } N\ A \leq K$

<proof>

lemma

assumes $F \neq \perp \forall_F x$ in F . $\text{measure } (Ni\ x)\ A \leq K \text{ measure } N\ A \leq K$

shows *Liminf-measure-bounded*: $\text{Liminf } F\ (\lambda i. \text{measure } (Ni\ i)\ A) < \infty\ 0 \leq \text{Liminf } F\ (\lambda i. \text{measure } (Ni\ i)\ A)$

and *Limsup-measure-bounded*: $\text{Limsup } F\ (\lambda i. \text{measure } (Ni\ i)\ A) < \infty\ 0 \leq \text{Limsup } F\ (\lambda i. \text{measure } (Ni\ i)\ A)$

<proof>

lemma *mweak-conv1*:

fixes $f:: 'a \Rightarrow \text{real}$

assumes *mweak-conv* $Ni\ N\ F$

and *uniformly-continuous-map Self euclidean-metric* f

shows $(\exists B. \forall x \in M. |f\ x| \leq B) \implies ((\lambda n. \text{integral}^L\ (Ni\ n)\ f) \longrightarrow \text{integral}^L\ N\ f)\ F$

<proof>

lemma *mweak-conv2*:

assumes $\bigwedge f:: 'a \Rightarrow \text{real}$. *uniformly-continuous-map Self euclidean-metric* $f \implies (\exists B. \forall x \in M. |f\ x| \leq B)$

$\implies ((\lambda n. \text{integral}^L\ (Ni\ n)\ f) \longrightarrow \text{integral}^L\ N\ f)\ F$

and *closedin mtopology* A

shows $\text{Limsup } F\ (\lambda x. \text{ereal } (\text{measure } (Ni\ x)\ A)) \leq \text{ereal } (\text{measure } N\ A)$

<proof>

lemma *mweak-conv3*:

assumes $\bigwedge A$. *closedin mtopology* $A \implies \text{Limsup } F\ (\lambda n. \text{measure } (Ni\ n)\ A) \leq \text{measure } N\ A$

and $((\lambda n. \text{measure } (Ni\ n)\ M) \longrightarrow \text{measure } N\ M)\ F$

and *openin mtopology* U

shows $\text{measure } N\ U \leq \text{Liminf } F\ (\lambda n. \text{measure } (Ni\ n)\ U)$

<proof>

lemma *mweak-conv3'*:

assumes $\bigwedge U$. *openin mtopology* $U \implies \text{measure } N\ U \leq \text{Liminf } F\ (\lambda n. \text{measure } (Ni\ n)\ U)$

and $((\lambda n. \text{measure } (Ni\ n)\ M) \longrightarrow \text{measure } N\ M)\ F$

and *closedin mtopology* A

shows $Limsup F (\lambda n. measure (Ni n) A) \leq measure N A$
 ⟨proof⟩

lemma *mweak-conv4*:

assumes $\bigwedge A. closedin\ mtopology\ A \implies Limsup F (\lambda n. measure (Ni n) A) \leq measure N A$

and $\bigwedge U. openin\ mtopology\ U \implies measure N U \leq Liminf F (\lambda n. measure (Ni n) U)$

and [measurable]: $A \in sets\ (borel-of\ mtopology)$

and $measure N (mtopology\ frontier-of\ A) = 0$

shows $((\lambda n. measure (Ni n) A) \longrightarrow measure N A) F$
 ⟨proof⟩

lemma *mweak-conv5*:

assumes $\bigwedge A. A \in sets\ (borel-of\ mtopology) \implies measure N (mtopology\ frontier-of\ A) = 0$

$\implies ((\lambda n. measure (Ni n) A) \longrightarrow measure N A) F$

shows *mweak-conv Ni N F*
 ⟨proof⟩

lemma *mweak-conv-eq*: *mweak-conv Ni N F*

$\longleftrightarrow (\forall f::'a \Rightarrow real. continuous-map\ mtopology\ euclidean\ f \longrightarrow (\exists B. \forall x \in M. |f x| \leq B))$

$\longrightarrow ((\lambda n. \int x. f x \partial Ni n) \longrightarrow (\int x. f x \partial N)) F$

⟨proof⟩

lemma *mweak-conv-eq1*: *mweak-conv Ni N F*

$\longleftrightarrow (\forall f::'a \Rightarrow real. uniformly-continuous-map\ Self\ euclidean-metric\ f \longrightarrow (\exists B. \forall x \in M. |f x| \leq B))$

$\longrightarrow ((\lambda n. \int x. f x \partial Ni n) \longrightarrow (\int x. f x \partial N)) F$

⟨proof⟩

lemma *mweak-conv-eq2*: *mweak-conv Ni N F*

$\longleftrightarrow ((\lambda n. measure (Ni n) M) \longrightarrow measure N M) F \wedge (\forall A. closedin\ mtopology\ A$

$\longrightarrow Limsup F (\lambda n. measure (Ni n) A) \leq measure N A)$

⟨proof⟩

lemma *mweak-conv-eq3*: *mweak-conv Ni N F*

$\longleftrightarrow ((\lambda n. measure (Ni n) M) \longrightarrow measure N M) F \wedge$

$(\forall U. openin\ mtopology\ U \longrightarrow measure N U \leq Liminf F (\lambda n. measure (Ni n) U))$

⟨proof⟩

lemma *mweak-conv-eq4*: *mweak-conv Ni N F*

$\longleftrightarrow (\forall A \in sets\ (borel-of\ mtopology). measure N (mtopology\ frontier-of\ A) = 0$

$\longrightarrow ((\lambda n. measure (Ni n) A) \longrightarrow measure N A) F)$

⟨proof⟩

corollary *mweak-conv-imp-limit-space:*

assumes *mweak-conv Ni N F*

shows $((\lambda i. \text{measure } (Ni\ i)\ M) \longrightarrow \text{measure } N\ M)\ F$

<proof>

end

lemma

assumes *metrizable-space X*

and $\forall_F i \text{ in } F. \text{sets } (Ni\ i) = \text{sets } (\text{borel-of } X) \forall_F i \text{ in } F. \text{finite-measure } (Ni\ i)$

and $\text{sets } N = \text{sets } (\text{borel-of } X) \text{finite-measure } N$

shows *weak-conv-on-eq1:*

weak-conv-on Ni N F X

$\longleftrightarrow ((\lambda n. \text{measure } (Ni\ n)\ (\text{topspace } X)) \longrightarrow \text{measure } N\ (\text{topspace } X))\ F$

$\wedge (\forall A. \text{closedin } X\ A \longrightarrow \text{Limsup } F\ (\lambda n. \text{measure } (Ni\ n)\ A) \leq \text{measure } N$

$A)$ **(is ?eq1)**

and *weak-conv-on-eq2:*

weak-conv-on Ni N F X

$\longleftrightarrow ((\lambda n. \text{measure } (Ni\ n)\ (\text{topspace } X)) \longrightarrow \text{measure } N\ (\text{topspace } X))\ F$

$\wedge (\forall U. \text{openin } X\ U \longrightarrow \text{measure } N\ U \leq \text{Liminf } F\ (\lambda n. \text{measure } (Ni\ n)$

$U))$ **(is ?eq2)**

and *weak-conv-on-eq3:*

weak-conv-on Ni N F X

$\longleftrightarrow (\forall A \in \text{sets } (\text{borel-of } X). \text{measure } N\ (X\ \text{frontier-of } A) = 0$

$\longrightarrow ((\lambda n. \text{measure } (Ni\ n)\ A) \longrightarrow \text{measure } N\ A)\ F)$ **(is ?eq3)**

<proof>

end

4 The Lévy-Prokhorov Metric

theory *Levy-Prokhorov-Distance*

imports *Lemmas-Levy-Prokhorov General-Weak-Convergence*

begin

4.1 The Lévy-Prokhorov Metric

lemma *LPm-ne'*:

assumes *finite-measure M finite-measure N*

shows $\exists e > 0. \forall A\ B\ C\ D. \text{measure } M\ A \leq \text{measure } N\ (B\ A\ e) + e \wedge \text{measure } N\ C \leq \text{measure } M\ (D\ C\ e) + e$

<proof>

locale *Levy-Prokhorov = Metric-space*

begin

definition $\mathcal{P} \equiv \{N. \text{sets } N = \text{sets } (\text{borel-of } \text{mtopology}) \wedge \text{finite-measure } N\}$

lemma *inP-D:*

assumes $N \in \mathcal{P}$
shows *finite-measure* N sets $N =$ sets (borel-of mtopology) space $N = M$
 ⟨proof⟩

declare *inP-D(2)*[measurable-cong]

lemma *inP-I*: sets $N =$ sets (borel-of mtopology) \implies *finite-measure* $N \implies N \in \mathcal{P}$
 ⟨proof⟩

lemma *inP-iff*: $N \in \mathcal{P} \iff$ sets $N =$ sets (borel-of mtopology) \wedge *finite-measure* N
 ⟨proof⟩

lemma *M-empty-P*:
assumes $M = \{\}$
shows $\mathcal{P} = \{\} \vee \mathcal{P} = \{\text{count-space } \{\}\}$
 ⟨proof⟩

lemma *M-empty-P'*:
assumes $M = \{\}$
shows $\mathcal{P} = \{\} \vee \mathcal{P} = \{\text{null-measure (borel-of mtopology)}\}$
 ⟨proof⟩

lemma *inP-mweak-conv-fin-all*:
assumes $\bigwedge i. Ni \ i \in \mathcal{P} \ N \in \mathcal{P}$
shows *mweak-conv-fin* $M \ d \ Ni \ N \ F$
 ⟨proof⟩

lemma *inP-mweak-conv-fin*:
assumes $\forall_F \ i \ in \ F. Ni \ i \in \mathcal{P} \ N \in \mathcal{P}$
shows *mweak-conv-fin* $M \ d \ Ni \ N \ F$
 ⟨proof⟩

definition *LPm* :: 'a measure \Rightarrow 'a measure \Rightarrow real **where**
 $LPm \ N \ L \equiv$
 if $N \in \mathcal{P} \wedge L \in \mathcal{P}$ then
 $(\bigcap \{e. e > 0 \wedge (\forall A \in \text{sets (borel-of mtopology)}. \text{measure } N \ A \leq \text{measure } L (\bigcup a \in A. \text{mball } a \ e) + e \wedge \text{measure } L \ A \leq \text{measure } N (\bigcup a \in A. \text{mball } a \ e) + e)\})$
 else 0

lemma *bdd-below-Levy-Prokhorov*:
bdd-below $\{e. e > 0 \wedge (\forall A \in \text{sets (borel-of mtopology)}. \text{measure } N \ A \leq \text{measure } L (\bigcup a \in A. \text{mball } a \ e) + e \wedge \text{measure } L \ A \leq \text{measure } N (\bigcup a \in A. \text{mball } a \ e) + e)\}$
 ⟨proof⟩

lemma *LPm-ne*:
assumes $N \in \mathcal{P} \ L \in \mathcal{P}$

shows $\{e. e > 0 \wedge (\forall A \in \text{sets } (\text{borel-of } \text{mtopology}))$
 $\text{measure } N A \leq \text{measure } L (\bigcup a \in A. \text{mball } a e) + e \wedge$
 $\text{measure } L A \leq \text{measure } N (\bigcup a \in A. \text{mball } a e) + e\}$
 $\neq \{\}$
 $\langle \text{proof} \rangle$

lemma *LPm-imp-le*:

assumes $e > 0$
and $\bigwedge B. B \in \text{sets } (\text{borel-of } \text{mtopology}) \implies \text{measure } L B \leq \text{measure } N (\bigcup a \in B.$
 $\text{mball } a e) + e$
and $\bigwedge B. B \in \text{sets } (\text{borel-of } \text{mtopology}) \implies \text{measure } N B \leq \text{measure } L (\bigcup a \in B.$
 $\text{mball } a e) + e$
shows $\text{LPm } L N \leq e$
 $\langle \text{proof} \rangle$

lemma *LPm-le-max-measure*: $\text{LPm } L N \leq \max (\text{measure } L (\text{space } L)) (\text{measure } N (\text{space } N))$
 $\langle \text{proof} \rangle$

lemma *LPm-less-then*:

assumes $N \in \mathcal{P}$ **and** $L \in \mathcal{P}$
and $\text{LPm } N L < e$ $A \in \text{sets } (\text{borel-of } \text{mtopology})$
shows $\text{measure } N A \leq \text{measure } L (\bigcup a \in A. \text{mball } a e) + e$ $\text{measure } L A \leq$
 $\text{measure } N (\bigcup a \in A. \text{mball } a e) + e$
 $\langle \text{proof} \rangle$

lemma *LPm-nonneg:0* $\leq \text{LPm } N L$
 $\langle \text{proof} \rangle$

lemma *LPm-open*: $\text{LPm } L N = (\text{if } L \in \mathcal{P} \wedge N \in \mathcal{P} \text{ then}$
 $(\prod \{e. e > 0 \wedge (\forall A \in \{U. \text{openin } \text{mtopology } U\}.$
 $\text{measure } L A \leq \text{measure } N (\bigcup a \in A. \text{mball}$
 $a e) + e \wedge$
 $\text{measure } N A \leq \text{measure } L (\bigcup a \in A. \text{mball}$
 $a e) + e\})$
 $\text{else } 0)$
 $\langle \text{proof} \rangle$

lemma *LPm-closed*: $\text{LPm } L N = (\text{if } L \in \mathcal{P} \wedge N \in \mathcal{P} \text{ then}$
 $(\prod \{e. e > 0 \wedge (\forall A \in \{U. \text{closedin } \text{mtopology } U\}.$
 $\text{measure } L A \leq \text{measure } N (\bigcup a \in A. \text{mball}$
 $a e) + e \wedge$
 $\text{measure } N A \leq \text{measure } L (\bigcup a \in A. \text{mball}$
 $a e) + e\})$
 $\text{else } 0)$
 $\langle \text{proof} \rangle$

lemma *LPm-compact*:

assumes *separable-space mtopology mcomplete*

shows $LPm\ L\ N = (if\ L \in \mathcal{P} \wedge N \in \mathcal{P}\ then$
 $(\prod \{e. e > 0 \wedge (\forall A \in \{U. compactin\ mtopology\ U}\.$
 $measure\ L\ A \leq measure\ N\ (\bigcup a \in A. mball\ a\ e)$
 $+ e \wedge$
 $+ e)\})$
 $else\ 0)$
 $\langle proof \rangle$

sublocale $LPm: Metric-space\ \mathcal{P}\ LPm$
 $\langle proof \rangle$

4.2 Convergence and Weak Convergence

lemma *converge-imp-mweak-conv*:
assumes $limitin\ LPm.mtopology\ Ni\ N\ F$
shows $mweak-conv\ Ni\ N\ F$
 $\langle proof \rangle$

lemma *mweak-conv-imp-converge*:
assumes $separable-space\ mtopology$
and $mweak-conv\ Ni\ N\ F$
shows $limitin\ LPm.mtopology\ Ni\ N\ F$
 $\langle proof \rangle$

corollary *conv-iff-mweak-conv: separable-space mtopology \implies limitin LPm.mtopology*
 $Ni\ N\ F \iff mweak-conv\ Ni\ N\ F$
 $\langle proof \rangle$

4.3 Separability

lemma *LPm-countable-base*:
assumes $ai:mdense\ (range\ ai)$
shows $LPm.mdense$
 $((\lambda(k,bi). sum-measure$
 $(borel-of\ mtopology)\ \{..k\}$
 $(\lambda i. scale-measure\ (ennreal\ (bi\ i))\ (return\ (borel-of\ mtopology)$
 $(ai\ i))))$
 $\langle (SIGMA\ k:(UNIV\ ::\ nat\ set). (\{..k\} \rightarrow_E\ \mathbb{Q} \cap \{0..\})) \rangle$ **(is** $LPm.mdense$
 $?D)$
 $\langle proof \rangle$

lemma *separable-LPm*:
assumes $separable-space\ mtopology$
shows $separable-space\ LPm.mtopology$
 $\langle proof \rangle$

lemma *closedin-bounded-measures*:
 $closedin\ LPm.mtopology\ \{N. sets\ N = sets\ (borel-of\ mtopology) \wedge N\ (space\ N)$
 $\leq\ ennreal\ r\}$

<proof>

lemma *closedin-subprobs:*

closedin LPm.mtopology {N. subprob-space N \wedge sets N = sets (borel-of mtopology)}

<proof>

lemma *closedin-probs: closedin LPm.mtopology {N. prob-space N \wedge sets N = sets (borel-of mtopology)}*

<proof>

4.4 The Lévy-Prokhorov Metric and Topology of Weak Convergence

lemma *weak-conv-topology-le-LPm-topology:*

assumes *openin (weak-conv-topology mtopology) S*

shows *openin LPm.mtopology S*

<proof>

lemma *LPm-topology-eq-weak-conv-topology:*

assumes *separable-space mtopology*

shows *LPm.mtopology = weak-conv-topology mtopology*

<proof>

end

corollary

assumes *metrizable-space X separable-space X*

shows *metrizable-weak-conv-topology:metrizable-space (weak-conv-topology X)*

and *separable-weak-conv-topology:separable-space (weak-conv-topology X)*

<proof>

end

5 Prokhorov's Theorem

theory *Prokhorov-Theorem*

imports *Levy-Prokhorov-Distance*

Alaoglu-Theorem

begin

5.1 Prokhorov's Theorem

context *Levy-Prokhorov*

begin

lemma *relatively-compact-imp-tight-LP:*

assumes $\Gamma \subseteq \mathcal{P}$ *separable-space mtopology mcomplete*

and *compactin LPm.mtopology (LPm.mtopology closure-of Γ)*

shows *tight-on-set mtopology* Γ
 ⟨proof⟩

lemma *mcompact-imp-LPmcompact:*

assumes *compact-space mtopology*
shows *compactin LPm.mtopology* $\{N. \text{sets } N = \text{sets (borel-of mtopology)} \wedge N$
*(space } $N) \leq \text{ennreal } r\}$
(is compactin - ?N)
 ⟨proof⟩*

lemma *tight-imp-relatively-compact-LP:*

assumes $\Gamma \subseteq \{N. \text{sets } N = \text{sets (borel-of mtopology)} \wedge N$ *(space } $N) \leq \text{ennreal } r\}$
separable-space mtopology
and *tight-on-set mtopology* Γ
shows *compactin LPm.mtopology* *(LPm.mtopology closure-of* $\Gamma)$
 ⟨proof⟩*

corollary *Prokhorov-theorem-LP:*

assumes $\Gamma \subseteq \{N. \text{sets } N = \text{sets (borel-of mtopology)} \wedge \text{emeasure } N$ *(space } $N) \leq \text{ennreal } r\}$
and *separable-space mtopology mcomplete*
shows *compactin LPm.mtopology* *(LPm.mtopology closure-of* $\Gamma) \longleftrightarrow$ *tight-on-set*
mtopology Γ
 ⟨proof⟩*

5.2 Completeness of the Lévy-Prokhorov Metric

lemma *mcomplete-tight-on-set:*

assumes $\Gamma \subseteq \mathcal{P}$ *mcomplete*
and $\bigwedge e f. e > 0 \implies f > 0$
 $\implies \exists \text{an } n. \text{an } \{..n::\text{nat}\} \subseteq M \wedge (\forall N \in \Gamma. \text{measure } N (M - (\bigcup_{i \leq n.}$
*mball (an } $i) f)) \leq e)$
shows *tight-on-set mtopology* Γ
 ⟨proof⟩*

lemma *mcomplete-LPmcomplete:*

assumes *mcomplete separable-space mtopology*
shows *LPm.mcomplete*
 ⟨proof⟩

5.3 Equivalence of Separability, Completeness, and Compactness

lemma *return-inP[simp]:return (borel-of mtopology) x* $\in \mathcal{P}$
 ⟨proof⟩

lemma *LPm-return-eq:*

assumes $x \in M$ $y \in M$
shows *LPm (return (borel-of mtopology) x) (return (borel-of mtopology) y) =*

min 1 (d x y)
<proof>

corollary *LPm-return-eq-capped-dist:*

assumes $x \in M \ y \in M$
shows $LPm \ (return \ (borel-of \ mtopology) \ x)(return \ (borel-of \ mtopology) \ y) =$
capped-dist 1 x y
<proof>

corollary *MCauchy-iff-MCauchy-return:*

assumes $range \ xn \subseteq M$
shows $MCauchy \ xn \longleftrightarrow LPm.MCauchy \ (\lambda n. \ return \ (borel-of \ mtopology) \ (xn \ n))$
<proof>

lemma *conv-conv-return:*

assumes *limitin mtopology xn x sequentially*
shows *limitin LPm.mtopology (λn. return (borel-of mtopology) (xn n)) (return*
(borel-of mtopology) x) sequentially
<proof>

lemma *conv-iff-conv-return:*

assumes $range \ xn \subseteq M \ x \in M$
shows *limitin mtopology xn x sequentially*
 $\longleftrightarrow \ limitin \ LPm.mtopology \ (\lambda n. \ return \ (borel-of \ mtopology) \ (xn \ n))$
 $(return \ (borel-of \ mtopology) \ x) \ sequentially$
<proof>

lemma *continuous-map-return: continuous-map mtopology LPm.mtopology (λx.*
return (borel-of mtopology) x)
<proof>

lemma *homeomorphic-map-return:*

homeomorphic-map mtopology
 $(subtopology \ LPm.mtopology \ ((\lambda x. \ return \ (borel-of \ mtopology) \ x) \ 'M))$
 $(\lambda x. \ return \ (borel-of \ mtopology) \ x)$
<proof>

corollary *homeomorphic-space-mtopology-return:*

mtopology homeomorphic-space (subtopology LPm.mtopology ((λx. return (borel-of
mtopology) x) ' M))
<proof>

lemma *closedin-returnM: closedin LPm.mtopology ((λx. return (borel-of mtopol-*
ogy) x) ' M)
<proof>

corollary *separable-iff-LPm-separable: separable-space mtopology \longleftrightarrow separable-space*
LPm.mtopology

<proof>

corollary *LPmcomplete-mcomplete:*

assumes *LPm.mcomplete*

shows *mcomplete*

<proof>

corollary *mcomplete-iff-LPmcomplete: separable-space mtopology \implies mcomplete*

\longleftarrow LPm.mcomplete

<proof>

lemma *LPmcompact-imp-mcompact: compact-space LPm.mtopology \implies compact-space mtopology*

<proof>

end

corollary *Polish-space-weak-conv-topology:*

assumes *Polish-space X*

shows *Polish-space (weak-conv-topology X)*

<proof>

5.4 Prokhorov Theorem for Topology of Weak Convergence

lemma *relatively-compact-imp-tight:*

assumes *Polish-space X $\Gamma \subseteq \{N. \text{sets } N = \text{sets (borel-of } X) \wedge \text{finite-measure } N\}$*

and *compactin (weak-conv-topology X) (weak-conv-topology X closure-of Γ)*

shows *tight-on-set X Γ*

<proof>

lemma *tight-imp-relatively-compact:*

assumes *metrizable-space X separable-space X*

$\Gamma \subseteq \{N. N (\text{space } N) \leq \text{ennreal } r \wedge \text{sets } N = \text{sets (borel-of } X)\}$

and *tight-on-set X Γ*

shows *compactin (weak-conv-topology X) (weak-conv-topology X closure-of Γ)*

<proof>

lemma *Prokhorov:*

assumes *Polish-space X $\Gamma \subseteq \{N. N (\text{space } N) \leq \text{ennreal } r \wedge \text{sets } N = \text{sets (borel-of } X)\}$*

shows *tight-on-set X $\Gamma \longleftrightarrow$ compactin (weak-conv-topology X) (weak-conv-topology X closure-of Γ)*

<proof>

corollary *tight-on-set-imp-convergent-subsequence:*

fixes *Ni :: nat \Rightarrow - measure*

assumes *metrizable-space X separable-space X*

and *tight-on-set X (range Ni) $\bigwedge i. (Ni i) (\text{space } (Ni i)) \leq \text{ennreal } r$*

shows $\exists a N. \text{strict-mono } a \wedge \text{finite-measure } N \wedge \text{sets } N = \text{sets (borel-of } X)$
 $\wedge N (\text{space } N) \leq \text{ennreal } r \wedge \text{weak-conv-on } (Ni \circ a) N \text{ sequentially } X$
 ⟨proof⟩

end

theory *Space-of-Finite-Measures*
imports *Prokhorov-Theorem*
begin

6 Measurable Space of Finite Measures

6.1 Measurable Space of Finite Measures

We define the measurable space of all finite measures in the same way as *subprob-algebra*.

definition *finite-measure-algebra* :: 'a measure \Rightarrow 'a measure measure **where**
finite-measure-algebra $K =$
 $(\text{SUP } A \in \text{sets } K. \text{vimage-algebra } \{M. \text{finite-measure } M \wedge \text{sets } M = \text{sets } K\}$
 $(\lambda M. \text{emeasure } M A) \text{ borel})$

lemma *space-finite-measure-algebra*:
 $\text{space (finite-measure-algebra } A) = \{M. \text{finite-measure } M \wedge \text{sets } M = \text{sets } A\}$
 ⟨proof⟩

lemma *finite-measure-algebra-cong*: $\text{sets } M = \text{sets } N \Longrightarrow \text{finite-measure-algebra } M = \text{finite-measure-algebra } N$
 ⟨proof⟩

lemma *measurable-emeasure-finite-measure-algebra[measurable]*:
 $a \in \text{sets } A \Longrightarrow (\lambda M. \text{emeasure } M a) \in \text{borel-measurable (finite-measure-algebra } A)$
 ⟨proof⟩

lemma *measurable-measure-finite-measure-algebra[measurable]*:
 $a \in \text{sets } A \Longrightarrow (\lambda M. \text{measure } M a) \in \text{borel-measurable (finite-measure-algebra } A)$
 ⟨proof⟩

lemma *finite-measure-measurableD*:
assumes $N: N \in \text{measurable } M (\text{finite-measure-algebra } S)$ **and** $x: x \in \text{space } M$
shows $\text{space } (N x) = \text{space } S$
and $\text{sets } (N x) = \text{sets } S$
and $\text{measurable } (N x) K = \text{measurable } S K$
and $\text{measurable } K (N x) = \text{measurable } K S$
 ⟨proof⟩

⟨ML⟩

context

fixes $K M N$ **assumes** $K: K \in \text{measurable } M \text{ (finite-measure-algebra } N)$
begin

lemma *finite-measure-space-kernel*: $a \in \text{space } M \implies \text{finite-measure } (K a)$
<proof>

lemma *sets-finite-kernel*: $a \in \text{space } M \implies \text{sets } (K a) = \text{sets } N$
<proof>

lemma *measurable-emeasure-finite-kernel*[*measurable*]:
 $A \in \text{sets } N \implies (\lambda a. \text{emeasure } (K a) A) \in \text{borel-measurable } M$
<proof>

end

lemma *measurable-finite-measure-algebra*:
 $(\bigwedge a. a \in \text{space } M \implies \text{finite-measure } (K a)) \implies$
 $(\bigwedge a. a \in \text{space } M \implies \text{sets } (K a) = \text{sets } N) \implies$
 $(\bigwedge A. A \in \text{sets } N \implies (\lambda a. \text{emeasure } (K a) A) \in \text{borel-measurable } M) \implies$
 $K \in \text{measurable } M \text{ (finite-measure-algebra } N)$
<proof>

lemma *measurable-finite-markov*:
 $K \in \text{measurable } M \text{ (finite-measure-algebra } M) \longleftrightarrow$
 $(\forall x \in \text{space } M. \text{finite-measure } (K x) \wedge \text{sets } (K x) = \text{sets } M) \wedge$
 $(\forall A \in \text{sets } M. (\lambda x. \text{emeasure } (K x) A) \in \text{measurable } M \text{ borel})$
<proof>

lemma *measurable-finite-measure-algebra-generated*:
assumes *eq*: $\text{sets } N = \text{sigma-sets } \Omega G$ **and** *Int-stable* $G G \subseteq \text{Pow } \Omega$
assumes *subsp*: $\bigwedge a. a \in \text{space } M \implies \text{finite-measure } (K a)$
assumes *sets*: $\bigwedge a. a \in \text{space } M \implies \text{sets } (K a) = \text{sets } N$
assumes $\bigwedge A. A \in G \implies (\lambda a. \text{emeasure } (K a) A) \in \text{borel-measurable } M$
assumes $\Omega: (\lambda a. \text{emeasure } (K a) \Omega) \in \text{borel-measurable } M$
shows $K \in \text{measurable } M \text{ (finite-measure-algebra } N)$
<proof>

lemma *space-finite-measure-algebra-empty*: $\text{space } N = \{\} \implies \text{space (finite-measure-algebra } N) = \{\text{null-measure } N\}$
<proof>

lemma *sets-subprob-algebra-restrict*:
 $\text{sets (subprob-algebra } M) = \text{sets (restrict-space (finite-measure-algebra } M) \{N. \text{subprob-space } N\})$
(is $\text{sets } ?L = \text{sets } ?R)$
<proof>

6.2 Equivalence between Spaces of Finite Measures

Corollary 17.21 [2].

lemma(in *Levy-Prokhorov*) *openin-lower-semicontinuous*:
assumes *openin mtopology U*
shows *lower-semicontinuous-map LPm.mtopology (λN . measure N U)*
<proof>

lemma(in *Levy-Prokhorov*) *closedin-upper-semicontinuous*:
assumes *closedin mtopology A*
shows *upper-semicontinuous-map LPm.mtopology (λN . measure N A)*
<proof>

context *Levy-Prokhorov*
begin

We show that the measurable space generated from *LPm.mtopology* is equal to *finite-measure-algebra (borel-of LPm.mtopology)*.

lemma *sets-LPm1*: *sets (finite-measure-algebra (borel-of mtopology))*
 \subseteq *sets (borel-of LPm.mtopology) (is sets ?Giry \subseteq sets ?Levy)*
<proof>

lemma *sets-LPm2*:
assumes *mcomplete separable-space mtopology*
shows *sets (borel-of LPm.mtopology) \subseteq sets (finite-measure-algebra (borel-of mtopology))*
(is sets ?Levy \subseteq sets ?Giry)
<proof>

corollary *sets-LPm-eq-sets-finite-measure-algebra*:
assumes *mcomplete separable-space mtopology*
shows *sets (borel-of LPm.mtopology) = sets (finite-measure-algebra (borel-of mtopology))*
<proof>

end

corollary *weak-conv-topology-eq-finite-measure-algebra*:
assumes *Polish-space X*
shows *sets (borel-of (weak-conv-topology X)) = sets (finite-measure-algebra (borel-of X))*
<proof>

corollary *weak-conv-topology-eq-subprob-algebra*:
assumes *Polish-space X*
shows *sets (borel-of (subtopology (weak-conv-topology X) {N. subprob-space N \wedge sets N = sets (borel-of X)}))*
 $=$ *sets (subprob-algebra (borel-of X)) (is ?lhs = ?rhs)*
<proof>

corollary *weak-conv-topology-eq-prob-algebra:*

assumes *Polish-space X*

shows *sets (borel-of (subtopology (weak-conv-topology X) {N. prob-space N ∧ sets N = sets (borel-of X)}))*

= sets (prob-algebra (borel-of X)) (is ?lhs = ?rhs)

<proof>

6.3 Standardness

lemma *closedin-weak-conv-topology-r:*

closedin (weak-conv-topology X) {N. sets N = sets (borel-of X) ∧ N (space N) ≤ ennreal r}

<proof>

lemma *closedin-weak-conv-topology-subprob:*

closedin (weak-conv-topology X) {N. subprob-space N ∧ sets N = sets (borel-of X)}

<proof>

lemma *closedin-weak-conv-topology-prob:*

closedin (weak-conv-topology X) {N. prob-space N ∧ sets N = sets (borel-of X)}

<proof>

corollary

assumes *standard-borel M*

shows *standard-borel-finite-measure-algebra: standard-borel (finite-measure-algebra M)*

and *standard-borel-ne-finite-measure-algebra: standard-borel-ne (finite-measure-algebra M)*

and *standard-borel-subprob-algebra: standard-borel (subprob-algebra M)*

and *standard-borel-prob-algebra: standard-borel (prob-algebra M)*

<proof>

corollary

assumes *standard-borel-ne M*

shows *standard-borel-ne-subprob-algebra: standard-borel-ne (subprob-algebra M)*

and *standard-borel-ne-prob-algebra: standard-borel-ne (prob-algebra M)*

<proof>

end

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