# Lebesgue-Stieltjes Integral

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**Abstract**

This entry formalizes some basic facts and lemmas relating to the integration with respect to the Lebesgue-Stieltjes measure (interval measure). It includes the well-known formula to calculate the Lebesgue-Stieltjes integral:

$$\int g(x)\, dF(x) = \int g(x) F'(x)\, dx.$$


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**theory** *Preliminaries-LSI*
  **imports** *HOL−Library.Rewrite HOL−Analysis.Analysis*
**begin**

**context** *order-topology*
**begin**

**lemma**
  **assumes** $a < b$
  **shows** *at-within-Ioo-at-right*: *at a within* $\{a<..<b\}$ = *at-right a* **and**
    *at-within-Ioo-at-left*: *at b within* $\{a<..<b\}$ = *at-left b*
    ⟨*proof*⟩

**end**

**lemma** *Int-atLeastAtMost-Unbounded*[*simp*]: $\{a..\}$ *Int* $\{..b\} = \{a..b\}$
  ⟨*proof*⟩

**lemma** *Int-greaterThanAtMost-Unbounded*[*simp*]: $\{a<..\}$ *Int* $\{..b\} = \{a<..b\}$
  ⟨*proof*⟩

**lemma** *Int-atLeastLessThan-Unbounded*[*simp*]: $\{a..\}$ *Int* $\{..<b\} = \{a..<b\}$
  $\langle proof \rangle$

**lemma** *Int-greaterThanLessThan-Unbounded*[*simp*]: $\{a<..\}$ *Int* $\{..<b\} = \{a<..<b\}$
  $\langle proof \rangle$

**lemma** *constant-on-empty*[*simp*]: *f constant-on* $\{\}$
  $\langle proof \rangle$

**lemma** *constant-on-Un*:
  **assumes** *f constant-on A f constant-on B* $A \cap B \neq \{\}$
  **shows** *f constant-on* $A \cup B$
$\langle proof \rangle$

**lemma** *differentiable-transform-open*:
  **assumes** *f differentiable* (*at x*)
    **and** $x \in s$
    **and** *open s*
    **and** $\bigwedge x'. \ x' \in s \Longrightarrow f\ x' = g\ x'$
  **shows** *g differentiable* (*at x*)
  $\langle proof \rangle$

**lemma** *differentiable-eq-field-differentiable-real*:
  **fixes** $f$ :: *real* $\Rightarrow$ *real*
  **shows** *f differentiable F* $\longleftrightarrow$ *f field-differentiable F*
  $\langle proof \rangle$

**lemma** *differentiable-on-eq-field-differentiable-real*:
  **fixes** $f$ :: *real* $\Rightarrow$ *real*
  **shows** *f differentiable-on s* $\longleftrightarrow$ ($\forall x \in s$. *f field-differentiable* (*at x within s*))
  $\langle proof \rangle$

**lemma** *set-borel-measurable-UNIV*[*simp*]:
  **fixes** $f$ :: *'a* :: *real-vector* $\Rightarrow$ *real*
  **shows** *set-borel-measurable M UNIV f* $\longleftrightarrow$ $f \in$ *borel-measurable M*
  $\langle proof \rangle$

**lemma** *deriv-measurable-real*:
  **fixes** $f$ :: *real* $\Rightarrow$ *real*
  **assumes** *f differentiable-on S open S f* $\in$ *borel-measurable borel*
  **shows** *set-borel-measurable borel S* (*deriv f*)
$\langle proof \rangle$

**corollary** *deriv-measurable-real-UNIV*:
  **fixes** $f$ :: *real* $\Rightarrow$ *real*

**assumes** *f differentiable-on UNIV f* $\in$ *borel-measurable borel*
**shows** *deriv f* $\in$ *borel-measurable borel*
$\langle proof \rangle$

**lemma** *piecewise-differentiable-on-deriv-measurable-real*:
**fixes** *f* :: *real* $\Rightarrow$ *real*
**assumes** *f piecewise-differentiable-on S open S f* $\in$ *borel-measurable borel*
**shows** *set-borel-measurable borel S* (*deriv f*)
$\langle proof \rangle$

**corollary** *piecewise-differentiable-on-deriv-measurable-real-UNIV*:
**fixes** *f* :: *real* $\Rightarrow$ *real*
**assumes** *f piecewise-differentiable-on UNIV f* $\in$ *borel-measurable borel*
**shows** (*deriv f*) $\in$ *borel-measurable borel*
$\langle proof \rangle$

**lemma** *einterval-empty*:
**fixes** *a b* :: *ereal*
**assumes** $a \geq b$
**shows** *einterval a b* = {}
$\langle proof \rangle$

**lemma** *einterval-split*:
**fixes** *a b* :: *ereal* **and** *s*::*real*
**assumes** *s* $\in$ *einterval a b*
**shows** *einterval a b* $-$ {*s*} = *einterval a s* $\cup$ *einterval s b*
$\langle proof \rangle$

**lemma** *einterval-Ioc-approximation*:
**fixes** *a b* :: *ereal*
**assumes** $a < b$
**obtains** *u l* :: *nat* $\Rightarrow$ *real* **where**
*einterval a b* = ($\bigcup$*i*. {*l i* <.. *u i*})
*incseq u decseq l* $\bigwedge$*i*. *l i* < *u i* $\bigwedge$*i*. *a* < *l i* $\bigwedge$*i*. *u i* < *b*
$l \longrightarrow a$ $u \longrightarrow b$
$\langle proof \rangle$

**lemma** *measure-eqI-Ioc*:
**fixes** *M N* :: *real measure*
**assumes** *sets*: *sets M* = *sets borel sets N* = *sets borel*
**assumes** *fin*: $\bigwedge$*a b*. $a \leq b \Longrightarrow$ *emeasure M* {*a*<..*b*} < $\infty$
**assumes** *eq*: $\bigwedge$*a b*. $a \leq b \Longrightarrow$ *emeasure M* {*a*<..*b*} = *emeasure N* {*a*<..*b*}
**shows** *M* = *N*
$\langle proof \rangle$

**lemma** *measure-einterval-eqI-Ioc*:
**fixes** *M N* :: *real measure* **and** *a b* :: *ereal*

**assumes** *Mborel*: $sets\ M = sets\ borel$ **and** *Nborel*: $sets\ N = sets\ borel$ **and**
$\bigwedge s\ t.\ a < ereal\ s \wedge s \leq t \wedge ereal\ t < b \Longrightarrow emeasure\ M\ \{s<..t\} \neq \infty$ **and**
$\bigwedge s\ t.\ a < ereal\ s \wedge s \leq t \wedge ereal\ t < b \Longrightarrow emeasure\ M\ \{s<..t\} = emeasure\ N\ \{s<..t\}$
**shows** $restrict\text{-}space\ M\ (einterval\ a\ b) = restrict\text{-}space\ N\ (einterval\ a\ b)$
⟨*proof*⟩

**lemma** *nn-integral-disjoint-pair2*:
**assumes** $B \in sets\ M\ C \in sets\ M\ B \cap C = \{\}$ **and**
[*measurable*]: $(\lambda x.\ f\ x * indicator\ B\ x) \in borel\text{-}measurable\ M$ **and**
[*measurable*]: $(\lambda x.\ f\ x * indicator\ C\ x) \in borel\text{-}measurable\ M$
**shows** $(\int^+ x \in B \cup C.\ f\ x\ \partial M) = (\int^+ x \in B.\ f\ x\ \partial M) + (\int^+ x \in C.\ f\ x\ \partial M)$
⟨*proof*⟩

**lemma** *set-nn-integral-interval-measure-bounded-finite*:
**fixes** $F :: real \Rightarrow real$ **and** $h :: real \Rightarrow ennreal$ **and** $A :: real\ set$ **and** $M::real$
**assumes** $bounded\ A\ \bigwedge x.\ x \in A \Longrightarrow h\ x \leq M\ A \in sets\ borel$ **and**
$mono\ F\ \bigwedge x.\ continuous\ (at\text{-}right\ x)\ F$
**shows** $(\int^+ x{\in}A.\ h\ x\ \partial(interval\text{-}measure\ F)) < \infty$
⟨*proof*⟩

**end**
**theory** *Lebesgue-Stieltjes-Integral*
**imports** *Wlog.Wlog Preliminaries-LSI*
**begin**

# 1 Interval Measure Integral

## 1.1 Basic Calculations

**lemma** *interval-measure-const-null*:
**fixes** $c::real$
**shows** $interval\text{-}measure\ (\lambda\text{-}.\ c) = null\text{-}measure\ lborel$
⟨*proof*⟩

**lemma** *interval-measure-singleton*:
**fixes** $F :: real \Rightarrow real$ **and** $s::real$
**assumes** $mono\ F\ \bigwedge x.\ continuous\ (at\text{-}right\ x)\ F$
**shows** $(interval\text{-}measure\ F)\ \{s\} = F\ s - Lim\ (at\text{-}left\ s)\ F$
⟨*proof*⟩

**lemma** *interval-measure-singleton-continuous*:
**fixes** $F :: real \Rightarrow real$ **and** $s::real$
**assumes** $mono\ F\ \bigwedge x.\ continuous\ (at\text{-}right\ x)\ F\ isCont\ F\ s$
**shows** $(interval\text{-}measure\ F)\ \{s\} = 0$
⟨*proof*⟩

## 1.2 Changing the Underlying Function

**lemma** *einterval-nn-integral-interval-measure-cong*:
**fixes** $F\ G :: real \Rightarrow real$ **and** $h :: real \Rightarrow ennreal$ **and** $a\ b :: ereal$
**assumes** $mono\ F\ \bigwedge x.\ continuous\ (at\text{-}right\ x)\ F$ **and**
$mono\ G\ \bigwedge x.\ continuous\ (at\text{-}right\ x)\ G$ **and**
$(F - G)\ constant\text{-}on\ (einterval\ a\ b)$ **and**
$h \in borel\text{-}measurable\ borel$
**shows** $(\int^+ x{\in}(einterval\ a\ b).\ h\ x\ \partial(interval\text{-}measure\ F)) =$
$(\int^+ x{\in}(einterval\ a\ b).\ h\ x\ \partial(interval\text{-}measure\ G))$
⟨*proof*⟩

**corollary** *Ioo-nn-integral-interval-measure-cong*:
**fixes** $F\ G :: real \Rightarrow real$ **and** $h :: real \Rightarrow ennreal$ **and** $r\ s :: real$
**assumes** $mono\ F\ \bigwedge x.\ continuous\ (at\text{-}right\ x)\ F$ **and**
$mono\ G\ \bigwedge x.\ continuous\ (at\text{-}right\ x)\ G$ **and**
$(F - G)\ constant\text{-}on\ \{r{<}..{<}s\}$ **and**
$h \in borel\text{-}measurable\ borel$
**shows** $(\int^+ x{\in}\{r{<}..{<}s\}.\ h\ x\ \partial(interval\text{-}measure\ F)) = (\int^+ x{\in}\{r{<}..{<}s\}.\ h\ x\ \partial(interval\text{-}measure\ G))$
⟨*proof*⟩

**corollary** *Ioi-nn-integral-interval-measure-cong*:
**fixes** $F\ G :: real \Rightarrow real$ **and** $h :: real \Rightarrow ennreal$ **and** $r::real$
**assumes** $mono\ F\ \bigwedge x.\ continuous\ (at\text{-}right\ x)\ F$ **and**
$mono\ G\ \bigwedge x.\ continuous\ (at\text{-}right\ x)\ G$ **and**
$(F - G)\ constant\text{-}on\ \{r{<}..\}$ **and**
$h \in borel\text{-}measurable\ borel$
**shows** $(\int^+ x{\in}\{r{<}..\}.\ h\ x\ \partial(interval\text{-}measure\ F)) = (\int^+ x{\in}\{r{<}..\}.\ h\ x\ \partial(interval\text{-}measure\ G))$
⟨*proof*⟩

**corollary** *Iio-nn-integral-interval-measure-cong*:
**fixes** $F\ G :: real \Rightarrow real$ **and** $h :: real \Rightarrow ennreal$ **and** $s::real$
**assumes** $mono\ F\ \bigwedge x.\ continuous\ (at\text{-}right\ x)\ F$ **and**
$mono\ G\ \bigwedge x.\ continuous\ (at\text{-}right\ x)\ G$ **and**
$(F - G)\ constant\text{-}on\ \{..{<}s\}$ **and**
$h \in borel\text{-}measurable\ borel$
**shows** $(\int^+ x{\in}\{..{<}s\}.\ h\ x\ \partial(interval\text{-}measure\ F)) = (\int^+ x{\in}\{..{<}s\}.\ h\ x\ \partial(interval\text{-}measure\ G))$
⟨*proof*⟩

**corollary** *nn-integral-interval-measure-cong*:
**fixes** $F\ G :: real \Rightarrow real$ **and** $h :: real \Rightarrow ennreal$
**assumes** $mono\ F\ \bigwedge x.\ continuous\ (at\text{-}right\ x)\ F$ **and**
$mono\ G\ \bigwedge x.\ continuous\ (at\text{-}right\ x)\ G$ **and**
$(F - G)\ constant\text{-}on\ UNIV$ **and**
$h \in borel\text{-}measurable\ borel$
**shows** $(\int^+ x.\ h\ x\ \partial(interval\text{-}measure\ F)) = (\int^+ x.\ h\ x\ \partial(interval\text{-}measure\ G))$
⟨*proof*⟩

**lemma** *singleton-nn-integral-interval-measure-cong*:
**fixes** *F G* :: *real* ⇒ *real* **and** *h* :: *real* ⇒ *ennreal* **and** *s*::*real*
**assumes** *mono F* ⋀*x. continuous* (*at-right x*) *F* **and**
*mono G* ⋀*x. continuous* (*at-right x*) *G* **and**
*F s* − *Lim* (*at-left s*) *F* = *G s* − *Lim* (*at-left s*) *G* **and**
*h* ∈ *borel-measurable borel*
**shows** (∫ ⁺*x*∈{*s*}. *h x* ∂(*interval-measure F*)) = (∫ ⁺*x*∈{*s*}. *h x* ∂(*interval-measure G*))
⟨*proof*⟩

**lemma** *singleton-const-nn-integral-interval-measure-cong*:
**fixes** *F G* :: *real* ⇒ *real* **and** *h* :: *real* ⇒ *ennreal* **and** *r s* :: *real*
**assumes** *mono F* ⋀*x. continuous* (*at-right x*) *F* **and**
*mono G* ⋀*x. continuous* (*at-right x*) *G* **and**
(*F* − *G*) *constant-on* {*r*<..*s*} **and** *r* < *s* **and**
*h* ∈ *borel-measurable borel*
**shows** (∫ ⁺*x*∈{*s*}. *h x* ∂(*interval-measure F*)) = (∫ ⁺*x*∈{*s*}. *h x* ∂(*interval-measure G*))
⟨*proof*⟩

**lemma** *Ioc-nn-integral-interval-measure-cong*:
**fixes** *F G* :: *real* ⇒ *real* **and** *h* :: *real* ⇒ *ennreal* **and** *r s* :: *real*
**assumes** *mono F* ⋀*x. continuous* (*at-right x*) *F* **and**
*mono G* ⋀*x. continuous* (*at-right x*) *G* **and**
(*F* − *G*) *constant-on* {*r*<..*s*} **and**
*h* ∈ *borel-measurable borel*
**shows** (∫ ⁺*x*∈{*r*<..*s*}. *h x* ∂(*interval-measure F*)) = (∫ ⁺*x*∈{*r*<..*s*}. *h x* ∂(*interval-measure G*))
⟨*proof*⟩

**lemma** *Iic-nn-integral-interval-measure-cong*:
**fixes** *F G* :: *real* ⇒ *real* **and** *h* :: *real* ⇒ *ennreal* **and** *s*::*real*
**assumes** *mono F* ⋀*x. continuous* (*at-right x*) *F* **and**
*mono G* ⋀*x. continuous* (*at-right x*) *G* **and**
(*F* − *G*) *constant-on* {..*s*} **and**
*h* ∈ *borel-measurable borel*
**shows** (∫ ⁺*x*∈{..*s*}. *h x* ∂(*interval-measure F*)) = (∫ ⁺*x*∈{..*s*}. *h x* ∂(*interval-measure G*))
⟨*proof*⟩

**lemma** *Ico-nn-integral-interval-measure-cong*:
**fixes** *F G* :: *real* ⇒ *real* **and** *h* :: *real* ⇒ *ennreal* **and** *r s* :: *real*
**assumes** *mono F* ⋀*x. continuous* (*at-right x*) *F* **and**
*mono G* ⋀*x. continuous* (*at-right x*) *G* **and**
(*F* − *G*) *constant-on* {*r*<..<*s*} **and**
*F r* − *Lim* (*at-left r*) *F* = *G r* − *Lim* (*at-left r*) *G* **and**
*h* ∈ *borel-measurable borel*
**shows** (∫ ⁺*x*∈{*r*..<*s*}. *h x* ∂(*interval-measure F*)) = (∫ ⁺*x*∈{*r*..<*s*}. *h x* ∂(*interval-measure*

*G*))
⟨*proof*⟩

**corollary** *Ico-Cont-nn-integral-interval-measure-cong*:
**fixes** $F\ G :: real \Rightarrow real$ **and** $h :: real \Rightarrow ennreal$ **and** $r\ s :: real$
**assumes** *mono F* $\bigwedge x$. *continuous* (*at-right x*) *F* **and**
*mono G* $\bigwedge x$. *continuous* (*at-right x*) *G* **and**
$(F - G)$ *constant-on* $\{r<..<s\}$ **and**
*isCont F r isCont G r* **and**
$h \in$ *borel-measurable borel*
**shows** $(\int^+ x\in\{r..<s\}.\ h\ x\ \partial(\textit{interval-measure}\ F)) = (\int^+ x\in\{r..<s\}.\ h\ x\ \partial(\textit{interval-measure}\ G))$
⟨*proof*⟩

**lemma** *Ici-nn-integral-interval-measure-cong*:
**fixes** $F\ G :: real \Rightarrow real$ **and** $h :: real \Rightarrow ennreal$ **and** $r::real$
**assumes** *mono F* $\bigwedge x$. *continuous* (*at-right x*) *F* **and**
*mono G* $\bigwedge x$. *continuous* (*at-right x*) *G* **and**
$(F - G)$ *constant-on* $\{r<..\}$ **and**
$F\ r - Lim\ (\textit{at-left}\ r)\ F = G\ r - Lim\ (\textit{at-left}\ r)\ G$ **and**
$h \in$ *borel-measurable borel*
**shows** $(\int^+ x\in\{r..\}.\ h\ x\ \partial(\textit{interval-measure}\ F)) = (\int^+ x\in\{r..\}.\ h\ x\ \partial(\textit{interval-measure}\ G))$
⟨*proof*⟩

**corollary** *Ici-Cont-nn-integral-interval-measure-cong*:
**fixes** $F\ G :: real \Rightarrow real$ **and** $h :: real \Rightarrow ennreal$ **and** $r::real$
**assumes** *mono F* $\bigwedge x$. *continuous* (*at-right x*) *F* **and**
*mono G* $\bigwedge x$. *continuous* (*at-right x*) *G* **and**
$(F - G)$ *constant-on* $\{r<..\}$ **and**
*isCont F r isCont G r* **and**
$h \in$ *borel-measurable borel*
**shows** $(\int^+ x\in\{r..\}.\ h\ x\ \partial(\textit{interval-measure}\ F)) = (\int^+ x\in\{r..\}.\ h\ x\ \partial(\textit{interval-measure}\ G))$
⟨*proof*⟩

**lemma** *Icc-nn-integral-interval-measure-cong*:
**fixes** $F\ G :: real \Rightarrow real$ **and** $h :: real \Rightarrow ennreal$ **and** $r\ s :: real$
**assumes** *mono F* $\bigwedge x$. *continuous* (*at-right x*) *F* **and**
*mono G* $\bigwedge x$. *continuous* (*at-right x*) *G* **and**
$(F - G)$ *constant-on* $\{r<..s\}$ **and**
$F\ r - Lim\ (\textit{at-left}\ r)\ F = G\ r - Lim\ (\textit{at-left}\ r)\ G$ **and**
$h \in$ *borel-measurable borel*
**shows** $(\int^+ x\in\{r..s\}.\ h\ x\ \partial(\textit{interval-measure}\ F)) = (\int^+ x\in\{r..s\}.\ h\ x\ \partial(\textit{interval-measure}\ G))$
⟨*proof*⟩

**corollary** *Icc-Cont-nn-integral-interval-measure-cong*:
**fixes** $F\ G :: real \Rightarrow real$ **and** $h :: real \Rightarrow ennreal$ **and** $r\ s :: real$

**assumes** *mono F* $\bigwedge x$. *continuous* (*at-right x*) *F* **and**
*mono G* $\bigwedge x$. *continuous* (*at-right x*) *G* **and**
$(F - G)$ *constant-on* $\{r<..s\}$ **and**
*isCont F r isCont G r* **and**
$h \in$ *borel-measurable borel*
**shows** $(\int^+ x \in \{r..s\}.\ h\ x\ \partial(\textit{interval-measure}\ F)) = (\int^+ x \in \{r..s\}.\ h\ x\ \partial(\textit{interval-measure}\ G))$
⟨*proof*⟩

## 1.3 Restricting the Integral

**lemma** *nn-integral-interval-measure-Ici*:
**fixes** $F$ :: *real* ⇒ *real* **and** $g$ :: *real* ⇒ *ennreal* **and** $r$::*real*
**assumes** *mono F* $\bigwedge x$. *continuous* (*at-right x*) *F* $g \in$ *borel-measurable borel* **and**
*F constant-on* $\{..<r\}$
**shows** $(\int^+ x.\ g\ x\ \partial(\textit{interval-measure}\ F)) = (\int^+ x \in \{r..\}.\ g\ x\ \partial(\textit{interval-measure}\ F))$
⟨*proof*⟩

**lemma** *nn-integral-interval-measure-Ioi*:
**fixes** $F$ :: *real* ⇒ *real* **and** $g$ :: *real* ⇒ *ennreal* **and** $r$::*real*
**assumes** *mono F* $\bigwedge x$. *continuous* (*at-right x*) *F* $g \in$ *borel-measurable borel* **and**
*F constant-on* $\{..<r\}$ *isCont F r*
**shows** $(\int^+ x.\ g\ x\ \partial(\textit{interval-measure}\ F)) = (\int^+ x \in \{r<..\}.\ g\ x\ \partial(\textit{interval-measure}\ F))$
⟨*proof*⟩

**lemma** *nn-integral-interval-measure-Iic*:
**fixes** $F$ :: *real* ⇒ *real* **and** $g$ :: *real* ⇒ *ennreal* **and** $s$::*real*
**assumes** *mono F* $\bigwedge x$. *continuous* (*at-right x*) *F* $g \in$ *borel-measurable borel* **and**
*F constant-on* $\{s<..\}$
**shows** $(\int^+ x.\ g\ x\ \partial(\textit{interval-measure}\ F)) = (\int^+ x \in \{..s\}.\ g\ x\ \partial(\textit{interval-measure}\ F))$
⟨*proof*⟩

**lemma** *nn-integral-interval-measure-Iio*:
**fixes** $F$ :: *real* ⇒ *real* **and** $g$ :: *real* ⇒ *ennreal* **and** $s$::*real*
**assumes** *mono F* $\bigwedge x$. *continuous* (*at-right x*) *F* $g \in$ *borel-measurable borel* **and**
*F constant-on* $\{s<..\}$ *isCont F s*
**shows** $(\int^+ x.\ g\ x\ \partial(\textit{interval-measure}\ F)) = (\int^+ x \in \{..<s\}.\ g\ x\ \partial(\textit{interval-measure}\ F))$
⟨*proof*⟩

**lemma** *nn-integral-interval-measure-Icc*:
**fixes** $F$ :: *real* ⇒ *real* **and** $g$ :: *real* ⇒ *ennreal* **and** $r\ s$ :: *real*
**assumes** *mono F* $\bigwedge x$. *continuous* (*at-right x*) *F* $g \in$ *borel-measurable borel* **and**
*F constant-on* $\{..<r\}$ *F constant-on* $\{s<..\}$
**shows** $(\int^+ x.\ g\ x\ \partial(\textit{interval-measure}\ F)) = (\int^+ x \in \{r..s\}.\ g\ x\ \partial(\textit{interval-measure}\ F))$

⟨*proof*⟩

**lemma** *nn-integral-interval-measure-Ioc*:
**fixes** $F$ :: *real* ⇒ *real* **and** $g$ :: *real* ⇒ *ennreal* **and** $r$ $s$ :: *real*
**assumes** *mono F* $\bigwedge x$. *continuous* (*at-right x*) *F* $g \in$ *borel-measurable borel* **and**
*F constant-on* {..<*r*} *F constant-on* {*s*<..} *isCont F rf*
**shows** ($\int^{+}x$. *g x* ∂(*interval-measure F*)) = ($\int^{+}x$∈{*r*<..*s*}. *g x* ∂(*interval-measure F*))
⟨*proof*⟩

**lemma** *nn-integral-interval-measure-Ico*:
**fixes** $F$ :: *real* ⇒ *real* **and** $g$ :: *real* ⇒ *ennreal* **and** $r$ $s$ :: *real*
**assumes** *mono F* $\bigwedge x$. *continuous* (*at-right x*) *F* $g \in$ *borel-measurable borel*
*F constant-on* {..<*r*} *F constant-on* {*s*<..} *isCont F s*
**shows** ($\int^{+}x$. *g x* ∂(*interval-measure F*)) = ($\int^{+}x$∈{*r*..<*s*}. *g x* ∂(*interval-measure F*))
⟨*proof*⟩

**lemma** *nn-integral-interval-measure-Ioo*:
**fixes** $F$ :: *real* ⇒ *real* **and** $g$ :: *real* ⇒ *ennreal* **and** $r$ $s$ :: *real*
**assumes** *mono F* $\bigwedge x$. *continuous* (*at-right x*) *F* $g \in$ *borel-measurable borel* **and**
*F constant-on* {..<*r*} *F constant-on* {*s*<..} *isCont F r isCont F s*
**shows** ($\int^{+}x$. *g x* ∂(*interval-measure F*)) = ($\int^{+}x$∈{*r*<..<*s*}. *g x* ∂(*interval-measure F*))
⟨*proof*⟩

## 1.4 Calculation by the Derivative

**proposition** *set-nn-integral-interval-measure-deriv*:
**fixes** $F$ :: *real* ⇒ *real* **and** $g$ :: *real* ⇒ *ennreal* **and** $a$ $b$ :: *ereal* **and** $A$ :: *real set*
**assumes** *mono F* $\bigwedge x$. *continuous* (*at-right x*) *F F differentiable-on* (*einterval a b*) **and**
*g-msr*: $g \in$ *borel-measurable lborel* **and**
$A \in$ *sets borel* $A \subseteq$ *einterval a b*
**shows** ($\int^{+}x$∈*A*. *g x* ∂(*interval-measure F*)) = ($\int^{+}x$∈*A*. *g x* ∗ *deriv F x* ∂*lborel*)
⟨*proof*⟩

**corollary** *nn-integral-interval-measure-deriv*:
**fixes** $F$ :: *real* ⇒ *real* **and** $g$ :: *real* ⇒ *ennreal*
**assumes** *mono F* $\bigwedge x$. *continuous* (*at-right x*) *F F differentiable-on UNIV* **and**
$g \in$ *borel-measurable lborel*
**shows** ($\int^{+}x$. *g x* ∂(*interval-measure F*)) = ($\int^{+}x$. *g x* ∗ *deriv F x* ∂*lborel*)
⟨*proof*⟩

**corollary** *Ioi-nn-integral-interval-measure-deriv*:
**fixes** $F$ :: *real* ⇒ *real* **and** $g$ :: *real* ⇒ *ennreal* **and** *r*::*real*
**assumes** *mono F* $\bigwedge x$. *continuous* (*at-right x*) *F F differentiable-on* {*r*<..} **and**
$g \in$ *borel-measurable lborel*
**shows** ($\int^{+}x$∈{*r*<..}. *g x* ∂(*interval-measure F*)) = ($\int^{+}x$∈{*r*<..}. *g x* ∗ *deriv F*

*x* ∂*lborel*)
⟨*proof*⟩

**corollary** *Iio-nn-integral-interval-measure-deriv*:
**fixes** *F* :: *real* ⇒ *real* **and** *g* :: *real* ⇒ *ennreal* **and** *s*::*real*
**assumes** *mono F* ⋀*x*. *continuous* (*at-right x*) *F F differentiable-on* {..<*s*} **and**
*g* ∈ *borel-measurable lborel*
**shows** (∫⁺*x*∈{..<*s*}. *g x* ∂(*interval-measure F*)) = (∫⁺*x*∈{..<*s*}. *g x* ∗ *deriv F x* ∂*lborel*)
⟨*proof*⟩

**corollary** *Ioo-nn-integral-interval-measure-deriv*:
**fixes** *F* :: *real* ⇒ *real* **and** *g* :: *real* ⇒ *ennreal* **and** *r s* :: *real*
**assumes** *mono F* ⋀*x*. *continuous* (*at-right x*) *F F differentiable-on* {*r*<..<*s*} **and**
*g* ∈ *borel-measurable lborel*
**shows** (∫⁺*x*∈{*r*<..<*s*}. *g x* ∂(*interval-measure F*)) = (∫⁺*x*∈{*r*<..<*s*}. *g x* ∗ *deriv F x* ∂*lborel*)
⟨*proof*⟩

**lemma** *set-nn-integral-finite-nondifferentiable-interval-measure-deriv*:
**fixes** *F* :: *real* ⇒ *real* **and** *g* :: *real* ⇒ *ennreal* **and** *a b* :: *ereal* **and** *S* :: *real set*
**assumes** *mono F* ⋀*x*. *continuous* (*at-right x*) *F g* ∈ *borel-measurable lborel* **and**
*cont*: *continuous-on* (*einterval a b*) *F* **and**
*diff*: *F differentiable-on einterval a b* − *S* **and**
*fin*: *finite S*
**shows** (∫⁺*x* ∈ *einterval a b*. *g x* ∂(*interval-measure F*)) =
(∫⁺*x* ∈ *einterval a b*. *g x* ∗ *deriv F x* ∂*lborel*)
⟨*proof*⟩

**proposition** *set-nn-integral-piecewise-differentiable-interval-measure-deriv*:
**fixes** *F* :: *real* ⇒ *real* **and** *g* :: *real* ⇒ *ennreal* **and** *a b* :: *ereal*
**assumes** *mono F* ⋀*x*. *continuous* (*at-right x*) *F F piecewise-differentiable-on* (*einterval a b*)
*g* ∈ *borel-measurable lborel*
**shows** (∫⁺*x* ∈ *einterval a b*. *g x* ∂(*interval-measure F*)) =
(∫⁺*x* ∈ *einterval a b*. *g x* ∗ *deriv F x* ∂*lborel*)
⟨*proof*⟩

**corollary** *nn-integral-piecewise-differentiable-interval-measure-deriv*:
**fixes** *F* :: *real* ⇒ *real* **and** *g* :: *real* ⇒ *ennreal*
**assumes** *mono F* ⋀*x*. *continuous* (*at-right x*) *F F piecewise-differentiable-on* *UNIV*
*g* ∈ *borel-measurable lborel*
**shows** (∫⁺*x*. *g x* ∂(*interval-measure F*)) = (∫⁺*x*. *g x* ∗ *deriv F x* ∂*lborel*)
⟨*proof*⟩

**corollary** *Ioi-nn-integral-piecewise-differentiable-interval-measure-deriv*:
**fixes** *F* :: *real* ⇒ *real* **and** *g* :: *real* ⇒ *ennreal* **and** *r*::*real*

**assumes** *mono F* $\bigwedge x$. *continuous* (*at-right x*) *F F piecewise-differentiable-on* $\{r<..\}$
*g* ∈ *borel-measurable lborel*
**shows** ($\int^+ x \in \{r<..\}$. *g x* ∂(*interval-measure F*)) = ($\int^+ x \in \{r<..\}$. *g x* ∗ *deriv F x* ∂*lborel*)
⟨*proof*⟩

**corollary** *Iio-nn-integral-piecewise-differentiable-interval-measure-deriv*:
**fixes** *F* :: *real* ⇒ *real* **and** *g* :: *real* ⇒ *ennreal* **and** *s*::*real*
**assumes** *mono F* $\bigwedge x$. *continuous* (*at-right x*) *F F piecewise-differentiable-on* $\{..<s\}$
*g* ∈ *borel-measurable lborel*
**shows** ($\int^+ x \in \{..<s\}$. *g x* ∂(*interval-measure F*)) = ($\int^+ x \in \{..<s\}$. *g x* ∗ *deriv F x* ∂*lborel*)
⟨*proof*⟩

**corollary** *Ioo-nn-integral-piecewise-differentiable-interval-measure-deriv*:
**fixes** *F* :: *real* ⇒ *real* **and** *g* :: *real* ⇒ *ennreal* **and** *r s* :: *real*
**assumes** *mono F* $\bigwedge x$. *continuous* (*at-right x*) *F F piecewise-differentiable-on* $\{r<..<s\}$
*g* ∈ *borel-measurable lborel*
**shows** ($\int^+ x \in \{r<..<s\}$. *g x* ∂(*interval-measure F*)) = ($\int^+ x \in \{r<..<s\}$. *g x* ∗ *deriv F x* ∂*lborel*)
⟨*proof*⟩

**end**