

# Lebesgue-Stieltjes Integral

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## Abstract

This entry formalizes some basic facts and lemmas relating to the integration with respect to the Lebesgue-Stieltjes measure (interval measure). It includes the well-known formula to calculate the Lebesgue-Stieltjes integral:

$$\int g(x) dF(x) = \int g(x)F'(x) dx.$$

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**theory** *Preliminaries-LSI*

**imports** *HOL-Library.Rewrite HOL-Analysis.Analysis*

**begin**

**context** *order-topology*

**begin**

**lemma**

**assumes**  $a < b$

**shows** *at-within-Ioo-at-right: at a within {a<..**and***

*at-within-Ioo-at-left: at b within {a<..**at-left***

**using** *assms at-within-nhd*

**apply** (*metis inf.right-idem open-lessThan greaterThanLessThan-eq lessThan-iff*)

**using** *assms at-within-nhd*

**by** (*smt (verit, ccfv-SIG) inf.idem inf-aci*

*greaterThanLessThan-def greaterThan-iff open-greaterThan*)

**end**

**lemma** *Int-atLeastAtMost-Unbounded[simp]: {a..} Int {..b} = {a..b}*

by *auto*

**lemma** *Int-greaterThanAtMost-Unbounded[simp]*:  $\{a<..\} \text{ Int } \{..b\} = \{a<..b\}$   
by *auto*

**lemma** *Int-atLeastLessThan-Unbounded[simp]*:  $\{a..\} \text{ Int } \{..<b\} = \{a..<b\}$   
by *auto*

**lemma** *Int-greaterThanLessThan-Unbounded[simp]*:  $\{a<..\} \text{ Int } \{..<b\} = \{a<..<b\}$   
by *auto*

**lemma** *constant-on-empty[simp]*:  $f \text{ constant-on } \{\}$   
by (*simp add: constant-on-def*)

**lemma** *constant-on-Un*:

assumes  $f \text{ constant-on } A$   $f \text{ constant-on } B$   $A \cap B \neq \{\}$   
shows  $f \text{ constant-on } A \cup B$

**proof** –

from *assms* obtain  $c$  where  $c \in A \cap B$  by *auto*

with *assms* have  $\bigwedge x. x \in A \cup B \implies f x = f c$  **unfolding** *constant-on-def* by *auto*

thus *?thesis* **unfolding** *constant-on-def* by *simp*

**qed**

**lemma** *differentiable-transform-open*:

assumes  $f \text{ differentiable (at } x)$

and  $x \in s$

and *open*  $s$

and  $\bigwedge x'. x' \in s \implies f x' = g x'$

shows  $g \text{ differentiable (at } x)$

**using** *assms* *has-derivative-transform-within-open at-within-open* **unfolding** *differentiable-def*

by *blast*

**lemma** *differentiable-eq-field-differentiable-real*:

fixes  $f :: \text{real} \Rightarrow \text{real}$

shows  $f \text{ differentiable } F \iff f \text{ field-differentiable } F$

**unfolding** *field-differentiable-def differentiable-def has-real-derivative*

**using** *has-real-derivative-iff* by *presburger*

**lemma** *differentiable-on-eq-field-differentiable-real*:

fixes  $f :: \text{real} \Rightarrow \text{real}$

shows  $f \text{ differentiable-on } s \iff (\forall x \in s. f \text{ field-differentiable (at } x \text{ within } s))$

**unfolding** *differentiable-on-def* **using** *differentiable-eq-field-differentiable-real* by *simp*

**lemma** *set-borel-measurable-UNIV*[simp]:  
**fixes**  $f :: 'a :: \text{real-vector} \Rightarrow \text{real}$   
**shows** *set-borel-measurable*  $M \text{ UNIV } f \longleftrightarrow f \in \text{borel-measurable } M$   
**unfolding** *set-borel-measurable-def* **by** *simp*

**lemma** *deriv-measurable-real*:  
**fixes**  $f :: \text{real} \Rightarrow \text{real}$   
**assumes**  $f$  *differentiable-on*  $S$  *open*  $S$   $f \in \text{borel-measurable borel}$   
**shows** *set-borel-measurable borel*  $S$   $(\text{deriv } f)$   
**proof** –  
**have**  $\bigwedge x. x \in S \implies \text{deriv } f x = \lim (\lambda i. (f (x + 1 / \text{Suc } i) - f x) / (1 / \text{Suc } i))$   
**proof** –  
**fix**  $x$  **assume**  $x \in S$ :  $x \in S$   
**hence**  $f$  *field-differentiable* (at  $x$  within  $S$ )  
**using** *differentiable-on-eq-field-differentiable-real assms* **by** *simp*  
**hence**  $f$  *has-field-derivative*  $(\text{deriv } f x)$  (at  $x$ )  
**using** *assms DERIV-deriv-iff-field-differentiable x-S at-within-open* **by** *force*  
**hence**  $(\lambda h. (f (x+h) - f x) / h) \rightarrow 0 \implies \text{deriv } f x$  **using** *DERIV-def* **by** *auto*  
**hence**  $\forall d. (\forall i. d i \in \text{UNIV} - \{0::\text{real}\}) \longrightarrow d \longrightarrow 0 \longrightarrow$   
 $((\lambda h. (f (x+h) - f x) / h) \circ d) \longrightarrow \text{deriv } f x$   
**using** *tendsto-at-iff-sequentially* **by** *blast*  
**moreover** **have**  $\forall i. 1 / \text{Suc } i \in \text{UNIV} - \{0::\text{real}\}$  **by** *simp*  
**moreover** **have**  $(\lambda i. 1 / \text{Suc } i) \longrightarrow 0$  **using** *LIMSEQ-Suc lim-const-over-n*  
**by** *blast*  
**ultimately** **have**  $((\lambda h. (f (x+h) - f x) / h) \circ (\lambda i. 1 / \text{Suc } i)) \longrightarrow \text{deriv } f x$   
**by** *auto*  
**thus**  $\text{deriv } f x = \lim (\lambda i. (f (x + 1 / \text{Suc } i) - f x) / (1 / \text{Suc } i))$   
**unfolding** *comp-def* **by** *(simp add: limI)*  
**qed**  
**moreover** **have**  $(\lambda x. \text{indicator } S x * \lim (\lambda i. (f (x + 1 / \text{Suc } i) - f x) / (1 / \text{Suc } i)))$   
 $\in \text{borel-measurable borel}$   
**using** *assms* **by** *(measurable, simp, measurable)*  
**ultimately** **show** *?thesis*  
**unfolding** *set-borel-measurable-def real-scaleR-def*  
**by** *(smt (verit) indicator-simps measurable-cong mult-eq-0-iff)*  
**qed**

**corollary** *deriv-measurable-real-UNIV*:  
**fixes**  $f :: \text{real} \Rightarrow \text{real}$   
**assumes**  $f$  *differentiable-on*  $\text{UNIV}$   $f \in \text{borel-measurable borel}$   
**shows**  $\text{deriv } f \in \text{borel-measurable borel}$   
**using** *deriv-measurable-real*[**where**  $S = \text{UNIV}$ ] *assms* **by** *simp*

**lemma** *piecewise-differentiable-on-deriv-measurable-real*:  
**fixes**  $f :: \text{real} \Rightarrow \text{real}$   
**assumes**  $f$  *piecewise-differentiable-on*  $S$  *open*  $S$   $f \in \text{borel-measurable borel}$   
**shows** *set-borel-measurable borel*  $S$   $(\text{deriv } f)$

**proof** –

**from** *assms* **obtain**  $T$  **where** *fin*: *finite*  $T$  **and**  
  *diff*:  $\bigwedge x. x \in S - T \implies f$  *differentiable at*  $x$  *within*  $S$   
  **unfolding** *piecewise-differentiable-on-def* **by** *blast*  
**with** *assms* **have** *open*  $(S - T)$  **using** *finite-imp-closed* **by** *blast*  
**moreover** *hence*  $f$  *differentiable-on*  $(S - T)$   
  **unfolding** *differentiable-on-def* **using** *assms* **by** (*metis* *Diff-iff at-within-open*  
*diff*)  
**ultimately** *have* *set-borel-measurable* *borel*  $(S - T)$  (*deriv*  $f$ )  
  **by** (*intro* *deriv-measurable-real*; *simp* *add*: *assms*)  
**thus** *?thesis*  
  **unfolding** *set-borel-measurable-def* *real-scaleR-def*  
  **by** (*rule* *measurable-discrete-difference*[**where**  $X=T$ ])  
  (*simp-all* *add*: *countable-finite* *fin* *indicator-diff*)

**qed**

**corollary** *piecewise-differentiable-on-deriv-measurable-real-UNIV*:

**fixes**  $f :: \text{real} \Rightarrow \text{real}$   
  **assumes**  $f$  *piecewise-differentiable-on*  $UNIV$   $f \in \text{borel-measurable borel}$   
  **shows**  $(\text{deriv } f) \in \text{borel-measurable borel}$   
  **using** *piecewise-differentiable-on-deriv-measurable-real*[**where**  $S=UNIV$ ] *assms*  
**by** *simp*

**lemma** *einterval-empty*:

**fixes**  $a b :: \text{ereal}$   
  **assumes**  $a \geq b$   
  **shows** *einterval*  $a b = \{\}$   
  **using** *assms* *einterval-iff* **by** *auto*

**lemma** *einterval-split*:

**fixes**  $a b :: \text{ereal}$  **and**  $s :: \text{real}$   
  **assumes**  $s \in \text{einterval } a b$   
  **shows** *einterval*  $a b - \{s\} = \text{einterval } a s \cup \text{einterval } s b$

**proof**

**show** *einterval*  $a b - \{s\} \subseteq \text{einterval } a (\text{ereal } s) \cup \text{einterval } (\text{ereal } s) b$   
  **using** *assms* *einterval-iff* **by** *auto*  
  **show** *einterval*  $a (\text{ereal } s) \cup \text{einterval } (\text{ereal } s) b \subseteq \text{einterval } a b - \{s\}$   
  **using** *assms* *einterval-iff*  
  **by** (*smt* (*verit*, *ccfv-threshold*) *Diff-iff* *Diff-insert-absorb* *Un-iff*  
  *order.strict-trans* *less-ereal.simps* *subsetI*)

**qed**

**lemma** *einterval-Ioc-approximation*:

**fixes**  $a b :: \text{ereal}$   
  **assumes**  $a < b$   
  **obtains**  $u l :: \text{nat} \Rightarrow \text{real}$  **where**  
  *einterval*  $a b = (\bigcup i. \{l\ i <.. u\ i\})$

$$\begin{aligned} & \text{incseq } u \text{ decseq } l \wedge i. l \ i < u \ i \wedge i. a < l \ i \wedge i. u \ i < b \\ & l \longrightarrow a \ u \longrightarrow b \end{aligned}$$

**proof** –

**from** *assms* **obtain**  $u \ l :: \text{nat} \Rightarrow \text{real}$  **where**

*einterval*  $a \ b = (\bigcup i. \{l \ i .. u \ i\})$

*incseq*  $u$  *decseq*  $l \wedge i. l \ i < u \ i \wedge i. a < \text{ereal } (l \ i) \wedge i. \text{ereal } (u \ i) < b$   
 $(\lambda i. \text{ereal } (l \ i)) \longrightarrow a \ (\lambda i. \text{ereal } (u \ i)) \longrightarrow b$

**using** *einterval-Icc-approximation* **by** (*smt* (*verit*) *Sup.SUP-cong*)

**moreover** **have**  $(\bigcup i. \{l \ i .. u \ i\}) = (\bigcup i. \{l \ i <.. u \ i\})$

**proof**

**show**  $(\bigcup i. \{l \ i .. u \ i\}) \subseteq (\bigcup i. \{l \ i <.. u \ i\})$

**proof**

**fix**  $x$  **assume**  $x \in (\bigcup i. \{l \ i .. u \ i\})$

**from** *this* **obtain**  $i$  **where**  $x_i: x \in \{l \ i .. u \ i\}$  **by** *blast*

**obtain**  $j$  **where**  $j \geq i \ l \ j < l \ i$

**proof** (*cases*  $\langle a = -\infty \rangle$ )

**case** *True*

**with**  $\langle (\lambda i. \text{ereal } (l \ i)) \longrightarrow a \rangle$  **obtain**  $k$  **where**  $l \ k \leq \text{ereal } (l \ i - 1)$

**using** *Lim-MInfty* **by** (*meson nle-le*)

**hence**  $l \ k < l \ i$  **by** *simp*

**hence**  $l \ (\text{max } i \ k + 1) < l \ i$

**using**  $\langle \text{decseq } l \rangle$  **by** (*smt* (*verit*, *del-insts*) *decseq-defle-add1 max.cobounded2*)

**moreover** **have**  $\text{max } i \ k + 1 \geq i$  **by** *simp*

**ultimately** **show** *?thesis* **using** *that* **by** *simp*

**next**

**case** *False*

**with** *assms* **obtain**  $ar$  **where**  $aar: a = \text{ereal } ar$  **using** *less-ereal.simps* **by**  
(*metis ereal-cases*)

**with**  $\langle (\lambda i. \text{ereal } (l \ i)) \longrightarrow a \rangle$  **have**  $l \longrightarrow ar$  **by** *auto*

**moreover** **have**  $0 < l \ i - ar$  **using**  $\langle \wedge i. a < \text{ereal } (l \ i) \rangle$   $aar$  **by** *simp*

**ultimately** **obtain**  $k$  **where**  $\wedge h. h \geq k \implies \text{norm } (l \ h - ar) < l \ i - ar$

**using** *LIMSEQ-D* **by** *blast*

**hence**  $\wedge h. h \geq k \implies l \ h < l \ i$  **by** *force*

**hence**  $l \ (\text{max } i \ k + 1) < l \ i$  **by** *simp*

**moreover** **have**  $\text{max } i \ k + 1 \geq i$  **by** *simp*

**ultimately** **show** *?thesis* **using** *that* **by** *simp*

**qed**

**hence**  $x \in \{l \ j <.. u \ j\}$  **using** *xi assms*  $\langle \text{incseq } u \rangle$  *monoD* **by** *fastforce*

**thus**  $x \in (\bigcup i. \{l \ i <.. u \ i\})$  **by** *blast*

**qed**

**next**

**have**  $\wedge i. \{l \ i <.. u \ i\} \subseteq \{l \ i .. u \ i\}$  **by** *force*

**thus**  $(\bigcup i. \{l \ i <.. u \ i\}) \subseteq (\bigcup i. \{l \ i .. u \ i\})$  **by** *blast*

**qed**

**ultimately** **show** *?thesis* **using** *that* **by** *simp*

**qed**

**lemma** *measure-eqI-Ioc*:

```

fixes  $M N$  :: real measure
assumes sets:  $sets\ M = sets\ borel\ sets\ N = sets\ borel$ 
assumes fin:  $\bigwedge a\ b. a \leq b \implies emeasure\ M\ \{a<..b\} < \infty$ 
assumes eq:  $\bigwedge a\ b. a \leq b \implies emeasure\ M\ \{a<..b\} = emeasure\ N\ \{a<..b\}$ 
shows  $M = N$ 
proof (rule measure-eqI-generator-eq-countable)
  let  $?Ioc = \lambda(a::real,b::real). \{a<..b\}$  let  $?E = range\ ?Ioc$ 
  show Int-stable  $?E$  using Int-stable-def Int-greaterThanAtMost by force
  show  $?E \subseteq Pow\ UNIV$   $sets\ M = sigma\text{-sets}\ UNIV\ ?E$   $sets\ N = sigma\text{-sets}\ UNIV\ ?E$ 
  unfolding sets by (auto simp add: borel-sigma-sets-Ioc)
  show  $\bigwedge I. I \in ?E \implies emeasure\ M\ I = emeasure\ N\ I$ 
  proof -
    fix  $I$  assume  $I \in ?E$ 
    then obtain  $a\ b$  where  $I = \{a<..b\}$  by auto
    thus  $emeasure\ M\ I = emeasure\ N\ I$  by (smt (verit, best) eq greaterThanAtMost-empty)
  qed
  show  $?Ioc\ ' (Rats \times Rats) \subseteq ?E$  ( $\bigcup i \in (Rats \times Rats). ?Ioc\ i = UNIV$ )
    using Rats-no-bot-less Rats-no-top-le by auto
  show countable ( $?Ioc\ ' (Rats \times Rats)$ ) using countable-rat by blast
  show  $\bigwedge I. I \in ?Ioc\ ' (Rats \times Rats) \implies emeasure\ M\ I \neq \infty$ 
  proof -
    fix  $I$  assume  $I \in ?Ioc\ ' (Rats \times Rats)$ 
    then obtain  $a\ b$  where  $(a,b) \in (Rats \times Rats)$   $I = \{a<..b\}$  by blast
    thus  $emeasure\ M\ I \neq \infty$  by (smt (verit, best) Ioc-inj fin order.strict-implies-not-eq)
  qed
qed

```

**lemma** *measure-einterval-eqI-Ioc*:

```

fixes  $M N$  :: real measure and  $a\ b$  :: ereal
assumes Mborel:  $sets\ M = sets\ borel$  and Nborel:  $sets\ N = sets\ borel$  and
   $\bigwedge s\ t. a < ereal\ s \wedge s \leq t \wedge ereal\ t < b \implies emeasure\ M\ \{s<..t\} \neq \infty$  and
   $\bigwedge s\ t. a < ereal\ s \wedge s \leq t \wedge ereal\ t < b \implies emeasure\ M\ \{s<..t\} = emeasure\ N\ \{s<..t\}$ 
shows  $restrict\text{-space}\ M\ (einterval\ a\ b) = restrict\text{-space}\ N\ (einterval\ a\ b)$ 
proof (cases <a < b>)

```

**case**  $ab$ : *True*

**let**  $?Iab = einterval\ a\ b$

**let**  $?Iocs = \{\{s<..t\} \mid s\ t :: real. a < ereal\ s \wedge s \leq t \wedge ereal\ t < b\}$

**have**  $Iocs\text{-}Iab$ :  $?Iocs \subseteq Pow\ ?Iab$

**by** *safe (metis einterval-iff greaterThanAtMost-iff ereal-less-le le-ereal-less less-eq-ereal-def)*

**obtain**  $u\ l$  :: *nat*  $\Rightarrow$  *real* **where**

$Iab\text{-}Un$ :  $?Iab = (\bigcup i. \{l\ i <.. u\ i\})$  **and**

$lu$ :  $\bigwedge i. l\ i < u\ i$  **and**  $al$ :  $\bigwedge i. a < l\ i$  **and**  $ub$ :  $\bigwedge i. u\ i < b$

```

using einterval-Ioc-approximation ab by (smt (verit) Inf.INF-cong)

let ?Iocn =  $\lambda n::nat. \{l\ n <.. u\ n\}$ 

have Iabsigma:  $\bigwedge L :: \text{real measure. sets } L = \text{sets borel} \implies$ 
  sets (restrict-space L ?Iab) = sigma-sets ?Iab ?Iocs
proof –
  fix L :: real measure assume asm: sets L = sets borel
  have sets (restrict-space L ?Iab) = (inter ?Iab) ‘ sigma-sets UNIV (range
( $\lambda(s,t) \Rightarrow \{s <.. t\}$ ))
  by (rewrite sets-restrict-space, rewrite asm, rewrite borel-sigma-sets-Ioc) simp
  also have ... = sigma-sets ?Iab ((inter ?Iab) ‘  $\{\{s <.. t\} \mid s\ t.\ \text{True}\}$ )
  apply (rewrite restricted-sigma)
  apply (metis Pow-UNIV borel-einterval borel-sigma-sets-Ioc sets-measure-of
top-greatest)
  unfolding image-def by simp-all
  also have ... = sigma-sets ?Iab ?Iocs
proof
  have (inter ?Iab) ‘  $\{\{s <.. t\} \mid s\ t.\ \text{True}\} \subseteq \text{sigma-sets ?Iab ?Iocs}$ 
proof
  fix A assume  $A \in (\text{inter ?Iab}) ‘ \{\{s <.. t\} \mid s\ t.\ \text{True}\}$ 
  from this obtain s t where A =  $\text{?Iab} \cap \{s <.. t\}$  by blast
  also have ... =  $(\bigcup n. \{l\ n <.. u\ n\} \cap \{s <.. t\})$  using Aabst Iab-Un UN-simps
by force
  also have ... =  $(\bigcup n. \{\max(l\ n)\ s <.. \min(u\ n)\ t\})$  by simp
  moreover have
 $\bigwedge n. \{\max(l\ n)\ s <.. \min(u\ n)\ t\} \in \text{sigma-sets ?Iab ?Iocs}$ 
proof –
  fix n
  show  $\{\max(l\ n)\ s <.. \min(u\ n)\ t\} \in \text{sigma-sets ?Iab ?Iocs}$ 
proof (cases  $\langle \max(l\ n)\ s \leq \min(u\ n)\ t \rangle$ )
  case True
  moreover have  $a < \text{ereal}(\max(l\ n)\ s)$  using al by (metis less-ereal-le
max-def)
  moreover have  $\text{ereal}(\min(u\ n)\ t) < b$  using ub by (smt (verit)
ereal-less-le)
  ultimately have  $\{\max(l\ n)\ s <.. \min(u\ n)\ t\} \in \text{?Iocs}$  by blast
  thus ?thesis ..
  next
  case False
  hence  $\{\max(l\ n)\ s <.. \min(u\ n)\ t\} = \{\}$  by auto
  thus ?thesis by (metis sigma-sets.Empty)
  qed
qed
ultimately show  $A \in \text{sigma-sets ?Iab ?Iocs}$  by (simp add: sigma-sets.Union)
qed
thus sigma-sets ?Iab ((inter ?Iab) ‘  $\{\{s <.. t\} \mid s\ t.\ \text{True}\}$ )  $\subseteq \text{sigma-sets ?Iab}$ 
?Iocs
by (rule sigma-sets-mono)

```

**next**  
**have**  $?Iocs \subseteq \text{sigma-sets } ?Iab ((\text{inter } ?Iab) \text{ ' } \{\{s <..t\} \mid s t. \text{True}\})$   
**proof**  
**fix**  $A$  **assume**  $A \in ?Iocs$   
**from this obtain**  $s t$  **where**  $a < \text{ereal } s \wedge s \leq t \wedge \text{ereal } t < b$  **and**  $A = \{s <..t\}$  **by** *blast*  
**hence**  $A = ?Iab \cap \{s <..t\}$  **using** *Iocs-Iab* **by** *blast*  
**hence**  $A \in (\text{inter } ?Iab) \text{ ' } \{\{s <..t\} \mid s t. \text{True}\}$  **by** *auto*  
**thus**  $A \in \text{sigma-sets } ?Iab ((\text{inter } ?Iab) \text{ ' } \{\{s <..t\} \mid s t. \text{True}\})$  **by** *blast*  
**qed**  
**thus**  $\text{sigma-sets } ?Iab ?Iocs \subseteq \text{sigma-sets } ?Iab ((\text{inter } ?Iab) \text{ ' } \{\{s <..t\} \mid s t. \text{True}\})$   
*True*)  
**by** (*rule sigma-sets-mono*)  
**qed**  
**finally show**  $\text{sets } (\text{restrict-space } L ?Iab) = \text{sigma-sets } ?Iab ?Iocs$  .  
**qed**

**have** *Int-stable ?Iocs*  
**proof** (*rule Int-stableI*)  
**fix**  $A B$  **assume**  $A \in ?Iocs B \in ?Iocs$   
**from this obtain**  $sA tA sB tB$  **where**  
 $asA: a < \text{ereal } sA$  **and**  $sAtA: sA \leq tA$  **and**  $tAb: \text{ereal } tA < b$  **and**  $A = \{sA <.. tA\}$  **and**  
 $asB: a < \text{ereal } sB$  **and**  $sBtB: sB \leq tB$  **and**  $eBb: \text{ereal } tB < b$  **and**  $B = \{sB <.. tB\}$   
**by** *blast*  
**hence**  $AB: A \cap B = \{\max sA sB <.. \min tA tB\}$  **using** *Int-greaterThanAtMost*  
**by** *blast*  
**thus**  $A \cap B \in ?Iocs$   
**proof** (*cases <max sA sB ≤ min tA tB>*)  
**case** *True*  
**moreover have**  $a < \text{ereal } (\max sA sB) \text{ereal } (\min tA tB) < b$   
**using**  $asA tAb asB eBb$  **by** *linarith+*  
**ultimately show** *?thesis* **using**  $AB$  **by** *blast*  
**next**  
**case** *False*  
**hence**  $A \cap B = \{\}$  **using**  $AB$  **by** *force*  
**thus**  $A \cap B \in ?Iocs$  **using**  $asA$  *less-ereal-le sAtA tAb* **by** *fastforce*  
**qed**  
**qed**  
**moreover note** *Iocs-Iab*  
**moreover have**  $\bigwedge A. A \in ?Iocs \implies$   
 $\text{emeasure } (\text{restrict-space } M ?Iab) A = \text{emeasure } (\text{restrict-space } N ?Iab) A$   
**proof** –  
**fix**  $A$  **assume**  $asm: A \in ?Iocs$   
**hence**  $[simp]: A \subseteq ?Iab$  **using** *Iocs-Iab* **by** *blast*  
**have**  $\text{emeasure } (\text{restrict-space } M ?Iab) A = \text{emeasure } M A$   
**by** (*rewrite emeasure-restrict-space; simp add: assms*)  
**also have**  $\dots = \text{emeasure } N A$  **using**  $assms$   $asm$  **by** *force*

**also have**  $\dots = \text{emeasure } (\text{restrict-space } N \ ?Iab) \ A$   
**by**  $(\text{rewrite } \text{emeasure-restrict-space}; \text{ simp } \text{add: } \text{assms})$   
**finally show**  $\text{emeasure } (\text{restrict-space } M \ ?Iab) \ A = \text{emeasure } (\text{restrict-space } N \ ?Iab) \ A$  .  
**qed**  
**moreover have**  $\text{sets } (\text{restrict-space } M \ ?Iab) = \text{sigma-sets } \ ?Iab \ ?Iocs$  **using**  $\text{Iab-sigma } \text{assms}$  **by**  $\text{simp}$   
**moreover have**  $\text{sets } (\text{restrict-space } N \ ?Iab) = \text{sigma-sets } \ ?Iab \ ?Iocs$  **using**  $\text{Iab-sigma } \text{assms}$  **by**  $\text{simp}$   
**moreover have**  $\text{rng } Iocn: \text{range } \ ?Iocn \subseteq \ ?Iocs$   
**using**  $\text{al } \text{lu } \text{ub}$  **by**  $(\text{smt } (\text{verit}) \ \text{image-subset-iff } \text{mem-Collect-eq})$   
**moreover have**  $(\bigcup n. \ ?Iocn \ n) = \ ?Iab$  **using**  $\text{Iab-Un}$  **by**  $\text{simp}$   
**moreover have**  $\bigwedge n. \ \text{emeasure } (\text{restrict-space } M \ ?Iab) \ (\ ?Iocn \ n) \neq \infty$   
**proof** –  
**fix**  $n::\text{nat}$   
**have**  $\{l \ n <.. \ u \ n\} \subseteq \ ?Iab$  **using**  $\text{Iab-Un}$  **by**  $\text{blast}$   
**moreover have**  $\text{emeasure } M \ \{l \ n <.. \ u \ n\} \neq \infty$  **using**  $\text{assms } \text{al } \text{lu } \text{ub } \text{less-imp-le}$   
**by**  $\text{metis}$   
**ultimately show**  $\text{emeasure } (\text{restrict-space } M \ ?Iab) \ (\ ?Iocn \ n) \neq \infty$   
**by**  $(\text{rewrite } \text{emeasure-restrict-space}; \text{ simp } \text{add: } \text{assms})$   
**qed**  
**ultimately show**  $\text{restrict-space } M \ ?Iab = \text{restrict-space } N \ ?Iab$   
**by**  $(\text{rule } \text{measure-eqI-generator-eq}[\text{where } A = \ ?Iocn]; \text{ simp})$

**next**

**case**  $\text{False}$   
**hence**  $\text{einterval } a \ b = \{\}$  **using**  $\text{einterval-empty}$  **by**  $\text{simp}$   
**thus**  $\ ?thesis$  **by**  $(\text{metis } \text{sets.empty-sets } \text{space-empty-eq-bot } \text{space-restrict-space2})$

**qed**

**lemma**  $\text{nn-integral-disjoint-pair2}$ :

**assumes**  $B \in \text{sets } M \ C \in \text{sets } M \ B \cap C = \{\}$  **and**  
 $[\text{measurable}]: (\lambda x. \ f \ x * \ \text{indicator } B \ x) \in \text{borel-measurable } M$  **and**  
 $[\text{measurable}]: (\lambda x. \ f \ x * \ \text{indicator } C \ x) \in \text{borel-measurable } M$   
**shows**  $(\int^{+x} \in B \cup C. \ f \ x \ \partial M) = (\int^{+x} \in B. \ f \ x \ \partial M) + (\int^{+x} \in C. \ f \ x \ \partial M)$   
**proof** –  
**have**  $[\text{measurable}]: (\lambda x. \ f \ x * \ \text{indicator } (B \cup C) \ x) \in \text{borel-measurable } M$   
**using**  $\text{assms}$  **apply**  $(\text{rewrite } \text{indicator-add}[\text{THEN } \text{sym}], \text{ simp})$   
**apply**  $(\text{rewrite } \text{distrib-left})$  **by**  $\text{simp}$   
**have**  $(\int^{+x} \in B \cup C. \ f \ x \ \partial M) = (\int^{+x} \in B \cup C. \ f \ x * \ \text{indicator } (B \cup C) \ x \ \partial M)$   
**by**  $(\text{metis } \text{indicator-simps } \text{mult-1-right } \text{mult-eq-0-iff})$   
**also have**  $\dots = (\int^{+x \in B. \ f \ x * \ \text{indicator } (B \cup C) \ x \ \partial M) + (\int^{+x \in C. \ f \ x * \ \text{indicator } (B \cup C) \ x \ \partial M)$   
**using**  $\text{assms}$  **by**  $(\text{rewrite } \text{nn-integral-disjoint-pair}; \text{ simp})$   
**moreover have**  $(\int^{+x \in B. \ f \ x * \ \text{indicator } (B \cup C) \ x \ \partial M) = (\int^{+x \in B. \ f \ x \ \partial M)$   
**by**  $(\text{metis } (\text{mono-tags}, \text{lifting}) \ \text{indicator-inter-arith } \text{inf-sup-absorb } \text{mult.assoc } \text{mult commute})$

**moreover have**  $(\int^{+x \in C}. f x * \text{indicator } (B \cup C) x \partial M) = (\int^{+x \in C}. f x \partial M)$   
**by** (*metis (no-types, lifting) indicator-inter-arith inf-sup-absorb mult.commute*  
*mult.left-commute sup-commute*)  
**ultimately show** *?thesis* **by** *simp*  
**qed**

**lemma** *set-nn-integral-interval-measure-bounded-finite:*

**fixes**  $F :: \text{real} \Rightarrow \text{real}$  **and**  $h :: \text{real} \Rightarrow \text{ennreal}$  **and**  $A :: \text{real set}$  **and**  $M :: \text{real}$   
**assumes** *bounded A*  $\bigwedge x. x \in A \implies h x \leq M$   $A \in \text{sets borel}$  **and**  
*mono F*  $\bigwedge x. \text{continuous (at-right } x) F$   
**shows**  $(\int^{+x \in A}. h x \partial(\text{interval-measure } F)) < \infty$

**proof** –

**let**  $?IMF = \text{interval-measure } F$   
**obtain**  $a$  **where**  $a\text{-ub}: \bigwedge x. x \in A \implies |x| \leq a$   
**using** *bounded-iff assms* **by** *force*  
**define**  $c$  **where**  $c \equiv |a| + 1$   
**have** [*simp*]:  $c > 0$   
**unfolding** *c-def* **by** *auto*  
**have**  $Ac: A \subseteq \{-c <..c\}$   
**using** *a-ub c-def* **by** *force*  
**have**  $(\int^{+x \in A}. h x \partial ?IMF) \leq (\int^{+x \in A}. M \partial ?IMF)$   
**using** *nn-set-integral-mono assms* **by** (*simp add: indicator-def nn-integral-mono*)  
**also have**  $\dots \leq (\int^{+x \in \{-c <..c\}}. M \partial ?IMF)$   
**using** *nn-set-integral-set-mono Ac* **by** *force*  
**also have**  $\dots = M * \text{emeasure } ?IMF \{-c <..c\}$   
**by** (*rewrite nn-integral-cmult-indicator; simp*)  
**also have**  $\dots \leq |M| * \text{emeasure } ?IMF \{-c <..c\}$   
**using** *abs-ge-self* **by** (*simp add: mult-right-mono*)  
**also have**  $\text{emeasure } ?IMF \{-c <..c\} = F c - F (-c)$   
**by** (*simp add: emeasure-interval-measure-Ioc-eq ennreal-eq-0-iff monoD assms*)  
**finally have**  $(\int^{+x \in A}. h x \partial ?IMF) \leq |M| * (F c - F (-c))$   
**by** (*simp add: ennreal-mult*)  
**then show** *?thesis*  
**by** (*simp add: le-less-trans*)

**qed**

**end**

**theory** *Lebesgue-Stieltjes-Integral*

**imports** *Wlog.Wlog Preliminaries-LSI*

**begin**

## 1 Interval Measure Integral

### 1.1 Basic Calculations

**lemma** *interval-measure-const-null:*

**fixes**  $c :: \text{real}$   
**shows** *interval-measure*  $(\lambda-. c) = \text{null-measure lborel}$   
**proof** –

**have**  $\bigwedge s t. s \leq t \implies \text{emeasure } (\text{interval-measure } (\lambda-. c)) \{s <..t\} = 0$   
**using** *emeasure-interval-measure-Ioc* **by** *force*  
**then show** *?thesis* **by** (*intro measure-eqI-Ioc; simp*)  
**qed**

**lemma** *interval-measure-singleton*:  
**fixes**  $F :: \text{real} \implies \text{real}$  **and**  $s :: \text{real}$   
**assumes** *mono F*  $\bigwedge x. \text{continuous } (\text{at-right } x) F$   
**shows**  $(\text{interval-measure } F) \{s\} = F s - \text{Lim } (\text{at-left } s) F$   
**proof** –  
**let**  $?dF = \text{interval-measure } F$   
**let**  $?I = \lambda n :: \text{nat}. \{s - 1 / (n+1) <..s\}$   
**have**  $\text{limsn}: (\lambda n. s - 1 / (\text{real } n + 1)) \longrightarrow s$  **by** *real-asymp*  
**have**  $\{s\} = (\bigcap (\text{range } ?I))$   
**proof**  
**have**  $\bigwedge n. s \in ?I n$  **by** *simp*  
**thus**  $\{s\} \subseteq (\bigcap (\text{range } ?I))$  **by** *simp*  
**show**  $(\bigcap (\text{range } ?I)) \subseteq \{s\}$   
**proof**  
**fix**  $x$  **assume**  $x \in (\bigcap (\text{range } ?I))$   
**hence**  $x-I: \bigwedge n. x \in ?I n$  **by** *simp*  
**hence**  $\bigwedge n. s - 1 / (\text{real } n + 1) < x$  **by** *simp*  
**hence**  $s \leq x$   
**apply** (*intro LIMSEQ-le-const2[OF limsn]*)  
**using** *less-eq-real-def* **by** (*intro exI[of - 0]*) *simp*  
**moreover** **have**  $x \leq s$  **using**  $x-I$  **by** *simp*  
**ultimately show**  $x \in \{s\}$  **by** *simp*  
**qed**  
**qed**  
**moreover** **have**  $(\lambda n. \text{emeasure } ?dF (?I n)) \longrightarrow \text{emeasure } ?dF (\bigcap (\text{range } ?I))$   
**proof** (*rule Lim-emeasure-decseq*)  
**show**  $\text{range } ?I \subseteq \text{sets } ?dF$  **by** (*rewrite sets-interval-measure*) *force*  
**show** *monotone*  $(\leq) (\lambda J K. K \subseteq J) ?I$   
**proof** (*rule monotoneI*)  
**fix**  $m n :: \text{nat}$  **assume**  $m \leq n$   
**hence**  $s - 1 / (\text{real } m + 1) \leq s - 1 / (\text{real } n + 1)$   
**by** (*metis add.commute of-nat-Suc Suc-le-mono diff-left-mono inverse-of-nat-le Suc-not-Zero*)  
**thus**  $?I n \subseteq ?I m$  **by** *force*  
**qed**  
**show**  $\bigwedge n. \text{emeasure } ?dF (?I n) \neq \infty$   
**using** *emeasure-interval-measure-Ioc* **by** (*simp add: assms monoD*)  
**qed**  
**moreover** **have**  $(\lambda n. ?dF (?I n)) \longrightarrow F s - \text{Lim } (\text{at-left } s) F$   
**proof** –  
**have**  $(F \longrightarrow \text{Sup } (F \text{ ‘ } \{..<s\})) (\text{at-left } s)$   
**using** *Lim-left-bound*[*of UNIV s F F s*] *monoD*[*OF ‹mono F›*] **by** *simp*  
**hence**  $(F \longrightarrow \text{Lim } (\text{at-left } s) F) (\text{at-left } s)$  **by** (*simp add: tendsto-Lim*)

**hence**  $(\lambda n::nat. F (s - 1 / (real\ n + 1))) \longrightarrow Lim\ (at\text{-}left\ s)\ F$   
**apply** *(rewrite in asm tendsto-at-iff-sequentially)*  
**apply** *(drule spec[of -  $\lambda n. s - 1 / (real\ n + 1)$ ])*  
**unfolding comp-def using limsn by simp**  
**hence**  $(\lambda n::nat. ennreal\ (F\ s - F\ (s - 1 / (real\ n + 1))))$   
 $\longrightarrow\ ennreal\ (F\ s - Lim\ (at\text{-}left\ s)\ F)$   
**by** *(intro tendsto-intros)*  
**thus ?thesis by** *(rewrite emeasure-interval-measure-Ioc; simp add: assms monoD)*  
**qed**  
**ultimately show ?thesis using LIMSEQ-unique by simp**  
**qed**

**lemma interval-measure-singleton-continuous:**  
**fixes**  $F :: real \Rightarrow real$  **and**  $s::real$   
**assumes**  $mono\ F \wedge x. continuous\ (at\text{-}right\ x)\ F$  **isCont**  $F\ s$   
**shows**  $(interval\text{-}measure\ F)\ \{s\} = 0$   
**proof** –  
**have**  $Lim\ (at\text{-}left\ s)\ F = F\ s$   
**using** *assms continuous-at-imp-continuous-within continuous-within*  
*tendsto-Lim trivial-limit-at-left-real* **by blast**  
**with interval-measure-singleton assms show ?thesis by simp**  
**qed**

## 1.2 Changing the Underlying Function

**lemma einterval-nn-integral-interval-measure-cong:**  
**fixes**  $F\ G :: real \Rightarrow real$  **and**  $h :: real \Rightarrow ennreal$  **and**  $a\ b :: ereal$   
**assumes**  $mono\ F \wedge x. continuous\ (at\text{-}right\ x)\ F$  **and**  
 $mono\ G \wedge x. continuous\ (at\text{-}right\ x)\ G$  **and**  
 $(F - G)\ constant\text{-}on\ (einterval\ a\ b)$  **and**  
 $h \in borel\text{-}measurable\ borel$   
**shows**  $(\int^{+x \in (einterval\ a\ b). h\ x}\ \partial(interval\text{-}measure\ F)) =$   
 $(\int^{+x \in (einterval\ a\ b). h\ x}\ \partial(interval\text{-}measure\ G))$   
**proof** –  
**let**  $?IM = interval\text{-}measure$   
**let**  $?Iab = einterval\ a\ b$   
**from assms obtain c where**  $FG\text{-}c: \wedge x. x \in ?Iab \implies F\ x - G\ x = c$   
**unfolding constant-on-def by force**  
**have**  $\wedge s\ t. a < ereal\ s \wedge s \leq t \wedge ereal\ t < b \implies emeasure\ (?IM\ F)\ \{s<..t\} \neq$   
 $\infty$   
**using assms unfolding mono-def by** *(rewrite emeasure-interval-measure-Ioc;*  
*simp)*  
**moreover have**  $\wedge s\ t. a < ereal\ s \wedge s \leq t \wedge ereal\ t < b \implies$   
 $emeasure\ (?IM\ F)\ \{s<..t\} = emeasure\ (?IM\ G)\ \{s<..t\}$   
**proof** –  
**fix**  $s\ t$  **assume**  $astb: a < ereal\ s \wedge s \leq t \wedge ereal\ t < b$   
**hence**  $F\ t - F\ s = G\ t - G\ s$   
**using**  $FG\text{-}c$  **by** *(smt (verit) einterval-iff ereal-less-le less-ereal-le)*  
**thus**  $emeasure\ (?IM\ F)\ \{s<..t\} = emeasure\ (?IM\ G)\ \{s<..t\}$

**using** *assms astb unfolding mono-def* **by** (*rewrite emeasure-interval-measure-Ioc;*  
*simp*)<sup>+</sup>  
**qed**  
**ultimately have** *restrict-space (?IM F) ?Iab = restrict-space (?IM G) ?Iab*  
**by** (*intro measure-einterval-eqI-Ioc; simp*)  
**thus** *?thesis using nn-integral-restrict-space[THEN sym]*  
**by** (*rewrite nn-integral-restrict-space[THEN sym], simp*)<sup>+</sup> *simp*  
**qed**

**corollary** *Ioo-nn-integral-interval-measure-cong:*

**fixes**  $F G :: \text{real} \Rightarrow \text{real}$  **and**  $h :: \text{real} \Rightarrow \text{ennreal}$  **and**  $r s :: \text{real}$   
**assumes** *mono F  $\wedge$  x. continuous (at-right x) F* **and**  
*mono G  $\wedge$  x. continuous (at-right x) G* **and**  
*(F - G) constant-on {r<.. **and**  
*h  $\in$  borel-measurable borel*  
**shows**  $(\int^+ x \in \{r < .. < s\}. h x \partial(\text{interval-measure } F)) = (\int^+ x \in \{r < .. < s\}. h x$   
 $\partial(\text{interval-measure } G))$   
**using** *einterval-eq-Icc assms einterval-nn-integral-interval-measure-cong*  
**by** (*metis (mono-tags, lifting) nn-integral-cong*)*

**corollary** *Ioi-nn-integral-interval-measure-cong:*

**fixes**  $F G :: \text{real} \Rightarrow \text{real}$  **and**  $h :: \text{real} \Rightarrow \text{ennreal}$  **and**  $r :: \text{real}$   
**assumes** *mono F  $\wedge$  x. continuous (at-right x) F* **and**  
*mono G  $\wedge$  x. continuous (at-right x) G* **and**  
*(F - G) constant-on {r<..}* **and**  
*h  $\in$  borel-measurable borel*  
**shows**  $(\int^+ x \in \{r < ..\}. h x \partial(\text{interval-measure } F)) = (\int^+ x \in \{r < ..\}. h x \partial(\text{interval-measure}$   
 $G))$   
**using** *einterval-eq-Ici assms einterval-nn-integral-interval-measure-cong*  
**by** (*metis (mono-tags, lifting) nn-integral-cong*)

**corollary** *Iio-nn-integral-interval-measure-cong:*

**fixes**  $F G :: \text{real} \Rightarrow \text{real}$  **and**  $h :: \text{real} \Rightarrow \text{ennreal}$  **and**  $s :: \text{real}$   
**assumes** *mono F  $\wedge$  x. continuous (at-right x) F* **and**  
*mono G  $\wedge$  x. continuous (at-right x) G* **and**  
*(F - G) constant-on {.. **and**  
*h  $\in$  borel-measurable borel*  
**shows**  $(\int^+ x \in \{..  
 $G))$   
**using** *einterval-eq-Iic assms einterval-nn-integral-interval-measure-cong*  
**by** (*metis (mono-tags, lifting) nn-integral-cong*)$*

**corollary** *nn-integral-interval-measure-cong:*

**fixes**  $F G :: \text{real} \Rightarrow \text{real}$  **and**  $h :: \text{real} \Rightarrow \text{ennreal}$   
**assumes** *mono F  $\wedge$  x. continuous (at-right x) F* **and**  
*mono G  $\wedge$  x. continuous (at-right x) G* **and**  
*(F - G) constant-on UNIV* **and**  
*h  $\in$  borel-measurable borel*  
**shows**  $(\int^+ x. h x \partial(\text{interval-measure } F)) = (\int^+ x. h x \partial(\text{interval-measure } G))$

using *einterval-nn-integral-interval-measure-cong einterval-eq-UNIV assms*  
 by (*metis (mono-tags, lifting) mult-1-right*  
*indicator-eq-1-iff nn-integral-cong space-interval-measure*)

**lemma** *singleton-nn-integral-interval-measure-cong*:

**fixes**  $F G :: \text{real} \Rightarrow \text{real}$  **and**  $h :: \text{real} \Rightarrow \text{ennreal}$  **and**  $s :: \text{real}$

**assumes**  $\text{mono } F \wedge x. \text{continuous (at-right } x) F$  **and**

$\text{mono } G \wedge x. \text{continuous (at-right } x) G$  **and**

$F s - \text{Lim (at-left } s) F = G s - \text{Lim (at-left } s) G$  **and**

$h \in \text{borel-measurable borel}$

**shows**  $(\int^+ x \in \{s\}. h x \partial(\text{interval-measure } F)) = (\int^+ x \in \{s\}. h x \partial(\text{interval-measure } G))$

**using** *interval-measure-singleton assms* **by** (*rewrite nn-integral-indicator-singleton; simp*)

**lemma** *singleton-const-nn-integral-interval-measure-cong*:

**fixes**  $F G :: \text{real} \Rightarrow \text{real}$  **and**  $h :: \text{real} \Rightarrow \text{ennreal}$  **and**  $r s :: \text{real}$

**assumes**  $\text{mono } F \wedge x. \text{continuous (at-right } x) F$  **and**

$\text{mono } G \wedge x. \text{continuous (at-right } x) G$  **and**

$(F - G) \text{ constant-on } \{r <..s\}$  **and**  $r < s$  **and**

$h \in \text{borel-measurable borel}$

**shows**  $(\int^+ x \in \{s\}. h x \partial(\text{interval-measure } F)) = (\int^+ x \in \{s\}. h x \partial(\text{interval-measure } G))$

**proof** –

**have** [*simp*]:  $\{..<s\} \cap \{r<..\} = \{r<.. **using** *assms* **by** *force*$

**have** [*simp*]: *at s within*  $\{r<..$

**using** *at-within-Ioo-at-left assms trivial-limit-at-left-real* **by** *metis*

**from** *assms* **obtain**  $c :: \text{real}$  **where**  $FGc: \wedge x. x \in \{r<..$

**unfolding** *constant-on-def* **by** *force*

**have**  $\text{Lim (at } s \text{ within } \{r<..$

**proof** –

**have**  $(F \longrightarrow \text{Sup } (F \text{ ' } (\{..<s\} \cap \{r<..\})))$  (*at s within*  $\{..<s\} \cap \{r<..\}$ )

**apply** (*rule Lim-left-bound*[**where**  $I=\{r<..\}$  **and**  $K=F s$ ])

**using** *assms monoD* **by** *force+*

**thus** *?thesis* **by** (*intro tendsto-Lim; simp*)

**qed**

**moreover** **have**  $\text{Lim (at } s \text{ within } \{r<..$

**proof** –

**have**  $(G \longrightarrow \text{Sup } (G \text{ ' } (\{..<s\} \cap \{r<..\})))$  (*at s within*  $\{..<s\} \cap \{r<..\}$ )

**apply** (*rule Lim-left-bound*[**where**  $I=\{r<..\}$  **and**  $K=G s$ ])

**using** *assms monoD* **by** *force+*

**thus** *?thesis* **by** (*intro tendsto-Lim; simp*)

**qed**

**moreover** **have**  $\text{Sup } (F \text{ ' } \{r<..$

**proof** –

**have** *bdd-above*  $(G \text{ ' } \{r<.. **using** *assms* **by** (*metis bdd-above-Ioo bdd-above-image-mono*)$

**moreover** **have**  $\wedge x. x \in \{r<.. **using** *FGc* **by** *force*$

**ultimately** **show** *?thesis* **using** *FGc assms* **by** (*rewrite Sup-add-eq*[*THEN sym*]; *simp*)

**qed**  
**ultimately have**  $\text{Lim (at-left } s) F - \text{Lim (at-left } s) G = c$   
**using** *at-within-Ioo-at-left assms* **by force**  
**hence**  $F s - \text{Lim (at-left } s) F = G s - \text{Lim (at-left } s) G$  **using** *FGc assms* **by**  
*fastforce*  
**thus** *?thesis* **by** (*intro singleton-nn-integral-interval-measure-cong; simp add:*  
*assms*)  
**qed**

**lemma** *Ioc-nn-integral-interval-measure-cong*:  
**fixes**  $F G :: \text{real} \Rightarrow \text{real}$  **and**  $h :: \text{real} \Rightarrow \text{ennreal}$  **and**  $r s :: \text{real}$   
**assumes** *mono F*  $\wedge x.$  *continuous (at-right x) F* **and**  
*mono G*  $\wedge x.$  *continuous (at-right x) G* **and**  
 $(F - G)$  *constant-on*  $\{r <..s\}$  **and**  
 $h \in \text{borel-measurable borel}$   
**shows**  $(\int^{+x \in \{r <..s\}}. h x \partial(\text{interval-measure } F)) = (\int^{+x \in \{r <..s\}}. h x \partial(\text{interval-measure } G))$   
**proof** –  
**let**  $?IM = \text{interval-measure}$   
**show** *?thesis*  
**proof** (*cases*  $\langle r < s \rangle$ )  
**case** *True*  
**hence** [*simp*]: *insert s*  $\{r <..<s\} = \{r <..s\}$  **by force**  
**have**  $(\int^{+x \in \{r <..<s\}}. h x \partial(?IM F)) = (\int^{+x \in \{r <..<s\}}. h x \partial(?IM G))$   
**proof** –  
**have**  $\{r <..<s\} = \text{einterval (ereal } r) (\text{ereal } s)$  **by** *simp*  
**moreover have**  $(F - G)$  *constant-on*  $(\text{einterval (ereal } r) (\text{ereal } s))$   
**proof** –  
**have**  $(\text{einterval (ereal } r) (\text{ereal } s)) \subseteq \{r <..s\}$  **by force**  
**with** *assms constant-on-subset* **show** *?thesis* **by** *simp*  
**qed**  
**ultimately show** *?thesis* **using** *assms einterval-nn-integral-interval-measure-cong*  
**by** *presburger*  
**qed**  
**moreover have**  $(\int^{+x \in \{s\}}. h x \partial(?IM F)) = (\int^{+x \in \{s\}}. h x \partial(?IM G))$   
**by** (*rule singleton-const-nn-integral-interval-measure-cong*[**where**  $r=r$ ]; *simp*  
*add: assms True*)  
**ultimately have**  $(\int^{+x \in \{r <..<s\}}. h x \partial(?IM F)) + (\int^{+x \in \{s\}}. h x \partial(?IM F)) =$   
 $(\int^{+x \in \{r <..<s\}}. h x \partial(?IM G)) + (\int^{+x \in \{s\}}. h x \partial(?IM G))$   
**by** *simp*  
**moreover have**  
 $\wedge H. (\int^{+x \in \{r <..<s\}}. h x \partial(?IM H)) + (\int^{+x \in \{s\}}. h x \partial(?IM H)) =$   
 $(\int^{+x \in \{r <..s\}}. h x \partial(?IM H))$   
**by** (*rewrite nn-integral-disjoint-pair*[*THEN sym*]; *measurable; simp add: assms*)  
**ultimately show** *?thesis* **by** *simp*  
**next**  
**case** *False*  
**hence**  $\{r <..s\} = \{\}$  **using** *greaterThanAtMost-empty-iff* **by** *simp*

thus ?thesis by simp  
qed  
qed

**lemma** *Iic-nn-integral-interval-measure-cong*:

fixes  $F G :: \text{real} \Rightarrow \text{real}$  and  $h :: \text{real} \Rightarrow \text{ennreal}$  and  $s :: \text{real}$

assumes  $\text{mono } F \wedge x. \text{continuous (at-right } x) F$  and

$\text{mono } G \wedge x. \text{continuous (at-right } x) G$  and

$(F - G) \text{ constant-on } \{..s\}$  and

$h \in \text{borel-measurable borel}$

shows  $(\int^{+x \in \{..s\}}. h x \partial(\text{interval-measure } F)) = (\int^{+x \in \{..s\}}. h x \partial(\text{interval-measure } G))$

**proof** –

let  $?IM = \text{interval-measure}$

have  $[simp]: \text{insert } s \{..<s\} = \{..s\}$  by force

have  $(\int^{+x \in \{..<s\}}. h x \partial(?IM F)) = (\int^{+x \in \{..<s\}}. h x \partial(?IM G))$

**proof** –

have  $\{..<s\} = \text{einterval } (-\infty) (\text{ereal } s)$  by simp

moreover have  $(F - G) \text{ constant-on } (\text{einterval } (-\infty) (\text{ereal } s))$

**proof** –

have  $(\text{einterval } (-\infty) (\text{ereal } s)) \subseteq \{..s\}$  by force

with *assms constant-on-subset* show ?thesis by simp

qed

ultimately show ?thesis using *assms einterval-nn-integral-interval-measure-cong*

by *presburger*

qed

moreover have  $(\int^{+x \in \{s\}}. h x \partial(?IM F)) = (\int^{+x \in \{s\}}. h x \partial(?IM G))$

**proof** –

have  $(F - G) \text{ constant-on } \{s-1 <..s\}$

using *assms constant-on-subset* by (*metis atMost-iff greaterThanAtMost-iff subset-eq*)

thus ?thesis

by (*intro singleton-const-nn-integral-interval-measure-cong*[**where**  $r=s-1$ ];

*simp add: assms*)

qed

ultimately have  $(\int^{+x \in \{..<s\}}. h x \partial(?IM F)) + (\int^{+x \in \{s\}}. h x \partial(?IM F)) =$   
 $(\int^{+x \in \{..<s\}}. h x \partial(?IM G)) + (\int^{+x \in \{s\}}. h x \partial(?IM G))$

by *simp*

moreover have

$\wedge H. (\int^{+x \in \{..<s\}}. h x \partial(?IM H)) + (\int^{+x \in \{s\}}. h x \partial(?IM H)) = (\int^{+x \in \{..s\}}. h x \partial(?IM H))$

by (*rewrite nn-integral-disjoint-pair*[*THEN sym*]; *measurable; simp add: assms*)

ultimately show ?thesis by *simp*

qed

**lemma** *Ico-nn-integral-interval-measure-cong*:

fixes  $F G :: \text{real} \Rightarrow \text{real}$  and  $h :: \text{real} \Rightarrow \text{ennreal}$  and  $r s :: \text{real}$

assumes  $\text{mono } F \wedge x. \text{continuous (at-right } x) F$  and

$\text{mono } G \wedge x. \text{continuous (at-right } x) G$  and

**( $F - G$ ) constant-on  $\{r <..<s\}$  and**  
 **$F r - \text{Lim (at-left } r) F = G r - \text{Lim (at-left } r) G$  and**  
 **$h \in \text{borel-measurable borel}$**   
**shows  $(\int^+ x \in \{r <..<s\}. h x \partial(\text{interval-measure } F)) = (\int^+ x \in \{r <..<s\}. h x \partial(\text{interval-measure } G))$**   
**proof –**  
**let  $?IM = \text{interval-measure}$**   
**show  $?thesis$**   
**proof (cases  $\langle r < s \rangle$ )**  
**case  $\text{True}$**   
**hence [simp]: insert  $r \{r <..<s\} = \{r <..<s\}$  by force**  
**have  $\bigwedge H. (\int^+ x \in \{r <..<s\}. h x \partial(?IM H)) = (\int^+ x \in \{r\}. h x \partial(?IM H)) + (\int^+ x \in \{r <..<s\}. h x \partial(?IM H))$**   
**by (rewrite nn-integral-disjoint-pair[THEN sym]; measurable; simp add: assms)**  
**moreover have  $(\int^+ x \in \{r\}. h x \partial(?IM F)) = (\int^+ x \in \{r\}. h x \partial(?IM G))$**   
**by (rewrite singleton-nn-integral-interval-measure-cong; measurable; simp add: assms)**  
**moreover have  $(\int^+ x \in \{r <..<s\}. h x \partial(?IM F)) = (\int^+ x \in \{r <..<s\}. h x \partial(?IM G))$**   
**apply (rewrite einterval-eq-Icc[THEN sym])+**  
**by (rewrite einterval-nn-integral-interval-measure-cong; measurable; simp add: assms)**  
**ultimately show  $?thesis$  by simp**  
**next**  
**case  $\text{False}$**   
**thus  $?thesis$  by simp**  
**qed**  
**qed**

**corollary  $\text{Ico-Cont-nn-integral-interval-measure-cong}$ :**  
**fixes  $F G :: \text{real} \Rightarrow \text{real}$  and  $h :: \text{real} \Rightarrow \text{ennreal}$  and  $r s :: \text{real}$**   
**assumes  $\text{mono } F \bigwedge x. \text{continuous (at-right } x) F$  and**  
 **$\text{mono } G \bigwedge x. \text{continuous (at-right } x) G$  and**  
 **$(F - G)$  constant-on  $\{r <..<s\}$  and**  
 **$\text{isCont } F r \text{ isCont } G r$  and**  
 **$h \in \text{borel-measurable borel}$**   
**shows  $(\int^+ x \in \{r <..<s\}. h x \partial(\text{interval-measure } F)) = (\int^+ x \in \{r <..<s\}. h x \partial(\text{interval-measure } G))$**   
**proof –**  
**from assms have  $F r - \text{Lim (at-left } r) F = G r - \text{Lim (at-left } r) G$**   
**by (metis add-0 continuous-at-imp-continuous-at-within continuous-within eq-diff-eq tendsto-Lim trivial-limit-at-left-real)**  
**thus  $?thesis$**   
**using assms by (intro Ico-nn-integral-interval-measure-cong; simp)**  
**qed**

**lemma  $\text{Ici-nn-integral-interval-measure-cong}$ :**  
**fixes  $F G :: \text{real} \Rightarrow \text{real}$  and  $h :: \text{real} \Rightarrow \text{ennreal}$  and  $r :: \text{real}$**   
**assumes  $\text{mono } F \bigwedge x. \text{continuous (at-right } x) F$  and**

**mono**  $G \wedge x$ . *continuous (at-right x)*  $G$  **and**  
 $(F - G)$  *constant-on {r<..}* **and**  
 $F r - \text{Lim (at-left r)}$   $F = G r - \text{Lim (at-left r)}$   $G$  **and**  
 $h \in \text{borel-measurable borel}$   
**shows**  $(\int^+ x \in \{r..\}. h x \partial(\text{interval-measure } F)) = (\int^+ x \in \{r..\}. h x \partial(\text{interval-measure } G))$   
**proof** –  
**let**  $?IM = \text{interval-measure}$   
**have**  $[simp]: \text{insert } r \{r<..\} = \{r..\}$  **by force**  
**have**  $\wedge H. (\int^+ x \in \{r..\}. h x \partial(?IM H)) = (\int^+ x \in \{r\}. h x \partial(?IM H)) + (\int^+ x \in \{r<..\}. h x \partial(?IM H))$   
**by**  $(\text{rewrite nn-integral-disjoint-pair}[THEN sym]; \text{measurable}; \text{simp add: assms})$   
**moreover have**  $(\int^+ x \in \{r\}. h x \partial(?IM F)) = (\int^+ x \in \{r\}. h x \partial(?IM G))$   
**by**  $(\text{rewrite singleton-nn-integral-interval-measure-cong}; \text{measurable}; \text{simp add: assms})$   
**moreover have**  $(\int^+ x \in \{r<..\}. h x \partial(?IM F)) = (\int^+ x \in \{r<..\}. h x \partial(?IM G))$   
**apply**  $(\text{rewrite einterval-eq-Ici}[THEN sym])+$   
**by**  $(\text{rewrite einterval-nn-integral-interval-measure-cong}; \text{measurable}; \text{simp add: assms})$   
**ultimately show**  $?thesis$  **by simp**  
**qed**

**corollary** *Ici-Cont-nn-integral-interval-measure-cong*:  
**fixes**  $F G :: \text{real} \Rightarrow \text{real}$  **and**  $h :: \text{real} \Rightarrow \text{ennreal}$  **and**  $r :: \text{real}$   
**assumes** **mono**  $F \wedge x$ . *continuous (at-right x)*  $F$  **and**  
**mono**  $G \wedge x$ . *continuous (at-right x)*  $G$  **and**  
 $(F - G)$  *constant-on {r<..}* **and**  
 $\text{isCont } F r \text{ isCont } G r$  **and**  
 $h \in \text{borel-measurable borel}$   
**shows**  $(\int^+ x \in \{r..\}. h x \partial(\text{interval-measure } F)) = (\int^+ x \in \{r..\}. h x \partial(\text{interval-measure } G))$   
**proof** –  
**from**  $assms$  **have**  $F r - \text{Lim (at-left r)}$   $F = G r - \text{Lim (at-left r)}$   $G$   
**by**  $(\text{metis continuous-at-imp-continuous-within continuous-within tendsto-Lim trivial-limit-at-left-real diff-self})$   
**thus**  $?thesis$   
**using**  $assms$  **by**  $(\text{intro Ici-nn-integral-interval-measure-cong}; \text{simp})$   
**qed**

**lemma** *Icc-nn-integral-interval-measure-cong*:  
**fixes**  $F G :: \text{real} \Rightarrow \text{real}$  **and**  $h :: \text{real} \Rightarrow \text{ennreal}$  **and**  $r s :: \text{real}$   
**assumes** **mono**  $F \wedge x$ . *continuous (at-right x)*  $F$  **and**  
**mono**  $G \wedge x$ . *continuous (at-right x)*  $G$  **and**  
 $(F - G)$  *constant-on {r<..s}* **and**  
 $F r - \text{Lim (at-left r)}$   $F = G r - \text{Lim (at-left r)}$   $G$  **and**  
 $h \in \text{borel-measurable borel}$   
**shows**  $(\int^+ x \in \{r..s\}. h x \partial(\text{interval-measure } F)) = (\int^+ x \in \{r..s\}. h x \partial(\text{interval-measure } G))$   
**proof** –

**let**  $?IM = interval\text{-}measure$   
**show**  $?thesis$   
**proof** ( $cases \langle r \leq s \rangle$ )  
  **case**  $True$   
    **hence** [ $simp$ ]:  $insert\ r\ \{r <..s\} = \{r..s\}$  **by**  $force$   
    **have**  $\bigwedge H. (\int^{+x \in \{r..s\}}. h\ x\ \partial(?IM\ H)) = (\int^{+x \in \{r\}}. h\ x\ \partial(?IM\ H)) +$   
     $(\int^{+x \in \{r <..s\}}. h\ x\ \partial(?IM\ H))$   
    **by** ( $rewrite\ nn\text{-}integral\text{-}disjoint\text{-}pair[THEN\ sym];\ measurable;\ simp\ add:\ assms$ )  
    **moreover** **have**  $(\int^{+x \in \{r\}}. h\ x\ \partial(?IM\ F)) = (\int^{+x \in \{r\}}. h\ x\ \partial(?IM\ G))$   
    **by** ( $rewrite\ singleton\text{-}nn\text{-}integral\text{-}interval\text{-}measure\text{-}cong;\ measurable;\ simp\ add:\$   
     $assms$ )  
    **moreover** **have**  $(\int^{+x \in \{r <..s\}}. h\ x\ \partial(?IM\ F)) = (\int^{+x \in \{r <..s\}}. h\ x\ \partial(?IM$   
     $G))$   
    **by** ( $rewrite\ Ioc\text{-}nn\text{-}integral\text{-}interval\text{-}measure\text{-}cong;\ measurable;\ simp\ add:\$   
     $assms$ )  
    **ultimately** **show**  $?thesis$  **by**  $simp$   
  **next**  
  **case**  $False$   
  **thus**  $?thesis$  **by**  $simp$   
  **qed**  
**qed**

**corollary**  $Icc\text{-}Cont\text{-}nn\text{-}integral\text{-}interval\text{-}measure\text{-}cong$ :  
**fixes**  $F\ G :: real \Rightarrow real$  **and**  $h :: real \Rightarrow ennreal$  **and**  $r\ s :: real$   
**assumes**  $mono\ F \bigwedge x. continuous\ (at\text{-}right\ x)\ F$  **and**  
   $mono\ G \bigwedge x. continuous\ (at\text{-}right\ x)\ G$  **and**  
   $(F - G)\ constant\text{-}on\ \{r <..s\}$  **and**  
   $isCont\ F\ r\ isCont\ G\ r$  **and**  
   $h \in borel\text{-}measurable\ borel$   
**shows**  $(\int^{+x \in \{r..s\}}. h\ x\ \partial(interval\text{-}measure\ F)) = (\int^{+x \in \{r..s\}}. h\ x\ \partial(interval\text{-}measure$   
 $G))$   
**proof** –  
  **from**  $assms$  **have**  $F\ r - Lim\ (at\text{-}left\ r)\ F = G\ r - Lim\ (at\text{-}left\ r)\ G$   
  **by** ( $metis\ add\text{-}0\ continuous\text{-}at\text{-}imp\text{-}continuous\text{-}at\text{-}within\ continuous\text{-}within$   
   $eq\text{-}diff\text{-}eq\ tendsto\text{-}Lim\ trivial\text{-}limit\text{-}at\text{-}left\text{-}real$ )  
  **thus**  $?thesis$   
  **using**  $assms$  **by** ( $intro\ Icc\text{-}nn\text{-}integral\text{-}interval\text{-}measure\text{-}cong;\ simp$ )  
**qed**

### 1.3 Restricting the Integral

**lemma**  $nn\text{-}integral\text{-}interval\text{-}measure\text{-}Ici$ :  
**fixes**  $F :: real \Rightarrow real$  **and**  $g :: real \Rightarrow ennreal$  **and**  $r :: real$   
**assumes**  $mono\ F \bigwedge x. continuous\ (at\text{-}right\ x)\ F$   $g \in borel\text{-}measurable\ borel$  **and**  
   $F\ constant\text{-}on\ \{..<r\}$   
**shows**  $(\int^{+x}. g\ x\ \partial(interval\text{-}measure\ F)) = (\int^{+x \in \{r.. \}}. g\ x\ \partial(interval\text{-}measure$   
 $F))$   
**proof** –  
  **let**  $?IMF = interval\text{-}measure\ F$

**have**  $\{..<r\} \cup \{r..\} = UNIV$  **by** *auto*  
**moreover have**  $\{..<r\} \cap \{r..\} = \{\}$  **by** *auto*  
**ultimately have**  $(\int^{+x}. g x \partial?IMF) = (\int^{+x \in \{..<r\}}. g x \partial?IMF) + (\int^{+x \in \{r..\}}. g x \partial?IMF)$   
**using** *assms* **by** (*rewrite nn-integral-disjoint-pair*[*THEN sym*]; *simp*)  
**also have**  $(\int^{+x \in \{..<r\}}. g x \partial?IMF) = 0$   
**using** *assms interval-measure-const-null mono-on-const*  
**by** (*rewrite Ioi-nn-integral-interval-measure-cong*[**where**  $G=\lambda-.0$ ]; *simp add: fun-diff-def*)  
**finally show** *?thesis* **by** *simp*  
**qed**

**lemma** *nn-integral-interval-measure-Ioi*:  
**fixes**  $F :: real \Rightarrow real$  **and**  $g :: real \Rightarrow ennreal$  **and**  $r::real$   
**assumes** *mono F*  $\wedge x.$  *continuous (at-right x) F*  $g \in$  *borel-measurable borel* **and**  
 $F$  *constant-on*  $\{..<r\}$  *isCont F r*  
**shows**  $(\int^{+x}. g x \partial(\text{interval-measure } F)) = (\int^{+x \in \{r<..\}}. g x \partial(\text{interval-measure } F))$

**proof** –  
**let**  $?IMF = \text{interval-measure } F$   
**have** [*simp*]: *insert r*  $\{r<..\} = \{r..\}$  **by** *force*  
**have**  $(\int^{+x}. g x \partial?IMF) = (\int^{+x \in \{r..\}}. g x \partial?IMF)$   
**by** (*rule nn-integral-interval-measure-Ici*; *simp add: assms*)  
**also have**  $\dots = (\int^{+x \in \{r\}}. g x \partial?IMF) + (\int^{+x \in \{r<..\}}. g x \partial?IMF)$   
**using** *assms* **by** (*rewrite nn-integral-disjoint-pair*[*THEN sym*]; *simp*)  
**also have**  $(\int^{+x \in \{r\}}. g x \partial?IMF) = 0$  **using** *assms interval-measure-singleton-continuous*  
**by** *simp*  
**finally show** *?thesis* **by** *simp*  
**qed**

**lemma** *nn-integral-interval-measure-Ici*:  
**fixes**  $F :: real \Rightarrow real$  **and**  $g :: real \Rightarrow ennreal$  **and**  $s::real$   
**assumes** *mono F*  $\wedge x.$  *continuous (at-right x) F*  $g \in$  *borel-measurable borel* **and**  
 $F$  *constant-on*  $\{s<..\}$   
**shows**  $(\int^{+x}. g x \partial(\text{interval-measure } F)) = (\int^{+x \in \{..s\}}. g x \partial(\text{interval-measure } F))$

**proof** –  
**let**  $?IMF = \text{interval-measure } F$   
**have**  $\{..s\} \cup \{s<..\} = UNIV$  **by** *auto*  
**moreover have**  $\{..s\} \cap \{s<..\} = \{\}$  **by** *auto*  
**ultimately have**  $(\int^{+x}. g x \partial?IMF) = (\int^{+x \in \{..s\}}. g x \partial?IMF) + (\int^{+x \in \{s<..\}}. g x \partial?IMF)$   
**using** *assms* **by** (*rewrite nn-integral-disjoint-pair*[*THEN sym*]; *simp*)  
**also have**  $(\int^{+x \in \{s<..\}}. g x \partial?IMF) = 0$   
**using** *assms interval-measure-const-null mono-on-const*  
**by** (*rewrite Ioi-nn-integral-interval-measure-cong*[**where**  $G=\lambda-.0$ ]; *simp add: fun-diff-def*)  
**finally show** *?thesis* **by** *simp*  
**qed**

**lemma** *nn-integral-interval-measure-Iio*:  
**fixes**  $F :: \text{real} \Rightarrow \text{real}$  **and**  $g :: \text{real} \Rightarrow \text{ennreal}$  **and**  $s :: \text{real}$   
**assumes**  $\text{mono } F \wedge x. \text{continuous (at-right } x) F$   $g \in \text{borel-measurable borel}$  **and**  
 $F \text{ constant-on } \{s < ..\}$   $\text{isCont } F \ s$   
**shows**  $(\int^+ x. g \ x \ \partial(\text{interval-measure } F)) = (\int^+ x \in \{.. < s\}. g \ x \ \partial(\text{interval-measure } F))$

**proof** –  
**let**  $?IMF = \text{interval-measure } F$   
**have**  $[simp]: \text{insert } s \ \{.. < s\} = \{..s\}$  **by force**  
**have**  $(\int^+ x. g \ x \ \partial ?IMF) = (\int^+ x \in \{..s\}. g \ x \ \partial ?IMF)$   
**by**  $(\text{rule nn-integral-interval-measure-Iic; simp add: assms})$   
**also have**  $\dots = (\int^+ x \in \{s\}. g \ x \ \partial ?IMF) + (\int^+ x \in \{.. < s\}. g \ x \ \partial ?IMF)$   
**using**  $\text{assms by (rewrite nn-integral-disjoint-pair[THEN sym]; simp)}$   
**also have**  $(\int^+ x \in \{s\}. g \ x \ \partial ?IMF) = 0$   
**using**  $\text{assms interval-measure-singleton-continuous by simp}$   
**finally show**  $?thesis$  **by simp**

**qed**

**lemma** *nn-integral-interval-measure-Icc*:  
**fixes**  $F :: \text{real} \Rightarrow \text{real}$  **and**  $g :: \text{real} \Rightarrow \text{ennreal}$  **and**  $r \ s :: \text{real}$   
**assumes**  $\text{mono } F \wedge x. \text{continuous (at-right } x) F$   $g \in \text{borel-measurable borel}$  **and**  
 $F \text{ constant-on } \{.. < r\}$   $F \text{ constant-on } \{s < ..\}$   
**shows**  $(\int^+ x. g \ x \ \partial(\text{interval-measure } F)) = (\int^+ x \in \{r..s\}. g \ x \ \partial(\text{interval-measure } F))$

**proof** –  
**let**  $?IMF = \text{interval-measure } F$   
**show**  $?thesis$   
**proof**  $(\text{cases } \langle r \leq s \rangle)$   
**case**  $\text{True}$   
**have**  $(\int^+ x. g \ x \ \partial ?IMF) = (\int^+ x \in \{r..s\}. g \ x \ \partial ?IMF)$   
**by**  $(\text{rule nn-integral-interval-measure-Ici; simp add: assms})$   
**also have**  $\dots = (\int^+ x \in \{r..s\}. g \ x \ \partial ?IMF)$   
**proof** –  
**have**  $\{r..s\} = \{r..s\} \cup \{s < ..\}$  **using**  $\text{assms using True by force}$   
**moreover have**  $\{r..s\} \cap \{s < ..\} = \{\}$  **by auto**  
**ultimately have**  $(\int^+ x \in \{r..s\}. g \ x \ \partial ?IMF) = (\int^+ x \in \{r..s\}. g \ x \ \partial ?IMF) +$   
 $(\int^+ x \in \{s < ..\}. g \ x \ \partial ?IMF)$   
**using**  $\text{assms by (rewrite nn-integral-disjoint-pair[THEN sym]; simp)}$   
**also have**  $(\int^+ x \in \{s < ..\}. g \ x \ \partial ?IMF) = 0$   
**using**  $\text{assms interval-measure-const-null mono-on-const}$   
**by**  $(\text{rewrite Ioi-nn-integral-interval-measure-cong[where } G = \lambda. 0\text{]; simp add: fun-diff-def})$   
**finally show**  $?thesis$  **by simp**

**qed**

**finally show**  $?thesis$  .

**next**  
**case**  $\text{False}$   
**hence**  $[simp]: \{s < ..\} \cup \{.. < r\} = \text{UNIV}$  **and**  $[simp]: \{r < ..s\} = \{\}$  **using False**

**by** *force+*  
**have**  $F$  *constant-on*  $(\{s<..\} \cup \{..<r\})$  **using** *assms constant-on-Un False* **by**  
*force*  
**hence** [*simp*]:  $F$  *constant-on UNIV* **by** *simp*  
**hence** [*simp*]: *isCont*  $F$   $r$   
**by** (*metis (no-types, lifting) ext UNIV-I constant-on-def continuous-const*)  
**have**  $(\int^{+x}. g \ x \ \partial ?IMF) = 0$   
**by** (*rewrite nn-integral-interval-measure-cong[where G= $\lambda$ -.0]*;  
*simp add: fun-diff-def interval-measure-const-null mono-on-const assms*)  
**also have**  $\dots = (\int^{+x \in \{r..s\}}. g \ x \ \partial ?IMF)$   
**by** (*rewrite Icc-Cont-nn-integral-interval-measure-cong[where G= $\lambda$ -.0]*;  
*simp add: assms fun-diff-def interval-measure-const-null mono-on-const*)  
**finally show** *?thesis* .  
**qed**  
**qed**

**lemma** *nn-integral-interval-measure-Ioc*:

**fixes**  $F :: \text{real} \Rightarrow \text{real}$  **and**  $g :: \text{real} \Rightarrow \text{ennreal}$  **and**  $r \ s :: \text{real}$   
**assumes** *mono*  $F \ \wedge x.$  *continuous (at-right x)*  $F \ g \in$  *borel-measurable borel* **and**  
 $F$  *constant-on*  $\{..<r\}$   $F$  *constant-on*  $\{s<..\}$  *isCont*  $F$   $r$   
**shows**  $(\int^{+x}. g \ x \ \partial(\text{interval-measure } F)) = (\int^{+x \in \{r<..s\}}. g \ x \ \partial(\text{interval-measure } F))$   
**proof** –  
**let**  $?IMF = \text{interval-measure } F$   
**show** *?thesis*  
**proof** (*cases*  $\langle r \leq s \rangle$ )  
**case** *True*  
**hence** [*simp*]: *insert*  $r \ \{r<..s\} = \{r..s\}$  **by** *force*  
**have**  $(\int^{+x}. g \ x \ \partial ?IMF) = (\int^{+x \in \{r..s\}}. g \ x \ \partial ?IMF)$   
**by** (*rule nn-integral-interval-measure-Icc; simp add: assms*)  
**also have**  $\dots = (\int^{+x \in \{r\}}. g \ x \ \partial ?IMF) + (\int^{+x \in \{r<..s\}}. g \ x \ \partial ?IMF)$   
**using** *assms* **by** (*rewrite nn-integral-disjoint-pair[THEN sym]; simp*)  
**also have**  $\dots = (\int^{+x \in \{r<..s\}}. g \ x \ \partial ?IMF)$   
**using** *assms interval-measure-singleton-continuous* **by** *simp*  
**finally show** *?thesis* .  
**next**  
**case** *False*  
**hence**  $F$  *constant-on*  $(\{s<..\} \cup \{..<r\})$  **by** (*intro constant-on-Un; simp add:*  
*assms*)  
**moreover have**  $\{s<..\} \cup \{..<r\} = \text{UNIV}$  **using** *False* **by** *force*  
**ultimately have** [*simp*]:  $F$  *constant-on UNIV* **by** *simp*  
**hence** [*simp*]:  $F$  *constant-on*  $\{r<..s\}$  **using** *constant-on-subset* **by** *blast*  
**have**  $(\int^{+x}. g \ x \ \partial ?IMF) = 0$   
**by** (*rewrite nn-integral-interval-measure-cong[where G= $\lambda$ -.0]*;  
*simp add: fun-diff-def interval-measure-const-null mono-on-const assms*)  
**also have**  $\dots = (\int^{+x \in \{r<..s\}}. g \ x \ \partial ?IMF)$   
**by** (*rewrite Ioc-nn-integral-interval-measure-cong[where G= $\lambda$ -.0]*;  
*simp add: assms fun-diff-def interval-measure-const-null mono-on-const*)  
**finally show** *?thesis* .

qed  
qed

**lemma** *nn-integral-interval-measure-Ico:*

**fixes**  $F :: \text{real} \Rightarrow \text{real}$  **and**  $g :: \text{real} \Rightarrow \text{ennreal}$  **and**  $r\ s :: \text{real}$

**assumes**  $\text{mono } F \wedge x. \text{continuous (at-right } x) F$   $g \in \text{borel-measurable borel}$

$F \text{ constant-on } \{..<r\}$   $F \text{ constant-on } \{s<..\}$   $\text{isCont } F\ s$

**shows**  $(\int^+ x. g\ x\ \partial(\text{interval-measure } F)) = (\int^+ x \in \{r..<s\}. g\ x\ \partial(\text{interval-measure } F))$

**proof** –

**let**  $?IMF = \text{interval-measure } F$

**show**  $?thesis$

**proof** (*cases*  $\langle r \leq s \rangle$ )

**case** *True*

**hence**  $[\text{simp}]: \text{insert } s\ \{r..<s\} = \{r..s\}$  **by** *force*

**have**  $(\int^+ x. g\ x\ \partial?IMF) = (\int^+ x \in \{r..s\}. g\ x\ \partial?IMF)$

**by** (*rule nn-integral-interval-measure-Icc; simp add: assms*)

**also have**  $\dots = (\int^+ x \in \{r..<s\}. g\ x\ \partial?IMF) + (\int^+ x \in \{s\}. g\ x\ \partial?IMF)$

**using** *assms* **by** (*rewrite nn-integral-disjoint-pair[THEN sym]; simp*)

**also have**  $\dots = (\int^+ x \in \{r..<s\}. g\ x\ \partial?IMF)$

**using** *interval-measure-singleton-continuous assms* **by** *simp*

**finally show**  $?thesis$  .

**next**

**case** *False*

**hence**  $F \text{ constant-on } (\{s<..\} \cup \{..<r\})$  **by** (*intro constant-on-Un; simp add: assms*)

**moreover have**  $\{s<..\} \cup \{..<r\} = \text{UNIV}$  **using** *False* **by** *force*

**ultimately have**  $[\text{simp}]: F \text{ constant-on UNIV}$  **by** *simp*

**hence**  $[\text{simp}]: \text{isCont } F\ r$

**by** (*metis (no-types, lifting) ext UNIV-I constant-on-def continuous-const*)

**have**  $[\text{simp}]: F \text{ constant-on } \{r<..\<s\}$  **using** *constant-on-subset[of F UNIV]* **by** *simp*

**have**  $(\int^+ x. g\ x\ \partial?IMF) = 0$

**by** (*rewrite nn-integral-interval-measure-cong[where G= $\lambda$ -.0];*

*simp add: fun-diff-def interval-measure-const-null mono-on-const assms*)

**also have**  $\dots = (\int^+ x \in \{r..<s\}. g\ x\ \partial?IMF)$

**by** (*rewrite Ico-Cont-nn-integral-interval-measure-cong[where G= $\lambda$ -.0];*

*simp add: assms fun-diff-def interval-measure-const-null mono-on-const*)

**finally show**  $?thesis$  .

qed

qed

**lemma** *nn-integral-interval-measure-Ioo:*

**fixes**  $F :: \text{real} \Rightarrow \text{real}$  **and**  $g :: \text{real} \Rightarrow \text{ennreal}$  **and**  $r\ s :: \text{real}$

**assumes**  $\text{mono } F \wedge x. \text{continuous (at-right } x) F$   $g \in \text{borel-measurable borel}$  **and**

$F \text{ constant-on } \{..<r\}$   $F \text{ constant-on } \{s<..\}$   $\text{isCont } F\ r$   $\text{isCont } F\ s$

**shows**  $(\int^+ x. g\ x\ \partial(\text{interval-measure } F)) = (\int^+ x \in \{r<..\<s\}. g\ x\ \partial(\text{interval-measure } F))$

**proof** –

**let**  $?IMF = \text{interval-measure } F$   
**show**  $?thesis$   
**proof** (*cases*  $\langle r < s \rangle$ )  
  **case** *True*  
    **hence**  $[simp]: \text{insert } r \{r <..<s\} = \{r..<s\}$  **using** *assms* **by** *force*  
    **have**  $(\int^{+x}. g \ x \ \partial ?IMF) = (\int^{+x \in \{r..<s\}}. g \ x \ \partial ?IMF)$   
    **using** *assms* **by** (*intro nn-integral-interval-measure-Ico; simp*)  
    **also have**  $\dots = (\int^{+x \in \{r\}}. g \ x \ \partial ?IMF) + (\int^{+x \in \{r <..<s\}}. g \ x \ \partial ?IMF)$   
    **using** *assms* **by** (*rewrite nn-integral-disjoint-pair[THEN sym]; simp*)  
    **also have**  $\dots = (\int^{+x \in \{r <..<s\}}. g \ x \ \partial ?IMF)$   
    **using** *assms interval-measure-singleton-continuous* **by** *simp*  
    **finally show**  $?thesis$  .  
  **next**  
  **case** *False*  
    **hence**  $\{..r\} \cap \{s..\} \neq \{\}$  **by** *auto*  
    **moreover have**  $F$  *constant-on*  $\{..r\}$   
    **proof** -  
      **from** *assms* **obtain**  $y$  **where**  $Fy: \bigwedge x. x \in \{..<r\} \implies F x = y$  **unfolding**  
      *constant-on-def* **by** *blast*  
      **have**  $(F \longrightarrow F r)$  (*at-left*  $r$ ) **using** *assms continuous-within filterlim-at-split*  
      **by** *blast*  
      **hence**  $((\lambda-. y) \longrightarrow F r)$  (*at-left*  $r$ )  
      **by** (*rewrite tendsto-cong; simp add: eventually-at-left-field lt-ex Fy*)  
      **moreover have**  $((\lambda-. y) \longrightarrow y)$  (*at-left*  $r$ ) **by** (*rule tendsto-const*)  
      **ultimately have**  $F r = y$  **by** (*intro tendsto-unique; auto*)  
      **with** *assms* **show**  $?thesis$   
      **unfolding** *constant-on-def* **using**  $Fy$  *less-eq-real-def* **by** (*intro exI[of - y]*)  
    *auto*  
    **qed**  
    **moreover have**  $F$  *constant-on*  $\{s..\}$   
    **proof** -  
      **from** *assms* **obtain**  $y$  **where**  $Fy: \bigwedge x. x \in \{s<..\} \implies F x = y$  **unfolding**  
      *constant-on-def* **by** *blast*  
      **have**  $(F \longrightarrow F s)$  (*at-right*  $s$ ) **using** *assms continuous-within filterlim-at-split*  
      **by** *blast*  
      **hence**  $((\lambda-. y) \longrightarrow F s)$  (*at-right*  $s$ )  
      **by** (*rewrite tendsto-cong; simp add: eventually-at-right-field gt-ex Fy*)  
      **moreover have**  $((\lambda-. y) \longrightarrow y)$  (*at-right*  $r$ ) **by** (*rule tendsto-const*)  
      **ultimately have**  $F s = y$  **by** (*intro tendsto-unique; auto*)  
      **with** *assms* **show**  $?thesis$   
      **unfolding** *constant-on-def* **using**  $Fy$  *less-eq-real-def* **by** (*intro exI[of - y]*)  
    *auto*  
    **qed**  
    **ultimately have**  $F$  *constant-on*  $\{..r\} \cup \{s..\}$  **by** (*intro constant-on-Un; simp*)  
    **moreover have**  $\{..r\} \cup \{s..\} = UNIV$  **using** *False* **by** *auto*  
    **ultimately have**  $[simp]: F$  *constant-on*  $UNIV$  **by** *simp*  
    **hence**  $[simp]: F$  *constant-on*  $\{r <..<s\}$  **using** *constant-on-subset* **by** *blast*  
    **hence**  $(\int^{+t}. g \ t \ \partial ?IMF) = 0$   
    **by** (*rewrite nn-integral-interval-measure-cong[where G= $\lambda-.0$ ]*);

*simp add: fun-diff-def interval-measure-const-null mono-on-const assms*  
**also have** ... =  $(\int^{+t \in \{r < \cdot < s\}}. g \ t \ \partial ?IMF)$   
**by** (*rewrite Ioo-nn-integral-interval-measure-cong*[**where**  $G = \lambda \cdot . 0$ ]);  
*simp add: assms fun-diff-def interval-measure-const-null mono-on-const*  
**finally show** *?thesis* .  
**qed**  
**qed**

## 1.4 Calculation by the Derivative

**proposition** *set-nn-integral-interval-measure-deriv:*

**fixes**  $F :: \text{real} \Rightarrow \text{real}$  **and**  $g :: \text{real} \Rightarrow \text{ennreal}$  **and**  $a \ b :: \text{ereal}$  **and**  $A :: \text{real set}$   
**assumes** *mono F*  $\wedge$  *continuous (at-right x) F* *F differentiable-on (einterval a b)* **and**

*g-msr: g*  $\in$  *borel-measurable lborel* **and**

$A \in$  *sets borel*  $A \subseteq$  *einterval a b*

**shows**  $(\int^{+x \in A}. g \ x \ \partial(\text{interval-measure } F)) = (\int^{+x \in A}. g \ x * \text{deriv } F \ x \ \partial \text{lborel})$

**proof** –

**let** *?Iab* = *einterval a b*

**let** *?IM* = *interval-measure*

**wlog**  $A = ?Iab$  **generalizing**  $A \ g$  **keeping** *g-msr*

**proof** –

**from** *assms* **have** *indicator A*  $\in$  *borel-measurable lborel* **by** *measurable*

**with** *assms* **have** *gA-msr: ( $\lambda x. g \ x * \text{indicator } A \ x$ )*  $\in$  *borel-measurable lborel*

**by** *measurable*

**have**  $(\int^{+x \in A}. g \ x \ \partial(?IM \ F)) = (\int^{+x \in ?Iab}. g \ x * \text{indicator } A \ x \ \partial(?IM \ F))$

**using** *assms* **by** (*simp add: mult-ac mult-indicator-subset*)

**also have** ... =  $(\int^{+x \in ?Iab}. g \ x * \text{indicator } A \ x * \text{deriv } F \ x \ \partial \text{lborel})$

**using** *hypothesis gA-msr* **by** *simp*

**also have** ... =  $(\int^{+x \in ?Iab}. g \ x * \text{deriv } F \ x * \text{indicator } A \ x \ \partial \text{lborel})$

**using** *mult.commute mult.assoc* **by** (*metis (no-types, lifting)*)

**also have** ... =  $(\int^{+x \in A}. g \ x * \text{deriv } F \ x \ \partial \text{lborel})$

**using** *assms* **by** (*simp add: mult.assoc mult-indicator-subset*)

**finally show** *?thesis* .

**qed**

**hence** *A-I: A = ?Iab* **by** *simp*

**thus** *?thesis*

**proof** (*cases*  $\langle a < b \rangle$ )

**case** *True*

**let** *?dFIab* =  $\lambda x. \text{deriv } F \ x * \text{indicator } ?Iab \ x$

**have** *dFsmr: set-borel-measurable borel (?Iab) (deriv F)*

**by** (*rule deriv-measurable-real; simp add: assms borel-measurable-mono*)

**hence** [*measurable*]:  $(\lambda x. \text{ennreal } (?dFIab \ x)) \in$  *borel-measurable borel*

**using** *assms* **unfolding** *set-borel-measurable-def* **by** (*rewrite mult.commute*)

*simp*

**have** *IMFst:  $\bigwedge s \ t. a < \text{ereal } s \wedge s \leq t \wedge \text{ereal } t < b \implies \text{emeasure } (?IM \ F)$*

$\{s <..t\} = F t - F s$   
**using** *assms monoD* **by** (*rewrite emeasure-interval-measure-Ioc; force*)  
**moreover have**  $\bigwedge s t. a < \text{ereal } s \wedge s \leq t \wedge \text{ereal } t < b \implies$   
 $\text{emeasure } (?IM F) \{s <..t\} = \text{emeasure } (\text{density lborel } ?dFIab) \{s <..t\}$   
**proof** –  
**fix**  $s t :: \text{real}$   
**assume** *asm*:  $a < \text{ereal } s \wedge s \leq t \wedge \text{ereal } t < b$   
**hence**  $\text{emeasure } (?IM F) \{s <..t\} = F t - F s$  **by** (*rule IMFst*)  
**also have**  $\dots = \int^+ x. \text{ennreal } (?dFIab x) * \text{indicator } \{s..t\} x \partial \text{lborel}$   
**proof** –  
**have** [*simp*]:  $\bigwedge x. x \in \{s..t\} \implies x \in ?Iab$   
**using** *asm einterval-iff ereal-less-le less-ereal-le* **by** *force*  
**hence** [*simp*]:  $\bigwedge x. s \leq x \wedge x \leq t \implies F$  *differentiable at x within ?Iab*  
**by** (*intro differentiable-onD; simp add: assms*)  
**have**  $\bigwedge x. x \in \{s..t\} \implies \text{DERIV } F x :> \text{deriv } F x$   
**apply** (*rewrite DERIV-deriv-iff-real-differentiable*)  
**by** (*rewrite at-within-open[where S=?Iab, THEN sym]; simp*)  
**moreover hence**  $\bigwedge x. x \in \{s..t\} \implies 0 \leq \text{deriv } F x$   
**using** *assms mono-imp-mono-on mono-on-imp-deriv-nonneg* **by** (*metis UNIV-I interior-UNIV*)  
**moreover have**  $(\lambda x. \text{deriv } F x * \text{indicator } ?Iab x) \in \text{borel-measurable borel}$   
**using** *dFmsr unfolding set-borel-measurable-def* **by** (*simp add: mult.commute*)  
**ultimately show** *?thesis* **by** (*rewrite nn-integral-FTC-Icc; simp add: asm*)  
**qed**  
**also have**  $\dots = \int^+ x. \text{ennreal } (?dFIab x) * \text{indicator } \{s <..t\} x \partial \text{lborel}$   
**proof** –  
**have** *sym-diff*  $\{s..t\} \{s <..t\} = \{s\}$  **using** *asm* **by** *force*  
**thus** *?thesis* **by** (*intro nn-integral-null-delta*) *auto*  
**qed**  
**also have**  $\dots = \text{emeasure } (\text{density lborel } ?dFIab) \{s <..t\}$   
**by** (*rewrite emeasure-density; simp add: assms True*)  
**finally show**  $\text{emeasure } (?IM F) \{s <..t\} = \text{emeasure } (\text{density lborel } ?dFIab)$   
 $\{s <..t\}$  .  
**qed**  
**ultimately have** *IMdsty*:  
 $\text{restrict-space } (?IM F) ?Iab = \text{restrict-space } (\text{density lborel } ?dFIab) ?Iab$   
**by** (*intro measure-einterval-eqI-Ioc; simp*)  
**show** *?thesis*  
**proof** –  
**have**  $(\int^+_{x \in ?Iab} g x \partial (?IM F)) = (\int^+ x. g x \partial (\text{restrict-space } (?IM F) ?Iab))$   
**by** (*rewrite nn-integral-restrict-space[THEN sym]; simp*)  
**also have**  $\dots = (\int^+ x. g x \partial (\text{restrict-space } (\text{density lborel } ?dFIab) ?Iab))$   
**using** *IMdsty* **by** *simp*  
**also have**  $\dots = (\int^+ x. g x * \text{indicator } ?Iab x \partial (\text{density lborel } ?dFIab))$   
**by** (*rewrite nn-integral-restrict-space; simp*)  
**also have**  $\dots =$   
 $(\int^+ x. \text{ennreal } (\text{deriv } F x * \text{indicator } ?Iab x) * (g x * \text{indicator } ?Iab x)$   
 $\partial \text{lborel})$   
**using** *g-msr* **by** (*rewrite nn-integral-density; simp*)

**also have**  $\dots = (\int^{+x \in ?Iab}. g \ x * \text{ennreal} (\text{deriv } F \ x) \ \partial \text{lborel})$   
**by** (*rule nn-integral-cong*) (*simp add: mult.left-commute indicator-times-eq-if*)  
**finally show** *?thesis* **using** *A-I* **by** *simp*  
**qed**  
**next**  
**case** *False*  
**hence**  $einterval \ a \ b = \{\}$  **using** *einterval-empty* **by** *simp*  
**hence**  $A = \{\}$  **using** *assms A-I* **by** *simp*  
**thus** *?thesis* **by** *simp*  
**qed**

**qed**

**corollary** *nn-integral-interval-measure-deriv*:

**fixes**  $F :: \text{real} \Rightarrow \text{real}$  **and**  $g :: \text{real} \Rightarrow \text{ennreal}$   
**assumes** *mono F*  $\wedge x.$  *continuous (at-right x) F F* *differentiable-on UNIV* **and**  
 $g \in \text{borel-measurable lborel}$   
**shows**  $(\int^{+x}. g \ x \ \partial(\text{interval-measure } F)) = (\int^{+x}. g \ x * \text{deriv } F \ x \ \partial \text{lborel})$   
**using** *set-nn-integral-interval-measure-deriv einterval-eq-UNIV indicator-UNIV*  
*assms*  
**by** (*metis (mono-tags, lifting) mult.right-neutral nn-integral-cong*  
*space-in-borel verit-eq-simplify(6)*)

**corollary** *Ioi-nn-integral-interval-measure-deriv*:

**fixes**  $F :: \text{real} \Rightarrow \text{real}$  **and**  $g :: \text{real} \Rightarrow \text{ennreal}$  **and**  $r :: \text{real}$   
**assumes** *mono F*  $\wedge x.$  *continuous (at-right x) F F* *differentiable-on {r<..}* **and**  
 $g \in \text{borel-measurable lborel}$   
**shows**  $(\int^{+x \in \{r<..\}}. g \ x \ \partial(\text{interval-measure } F)) = (\int^{+x \in \{r<..\}}. g \ x * \text{deriv } F \ x \ \partial \text{lborel})$   
**proof** –  
**let**  $?I = einterval$   
**let**  $?IMF = interval-measure \ F$   
**have**  $(\int^{+x \in \{r<..\}}. g \ x \ \partial ?IMF) = (\int^{+x \in (?I \ (ereal \ r) \ \infty)}. g \ x \ \partial ?IMF)$  **by** *simp*  
**also have**  $\dots = (\int^{+x \in (?I \ (ereal \ r) \ \infty)}. g \ x * \text{ennreal} (\text{deriv } F \ x) \ \partial \text{lborel})$   
**using** *assms* **by** (*rewrite set-nn-integral-interval-measure-deriv[of - ereal r \infty]*;  
*simp*)  
**also have**  $\dots = (\int^{+x \in \{r<..\}}. g \ x * \text{ennreal} (\text{deriv } F \ x) \ \partial \text{lborel})$  **by** *simp*  
**finally show** *?thesis* .  
**qed**

**corollary** *Iio-nn-integral-interval-measure-deriv*:

**fixes**  $F :: \text{real} \Rightarrow \text{real}$  **and**  $g :: \text{real} \Rightarrow \text{ennreal}$  **and**  $s :: \text{real}$   
**assumes** *mono F*  $\wedge x.$  *continuous (at-right x) F F* *differentiable-on {..<s}* **and**  
 $g \in \text{borel-measurable lborel}$   
**shows**  $(\int^{+x \in \{..<s\}}. g \ x \ \partial(\text{interval-measure } F)) = (\int^{+x \in \{..<s\}}. g \ x * \text{deriv } F \ x \ \partial \text{lborel})$   
**proof** –  
**let**  $?I = einterval$   
**let**  $?IMF = interval-measure \ F$

**have**  $(\int^{+x \in \{..<s\}}. g \ x \ \partial ?IMF) = (\int^{+x \in (?I \ (-\infty) \ (ereal \ s))}. g \ x \ \partial ?IMF)$  **by**  
*simp*  
**also have**  $\dots = (\int^{+x \in (?I \ (-\infty) \ (ereal \ s))}. g \ x \ * \ ennreal \ (deriv \ F \ x) \ \partial lborel)$   
**using** *assms* **by** *(rewrite set-nn-integral-interval-measure-deriv[of - -\infty ereal s]; simp)*  
**also have**  $\dots = (\int^{+x \in \{..<s\}}. g \ x \ * \ deriv \ F \ x \ \partial lborel)$  **by** *simp*  
**finally show** *?thesis* .  
**qed**

**corollary** *Ioo-nn-integral-interval-measure-deriv:*

**fixes**  $F :: real \Rightarrow real$  **and**  $g :: real \Rightarrow ennreal$  **and**  $r \ s :: real$   
**assumes** *mono F  $\wedge$  x. continuous (at-right x) F F differentiable-on {r<..~~}~~*  
**and**

$g \in borel\text{-measurable } lborel$   
**shows**  $(\int^{+x \in \{r<..$

**proof** –

**let**  $?I = einterval$   
**let**  $?IMF = interval\text{-measure } F$   
**have**  $(\int^{+x \in \{r<..  
**by** *simp*  
**also have**  $\dots = (\int^{+x \in (?I \ (ereal \ r) \ (ereal \ s))}. g \ x \ * \ ennreal \ (deriv \ F \ x) \ \partial lborel)$   
**using** *assms* **by** *(rewrite set-nn-integral-interval-measure-deriv[of - ereal r ereal s]; simp)*  
**also have**  $\dots = (\int^{+x \in \{r<.. **by** *simp*  
**finally show** *?thesis* .  
**qed**$$

**lemma** *set-nn-integral-finite-nondifferentiable-interval-measure-deriv:*

**fixes**  $F :: real \Rightarrow real$  **and**  $g :: real \Rightarrow ennreal$  **and**  $a \ b :: ereal$  **and**  $S :: real \ set$   
**assumes** *mono F  $\wedge$  x. continuous (at-right x) F g  $\in$  borel-measurable lborel and*  
*cont: continuous-on (einterval a b) F and*  
*diff: F differentiable-on einterval a b - S and*  
*fin: finite S*  
**shows**  $(\int^{+x \in einterval \ a \ b}. g \ x \ \partial (interval\text{-measure } F)) =$   
 $(\int^{+x \in einterval \ a \ b}. g \ x \ * \ deriv \ F \ x \ \partial lborel)$   
**using** *fin cont diff*  
**proof** *(induction S arbitrary: a b rule: finite-psubset-induct)*

**let**  $?I = einterval$   
**let**  $?IMF = interval\text{-measure } F$

**have** *[measurable]: F  $\in$  borel-measurable borel*  
**using** *assms borel-measurable-mono* **by** *blast*

**case** *IH: (psubset S)*

**thus** *?case*

**proof** *(cases S)*

**case** *emptyI*

**thus** *?thesis* **using** *assms set-nn-integral-interval-measure-deriv IH* **by** *simp*  
**next**  
**case** (*insertI S' s*)  
**hence** *Ss: S - {s} ⊂ S ∧ finite (S - {s})* **by** *force+*  
**have** [*simp*]: *?I a (ereal s) ∩ ?I (ereal s) b = {}* **using** *einterval-iff* **by** *force*  
**show** *?thesis*  
**proof** (*cases ‹s ∈ ?I a b›*)  
  
**case** *True*  
  
**hence** [*simp*]: *a < b* **using** *einterval-iff* **by** *force*  
**from** *True* **have** *Iabs: ?I a b - {s} = ?I a (ereal s) ∪ ?I (ereal s) b*  
**by** (*rule einterval-split*)  
**hence** [*simp*]: *?I a (ereal s) ⊆ ?I a b* **and** [*simp*]: *?I (ereal s) b ⊆ ?I a b* **by**  
*auto*  
  
**have** *F* *piecewise-differentiable-on ?I a b*  
**using** *IH unfolding differentiable-on-def piecewise-differentiable-on-def*  
**by** (*metis DiffE at-within-open finite-imp-closed open-Diff open-einterval*)  
**hence** [*measurable*]: *set-borel-measurable borel (?I a b) (deriv F)*  
**by** (*rule piecewise-differentiable-on-deriv-measurable-real; simp add: assms*)  
**hence** [*measurable*]:  $\bigwedge A. A \subseteq ?I a b \implies A \in \text{sets borel} \implies$   
 $(\lambda x. g x * \text{ennreal (deriv F x)} * \text{indicator A x}) \in \text{borel-measurable borel}$   
**proof** –  
**fix** *A* **assume** *A ⊆ ?I a b* **and** *A ∈ sets borel*  
**hence** *set-borel-measurable borel A (deriv F)*  
**using** *set-borel-measurable-subset[of - ?I a b - A]* **by** *measurable*  
**hence**  $(\lambda x. \text{ennreal (deriv F x)} * \text{indicator A x}) \in \text{borel-measurable borel}$   
**apply** (*rewrite mult.commute, rewrite indicator-mult-ennreal*)  
**unfolding** *set-borel-measurable-def* **by** *simp*  
**thus**  $(\lambda x. g x * \text{ennreal (deriv F x)} * \text{indicator A x}) \in \text{borel-measurable borel}$   
**using** *assms measurable-lborel1* **by** (*rewrite mult.assoc; simp*)  
**qed**  
  
**let** *?Sl = S ∩ ?I a (ereal s)*  
**have** *?Sl ⊂ S ∧ finite ?Sl* **using** *Ss einterval-iff* **by** *force+*  
**moreover** **have** *continuous-on (?I a (ereal s)) F*  
**using** *IH True*  
**by** (*smt (verit) at-within-open order.strict-trans*  
*continuous-on-eq-continuous-within einterval-iff open-einterval*)  
**moreover** **have** *F* *differentiable-on ?I a (ereal s) - ?Sl*  
**apply** (*rule differentiable-on-subset[of - ?I a b - S], simp add: IH*)  
**using** *True einterval-iff eereal-less-le* **by** *fastforce*  
**ultimately** **have**  
 $II: (\int^{+x} \in ?I a (ereal s). g x \partial ?IMF) =$   
 $(\int^{+x} \in ?I a (ereal s). g x * \text{ennreal (deriv F x)} \partial \text{lborel})$   
**using** *IH assms* **by** *simp*  
  
**let** *?Sr = S ∩ ?I (ereal s) b*

```

have ?Sr  $\subset S \wedge$  finite ?Sr using Ss einterval-iff by force+
moreover have continuous-on (?I (ereal s) b) F using IH True
  by (smt (verit) at-within-open order.strict-trans
    continuous-on-eq-continuous-within einterval-iff open-einterval)
moreover have F differentiable-on ?I (ereal s) b - ?Sr
  apply (rule differentiable-on-subset[of - ?I a b - S], simp add: IH)
  using True einterval-iff order.strict-trans by fastforce
ultimately have
  Ir:  $(\int^+ x \in ?I (ereal s) b. g x \partial ?IMF) =$ 
     $(\int^+ x \in ?I (ereal s) b. g x * ennreal (deriv F x) \partial lborel)$ 
  using IH assms by simp

have isCont F s
  using True IH by (metis at-within-open continuous-on-eq-continuous-within
open-einterval)
hence emeasure ?IMF {s} = ennreal (deriv F s) * emeasure lborel {s}
  using assms by (rewrite interval-measure-singleton-continuous; simp)
hence Is:  $(\int^+ x \in \{s\}. g x \partial ?IMF) = (\int^+ x \in \{s\}. g x * deriv F x \partial lborel)$ 
  by (rewrite nn-integral-indicator-singleton; simp)

show ?thesis
proof -
  have [simp]: insert s (?I a b) = ?I a b using insert-absorb True by force
  have  $(\int^+ x \in ?I a b. g x \partial ?IMF) = (\int^+ x \in ?I a b - \{s\}. g x \partial ?IMF) +$ 
 $(\int^+ x \in \{s\}. g x \partial ?IMF)$ 
  using True assms by (rewrite nn-integral-disjoint-pair[THEN sym]; simp)
  also have ... =  $(\int^+ x \in ?I a (ereal s). g x \partial ?IMF) +$ 
 $(\int^+ x \in ?I (ereal s) b. g x \partial ?IMF) + (\int^+ x \in \{s\}. g x \partial ?IMF)$ 
  using True assms Iabs by (rewrite in  $\sqcup + -$  nn-integral-disjoint-pair[THEN
sym]; simp)
  also have ... =  $(\int^+ x \in ?I a (ereal s). g x * ennreal (deriv F x) \partial lborel) +$ 
 $(\int^+ x \in ?I (ereal s) b. g x * ennreal (deriv F x) \partial lborel) +$ 
 $(\int^+ x \in \{s\}. g x * ennreal (deriv F x) \partial lborel)$ 
  using II Ir Is by simp
  also have ... =  $(\int^+ x \in ?I a b - \{s\}. g x * ennreal (deriv F x) \partial lborel) +$ 
 $(\int^+ x \in \{s\}. g x * ennreal (deriv F x) \partial lborel)$ 
  by (rewrite nn-integral-disjoint-pair2[THEN sym]; simp add: Iabs)
  also have ... =  $(\int^+ x \in ?I a b. g x * ennreal (deriv F x) \partial lborel)$ 
  by (rewrite nn-integral-disjoint-pair2[THEN sym]; simp add: True)
  finally show ?thesis .
qed

next

case False
hence ?I a b - S = ?I a b - (S - {s}) by blast
with IH have F differentiable-on ?I a b - (S - {s}) by simp
then show ?thesis using assms IH Ss by simp

```

qed  
 qed  
 qed

**proposition** *set-nn-integral-piecewise-differentiable-interval-measure-deriv*:

**fixes**  $F :: \text{real} \Rightarrow \text{real}$  **and**  $g :: \text{real} \Rightarrow \text{ennreal}$  **and**  $a\ b :: \text{ereal}$

**assumes**  $\text{mono } F \wedge x. \text{continuous (at-right } x) F F \text{ piecewise-differentiable-on (einterval } a\ b)$

$g \in \text{borel-measurable lborel}$

**shows**  $(\int^+ x \in \text{einterval } a\ b. g\ x\ \partial(\text{interval-measure } F)) =$   
 $(\int^+ x \in \text{einterval } a\ b. g\ x * \text{deriv } F\ x\ \partial\text{lborel})$

**proof** –

**let**  $?Iab = \text{einterval } a\ b$

**from** *assms* **have**  $\text{continuous-on } ?Iab\ F$  **by** (*intro piecewise-differentiable-on-imp-continuous-on*)

**moreover from** *assms* **obtain**  $S$  **where**  $\text{finite } S$  **and**  $\wedge x. x \in ?Iab - S \implies F$

*differentiable (at } x)*

**unfolding** *piecewise-differentiable-on-def* **by** (*metis DiffE at-within-open open-einterval*)

**moreover hence**  $F$  *differentiable-on*  $?Iab - S$

**using** *differentiable-at-imp-differentiable-on* **by** *blast*

**ultimately show** *?thesis*

**using** *assms* **by** (*intro set-nn-integral-finite-nondifferentiable-interval-measure-deriv*)

*simp*

qed

**corollary** *nn-integral-piecewise-differentiable-interval-measure-deriv*:

**fixes**  $F :: \text{real} \Rightarrow \text{real}$  **and**  $g :: \text{real} \Rightarrow \text{ennreal}$

**assumes**  $\text{mono } F \wedge x. \text{continuous (at-right } x) F F \text{ piecewise-differentiable-on UNIV}$

$g \in \text{borel-measurable lborel}$

**shows**  $(\int^+ x. g\ x\ \partial(\text{interval-measure } F)) = (\int^+ x. g\ x * \text{deriv } F\ x\ \partial\text{lborel})$

**using** *set-nn-integral-piecewise-differentiable-interval-measure-deriv*

*einterval-eq-UNIV indicator-UNIV assms*

**by** (*metis (mono-tags, lifting) mult.right-neutral nn-integral-cong*)

**corollary** *Ioi-nn-integral-piecewise-differentiable-interval-measure-deriv*:

**fixes**  $F :: \text{real} \Rightarrow \text{real}$  **and**  $g :: \text{real} \Rightarrow \text{ennreal}$  **and**  $r :: \text{real}$

**assumes**  $\text{mono } F \wedge x. \text{continuous (at-right } x) F F \text{ piecewise-differentiable-on } \{r < ..\}$

$g \in \text{borel-measurable lborel}$

**shows**  $(\int^+ x \in \{r < ..\}. g\ x\ \partial(\text{interval-measure } F)) = (\int^+ x \in \{r < ..\}. g\ x * \text{deriv } F\ x\ \partial\text{lborel})$

**using** *set-nn-integral-piecewise-differentiable-interval-measure-deriv assms*

**by** (*metis (mono-tags, lifting) einterval-eq-Ici nn-integral-cong*)

**corollary** *Iio-nn-integral-piecewise-differentiable-interval-measure-deriv*:

**fixes**  $F :: \text{real} \Rightarrow \text{real}$  **and**  $g :: \text{real} \Rightarrow \text{ennreal}$  **and**  $s :: \text{real}$

**assumes**  $\text{mono } F \wedge x. \text{continuous (at-right } x) F F \text{ piecewise-differentiable-on } \{.. < s\}$

$g \in \text{borel-measurable lborel}$

**shows**  $(\int^{+x \in \{..<s\}}. g \ x \ \partial(\text{interval-measure } F)) = (\int^{+x \in \{..<s\}}. g \ x \ * \ \text{deriv } F \ x \ \partial\text{lborel})$   
**using** *set-nn-integral-piecewise-differentiable-interval-measure-deriv assms*  
**by** (*metis (mono-tags, lifting) einterval-eq-Iic nn-integral-cong*)

**corollary** *Ioo-nn-integral-piecewise-differentiable-interval-measure-deriv*:  
**fixes**  $F :: \text{real} \Rightarrow \text{real}$  **and**  $g :: \text{real} \Rightarrow \text{ennreal}$  **and**  $r \ s :: \text{real}$   
**assumes**  $\text{mono } F \ \wedge \ x. \text{continuous (at-right } x) \ F \ F \ \text{piecewise-differentiable-on } \{r < .. < s\}$   
 $g \in \text{borel-measurable lborel}$   
**shows**  $(\int^{+x \in \{r < .. < s\}}. g \ x \ \partial(\text{interval-measure } F)) = (\int^{+x \in \{r < .. < s\}}. g \ x \ * \ \text{deriv } F \ x \ \partial\text{lborel})$   
**using** *set-nn-integral-piecewise-differentiable-interval-measure-deriv assms*  
**by** (*metis (mono-tags, lifting) einterval-eq-Icc nn-integral-cong*)

**end**