

The Correctness of Launchbury’s Natural Semantics for Lazy Evaluation

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In his seminal paper “Natural Semantics for Lazy Evaluation” [Lau93], John Launchbury proves his semantics correct with respect to a denotational semantics, and outlines an adequacy proof. We have formalized both semantics and machine-checked the correctness proof, clarifying some details. Furthermore, we provide a new and more direct adequacy proof that does not require intermediate operational semantics.

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1 Introduction

The Natural Semantics for Lazy Evaluation [Lau93] created by John Launchbury in 1992 is often taken as the base for formal treatments of call-by-need evaluation, either to prove properties of lazy evaluation or as a base to describe extensions of the language or the implementation of the language. Therefore, assurance about the correctness and adequacy of the semantics is important in this field of research. Launchbury himself supports his semantics by defining a standard denotational semantics to prove both correctness and adequacy.

Although his proofs are already on the more rigorous side for pen-and-paper proofs, they have not yet been verified by transforming them to machine-checked proofs. The present work fills this gap by formalizing both semantics in the proof assistant Isabelle and proving both correctness and adequacy.

Our correctness formal proof is very close to the original proof. This is possible if the operator \sqcup is understood as a right-sided update. If we were to understand \sqcup as the least upper bound, then Theorem 2 in [Lau93], which is the generalization of the correctness statement used for Launchbury’s inductive proof, is wrong. The main correctness result still holds, but needs a different proof; this is discussed in greater detail in [Bre13].

Launchbury outlines an adequacy proof via an intermediate operational semantics and resourced denotational semantics. The alternative operational semantics uses indirection instead of substitution for applications, does not update variable results and does not perform blackholing during evaluation of a variable. The equivalence of these two operational semantics is hard and tricky to prove. We found a direct proof for the adequacy of the original operational semantics and the (slightly modified) resourced denotational semantics. This is, as far as we know, the first complete and rigorous proof of adequacy of Launchbury’s semantics.

In this development we extend Launchbury’s syntax and semantics with boolean values and an if-then-else construct, in order to base a subsequent work [?] on this. This extension does not affect the validity of the proven theorems, and the extra cases can simply be ignored if one is interested in the plain semantics. The next introductory section does exactly that. Unfortunately, such meta-level arguments are not easily implemented inside a theorem prover.

Our contributions are:

- We define the natural and denotational semantics given by Launchbury in the theorem prover Isabelle.
- We demonstrate how to use both the Nominal package (to handle name binding) [UK12] and the HOLCF [Huf12] package (for the domain-theoretic aspects) in the same development.
- We verify Launchbury’s proof of correctness.

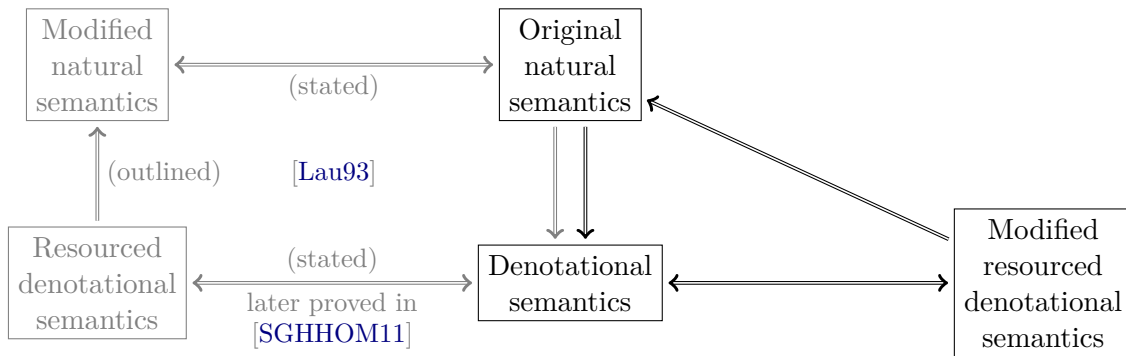
- We provide a new and more direct proof of adequacy.
- In order to do so, we formalize parts of [SGHHOM11], fixing a mistake in the proof.

1.1 Main definitions and theorems

For your convenience, the main definitions and theorems of the present work are assembled in this section. The following formulas are mechanically pretty-printed versions of the statements as defined resp. proven in Isabelle. Free variables are all-quantified. Some type conversion functions (like $set::'a\ list \Rightarrow 'a\ set$) are omitted. The relations \sharp and \sharp^* come from the Nominal package and express freshness of the variables on the left with regard to the expressions on the right.

1.1.1 The big picture

The following picture gives an overview of the different semantics. Elements printed in black are formally defined and proved in the present work, while the gray square on the left shows the proofs and propositions in Launchbury's original work [Lau93].



1.1.2 Expressions

The type var of variables is abstract and provided by the Nominal package. All we know about it is that it is countably infinite. Expressions of type exp are given by the following grammar:

$e ::= \lambda x. e$	lambda abstraction
$ e x$	application
$ x$	variable
$ let\ as\ in\ e$	recursive let

In the introduction we pretty-print expressions to resemble the notation in [Lau93] and omit the constructor names *Var*, *App*, *Lam* and *Let*. In the actual theories, these are visible. These expressions are, due to the machinery of the Nominal package, actually alpha-equivalency classes, so $\lambda x. x = \lambda y. y$ holds provably. This differs from Launchbury's original definition, which expects distinctly-named expressions and performs explicit alpha-renaming in the semantics.

The type *heap* is an abbreviation for $(var \times exp)$ *list*. These are *not* alpha-equivalency classes, i.e. we manage the bindings in heaps explicitly.

1.1.3 The natural semantics

Launchbury's original semantics, extended with some technical overhead related to name binding (following [Ses97]), is defined as follows:

$$\begin{array}{c}
\frac{}{\Gamma : \lambda x. e \Downarrow_L \Gamma : \lambda x. e} \quad \text{LAMBDA} \\
\\
\frac{y \# (\Gamma, e, x, L, \Delta, \Theta, z) \quad \Gamma : e \Downarrow_L \Delta : \lambda y. e' \quad \Delta : e'[y := x] \Downarrow_L \Theta : z}{\Gamma : e x \Downarrow_L \Theta : z} \quad \text{APPLICATION} \\
\\
\frac{(x, e) \in \Gamma \quad \Gamma \setminus x : e \Downarrow_x . L \Delta : z}{\Gamma : x \Downarrow_L (x, z) \cdot \Delta : z} \quad \text{VARIABLE} \\
\\
\frac{dom \Delta \#* (\Gamma, L) \quad \Delta @ \Gamma : body \Downarrow_L \Theta : z}{\Gamma : let \Delta in body \Downarrow_L \Theta : z} \quad \text{LET}
\end{array}$$

1.1.4 The denotational semantics

The value domain of the denotational semantics is the initial solution to

$$D = [D \rightarrow D]_{\perp}$$

as introduced in [Abr90]. The type *Value*, together with the bottom value $\perp :: Value$, the injection $Fn :: (Value \rightarrow Value) \rightarrow Value$ and the projection $_ \Downarrow Fn _ :: Value \rightarrow Value \rightarrow Value$, is constructed as a pointed chain-complete partial order from this equation by the HOLCF package. The type of semantic environments is $var \Rightarrow Value$.

The semantics of an expression $e :: exp$ in an environment $\rho :: var \Rightarrow Value$ is written $\llbracket e \rrbracket_{\rho} :: Value$ and defined by the following equations:

$$\begin{aligned}
\llbracket \lambda x. e \rrbracket_{\rho} &= Fn \cdot (\Lambda v. \llbracket e \rrbracket_{\rho(x := v)}) \\
\llbracket e x \rrbracket_{\rho} &= \llbracket e \rrbracket_{\rho} \Downarrow Fn \rho x \\
\llbracket x \rrbracket_{\rho} &= \rho x \\
\llbracket let \Gamma in body \rrbracket_{\rho} &= \llbracket body \rrbracket_{\llbracket \Gamma \rrbracket_{\rho}}
\end{aligned}$$

The expression $\llbracket \Gamma \rrbracket_{\varrho}$ maps the evaluation function over a heap, returning an environment:

$$\begin{aligned} (\llbracket \Gamma \rrbracket_{\varrho}) v &= \llbracket e \rrbracket_{\varrho} && \text{if } (v, e) \in \Gamma \\ (\llbracket \Gamma \rrbracket_{\varrho}) v &= \perp && \text{if } v \notin \text{dom } \Gamma \end{aligned}$$

The semantics $\{\Gamma\}_{\varrho} :: \text{var} \Rightarrow \text{Value}$ of a heap $\Gamma :: \text{heap}$ in an environment $\varrho :: \text{var} \Rightarrow \text{Value}$ is defined by the recursive equation

$$\{\Gamma\}_{\varrho} = \varrho \text{ ++}_{\text{dom } \Gamma} \llbracket \Gamma \rrbracket_{\{\Gamma\}_{\varrho}}$$

where

$$\begin{aligned} (f \text{ ++}_A g) a &= f a && \text{if } a \notin A \\ (f \text{ ++}_A g) a &= g a && \text{if } a \in A. \end{aligned}$$

The semantics of the heap in the empty environment \perp is abbreviated as $\{\Gamma\}$.

1.1.5 Correctness and Adequacy

The statement of correctness reads: If $\Gamma : e \Downarrow_L \Delta : v$ and, as a side condition, $fv(\Gamma, e) \subseteq L \cup \text{dom } \Gamma$ holds, then

$$\llbracket e \rrbracket_{\{\Gamma\}_{\varrho}} = \llbracket v \rrbracket_{\{\Delta\}_{\varrho}}.$$

The statement of adequacy reads:

$$\text{If } \llbracket e \rrbracket_{\{\Gamma\}} \neq \perp \text{ then } \exists \Delta v. \Gamma : e \Downarrow_S \Delta : v.$$

1.2 Differences to our previous work

We have previously published [Bre13] of which the present work is a continuation. They differ in scope and focus:

1.2.1 The treatment of \sqcup

In [Bre13], the question of the precise meaning of \sqcup is discussed in detail. The original paper is not clear about whether this operator denotes the least upper bound, or the right-sided override operator. A lemma stated in [Lau93] only holds if \sqcup is the least upper bound, but with that definition, Launchbury's Theorem 2 – the generalized correctness theorem – is false; a counter-example is given in [Bre13].

We came up with an alternative operational semantics that keeps more of the evaluation context in the judgments and allows the correctness theorem to be proved inductively without the problematic generalization. We proved the two operational semantics equivalent and thus obtained the (non-generalized) correctness of Launchbury’s semantics.

We also showed that if one takes \sqcup to be the update operator, Theorem 2 holds and the proof goes through as it is. Furthermore, we showed that the resulting denotational semantics are identical for expressions, and can differ only for heaps. Therefore, the question of the precise meaning of \sqcup can be considered of little importance and for the present work we solely work with right sided updates. We also avoid the ambiguous syntax \sqcup and write $_ ++ _ _$ instead (the index indicates on what set the function on the right overrides the function on the left). The alternative operational semantics is not included in this work.

1.2.2 The types of environments

Another difference is the choice of the type for environments, which map variables to semantics values. A naive choice is $var \Rightarrow Value$, but this causes problems when defining the value semantics, for which

$$\llbracket \lambda x. e \rrbracket_{\rho} = Fn \cdot (\Lambda v. \llbracket e \rrbracket_{\rho(x := v)})$$

is a defining equation. The argument on the left hand side is the representative of an equivalence class (defined using the Nominal package), so this is only allowed if the right hand side is indeed independent of the actual choice of x . This is shown most commonly and easily if x is fresh in all the other arguments ($x \# \rho$), and indeed the Nominal package allows us to specify this as a side condition to the defining equation, which is what we did in [Bre13].

But this convenience comes as a price: Such side-conditions are only allowed if the argument has finite support (otherwise there might no variable fulfilling $x \# \rho$). More precisely: The type of the argument must be a member of the *fs* typeclass provided by the Nominal package. The type $var \Rightarrow Value$ cannot be made a member of this class, as there obviously are elements that have infinite support. The fix here was to introduce a new type constructor, *fmap*, for partial functions with finite domain. This is fine: Only functions with finite domain matter in our formalisation.

The introduction of *fmap* had further consequences. The main type class of the HOLCF package, which we use to define domains and continuous functions on them, is the class *cpo*, of chain-complete partial orders. With the usual ordering on partial functions, $(var, Value)$ *fmap* cannot be a member of this class. The fix here is to use a different ordering and only let elements be comparable that have the same domain. In our formalisation, the domain is always known (e.g. all variables bound on some heap), so this worked out.

But not without causing yet another issue: With this ordering, $(var, Value)$ *fmap* is a *cpo*, but lacks a bottom element, i.e. now it is no *pcpo*, and HOLCF’s built-in operator

$\mu x. f x$ for expressing least fixed-points, as they occur in the semantics of heaps, is not available. Furthermore, \sqcup is not a total function, i.e. defined only on a subset of all possible arguments. The solution was a rather convoluted set of theories that formalize functions that are continuous on a specific set, fixed-points on such sets etc.

In the present work, this problem is solved in a much more elegant way. Using a small trick we defined the semantics functions so that

$$\llbracket \lambda x. e \rrbracket_{\varrho} = Fn \cdot (\Lambda v. \llbracket e \rrbracket_{\varrho(x := v)})$$

holds unconditionally. The actual, technical definition is

$$\llbracket \lambda x. e \rrbracket_{\varrho} = Fn \cdot (\Lambda v. \llbracket e \rrbracket_{\varrho|_{fv(\lambda x. e)}(x := v)})$$

where the right-hand-side can be shown to be invariant of the choice of x , as $x \notin fv(\lambda x. e)$. Once the function is defined, the equality $\llbracket e \rrbracket_{\varrho} = \llbracket e \rrbracket_{\varrho|_{fv e}}$ can be proved. With that, the desired equation for $\llbracket \lambda x. e \rrbracket_{\varrho}$ follows. The same trick is applied to the equation for $\llbracket \text{let } \Gamma \text{ in body} \rrbracket_{\varrho}$.

This allows us to use the type $var \Rightarrow Value$ for the semantic environments and considerably simplifies the formalization compared to [Bre13].

1.2.3 No type *assn*

The nominal package provides means to define types that are alpha-equivalence classes, and we use that to define our type *exp*, which contains a constructor *let binds in expr*. The desired type of the parameter for the binding is $(var \times exp) \text{ list}$, but the Nominal package does not support such nested recursion, and requires a mutual recursive definition with a custom type (*assn*) with constructors *ANil* and *ACons* that is isomorphic to $(var \times exp) \text{ list}$. In [Bre13], this type and conversion functions from and to $(var \times exp) \text{ list}$ cluttered the whole development. In the present work we improved this by defining the type with a “temporary” constructor $LetA::assn \Rightarrow exp \Rightarrow exp$. Afterwards we define conversion functions and the desired constructor $Let::(var \times exp) \text{ list} \Rightarrow exp \Rightarrow exp$, and re-state all lemmas produced by the Nominal package (such as type exhaustiveness, distinctiveness of constructors and the induction rules) with that constructor. From that point on, the development is free of the crutch *assn*.

In short, the notable changes in this work over [Bre13] are:

- We consider \sqcup to be a right-sided update and do not discuss neither the problem with \sqcup denoting the least upper bound, nor possible solutions.
- This, a simpler choice for the type of semantic environments and a better definition of the type for terms, considerably simplifies the work.
- Most importantly, this work contains a complete and formal proof of the adequacy of Launchbury’s semantics.

1.3 Related work

Lidia Sánchez-Gil, Mercedes Hidalgo-Herrero and Yolanda Ortega-Mallén have worked on formal aspects of Launchbury’s semantics as well.

They identified a step in his adequacy proof relating the standard and the resourced denotational semantics that is not as trivial as it seems at first and worked out a detailed pen-and-paper proof [SGHHOM11], where they first construct a similarity relation $_ \triangleleft _$ between the standard semantic domain (*Value*) and the resourced domain (*CValue*) and show that the denotation semantics yield similar results ($\varrho \triangleleft^* \sigma \implies \llbracket e \rrbracket_\varrho \triangleleft (\mathcal{N} \llbracket e \rrbracket_\sigma) \cdot C^\infty$), which is one step in the adequacy proof. We formalized this (Sections 8.1 and 8.2), identifying and fixing a mistake in the paper (Lemma 2.3(3) does not hold; the problem can be fixed by applying an extra round of take-induction in the proof of Proposition 9).

Currently, they are working on completing the adequacy proof as outlined by Launchbury, i.e. by going via the alternative natural semantics given in [Lau93], which differs from the semantics above in that the application rule works with an indirection on the heap instead of a substitution and that the variable rule has no blackholing and no update. In [SGHHOM14], they relate the original semantics with one where indirections have been introduced. The next step, modifying the variable rule, is under development. Once that is done they can close the loop and have completed Launchbury’s work.

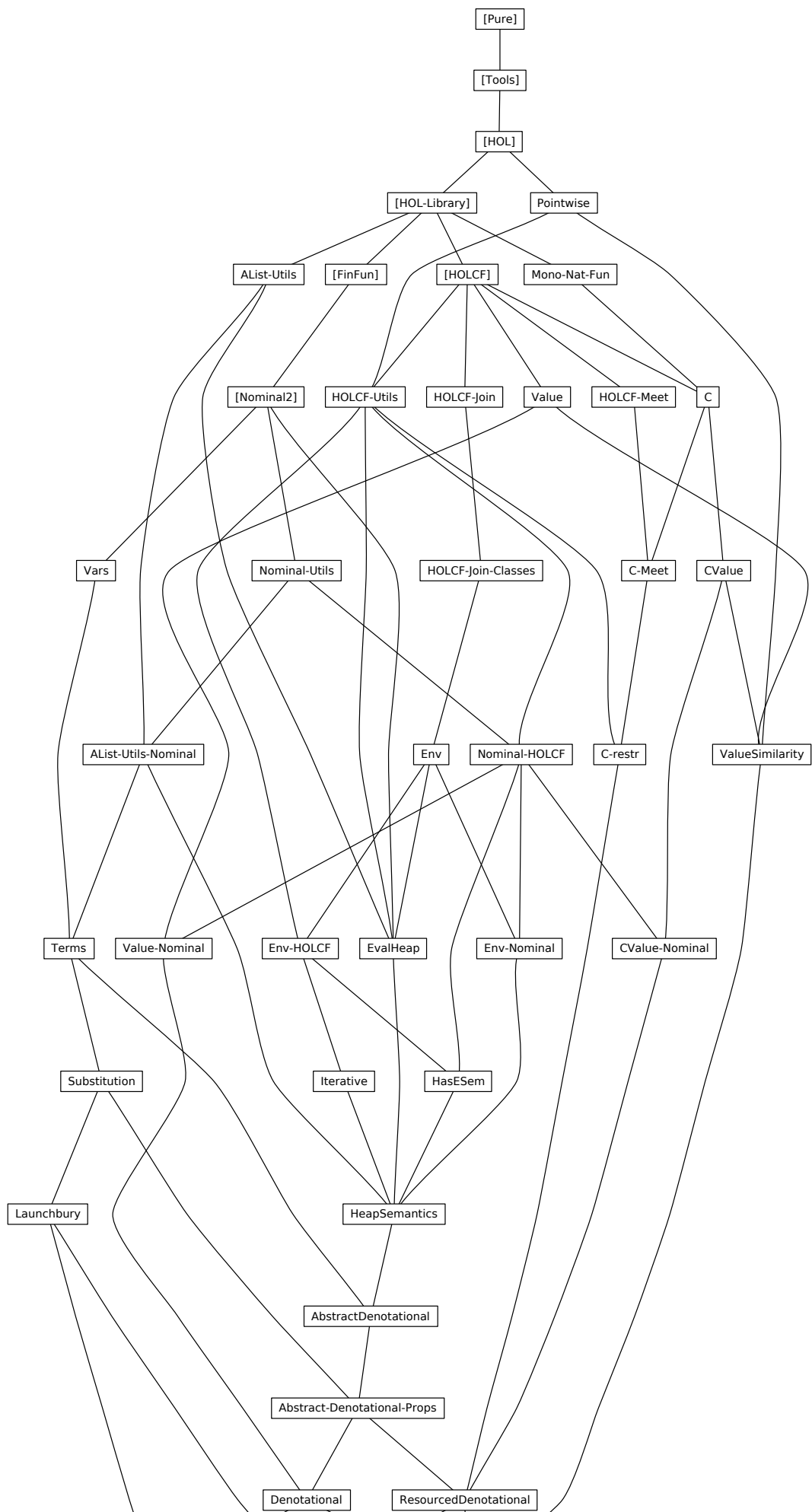
This work proves the adequacy as stated by Launchbury as well, but in contrast to his proof outline no alternative operational semantics is introduced. The problems of indirection vs. substitution and of blackholing is solved on the denotational side instead, which turned out to be much easier than proving the various operational semantics to be equivalent.

1.4 Theory overview

The following chapters contain the complete Isabelle theories, with one section per theory. Their interdependencies are visualized in Figure 1.

Chapter 2 contains auxiliary theories, not necessarily tied to Launchbury’s semantics. The base theories are kept independent of Nominal and HOLCF where possible, the lemmas combining them are in theories of their own, creatively named by appending *-Nominal* resp. *-HOLCF*. You will find these theories:

- A definition for lifting a relation point-wise (*Pointwise*).
- A collection of definition related to associative lists (*AList-Utils*, *AList-Utils-Nominal*).
- A characterization of monotonous functions $\mathbb{N} \rightarrow \mathbb{N}$ (*Mono-Nat-Fun*).
- General utility functions extending Nominal (*Nominal-Utils*).



- General utility functions extending HOLCF (*HOLCF-Utils*).
- Binary meets in the context of HOLCF (*HOLCF-Meet*).
- A theory combining notions from HOLCF and Nominal, e.g. continuity of permutation (*Nominal-HOLCF*).
- A theory for working with pcpo-valued functions as semantic environments (*Env*, *Env-Nominal*, *Env-HOLCF*).
- A function *evalHeap* that converts between associative lists and functions. (*Eval-Heap*)

Chapter 3 defines the syntax and Launchbury’s natural semantics.

Chapter 4 sets the stage for the denotational semantics by defining a locale *semantic-domain* for denotational domains, and an instantiation for the standard domain.

Chapter 5 defines the denotational semantics. It also introduces the locale *has-ESem* which abstracts over the value semantics when defining the semantics of heaps.

Chapter 6 defines the resourced denotational semantics.

Chapter 7 proves the correctness of Launchbury’s semantics with regard to both denotational semantics. We need the correctness with regard to the resourced semantics in the adequacy proof.

Chapter 8 proves the two denotational semantics related, which is used in

Chapter 9, where finally the adequacy is proved.

1.5 Acknowledgements

I’d like to thank Lidia Sánchez-Gil, Mercedes Hidalgo-Herrero and Yolanda Ortega-Mallén for inviting me to Madrid to discuss our respective approaches.

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2 Auxiliary theories

2.1 Pointwise

theory *Pointwise* imports *Main* begin

Lifting a relation to a function.

definition *pointwise* **where** *pointwise* $P\ m\ m' = (\forall\ x.\ P\ (m\ x)\ (m'\ x))$

lemma *pointwiseI*[*intro*]: $(\bigwedge\ x.\ P\ (m\ x)\ (m'\ x)) \implies \textit{pointwise}\ P\ m\ m'$ **unfolding** *pointwise-def*
by *blast*

end

2.2 AList-Utills

theory *AList-Utills*

imports *Main* *HOL-Library.AList*

begin

declare *implies-True-equals* [*simp*] *False-implies-equals*[*simp*]

We want to have *delete* and *update* back in the namespace.

abbreviation *delete* **where** *delete* \equiv *AList.delete*

abbreviation *update* **where** *update* \equiv *AList.update*

abbreviation *restrictA* **where** *restrictA* \equiv *AList.restrict*

abbreviation *clearjunk* **where** *clearjunk* \equiv *AList.clearjunk*

lemmas *restrict-eq* = *AList.restrict-eq*

and *delete-eq* = *AList.delete-eq*

lemma *restrictA-append*: $\textit{restrictA}\ S\ (a\ @\ b) = \textit{restrictA}\ S\ a\ @\ \textit{restrictA}\ S\ b$

unfolding *restrict-eq* **by** (*rule filter-append*)

lemma *length-restrictA-le*: $\textit{length}\ (\textit{restrictA}\ S\ a) \leq \textit{length}\ a$

by (*metis length-filter-le restrict-eq*)

2.2.1 The domain of an associative list

definition *domA*

where *domA* $h = \textit{fst}\ 'set\ h$

lemma *domA-append*[*simp*]: $\textit{domA}\ (a\ @\ b) = \textit{domA}\ a\ \cup\ \textit{domA}\ b$

and [*simp*]: $\textit{domA}\ ((v,e)\ \#\ h) = \textit{insert}\ v\ (\textit{domA}\ h)$

and [*simp*]: $\textit{domA}\ (p\ \#\ h) = \textit{insert}\ (\textit{fst}\ p)\ (\textit{domA}\ h)$

and [*simp*]: $\textit{domA}\ [] = \{\}$

by (*auto simp add: domA-def*)

lemma *domA-from-set*:

$(x, e) \in \text{set } h \implies x \in \text{domA } h$

by (*induct h, auto*)

lemma *finite-domA[simp]*:

$\text{finite } (\text{domA } \Gamma)$

by (*auto simp add: domA-def*)

lemma *domA-delete[simp]*:

$\text{domA } (\text{delete } x \Gamma) = \text{domA } \Gamma - \{x\}$

by (*induct \Gamma auto*)

lemma *domA-restrictA[simp]*:

$\text{domA } (\text{restrictA } S \Gamma) = \text{domA } \Gamma \cap S$

by (*induct \Gamma auto*)

lemma *delete-not-domA[simp]*:

$x \notin \text{domA } \Gamma \implies \text{delete } x \Gamma = \Gamma$

by (*induct \Gamma auto*)

lemma *deleted-not-domA*: $x \notin \text{domA } (\text{delete } x \Gamma)$

by (*induct \Gamma auto*)

lemma *dom-map-of-conv-domA*:

$\text{dom } (\text{map-of } \Gamma) = \text{domA } \Gamma$

by (*induct \Gamma (auto simp add: dom-if)*)

lemma *domA-map-of-Some-the*:

$x \in \text{domA } \Gamma \implies \text{map-of } \Gamma x = \text{Some } (\text{the } (\text{map-of } \Gamma x))$

by (*induct \Gamma (auto simp add: dom-if)*)

lemma *domA-clearjunk[simp]*: $\text{domA } (\text{clearjunk } \Gamma) = \text{domA } \Gamma$

unfolding *domA-def* **using** *dom-clearjunk*.

lemma *the-map-option-domA[simp]*: $x \in \text{domA } \Gamma \implies \text{the } (\text{map-option } f (\text{map-of } \Gamma x)) = f$
 $(\text{the } (\text{map-of } \Gamma x))$

by (*induction \Gamma auto*)

lemma *map-of-domAD*: $\text{map-of } \Gamma x = \text{Some } e \implies x \in \text{domA } \Gamma$

using *dom-map-of-conv-domA* **by** *fastforce*

lemma *restrictA-noop*: $\text{domA } \Gamma \subseteq S \implies \text{restrictA } S \Gamma = \Gamma$

unfolding *restrict-eq* **by** (*induction \Gamma auto*)

lemma *restrictA-cong*:

$(\bigwedge x. x \in \text{domA } m1 \implies x \in V \longleftrightarrow x \in V') \implies m1 = m2 \implies \text{restrictA } V m1 = \text{restrictA}$
 $V' m2$

unfolding *restrict-eq* **by** (*induction m1 arbitrary: m2 auto*)

2.2.2 Other lemmas about associative lists

lemma *delete-set-none*: $(\text{map-of } l)(x := \text{None}) = \text{map-of } (\text{delete } x \ l)$

proof *(induct l)*
case *Nil* **thus** *?case* **by** *simp*
case *(Cons l ls)*
from *this[symmetric]*
show *?case*
by *(cases fst l = x) auto*
qed

lemma *list-size-delete[simp]*: $\text{size-list size } (\text{delete } x \ l) < \text{Suc } (\text{size-list size } l)$
by *(induct l) auto*

lemma *delete-append[simp]*: $\text{delete } x \ (l1 \ @ \ l2) = \text{delete } x \ l1 \ @ \ \text{delete } x \ l2$
unfolding *AList.delete-eq* **by** *simp*

lemma *map-of-delete-insert*:
assumes $\text{map-of } \Gamma \ x = \text{Some } e$
shows $\text{map-of } ((x,e) \# \text{delete } x \ \Gamma) = \text{map-of } \Gamma$
using *assms* **by** *(induct \Gamma) (auto split:prod.split)*

lemma *map-of-delete-iff[simp]*: $\text{map-of } (\text{delete } x \ \Gamma) \ xa = \text{Some } e \iff (\text{map-of } \Gamma \ xa = \text{Some } e) \wedge xa \neq x$
by *(metis delete-conv fun-upd-same map-of-delete option.distinct(1))*

lemma *map-add-domA[simp]*:
 $x \in \text{domA } \Gamma \implies (\text{map-of } \Delta \ ++ \ \text{map-of } \Gamma) \ x = \text{map-of } \Gamma \ x$
 $x \notin \text{domA } \Gamma \implies (\text{map-of } \Delta \ ++ \ \text{map-of } \Gamma) \ x = \text{map-of } \Delta \ x$
apply *(metis dom-map-of-conv-domA map-add-dom-app-simps(1))*
apply *(metis dom-map-of-conv-domA map-add-dom-app-simps(3))*
done

lemma *set-delete-subset*: $\text{set } (\text{delete } k \ al) \subseteq \text{set } al$
by *(auto simp add: delete-eq)*

lemma *dom-delete-subset*: $\text{snd } ' \ \text{set } (\text{delete } k \ al) \subseteq \text{snd } ' \ \text{set } al$
by *(auto simp add: delete-eq)*

lemma *map-ran-cong[fundef-cong]*:
 $\llbracket \bigwedge x . x \in \text{set } m1 \implies f1 \ (fst \ x) \ (snd \ x) = f2 \ (fst \ x) \ (snd \ x) ; m1 = m2 \rrbracket$
 $\implies \text{map-ran } f1 \ m1 = \text{map-ran } f2 \ m2$
by *(induction m1 arbitrary: m2) auto*

lemma *domA-map-ran[simp]*: $\text{domA } (\text{map-ran } f \ m) = \text{domA } m$
unfolding *domA-def* **by** *(rule dom-map-ran)*

lemma *map-ran-delete*:
 $\text{map-ran } f \ (\text{delete } x \ \Gamma) = \text{delete } x \ (\text{map-ran } f \ \Gamma)$

by (induction Γ) auto

lemma *map-ran-restrictA*:

$map\text{-}ran\ f\ (restrictA\ V\ \Gamma) = restrictA\ V\ (map\text{-}ran\ f\ \Gamma)$

by (induction Γ) auto

lemma *map-ran-append*:

$map\text{-}ran\ f\ (\Gamma @ \Delta) = map\text{-}ran\ f\ \Gamma @ map\text{-}ran\ f\ \Delta$

by (induction Γ) auto

2.2.3 Syntax for map comprehensions

definition *mapCollect* :: ('a \Rightarrow 'b \Rightarrow 'c) \Rightarrow ('a \rightarrow 'b) \Rightarrow 'c set

where $mapCollect\ f\ m = \{f\ k\ v \mid k\ v . m\ k = Some\ v\}$

syntax

-*MapCollect* :: 'c \Rightarrow pttm \Rightarrow pttm \Rightarrow 'a \rightarrow 'b \Rightarrow 'c set ((1{- |/-/+/-/∈/-/})

translations

$\{e \mid k \mapsto v \in m\} == CONST\ mapCollect\ (\lambda k\ v . e)\ m$

lemma *mapCollect-empty[simp]*: $\{f\ k\ v \mid k \mapsto v \in Map.empty\} = \{\}$

unfolding *mapCollect-def* by *simp*

lemma *mapCollect-const[simp]*:

$m \neq Map.empty \implies \{e \mid k \mapsto v \in m\} = \{e\}$

unfolding *mapCollect-def* by *auto*

lemma *mapCollect-cong[fundef-cong]*:

$(\bigwedge k\ v . m1\ k = Some\ v \implies f1\ k\ v = f2\ k\ v) \implies m1 = m2 \implies mapCollect\ f1\ m1 = mapCollect\ f2\ m2$

unfolding *mapCollect-def* by *force*

lemma *mapCollectE[elim!]*:

assumes $x \in \{f\ k\ v \mid k \mapsto v \in m\}$

obtains $k\ v$ where $m\ k = Some\ v$ and $x = f\ k\ v$

using *assms* by (auto simp add: *mapCollect-def*)

lemma *mapCollectI[intro]*:

assumes $m\ k = Some\ v$

shows $f\ k\ v \in \{f\ k\ v \mid k \mapsto v \in m\}$

using *assms* by (auto simp add: *mapCollect-def*)

lemma *ball-mapCollect[simp]*:

$(\forall x \in \{f\ k\ v \mid k \mapsto v \in m\} . P\ x) \iff (\forall k\ v . m\ k = Some\ v \implies P\ (f\ k\ v))$

by (auto simp add: *mapCollect-def*)

lemma *image-mapCollect[simp]*:

$g\ ` \{f\ k\ v \mid k \mapsto v \in m\} = \{g\ (f\ k\ v) \mid k \mapsto v \in m\}$

by (auto simp add: mapCollect-def)

lemma mapCollect-map-upd[simp]:

mapCollect f (m(k↦v)) = insert (f k v) (mapCollect f (m(k := None)))

unfolding mapCollect-def **by** auto

definition mapCollectFilter :: ('a ⇒ 'b ⇒ (bool × 'c)) ⇒ ('a → 'b) ⇒ 'c set

where mapCollectFilter f m = {snd (f k v) | k v . m k = Some v ∧ fst (f k v)}

syntax

-MapCollectFilter :: 'c ⇒ pptrn ⇒ pptrn ⇒ ('a → 'b) ⇒ bool ⇒ 'c set ((1{- ||-/↦-/∈/-/./-})

translations

{e | k↦v ∈ m . P } == CONST mapCollectFilter (λk v. (P,e)) m

lemma mapCollectFilter-const-False[simp]:

{e | k↦v ∈ m . False } = {}

unfolding mapCollect-def mapCollectFilter-def **by** simp

lemma mapCollectFilter-const-True[simp]:

{e | k↦v ∈ m . True } = {e | k↦v ∈ m}

unfolding mapCollect-def mapCollectFilter-def **by** simp

end

2.3 Mono-Nat-Fun

theory Mono-Nat-Fun

imports HOL-Library.Infinite-Set

begin

The following lemma proves that a monotonous function from and to the natural numbers is either eventually constant or unbounded.

lemma nat-mono-characterization:

fixes f :: nat ⇒ nat

assumes mono f

obtains n **where** $\bigwedge m . n \leq m \implies f n = f m \mid \bigwedge m . \exists n . m \leq f n$

proof (cases finite (range f))

case True

from Max-in[OF True]

obtain n **where** Max: f n = Max (range f) **by** auto

show thesis

proof(rule that(1))

fix m

assume n ≤ m

```

    hence  $f\ n \leq f\ m$  using  $\langle mono\ f \rangle$  by (metis monoD)
    also
    have  $f\ m \leq f\ n$  unfolding Max by (rule Max-ge[OF True rangeI])
    finally
    show  $f\ n = f\ m$ .
  qed
next
  case False
  thus thesis by (fastforce intro: that(2) simp add: infinite-nat-iff-unbounded-le)
qed
end

```

2.4 Nominal-Utills

```

theory Nominal-Utills
imports Nominal2.Nominal2 HOL-Library.AList
begin

```

2.4.1 Lemmas helping with equivariance proofs

```

lemma perm-rel-lemma:
  assumes  $\bigwedge \pi\ x\ y. r\ (\pi \cdot x)\ (\pi \cdot y) \implies r\ x\ y$ 
  shows  $r\ (\pi \cdot x)\ (\pi \cdot y) \longleftrightarrow r\ x\ y$  (is  $?l \longleftrightarrow ?r$ )
by (metis (full-types) assms permute-minus-cancel(2))

```

```

lemma perm-rel-lemma2:
  assumes  $\bigwedge \pi\ x\ y. r\ x\ y \implies r\ (\pi \cdot x)\ (\pi \cdot y)$ 
  shows  $r\ x\ y \longleftrightarrow r\ (\pi \cdot x)\ (\pi \cdot y)$  (is  $?l \longleftrightarrow ?r$ )
by (metis (full-types) assms permute-minus-cancel(2))

```

```

lemma fun-eqvtI:
  assumes f-eqvt[eqvt]:  $(\bigwedge p\ x. p \cdot (f\ x) = f\ (p \cdot x))$ 
  shows  $p \cdot f = f$  by perm-simp rule

```

```

lemma eqvt-at-apply:
  assumes eqvt-at  $f\ x$ 
  shows  $(p \cdot f)\ x = f\ x$ 
by (metis (opaque-lifting, no-types) assms eqvt-at-def permute-fun-def permute-minus-cancel(1))

```

```

lemma eqvt-at-apply':
  assumes eqvt-at  $f\ x$ 
  shows  $p \cdot f\ x = f\ (p \cdot x)$ 
by (metis (opaque-lifting, no-types) assms eqvt-at-def)

```

```

lemma eqvt-at-apply'':
  assumes eqvt-at  $f\ x$ 
  shows  $(p \cdot f)\ (p \cdot x) = f\ (p \cdot x)$ 
by (metis (opaque-lifting, no-types) assms eqvt-at-def permute-fun-def permute-minus-cancel(1))

```

lemma *size-list-eqvt*[*eqvt*]: $p \cdot \text{size-list } f \ x = \text{size-list } (p \cdot f) \ (p \cdot x)$
proof (*induction x*)
 case (*Cons x xs*)
 have $f \ x = p \cdot (f \ x)$ **by** (*simp add: permute-pure*)
 also have $\dots = (p \cdot f) \ (p \cdot x)$ **by** *simp*
 with *Cons*
 show *?case* **by** (*auto simp add: permute-pure*)
qed *simp*

2.4.2 Freshness via equivariance

lemma *eqvt-fresh-cong1*: $(\bigwedge p \ x. p \cdot (f \ x) = f \ (p \cdot x)) \implies a \ \sharp \ x \implies a \ \sharp \ f \ x$
 apply (*rule fresh-fun-eqvt-app*[*of f*])
 apply (*rule eqvtI*)
 apply (*rule eq-reflection*)
 apply (*rule ext*)
 apply (*metis permute-fun-def permute-minus-cancel*(1))
 apply *assumption*
 done

lemma *eqvt-fresh-cong2*:
 assumes *eqvt*: $(\bigwedge p \ x \ y. p \cdot (f \ x \ y) = f \ (p \cdot x) \ (p \cdot y))$
 and *fresh1*: $a \ \sharp \ x$ **and** *fresh2*: $a \ \sharp \ y$
 shows $a \ \sharp \ f \ x \ y$
proof –
 have *eqvt* $(\lambda \ (x,y). f \ x \ y)$
 using *eqvt*
 apply –
 apply (*auto simp add: eqvt-def*)
 apply (*rule ext*)
 apply *auto*
 by (*metis permute-minus-cancel*(1))
 moreover
 have $a \ \sharp \ (x, y)$ **using** *fresh1 fresh2* **by** *auto*
 ultimately
 have $a \ \sharp \ (\lambda \ (x,y). f \ x \ y) \ (x, y)$ **by** (*rule fresh-fun-eqvt-app*)
 thus *?thesis* **by** *simp*
qed

lemma *eqvt-fresh-star-cong1*:
 assumes *eqvt*: $(\bigwedge p \ x. p \cdot (f \ x) = f \ (p \cdot x))$
 and *fresh1*: $a \ \sharp^* \ x$
 shows $a \ \sharp^* \ f \ x$
 by (*metis fresh-star-def eqvt-fresh-cong1 assms*)

lemma *eqvt-fresh-star-cong2*:
 assumes *eqvt*: $(\bigwedge p \ x \ y. p \cdot (f \ x \ y) = f \ (p \cdot x) \ (p \cdot y))$

and $\text{fresh1}: a \#* x$ and $\text{fresh2}: a \#* y$
 shows $a \#* f x y$
 by (*metis fresh-star-def eqvt-fresh-cong2 assms*)

lemma *eqvt-fresh-cong3*:

assumes *eqvt*: $(\bigwedge p x y z. p \cdot (f x y z) = f (p \cdot x) (p \cdot y) (p \cdot z))$
 and $\text{fresh1}: a \# x$ and $\text{fresh2}: a \# y$ and $\text{fresh3}: a \# z$
 shows $a \# f x y z$

proof–

have *eqvt* $(\lambda (x,y,z). f x y z)$
 using *eqvt*
 apply –
 apply (*auto simp add: eqvt-def*)
 apply (*rule ext*)
 apply *auto*
 by (*metis permute-minus-cancel(1)*)

moreover

have $a \# (x, y, z)$ using fresh1 fresh2 fresh3 by *auto*
 ultimately
 have $a \# (\lambda (x,y,z). f x y z) (x, y, z)$ by (*rule fresh-fun-eqvt-app*)
 thus *?thesis* by *simp*

qed

lemma *eqvt-fresh-star-cong3*:

assumes *eqvt*: $(\bigwedge p x y z. p \cdot (f x y z) = f (p \cdot x) (p \cdot y) (p \cdot z))$
 and $\text{fresh1}: a \#* x$ and $\text{fresh2}: a \#* y$ and $\text{fresh3}: a \#* z$
 shows $a \#* f x y z$
 by (*metis fresh-star-def eqvt-fresh-cong3 assms*)

2.4.3 Additional simplification rules

lemma *not-self-fresh[simp]*: $\text{atom } x \# x \longleftrightarrow \text{False}$
 by (*metis fresh-at-base(2)*)

lemma *fresh-star-singleton*: $\{ x \} \#* e \longleftrightarrow x \# e$
 by (*simp add: fresh-star-def*)

2.4.4 Additional equivariance lemmas

lemma *eqvt-cases*:

fixes $f x \pi$
 assumes *eqvt*: $\bigwedge x. \pi \cdot f x = f (\pi \cdot x)$
 obtains $f x f (\pi \cdot x) \mid \neg f x \quad \neg f (\pi \cdot x)$
 using *assms[symmetric]*
 by (*cases f x*) *auto*

lemma *range-eqvt*: $\pi \cdot \text{range } Y = \text{range } (\pi \cdot Y)$
 unfolding *image-eqvt UNIV-eqvt ..*

lemma *case-option-eqvt[eqvt]*:

$\pi \cdot \text{case-option } d f x = \text{case-option } (\pi \cdot d) (\pi \cdot f) (\pi \cdot x)$
by(cases x)(simp-all)

lemma *supp-option-eqv*:

$\text{supp } (\text{case-option } d f x) \subseteq \text{supp } d \cup \text{supp } f \cup \text{supp } x$
apply (cases x)
apply (auto simp add: supp-Some)
apply (metis (mono-tags) Un-iff subsetCE supp-fun-app)
done

lemma *funpow-eqv[simp,eqvt]*:

$\pi \cdot ((f :: 'a \Rightarrow 'a::pt) \text{ } \sim\sim n) = (\pi \cdot f) \text{ } \sim\sim (\pi \cdot n)$
apply (induct n)
apply simp
apply (rule ext)
apply simp
apply perm-simp
apply simp
done

lemma *delete-eqv[eqvt]*:

$\pi \cdot \text{AList.delete } x \Gamma = \text{AList.delete } (\pi \cdot x) (\pi \cdot \Gamma)$
by (induct Γ , auto)

lemma *restrict-eqv[eqvt]*:

$\pi \cdot \text{AList.restrict } S \Gamma = \text{AList.restrict } (\pi \cdot S) (\pi \cdot \Gamma)$
unfolding *AList.restrict-eq* **by** perm-simp rule

lemma *supp-restrict*:

$\text{supp } (\text{AList.restrict } S \Gamma) \subseteq \text{supp } \Gamma$
by (induction Γ) (auto simp add: supp-Pair supp-Cons)

lemma *clearjunk-eqv[eqvt]*:

$\pi \cdot \text{AList.clearjunk } \Gamma = \text{AList.clearjunk } (\pi \cdot \Gamma)$
by (induction Γ rule: clearjunk.induct) auto

lemma *map-ran-eqv[eqvt]*:

$\pi \cdot \text{map-ran } f \Gamma = \text{map-ran } (\pi \cdot f) (\pi \cdot \Gamma)$
by (induct Γ , auto)

lemma *dom-perm*:

$\text{dom } (\pi \cdot f) = \pi \cdot (\text{dom } f)$
unfolding *dom-def* **by** (perm-simp) (simp)

lemmas *dom-perm-rev[simp,eqvt]* = *dom-perm[symmetric]*

lemma *ran-perm[simp]*:

$\pi \cdot (\text{ran } f) = \text{ran } (\pi \cdot f)$
unfolding *ran-def* **by** (perm-simp) (simp)

lemma *map-add-eqv*[*eqvt*]:

$\pi \cdot (m1 ++ m2) = (\pi \cdot m1) ++ (\pi \cdot m2)$
unfolding *map-add-def*
by (*perm-simp*, *rule*)

lemma *map-of-eqv*[*eqvt*]:

$\pi \cdot \text{map-of } l = \text{map-of } (\pi \cdot l)$
apply (*induct l*)
apply (*simp add: permute-fun-def*)
apply *simp*
apply *perm-simp*
apply *auto*
done

lemma *concat-eqv*[*eqvt*]: $\pi \cdot \text{concat } l = \text{concat } (\pi \cdot l)$

by (*induction l*)(*auto simp add: append-eqv*)

lemma *tranclp-eqv*[*eqvt*]: $\pi \cdot \text{tranclp } P v_1 v_2 = \text{tranclp } (\pi \cdot P) (\pi \cdot v_1) (\pi \cdot v_2)$

unfolding *tranclp-def* **by** *perm-simp rule*

lemma *rtranclp-eqv*[*eqvt*]: $\pi \cdot \text{rtranclp } P v_1 v_2 = \text{rtranclp } (\pi \cdot P) (\pi \cdot v_1) (\pi \cdot v_2)$

unfolding *rtranclp-def* **by** *perm-simp rule*

lemma *Set-filter-eqv*[*eqvt*]: $\pi \cdot \text{Set.filter } P S = \text{Set.filter } (\pi \cdot P) (\pi \cdot S)$

unfolding *Set.filter-def*
by *perm-simp rule*

lemma *Sigma-eqv'*[*eqvt*]: $\pi \cdot \text{Sigma} = \text{Sigma}$

apply (*rule ext*)
apply (*rule ext*)
apply (*subst permute-fun-def*)
apply (*subst permute-fun-def*)
unfolding *Sigma-def*
apply *perm-simp*
apply (*simp add: permute-self*)
done

lemma *override-on-eqv*[*eqvt*]:

$\pi \cdot (\text{override-on } m1 m2 S) = \text{override-on } (\pi \cdot m1) (\pi \cdot m2) (\pi \cdot S)$
by (*auto simp add: override-on-def*)

lemma *card-eqv*[*eqvt*]:

$\pi \cdot (\text{card } S) = \text{card } (\pi \cdot S)$

by (*cases finite S*, *induct rule: finite-induct*) (*auto simp add: card-insert-if mem-permute-iff permute-pure*)

lemma *Projl-permute*:
assumes $a: \exists y. f = \text{Inl } y$
shows $(p \cdot (\text{Sum-Type.proj1 } f)) = \text{Sum-Type.proj1 } (p \cdot f)$
using a **by** *auto*

lemma *Projr-permute*:
assumes $a: \exists y. f = \text{Inr } y$
shows $(p \cdot (\text{Sum-Type.proj2 } f)) = \text{Sum-Type.proj2 } (p \cdot f)$
using a **by** *auto*

2.4.5 Freshness lemmas

lemma *fresh-list-elem*:
assumes $a \# \Gamma$
and $e \in \text{set } \Gamma$
shows $a \# e$
using *assms*
by(*induct* Γ)(*auto simp add: fresh-Cons*)

lemma *set-not-fresh*:
 $x \in \text{set } L \implies \neg(\text{atom } x \# L)$
by (*metis fresh-list-elem not-self-fresh*)

lemma *pure-fresh-star[simp]*: $a \#* (x :: 'a :: \text{pure})$
by (*simp add: fresh-star-def pure-fresh*)

lemma *supp-set-mem*: $x \in \text{set } L \implies \text{supp } x \subseteq \text{supp } L$
by (*induct* L) (*auto simp add: supp-Cons*)

lemma *set-supp-mono*: $\text{set } L \subseteq \text{set } L2 \implies \text{supp } L \subseteq \text{supp } L2$
by (*induct* L)(*auto simp add: supp-Cons supp-Nil dest:supp-set-mem*)

lemma *fresh-star-at-base*:
fixes $x :: 'a :: \text{at-base}$
shows $S \#* x \iff \text{atom } x \notin S$
by (*metis fresh-at-base(2) fresh-star-def*)

2.4.6 Freshness and support for subsets of variables

lemma *supp-mono*: $\text{finite } (B :: 'a :: \text{fs set}) \implies A \subseteq B \implies \text{supp } A \subseteq \text{supp } B$
by (*metis infinite-super subset-Un-eq supp-of-finite-union*)

lemma *fresh-subset*:
 $\text{finite } B \implies x \# (B :: 'a :: \text{at-base set}) \implies A \subseteq B \implies x \# A$
by (*auto dest:supp-mono simp add: fresh-def*)

lemma *fresh-star-subset*:
 $\text{finite } B \implies x \#* (B :: 'a :: \text{at-base set}) \implies A \subseteq B \implies x \#* A$
by (*metis fresh-star-def fresh-subset*)

lemma *fresh-star-set-subset*:

$x \#* (B :: 'a::at-base \text{ list}) \implies \text{set } A \subseteq \text{set } B \implies x \#* A$
by (*metis fresh-star-set fresh-star-subset[OF finite-set]*)

2.4.7 The set of free variables of an expression

definition *fv* :: $'a::pt \Rightarrow 'b::at-base \text{ set}$

where $fv \ e = \{v. \text{atom } v \in \text{supp } e\}$

lemma *fv-eqv[simp,eqvt]*: $\pi \cdot (fv \ e) = fv \ (\pi \cdot e)$

unfolding *fv-def* **by** *simp*

lemma *fv-Nil[simp]*: $fv \ [] = \{\}$

by (*auto simp add: fv-def supp-Nil*)

lemma *fv-Cons[simp]*: $fv \ (x \# \ xs) = fv \ x \cup fv \ xs$

by (*auto simp add: fv-def supp-Cons*)

lemma *fv-Pair[simp]*: $fv \ (x, y) = fv \ x \cup fv \ y$

by (*auto simp add: fv-def supp-Pair*)

lemma *fv-append[simp]*: $fv \ (x @ y) = fv \ x \cup fv \ y$

by (*auto simp add: fv-def supp-append*)

lemma *fv-at-base[simp]*: $fv \ a = \{a::'a::at-base\}$

by (*auto simp add: fv-def supp-at-base*)

lemma *fv-pure[simp]*: $fv \ (a::'a::pure) = \{\}$

by (*auto simp add: fv-def pure-supp*)

lemma *fv-set-at-base[simp]*: $fv \ (l :: ('a :: at-base) \text{ list}) = \text{set } l$

by (*induction l*) *auto*

lemma *flip-not-fv*: $a \notin fv \ x \implies b \notin fv \ x \implies (a \leftrightarrow b) \cdot x = x$

by (*metis flip-def fresh-def fv-def mem-Collect-eq swap-fresh-fresh*)

lemma *fv-not-fresh*: $\text{atom } x \# e \longleftrightarrow x \notin fv \ e$

unfolding *fv-def fresh-def* **by** *blast*

lemma *fresh-fv*: $\text{finite } (fv \ e :: 'a \text{ set}) \implies \text{atom } (x :: ('a::at-base)) \# (fv \ e :: 'a \text{ set}) \longleftrightarrow \text{atom } x \# e$

unfolding *fv-def fresh-def*

by (*auto simp add: supp-finite-set-at-base*)

lemma *finite-fv[simp]*: $\text{finite } (fv \ (e::'a::fs) :: ('b::at-base) \text{ set})$

proof –

have $\text{finite } (\text{supp } e)$ **by** (*metis finite-supp*)

hence $\text{finite } (\text{atom } - ' \text{supp } e :: 'b \text{ set})$

apply (*rule finite-vimageI*)

apply (*rule inj-onI*)

apply (*simp*)

done

moreover

have $(\text{atom } - ' \text{supp } e :: 'b \text{ set}) = fv \ e$ **unfolding** *fv-def* **by** *auto*

ultimately
 show *?thesis* by *simp*
 qed

definition *fv-list* :: 'a::fs \Rightarrow 'b::at-base list
 where *fv-list* e = (SOME l. set l = *fv* e)

lemma *set-fv-list[simp]*: set (*fv-list* e) = (*fv* e :: ('b::at-base) set)
 proof–

have *finite* (*fv* e :: 'b set) by (rule *finite-fv*)
 from *finite-list[OF finite-fv]*
 obtain l where set l = (*fv* e :: 'b set)..
 thus *?thesis*
 unfolding *fv-list-def* by (rule *someI*)

qed

lemma *fresh-fv-list[simp]*:

a $\#$ (*fv-list* e :: 'b::at-base list) \longleftrightarrow a $\#$ (*fv* e :: 'b::at-base set)

proof–

have a $\#$ (*fv-list* e :: 'b::at-base list) \longleftrightarrow a $\#$ set (*fv-list* e :: 'b::at-base list)
 by (rule *fresh-set[symmetric]*)
 also have ... \longleftrightarrow a $\#$ (*fv* e :: 'b::at-base set) by *simp*
 finally show *?thesis*.

qed

2.4.8 Other useful lemmas

lemma *pure-permute-id*: *permute* p = (λ x. (x::'a::pure))
 by rule (*simp add: permute-pure*)

lemma *supp-set-elim-finite*:

assumes *finite* S
 and (m::'a::fs) \in S
 and y \in *supp* m
 shows y \in *supp* S
 using *assms supp-of-finite-sets*
 by *auto*

lemmas *fresh-star-Cons* = *fresh-star-list*(2)

lemma *mem-permute-set*:

shows x \in p \cdot S \longleftrightarrow ($-$ p \cdot x) \in S
 by (*metis mem-permute-iff permute-minus-cancel*(2))

lemma *flip-set-both-not-in*:

assumes x \notin S and x' \notin S
 shows ((x' \leftrightarrow x) \cdot S) = S
 unfolding *permute-set-def*
 by (*auto*) (*metis assms flip-at-base-simps*(3))+

lemma *inj-atom*: *inj atom* **by** (*metis atom-eq-iff injI*)

lemmas *image-Int*[*OF inj-atom, simp*]

lemma *eqvt-uncurry*: *eqvt f* \implies *eqvt (case-prod f)*
unfolding *eqvt-def*
by *perm-simp simp*

lemma *supp-fun-app-eqvt2*:
assumes *a*: *eqvt f*
shows *supp (f x y)* \subseteq *supp x* \cup *supp y*

proof–

from *supp-fun-app-eqvt*[*OF eqvt-uncurry [OF a]*]
have *supp (case-prod f (x,y))* \subseteq *supp (x,y)*.
thus *?thesis* **by** (*simp add: supp-Pair*)

qed

lemma *supp-fun-app-eqvt3*:
assumes *a*: *eqvt f*
shows *supp (f x y z)* \subseteq *supp x* \cup *supp y* \cup *supp z*

proof–

from *supp-fun-app-eqvt2*[*OF eqvt-uncurry [OF a]*]
have *supp (case-prod f (x,y) z)* \subseteq *supp (x,y)* \cup *supp z*.
thus *?thesis* **by** (*simp add: supp-Pair*)

qed

lemma *permute-0*[*simp*]: *permute 0* = ($\lambda x. x$)

by *auto*

lemma *permute-comp*[*simp*]: *permute x* \circ *permute y* = *permute (x + y)* **by** *auto*

lemma *map-permute*: *map (permute p)* = *permute p*

apply *rule*

apply (*induct-tac x*)

apply *auto*

done

lemma *fresh-star-restrictA*[*intro*]: *a* $\sharp^* \Gamma \implies a \sharp^* AList.restrict V \Gamma$

by (*induction* Γ) (*auto simp add: fresh-star-Cons*)

lemma *Abs-lst-Nil-eq*[*simp*]: $[\] lst. (x::'a::fs) = [xs] lst. x' \longleftrightarrow (([\],x) = (xs, x'))$

apply *rule*

apply (*frule Abs-lst-fcb2*[**where** *f* = $\lambda x y. \cdot (x,y)$ **and** *as* = $[\]$ **and** *bs* = *xs* **and** *c* = $()$])

apply (*auto simp add: fresh-star-def*)

done

lemma *Abs-lst-Nil-eq2[simp]*: $[xs]lst. (x::'a::fs) = [[]]lst. x' \longleftrightarrow ((xs,x) = ([], x^{\wedge}))$
by (*subst eq-commute*) *auto*

end

2.5 AList-Utills-Nominal

theory *AList-Utills-Nominal*
imports *AList-Utills Nominal-Utills*
begin

2.5.1 Freshness lemmas related to associative lists

lemma *domA-not-fresh*:
 $x \in domA \Gamma \implies \neg(atom\ x \# \Gamma)$
by (*induct* Γ , *auto simp add: fresh-Cons fresh-Pair*)

lemma *fresh-delete*:
assumes $a \# \Gamma$
shows $a \# delete\ v\ \Gamma$
using *assms*
by(*induct* Γ)(*auto simp add: fresh-Cons*)

lemma *fresh-star-delete*:
assumes $S \#* \Gamma$
shows $S \#* delete\ v\ \Gamma$
using *assms fresh-delete unfolding fresh-star-def by fastforce*

lemma *fv-delete-subset*:
 $fv\ (delete\ v\ \Gamma) \subseteq fv\ \Gamma$
using *fresh-delete unfolding fresh-def fv-def by auto*

lemma *fresh-heap-expr*:
assumes $a \# \Gamma$
and $(x,e) \in set\ \Gamma$
shows $a \# e$
using *assms*
by (*metis fresh-list-elem fresh-Pair*)

lemma *fresh-heap-expr'*:
assumes $a \# \Gamma$
and $e \in snd\ 'set\ \Gamma$
shows $a \# e$
using *assms*
by (*induct* Γ , *auto simp add: fresh-Cons fresh-Pair*)

lemma *fresh-star-heap-expr'*:
assumes $S \#^* \Gamma$
and $e \in \text{snd } \text{'set } \Gamma$
shows $S \#^* e$
using *assms*
by (*metis fresh-star-def fresh-heap-expr'*)

lemma *fresh-map-of*:
assumes $x \in \text{dom}A \Gamma$
assumes $a \# \Gamma$
shows $a \# \text{the } (\text{map-of } \Gamma x)$
using *assms*
by (*induct* Γ)(*auto simp add: fresh-Cons fresh-Pair*)

lemma *fresh-star-map-of*:
assumes $x \in \text{dom}A \Gamma$
assumes $a \#^* \Gamma$
shows $a \#^* \text{the } (\text{map-of } \Gamma x)$
using *assms* **by** (*simp add: fresh-star-def fresh-map-of*)

lemma *domA-fv-subset*: $\text{dom}A \Gamma \subseteq \text{fv } \Gamma$
by (*induction* Γ) *auto*

lemma *map-of-fv-subset*: $x \in \text{dom}A \Gamma \implies \text{fv } (\text{the } (\text{map-of } \Gamma x)) \subseteq \text{fv } \Gamma$
by (*induction* Γ) *auto*

lemma *map-of-Some-fv-subset*: $\text{map-of } \Gamma x = \text{Some } e \implies \text{fv } e \subseteq \text{fv } \Gamma$
by (*metis domA-from-set map-of-fv-subset map-of-SomeD option.sel*)

2.5.2 Equivariance lemmas

lemma *domA[eqvt]*:
 $\pi \cdot \text{dom}A \Gamma = \text{dom}A (\pi \cdot \Gamma)$
by (*simp add: domA-def*)

lemma *mapCollect[eqvt]*:
 $\pi \cdot \text{mapCollect } f m = \text{mapCollect } (\pi \cdot f) (\pi \cdot m)$
unfolding *mapCollect-def*
by *perm-simp rule*

2.5.3 Freshness and distinctness

lemma *fresh-distinct*:
assumes $\text{atom } \text{' } S \#^* \Gamma$
shows $S \cap \text{dom}A \Gamma = \{\}$
proof–
{ **fix** x
assume $x \in S$
moreover

```

    assume  $x \in \text{dom}A \Gamma$ 
    hence  $\text{atom } x \in \text{supp } \Gamma$ 
      by (induct  $\Gamma$ )(auto simp add: supp-Cons domA-def supp-Pair supp-at-base)
    ultimately
    have False
      using assms
      by (simp add: fresh-star-def fresh-def)
  }
  thus  $S \cap \text{dom}A \Gamma = \{\}$  by auto
qed

```

```

lemma fresh-distinct-list:
  assumes  $\text{atom } 'S \#* l$ 
  shows  $S \cap \text{set } l = \{\}$ 
  using assms
  by (metis disjoint-iff-not-equal fresh-list-elem fresh-star-def image-eqI not-self-fresh)

```

```

lemma fresh-distinct-fv:
  assumes  $\text{atom } 'S \#* l$ 
  shows  $S \cap \text{fv } l = \{\}$ 
  using assms
  by (metis disjoint-iff-not-equal fresh-star-def fv-not-fresh image-eqI)

```

2.5.4 Pure codomains

```

lemma domA-fv-pure:
  fixes  $\Gamma :: ('a::\text{at-base} \times 'b::\text{pure}) \text{ list}$ 
  shows  $\text{fv } \Gamma = \text{dom}A \Gamma$ 
  apply (induct  $\Gamma$ )
  apply simp
  apply (case-tac a)
  apply (simp)
  done

```

```

lemma domA-fresh-pure:
  fixes  $\Gamma :: ('a::\text{at-base} \times 'b::\text{pure}) \text{ list}$ 
  shows  $x \in \text{dom}A \Gamma \longleftrightarrow \neg(\text{atom } x \# \Gamma)$ 
  unfolding domA-fv-pure[symmetric]
  by (auto simp add: fv-def fresh-def)

```

end

2.6 HOLCF-Utills

```

theory HOLCF-Utills
  imports HOLCF Pointwise
begin

```

```

default-sort type

```

lemmas *cont-fun*[*simp*]
lemmas *cont2cont-fun*[*simp*]

lemma *cont-compose2*:
assumes $\bigwedge y. \text{cont } (\lambda x. c \ x \ y)$
assumes $\bigwedge x. \text{cont } (\lambda y. c \ x \ y)$
assumes *cont f*
assumes *cont g*
shows *cont* $(\lambda x. c \ (f \ x) \ (g \ x))$
by (*intro cont-apply*[*OF assms(4) assms(2)*]
cont2cont-fun[*OF cont-compose*[*OF - assms(3)*]]
cont2cont-lambda[*OF assms(1)*]])

lemma *pointwise-adm*:
fixes *P* :: '*a*::*pcpo* \Rightarrow '*b*::*pcpo* \Rightarrow *bool*
assumes *adm* $(\lambda x. P \ (fst \ x) \ (snd \ x))$
shows *adm* $(\lambda m. \text{pointwise } P \ (fst \ m) \ (snd \ m))$
proof (*rule admI, goal-cases*)
case *prems*: (*1 Y*)
show ?*case*
apply (*rule pointwiseI*)
apply (*rule admD*[*OF adm-subst*[**where** *t* = $\lambda p. (fst \ p \ x, snd \ p \ x)$ **for** *x*, *OF - assms*,
simplified] $\langle chain \ Y \rangle$])
using *prems(2)* **unfolding** *pointwise-def* **apply** *auto*
done
qed

lemma *cfun-beta-Pair*:
assumes *cont* $(\lambda p. f \ (fst \ p) \ (snd \ p))$
shows *csplit*. $(\Lambda a \ b. f \ a \ b) \cdot (x, y) = f \ x \ y$
apply *simp*
apply (*subst beta-cfun*)
apply (*rule cont2cont-LAM'*)
apply (*rule assms*)
apply (*rule beta-cfun*)
apply (*rule cont2cont-fun*)
using *assms*
unfolding *prod-cont-iff*
apply *auto*
done

lemma *fun-upd-mono*:
 $\rho1 \sqsubseteq \rho2 \Longrightarrow v1 \sqsubseteq v2 \Longrightarrow \rho1(x := v1) \sqsubseteq \rho2(x := v2)$
apply (*rule fun-belowI*)
apply (*case-tac xa = x*)
apply *simp*
apply (*auto elim:fun-belowD*)

done

lemma *fun-upd-cont*[*simp, cont2cont*]:
 assumes *cont f* **and** *cont h*
 shows *cont* ($\lambda x. (f x)(v := h x) :: 'a \Rightarrow 'b::pcpo$)
 by (*rule cont2cont-lambda*)(*auto simp add: assms*)

lemma *fun-upd-belowI*:
 assumes $\bigwedge z. z \neq x \implies \varrho z \sqsubseteq \varrho' z$
 assumes $y \sqsubseteq \varrho' x$
 shows $\varrho(x := y) \sqsubseteq \varrho'$
 apply (*rule fun-belowI*)
 using *assms*
 apply (*case-tac xa = x*)
 apply *auto*
 done

lemma *cont-if-else-above*:
 assumes *cont f*
 assumes *cont g*
 assumes $\bigwedge x. f x \sqsubseteq g x$
 assumes $\bigwedge x y. x \sqsubseteq y \implies P y \implies P x$
 assumes *adm P*
 shows *cont* ($\lambda x. \text{if } P x \text{ then } f x \text{ else } g x$) (**is** *cont ?I*)

proof(*intro contI2 monofunI*)
 fix *x y* :: '*a*
 assume $x \sqsubseteq y$
 with *assms*(4)[*OF this*]
 show $?I x \sqsubseteq ?I y$
 apply (*auto*)
 apply (*rule cont2monofunE*[*OF assms*(1)], *assumption*)
 apply (*rule below-trans*[*OF cont2monofunE*[*OF assms*(1)] *assms*(3)], *assumption*)
 apply (*rule cont2monofunE*[*OF assms*(2)], *assumption*)
 done

next
 fix *Y* :: *nat* \Rightarrow '*a*
 assume *chain Y*
 assume *chain* ($\lambda i. ?I (Y i)$)

have *ch-f*: $f (\bigsqcup i. Y i) \sqsubseteq (\bigsqcup i. f (Y i))$ **by** (*metis* $\langle \text{chain } Y \rangle$ *assms*(1) *below-refl cont2contlubE*)

show $?I (\bigsqcup i. Y i) \sqsubseteq (\bigsqcup i. ?I (Y i))$
proof(*cases* $\forall i. P (Y i)$)
 case *True* **hence** $P (\bigsqcup i. Y i)$ **by** (*metis* $\langle \text{chain } Y \rangle$ *adm-def assms*(5))
 with *True ch-f* **show** *?thesis* **by** *auto*
next
 case *False*


```

then obtain  $j$  where  $\neg P (Y j)$  by auto
hence  $*$ :  $\forall i \geq j. \neg P (Y i) \neg P (\bigsqcup i. Y i)$ 
  apply (auto)
  apply (metis assms(4) chain-mono[OF  $\langle$ chain  $Y \rangle$ ])
  apply (metis assms(4) is-ub-theLub[OF  $\langle$ chain  $Y \rangle$ ])
  done

have  $?I (\bigsqcup i. Y i) = g (\bigsqcup i. Y i)$  using  $*$  by simp
also have  $\dots = g (\bigsqcup i. Y (i + j))$  by (metis lub-range-shift[OF  $\langle$ chain  $Y \rangle$ ])
also have  $\dots = (\bigsqcup i. (g (Y (i + j))))$  by (rule cont2contLubE[OF assms(2) chain-shift[OF
 $\langle$ chain  $Y \rangle$ ]])
  also have  $\dots = (\bigsqcup i. (?I (Y (i + j))))$  using  $*$  by auto
  also have  $\dots = (\bigsqcup i. (?I (Y i)))$  by (metis lub-range-shift[OF  $\langle$ chain  $(\lambda i. ?I (Y i)) \rangle$ ])
  finally show ?thesis by simp
qed
qed

```

```

fun up2option :: 'a::cpo $\perp$   $\Rightarrow$  'a option
  where up2option Ibottom = None
  | up2option (Iup  $a$ ) = Some  $a$ 

```

```

lemma up2option-simps[simp]:
  up2option  $\perp$  = None
  up2option (up· $x$ ) = Some  $x$ 
  unfolding up-def by (simp-all add: cont-Iup inst-up-pcpo)

```

```

fun option2up :: 'a option  $\Rightarrow$  'a::cpo $\perp$ 
  where option2up None =  $\perp$ 
  | option2up (Some  $a$ ) = up· $a$ 

```

```

lemma option2up-up2option[simp]:
  option2up (up2option  $x$ ) =  $x$ 
  by (cases  $x$ ) auto

```

```

lemma up2option-option2up[simp]:
  up2option (option2up  $x$ ) =  $x$ 
  by (cases  $x$ ) auto

```

```

lemma adm-subst2: cont  $f \Longrightarrow$  cont  $g \Longrightarrow$  adm  $(\lambda x. f (fst x) = g (snd x))$ 
  apply (rule admI)
  apply (simp add:
    cont2contLubE[where  $f = f$ ] cont2contLubE[where  $f = g$ ]
    cont2contLubE[where  $f = snd$ ] cont2contLubE[where  $f = fst$ ]
  )
  done

```

2.6.1 Composition of fun and cfun

```

lemma cont2cont-comp [simp, cont2cont]:
  assumes cont  $f$ 

```

```

assumes  $\bigwedge x. \text{cont } (f x)$ 
assumes  $\text{cont } g$ 
shows  $\text{cont } (\lambda x. (f x) \circ (g x))$ 
unfolding comp-def
by (rule cont2cont-lambda)
      (intro cont2cont  $\langle \text{cont } g \rangle \langle \text{cont } f \rangle \text{cont-compose2}[\text{OF cont2cont-fun}[\text{OF assms}(1)] \text{assms}(2)]$ 
cont2cont-fun)

```

```

definition cfun-comp :: ('a::pcpo  $\rightarrow$  'b::pcpo)  $\rightarrow$  ('c::type  $\Rightarrow$  'a)  $\rightarrow$  ('c::type  $\Rightarrow$  'b)
where cfun-comp = ( $\bigwedge f \ \varrho. (\lambda x. f \cdot x) \circ \varrho$ )

```

```

lemma [simp]: cfun-comp. $f \cdot (\varrho(x := v)) = (cfun-comp.f \cdot \varrho)(x := f \cdot v)$ 
unfolding cfun-comp-def by auto

```

```

lemma cfun-comp-app[simp]: (cfun-comp. $f \cdot \varrho$ )  $x = f \cdot (\varrho x)$ 
unfolding cfun-comp-def by auto

```

```

lemma fix-eq-fix:
   $f \cdot (\text{fix } g) \sqsubseteq \text{fix } g \implies g \cdot (\text{fix } f) \sqsubseteq \text{fix } f \implies \text{fix } f = \text{fix } g$ 
by (metis fix-least-below below-antisym)

```

2.6.2 Additional transitivity rules

These collect side-conditions of the form $\text{cont } f$, so the usual way to discharge them is to write *by this* (*intro cont2cont*) $+$ at the end.

```

lemma below-trans-cong[trans]:
   $a \sqsubseteq f x \implies x \sqsubseteq y \implies \text{cont } f \implies a \sqsubseteq f y$ 
by (metis below-trans cont2monofunE)

```

```

lemma not-bot-below-trans[trans]:
   $a \neq \perp \implies a \sqsubseteq b \implies b \neq \perp$ 
by (metis below-bottom-iff)

```

```

lemma not-bot-below-trans-cong[trans]:
   $f a \neq \perp \implies a \sqsubseteq b \implies \text{cont } f \implies f b \neq \perp$ 
by (metis below-bottom-iff cont2monofunE)

```

end

2.7 HOLCF-Meet

```

theory HOLCF-Meet
imports HOLCF
begin

```

This theory defines the \sqcap operator on HOLCF domains, and introduces a type class for domains where all finite meets exist.

2.7.1 Towards meets: Lower bounds

context *po*

begin

definition *is-lb* :: 'a set \Rightarrow 'a \Rightarrow bool (infix $>|$ 55) **where**
 $S >| x \longleftrightarrow (\forall y \in S. x \sqsubseteq y)$

lemma *is-lbI*: $(!!x. x \in S \implies l \sqsubseteq x) \implies S >| l$
by (*simp add: is-lb-def*)

lemma *is-lbD*: $[|S >| l; x \in S|] \implies l \sqsubseteq x$
by (*simp add: is-lb-def*)

lemma *is-lb-empty* [*simp*]: $\{\} >| l$
unfolding *is-lb-def* **by** *fast*

lemma *is-lb-insert* [*simp*]: $(\text{insert } x \ A) >| y = (y \sqsubseteq x \wedge A >| y)$
unfolding *is-lb-def* **by** *fast*

lemma *is-lb-downward*: $[|S >| l; y \sqsubseteq l|] \implies S >| y$
unfolding *is-lb-def* **by** (*fast intro: below-trans*)

2.7.2 Greatest lower bounds

definition *is-glb* :: 'a set \Rightarrow 'a \Rightarrow bool (infix $>>|$ 55) **where**
 $S >>| x \longleftrightarrow S >| x \wedge (\forall u. S >| u \longrightarrow u \sqsubseteq x)$

definition *glb* :: 'a set \Rightarrow 'a (\lceil - [60]60) **where**
 $\text{glb } S = (\text{THE } x. S >>| x)$

Access to the definition as inference rule

lemma *is-glbD1*: $S >>| x \implies S >| x$
unfolding *is-glb-def* **by** *fast*

lemma *is-glbD2*: $[|S >>| x; S >| u|] \implies u \sqsubseteq x$
unfolding *is-glb-def* **by** *fast*

lemma (in *po*) *is-glbI*: $[|S >| x; !!u. S >| u \implies u \sqsubseteq x|] \implies S >>| x$
unfolding *is-glb-def* **by** *fast*

lemma *is-glb-above-iff*: $S >>| x \implies u \sqsubseteq x \longleftrightarrow S >| u$
unfolding *is-glb-def is-lb-def* **by** (*metis below-trans*)

glbs are unique

lemma *is-glb-unique*: $[|S >>| x; S >>| y|] \implies x = y$
unfolding *is-glb-def is-lb-def* **by** (*blast intro: below-antisym*)

technical lemmas about *glb* and ($>>|$)

lemma *is-glb-glb*: $M \gg | x \implies M \gg | \text{glb } M$
unfolding *glb-def* **by** (*rule theI [OF - is-glb-unique]*)

lemma *glb-eqI*: $M \gg | l \implies \text{glb } M = l$
by (*rule is-glb-unique [OF is-glb-glb]*)

lemma *is-glb-singleton*: $\{x\} \gg | x$
by (*simp add: is-glb-def*)

lemma *glb-singleton [simp]*: $\text{glb } \{x\} = x$
by (*rule is-glb-singleton [THEN glb-eqI]*)

lemma *is-glb-bin*: $x \sqsubseteq y \implies \{x, y\} \gg | x$
by (*simp add: is-glb-def*)

lemma *glb-bin*: $x \sqsubseteq y \implies \text{glb } \{x, y\} = x$
by (*rule is-glb-bin [THEN glb-eqI]*)

lemma *is-glb-maximal*: $[|S \gg | x; x \in S|] \implies S \gg | x$
by (*erule is-glbI, erule (1) is-lbD*)

lemma *glb-maximal*: $[|S \gg | x; x \in S|] \implies \text{glb } S = x$
by (*rule is-glb-maximal [THEN glb-eqI]*)

lemma *glb-above*: $S \gg | z \implies x \sqsubseteq \text{glb } S \iff S \gg | x$
by (*metis glb-eqI is-glb-above-iff*)

end

lemma (*in cpo*) *Meet-insert*: $S \gg | l \implies \{x, l\} \gg | l2 \implies \text{insert } x S \gg | l2$
apply (*rule is-glbI*)
apply (*metis is-glb-above-iff is-glb-def is-lb-insert*)
by (*metis is-glb-above-iff is-glb-def is-glb-singleton is-lb-insert*)

Binary, hence finite meets.

class *Finite-Meet-cpo* = *cpo* +
assumes *binary-meet-exists*: $\exists l. l \sqsubseteq x \wedge l \sqsubseteq y \wedge (\forall z. z \sqsubseteq x \longrightarrow z \sqsubseteq y \longrightarrow z \sqsubseteq l)$
begin

lemma *binary-meet-exists'*: $\exists l. \{x, y\} \gg | l$
using *binary-meet-exists[of x y]*
unfolding *is-glb-def is-lb-def*
by *auto*

lemma *finite-meet-exists*:
assumes $S \neq \{\}$
and *finite S*
shows $\exists x. S \gg | x$
using $\langle S \neq \{\} \rangle$
apply (*induct rule: finite-induct[OF \langle finite S \rangle]*)

```

apply (erule notE, rule refl)[1]
apply (case-tac F = {})
apply (metis is-glb-singleton)
apply (metis Meet-insert binary-meet-exists')
done
end

```

definition *meet* :: 'a::cpo \Rightarrow 'a \Rightarrow 'a (**infix** \sqcap 80) **where**
 $x \sqcap y = (\text{if } \exists z. \{x, y\} \gg | z \text{ then } \text{glb } \{x, y\} \text{ else } x)$

lemma *meet-def'*: (x::'a::Finite-Meet-cpo) \sqcap y = *glb* {x, y}
unfolding *meet-def* **by** (metis binary-meet-exists')

lemma *meet-comm*: (x::'a::Finite-Meet-cpo) \sqcap y = y \sqcap x **unfolding** *meet-def'* **by** (metis insert-commute)

lemma *meet-bot1*[simp]:
fixes y :: 'a :: {Finite-Meet-cpo,pcpo}
shows ($\perp \sqcap y$) = \perp **unfolding** *meet-def'* **by** (metis minimal-po-class.glb-bin)

lemma *meet-bot2*[simp]:
fixes x :: 'a :: {Finite-Meet-cpo,pcpo}
shows (x \sqcap \perp) = \perp **by** (metis meet-bot1 meet-comm)

lemma *meet-below1*[intro]:
fixes x y :: 'a :: Finite-Meet-cpo
assumes $x \sqsubseteq z$
shows (x \sqcap y) \sqsubseteq z **unfolding** *meet-def'* **by** (metis assms binary-meet-exists' below-trans glb-eqI is-glbD1 is-lb-insert)

lemma *meet-below2*[intro]:
fixes x y :: 'a :: Finite-Meet-cpo
assumes $y \sqsubseteq z$
shows (x \sqcap y) \sqsubseteq z **unfolding** *meet-def'* **by** (metis assms binary-meet-exists' below-trans glb-eqI is-glbD1 is-lb-insert)

lemma *meet-above-iff*:
fixes x y z :: 'a :: Finite-Meet-cpo
shows $z \sqsubseteq x \sqcap y \longleftrightarrow z \sqsubseteq x \wedge z \sqsubseteq y$

proof –
obtain g **where** {x,y} $\gg |$ g **by** (metis binary-meet-exists')
thus ?thesis
unfolding *meet-def'* **by** (simp add: glb-above)
qed

lemma *below-meet*[simp]:
fixes x y :: 'a :: Finite-Meet-cpo
assumes $x \sqsubseteq z$
shows (x \sqcap z) = x **by** (metis assms glb-bin meet-def')

lemma *below-meet2*[simp]:

fixes $x y :: 'a :: \text{Finite-Meet-cpo}$
assumes $z \sqsubseteq x$
shows $(x \sqcap z) = z$ **by** $(\text{metis assms below-meet meet-comm})$

lemma *meet-aboveI*:
fixes $x y z :: 'a :: \text{Finite-Meet-cpo}$
shows $z \sqsubseteq x \implies z \sqsubseteq y \implies z \sqsubseteq x \sqcap y$ **by** $(\text{simp add: meet-above-iff})$

lemma *is-meetI*:
fixes $x y z :: 'a :: \text{Finite-Meet-cpo}$
assumes $z \sqsubseteq x$
assumes $z \sqsubseteq y$
assumes $\bigwedge a. [a \sqsubseteq x ; a \sqsubseteq y] \implies a \sqsubseteq z$
shows $x \sqcap y = z$
by $(\text{metis assms below-antisym meet-above-iff below-refl})$

lemma *meet-assoc[simp]*: $((x :: 'a :: \text{Finite-Meet-cpo}) \sqcap y) \sqcap z = x \sqcap (y \sqcap z)$
apply (rule is-meetI)
apply $(\text{metis below-refl meet-above-iff})$
apply $(\text{metis below-refl meet-below2})$
apply $(\text{metis meet-above-iff})$
done

lemma *meet-self[simp]*: $r \sqcap r = (r :: 'a :: \text{Finite-Meet-cpo})$
by $(\text{metis below-refl is-meetI})$

lemma *[simp]*: $(r :: 'a :: \text{Finite-Meet-cpo}) \sqcap (r \sqcap x) = r \sqcap x$
by $(\text{metis below-refl is-meetI meet-below1})$

lemma *meet-monofun1*:
fixes $y :: 'a :: \text{Finite-Meet-cpo}$
shows *monofun* $(\lambda x. (x \sqcap y))$
by $(\text{rule monofunI})(\text{auto simp add: meet-above-iff})$

lemma *chain-meet1*:
fixes $y :: 'a :: \text{Finite-Meet-cpo}$
assumes *chain* Y
shows *chain* $(\lambda i. Y i \sqcap y)$
by $(\text{rule chainI})(\text{auto simp add: meet-above-iff intro: chainI chainE[OF assms]})$

class *cont-binary-meet* = *Finite-Meet-cpo* +
assumes *meet-cont'*: $\text{chain } Y \implies (\bigsqcap i. Y i) \sqcap y = (\bigsqcap i. Y i \sqcap y)$

lemma *meet-cont1*:
fixes $y :: 'a :: \text{cont-binary-meet}$
shows *cont* $(\lambda x. (x \sqcap y))$
by $(\text{rule contI2[OF meet-monofun1]})(\text{simp add: meet-cont'})$

lemma *meet-cont2*:

```

fixes  $x :: 'a :: cont-binary-meet$ 
shows  $cont (\lambda y. (x \sqcap y))$  by (subst meet-comm, rule meet-cont1)

```

```

lemma meet-cont[cont2cont,simp]: cont f  $\implies$  cont g  $\implies$  cont ( $\lambda x. (fx \sqcap (g x :: 'a :: cont-binary-meet))$ )
apply (rule cont2cont-case-prod[where g =  $\lambda x. (fx, gx)$  and f =  $\lambda p x y . x \sqcap y$ , simplified])
apply (rule meet-cont1)
apply (rule meet-cont2)
apply (metis cont2cont-Pair)
done

```

end

2.8 Nominal-HOLCF

```

theory Nominal-HOLCF
imports
  Nominal-Utils HOLCF-Utils
begin

```

2.8.1 Type class of continuous permutations and variations thereof

```

class cont-pt =
  cpo +
  pt +
  assumes perm-cont:  $\bigwedge p. cont ((permute\ p) :: 'a :: \{cpo, pt\} \Rightarrow 'a)$ 

```

```

class discr-pt =
  discrete-cpo +
  pt

```

```

class pcpo-pt =
  cont-pt +
  pcpo

```

```

instance pcpo-pt  $\subseteq$  cont-pt
by standard (auto intro: perm-cont)

```

```

instance discr-pt  $\subseteq$  cont-pt
by standard auto

```

```

lemma (in cont-pt) perm-cont-simp[simp]:  $\pi \cdot x \sqsubseteq \pi \cdot y \iff x \sqsubseteq y$ 
apply rule
apply (drule cont2monofunE[OF perm-cont, of - -  $-\pi$ ], simp)[1]
apply (erule cont2monofunE[OF perm-cont, of - -  $\pi$ ])
done

```

```

lemma (in cont-pt) perm-below-to-right:  $\pi \cdot x \sqsubseteq y \iff x \sqsubseteq - \pi \cdot y$ 
by (metis perm-cont-simp pt-class.permute-minus-cancel(2))

```

lemma *perm-is-ub-simp*[simp]: $\pi \cdot S <| \pi \cdot (x::'a::\text{cont-pt}) \longleftrightarrow S <| x$
by (*auto simp add: is-ub-def permute-set-def*)

lemma *perm-is-ub-eqvt*[simp,eqvt]: $S <| (x::'a::\text{cont-pt}) \implies \pi \cdot S <| \pi \cdot x$
by *simp*

lemma *perm-is-lub-simp*[simp]: $\pi \cdot S <<| \pi \cdot (x::'a::\text{cont-pt}) \longleftrightarrow S <<| x$
apply (*rule perm-rel-lemma*)
by (*metis is-lubI is-lubD1 is-lubD2 perm-cont-simp perm-is-ub-simp*)

lemma *perm-is-lub-eqvt*[simp,eqvt]: $S <<| (x::'a::\text{cont-pt}) \implies \pi \cdot S <<| \pi \cdot x$
by *simp*

lemmas *perm-cont2cont*[simp,cont2cont] = *cont-compose*[OF *perm-cont*]

lemma *perm-still-cont*: $\text{cont} (\pi \cdot f) = \text{cont} (f :: ('a :: \text{cont-pt}) \Rightarrow ('b :: \text{cont-pt}))$

proof

have $\text{imp}:\bigwedge (f :: 'a \Rightarrow 'b) \pi. \text{cont } f \implies \text{cont} (\pi \cdot f)$

unfolding *permute-fun-def*

by (*metis cont-compose perm-cont*)

show $\text{cont } f \implies \text{cont} (\pi \cdot f)$ **using** *imp*[of *f* π].

show $\text{cont} (\pi \cdot f) \implies \text{cont} (f)$ **using** *imp*[of $\pi \cdot f$ $-\pi$] **by** *simp*

qed

lemma *perm-bottom*[simp,eqvt]: $\pi \cdot \perp = (\perp::'a::\{\text{cont-pt},\text{pcpo}\})$

proof–

have $\perp \sqsubseteq -\pi \cdot (\perp::'a::\{\text{cont-pt},\text{pcpo}\})$ **by** *simp*

hence $\pi \cdot \perp \sqsubseteq \pi \cdot (-\pi \cdot (\perp::'a::\{\text{cont-pt},\text{pcpo}\}))$ **by**(*rule cont2monofunE*[OF *perm-cont*])

hence $\pi \cdot \perp \sqsubseteq (\perp::'a::\{\text{cont-pt},\text{pcpo}\})$ **by** *simp*

thus $\pi \cdot \perp = (\perp::'a::\{\text{cont-pt},\text{pcpo}\})$ **by** *simp*

qed

lemma *bot-supp*[simp]: $\text{supp} (\perp :: 'a :: \text{pcpo-pt}) = \{\}$

by (*rule supp-fun-eqvt*) (*simp add: eqvt-def*)

lemma *bot-fresh*[simp]: $a \# (\perp :: 'a :: \text{pcpo-pt})$

by (*simp add: fresh-def*)

lemma *bot-fresh-star*[simp]: $a \#* (\perp :: 'a :: \text{pcpo-pt})$

by (*simp add: fresh-star-def*)

lemma *below-eqvt* [eqvt]:

$\pi \cdot (x \sqsubseteq y) = (\pi \cdot x \sqsubseteq \pi \cdot (y::'a::\text{cont-pt}))$ **by** (*auto simp add: permute-pure*)

lemma *lub-eqvt*[simp]:

$(\exists z. S <<| (z::'a::\{\text{cont-pt}\})) \implies \pi \cdot \text{lub } S = \text{lub} (\pi \cdot S)$

by (*metis lub-eqI perm-is-lub-simp*)


```

lemma chain-eqvt[eqvt]:
  fixes F :: nat => 'a::cont-pt
  shows chain F ==> chain ( $\pi \cdot F$ )
  apply (rule chainI)
  apply (drule-tac i = i in chainE)
  apply (subst (asm) perm-cont-simp[symmetric, where  $\pi = \pi$ ])
  by (metis permute-fun-app-eq permute-pure)

```

2.8.2 Instance for *cfun*

```

instantiation cfun :: (cont-pt, cont-pt) pt
begin
  definition p · (f :: 'a → 'b) = ( $\Lambda x. p \cdot (f \cdot (- p \cdot x))$ )

```

```

  instance
  apply standard
  apply (simp add: permute-cfun-def eta-cfun)
  apply (simp add: permute-cfun-def cfun-eqI minus-add)
  done
end

```

```

lemma permute-cfun-eq: permute p = ( $\lambda f. (Abs-cfun (permute p)) \circ f \circ (Abs-cfun (permute (-p)))$ )
by (rule, rule cfun-eqI, auto simp add: permute-cfun-def)

```

```

lemma Cfun-app-eqvt[eqvt]:
   $\pi \cdot (f \cdot x) = (\pi \cdot f) \cdot (\pi \cdot x)$ 
  unfolding permute-cfun-def
  by auto

```

```

lemma permute-Lam: cont f ==> p · ( $\Lambda x. f x$ ) = ( $\Lambda x. (p \cdot f) x$ )
  apply (rule cfun-eqI)
  unfolding permute-cfun-def
  by (metis Abs-cfun-inverse2 eqvt-lambda unpermute-def )

```

```

lemma Abs-cfun-eqvt: cont f ==> (p · Abs-cfun) f = Abs-cfun f
  apply (subst permute-fun-def)
  by (metis permute-Lam perm-still-cont permute-minus-cancel(1))

```

```

lemma cfun-eqvtI: ( $\Lambda x. p \cdot (f \cdot x) = f' \cdot (p \cdot x)$ ) ==> p · f = f'
  by (metis Cfun-app-eqvt cfun-eqI permute-minus-cancel(1))

```

```

lemma ID-eqvt[eqvt]:  $\pi \cdot ID = ID$ 
  unfolding ID-def
  apply perm-simp
  apply (simp add: Abs-cfun-eqvt)
  done

```

```

instance cfun :: (cont-pt, cont-pt) cont-pt

```

by standard (subst permute-cfun-eq, auto)

instance cfun :: ({pure,cont-pt}, {pure,cont-pt}) pure
by standard (auto simp add: permute-cfun-def permute-pure Cfun.cfun.Rep-cfun-inverse)

instance cfun :: (cont-pt, pcpo-pt) pcpo-pt
by standard

2.8.3 Instance for fun

lemma permute-fun-eq: permute p = ($\lambda f. (permute p) \circ f \circ (permute (-p))$)
by (rule, rule, metis comp-apply eqvt-lambda unpermute-def)

instance fun :: (pt, cont-pt) cont-pt
apply standard
apply (rule cont2cont-lambda)
apply (subst permute-fun-def)
apply (rule perm-cont2cont)
apply (rule cont-fun)
done

lemma fix-eqvt[eqvt]:
 $\pi \cdot fix = (fix :: ('a \rightarrow 'a) \rightarrow 'a :: \{cont-pt, pcpo\})$
apply (rule cfun-eqI)
apply (subst permute-cfun-def)
apply simp
apply (rule parallel-fix-ind[OF adm-subst2])
apply (auto simp add: permute-self)
done

2.8.4 Instance for u

instantiation u :: (cont-pt) pt
begin
definition p · (x :: 'a u) = fup·($\Lambda x. up \cdot (p \cdot x)$)·x
instance
apply standard
apply (case-tac x) apply (auto simp add: permute-u-def)
apply (case-tac x) apply (auto simp add: permute-u-def)
done
end

instance u :: (cont-pt) cont-pt

proof
fix p

have permute p = ($\lambda x. fup \cdot (\Lambda x. up \cdot (p \cdot x)) \cdot (x :: 'a u)$)
by (rule ext, rule permute-u-def)
moreover have cont ($\lambda x. fup \cdot (\Lambda x. up \cdot (p \cdot x)) \cdot (x :: 'a u)$) by simp
ultimately show cont (permute p :: 'a u \Rightarrow 'a u) by simp

qed

instance $u :: (\text{cont-pt}) \text{pcpo-pt} ..$

class $\text{pure-cont-pt} = \text{pure} + \text{cont-pt}$

instance $u :: (\text{pure-cont-pt}) \text{pure}$
 apply *standard*
 apply (*case-tac x*)
 apply (*auto simp add: permute-u-def permute-pure*)
 done

lemma $\text{up-eqvt}[eqvt]: \pi \cdot \text{up} = \text{up}$
 apply (*rule cfun-eqI*)
 apply (*subst permute-cfun-def, simp*)
 apply (*simp add: permute-u-def*)
 done

lemma $\text{fup-eqvt}[eqvt]: \pi \cdot \text{fup} = \text{fup}$
 apply (*rule cfun-eqI*)
 apply (*rule cfun-eqI*)
 apply (*subst permute-cfun-def, simp*)
 apply (*subst permute-cfun-def, simp*)
 apply (*case-tac xa*)
 apply *simp*
 apply (*simp add: permute-self*)
 done

2.8.5 Instance for *lift*

instantiation $\text{lift} :: (\text{pt}) \text{pt}$

begin
 definition $p \cdot (x :: 'a \text{ lift}) = \text{case-lift } \perp (\lambda x. \text{Def } (p \cdot x)) x$
 instance
 apply *standard*
 apply (*case-tac x*) apply (*auto simp add: permute-lift-def*)
 apply (*case-tac x*) apply (*auto simp add: permute-lift-def*)
 done
end

instance $\text{lift} :: (\text{pt}) \text{cont-pt}$

proof

fix p

have $\text{permute } p = (\lambda x. \text{case-lift } \perp (\lambda x. \text{Def } (p \cdot x)) (x :: 'a \text{ lift}))$

by (*rule ext, rule permute-lift-def*)

moreover have $\text{cont } (\lambda x. \text{case-lift } \perp (\lambda x. \text{Def } (p \cdot x)) (x :: 'a \text{ lift}))$ by *simp*

ultimately show $\text{cont } (\text{permute } p :: 'a \text{ lift} \Rightarrow 'a \text{ lift})$ by *simp*

qed

instance *lift* :: (*pt*) *pcpo-pt* ..

instance *lift* :: (*pure*) *pure*
 apply *standard*
 apply (*case-tac* *x*)
 apply (*auto simp add: permute-lift-def permute-pure*)
 done

lemma *Def-eqvt*[*eqvt*]: $\pi \cdot (\text{Def } x) = \text{Def } (\pi \cdot x)$
 by (*simp add: permute-lift-def*)

lemma *case-lift-eqvt*[*eqvt*]: $\pi \cdot \text{case-lift } d f x = \text{case-lift } (\pi \cdot d) (\pi \cdot f) (\pi \cdot x)$
 by (*cases* *x*) (*auto simp add: permute-self*)

2.8.6 Instance for *prod*

instance *prod* :: (*cont-pt*, *cont-pt*) *cont-pt*

proof

fix *p*

have *permute* *p* = $(\lambda (x :: ('a, 'b) \text{prod}). (p \cdot \text{fst } x, p \cdot \text{snd } x))$ **by** *auto*
 moreover have *cont* ... **by** (*intro cont2cont*)

ultimately show *cont* (*permute* *p* :: ('a,'b) *prod* \Rightarrow ('a,'b) *prod*) **by** *simp*
qed

end

2.9 Env

theory *Env*

imports *Main HOLCF-Join-Classes*

begin

default-sort *type*

Our type for environments is a function with a pcpo as the co-domain; this theory collects related definitions.

2.9.1 The domain of a pcpo-valued function

definition *edom* :: ('key \Rightarrow 'value::pcpo) \Rightarrow 'key *set*
 where *edom* *m* = $\{x. m \ x \neq \perp\}$

lemma *bot-edom*[*simp*]: *edom* $\perp = \{\}$ **by** (*simp add: edom-def*)

lemma *bot-edom2*[simp]: $\text{edom } (\lambda \cdot \perp) = \{\}$ **by** (*simp add: edom-def*)

lemma *edomIff*: $(a \in \text{edom } m) = (m \ a \neq \perp)$ **by** (*simp add: edom-def*)

lemma *edom-iff2*: $(m \ a = \perp) \longleftrightarrow (a \notin \text{edom } m)$ **by** (*simp add: edom-def*)

lemma *edom-empty-iff-bot*: $\text{edom } m = \{\} \longleftrightarrow m = \perp$
by (*metis below-bottom-iff bot-edom edomIff empty-iff fun-belowI*)

lemma *lookup-not-edom*: $x \notin \text{edom } m \implies m \ x = \perp$ **by** (*auto iff:edomIff*)

lemma *lookup-edom*[simp]: $m \ x \neq \perp \implies x \in \text{edom } m$ **by** (*auto iff:edomIff*)

lemma *edom-mono*: $x \sqsubseteq y \implies \text{edom } x \subseteq \text{edom } y$
unfolding *edom-def*
by *auto* (*metis below-bottom-iff fun-belowD*)

lemma *edom-subset-adm*[simp]:
adm $(\lambda ae'. \text{edom } ae' \subseteq S)$
apply (*rule admI*)
apply *rule*
apply (*subst (asm) edom-def*) **back**
apply *simp*
apply (*subst (asm) lub-fun*) **apply** *assumption*
apply (*subst (asm) lub-eq-bottom-iff*)
apply (*erule ch2ch-fun*)
unfolding *not-all*
apply (*erule exE*)
apply (*rule subsetD*)
apply (*rule allE*) **apply** *assumption* **apply** *assumption*
unfolding *edom-def*
apply *simp*
done

2.9.2 Updates

lemma *edom-fun-upd-subset*: $\text{edom } (h \ (x := v)) \subseteq \text{insert } x \ (\text{edom } h)$
by (*auto simp add: edom-def*)

declare *fun-upd-same*[simp] *fun-upd-other*[simp]

2.9.3 Restriction

definition *env-restr* :: $'a \text{ set} \Rightarrow ('a \Rightarrow 'b::\text{pcpo}) \Rightarrow ('a \Rightarrow 'b)$
where *env-restr* $S \ m = (\lambda x. \text{if } x \in S \text{ then } m \ x \text{ else } \perp)$

abbreviation *env-restr-rev* (**infixl** $f|'$ 110)
where *env-restr-rev* $m \ S \equiv \text{env-restr } S \ m$

notation (*latex output*) *env-restr-rev* $(-|_)$

lemma *env-restr-empty-iff*[simp]: $m f|' S = \perp \iff \text{edom } m \cap S = \{\}$
apply (*auto simp add: edom-def env-restr-def lambda-strict[symmetric] split:if-splits*)
apply *metis*
apply (*fastforce simp add: edom-def env-restr-def lambda-strict[symmetric] split:if-splits*)
done
lemmas *env-restr-empty* = *iffD2[OF env-restr-empty-iff, simp]*

lemma *lookup-env-restr*[simp]: $x \in S \implies (m f|' S) x = m x$
by (*fastforce simp add: env-restr-def*)

lemma *lookup-env-restr-not-there*[simp]: $x \notin S \implies (\text{env-restr } S m) x = \perp$
by (*fastforce simp add: env-restr-def*)

lemma *lookup-env-restr-eq*: $(m f|' S) x = (\text{if } x \in S \text{ then } m x \text{ else } \perp)$
by *simp*

lemma *env-restr-eqI*: $(\bigwedge x. x \in S \implies m_1 x = m_2 x) \implies m_1 f|' S = m_2 f|' S$
by (*auto simp add: lookup-env-restr-eq*)

lemma *env-restr-eqD*: $m_1 f|' S = m_2 f|' S \implies x \in S \implies m_1 x = m_2 x$
by (*auto dest!: fun-cong[where x = x]*)

lemma *env-restr-belowI*: $(\bigwedge x. x \in S \implies m_1 x \sqsubseteq m_2 x) \implies m_1 f|' S \sqsubseteq m_2 f|' S$
by (*auto intro: fun-belowI simp add: lookup-env-restr-eq*)

lemma *env-restr-belowD*: $m_1 f|' S \sqsubseteq m_2 f|' S \implies x \in S \implies m_1 x \sqsubseteq m_2 x$
by (*auto dest!: fun-belowD[where x = x]*)

lemma *env-restr-env-restr*[simp]:
 $x f|' d2 f|' d1 = x f|' (d1 \cap d2)$
by (*fastforce simp add: env-restr-def*)

lemma *env-restr-env-restr-subset*:
 $d1 \subseteq d2 \implies x f|' d2 f|' d1 = x f|' d1$
by (*metis Int-absorb2 env-restr-env-restr*)

lemma *env-restr-useless*: $\text{edom } m \subseteq S \implies m f|' S = m$
by (*rule ext*) (*auto simp add: lookup-env-restr-eq dest!: subsetD*)

lemma *env-restr-UNIV*[simp]: $m f|' \text{UNIV} = m$
by (*rule env-restr-useless*) *simp*

lemma *env-restr-fun-upd*[simp]: $x \in S \implies m1(x := v) f|' S = (m1 f|' S)(x := v)$
apply (*rule ext*)
apply (*case-tac xa = x*)
apply (*auto simp add: lookup-env-restr-eq*)
done

lemma *env-restr-fun-upd-other*[simp]: $x \notin S \implies m1(x := v) f|' S = m1 f|' S$
apply (*rule ext*)
apply (*case-tac xa = x*)
apply (*auto simp add: lookup-env-restr-eq*)
done

lemma *env-restr-eq-subset*:
assumes $S \subseteq S'$
and $m1 f|' S' = m2 f|' S'$
shows $m1 f|' S = m2 f|' S$
using *assms*
by (*metis env-restr-env-restr le-iff-inf*)

lemma *env-restr-below-subset*:
assumes $S \subseteq S'$
and $m1 f|' S' \sqsubseteq m2 f|' S'$
shows $m1 f|' S \sqsubseteq m2 f|' S$
using *assms*
by (*auto intro!: env-restr-belowI dest!: env-restr-belowD*)

lemma *edom-env*[simp]:
 $edom (m f|' S) = edom m \cap S$
unfolding *edom-def env-restr-def* **by** *auto*

lemma *env-restr-below-self*: $f f|' S \sqsubseteq f$
by (*rule fun-belowI*) (*auto simp add: env-restr-def*)

lemma *env-restr-below-trans*:
 $m1 f|' S1 \sqsubseteq m2 f|' S1 \implies m2 f|' S2 \sqsubseteq m3 f|' S2 \implies m1 f|' (S1 \cap S2) \sqsubseteq m3 f|' (S1 \cap S2)$
by (*auto intro!: env-restr-belowI dest!: env-restr-belowD elim: below-trans*)

lemma *env-restr-cont*: *cont* (*env-restr* S)
apply (*rule cont2cont-lambda*)
unfolding *env-restr-def*
apply (*intro cont2cont cont-fun*)
done

lemma *env-restr-mono*: $m1 \sqsubseteq m2 \implies m1 f|' S \sqsubseteq m2 f|' S$
by (*metis env-restr-belowI fun-belowD*)

lemma *env-restr-mono2*: $S2 \subseteq S1 \implies m f|' S2 \sqsubseteq m f|' S1$
by (*metis env-restr-below-self env-restr-env-restr-subset*)

lemmas *cont-compose*[*OF env-restr-cont, cont2cont, simp*]

lemma *env-restr-cong*: $(\bigwedge x. edom m \subseteq S \cap S' \cup -S \cap -S') \implies m f|' S = m f|' S'$
by (*rule ext*)(*auto simp add: lookup-env-restr-eq edom-def*)

2.9.4 Deleting

definition $env\text{-}delete :: 'a \Rightarrow ('a \Rightarrow 'b) \Rightarrow ('a \Rightarrow 'b::pcpo)$
where $env\text{-}delete\ x\ m = m(x := \perp)$

lemma $lookup\text{-}env\text{-}delete[simp]$:
 $x' \neq x \Longrightarrow env\text{-}delete\ x\ m\ x' = m\ x'$
by ($simp\ add: env\text{-}delete\text{-}def$)

lemma $lookup\text{-}env\text{-}delete\text{-}None[simp]$:
 $env\text{-}delete\ x\ m\ x = \perp$
by ($simp\ add: env\text{-}delete\text{-}def$)

lemma $edom\text{-}env\text{-}delete[simp]$:
 $edom\ (env\text{-}delete\ x\ m) = edom\ m - \{x\}$
by ($auto\ simp\ add: env\text{-}delete\text{-}def\ edom\text{-}def$)

lemma $edom\text{-}env\text{-}delete\text{-}subset$:
 $edom\ (env\text{-}delete\ x\ m) \subseteq edom\ m$ **by** $auto$

lemma $env\text{-}delete\text{-}fun\text{-}upd[simp]$:
 $env\text{-}delete\ x\ (m(x := v)) = env\text{-}delete\ x\ m$
by ($auto\ simp\ add: env\text{-}delete\text{-}def$)

lemma $env\text{-}delete\text{-}fun\text{-}upd2[simp]$:
 $(env\text{-}delete\ x\ m)(x := v) = m(x := v)$
by ($auto\ simp\ add: env\text{-}delete\text{-}def$)

lemma $env\text{-}delete\text{-}fun\text{-}upd3[simp]$:
 $x \neq y \Longrightarrow env\text{-}delete\ x\ (m(y := v)) = (env\text{-}delete\ x\ m)(y := v)$
by ($auto\ simp\ add: env\text{-}delete\text{-}def$)

lemma $env\text{-}delete\text{-}noop[simp]$:
 $x \notin edom\ m \Longrightarrow env\text{-}delete\ x\ m = m$
by ($auto\ simp\ add: env\text{-}delete\text{-}def\ edom\text{-}def$)

lemma $fun\text{-}upd\text{-}env\text{-}delete[simp]$: $x \in edom\ \Gamma \Longrightarrow (env\text{-}delete\ x\ \Gamma)(x := \Gamma\ x) = \Gamma$
by ($auto$)

lemma $env\text{-}restr\text{-}env\text{-}delete\text{-}other[simp]$: $x \notin S \Longrightarrow env\text{-}delete\ x\ m\ f|'S = m\ f|'S$
apply ($rule\ ext$)
apply ($auto\ simp\ add: lookup\text{-}env\text{-}restr\text{-}eq$)
by ($metis\ lookup\text{-}env\text{-}delete$)

lemma $env\text{-}delete\text{-}restr$: $env\text{-}delete\ x\ m = m\ f|'(-\{x\})$
by ($auto\ simp\ add: lookup\text{-}env\text{-}restr\text{-}eq$)

lemma $below\text{-}env\text{-}deleteI$: $f\ x = \perp \Longrightarrow f \sqsubseteq g \Longrightarrow f \sqsubseteq env\text{-}delete\ x\ g$
by ($metis\ env\text{-}delete\text{-}def\ env\text{-}delete\text{-}restr\ env\text{-}restr\text{-}mono\ fun\text{-}upd\text{-}triv$)

lemma *env-delete-below-cong*[intro]:
assumes $x \neq v \implies e1\ x \sqsubseteq e2\ x$
shows $env\ delete\ v\ e1\ x \sqsubseteq env\ delete\ v\ e2\ x$
using *assms* **unfolding** *env-delete-def* **by** *auto*

lemma *env-delete-env-restr-swap*:
 $env\ delete\ x\ (env\ restr\ S\ e) = env\ restr\ S\ (env\ delete\ x\ e)$
by (*metis* (*erased*, *opaque-lifting*) *env-delete-def env-restr-fun-upd env-restr-fun-upd-other fun-upd-triv lookup-env-restr-eq*)

lemma *env-delete-mono*:
 $m \sqsubseteq m' \implies env\ delete\ x\ m \sqsubseteq env\ delete\ x\ m'$
unfolding *env-delete-restr*
by (*rule env-restr-mono*)

lemma *env-delete-below-arg*:
 $env\ delete\ x\ m \sqsubseteq m$
unfolding *env-delete-restr*
by (*rule env-restr-below-self*)

2.9.5 Merging of two functions

We'd like to have some nice syntax for *override-on*.

abbreviation *override-on-syn* $(- \ ++_S \ - \ [100, 0, 100] \ 100)$ **where** $f1 \ ++_S \ f2 \equiv override\ on\ f1\ f2\ S$

lemma *override-on-bot*[simp]:
 $\perp \ ++_S \ m = m\ f|' \ S$
 $m \ ++_S \ \perp = m\ f|' \ (-S)$
by (*auto simp add: override-on-def env-restr-def*)

lemma *edom-override-on*[simp]: $edom\ (m1 \ ++_S \ m2) = (edom\ m1 \ - \ S) \cup (edom\ m2 \ \cap \ S)$
by (*auto simp add: override-on-def edom-def*)

lemma *lookup-override-on-eq*: $(m1 \ ++_S \ m2)\ x = (if\ x \in S\ then\ m2\ x\ else\ m1\ x)$
by (*cases x \notin S*) *simp-all*

lemma *override-on-upd-swap*:
 $x \notin S \implies \varrho(x := z) \ ++_S \ \varrho' = (\varrho \ ++_S \ \varrho')(x := z)$
by (*auto simp add: override-on-def edom-def*)

lemma *override-on-upd*:
 $x \in S \implies \varrho \ ++_S \ (\varrho'(x := z)) = (\varrho \ ++_S \ - \ \{x\} \ \varrho')(x := z)$
by (*auto simp add: override-on-def edom-def*)

lemma *env-restr-add*: $(m1 \ ++_{S2} \ m2)\ f|' \ S = m1\ f|' \ S \ ++_{S2} \ m2\ f|' \ S$
by (*auto simp add: override-on-def edom-def env-restr-def*)

lemma *env-delete-add*: $env\text{-delete } x (m1 ++_S m2) = env\text{-delete } x m1 ++_S - \{x\} env\text{-delete } x m2$

by (*auto simp add: override-on-def edom-def env-restr-def env-delete-def*)

2.9.6 Environments with binary joins

lemma *edom-join[simp]*: $edom (f \sqcup (g :: ('a::type \Rightarrow 'b::\{Finite-Join-cpo,pcpo\}))) = edom f \cup edom g$

unfolding *edom-def* **by** *auto*

lemma *env-delete-join[simp]*: $env\text{-delete } x (f \sqcup (g :: ('a::type \Rightarrow 'b::\{Finite-Join-cpo,pcpo\}))) = env\text{-delete } x f \sqcup env\text{-delete } x g$

by (*metis env-delete-def fun-upd-meet-simp*)

lemma *env-restr-join*:

fixes $m1 m2 :: 'a::type \Rightarrow 'b::\{Finite-Join-cpo,pcpo\}$

shows $(m1 \sqcup m2) f|' S = (m1 f|' S) \sqcup (m2 f|' S)$

by (*auto simp add: env-restr-def*)

lemma *env-restr-join2*:

fixes $m :: 'a::type \Rightarrow 'b::\{Finite-Join-cpo,pcpo\}$

shows $m f|' S \sqcup m f|' S' = m f|' (S \cup S')$

by (*auto simp add: env-restr-def*)

lemma *join-env-restr-UNIV*:

fixes $m :: 'a::type \Rightarrow 'b::\{Finite-Join-cpo,pcpo\}$

shows $S1 \cup S2 = UNIV \Longrightarrow (m f|' S1) \sqcup (m f|' S2) = m$

by (*fastforce simp add: env-restr-def*)

lemma *env-restr-split*:

fixes $m :: 'a::type \Rightarrow 'b::\{Finite-Join-cpo,pcpo\}$

shows $m = m f|' S \sqcup m f|' (- S)$

by (*simp add: env-restr-join2 Compl-partition*)

lemma *env-restr-below-split*:

$m f|' S \sqsubseteq m' \Longrightarrow m f|' (- S) \sqsubseteq m' \Longrightarrow m \sqsubseteq m'$

by (*metis ComplI fun-below-iff lookup-env-restr*)

2.9.7 Singleton environments

definition *esing* :: $'a \Rightarrow 'b::\{pcpo\} \rightarrow ('a \Rightarrow 'b)$

where $esing x = (\Lambda a. (\lambda y. (if x = y then a else \perp)))$

lemma *esing-bot[simp]*: $esing x \cdot \perp = \perp$

by (*rule ext*)(*simp add: esing-def*)

lemma *esing-simps[simp]*:

$(esing x \cdot n) x = n$

$x' \neq x \Longrightarrow (esing x \cdot n) x' = \perp$

by (*simp-all add: esing-def*)

lemma *esing-eq-up-iff*[simp]: $(\text{esing } x \cdot (\text{up } a)) \ y = \text{up} \cdot a' \longleftrightarrow (x = y \wedge a = a')$
by (*auto simp add: fun-below-iff esing-def*)

lemma *esing-below-iff*[simp]: $\text{esing } x \cdot a \sqsubseteq ae \longleftrightarrow a \sqsubseteq ae \ x$
by (*auto simp add: fun-below-iff esing-def*)

lemma *edom-esing-subset*: $\text{edom } (\text{esing } x \cdot n) \subseteq \{x\}$
unfolding *edom-def esing-def* **by** *auto*

lemma *edom-esing-up*[simp]: $\text{edom } (\text{esing } x \cdot (\text{up} \cdot n)) = \{x\}$
unfolding *edom-def esing-def* **by** *auto*

lemma *env-delete-esing*[simp]: $\text{env-delete } x \ (\text{esing } x \cdot n) = \perp$
unfolding *env-delete-def esing-def*
by *auto*

lemma *env-restr-esing*[simp]:
 $x \in S \implies \text{esing } x \cdot v \ f \upharpoonright S = \text{esing } x \cdot v$
by (*auto intro: env-restr-useless dest: subsetD[OF edom-esing-subset]*)

lemma *env-restr-esing2*[simp]:
 $x \notin S \implies \text{esing } x \cdot v \ f \upharpoonright S = \perp$
by (*auto dest: subsetD[OF edom-esing-subset]*)

lemma *esing-eq-iff*[simp]:
 $\text{esing } x \cdot v = \text{esing } x \cdot v' \longleftrightarrow v = v'$
by (*metis esing-simps(1)*)

end

2.10 Env-Nominal

theory *Env-Nominal*
imports *Env Nominal-Utills Nominal-HOLCF*
begin

2.10.1 Equivariance lemmas

lemma *edom-perm*:
fixes $f :: 'a::pt \Rightarrow 'b::\{pcpo-pt\}$
shows $\text{edom } (\pi \cdot f) = \pi \cdot (\text{edom } f)$
by (*simp add: edom-def*)

lemmas *edom-perm-rev*[simp,eqvt] = *edom-perm*[symmetric]

lemma *mem-edom-perm*[simp]:
fixes $\varrho :: 'a::at-base \Rightarrow 'b::\{pcpo-pt\}$

shows $xa \in \text{edom } (p \cdot \varrho) \longleftrightarrow - p \cdot xa \in \text{edom } \varrho$
by (*metis (mono-tags) edom-perm-rev mem-Collect-eq permute-set-eq*)

lemma *env-restr-eqvt*[*eqvt*]:
fixes $m :: 'a::pt \Rightarrow 'b::\{cont-pt,pcpo\}$
shows $\pi \cdot m f|' d = (\pi \cdot m) f|' (\pi \cdot d)$
by (*auto simp add: env-restr-def*)

lemma *env-delete-eqvt*[*eqvt*]:
fixes $m :: 'a::pt \Rightarrow 'b::\{cont-pt,pcpo\}$
shows $\pi \cdot \text{env-delete } x m = \text{env-delete } (\pi \cdot x) (\pi \cdot m)$
by (*auto simp add: env-delete-def*)

lemma *esing-eqvt*[*eqvt*]: $\pi \cdot (\text{esing } x) = \text{esing } (\pi \cdot x)$
unfolding *esing-def*
apply *perm-simp*
apply (*simp add: Abs-cfun-eqvt*)
done

2.10.2 Permutation and restriction

lemma *env-restr-perm*:
fixes $\varrho :: 'a::at-base \Rightarrow 'b::\{pcpo-pt,pure\}$
assumes $\text{supp } p \#* S$ **and** [*simp*]: *finite S*
shows $(p \cdot \varrho) f|' S = \varrho f|' S$
using *assms*
apply –
apply (*rule ext*)
apply (*case-tac x \in S*)
apply (*simp*)
apply (*subst permute-fun-def*)
apply (*simp add: permute-pure*)
apply (*subst perm-supp-eq*)
apply (*auto simp add:perm-supp-eq supp-minus-perm fresh-star-def fresh-def supp-set-elem-finite*)
done

lemma *env-restr-perm'*:
fixes $\varrho :: 'a::at-base \Rightarrow 'b::\{pcpo-pt,pure\}$
assumes $\text{supp } p \#* S$ **and** [*simp*]: *finite S*
shows $p \cdot (\varrho f|' S) = \varrho f|' S$
by (*simp add: perm-supp-eq[OF assms(1)] env-restr-perm[OF assms]*)

lemma *env-restr-flip*:
fixes $\varrho :: 'a::at-base \Rightarrow 'b::\{pcpo-pt,pure\}$
assumes $x \notin S$ **and** $x' \notin S$
shows $((x' \leftrightarrow x) \cdot \varrho) f|' S = \varrho f|' S$
using *assms*
apply –
apply *rule*

apply (*auto simp add: permute-flip-at env-restr-def split-if-splits*)
by (*metis eqvt-lambda flip-at-base-simps(3) minus-flip permute-pure unpermute-def*)

lemma *env-restr-flip'*:
fixes $\varrho :: 'a::at-base \Rightarrow 'b::\{pcpo-pt,pure\}$
assumes $x \notin S$ **and** $x' \notin S$
shows $(x' \leftrightarrow x) \cdot (\varrho f|'S) = \varrho f|'S$
by (*simp add: flip-set-both-not-in[OF assms] env-restr-flip[OF assms]*)

2.10.3 Pure codomains

lemma *edom-fv-pure*:
fixes $f :: ('a::at-base \Rightarrow 'b::\{pcpo,pure\})$
assumes *finite* (*edom f*)
shows $fv\ f \subseteq edom\ f$
using *assms*
proof (*induction edom f arbitrary: f*)
case *empty*
hence $f = \perp$ **unfolding** *edom-def* **by** *auto*
thus *?case* **by** (*auto simp add: fv-def fresh-def supp-def*)
next
case (*insert x S*)
have $f = (env-delete\ x\ f)(x := f\ x)$ **by** *auto*
hence $fv\ f \subseteq fv\ (env-delete\ x\ f) \cup fv\ x \cup fv\ (f\ x)$
using *eqvt-fresh-cong3*[**where** $f = fun-upd$ **and** $x = env-delete\ x\ f$ **and** $y = x$ **and** $z = f\ x$,
OF fun-upd-eqvt]
apply (*auto simp add: fv-def fresh-def*)
by (*metis fresh-def pure-fresh*)
also
from $\langle insert\ x\ S = edom\ f \rangle$ **and** $\langle x \notin S \rangle$
have $S = edom\ (env-delete\ x\ f)$ **by** *auto*
hence $fv\ (env-delete\ x\ f) \subseteq edom\ (env-delete\ x\ f)$ **by** (*rule insert*)
also
have $fv\ (f\ x) = \{\}$ **by** (*rule fv-pure*)
also
from $\langle insert\ x\ S = edom\ f \rangle$ **have** $x \in edom\ f$ **by** *auto*
hence $edom\ (env-delete\ x\ f) \cup fv\ x \cup \{\} \subseteq edom\ f$ **by** *auto*
finally
show *?case* **by** *this* (*intro Un-mono subset-refl*)
qed

end

2.11 Env-HOLCF

theory *Env-HOLCF*

```

imports Env HOLCF-Utills
begin

```

2.11.1 Continuity and pcpo-valued functions

lemma *override-on-belowI*:

```

assumes  $\bigwedge a. a \in S \implies y \sqsubseteq z \ a$ 
and  $\bigwedge a. a \notin S \implies x \sqsubseteq z \ a$ 
shows  $x \text{ ++}_S y \sqsubseteq z$ 
using assms
apply  $-$ 
apply (rule fun-belowI)
apply (case-tac xa \in S)
apply auto
done

```

lemma *override-on-cont1*: *cont* ($\lambda x. x \text{ ++}_S m$)
by (*rule cont2cont-lambda*) (*auto simp add: override-on-def*)

lemma *override-on-cont2*: *cont* ($\lambda x. m \text{ ++}_S x$)
by (*rule cont2cont-lambda*) (*auto simp add: override-on-def*)

lemma *override-on-cont2cont*[*simp*, *cont2cont*]:

```

assumes cont f
assumes cont g
shows cont ( $\lambda x. f \ x \text{ ++}_S g \ x$ )
by (rule cont-apply[OF assms(1) override-on-cont1 cont-compose[OF override-on-cont2 assms(2)]])

```

lemma *override-on-mono*:

```

assumes  $x1 \sqsubseteq (x2 :: 'a::\text{type} \Rightarrow 'b::\text{cpo})$ 
assumes  $y1 \sqsubseteq y2$ 
shows  $x1 \text{ ++}_S y1 \sqsubseteq x2 \text{ ++}_S y2$ 
by (rule below-trans[OF cont2monofunE[OF override-on-cont1 assms(1)] cont2monofunE[OF override-on-cont2 assms(2)]])

```

lemma *fun-upd-below-env-deleteI*:

```

assumes env-delete  $x \ \varrho \sqsubseteq \text{env-delete } x \ \varrho'$ 
assumes  $y \sqsubseteq \varrho' \ x$ 
shows  $\varrho(x := y) \sqsubseteq \varrho'$ 
using assms
apply (auto intro!: fun-upd-belowI simp add: env-delete-def)
by (metis fun-belowD fun-upd-other)

```

lemma *fun-upd-belowI2*:

```

assumes  $\bigwedge z. z \neq x \implies \varrho \ z \sqsubseteq \varrho' \ z$ 
assumes  $\varrho \ x \sqsubseteq y$ 
shows  $\varrho \sqsubseteq \varrho'(x := y)$ 
apply (rule fun-belowI)
using assms by auto

```

lemma *env-restr-belowI*:

assumes $\bigwedge x. x \in S \implies (m1\ f|' S)\ x \sqsubseteq (m2\ f|' S)\ x$
shows $m1\ f|' S \sqsubseteq m2\ f|' S$
apply (*rule fun-belowI*)
by (*metis assms below-bottom-iff lookup-env-restr-not-there*)

lemma *env-restr-belowI2*:

assumes $\bigwedge x. x \in S \implies m1\ x \sqsubseteq m2\ x$
shows $m1\ f|' S \sqsubseteq m2$
by (*rule fun-belowI*)
 (*simp add: assms env-restr-def*)

lemma *env-restr-below-itself*:

shows $m\ f|' S \sqsubseteq m$
apply (*rule fun-belowI*)
apply (*case-tac x \in S*)
apply *auto*
done

lemma *env-restr-cont*: *cont (env-restr S)*

apply (*rule cont2cont-lambda*)
apply (*case-tac y \in S*)
apply *auto*
done

lemma *env-restr-belowD*:

assumes $m1\ f|' S \sqsubseteq m2\ f|' S$
assumes $x \in S$
shows $m1\ x \sqsubseteq m2\ x$
using *fun-belowD[OF assms(1), where x = x] assms(2)* **by** *simp*

lemma *env-restr-eqD*:

assumes $m1\ f|' S = m2\ f|' S$
assumes $x \in S$
shows $m1\ x = m2\ x$
by (*metis assms(1) assms(2) lookup-env-restr*)

lemma *env-restr-below-subset*:

assumes $S \subseteq S'$
and $m1\ f|' S' \sqsubseteq m2\ f|' S'$
shows $m1\ f|' S \sqsubseteq m2\ f|' S$
using *assms*
by (*auto intro!: env-restr-belowI dest: env-restr-belowD*)

lemma *override-on-below-restrI*:

```

assumes  $x f|' (-S) \sqsubseteq z f|' (-S)$ 
and  $y f|' S \sqsubseteq z f|' S$ 
shows  $x ++_S y \sqsubseteq z$ 
using assms
by (auto intro: override-on-belowI dest:env-restr-belowD)

lemma fmap-below-add-restrI:
  assumes  $x f|' (-S) \sqsubseteq y f|' (-S)$ 
  and  $x f|' S \sqsubseteq z f|' S$ 
  shows  $x \sqsubseteq y ++_S z$ 
using assms
by (auto intro!: fun-belowI dest:env-restr-belowD simp add: lookup-override-on-eq)

lemmas env-restr-cont2cont[simp,cont2cont] = cont-compose[OF env-restr-cont]

lemma env-delete-cont: cont (env-delete  $x$ )
  apply (rule cont2cont-lambda)
  apply (case-tac  $y = x$ )
  apply auto
  done
lemmas env-delete-cont2cont[simp,cont2cont] = cont-compose[OF env-delete-cont]

end

```

2.12 EvalHeap

```

theory EvalHeap
  imports AList-Utils Env Nominal2.Nominal2 HOLCF-Utils
begin

```

2.12.1 Conversion from heaps to environments

```

fun
  evalHeap :: ('var × 'exp) list ⇒ ('exp ⇒ 'value::{pure,pcpo}) ⇒ 'var ⇒ 'value
where
  evalHeap [] - = ⊥
| evalHeap (( $x, e$ )# $h$ ) eval = (evalHeap  $h$  eval) ( $x :=$  eval  $e$ )

lemma cont2cont-evalHeap[simp, cont2cont]:
  ( $\bigwedge e . e \in \text{snd } 'set\ h \implies \text{cont } (\lambda \varrho . \text{eval } \varrho\ e) \implies \text{cont } (\lambda \varrho . \text{evalHeap } h\ (\text{eval } \varrho))$ )
  by(induct  $h$ , auto)

lemma evalHeap-eqvt[eqvt]:
   $\pi \cdot \text{evalHeap } h\ \text{eval} = \text{evalHeap } (\pi \cdot h)\ (\pi \cdot \text{eval})$ 
  by (induct  $h$ ) (auto simp add:fun-upd-eqvt simp del: fun-upd-apply)

lemma edom-evalHeap-subset:edom (evalHeap  $h$  eval)  $\subseteq \text{dom } A\ h$ 

```


by (*induct* h *eval* *rule:evalHeap.induct*) (*auto* *dest:subsetD*[*OF edom-fun-upd-subset*] *simp* *del:fun-upd-apply*)

lemma *evalHeap-cong*[*fundef-cong*]:

$\llbracket \text{heap1} = \text{heap2} ; (\bigwedge e. e \in \text{snd } \text{'set heap2} \implies \text{eval1 } e = \text{eval2 } e) \rrbracket$
 $\implies \text{evalHeap heap1 eval1} = \text{evalHeap heap2 eval2}$

by (*induct* heap2 eval2 *arbitrary:heap1* *rule:evalHeap.induct*, *auto*)

lemma *lookupEvalHeap*:

assumes $v \in \text{domA } h$

shows ($\text{evalHeap } h f$) $v = f$ (*the* (*map-of* h v))

using *assms*

by (*induct* h f *rule:evalHeap.induct*) *auto*

lemma *lookupEvalHeap'*:

assumes *map-of* Γ $v = \text{Some } e$

shows ($\text{evalHeap } \Gamma f$) $v = f e$

using *assms*

by (*induct* Γ f *rule:evalHeap.induct*) *auto*

lemma *lookupEvalHeap-other*[*simp*]:

assumes $v \notin \text{domA } \Gamma$

shows ($\text{evalHeap } \Gamma f$) $v = \perp$

using *assms*

by (*induct* Γ f *rule:evalHeap.induct*) *auto*

lemma *env-restr-evalHeap-noop*:

$\text{domA } h \subseteq S \implies \text{env-restr } S (\text{evalHeap } h \text{ eval}) = \text{evalHeap } h \text{ eval}$

apply (*rule ext*)

apply (*case-tac* $x \in S$)

apply (*auto* *simp* *add:lookupEvalHeap* *intro:lookupEvalHeap-other*)

done

lemma *env-restr-evalHeap-same*[*simp*]:

$\text{env-restr } (\text{domA } h) (\text{evalHeap } h \text{ eval}) = \text{evalHeap } h \text{ eval}$

by (*simp* *add:env-restr-evalHeap-noop*)

lemma *evalHeap-cong'*:

$\llbracket (\bigwedge x. x \in \text{domA } \text{heap} \implies \text{eval1 } (\text{the } (\text{map-of } \text{heap } x)) = \text{eval2 } (\text{the } (\text{map-of } \text{heap } x))) \rrbracket$
 $\implies \text{evalHeap } \text{heap } \text{eval1} = \text{evalHeap } \text{heap } \text{eval2}$

apply (*rule ext*)

apply (*case-tac* $x \in \text{domA } \text{heap}$)

apply (*auto* *simp* *add:lookupEvalHeap*)

done

lemma *lookupEvalHeapNotAppend*[*simp*]:

assumes $x \notin \text{domA } \Gamma$

shows ($\text{evalHeap } (\Gamma @ h) f$) $x = \text{evalHeap } h f x$

using *assms* **by** (*induct* Γ , *auto*)

lemma *evalHeap-delete[simp]*: $evalHeap (delete\ x\ \Gamma)\ eval = env\ delete\ x\ (evalHeap\ \Gamma\ eval)$
by (*induct* Γ) *auto*

lemma *evalHeap-mono*:
 $x \notin domA\ \Gamma \implies$
 $evalHeap\ \Gamma\ eval \sqsubseteq evalHeap\ ((x,\ e)\ \#\ \Gamma)\ eval$
apply *simp*
apply (*rule fun-belowI*)
apply (*case-tac* $xa \in domA\ \Gamma$)
apply (*case-tac* $xa = x$)
apply *auto*
done

2.12.2 Reordering lemmas

lemma *evalHeap-reorder*:
assumes $map\ of\ \Gamma = map\ of\ \Delta$
shows $evalHeap\ \Gamma\ h = evalHeap\ \Delta\ h$
proof (*rule ext*)
from *assms*
have $*$: $domA\ \Gamma = domA\ \Delta$ **by** (*metis dom-map-of-conv-domA*)

fix x
show $evalHeap\ \Gamma\ h\ x = evalHeap\ \Delta\ h\ x$
using *assms(1) **
apply (*cases* $x \in domA\ \Gamma$)
apply (*auto simp add: lookupEvalHeap*)
done
qed

lemma *evalHeap-reorder-head*:
assumes $x \neq y$
shows $evalHeap\ ((x,e1)\ \#\ (y,e2)\ \#\ \Gamma)\ eval = evalHeap\ ((y,e2)\ \#\ (x,e1)\ \#\ \Gamma)\ eval$
by (*rule evalHeap-reorder*) (*simp add: fun-upd-twist[OF assms]*)

lemma *evalHeap-reorder-head-append*:
assumes $x \notin domA\ \Gamma$
shows $evalHeap\ ((x,e)\ \#\ \Gamma\ @\ \Delta)\ eval = evalHeap\ (\Gamma\ @\ ((x,e)\ \#\ \Delta))\ eval$
by (*rule evalHeap-reorder*) (*simp, metis assms dom-map-of-conv-domA map-add-upd-left*)

lemma *evalHeap-subst-exp*:
assumes $eval\ e = eval\ e'$
shows $evalHeap\ ((x,e)\ \#\ \Gamma)\ eval = evalHeap\ ((x,e')\ \#\ \Gamma)\ eval$
by (*simp add: assms*)

end

3 Launchbury's natural semantics

3.1 Vars

```
theory Vars
imports Nominal2.Nominal2
begin
```

The type of variables is abstract and provided by the `Nominal` package. All we know is that it is countable.

```
atom-decl var

end
```

3.2 Terms

```
theory Terms
imports Nominal-Utils Vars AList-Utils-Nominal
begin
```

3.2.1 Expressions

This is the main data type of the development; our minimal lambda calculus with recursive let-bindings. It is created using the `nominal_datatype` command, which creates alpha-equivalence classes.

The package does not support nested recursion, so the bindings of the let cannot simply be of type (var, exp) list. Instead, the definition of lists have to be inlined here, as the custom type `assn`. Later we create conversion functions between these two types, define a properly typed `let` and redo the various lemmas in terms of that, so that afterwards, the type `assn` is no longer referenced.

```
nominal-datatype exp =
  Var var
| App exp var
| LetA as::assn body::exp binds bn as in body as
| Lam x::var body::exp binds x in body (Lam [-]. - [100, 100] 100)
| Bool bool
| IfThenElse exp exp exp (((-)/ ?(-)/ : (-)) [0, 0, 10] 10)
and assn =
  ANil | ACons var exp assn
binder
  bn :: assn  $\Rightarrow$  atom list
where bn ANil = [] | bn (ACons x t as) = (atom x) # (bn as)

notation (latex output) Terms.Var (-)
notation (latex output) Terms.App (- -)
```

notation (*latex output*) *Terms.Lam* ($\lambda\cdot$ - [100, 100] 100)

type-synonym *heap* = (*var* × *exp*) *list*

lemma *exp-assn-size-eqvt*[*eqvt*]: $p \cdot (\text{size} :: \text{exp} \Rightarrow \text{nat}) = \text{size}$
by (*metis exp-assn.size-eqvt(1) fun-eqvtI permute-pure*)

3.2.2 Rewriting in terms of heaps

We now work towards using *heap* instead of *assn*. All this could be skipped if Nominal supported nested recursion.

Conversion from *assn* to *heap*.

nominal-function *asToHeap* :: *assn* \Rightarrow *heap*
where *ANilToHeap*: *asToHeap* *ANil* = []
| *AConsToHeap*: *asToHeap* (*ACons* *v e as*) = (*v, e*) # *asToHeap as*
unfolding *eqvt-def asToHeap-graph-aux-def*
apply *rule*
apply *simp*
apply *rule*
apply(*case-tac x rule: exp-assn.exhaust(2)*)
apply *auto*
done
nominal-termination(*eqvt*) by *lexicographic-order*

lemma *asToHeap-eqvt*: *eqvt asToHeap*
unfolding *eqvt-def*
by (*auto simp add: permute-fun-def asToHeap.eqvt*)

The other direction.

fun *heapToAssn* :: *heap* \Rightarrow *assn*
where *heapToAssn* [] = *ANil*
| *heapToAssn* ((*v,e*)# Γ) = *ACons v e (heapToAssn Γ)*

declare *heapToAssn.simps*[*simp del*]

lemma *heapToAssn-eqvt*[*simp,eqvt*]: $p \cdot \text{heapToAssn } \Gamma = \text{heapToAssn } (p \cdot \Gamma)$
by (*induct Γ rule: heapToAssn.induct*)
(*auto simp add: heapToAssn.simps*)

lemma *bn-heapToAssn*: $\text{bn } (\text{heapToAssn } \Gamma) = \text{map } (\lambda x. \text{atom } (\text{fst } x)) \Gamma$
by (*induct rule: heapToAssn.induct*)
(*auto simp add: heapToAssn.simps exp-assn.bn-defs*)

lemma *set-bn-to-atom-domA*:
 $\text{set } (\text{bn } \text{as}) = \text{atom } \text{'domA } (\text{asToHeap } \text{as})$
by (*induct as rule: asToHeap.induct*)
(*auto simp add: exp-assn.bn-defs*)

They are inverse to each other.

lemma *heapToAssn-asToHeap*[simp]:
 $heapToAssn (asToHeap as) = as$
by (*induct rule: exp-assn.inducts*(2)[*of* $\lambda - . True$])
(auto simp add: heapToAssn.simps)

lemma *asToHeap-heapToAssn*[simp]:
 $asToHeap (heapToAssn as) = as$
by (*induct rule: heapToAssn.induct*)
(auto simp add: heapToAssn.simps)

lemma *heapToAssn-inject*[simp]:
 $heapToAssn x = heapToAssn y \longleftrightarrow x = y$
by (*metis asToHeap-heapToAssn*)

They are transparent to various notions from the Nominal package.

lemma *supp-heapToAssn*: $supp (heapToAssn \Gamma) = supp \Gamma$
by (*induct rule: heapToAssn.induct*)
(simp-all add: heapToAssn.simps exp-assn.supp supp-Nil supp-Cons supp-Pair sup-assoc)

lemma *supp-asToHeap*: $supp (asToHeap as) = supp as$
by (*induct as rule: asToHeap.induct*)
(simp-all add: exp-assn.supp supp-Nil supp-Cons supp-Pair sup-assoc)

lemma *fv-asToHeap*: $fv (asToHeap \Gamma) = fv \Gamma$
unfolding *fv-def* **by** (*auto simp add: supp-asToHeap*)

lemma *fv-heapToAssn*: $fv (heapToAssn \Gamma) = fv \Gamma$
unfolding *fv-def* **by** (*auto simp add: supp-heapToAssn*)

lemma [simp]: $size (heapToAssn \Gamma) = size-list (\lambda (v,e) . size e) \Gamma$
by (*induct rule: heapToAssn.induct*)
(simp-all add: heapToAssn.simps)

lemma *Lam-eq-same-var*[simp]: $Lam [y]. e = Lam [y]. e' \longleftrightarrow e = e'$
by *auto (metis fresh-PairD(2) obtain-fresh)*

Now we define the Let constructor in the form that we actually want.

hide-const *HOL.Let*

definition *Let* :: $heap \Rightarrow exp \Rightarrow exp$
where $Let \Gamma e = LetA (heapToAssn \Gamma) e$

notation (*latex output*) $Let (let - in -)$

abbreviation

$LetBe :: var \Rightarrow exp \Rightarrow exp \Rightarrow exp (let - be - in - [100,100,100] 100)$

where

$let x be t1 in t2 \equiv Let [(x,t1)] t2$

We rewrite all (relevant) lemmas about $LetA$ in terms of Let .

lemma *size-Let[simp]*: $size (Let \Gamma e) = size-list (\lambda p. size (snd p)) \Gamma + size e + Suc 0$
unfolding *Let-def* **by** (*auto simp add: split-beta'*)

lemma *Let-distinct[simp]*:

$Var v \neq Let \Gamma e$
 $Let \Gamma e \neq Var v$
 $App e v \neq Let \Gamma e'$
 $Lam [v]. e' \neq Let \Gamma e$
 $Let \Gamma e \neq Lam [v]. e'$
 $Let \Gamma e' \neq App e v$
 $Bool b \neq Let \Gamma e$
 $Let \Gamma e \neq Bool b$
 $(scrut ? e1 : e2) \neq Let \Gamma e$
 $Let \Gamma e \neq (scrut ? e1 : e2)$
unfolding *Let-def* **by** *simp-all*

lemma *Let-perm-simps[simp,eqvt]*:

$p \cdot Let \Gamma e = Let (p \cdot \Gamma) (p \cdot e)$
unfolding *Let-def* **by** *simp*

lemma *Let-supp*:

$supp (Let \Gamma e) = (supp e \cup supp \Gamma) - atom ' (domA \Gamma)$
unfolding *Let-def* **by** (*simp add: exp-assn.supp supp-heapToAssn bn-heapToAssn domA-def image-image*)

lemma *Let-fresh[simp]*:

$a \# Let \Gamma e = (a \# e \wedge a \# \Gamma \vee a \in atom ' (domA \Gamma))$
unfolding *fresh-def* **by** (*auto simp add: Let-supp*)

lemma *Abs-eq-cong*:

assumes $\bigwedge p. (p \cdot x = x') \longleftrightarrow (p \cdot y = y')$
assumes $supp y = supp x$
assumes $supp y' = supp x'$
shows $([a]lst. x = [a']lst. x') \longleftrightarrow ([a]lst. y = [a']lst. y')$
by (*simp add: Abs-eq-iff alpha-lst assms*)

lemma *Let-eq-iff[simp]*:

$(Let \Gamma e = Let \Gamma' e') = ([map (\lambda x. atom (fst x)) \Gamma]lst. (e, \Gamma) = [map (\lambda x. atom (fst x)) \Gamma']lst. (e', \Gamma'))$

unfolding *Let-def*
apply (*simp add: bn-heapToAssn*)
apply (*rule Abs-eq-cong*)
apply (*simp-all add: supp-Pair supp-heapToAssn*)
done

lemma *exp-strong-exhaust*:

fixes $c :: 'a :: fs$
assumes $\bigwedge var. y = Var var \implies P$

assumes $\bigwedge \text{exp var. } y = \text{App exp var} \implies P$
assumes $\bigwedge \Gamma \text{ exp. atom } \text{' domA } \Gamma \#* c \implies y = \text{Let } \Gamma \text{ exp} \implies P$
assumes $\bigwedge \text{var exp. } \{ \text{atom var} \} \#* c \implies y = \text{Lam } [\text{var}]. \text{exp} \implies P$
assumes $\bigwedge b. (y = \text{Bool } b) \implies P$
assumes $\bigwedge \text{scrut } e1 \text{ } e2. y = (\text{scrut } ? e1 : e2) \implies P$
shows P
apply (*rule exp-assn.strong-exhaust(1)[where y = y and c = c]*)
apply (*metis assms(1)*)
apply (*metis assms(2)*)
apply (*metis assms(3) set-bn-to-atom-domA Let-def heapToAssn-asToHeap*)
apply (*metis assms(4)*)
apply (*metis assms(5)*)
apply (*metis assms(6)*)
done

And finally the induction rules with *Let*.

lemma *exp-heap-induct[case-names Var App Let Lam Bool IfThenElse Nil Cons]:*

assumes $\bigwedge b \text{ var. } P1 \text{ (Var var)}$
assumes $\bigwedge \text{exp var. } P1 \text{ exp} \implies P1 \text{ (App exp var)}$
assumes $\bigwedge \Gamma \text{ exp. } P2 \text{ } \Gamma \implies P1 \text{ exp} \implies P1 \text{ (Let } \Gamma \text{ exp)}$
assumes $\bigwedge \text{var exp. } P1 \text{ exp} \implies P1 \text{ (Lam } [\text{var}]. \text{exp)}$
assumes $\bigwedge b. P1 \text{ (Bool } b)$
assumes $\bigwedge \text{scrut } e1 \text{ } e2. P1 \text{ scrut} \implies P1 \text{ } e1 \implies P1 \text{ } e2 \implies P1 \text{ (scrut } ? e1 : e2)$
assumes $P2 \text{ []}$
assumes $\bigwedge \text{var exp } \Gamma. P1 \text{ exp} \implies P2 \text{ } \Gamma \implies P2 \text{ ((var, exp)\#}\Gamma)$
shows $P1 \text{ } e \text{ and } P2 \text{ } \Gamma$

proof–

have $P1 \text{ } e \text{ and } P2 \text{ (asToHeap (heapToAssn } \Gamma))$
apply (*induct rule: exp-assn.inducts[of P1 λ assn. P2 (asToHeap assn)]*)
apply (*metis assms(1)*)
apply (*metis assms(2)*)
apply (*metis assms(3) Let-def heapToAssn-asToHeap*)
apply (*metis assms(4)*)
apply (*metis assms(5)*)
apply (*metis assms(6)*)
apply (*metis assms(7) asToHeap.simps(1)*)
apply (*metis assms(8) asToHeap.simps(2)*)
done

thus $P1 \text{ } e \text{ and } P2 \text{ } \Gamma$ **unfolding** *asToHeap-heapToAssn*.

qed

lemma *exp-heap-strong-induct[case-names Var App Let Lam Bool IfThenElse Nil Cons]:*

assumes $\bigwedge \text{var } c. P1 \text{ } c \text{ (Var var)}$
assumes $\bigwedge \text{exp var } c. (\bigwedge c. P1 \text{ } c \text{ exp}) \implies P1 \text{ } c \text{ (App exp var)}$
assumes $\bigwedge \Gamma \text{ exp } c. \text{atom } \text{' domA } \Gamma \#* c \implies (\bigwedge c. P2 \text{ } c \text{ } \Gamma) \implies (\bigwedge c. P1 \text{ } c \text{ exp}) \implies P1 \text{ } c \text{ (Let } \Gamma \text{ exp)}$
assumes $\bigwedge \text{var exp } c. \{ \text{atom var} \} \#* c \implies (\bigwedge c. P1 \text{ } c \text{ exp}) \implies P1 \text{ } c \text{ (Lam } [\text{var}]. \text{exp)}$
assumes $\bigwedge b \text{ } c. P1 \text{ } c \text{ (Bool } b)$
assumes $\bigwedge \text{scrut } e1 \text{ } e2 \text{ } c. (\bigwedge c. P1 \text{ } c \text{ scrut}) \implies (\bigwedge c. P1 \text{ } c \text{ } e1) \implies (\bigwedge c. P1 \text{ } c \text{ } e2) \implies P1$

c (*scrut* ? $e1 : e2$)
assumes $\bigwedge c. P2\ c \ []$
assumes $\bigwedge var\ exp\ \Gamma\ c. (\bigwedge c. P1\ c\ exp) \implies (\bigwedge c. P2\ c\ \Gamma) \implies P2\ c\ ((var,exp)\#\Gamma)$
fixes $c :: 'a :: fs$
shows $P1\ c\ e$ and $P2\ c\ \Gamma$

proof–

have $P1\ c\ e$ and $P2\ c$ (*asToHeap* (*heapToAssn* Γ))
apply (*induct rule: exp-assn.strong-induct*[of $P1\ \lambda\ c\ assn. P2\ c$ (*asToHeap* $assn$)])
apply (*metis assms*(1))
apply (*metis assms*(2))
apply (*metis assms*(3) *set-bn-to-atom-domA Let-def heapToAssn-asToHeap*)
apply (*metis assms*(4))
apply (*metis assms*(5))
apply (*metis assms*(6))
apply (*metis assms*(7) *asToHeap.simps*(1))
apply (*metis assms*(8) *asToHeap.simps*(2))
done

thus $P1\ c\ e$ and $P2\ c\ \Gamma$ **unfolding** *asToHeap-heapToAssn*.

qed

3.2.3 Nice induction rules

These rules can be used instead of the original induction rules, which require a separate goal for *assn*.

lemma *exp-induct*[*case-names Var App Let Lam Bool IfThenElse*]:

assumes $\bigwedge var. P\ (Var\ var)$
assumes $\bigwedge exp\ var. P\ exp \implies P\ (App\ exp\ var)$
assumes $\bigwedge \Gamma\ exp. (\bigwedge x. x \in domA\ \Gamma \implies P\ (the\ (map-of\ \Gamma\ x))) \implies P\ exp \implies P\ (Let\ \Gamma\ exp)$
assumes $\bigwedge var\ exp. P\ exp \implies P\ (Lam\ [var].\ exp)$
assumes $\bigwedge b. P\ (Bool\ b)$
assumes $\bigwedge scrut\ e1\ e2. P\ scrut \implies P\ e1 \implies P\ e2 \implies P\ (scrut\ ?\ e1 : e2)$
shows $P\ exp$
apply (*rule exp-heap-induct*[of $P\ \lambda\ \Gamma. (\forall x \in domA\ \Gamma. P\ (the\ (map-of\ \Gamma\ x)))$])
apply (*metis assms*(1))
apply (*metis assms*(2))
apply (*metis assms*(3))
apply (*metis assms*(4))
apply (*metis assms*(5))
apply (*metis assms*(6))
apply *auto*
done

lemma *exp-strong-induct-set*[*case-names Var App Let Lam Bool IfThenElse*]:

assumes $\bigwedge var\ c. P\ c\ (Var\ var)$
assumes $\bigwedge exp\ var\ c. (\bigwedge c. P\ c\ exp) \implies P\ c\ (App\ exp\ var)$
assumes $\bigwedge \Gamma\ exp\ c.$
atom $'\ domA\ \Gamma\ \#\ast\ c \implies (\bigwedge c\ x\ e. (x,e) \in set\ \Gamma \implies P\ c\ e) \implies (\bigwedge c. P\ c\ exp) \implies P\ c\ (Let$

$\Gamma \text{ exp}$
assumes $\bigwedge \text{var exp } c. \{atom \text{ var}\} \#* c \implies (\bigwedge c. P \ c \ \text{exp}) \implies P \ c \ (\text{Lam } [\text{var}]. \ \text{exp})$
assumes $\bigwedge b \ c. P \ c \ (\text{Bool } b)$
assumes $\bigwedge \text{scrut } e1 \ e2 \ c. (\bigwedge c. P \ c \ \text{scrut}) \implies (\bigwedge c. P \ c \ e1) \implies (\bigwedge c. P \ c \ e2) \implies P \ c \ (\text{scrut } ? \ e1 : e2)$
shows $P \ (c::'a::fs) \ \text{exp}$
apply $(rule \ \text{exp-heap-strong-induct}(1)[of \ P \ \lambda \ c \ \Gamma. (\forall (x,e) \in \text{set } \Gamma. P \ c \ e)])$
apply $(metis \ \text{assms}(1))$
apply $(metis \ \text{assms}(2))$
apply $(metis \ \text{assms}(3) \ \text{split-conv})$
apply $(metis \ \text{assms}(4))$
apply $(metis \ \text{assms}(5))$
apply $(metis \ \text{assms}(6))$
apply $auto$
done

lemma $\text{exp-strong-induct}[\text{case-names } \text{Var } \text{App } \text{Let } \text{Lam } \text{Bool } \text{IfThenElse}]$:
assumes $\bigwedge \text{var } c. P \ c \ (\text{Var } \text{var})$
assumes $\bigwedge \text{exp var } c. (\bigwedge c. P \ c \ \text{exp}) \implies P \ c \ (\text{App } \ \text{exp } \ \text{var})$
assumes $\bigwedge \Gamma \ \text{exp } c.$
 $atom \ ' \ \text{domA } \Gamma \ \#* \ c \implies (\bigwedge c \ x. x \in \text{domA } \Gamma \implies P \ c \ (\text{the } (\text{map-of } \Gamma \ x))) \implies (\bigwedge c. P \ c \ \text{exp}) \implies P \ c \ (\text{Let } \Gamma \ \text{exp})$
assumes $\bigwedge \text{var exp } c. \{atom \ \text{var}\} \#* c \implies (\bigwedge c. P \ c \ \text{exp}) \implies P \ c \ (\text{Lam } [\text{var}]. \ \text{exp})$
assumes $\bigwedge b \ c. P \ c \ (\text{Bool } b)$
assumes $\bigwedge \text{scrut } e1 \ e2 \ c. (\bigwedge c. P \ c \ \text{scrut}) \implies (\bigwedge c. P \ c \ e1) \implies (\bigwedge c. P \ c \ e2) \implies P \ c \ (\text{scrut } ? \ e1 : e2)$
shows $P \ (c::'a::fs) \ \text{exp}$
apply $(rule \ \text{exp-heap-strong-induct}(1)[of \ P \ \lambda \ c \ \Gamma. (\forall x \in \text{domA } \Gamma. P \ c \ (\text{the } (\text{map-of } \Gamma \ x)))]])$
apply $(metis \ \text{assms}(1))$
apply $(metis \ \text{assms}(2))$
apply $(metis \ \text{assms}(3))$
apply $(metis \ \text{assms}(4))$
apply $(metis \ \text{assms}(5))$
apply $(metis \ \text{assms}(6))$
apply $auto$
done

3.2.4 Testing alpha equivalence

lemma alpha-test :
shows $\text{Lam } [x]. (\text{Var } x) = \text{Lam } [y]. (\text{Var } y)$
by $(simp \ \text{add: } \text{Abs1-eq-iff } \text{fresh-at-base } \text{pure-fresh})$

lemma alpha-test2 :
shows $\text{let } x \ \text{be } (\text{Var } x) \ \text{in } (\text{Var } x) = \text{let } y \ \text{be } (\text{Var } y) \ \text{in } (\text{Var } y)$
by $(simp \ \text{add: } \text{fresh-Cons } \text{fresh-Nil } \text{Abs1-eq-iff } \text{fresh-Pair } \text{add: } \text{fresh-at-base } \text{pure-fresh})$

lemma alpha-test3 :

shows

$Let [(x, Var y), (y, Var x)] (Var x)$
 $=$
 $Let [(y, Var x), (x, Var y)] (Var y)$ (is $Let ?la ?lb = -$)
 by (simp add: *bn-heapToAssn Abs1-eq-iff fresh-Pair fresh-at-base*
Abs-swap2[of atom x (?lb, [(x, Var y), (y, Var x)]) [atom x, atom y] atom y])

3.2.5 Free variables

lemma *fv-supp-exp*: $supp e = atom \text{ ` } (fv (e::exp) :: var set)$ **and** *fv-supp-as*: $supp as = atom \text{ ` } (fv (as::assn) :: var set)$

by (*induction e and as rule:exp-assn.inducts*)
(auto simp add: fv-def exp-assn.supp supp-at-base pure-supp)

lemma *fv-supp-heap*: $supp (\Gamma::heap) = atom \text{ ` } (fv \Gamma :: var set)$

by (*metis fv-def fv-supp-as supp-heapToAssn*)

lemma *fv-Lam[simp]*: $fv (Lam [x]. e) = fv e - \{x\}$

unfolding *fv-def* **by** (*auto simp add: exp-assn.supp*)

lemma *fv-Var[simp]*: $fv (Var x) = \{x\}$

unfolding *fv-def* **by** (*auto simp add: exp-assn.supp supp-at-base pure-supp*)

lemma *fv-App[simp]*: $fv (App e x) = insert x (fv e)$

unfolding *fv-def* **by** (*auto simp add: exp-assn.supp supp-at-base*)

lemma *fv-Let[simp]*: $fv (Let \Gamma e) = (fv \Gamma \cup fv e) - domA \Gamma$

unfolding *fv-def* **by** (*auto simp add: Let-supp exp-assn.supp supp-at-base set-bn-to-atom-domA*)

lemma *fv-Bool[simp]*: $fv (Bool b) = \{\}$

unfolding *fv-def* **by** (*auto simp add: exp-assn.supp pure-supp*)

lemma *fv-IfThenElse[simp]*: $fv (scrut ? e1 : e2) = fv scrut \cup fv e1 \cup fv e2$

unfolding *fv-def* **by** (*auto simp add: exp-assn.supp*)

lemma *fv-delete-heap*:

assumes *map-of* $\Gamma x = Some e$

shows $fv (delete x \Gamma, e) \cup \{x\} \subseteq (fv (\Gamma, Var x) :: var set)$

proof –

have $fv (delete x \Gamma) \subseteq fv \Gamma$ **by** (*metis fv-delete-subset*)

moreover

have $(x, e) \in set \Gamma$ **by** (*metis assms map-of-SomeD*)

hence $fv e \subseteq fv \Gamma$ **by** (*metis assms domA-from-set map-of-fv-subset option.sel*)

moreover

have $x \in fv (Var x)$ **by** *simp*

ultimately show *?thesis* **by** *auto*

qed

3.2.6 Lemmas helping with nominal definitions

lemma *eqvt-lam-case*:

assumes $Lam [x]. e = Lam [x']. e'$

assumes $\bigwedge \pi . supp (-\pi) \#* (fv (Lam [x]. e) :: var set) \implies$

$supp \pi \#* (Lam [x]. e) \implies$

$F (\pi \cdot e) (\pi \cdot x) (Lam [x]. e) = F e x (Lam [x]. e)$
shows $F e x (Lam [x]. e) = F e' x' (Lam [x']. e')$
proof–
from *assms(1)*
have $[[atom\ x]]lst. (e, x) = [[atom\ x']]lst. (e', x')$ **by** *auto*
then obtain *p*
where $(supp\ (e, x) - \{atom\ x\}) \#* p$
and $[simp]: p \cdot x = x'$
and $[simp]: p \cdot e = e'$
unfolding *Abs-eq-iff(3) alpha-lst.simps* **by** *auto*

from $\langle - \#* p \rangle$
have $*$: $supp\ (-p) \#* (fv\ (Lam\ [x].\ e) :: var\ set)$
by (*auto simp add: fresh-star-def fresh-def supp-finite-set-at-base supp-Pair fv-supply-exp fv-supply-heap supp-minus-perm*)

from $\langle - \#* p \rangle$
have $**$: $supp\ p \#* Lam\ [x].\ e$
by (*auto simp add: fresh-star-def fresh-def supp-Pair fv-supply-exp*)

have $F\ e\ x\ (Lam\ [x].\ e) = F\ (p \cdot e)\ (p \cdot x)\ (Lam\ [x].\ e)$ **by** (*rule assms(2)[OF * **, symmetric]*)
also have $\dots = F\ e'\ x'\ (Lam\ [x'].\ e')$ **by** (*simp add: assms(1)*)
finally show *?thesis*.
qed

lemma *eqvt-let-case*:

assumes $Let\ as\ body = Let\ as'\ body'$
assumes $\bigwedge \pi.$
 $supp\ (-\pi) \#* (fv\ (Let\ as\ body) :: var\ set) \implies$
 $supp\ \pi \#* Let\ as\ body \implies$
 $F\ (\pi \cdot as)\ (\pi \cdot body)\ (Let\ as\ body) = F\ as\ body\ (Let\ as\ body)$
shows $F\ as\ body\ (Let\ as\ body) = F\ as'\ body'\ (Let\ as'\ body')$
proof–
from *assms(1)*
have $[map\ (\lambda\ p.\ atom\ (fst\ p))\ as]lst. (body, as) = [map\ (\lambda\ p.\ atom\ (fst\ p))\ as']lst. (body', as')$
by *auto*
then obtain *p*
where $(supp\ (body, as) - atom\ 'domA\ as) \#* p$
and $[simp]: p \cdot body = body'$
and $[simp]: p \cdot as = as'$
unfolding *Abs-eq-iff(3) alpha-lst.simps* **by** (*auto simp add: domA-def image-image*)

from $\langle - \#* p \rangle$
have $*$: $supp\ (-p) \#* (fv\ (Terms.Let\ as\ body) :: var\ set)$
by (*auto simp add: fresh-star-def fresh-def supp-finite-set-at-base supp-Pair fv-supply-exp*)

fv-supp-heap supp-minus-perm)

```

from ⟨- #* p⟩
have **: supp p #* Terms.Let as body
  by (auto simp add: fresh-star-def fresh-def supp-Pair fv-supp-exp fv-supp-heap )

  have F as body (Let as body) = F (p · as) (p · body) (Let as body) by (rule assms(2)[OF *
** , symmetric])
  also have ... = F as' body' (Let as' body') by (simp add: assms(1))
  finally show ?thesis.
qed

```

3.2.7 A smart constructor for lets

Certain program transformations might change the bound variables, possibly making it an empty list. This smart constructor avoids the empty let in the resulting expression. Semantically, it should not make a difference.

definition *SmartLet* :: *heap => exp => exp*
where *SmartLet* Γ *e* = (*if* $\Gamma = []$ *then* *e* *else* *Let* Γ *e*)

lemma *SmartLet-eqvt*[*eqvt*]: $\pi \cdot (\text{SmartLet } \Gamma \ e) = \text{SmartLet } (\pi \cdot \Gamma) (\pi \cdot e)$
unfolding *SmartLet-def* **by** *perm-simp rule*

lemma *SmartLet-supp*:
*supp (SmartLet Γ *e*) = (supp *e* \cup *supp* Γ) - *atom* ‘(*domA* Γ)*
unfolding *SmartLet-def* **using** *Let-supp* **by** (*auto simp add: supp-Nil*)

lemma *fv-SmartLet*[*simp*]: *fv (SmartLet Γ *e*) = (fv $\Gamma \cup$ *fv* *e*) - *domA* Γ*

unfolding *SmartLet-def* **by** *auto*

3.2.8 A predicate for value expressions

nominal-function *isLam* :: *exp* \Rightarrow *bool* **where**

```

isLam (Var x) = False |
isLam (Lam [x]. e) = True |
isLam (App e x) = False |
isLam (Let as e) = False |
isLam (Bool b) = False |
isLam (scrut ? e1 : e2) = False
unfolding isLam-graph-aux-def eqvt-def
apply simp
apply simp
apply (metis exp-strong-exhaust)
apply auto
done

```

nominal-termination (*eqvt*) **by** *lexicographic-order*

lemma *isLam-Lam*: *isLam (Lam [x]. e)* **by** *simp*

lemma *isLam-obtain-fresh*:
assumes *isLam* *z*
obtains *y e'*
where $z = (\text{Lam } [y]. e')$ **and** *atom* $y \# (c::'a::fs)$
using *assms* **by** (*nominal-induct* *z* *avoiding*: *c* *rule*:*exp-strong-induct*) *auto*

nominal-function *isVal* :: *exp* \Rightarrow *bool* **where**

isVal (*Var* *x*) = *False* |
isVal (*Lam* [*x*]. *e*) = *True* |
isVal (*App* *e* *x*) = *False* |
isVal (*Let* *as* *e*) = *False* |
isVal (*Bool* *b*) = *True* |
isVal (*scrut* ? *e1* : *e2*) = *False*
unfolding *isVal-graph-aux-def* *eqvt-def*
apply *simp*
apply *simp*
apply (*metis* *exp-strong-exhaust*)
apply *auto*
done

nominal-termination (*eqvt*) **by** *lexicographic-order*

lemma *isVal-Lam*: *isVal* (*Lam* [*x*]. *e*) **by** *simp*

lemma *isVal-Bool*: *isVal* (*Bool* *b*) **by** *simp*

3.2.9 The notion of thunks

definition *thunks* :: *heap* \Rightarrow *var set* **where**

thunks $\Gamma = \{x . \text{case map-of } \Gamma \text{ of } \text{Some } e \Rightarrow \neg \text{isVal } e \mid \text{None} \Rightarrow \text{False}\}$

lemma *thunks-Nil*[*simp*]: *thunks* [] = {} **by** (*auto* *simp* *add*: *thunks-def*)

lemma *thunks-domA*: *thunks* $\Gamma \subseteq \text{domA } \Gamma$

by (*induction* Γ) (*auto* *simp* *add*: *thunks-def*)

lemma *thunks-Cons*: *thunks* ((*x,e*)# Γ) = (*if isVal* *e* *then* *thunks* $\Gamma - \{x\}$ *else* *insert* *x* (*thunks* Γ))

by (*auto* *simp* *add*: *thunks-def*)

lemma *thunks-append*[*simp*]: *thunks* ($\Delta @ \Gamma$) = *thunks* $\Delta \cup (\text{thunks } \Gamma - \text{domA } \Delta)$

by (*induction* Δ) (*auto* *simp* *add*: *thunks-def*)

lemma *thunks-delete*[*simp*]: *thunks* (*delete* *x* Γ) = *thunks* $\Gamma - \{x\}$

by (*induction* Γ) (*auto* *simp* *add*: *thunks-def*)

lemma *thunksI*[*intro*]: *map-of* Γ *x* = *Some* *e* $\Longrightarrow \neg \text{isVal } e \Longrightarrow x \in \text{thunks } \Gamma$

by (*induction* Γ) (*auto* *simp* *add*: *thunks-def*)

lemma *thunksE*[*intro*]: $x \in \text{thunks } \Gamma \Longrightarrow \text{map-of } \Gamma \text{ } x = \text{Some } e \Longrightarrow \neg \text{isVal } e$

by (induction Γ) (auto simp add: thunks-def)

lemma *thunks-cong*: map-of $\Gamma = \text{map-of } \Delta \implies \text{thunks } \Gamma = \text{thunks } \Delta$
 by (simp add: thunks-def)

lemma *thunks-eqvt*[*eqvt*]:
 $\pi \cdot \text{thunks } \Gamma = \text{thunks } (\pi \cdot \Gamma)$
 unfolding *thunks-def*
 by *perm-simp rule*

3.2.10 Non-recursive Let bindings

definition *nonrec* :: heap \implies bool **where**
nonrec $\Gamma = (\exists x e. \Gamma = [(x,e)] \wedge x \notin \text{fv } e)$

lemma *nonrecE*:
 assumes *nonrec* Γ
 obtains *x e* **where** $\Gamma = [(x,e)]$ **and** $x \notin \text{fv } e$
 using *assms*
 unfolding *nonrec-def*
 by *blast*

lemma *nonrec-eqvt*[*eqvt*]:
nonrec $\Gamma \implies \text{nonrec } (\pi \cdot \Gamma)$
 apply (erule *nonrecE*)
 apply (auto simp add: *nonrec-def* *fv-def* *fresh-def*)
 apply (metis *fresh-at-base-permute-iff* *fresh-def*)
 done

lemma *exp-induct-rec*[*case-names* *Var* *App* *Let* *Let-nonrec* *Lam* *Bool* *IfThenElse*]:
 assumes $\bigwedge \text{var}. P (\text{Var } \text{var})$
 assumes $\bigwedge \text{exp } \text{var}. P \text{exp} \implies P (\text{App } \text{exp } \text{var})$
 assumes $\bigwedge \Gamma \text{exp}. \neg \text{nonrec } \Gamma \implies (\bigwedge x. x \in \text{domA } \Gamma \implies P (\text{the } (\text{map-of } \Gamma x))) \implies P \text{exp}$
 $\implies P (\text{Let } \Gamma \text{exp})$
 assumes $\bigwedge x e \text{exp}. x \notin \text{fv } e \implies P e \implies P \text{exp} \implies P (\text{let } x \text{ be } e \text{ in } \text{exp})$
 assumes $\bigwedge \text{var } \text{exp}. P \text{exp} \implies P (\text{Lam } [\text{var}]. \text{exp})$
 assumes $\bigwedge b. P (\text{Bool } b)$
 assumes $\bigwedge \text{scrut } e1 e2. P \text{scrut} \implies P e1 \implies P e2 \implies P (\text{scrut } ? e1 : e2)$
 shows *P exp*
 apply (rule *exp-induct*[of *P*])
 apply (metis *assms*(1))
 apply (metis *assms*(2))
 apply (case-tac *nonrec* Γ)
 apply (erule *nonrecE*)
 apply *simp*
 apply (metis *assms*(4))
 apply (metis *assms*(3))
 apply (metis *assms*(5))

apply (*metis assms(6)*)
apply (*metis assms(7)*)
done

lemma *exp-strong-induct-rec*[*case-names Var App Let Let-nonrec Lam Bool IfThenElse*]:

assumes $\bigwedge \text{var } c. P\ c\ (\text{Var } \text{var})$
assumes $\bigwedge \text{exp } \text{var } c. (\bigwedge c. P\ c\ \text{exp}) \implies P\ c\ (\text{App } \text{exp } \text{var})$
assumes $\bigwedge \Gamma\ \text{exp } c.$
 $\text{atom } ' \text{domA } \Gamma\ \#* c \implies \neg \text{nonrec } \Gamma \implies (\bigwedge c\ x. x \in \text{domA } \Gamma \implies P\ c\ (\text{the } (\text{map-of } \Gamma\ x)))$
 $\implies (\bigwedge c. P\ c\ \text{exp}) \implies P\ c\ (\text{Let } \Gamma\ \text{exp})$
assumes $\bigwedge x\ e\ \text{exp } c. \{\text{atom } x\} \#* c \implies x \notin \text{fv } e \implies (\bigwedge c. P\ c\ e) \implies (\bigwedge c. P\ c\ \text{exp}) \implies P\ c\ (\text{let } x\ \text{be } e\ \text{in } \text{exp})$
assumes $\bigwedge \text{var } \text{exp } c. \{\text{atom } \text{var}\} \#* c \implies (\bigwedge c. P\ c\ \text{exp}) \implies P\ c\ (\text{Lam } [\text{var}].\ \text{exp})$
assumes $\bigwedge b\ c. P\ c\ (\text{Bool } b)$
assumes $\bigwedge \text{scrut } e1\ e2\ c. (\bigwedge c. P\ c\ \text{scrut}) \implies (\bigwedge c. P\ c\ e1) \implies (\bigwedge c. P\ c\ e2) \implies P\ c\ (\text{scrut } ?\ e1 : e2)$
shows $P\ (c::'a::fs)\ \text{exp}$
apply (*rule exp-strong-induct[of P]*)
apply (*metis assms(1)*)
apply (*metis assms(2)*)
apply (*case-tac nonrec Γ*)
apply (*erule nonrecE*)
apply *simp*
apply (*metis assms(4)*)
apply (*metis assms(3)*)
apply (*metis assms(5)*)
apply (*metis assms(6)*)
apply (*metis assms(7)*)
done

lemma *exp-strong-induct-rec-set*[*case-names Var App Let Let-nonrec Lam Bool IfThenElse*]:

assumes $\bigwedge \text{var } c. P\ c\ (\text{Var } \text{var})$
assumes $\bigwedge \text{exp } \text{var } c. (\bigwedge c. P\ c\ \text{exp}) \implies P\ c\ (\text{App } \text{exp } \text{var})$
assumes $\bigwedge \Gamma\ \text{exp } c.$
 $\text{atom } ' \text{domA } \Gamma\ \#* c \implies \neg \text{nonrec } \Gamma \implies (\bigwedge c\ x\ e. (x,e) \in \text{set } \Gamma \implies P\ c\ e) \implies (\bigwedge c. P\ c\ \text{exp}) \implies P\ c\ (\text{Let } \Gamma\ \text{exp})$
assumes $\bigwedge x\ e\ \text{exp } c. \{\text{atom } x\} \#* c \implies x \notin \text{fv } e \implies (\bigwedge c. P\ c\ e) \implies (\bigwedge c. P\ c\ \text{exp}) \implies P\ c\ (\text{let } x\ \text{be } e\ \text{in } \text{exp})$
assumes $\bigwedge \text{var } \text{exp } c. \{\text{atom } \text{var}\} \#* c \implies (\bigwedge c. P\ c\ \text{exp}) \implies P\ c\ (\text{Lam } [\text{var}].\ \text{exp})$
assumes $\bigwedge b\ c. P\ c\ (\text{Bool } b)$
assumes $\bigwedge \text{scrut } e1\ e2\ c. (\bigwedge c. P\ c\ \text{scrut}) \implies (\bigwedge c. P\ c\ e1) \implies (\bigwedge c. P\ c\ e2) \implies P\ c\ (\text{scrut } ?\ e1 : e2)$
shows $P\ (c::'a::fs)\ \text{exp}$
apply (*rule exp-strong-induct-set(1)[of P]*)
apply (*metis assms(1)*)
apply (*metis assms(2)*)
apply (*case-tac nonrec Γ*)
apply (*erule nonrecE*)
apply *simp*

```

apply (metis assms(4))
apply (metis assms(3))
apply (metis assms(5))
apply (metis assms(6))
apply (metis assms(7))
done

```

3.2.11 Renaming a lambda-bound variable

```

lemma change-Lam-Variable:
  assumes  $y' \neq y \implies \text{atom } y' \# (e, y)$ 
  shows  $\text{Lam } [y]. e = \text{Lam } [y']. ((y \leftrightarrow y') \cdot e)$ 
proof(cases  $y' = y$ )
  case True thus ?thesis by simp
next
  case False
  from assms[OF this]
  have  $(y \leftrightarrow y') \cdot (\text{Lam } [y]. e) = \text{Lam } [y]. e$ 
  by -(rule flip-fresh-fresh, (simp add: fresh-Pair)+)
  moreover
  have  $(y \leftrightarrow y') \cdot (\text{Lam } [y]. e) = \text{Lam } [y']. ((y \leftrightarrow y') \cdot e)$ 
  by simp
  ultimately
  show  $\text{Lam } [y]. e = \text{Lam } [y']. ((y \leftrightarrow y') \cdot e)$  by (simp add: fresh-Pair)
qed

```

end

3.3 Substitution

```

theory Substitution
imports Terms
begin

```

Defining a substitution function on terms turned out to be slightly tricky.

```

fun
  subst-var :: var  $\Rightarrow$  var  $\Rightarrow$  var  $\Rightarrow$  var (-[::v=-] [1000,100,100] 1000)
where  $x[y ::v= z] = (\text{if } x = y \text{ then } z \text{ else } x)$ 

nominal-function (default case-sum ( $\lambda x. \text{Inl undefined}$ ) ( $\lambda x. \text{Inr undefined}$ ),
  invariant  $\lambda a r . (\forall \Gamma y z . ((a = \text{Inr } (\Gamma, y, z) \wedge \text{atom } \text{' domA } \Gamma \#* (y, z)) \longrightarrow$ 
   $\text{map } (\lambda x . \text{atom } (\text{fst } x)) (\text{Sum-Type.proj } r) = \text{map } (\lambda x . \text{atom } (\text{fst } x)) \Gamma)))$ 
  subst :: exp  $\Rightarrow$  var  $\Rightarrow$  var  $\Rightarrow$  exp (-[:::=] [1000,100,100] 1000)
and
  subst-heap :: heap  $\Rightarrow$  var  $\Rightarrow$  var  $\Rightarrow$  heap (-[::h=-] [1000,100,100] 1000)
where
  ( $\text{Var } x$ )[ $y ::= z$ ] =  $\text{Var } (x[y ::v= z])$ 

```



```

| (App e v)[y ::= z] = App (e[y ::= z]) (v[y ::= v= z])
| atom ‘ domA Γ #* (y,z) ==>
  (Let Γ body)[y ::= z] = Let (Γ[y ::h= z]) (body[y ::= z])
| atom x # (y,z) ==> (Lam [x].e)[y ::= z] = Lam [x].(e[y::=z])
| (Bool b)[y ::= z] = Bool b
| (scrut ? e1 : e2)[y ::= z] = (scrut[y ::= z] ? e1[y ::= z] : e2[y ::= z])
| [][y ::h= z] = []
| ((v,e)# Γ)[y ::h= z] = (v, e[y ::= z])# (Γ[y ::h= z])

```

proof goal-cases

have eqvt-at-subst: $\bigwedge e y z . \text{eqvt-at subst-subst-heap-sumC (Inl (e, y, z))} \implies \text{eqvt-at } (\lambda(a, b, c). \text{subst a b c}) (e, y, z)$

```

apply(simp add: eqvt-at-def subst-def)
apply(rule)
apply(subst Projl-permute)
apply(thin-tac -)+
apply (simp add: subst-subst-heap-sumC-def)
apply (simp add: THE-default-def)
apply (case-tac Ex1 (subst-subst-heap-graph (Inl (e, y, z))))
apply(simp)
apply(auto)[1]
apply (erule-tac x=x in allE)
apply simp
apply(cases rule: subst-subst-heap-graph.cases)
apply(assumption)
apply(rule-tac x=Sum-Type.projl x in exI)
apply(clarify)
apply (rule the1-equality)
apply blast
apply(simp (no-asm) only: sum.sel)
apply(rule-tac x=Sum-Type.projl x in exI)
apply(clarify)
apply (rule the1-equality)
apply blast
apply(simp (no-asm) only: sum.sel)
apply(rule-tac x=Sum-Type.projl x in exI)
apply(clarify)
apply (rule the1-equality)
apply blast
apply(simp (no-asm) only: sum.sel)
apply(rule-tac x=Sum-Type.projl x in exI)
apply(clarify)
apply (rule the1-equality)
apply blast
apply(simp (no-asm) only: sum.sel)
apply(rule-tac x=Sum-Type.projl x in exI)
apply(clarify)
apply (rule the1-equality)
apply blast

```

```

apply(simp (no-asm) only: sum.sel)
apply(rule-tac x=Sum-Type.proj1 x in exI)
apply(clarify)
apply (rule the1-equality)
apply blast
apply(simp (no-asm) only: sum.sel)
apply (metis Inr-not-Inl)
apply (metis Inr-not-Inl)
apply(simp)
apply(perm-simp)
apply(simp)
done

have eqvt-at-subst-heap:  $\bigwedge \Gamma y z . eqvt-at\ subst-subst-heap-sumC\ (Inr\ (\Gamma, y, z)) \implies eqvt-at$ 
( $\lambda(a, b, c). subst-heap\ a\ b\ c$ ) ( $\Gamma, y, z$ )
  apply(simp add: eqvt-at-def subst-heap-def)
  apply(rule)
  apply(subst Projr-permute)
  apply(thin-tac -)+
  apply (simp add: subst-subst-heap-sumC-def)
  apply (simp add: THE-default-def)
  apply (case-tac Ex1 (subst-subst-heap-graph (Inr ( $\Gamma, y, z$ ))))
  apply(simp)
  apply(auto)[1]
  apply (erule-tac x=x in allE)
  apply simp
  apply(cases rule: subst-subst-heap-graph.cases)
  apply(assumption)
  apply (metis (mono-tags) Inr-not-Inl)+
  apply(rule-tac x=Sum-Type.proj1 x in exI)
  apply(clarify)
  apply (rule the1-equality)
  apply auto[1]
  apply(simp (no-asm) only: sum.sel)

  apply(rule-tac x=Sum-Type.proj1 x in exI)
  apply(clarify)
  apply (rule the1-equality)
  apply auto[1]
  apply(simp (no-asm) only: sum.sel)

  apply(simp)
  apply(perm-simp)
  apply(simp)
done

{

case 1 thus ?case

```

unfolding *eqvt-def subst-subst-heap-graph-aux-def*
by *simp*

next case 2 **thus** ?*case*
by (*induct rule: subst-subst-heap-graph.induct*)(*auto simp add: exp-assn.bn-defs fresh-star-insert*)

next case *prems: (3 P x)* **show** ?*case*
proof(*cases x*)
case (*Inl a*) **thus** *P*
proof(*cases a*)
case (*fields a1 a2 a3*)
thus *P* **using** *Inl prems*
apply (*rule-tac y =a1 and c =(a2, a3) in exp-strong-exhaust*)
apply (*auto simp add: fresh-star-def*)
done
qed
next
case (*Inr a*) **thus** *P*
proof (*cases a*)
case (*fields a1 a2 a3*)
thus *P* **using** *Inr prems*
by (*metis heapToAssn.cases*)
qed
qed

next case (19 *e y2 z2 Γ e2 y z as2*) **thus** ?*case*
apply –
apply (*drule eqvt-at-subst*)
apply (*drule eqvt-at-subst-heap*)
apply (*simp only: meta-eq-to-obj-eq[OF subst-def, symmetric, unfolded fun-eq-iff]*
meta-eq-to-obj-eq[OF subst-heap-def, symmetric, unfolded fun-eq-iff])
apply (*auto simp add: Abs-fresh-iff*)
apply (*drule-tac*
c = (y, z) and
as = (map (λx. atom (fst x)) e) and
bs = (map (λx. atom (fst x)) e2) and
f = λ a b c . [a]lst. (subst (fst b) y z, subst-heap (snd b) y z) in Abs-lst-fcb2)
apply (*simp add: perm-supp-eq fresh-Pair fresh-star-def Abs-fresh-iff*)
apply (*metis domA-def image-image image-set*)
apply (*metis domA-def image-image image-set*)
apply (*simp add: eqvt-at-def, simp add: fresh-star-Pair perm-supp-eq*)
apply (*simp add: eqvt-at-def, simp add: fresh-star-Pair perm-supp-eq*)
apply (*simp add: eqvt-at-def*)
done

next case (25 *x2 y2 z2 e2 x y z e*) **thus** ?*case*

```

apply –
apply (drule eqvt-at-subst) +
apply (simp only: Abs-fresh-iff meta-eq-to-obj-eq[OF subst-def, symmetric, unfolded fun-eq-iff])

apply (simp add: eqvt-at-def)
apply rule
apply (erule-tac x = (x2 ↔ c) in allE)
apply (erule-tac x = (x ↔ c) in allE)
apply auto
done
}
qed(auto)

```

nominal-termination (*eqvt*) **by** *lexicographic-order*

lemma shows

```

True and bn-subst[simp]: domA (subst-heap Γ y z) = domA Γ
by(induct rule:subst-subst-heap.induct)
(auto simp add: exp-assn.bn-defs fresh-star-insert)

```

lemma *subst-noop[simp]:*

```

shows  $e[y ::= y] = e$  and  $\Gamma[y::h=y] = \Gamma$ 
by(induct e y y and Γ y y rule:subst-subst-heap.induct)
(auto simp add: fresh-star-Pair exp-assn.bn-defs)

```

lemma *subst-is-fresh[simp]:*

```

assumes  $\text{atom } y \# z$ 
shows
 $\text{atom } y \# e[y ::= z]$ 
and
 $\text{atom } \text{domA } \Gamma \#* y \implies \text{atom } y \# \Gamma[y::h=z]$ 
using assms
by(induct e y z and Γ y z rule:subst-subst-heap.induct)
(auto simp add: fresh-at-base fresh-star-Pair fresh-star-insert fresh-Nil fresh-Cons pure-fresh)

```

lemma

```

subst-pres-fresh: atom x # e ∨ x = y ⟹ atom x # z ⟹ atom x # e[y ::= z]
and
 $\text{atom } x \# \Gamma \vee x = y \implies \text{atom } x \# z \implies x \notin \text{domA } \Gamma \implies \text{atom } x \# (\Gamma[y::h= z])$ 
by(induct e y z and Γ y z rule:subst-subst-heap.induct)
(auto simp add: fresh-star-Pair exp-assn.bn-defs fresh-Cons fresh-Nil pure-fresh)

```

lemma *subst-fresh-noop: atom x # e ⟹ e[x ::= y] = e*

```

and subst-heap-fresh-noop: atom x # Γ ⟹ Γ[x::h= y] = Γ
by (nominal-induct e and Γ avoiding: x y rule:exp-heap-strong-induct)
(auto simp add: fresh-star-def fresh-Pair fresh-at-base fresh-Cons simp del: exp-assn.eq-iff)

```

lemma *supp-subst-eq: supp (e[y::=x]) = (supp e - {atom y}) ∪ (if atom y ∈ supp e then {atom*

$x\}$ else $\{\}$)
and $\text{atom } \text{'domA } \Gamma \#* y \implies \text{supp } (\Gamma[y::h=x]) = (\text{supp } \Gamma - \{\text{atom } y\}) \cup (\text{if } \text{atom } y \in \text{supp } \Gamma \text{ then } \{\text{atom } x\} \text{ else } \{\})$
by (*nominal-induct* e **and** Γ *avoiding: $x y$ rule:exp-heap-strong-induct*)
(auto simp add: fresh-star-def fresh-Pair supp-Nil supp-Cons supp-Pair fresh-Cons exp-assn.supp Let-supp supp-at-base pure-supp simp del: exp-assn.eq-iff)

lemma *supp-subst*: $\text{supp } (e[y::=x]) \subseteq (\text{supp } e - \{\text{atom } y\}) \cup \{\text{atom } x\}$
using *supp-subst-eq* **by** *auto*

lemma *fv-subst-eq*: $\text{fv } (e[y::=x]) = (\text{fv } e - \{y\}) \cup (\text{if } y \in \text{fv } e \text{ then } \{x\} \text{ else } \{\})$
and $\text{atom } \text{'domA } \Gamma \#* y \implies \text{fv } (\Gamma[y::h=x]) = (\text{fv } \Gamma - \{y\}) \cup (\text{if } y \in \text{fv } \Gamma \text{ then } \{x\} \text{ else } \{\})$
by (*nominal-induct* e **and** Γ *avoiding: $x y$ rule:exp-heap-strong-induct*)
(auto simp add: fresh-star-def fresh-Pair supp-Nil supp-Cons supp-Pair fresh-Cons exp-assn.supp Let-supp supp-at-base simp del: exp-assn.eq-iff)

lemma *fv-subst-subset*: $\text{fv } (e[y ::= x]) \subseteq (\text{fv } e - \{y\}) \cup \{x\}$
using *fv-subst-eq* **by** *auto*

lemma *fv-subst-int*: $x \notin S \implies y \notin S \implies \text{fv } (e[y ::= x]) \cap S = \text{fv } e \cap S$
by (*auto simp add: fv-subst-eq*)

lemma *fv-subst-int2*: $x \notin S \implies y \notin S \implies S \cap \text{fv } (e[y ::= x]) = S \cap \text{fv } e$
by (*auto simp add: fv-subst-eq*)

lemma *subst-swap-same*: $\text{atom } x \# e \implies (x \leftrightarrow y) \cdot e = e[y ::= x]$
and $\text{atom } x \# \Gamma \implies \text{atom } \text{'domA } \Gamma \#* y \implies (x \leftrightarrow y) \cdot \Gamma = \Gamma[y ::= h=x]$
by (*nominal-induct* e **and** Γ *avoiding: $x y$ rule:exp-heap-strong-induct*)
(auto simp add: fresh-star-Pair fresh-star-at-base fresh-Cons pure-fresh permute-pure simp del: exp-assn.eq-iff)

lemma *subst-subst-back*: $\text{atom } x \# e \implies e[y ::= x][x ::= y] = e$
and $\text{atom } x \# \Gamma \implies \text{atom } \text{'domA } \Gamma \#* y \implies \Gamma[y ::= h=x][x ::= h=y] = \Gamma$
by (*nominal-induct* e **and** Γ *avoiding: $x y$ rule:exp-heap-strong-induct*)
(auto simp add: fresh-star-Pair fresh-star-at-base fresh-star-Cons fresh-Cons exp-assn.bn-defs simp del: exp-assn.eq-iff)

lemma *subst-heap-delete[simp]*: $(\text{delete } x \Gamma)[y ::= h=z] = \text{delete } x (\Gamma[y ::= h=z])$
by (*induction* Γ) *auto*

lemma *subst-nil-iff[simp]*: $\Gamma[x ::= h=z] = [] \longleftrightarrow \Gamma = []$
by (*cases* Γ) *auto*

lemma *subst-SmartLet[simp]*:
 $\text{atom } \text{'domA } \Gamma \#* (y,z) \implies (\text{SmartLet } \Gamma \text{ body})[y ::= z] = \text{SmartLet } (\Gamma[y ::= h=z]) (\text{body}[y ::= z])$
unfolding *SmartLet-def* **by** *auto*

lemma *subst-let-be[simp]*:
 $atom\ x' \# y \implies atom\ x' \# x \implies (let\ x'\ be\ e\ in\ exp)[y::=x] = (let\ x'\ be\ e[y::=x]\ in\ exp[y::=x])$
by (*simp add: fresh-star-def fresh-Pair*)

lemma *isLam-subst[simp]*: $isLam\ e[x::=y] = isLam\ e$
by (*nominal-induct e avoiding: x y rule: exp-strong-induct*)
(auto simp add: fresh-star-Pair)

lemma *isVal-subst[simp]*: $isVal\ e[x::=y] = isVal\ e$
by (*nominal-induct e avoiding: x y rule: exp-strong-induct*)
(auto simp add: fresh-star-Pair)

lemma *thunks-subst[simp]*:
 $thunks\ \Gamma[y::h=x] = thunks\ \Gamma$
by (*induction \Gamma*) *(auto simp add: thunks-Cons)*

lemma *map-of-subst*:
 $map-of\ (\Gamma[x::h=y])\ k = map-option\ (\lambda\ e.\ e[x::=y])\ (map-of\ \Gamma\ k)$
by (*induction \Gamma*) *auto*

lemma *mapCollect-subst[simp]*:
 $\{e\ k\ v \mid k \mapsto v \in map-of\ \Gamma[x::h=y]\} = \{e\ k\ v[x::=y] \mid k \mapsto v \in map-of\ \Gamma\}$
by (*auto simp add: map-of-subst*)

lemma *subst-eq-Cons*:
 $\Gamma[x::h=y] = (x', e) \# \Delta \iff (\exists\ e'\ \Gamma'. \Gamma = (x', e') \# \Gamma' \wedge e'[x::=y] = e \wedge \Gamma'[x::h=y] = \Delta)$
by (*cases \Gamma*) *auto*

lemma *nonrec-subst*:
 $atom\ 'domA\ \Gamma \#* x \implies atom\ 'domA\ \Gamma \#* y \implies nonrec\ \Gamma[x::h=y] \iff nonrec\ \Gamma$
by (*auto simp add: nonrec-def fresh-star-def subst-eq-Cons fv-subst-eq*)

end

3.4 Launchbury

theory *Launchbury*
imports *Terms Substitution*
begin

3.4.1 The natural semantics

This is the semantics as in [Lau93], with two differences:

- Explicit freshness requirements for bound variables in the application and the Let rule.
- An additional parameter that stores variables that have to be avoided, but do not occur in the judgement otherwise, following [Ses97].

inductive

$reds :: heap \Rightarrow exp \Rightarrow var\ list \Rightarrow heap \Rightarrow exp \Rightarrow bool$
 $(- : - \Downarrow - : - [50,50,50,50] 50)$

where*Lambda:*
 $\Gamma : (Lam\ [x].\ e) \Downarrow_L \Gamma : (Lam\ [x].\ e)$
| *Application:* \llbracket
 $atom\ y \# (\Gamma, e, x, L, \Delta, \Theta, z) ;$
 $\Gamma : e \Downarrow_L \Delta : (Lam\ [y].\ e')$
 $\Delta : e'[y ::= x] \Downarrow_L \Theta : z$
 $\rrbracket \Rightarrow$
 $\Gamma : App\ e\ x \Downarrow_L \Theta : z$
| *Variable:* \llbracket
 $map-of\ \Gamma\ x = Some\ e; delete\ x\ \Gamma : e \Downarrow_{x\#L} \Delta : z$
 $\rrbracket \Rightarrow$
 $\Gamma : Var\ x \Downarrow_L (x, z) \# \Delta : z$
| *Let:* \llbracket
 $atom\ 'domA\ \Delta \#* (\Gamma, L);$
 $\Delta @ \Gamma : body \Downarrow_L \Theta : z$
 $\rrbracket \Rightarrow$
 $\Gamma : Let\ \Delta\ body \Downarrow_L \Theta : z$
| *Bool:*
 $\Gamma : Bool\ b \Downarrow_L \Gamma : Bool\ b$
| *IfThenElse:* \llbracket
 $\Gamma : scrut \Downarrow_L \Delta : (Bool\ b);$
 $\Delta : (if\ b\ then\ e_1\ else\ e_2) \Downarrow_L \Theta : z$
 $\rrbracket \Rightarrow$
 $\Gamma : (scrut\ ?\ e_1 : e_2) \Downarrow_L \Theta : z$
equivariance reds**nominal-inductive reds****avoids** *Application: y***by** (*auto simp add: fresh-star-def fresh-Pair*)**3.4.2 Example evaluations****lemma** *eval-test:*
 $\llbracket : (Let\ [(x, Lam\ [y].\ Var\ y)]\ (Var\ x)) \Downarrow_{\llbracket} [(x, Lam\ [y].\ Var\ y)] : (Lam\ [y].\ Var\ y)$
apply(*auto intro!: Lambda Application Variable Let**simp add: fresh-Pair fresh-Cons fresh-Nil fresh-star-def*)**done****lemma** *eval-test2:*
 $y \neq x \Rightarrow n \neq y \Rightarrow n \neq x \Rightarrow \llbracket : (Let\ [(x, Lam\ [y].\ Var\ y)]\ (App\ (Var\ x)\ x)) \Downarrow_{\llbracket} [(x, Lam\ [y].\ Var\ y)] : (Lam\ [y].\ Var\ y)$
by (*auto intro!: Lambda Application Variable Let simp add: fresh-Pair fresh-at-base fresh-Cons fresh-Nil fresh-star-def pure-fresh*)

3.4.3 Better introduction rules

This variant do not require freshness.

lemma *reds-ApplicationI*:

assumes $\Gamma : e \Downarrow_L \Delta : Lam [y]. e'$

assumes $\Delta : e'[y::=x] \Downarrow_L \Theta : z$

shows $\Gamma : App e x \Downarrow_L \Theta : z$

proof–

obtain $y' :: var$ **where** $atom\ y' \# (\Gamma, e, x, L, \Delta, \Theta, z, e')$ **by** (*rule obtain-fresh*)

have $a: Lam [y']. ((y' \leftrightarrow y) \cdot e') = Lam [y]. e'$

using $\langle atom\ y' \# \rightarrow \rangle$

by (*auto simp add: Abs1-eq-iff fresh-Pair fresh-at-base*)

have $b: ((y' \leftrightarrow y) \cdot e')[y'::=x] = e'[y::=x]$

proof(*cases x = y*)

case *True*

have $atom\ y' \# e'$ **using** $\langle atom\ y' \# \rightarrow \rangle$ **by** *simp*

thus *?thesis*

by (*simp add: True subst-swap-same subst-sbst-back*)

next

case *False*

hence $atom\ y \# x$ **by** *simp*

have $[simp]: (y' \leftrightarrow y) \cdot x = x$ **using** $\langle atom\ y \# \rightarrow \rangle \langle atom\ y' \# \rightarrow \rangle$

by (*simp add: flip-fresh-fresh fresh-Pair fresh-at-base*)

have $((y' \leftrightarrow y) \cdot e')[y'::=x] = (y' \leftrightarrow y) \cdot (e'[y::=x])$ **by** *simp*

also have $\dots = e'[y::=x]$

using $\langle atom\ y \# \rightarrow \rangle \langle atom\ y' \# \rightarrow \rangle$

by (*simp add: flip-fresh-fresh fresh-Pair fresh-at-base subst-pres-fresh*)

finally

show *?thesis*.

qed

have $atom\ y' \# (\Gamma, e, x, L, \Delta, \Theta, z)$ **using** $\langle atom\ y' \# \rightarrow \rangle$ **by** (*simp add: fresh-Pair*)

from *this assms[folded a b]*

show *?thesis ..*

qed

lemma *reds-SmartLet*: \llbracket

$atom\ 'domA\ \Delta \#* (\Gamma, L);$

$\Delta @ \Gamma : body \Downarrow_L \Theta : z$

$\rrbracket \implies$

$\Gamma : SmartLet\ \Delta\ body \Downarrow_L \Theta : z$

unfolding *SmartLet-def*

by (*auto intro: reds.Let*)

A single rule for values

lemma *reds-isValI*:
 $isVal\ z \implies \Gamma : z \Downarrow_L \Gamma : z$
by (*cases z rule:isVal.cases*) (*auto intro: reds.intros*)

3.4.4 Properties of the semantics

Heap entries are never removed.

lemma *reds-doesnt-forget*:
 $\Gamma : e \Downarrow_L \Delta : z \implies domA\ \Gamma \subseteq domA\ \Delta$
by(*induct rule: reds.induct*) *auto*

Live variables are not added to the heap.

lemma *reds-avoids-live'*:
assumes $\Gamma : e \Downarrow_L \Delta : z$
shows $(domA\ \Delta - domA\ \Gamma) \cap set\ L = \{\}$
using *assms*
by(*induct rule:reds.induct*)
(*auto dest: map-of-domAD fresh-distinct-list simp add: fresh-star-Pair*)

lemma *reds-avoids-live*:
 $\llbracket \Gamma : e \Downarrow_L \Delta : z;$
 $x \in set\ L;$
 $x \notin domA\ \Gamma$
 $\rrbracket \implies x \notin domA\ \Delta$
using *reds-avoids-live'* **by** *blast*

Fresh variables either stay fresh or are added to the heap.

lemma *reds-fresh*: $\llbracket \Gamma : e \Downarrow_L \Delta : z;$
 $atom\ (x::var)\ \#\ (\Gamma, e)$
 $\rrbracket \implies atom\ x\ \#\ (\Delta, z) \vee x \in (domA\ \Delta - set\ L)$
proof(*induct rule: reds.induct*)
case (*Lambda* $\Gamma\ x\ e$) **thus** *?case by auto*
next
case (*Application* $y\ \Gamma\ e\ x'\ L\ \Delta\ \Theta\ z\ e'$)
hence $atom\ x\ \#\ (\Delta, Lam\ [y].\ e') \vee x \in domA\ \Delta - set\ (x' \# L)$ **by** (*auto simp add:*
fresh-Pair)

thus *?case*
proof
assume $atom\ x\ \#\ (\Delta, Lam\ [y].\ e')$
hence $atom\ x\ \#\ e'[y ::= x']$
using *Application.prem*
by (*auto intro: subst-pres-fresh simp add: fresh-Pair*)
thus *?thesis using Application.hyps(5) <atom x # (Δ, Lam [y]. e')>* **by** *auto*
next
assume $x \in domA\ \Delta - set\ (x' \# L)$
thus *?thesis using reds-doesnt-forget[OF Application.hyps(4)] by auto*

```

qed
next

case (Variable  $\Gamma v e L \Delta z$ )
  have  $atom\ x \# \Gamma$  and  $atom\ x \# v$  using Variable.premis(1) by (auto simp add: fresh-Pair)
  from fresh-delete[OF this(1)]
  have  $atom\ x \# delete\ v\ \Gamma$ .
  moreover
  have  $v \in domA\ \Gamma$  using Variable.hyps(1) by (metis domA-from-set map-of-SomeD)
  from fresh-map-of[OF this <atom x #  $\Gamma$ >]
  have  $atom\ x \# the\ (map-of\ \Gamma\ v)$ .
  hence  $atom\ x \# e$  using <map-of  $\Gamma\ v = Some\ e>$  by simp
  ultimately
  have  $atom\ x \# (\Delta, z) \vee x \in domA\ \Delta - set\ (v \# L)$  using Variable.hyps(3) by (auto simp
add: fresh-Pair)
  thus ?case using <atom x # v> by (auto simp add: fresh-Pair fresh-Cons fresh-at-base)
next

case (Bool  $\Gamma b L$ )
  thus ?case by auto
next

case (IfThenElse  $\Gamma scrut L \Delta b e_1 e_2 \Theta z$ )
  from <atom x # ( $\Gamma, scrut\ ?\ e_1 : e_2$ )>
  have  $atom\ x \# (\Gamma, scrut)$  and  $atom\ x \# (e_1, e_2)$  by (auto simp add: fresh-Pair)
  from IfThenElse.hyps(2)[OF this(1)]
  show ?case
  proof
    assume  $atom\ x \# (\Delta, Bool\ b)$  with <atom x # ( $e_1, e_2$ )>
    have  $atom\ x \# (\Delta, if\ b\ then\ e_1\ else\ e_2)$  by auto
    from IfThenElse.hyps(4)[OF this]
    show ?thesis.
  next
    assume  $x \in domA\ \Delta - set\ L$ 
    with reds-doesnt-forget[OF < $\Delta : (if\ b\ then\ e_1\ else\ e_2) \Downarrow_L\ \Theta : z$ >]
    show ?thesis by auto
  qed
next

case (Let  $\Delta \Gamma L body \Theta z$ )
  show ?case
  proof (cases  $x \in domA\ \Delta$ )
    case False
      hence  $atom\ x \# \Delta$  using Let.premis by(auto simp add: fresh-Pair)
      show ?thesis
        apply(rule Let.hyps(3))
        using Let.premis <atom x #  $\Delta$ > False
        by (auto simp add: fresh-Pair fresh-append)
    next

```

```

case True
  hence  $x \notin \text{set } L$ 
  using Let(1)
  by (metis fresh-PairD(2) fresh-star-def image-eqI set-not-fresh)
  with True
  show ?thesis
  using reds-doesnt-forget[OF Let.hyps(2)] by auto
qed
qed

```

```

lemma reds-fresh-fv:  $\llbracket \Gamma : e \Downarrow_L \Delta : z;$ 
   $x \in \text{fv}(\Delta, z) \wedge (x \notin \text{dom}A \Delta \vee x \in \text{set } L)$ 
   $\rrbracket \implies x \in \text{fv}(\Gamma, e)$ 
using reds-fresh
unfolding fv-def fresh-def
by blast

```

```

lemma new-free-vars-on-heap:
  assumes  $\Gamma : e \Downarrow_L \Delta : z$ 
  shows  $\text{fv}(\Delta, z) - \text{dom}A \Delta \subseteq \text{fv}(\Gamma, e) - \text{dom}A \Gamma$ 
using reds-fresh-fv[OF assms(1)] reds-doesnt-forget[OF assms(1)] by auto

```

```

lemma reds-pres-closed:
  assumes  $\Gamma : e \Downarrow_L \Delta : z$ 
  and  $\text{fv}(\Gamma, e) \subseteq \text{set } L \cup \text{dom}A \Gamma$ 
  shows  $\text{fv}(\Delta, z) \subseteq \text{set } L \cup \text{dom}A \Delta$ 
using new-free-vars-on-heap[OF assms(1)] assms(2) by auto

```

Reducing the set of variables to avoid is always possible.

```

lemma reds-smaller-L:  $\llbracket \Gamma : e \Downarrow_L \Delta : z;$ 
   $\text{set } L' \subseteq \text{set } L$ 
   $\rrbracket \implies \Gamma : e \Downarrow_{L'} \Delta : z$ 
proof(nominal-induct avoiding : L' rule: reds.strong-induct)
case (Lambda  $\Gamma x e L L'$ )
  show ?case
  by (rule reds.Lambda)
next
case (Application  $y \Gamma e xa L \Delta \Theta z e' L'$ )
  from Application.hyps(10)[OF Application.prem] Application.hyps(12)[OF Application.prem]
  show ?case
  by (rule reds-ApplicationI)
next
case (Variable  $\Gamma xa e L \Delta z L'$ )
  have  $\text{set}(xa \# L') \subseteq \text{set}(xa \# L)$ 
  using Variable.prem by auto
  thus ?case
  by (rule reds.Variable[OF Variable(1) Variable.hyps(3)])
next
case (Bool  $b$ )

```

```

  show ?case..
next
case (IfThenElse  $\Gamma$  scrut  $L \Delta b e_1 e_2 \Theta z L'$ )
  thus ?case by (metis reds.IfThenElse)
next
case (Let  $\Delta \Gamma L$  body  $\Theta z L'$ )
  have atom ' domA  $\Delta \#*$  ( $\Gamma, L'$ )
  using Let(1-3) Let.premis
  by (auto simp add: fresh-star-Pair fresh-star-set-subset)
  thus ?case
  by (rule reds.Let[OF - Let.hyps(4)][OF Let.premis])
qed

```

Things are evaluated to a lambda expression, and the variable can be freely chose.

lemma *result-evaluated*:

```

 $\Gamma : e \Downarrow_L \Delta : z \implies isVal z$ 
by (induct  $\Gamma e L \Delta z$  rule:reds.induct) (auto dest: reds-doesnt-forget)

```

lemma *result-evaluated-fresh*:

```

assumes  $\Gamma : e \Downarrow_L \Delta : z$ 
obtains  $y e'$ 
where  $z = (Lam [y]. e')$  and  $atom y \# (c::'a::fs) \mid b$  where  $z = Bool b$ 
proof-
from assms
have  $isVal z$  by (rule result-evaluated)
hence  $(\exists y e'. z = Lam [y]. e' \wedge atom y \# c) \vee (\exists b. z = Bool b)$ 
by (nominal-induct  $z$  avoiding:  $c$  rule:exp-strong-induct) auto
thus thesis using that by blast
qed

```

end

4 Denotational domain

4.1 Value

```

theory Value
  imports HOLCF
begin

```

4.1.1 The semantic domain for values and environments

```

domain Value = Fn (lazy Value  $\rightarrow$  Value) | B (lazy bool discr)

```

```

fixrec Fn-project :: Value  $\rightarrow$  Value  $\rightarrow$  Value
  where Fn-project.(Fn.f) = f

```

abbreviation *Fn-project-abbr* (**infix** $\downarrow Fn$ 55)
where $f \downarrow Fn v \equiv Fn\text{-project}\cdot f\cdot v$

lemma [*simp*]:
 $\perp \downarrow Fn x = \perp$
 $(B\cdot b) \downarrow Fn x = \perp$
by (*fixrec-simp*)⁺

fixrec *B-project* :: *Value* \rightarrow *Value* \rightarrow *Value* \rightarrow *Value* **where**
B-project.(*B*·*db*)·*v*₁·*v*₂ = (*if undiscr db then v*₁ *else v*₂)

lemma [*simp*]:
B-project.(*B*·(*Discr b*))·*v*₁·*v*₂ = (*if b then v*₁ *else v*₂)
B-project· \perp ·*v*₁·*v*₂ = \perp
B-project.(*Fn*·*f*)·*v*₁·*v*₂ = \perp
by *fixrec-simp*⁺

A chain in the domain *Value* is either always bottom, or eventually *Fn* of another chain

lemma *Value-chainE*[*consumes 1, case-names bot B Fn*]:

assumes *chain Y*
obtains $Y = (\lambda \cdot \cdot \perp) \mid$
 $n\ b$ **where** $Y = (\lambda m. (\text{if } m < n \text{ then } \perp \text{ else } B\cdot b)) \mid$
 $n\ Y'$ **where** $Y = (\lambda m. (\text{if } m < n \text{ then } \perp \text{ else } Fn\cdot(Y' (m-n))))$ *chain Y'*

proof(*cases Y = (\lambda \cdot \cdot \perp)*)

case *True*

thus *?thesis* **by** (*rule that(1)*)

next

case *False*

hence $\exists i. Y\ i \neq \perp$ **by** *auto*

hence $\exists n. Y\ n \neq \perp \wedge (\forall m. Y\ m \neq \perp \longrightarrow m \geq n)$

by (*rule exE*)(*rule ex-has-least-nat*)

then obtain *n* **where** $Y\ n \neq \perp$ **and** $\forall m. m < n \longrightarrow Y\ m = \perp$ **by** *fastforce*

hence $(\exists f. Y\ n = Fn\cdot f) \vee (\exists b. Y\ n = B\cdot b)$ **by** (*metis Value.exhaust*)

thus *?thesis*

proof

assume $(\exists f. Y\ n = Fn\cdot f)$

then obtain *f* **where** $Y\ n = Fn\cdot f$ **by** *blast*

{

fix *i*

from $\langle \text{chain } Y \rangle$ **have** $Y\ n \sqsubseteq Y\ (i+n)$ **by** (*metis chain-mono le-add2*)

with $\langle Y\ n = \cdot \rangle$

have $\exists g. (Y\ (i+n) = Fn\cdot g)$

by (*metis Value.dist-les(1) Value.exhaust below-bottom-iff*)

}

then obtain *Y'* **where** $Y': \bigwedge i. Y\ (i+n) = Fn\cdot (Y'\ i)$ **by** *metis*

have $Y = (\lambda m. \text{if } m < n \text{ then } \perp \text{ else } Fn\cdot(Y' (m-n)))$

using $\langle \forall m. \rightarrow Y' \rangle$ **by** (*metis add-diff-inverse add commute*)

```

moreover
have chain  $Y'$  using  $\langle$ chain  $Y$  $\rangle$ 
  by (auto intro!:chainI elim: chainE simp add: Value.inverts[symmetric]  $Y'$ [symmetric]
simp del: Value.inverts)
  ultimately
  show ?thesis by (rule that(3))
next
assume  $(\exists b. Y\ n = B \cdot b)$ 
then obtain  $b$  where  $Y\ n = B \cdot b$  by blast
  {
    fix  $i$ 
    from  $\langle$ chain  $Y$  $\rangle$  have  $Y\ n \sqsubseteq Y\ (i+n)$  by (metis chain-mono le-add2)
    with  $\langle Y\ n = \cdot \rangle$ 
    have  $Y\ (i+n) = B \cdot b$ 
    by (metis Value.dist-les(2) Value.exhaust Value.inverts(2) below-bottom-iff discrete-cpo)
  }
hence  $Y': \bigwedge i. Y\ (i + n) = B \cdot b$  by metis

have  $Y = (\lambda m. \text{if } m < n \text{ then } \perp \text{ else } B \cdot b)$ 
  using  $\langle \forall m. \rightarrow Y' \rangle$  by (metis add-diff-inverse add commute)
thus ?thesis by (rule that(2))
qed
qed

end

```

4.2 Value-Nominal

```

theory Value-Nominal
imports Value Nominal-Utills Nominal-HOLCF
begin

```

Values are pure, i.e. contain no variables.

```

instantiation Value :: pure
begin
  definition  $p \cdot (v :: \text{Value}) = v$ 
instance
  apply standard
  apply (auto simp add: permute-Value-def)
  done
end

```

```

instance Value :: pcpo-pt
  by standard (simp add: pure-permute-id)

end

```

5 Denotational semantics

5.1 Iterative

```
theory Iterative
imports Env-HOLCF
begin
```

A setup for defining a fixed point of mutual recursive environments iteratively

```
locale iterative =
  fixes  $\rho :: 'a::type \Rightarrow 'b::pcpo$ 
  and  $e1 :: ('a \Rightarrow 'b) \rightarrow ('a \Rightarrow 'b)$ 
  and  $e2 :: ('a \Rightarrow 'b) \rightarrow 'b$ 
  and  $S :: 'a \text{ set}$  and  $x :: 'a$ 
  assumes  $ne: x \notin S$ 
begin
  abbreviation  $L == (\Lambda \rho'. (\rho \text{ ++}_S e1 \cdot \rho')(x := e2 \cdot \rho'))$ 
  abbreviation  $H == (\lambda \rho'. \Lambda \rho''. \rho' \text{ ++}_S e1 \cdot \rho'')$ 
  abbreviation  $R == (\Lambda \rho'. (\rho \text{ ++}_S (\text{fix} \cdot (H \rho')))(x := e2 \cdot \rho'))$ 
  abbreviation  $R' == (\Lambda \rho'. (\rho \text{ ++}_S (\text{fix} \cdot (H \rho')))(x := e2 \cdot (\text{fix} \cdot (H \rho'))))$ 
```

lemma *split-x*:

```
  fixes  $y$ 
  obtains  $y = x$  and  $y \notin S \mid y \in S$  and  $y \neq x \mid y \notin S$  and  $y \neq x$  using  $ne$  by blast
  lemmas below = fun-belowI[OF split-x, where  $y1 = \lambda x. x$ ]
  lemmas eq = ext[OF split-x, where  $y1 = \lambda x. x$ ]
```

lemma *lookup-fix[simp]*:

```
  fixes  $y$  and  $F :: ('a \Rightarrow 'b) \rightarrow ('a \Rightarrow 'b)$ 
  shows  $(\text{fix} \cdot F) y = (F \cdot (\text{fix} \cdot F)) y$ 
  by (subst fix-eq, rule)
```

lemma *R-S*: $\bigwedge y. y \in S \Longrightarrow (\text{fix} \cdot R) y = (e1 \cdot (\text{fix} \cdot (H (\text{fix} \cdot R)))) y$
 by (case-tac y rule: split-x) simp-all

lemma *R'-S*: $\bigwedge y. y \in S \Longrightarrow (\text{fix} \cdot R') y = (e1 \cdot (\text{fix} \cdot (H (\text{fix} \cdot R')))) y$
 by (case-tac y rule: split-x) simp-all

lemma *HR-is-R[simp]*: $\text{fix} \cdot (H (\text{fix} \cdot R)) = \text{fix} \cdot R$
 by (rule eq) simp-all

lemma *HR'-is-R'[simp]*: $\text{fix} \cdot (H (\text{fix} \cdot R')) = \text{fix} \cdot R'$
 by (rule eq) simp-all

lemma *H-noop*:

```
  fixes  $\rho' \rho''$ 
  assumes  $\bigwedge y. y \in S \Longrightarrow y \neq x \Longrightarrow (e1 \cdot \rho'') y \sqsubseteq \rho' y$ 
  shows  $H \rho' \cdot \rho'' \sqsubseteq \rho'$ 
  using assms
```

by $-(rule\ below, simp\ all)$

lemma *HL-is-L[simp]*: $fix \cdot (H (fix \cdot L)) = fix \cdot L$

proof (*rule below-antisym*)

show $fix \cdot (H (fix \cdot L)) \sqsubseteq fix \cdot L$

by (*rule fix-least-below[OF H-noop]*) *simp*

hence *: $e2 \cdot (fix \cdot (H (fix \cdot L))) \sqsubseteq e2 \cdot (fix \cdot L)$ **by** (*rule monofun-cfun-arg*)

show $fix \cdot L \sqsubseteq fix \cdot (H (fix \cdot L))$

by (*rule fix-least-below[OF below]*) (*simp-all add: ne **)

qed

lemma *iterative-override-on*:

shows $fix \cdot L = fix \cdot R$

proof(*rule below-antisym*)

show $fix \cdot R \sqsubseteq fix \cdot L$

by (*rule fix-least-below[OF below]*) *simp-all*

show $fix \cdot L \sqsubseteq fix \cdot R$

apply (*rule fix-least-below[OF below]*)

apply *simp*

apply (*simp del: lookup-fix add: R-S*)

apply *simp*

done

qed

lemma *iterative-override-on'*:

shows $fix \cdot L = fix \cdot R'$

proof(*rule below-antisym*)

show $fix \cdot R' \sqsubseteq fix \cdot L$

by (*rule fix-least-below[OF below]*) *simp-all*

show $fix \cdot L \sqsubseteq fix \cdot R'$

apply (*rule fix-least-below[OF below]*)

apply *simp*

apply (*simp del: lookup-fix add: R'-S*)

apply *simp*

done

qed

end

end

5.2 HasESem

theory *HasESem*

imports *Nominal-HOLCF Env-HOLCF*

begin

A locale to work abstract in the expression type and semantics.

```

locale has-ESem =
  fixes ESem :: 'exp::pt  $\Rightarrow$  ('var::at-base  $\Rightarrow$  'value)  $\rightarrow$  'value::{pure,pcpo}
begin
  abbreviation ESem-syn ( $\llbracket$  -  $\rrbracket$ - [0,0] 110) where  $\llbracket e \rrbracket_{\varrho} \equiv ESem\ e \cdot \varrho$ 
end

```

```

locale has-ignore-fresh-ESem = has-ESem +
  assumes fv-supp: supp e = atom ' (fv e :: 'b set)
  assumes ESem-considers-fv:  $\llbracket e \rrbracket_{\varrho} = \llbracket e \rrbracket_{\varrho} f | ' (fv\ e)$ 

```

end

5.3 HeapSemantics

theory *HeapSemantics*

```

imports EvalHeap AList-Utills-Nominal HasESem Iterative Env-Nominal
begin

```

5.3.1 A locale for heap semantics, abstract in the expression semantics

```

context has-ESem
begin

```

```

abbreviation EvalHeapSem-syn ( $\llbracket$  -  $\rrbracket$ - [0,0] 110)
  where EvalHeapSem-syn  $\Gamma\ \varrho \equiv evalHeap\ \Gamma\ (\lambda\ e.\ \llbracket e \rrbracket_{\varrho})$ 

```

definition

```

HSem :: ('var  $\times$  'exp) list  $\Rightarrow$  ('var  $\Rightarrow$  'value)  $\rightarrow$  ('var  $\Rightarrow$  'value)
  where HSem  $\Gamma = (\Lambda\ \varrho \cdot (\mu\ \varrho'.\ \varrho\ ++\ domA\ \Gamma\ \llbracket \Gamma \rrbracket_{\varrho'}))$ 

```

```

abbreviation HSem-syn ( $\{\!\!\{$  -  $\}\!\!\}$ - [0,60] 60)
  where  $\{\!\!\{ \Gamma \}\!\!\}_{\varrho} \equiv HSem\ \Gamma \cdot \varrho$ 

```

```

lemma HSem-def':  $\{\!\!\{ \Gamma \}\!\!\}_{\varrho} = (\mu\ \varrho'.\ \varrho\ ++\ domA\ \Gamma\ \llbracket \Gamma \rrbracket_{\varrho'})$ 
  unfolding HSem-def by simp

```

5.3.2 Induction and other lemmas about *HSem*

lemma *HSem-ind*:

```

assumes adm P
assumes P  $\perp$ 
assumes step:  $\bigwedge\ \varrho'.\ P\ \varrho' \Longrightarrow P\ (\varrho\ ++\ domA\ \Gamma\ \llbracket \Gamma \rrbracket_{\varrho'})$ 
shows P ( $\{\!\!\{ \Gamma \}\!\!\}_{\varrho}$ )
unfolding HSem-def'
apply (rule fix-ind[OF assms(1), OF assms(2)])
using step by simp

```

lemma *HSem-below*:

assumes ρ : $\bigwedge x. x \notin \text{dom} A \ h \implies \rho \ x \sqsubseteq r \ x$

assumes h : $\bigwedge x. x \in \text{dom} A \ h \implies \llbracket \text{the } (\text{map-of } h \ x) \rrbracket_r \sqsubseteq r \ x$

shows $\{\!\{h\}\!\}_\rho \sqsubseteq r$

proof (*rule HSem-ind, goal-cases*)

case 1 show *?case* **by** (*auto*)

next

case 2 show *?case* **by** (*rule minimal*)

next

case (*3* ρ')

show *?case*

by (*rule override-on-belowI*)

(*auto simp add: lookupEvalHeap below-trans[OF monofun-cfun-arg[OF $\rho' \sqsubseteq r$]] h*) *rho*)

qed

lemma *HSem-bot-below*:

assumes h : $\bigwedge x. x \in \text{dom} A \ h \implies \llbracket \text{the } (\text{map-of } h \ x) \rrbracket_r \sqsubseteq r \ x$

shows $\{\!\{h\}\!\}_\perp \sqsubseteq r$

using *assms*

by (*metis HSem-below fun-belowD minimal*)

lemma *HSem-bot-ind*:

assumes *adm* P

assumes $P \perp$

assumes *step*: $\bigwedge \rho'. P \ \rho' \implies P \ (\llbracket \Gamma \rrbracket_{\rho'})$

shows $P \ (\{\!\{\Gamma\}\!\}_\perp)$

apply (*rule HSem-ind[OF assms(1,2)]*)

apply (*drule assms(3)*)

apply *simp*

done

lemma *parallel-HSem-ind*:

assumes *adm* $(\lambda \rho'. P \ (\text{fst } \rho') \ (\text{snd } \rho'))$

assumes $P \perp \perp$

assumes *step*: $\bigwedge y \ z. P \ y \ z \implies$

$P \ (\rho_1 \ ++_{\text{dom} A \ \Gamma_1} \llbracket \Gamma_1 \rrbracket y) \ (\rho_2 \ ++_{\text{dom} A \ \Gamma_2} \llbracket \Gamma_2 \rrbracket z)$

shows $P \ (\{\!\{\Gamma_1\}\!\}_{\rho_1}) \ (\{\!\{\Gamma_2\}\!\}_{\rho_2})$

unfolding *HSem-def'*

apply (*rule parallel-fix-ind[OF assms(1), OF assms(2)]*)

using *step* **by** *simp*

lemma *HSem-eq*:

shows $\{\!\{\Gamma\}\!\}_\rho = \rho \ ++_{\text{dom} A \ \Gamma} \llbracket \Gamma \rrbracket_{\{\!\{\Gamma\}\!\}_\rho}$

unfolding *HSem-def'*

by (*subst fix-eq*) *simp*

lemma *HSem-bot-eq*:

shows $\{\!\{\Gamma\}\!\}_\perp = \llbracket \Gamma \rrbracket_{\{\!\{\Gamma\}\!\}_\perp}$

by (*subst HSem-eq*) *simp*

lemma *lookup-HSem-other*:

assumes $y \notin \text{dom}A \ h$
shows $\{\!|h|\!\}_\varrho \ y = \varrho \ y$
apply (*subst HSem-eq*)
using *assms* **by** *simp*

lemma *lookup-HSem-heap*:

assumes $y \in \text{dom}A \ h$
shows $\{\!|h|\!\}_\varrho \ y = \llbracket \text{the } (\text{map-of } h \ y) \rrbracket \{\!|h|\!\}_\varrho$
apply (*subst HSem-eq*)
using *assms* **by** (*simp add: lookupEvalHeap*)

lemma *HSem-edom-subset*: $\text{edom } (\{\!\Gamma\!\}_\varrho) \subseteq \text{edom } \varrho \cup \text{dom}A \ \Gamma$

apply *rule*
unfolding *edomIff*
apply (*case-tac x \in \text{dom}A \ \Gamma*)
apply (*auto simp add: lookup-HSem-other*)
done

lemma (**in** $-$) *env-restr-override-onI*: $-S2 \subseteq S \implies \text{env-restr } S \ \varrho1 \ ++_{S2} \ \varrho2 = \varrho1 \ ++_{S2} \ \varrho2$
by (*rule ext*) (*auto simp add: lookup-override-on-eq*)

lemma *HSem-restr*:

$\{\!|h|\!\}_\varrho \ f \!|' \ (- \ \text{dom}A \ h) = \{\!|h|\!\}_\varrho$
apply (*rule parallel-HSem-ind*)
apply *simp*
apply *auto[1]*
apply (*subst env-restr-override-onI*)
apply *simp-all*
done

lemma *HSem-restr-cong*:

assumes $\varrho \ f \!|' \ (- \ \text{dom}A \ h) = \varrho' \ f \!|' \ (- \ \text{dom}A \ h)$
shows $\{\!|h|\!\}_\varrho = \{\!|h|\!\}_{\varrho'}$
apply (*subst (1 2) HSem-restr[symmetric]*)
by (*simp add: assms*)

lemma *HSem-restr-cong-below*:

assumes $\varrho \ f \!|' \ (- \ \text{dom}A \ h) \sqsubseteq \varrho' \ f \!|' \ (- \ \text{dom}A \ h)$
shows $\{\!|h|\!\}_\varrho \sqsubseteq \{\!|h|\!\}_{\varrho'}$
by (*subst (1 2) HSem-restr[symmetric]*) (*rule monofun-cfun-arg[OF assms]*)

lemma *HSem-reorder*:

assumes $\text{map-of } \Gamma = \text{map-of } \Delta$
shows $\{\!\Gamma\!\}_\varrho = \{\!\Delta\!\}_\varrho$
by (*simp add: HSem-def' evalHeap-reorder[OF assms] assms dom-map-of-conv-domA[symmetric]*)

lemma *HSem-reorder-head*:

assumes $x \neq y$
shows $\{\!(x, e1)\#(y, e2)\#\Gamma\}\varrho = \{\!(y, e2)\#(x, e1)\#\Gamma\}\varrho$
proof–
have $set((x, e1)\#(y, e2)\#\Gamma) = set((y, e2)\#(x, e1)\#\Gamma)$
by *auto*
thus *?thesis*
unfolding *HSem-def evalHeap-reorder-head[OF assms]*
by (*simp add: domA-def*)
qed

lemma *HSem-reorder-head-append*:
assumes $x \notin domA \Gamma$
shows $\{\!(x, e)\#\Gamma @ \Delta\}\varrho = \{\!\Gamma @ ((x, e)\#\Delta)\}\varrho$
proof–
have $set((x, e)\#\Gamma @ \Delta) = set(\Gamma @ ((x, e)\#\Delta))$ **by** *auto*
thus *?thesis*
unfolding *HSem-def evalHeap-reorder-head-append[OF assms]*
by *simp*
qed

lemma *env-restr-HSem*:
assumes $domA \Gamma \cap S = \{\}$
shows $(\{\!\Gamma\}\varrho) f \mid S = \varrho f \mid S$
proof (*rule env-restr-eqI*)
fix x
assume $x \in S$
hence $x \notin domA \Gamma$ **using** *assms* **by** *auto*
thus $(\{\!\Gamma\}\varrho) x = \varrho x$
by (*rule lookup-HSem-other*)
qed

lemma *env-restr-HSem-noop*:
assumes $domA \Gamma \cap edom \varrho = \{\}$
shows $(\{\!\Gamma\}\varrho) f \mid edom \varrho = \varrho$
by (*simp add: env-restr-HSem[OF assms] env-restr-useless*)

lemma *HSem-Nil[simp]*: $\{\!\{\}\}\varrho = \varrho$
by (*subst HSem-eq, simp*)

5.3.3 Substitution

lemma *HSem-subst-exp*:
assumes $\bigwedge \varrho'. \llbracket e \rrbracket_{\varrho'} = \llbracket e' \rrbracket_{\varrho'}$
shows $\{\!(x, e)\#\Gamma\}\varrho = \{\!(x, e')\#\Gamma\}\varrho$
by (*rule parallel-HSem-ind*) (*auto simp add: assms evalHeap-subst-exp*)

lemma *HSem-subst-expr-below*:
assumes *below*: $\llbracket e1 \rrbracket \{\!(x, e2)\#\Gamma\}\varrho \sqsubseteq \llbracket e2 \rrbracket \{\!(x, e2)\#\Gamma\}\varrho$
shows $\{\!(x, e1)\#\Gamma\}\varrho \sqsubseteq \{\!(x, e2)\#\Gamma\}\varrho$

by (rule HSem-below) (auto simp add: lookup-HSem-heap below lookup-HSem-other)

lemma *HSem-subst-expr*:

assumes *below1*: $\llbracket e1 \rrbracket_{\rho} \{x, e2\} \# \Gamma \sqsubseteq \llbracket e2 \rrbracket_{\rho} \{x, e2\} \# \Gamma$

assumes *below2*: $\llbracket e2 \rrbracket_{\rho} \{x, e1\} \# \Gamma \sqsubseteq \llbracket e1 \rrbracket_{\rho} \{x, e1\} \# \Gamma$

shows $\{x, e1\} \# \Gamma \sqsubseteq \{x, e2\} \# \Gamma$

by (metis *assms HSem-subst-expr-below below-antisym*)

5.3.4 Re-calculating the semantics of the heap is idempotent

lemma *HSem-redo*:

shows $\{\Gamma\}(\{\Gamma @ \Delta\}_{\rho}) f |' (edom \rho \cup domA \Delta) = \{\Gamma @ \Delta\}_{\rho}$ (is ?LHS = ?RHS)

proof (rule *below-antisym*)

show ?LHS \sqsubseteq ?RHS

by (rule *HSem-below*)

(auto simp add: lookup-HSem-heap fun-belowD[OF *env-restr-below-itself*])

show ?RHS \sqsubseteq ?LHS

proof(rule *HSem-below*, *goal-cases*)

case (1 *x*)

thus ?*case*

by (*cases x* \notin *edom* ρ) (auto simp add: lookup-HSem-other *dest:lookup-not-edom*)

next

case *prems*: (2 *x*)

thus ?*case*

proof(*cases x* \in *domA* Γ)

case *True*

thus ?*thesis* **by** (auto simp add: lookup-HSem-heap)

next

case *False*

hence *delta*: *x* \in *domA* Δ **using** *prems* **by** *auto*

with *False* \langle ?LHS \sqsubseteq ?RHS \rangle

show ?*thesis* **by** (auto simp add: lookup-HSem-other lookup-HSem-heap *monofun-cfun-arg*)

qed

qed

qed

5.3.5 Iterative definition of the heap semantics

lemma *iterative-HSem*:

assumes *x* \notin *domA* Γ

shows $\{x, e\} \# \Gamma \sqsubseteq (\mu \rho'. (\rho ++_{domA \Gamma} (\{\Gamma\}_{\rho'})) (x := \llbracket e \rrbracket_{\rho'}))$

proof–

from *assms*

interpret *iterative*

where *e1* = $\Lambda \rho'. \llbracket \Gamma \rrbracket_{\rho'}$

and *e2* = $\Lambda \rho'. \llbracket e \rrbracket_{\rho'}$

and *S* = *domA* Γ

and *x* = *x* **by** *unfold-locales*

have $\llbracket (x, e) \# \Gamma \rrbracket_{\varrho} = \text{fix} \cdot L$
by (*simp add: HSem-def' override-on-upd ne*)
also have $\dots = \text{fix} \cdot R$
by (*rule iterative-override-on*)
also have $\dots = (\mu \varrho'. (\varrho \text{ ++ }_{\text{dom}A \Gamma} (\llbracket \Gamma \rrbracket_{\varrho'})) (x := \llbracket e \rrbracket_{\varrho'}))$
by (*simp add: HSem-def'*)
finally show *?thesis.*
qed

lemma *iterative-HSem'*:
assumes $x \notin \text{dom}A \Gamma$
shows $(\mu \varrho'. (\varrho \text{ ++ }_{\text{dom}A \Gamma} (\llbracket \Gamma \rrbracket_{\varrho'})) (x := \llbracket e \rrbracket_{\varrho'}))$
 $= (\mu \varrho'. (\varrho \text{ ++ }_{\text{dom}A \Gamma} (\llbracket \Gamma \rrbracket_{\varrho'})) (x := \llbracket e \rrbracket_{\llbracket \Gamma \rrbracket_{\varrho'}}))$

proof–
from *assms*
interpret *iterative*
where $e1 = \Lambda \varrho'. \llbracket \Gamma \rrbracket_{\varrho'}$
and $e2 = \Lambda \varrho'. \llbracket e \rrbracket_{\varrho'}$
and $S = \text{dom}A \Gamma$
and $x = x$ **by** *unfold-locales*

have $(\mu \varrho'. (\varrho \text{ ++ }_{\text{dom}A \Gamma} (\llbracket \Gamma \rrbracket_{\varrho'})) (x := \llbracket e \rrbracket_{\varrho'})) = \text{fix} \cdot R$
by (*simp add: HSem-def'*)
also have $\dots = \text{fix} \cdot L$
by (*rule iterative-override-on[symmetric]*)
also have $\dots = \text{fix} \cdot R'$
by (*rule iterative-override-on'*)
also have $\dots = (\mu \varrho'. (\varrho \text{ ++ }_{\text{dom}A \Gamma} (\llbracket \Gamma \rrbracket_{\varrho'})) (x := \llbracket e \rrbracket_{\llbracket \Gamma \rrbracket_{\varrho'}}))$
by (*simp add: HSem-def'*)
finally
show *?thesis.*

qed

5.3.6 Fresh variables on the heap are irrelevant

lemma *HSem-ignores-fresh-restr'*:
assumes $\text{fv } \Gamma \subseteq S$
assumes $\bigwedge x \varrho. x \in \text{dom}A \Gamma \implies \llbracket \text{the } (\text{map-of } \Gamma x) \rrbracket_{\varrho} = \llbracket \text{the } (\text{map-of } \Gamma x) \rrbracket_{\varrho} f|' (\text{fv } (\text{the } (\text{map-of } \Gamma x)))$
shows $(\llbracket \Gamma \rrbracket_{\varrho}) f|' S = \llbracket \Gamma \rrbracket_{\varrho} f|' S$
proof (*induction rule: parallel-HSem-ind[case-names adm base step]*)
case *adm* **thus** *?case* **by** *simp*
next
case *base*
show *?case* **by** *simp*
next
case (*step* $y z$)
have $\llbracket \Gamma \rrbracket_y = \llbracket \Gamma \rrbracket_z$

```

proof(rule evalHeap-cong^)
  fix x
  assume  $x \in \text{dom}A \Gamma$ 
  hence  $\text{fv } (the \ (map-of \ \Gamma \ x)) \subseteq \text{fv } \Gamma$  by (rule map-of-fv-subset)
  with  $assms(1)$ 
  have  $\text{fv } (the \ (map-of \ \Gamma \ x)) \cap S = \text{fv } (the \ (map-of \ \Gamma \ x))$  by auto
  with step
  have  $y \ f|' \ \text{fv } (the \ (map-of \ \Gamma \ x)) = z \ f|' \ \text{fv } (the \ (map-of \ \Gamma \ x))$  by auto
  with  $\langle x \in \text{dom}A \ \Gamma \rangle$ 
  show  $\llbracket the \ (map-of \ \Gamma \ x) \rrbracket_y = \llbracket the \ (map-of \ \Gamma \ x) \rrbracket_z$ 
    by (subst (1 2)  $assms(2)[OF \ \langle x \in \text{dom}A \ \Gamma \rangle]$ ) simp
  qed
  moreover
  have  $\text{dom}A \ \Gamma \subseteq S$  using domA-fv-subset  $assms(1)$  by auto
  ultimately
  show ?case by (simp add: env-restr-add env-restr-evalHeap-noop)
qed
end

```

5.3.7 Freshness

context has-ignore-fresh-ESem **begin**

lemma ESem-fresh-cong:

assumes $\varrho \ f|' \ (fv \ e) = \varrho' \ f|' \ (fv \ e)$

shows $\llbracket e \rrbracket_{\varrho} = \llbracket e \rrbracket_{\varrho'}$

by (metis $assms$ ESem-considers-fv)

lemma ESem-fresh-cong-subset:

assumes $\text{fv } e \subseteq S$

assumes $\varrho \ f|' \ S = \varrho' \ f|' \ S$

shows $\llbracket e \rrbracket_{\varrho} = \llbracket e \rrbracket_{\varrho'}$

by (rule ESem-fresh-cong[OF env-restr-eq-subset[OF $assms$]])

lemma ESem-fresh-cong-below:

assumes $\varrho \ f|' \ (fv \ e) \sqsubseteq \varrho' \ f|' \ (fv \ e)$

shows $\llbracket e \rrbracket_{\varrho} \sqsubseteq \llbracket e \rrbracket_{\varrho'}$

by (metis $assms$ ESem-considers-fv monofun-cfun-arg)

lemma ESem-fresh-cong-below-subset:

assumes $\text{fv } e \subseteq S$

assumes $\varrho \ f|' \ S \sqsubseteq \varrho' \ f|' \ S$

shows $\llbracket e \rrbracket_{\varrho} \sqsubseteq \llbracket e \rrbracket_{\varrho'}$

by (rule ESem-fresh-cong-below[OF env-restr-below-subset[OF $assms$]])

lemma ESem-ignores-fresh-restr:

assumes $\text{atom } 'S \ \sharp^* \ e$

shows $\llbracket e \rrbracket_{\varrho} = \llbracket e \rrbracket_{\varrho \ f|' \ (- \ S)}$

proof–

have $fv\ e \cap -\ S = fv\ e$ **using** *assms* **by** (*auto simp add: fresh-def fresh-star-def fv-supp*)
thus *?thesis* **by** (*subst (1 2) ESem-considers-fv*) *simp*
qed

lemma *ESem-ignores-fresh-restr'*:
assumes $atom\ 'e\ (edom\ \varrho - S) \#* e$
shows $\llbracket e \rrbracket_{\varrho} = \llbracket e \rrbracket_{\varrho} f|' S$

proof –

have $\llbracket e \rrbracket_{\varrho} = \llbracket e \rrbracket_{\varrho} f|' (- (edom\ \varrho - S))$
by (*rule ESem-ignores-fresh-restr[OF assms]*)
also have $\varrho f|' (- (edom\ \varrho - S)) = \varrho f|' S$
by (*rule ext*) (*auto simp add: lookup-env-restr-eq dest: lookup-not-edom*)
finally show *?thesis*.

qed

lemma *HSem-ignores-fresh-restr''*:

assumes $fv\ \Gamma \subseteq S$
shows $(\{\Gamma\}_{\varrho}) f|' S = \{\Gamma\}_{\varrho} f|' S$

by (*rule HSem-ignores-fresh-restr''[OF assms(1) ESem-considers-fv]*)

lemma *HSem-ignores-fresh-restr*:

assumes $atom\ 'S\ \#* \Gamma$
shows $(\{\Gamma\}_{\varrho}) f|' (- S) = \{\Gamma\}_{\varrho} f|' (- S)$

proof –

from *assms* **have** $fv\ \Gamma \subseteq - S$ **by** (*auto simp add: fv-def fresh-star-def fresh-def*)
thus *?thesis* **by** (*rule HSem-ignores-fresh-restr''*)

qed

lemma *HSem-fresh-cong-below*:

assumes $\varrho f|' ((S \cup fv\ \Gamma) - domA\ \Gamma) \sqsubseteq \varrho' f|' ((S \cup fv\ \Gamma) - domA\ \Gamma)$
shows $(\{\Gamma\}_{\varrho}) f|' S \sqsubseteq (\{\Gamma\}_{\varrho'}) f|' S$

proof –

from *assms*
have $\{\Gamma\}_{\varrho} f|' (S \cup fv\ \Gamma) \sqsubseteq \{\Gamma\}_{\varrho'} f|' (S \cup fv\ \Gamma)$
by (*auto intro: HSem-restr-cong-below simp add: Diff-eq inf-commute*)
hence $(\{\Gamma\}_{\varrho}) f|' (S \cup fv\ \Gamma) \sqsubseteq (\{\Gamma\}_{\varrho'}) f|' (S \cup fv\ \Gamma)$
by (*subst (1 2) HSem-ignores-fresh-restr''*) *simp-all*
thus *?thesis*
by (*rule env-restr-below-subset[OF Un-upper1]*)

qed

lemma *HSem-fresh-cong*:

assumes $\varrho f|' ((S \cup fv\ \Gamma) - domA\ \Gamma) = \varrho' f|' ((S \cup fv\ \Gamma) - domA\ \Gamma)$
shows $(\{\Gamma\}_{\varrho}) f|' S = (\{\Gamma\}_{\varrho'}) f|' S$

apply (*rule below-antisym*)

apply (*rule HSem-fresh-cong-below[OF eq-imp-below[OF assms]]*)

apply (*rule HSem-fresh-cong-below[OF eq-imp-below[OF assms[symmetric]]]*)

done

5.3.8 Adding a fresh variable to a heap does not affect its semantics

lemma *HSem-add-fresh'*:

assumes *fresh*: $atom\ x \# \Gamma$

assumes $x \notin edom\ \varrho$

assumes *step*: $\bigwedge e\ \varrho'.\ e \in snd\ \langle set\ \Gamma \rangle \implies \llbracket e \rrbracket_{\varrho'} = \llbracket e \rrbracket_{env\text{-}delete\ x\ \varrho'}$

shows $env\text{-}delete\ x\ (\{\{x, e\} \# \Gamma\} \varrho) = \{\Gamma\} \varrho$

proof (*rule parallel-HSem-ind, goal-cases*)

case 1 show ?case by simp

next

case 2 show ?case by auto

next

case prems: $(\exists y\ z)$

have $env\text{-}delete\ x\ \varrho = \varrho$ **using** $\langle x \notin edom\ \varrho \rangle$ **by** (*rule env-delete-noop*)

moreover

from *fresh* **have** $x \notin domA\ \Gamma$ **by** (*metis domA-not-fresh*)

hence $env\text{-}delete\ x\ (\llbracket (x, e) \# \Gamma \rrbracket_y) = \llbracket \Gamma \rrbracket_y$

by (*auto intro: env-delete-noop dest: subsetD[OF edom-evalHeap-subset]*)

moreover

have $\dots = \llbracket \Gamma \rrbracket_z$

apply (*rule evalHeap-cong[OF refl]*)

apply (*subst (1) step, assumption*)

using *prems(1)* **apply** *auto*

done

ultimately

show *?case* **using** $\langle x \notin domA\ \Gamma \rangle$

by (*simp add: env-delete-add*)

qed

lemma *HSem-add-fresh*:

assumes $atom\ x \# \Gamma$

assumes $x \notin edom\ \varrho$

shows $env\text{-}delete\ x\ (\{\{x, e\} \# \Gamma\} \varrho) = \{\Gamma\} \varrho$

proof(*rule HSem-add-fresh'[OF assms], goal-cases*)

case $(1\ e\ \varrho')$

assume $e \in snd\ \langle set\ \Gamma \rangle$

hence $atom\ x \# e$ **by** (*metis assms(1) fresh-heap-expr'*)

hence $x \notin fv\ e$ **by** (*simp add: fv-def fresh-def*)

thus *?case*

by (*rule ESem-fresh-cong[OF env-restr-env-delete-other[symmetric]]*)

qed

5.3.9 Mutual recursion with fresh variables

lemma *HSem-subset-below*:

assumes *fresh*: $atom\ \langle domA\ \Gamma \# * \Delta \rangle$

shows $\{\Delta\}(\varrho\ f|\langle -\ domA\ \Gamma \rangle) \sqsubseteq (\{\Delta @ \Gamma\} \varrho)\ f|\langle -\ domA\ \Gamma \rangle$

proof(*rule HSem-below*)

fix x

assume [*simp*]: $x \in domA\ \Delta$

with *assms* **have** *: *atom* ‘ *domA* Γ \sharp * *the* (*map-of* Δ *x*) **by** (*metis* *fresh-star-map-of*)
hence [*simp*]: $x \notin \text{domA } \Gamma$ **using** *fresh* ‘ $x \in \text{domA } \Delta$ ’ **by** (*metis* *fresh-star-def* *domA-not-fresh* *image-eqI*)
show $\llbracket \text{the } (\text{map-of } \Delta \ x) \rrbracket (\{\Delta @ \Gamma\}_\rho) f | ‘ (- \text{domA } \Gamma) \sqsubseteq ((\{\Delta @ \Gamma\}_\rho) f | ‘ (- \text{domA } \Gamma)) \ x$
by (*simp* *add: lookup-HSem-heap ESem-ignores-fresh-restr*[*OF* *, *symmetric*])
qed (*simp* *add: lookup-HSem-other lookup-env-restr-eq*)

In the following lemma we show that the semantics of fresh variables can be calculated together with the presently bound variables, or separately.

lemma *HSem-merge*:

assumes *fresh*: *atom* ‘ *domA* Γ \sharp * Δ
shows $\{\Gamma\}\{\Delta\}_\rho = \{\Gamma @ \Delta\}_\rho$
proof(*rule* *below-antisym*)
have *map-of-eq*: *map-of* ($\Delta @ \Gamma$) = *map-of* ($\Gamma @ \Delta$)
proof
fix *x*
show *map-of* ($\Delta @ \Gamma$) *x* = *map-of* ($\Gamma @ \Delta$) *x*
proof (*cases* $x \in \text{domA } \Gamma$)
case *True*
hence $x \notin \text{domA } \Delta$ **by** (*metis* *fresh-distinct* *fresh IntI* *equals0D*)
thus *map-of* ($\Delta @ \Gamma$) *x* = *map-of* ($\Gamma @ \Delta$) *x*
by (*simp* *add: map-add-dom-app-simps dom-map-of-conv-domA*)
next
case *False*
thus *map-of* ($\Delta @ \Gamma$) *x* = *map-of* ($\Gamma @ \Delta$) *x*
by (*simp* *add: map-add-dom-app-simps dom-map-of-conv-domA*)
qed
qed

show $\{\Gamma\}\{\Delta\}_\rho \sqsubseteq \{\Gamma @ \Delta\}_\rho$

proof(*rule* *HSem-below*)

fix *x*
assume [*simp*]: $x \notin \text{domA } \Gamma$

have $(\{\Delta\}_\rho) \ x = ((\{\Delta\}_\rho) f | ‘ (- \text{domA } \Gamma)) \ x$ **by** *simp*
also **have** $\dots = (\{\Delta\}_\rho) f | ‘ (- \text{domA } \Gamma)) \ x$
by (*rule* *arg-cong*[*OF* *HSem-ignores-fresh-restr*[*OF* *fresh*]])
also **have** $\dots \sqsubseteq ((\{\Delta @ \Gamma\}_\rho) f | ‘ (- \text{domA } \Gamma)) \ x$
by (*rule* *fun-belowD*[*OF* *HSem-subset-below*[*OF* *fresh*]])
also **have** $\dots = (\{\Delta @ \Gamma\}_\rho) \ x$ **by** *simp*
also **have** $\dots = (\{\Gamma @ \Delta\}_\rho) \ x$ **by** (*rule* *arg-cong*[*OF* *HSem-reorder*[*OF* *map-of-eq*]])
finally
show $(\{\Delta\}_\rho) \ x \sqsubseteq (\{\Gamma @ \Delta\}_\rho) \ x$.
qed (*auto* *simp* *add: lookup-HSem-heap lookup-env-restr-eq*)

have *: $\bigwedge x. x \in \text{domA } \Delta \implies x \notin \text{domA } \Gamma$
using *fresh* **by** (*auto* *simp* *add: fresh-Pair* *fresh-star-def* *domA-not-fresh*)

have *foo*: $\text{edom } \rho \cup \text{domA } \Delta \cup \text{domA } \Gamma - (\text{edom } \rho \cup \text{domA } \Delta \cup \text{domA } \Gamma) \cap - \text{domA } \Gamma =$

```

domA  $\Gamma$  by auto
have foo2:(edom  $\varrho \cup \text{domA } \Delta - (\text{edom } \varrho \cup \text{domA } \Delta) \cap - \text{domA } \Gamma) \subseteq \text{domA } \Gamma$  by auto

{ fix x
  assume  $x \in \text{domA } \Delta$ 
  hence *: atom ' domA  $\Gamma$  #* the (map-of  $\Delta$  x)
    by (rule fresh-star-map-of[OF - fresh])

  have [ the (map-of  $\Delta$  x) ]_{\{\Gamma\}\{\Delta\}\varrho} = [ the (map-of  $\Delta$  x) ]_{(\{\Gamma\}\{\Delta\}\varrho) f|' (- \text{domA } \Gamma)}
    by (rule ESem-ignores-fresh-restr[OF *])
  also have (\{\Gamma\}\{\Delta\}\varrho) f|' (- \text{domA } \Gamma) = (\{\Delta\}\varrho) f|' (- \text{domA } \Gamma)
    by (simp add: env-restr-HSem)
  also have [ the (map-of  $\Delta$  x) ]_{\dots} = [ the (map-of  $\Delta$  x) ]_{\{\Delta\}\varrho}
    by (rule ESem-ignores-fresh-restr[symmetric, OF *])
  finally
  have [ the (map-of  $\Delta$  x) ]_{\{\Gamma\}\{\Delta\}\varrho} = [ the (map-of  $\Delta$  x) ]_{\{\Delta\}\varrho}.
}
thus  $\{\Gamma @ \Delta\}\varrho \sqsubseteq \{\Gamma\}\{\Delta\}\varrho$ 
  by -(rule HSem-below, auto simp add: lookup-HSem-other lookup-HSem-heap *)
qed
end

```

5.3.10 Parallel induction

```

lemma parallel-HSem-ind-different-ESem:
  assumes adm ( $\lambda \varrho'. P$  (fst  $\varrho'$ ) (snd  $\varrho'$ ))
  assumes  $P \perp \perp$ 
  assumes  $\bigwedge y z. P$   $y$   $z \implies P$  ( $\varrho ++_{\text{domA } h}$  evalHeap  $h$  ( $\lambda e. \text{ESem1 } e \cdot y$ )) ( $\varrho' ++_{\text{domA } h}$  evalHeap  $h$  ( $\lambda e. \text{ESem2 } e \cdot z$ ))
  shows  $P$  (has-ESem.HSem ESem1  $h \cdot \varrho$ ) (has-ESem.HSem ESem2  $h$   $\cdot \varrho'$ )
proof -
  interpret HSem1: has-ESem ESem1.
  interpret HSem2: has-ESem ESem2.

  show ?thesis
    unfolding HSem1.HSem-def' HSem2.HSem-def'
    apply (rule parallel-fix-ind[OF assms(1)])
    apply (rule assms(2))
    apply simp
    apply (erule assms(3))
    done
qed

```

5.3.11 Congruence rule

```

lemma HSem-cong[fundef-cong]:
  [ ( $\bigwedge e. e \in \text{snd}$  ' set heap2  $\implies \text{ESem1 } e = \text{ESem2 } e$ ); heap1 = heap2 ]
   $\implies \text{has-ESem.HSem ESem1 heap1} = \text{has-ESem.HSem ESem2 heap2}$ 
  unfolding has-ESem.HSem-def

```

by (auto cong:evalHeap-cong)

5.3.12 Equivariance of the heap semantics

lemma *HSem-eqvt[eqvt]*:

$\pi \cdot \text{has-ESem.HSem ESem } \Gamma = \text{has-ESem.HSem } (\pi \cdot \text{ESem}) (\pi \cdot \Gamma)$

proof –

show *?thesis*

unfolding *has-ESem.HSem-def*

apply (*subst permute-Lam, simp*)

apply (*subst eqvt-lambda*)

apply (*simp add: Cfun-app-eqvt permute-Lam*)

done

qed

end

5.4 AbstractDenotational

theory *AbstractDenotational*

imports *HeapSemantics Terms*

begin

5.4.1 The denotational semantics for expressions

Because we need to define two semantics later on, we are abstract in the actual domain.

locale *semantic-domain* =

fixes *Fn* :: ('Value → 'Value) → ('Value::{pcpo-pt,pure})

fixes *Fn-project* :: 'Value → ('Value → 'Value)

fixes *B* :: bool *discr* → 'Value

fixes *B-project* :: 'Value → 'Value → 'Value → 'Value

fixes *tick* :: 'Value → 'Value

begin

nominal-function

ESem :: *exp* ⇒ (*var* ⇒ 'Value) → 'Value

where

ESem (*Lam* [*x*]. *e*) = (Λ *ρ*. *tick*·(*Fn*·(Λ *v*. *ESem* *e*·((*ρ* *f*|[′] *fv* (*Lam* [*x*]. *e*))(*x* := *v*))))

| *ESem* (*App* *e* *x*) = (Λ *ρ*. *tick*·(*Fn-project*·(*ESem* *e*·*ρ*)·(*ρ* *x*)))

| *ESem* (*Var* *x*) = (Λ *ρ*. *tick*·(*ρ* *x*))

| *ESem* (*Let* *as* *body*) = (Λ *ρ*. *tick*·(*ESem* *body*·(*has-ESem.HSem* *ESem* *as*·(*ρ* *f*|[′] *fv* (*Let* *as* *body*))))))

| *ESem* (*Bool* *b*) = (Λ *ρ*. *tick*·(*B*·(*Discr* *b*)))

| *ESem* (*scrut* ? *e1* : *e2*) = (Λ *ρ*. *tick*·((*B-project*·(*ESem* *scrut*·*ρ*))·(*ESem* *e1*·*ρ*)·(*ESem* *e2*·*ρ*)))

proof *goal-cases*

The following proofs discharge technical obligations generated by the Nominal package.

case 1 thus *?case*

```

unfolding eqvt-def ESem-graph-aux-def
apply rule
apply (perm-simp)
apply (simp add: Abs-cfun-eqvt)
apply (simp add: unpermute-def permute-pure)
done
next
case ( $\beta P x$ )
  thus ?case by (metis (poly-guards-query) exp-strong-exhaust)
next

case prems: ( $\lambda x e x' e'$ )
  from prems(5)
  show ?case
  proof (rule eqvt-lam-case)
    fix  $\pi :: \text{perm}$ 
    assume *:  $\text{supp } (-\pi) \#* (\text{fv } (\text{Lam } [x]. e) :: \text{var set})$ 
    { fix  $\varrho v$ 
      have  $\text{ESem-sumC } (\pi \cdot e) \cdot ((\varrho f |' \text{fv } (\text{Lam } [x]. e)) (\pi \cdot x := v)) = - \pi \cdot \text{ESem-sumC } (\pi \cdot e) \cdot ((\varrho f |' \text{fv } (\text{Lam } [x]. e)) (\pi \cdot x := v))$ 
      by (simp add: permute-pure)
      also have  $\dots = \text{ESem-sumC } e \cdot ((-\pi \cdot (\varrho f |' \text{fv } (\text{Lam } [x]. e))) (x := v))$  by (simp add: pemute-minus-self eqvt-at-apply[OF prems(1)])
      also have  $-\pi \cdot (\varrho f |' \text{fv } (\text{Lam } [x]. e)) = (\varrho f |' \text{fv } (\text{Lam } [x]. e))$  by (rule env-restr-perm'[OF *]) auto
      finally have  $\text{ESem-sumC } (\pi \cdot e) \cdot ((\varrho f |' \text{fv } (\text{Lam } [x]. e)) (\pi \cdot x := v)) = \text{ESem-sumC } e \cdot ((\varrho f |' \text{fv } (\text{Lam } [x]. e)) (x := v))$ 
    }
    thus  $(\Lambda \varrho. \text{tick} \cdot (\text{Fn} \cdot (\Lambda v. \text{ESem-sumC } (\pi \cdot e) \cdot ((\varrho f |' \text{fv } (\text{Lam } [x]. e)) (\pi \cdot x := v)))) = (\Lambda \varrho. \text{tick} \cdot (\text{Fn} \cdot (\Lambda v. \text{ESem-sumC } e \cdot ((\varrho f |' \text{fv } (\text{Lam } [x]. e)) (x := v))))$  by simp
  qed
next

case prems: ( $\lambda as \text{body } as' \text{body}'$ )
  from prems(9)
  show ?case
  proof (rule eqvt-let-case)
    fix  $\pi :: \text{perm}$ 
    assume *:  $\text{supp } (-\pi) \#* (\text{fv } (\text{Terms.Let } as \text{body}) :: \text{var set})$ 

    { fix  $\varrho$ 
      have  $\text{ESem-sumC } (\pi \cdot \text{body}) \cdot (\text{has-ESem.HSem } \text{ESem-sumC } (\pi \cdot as) \cdot (\varrho f |' \text{fv } (\text{Terms.Let } as \text{body})))$ 
       $= - \pi \cdot \text{ESem-sumC } (\pi \cdot \text{body}) \cdot (\text{has-ESem.HSem } \text{ESem-sumC } (\pi \cdot as) \cdot (\varrho f |' \text{fv } (\text{Terms.Let } as \text{body})))$ 
      by (rule permute-pure[symmetric])
      also have  $\dots = (- \pi \cdot \text{ESem-sumC } \text{body}) \cdot (\text{has-ESem.HSem } (- \pi \cdot \text{ESem-sumC } as) \cdot (- \pi \cdot \varrho f |' \text{fv } (\text{Terms.Let } as \text{body})))$ 
      by (simp add: pemute-minus-self)
    }
  
```

```

also have  $(- \pi \cdot ESem\text{-}sumC) \text{ body} = ESem\text{-}sumC \text{ body}$ 
  by (rule eqvt-at-apply[OF eqvt-at ESem-sumC body])
also have  $has\text{-}ESem.HSem (- \pi \cdot ESem\text{-}sumC) \text{ as} = has\text{-}ESem.HSem ESem\text{-}sumC \text{ as}$ 
  by (rule HSem-cong[OF eqvt-at-apply[OF prems(2)] refl])
also have  $- \pi \cdot \varrho f \mid' \text{fv} (Let \text{ as } \text{body}) = \varrho f \mid' \text{fv} (Let \text{ as } \text{body})$ 
  by (rule env-restr-perm'[OF *], simp)
finally have  $ESem\text{-}sumC (\pi \cdot \text{body}) \cdot (has\text{-}ESem.HSem ESem\text{-}sumC (\pi \cdot \text{as}) \cdot (\varrho f \mid' \text{fv} (Let \text{ as } \text{body}))) = ESem\text{-}sumC \text{ body} \cdot (has\text{-}ESem.HSem ESem\text{-}sumC \text{ as} \cdot (\varrho f \mid' \text{fv} (Let \text{ as } \text{body})))$ .
}
thus  $(\Lambda \varrho. tick \cdot (ESem\text{-}sumC (\pi \cdot \text{body}) \cdot (has\text{-}ESem.HSem ESem\text{-}sumC (\pi \cdot \text{as}) \cdot (\varrho f \mid' \text{fv} (Let \text{ as } \text{body})))))) =$ 
   $(\Lambda \varrho. tick \cdot (ESem\text{-}sumC \text{ body} \cdot (has\text{-}ESem.HSem ESem\text{-}sumC \text{ as} \cdot (\varrho f \mid' \text{fv} (Let \text{ as } \text{body}))))))$ 
by (simp only:)
  qed
qed auto

```

nominal-termination (in *semantic-domain*) (*no-eqvt*) **by** *lexicographic-order*

sublocale *has-ESem ESem*.

notation *ESem-syn* ($\llbracket - \rrbracket_- [60,60] 60$)

notation *EvalHeapSem-syn* ($\llbracket - \rrbracket_- [0,0] 110$)

notation *HSem-syn* ($\{\!\!-\!\!\} [60,60] 60$)

abbreviation *AHSem-bot* ($\{\!\!-\!\!\} [60] 60$) **where** $\{\!\!-\!\!\} \equiv \{\!\!-\!\!\} \perp$

end

end

5.5 Abstract-Denotational-Props

theory *Abstract-Denotational-Props*

imports *AbstractDenotational Substitution*

begin

context *semantic-domain*

begin

5.5.1 The semantics ignores fresh variables

lemma *ESem-considers-fv'*: $\llbracket e \rrbracket_{\varrho} = \llbracket e \rrbracket_{\varrho f \mid' (fv e)}$

proof (*induct e arbitrary: ϱ rule:exp-induct*)

case *Var*

show *?case by simp*

next

have [*simp*]: $\bigwedge S x. S \cap insert\ x\ S = S$ **by** *auto*

case *App*

show *?case*

by (*simp, subst (1 2) App, simp*)

next

```

  case (Lam x e)
  show ?case by simp
next
  case (IfThenElse scrut e1 e2)
  have [simp]: (fv scrut  $\cap$  (fv scrut  $\cup$  fv e1  $\cup$  fv e2)) = fv scrut by auto
  have [simp]: (fv e1  $\cap$  (fv scrut  $\cup$  fv e1  $\cup$  fv e2)) = fv e1 by auto
  have [simp]: (fv e2  $\cap$  (fv scrut  $\cup$  fv e1  $\cup$  fv e2)) = fv e2 by auto
  show ?case
    apply simp
    apply (subst (1 2) IfThenElse(1))
    apply (subst (1 2) IfThenElse(2))
    apply (subst (1 2) IfThenElse(3))
    apply (simp)
    done
next
  case (Let as e)

  have  $\llbracket e \rrbracket_{\{as\}\varrho} = \llbracket e \rrbracket_{(\{as\}\varrho) f|' (fv as \cup fv e)}$ 
    apply (subst (1 2) Let(2))
    apply simp
    done
  also
  have  $fv as \subseteq fv as \cup fv e$  by (rule inf-sup-ord(3))
  hence  $(\{as\}\varrho) f|' (fv as \cup fv e) = \{as\}\varrho f|' (fv as \cup fv e)$ 
    by (rule HSem-ignores-fresh-restr'[OF - Let(1)])
  also
  have  $\{as\}\varrho f|' (fv as \cup fv e) = \{as\}\varrho f|' (fv as \cup fv e - dom A as)$ 
    by (rule HSem-restr-cong) (auto simp add: lookup-env-restr-eq)
  finally
  show ?case by simp
qed auto

sublocale has-ignore-fresh-ESem ESem
  by standard (rule fv-supp-exp, rule ESem-considers-fv')

```

5.5.2 Nicer equations for ESem, without freshness requirements

```

lemma ESem-Lam[simp]:  $\llbracket Lam [x]. e \rrbracket_{\varrho} = tick \cdot (Fn \cdot (\Lambda v. \llbracket e \rrbracket_{\varrho(x := v)}))$ 
proof-
  have  $*$ :  $\bigwedge v. ((\varrho f|' (fv e - \{x\}))(x := v)) f|' fv e = (\varrho(x := v)) f|' fv e$ 
    by (rule ext) (auto simp add: lookup-env-restr-eq)

  have  $\llbracket Lam [x]. e \rrbracket_{\varrho} = \llbracket Lam [x]. e \rrbracket_{env-delete x \varrho}$ 
    by (rule ESem-fresh-cong) simp
  also have  $\dots = tick \cdot (Fn \cdot (\Lambda v. \llbracket e \rrbracket_{(\varrho f|' (fv e - \{x\}))(x := v)}))$ 
    by simp
  also have  $\dots = tick \cdot (Fn \cdot (\Lambda v. \llbracket e \rrbracket_{((\varrho f|' (fv e - \{x\}))(x := v)) f|' fv e}))$ 
    by (subst ESem-considers-fv, rule)
  also have  $\dots = tick \cdot (Fn \cdot (\Lambda v. \llbracket e \rrbracket_{\varrho(x := v) f|' fv e}))$ 

```

unfolding *..
also have $\dots = tick \cdot (Fn \cdot (\Lambda v. \llbracket e \rrbracket_{\varrho}(x := v)))$
unfolding *ESem-considers-fv[symmetric]*..
finally show *?thesis*.
qed
declare *ESem.simps(1)[simp del]*

lemma *ESem-Let[simp]*: $\llbracket Let\ as\ body \rrbracket_{\varrho} = tick \cdot (\llbracket body \rrbracket_{\{as\}_{\varrho}})$
proof–
have $\llbracket Let\ as\ body \rrbracket_{\varrho} = tick \cdot (\llbracket body \rrbracket_{\{as\}_{\varrho}}(\varrho\ f|\ 'fv\ (Let\ as\ body)))$
by *simp*
also have $\{as\}_{\varrho}\ f|\ 'fv(Let\ as\ body) = \{as\}_{\varrho}\ f|\ '(fv\ as \cup fv\ body)$
by (*rule HSem-restr-cong*) (*auto simp add: lookup-env-restr-eq*)
also have $\dots = (\{as\}_{\varrho})\ f|\ '(fv\ as \cup fv\ body)$
by (*rule HSem-ignores-fresh-restr'[symmetric, OF - ESem-considers-fv]*) *simp*
also have $\llbracket body \rrbracket_{\dots} = \llbracket body \rrbracket_{\{as\}_{\varrho}}$
by (*rule ESem-fresh-cong*) (*auto simp add: lookup-env-restr-eq*)
finally show *?thesis*.
qed
declare *ESem.simps(4)[simp del]*

5.5.3 Denotation of Substitution

lemma *ESem-subst-same*: $\varrho\ x = \varrho\ y \implies \llbracket e \rrbracket_{\varrho} = \llbracket e[x ::= y] \rrbracket_{\varrho}$
and
 $\varrho\ x = \varrho\ y \implies (\llbracket as \rrbracket_{\varrho}) = \llbracket as[x ::= h=y] \rrbracket_{\varrho}$
proof (*nominal-induct e and as avoiding: x y arbitrary: ρ and ρ rule:exp-heap-strong-induct*)
case *Var* **thus** *?case* **by** *auto*
next
case *App*
from *App(1)[OF App(2)] App(2)*
show *?case* **by** *auto*
next
case (*Let as exp x y ρ*)
from $\langle atom\ 'domA\ as\ \#* x \rangle \langle atom\ 'domA\ as\ \#* y \rangle$
have $x \notin domA\ as\ y \notin domA\ as$
by (*metis fresh-star-at-base imageI*)
hence $[simp]: domA\ (as[x ::= h=y]) = domA\ as$
by (*metis bn-subst*)

from $\langle \varrho\ x = \varrho\ y \rangle$
have $(\{as\}_{\varrho})\ x = (\{as\}_{\varrho})\ y$
using $\langle x \notin domA\ as \rangle \langle y \notin domA\ as \rangle$
by (*simp add: lookup-HSem-other*)
hence $\llbracket exp \rrbracket_{\{as\}_{\varrho}} = \llbracket exp[x ::= y] \rrbracket_{\{as\}_{\varrho}}$
by (*rule Let*)
moreover
from $\langle \varrho\ x = \varrho\ y \rangle$
have $\{as\}_{\varrho} = \{as[x ::= h=y]\}_{\varrho}$ **and** $(\{as\}_{\varrho})\ x = (\{as[x ::= h=y]\}_{\varrho})\ y$


```

apply (induction rule: parallel-HSem-ind)
apply (intro adm-lemmas cont2cont cont2cont-fun)
apply simp
apply simp
apply simp
apply (erule arg-cong[OF Let(3)])
using  $\langle x \notin \text{dom}A \text{ as} \rangle \langle y \notin \text{dom}A \text{ as} \rangle$ 
apply simp
done
ultimately
show ?case using Let(1,2,3) by (simp add: fresh-star-Pair)
next
case (Lam var exp x y ρ)
  from  $\langle \rho x = \rho y \rangle$ 
  have  $\bigwedge v. (\rho(\text{var} := v)) x = (\rho(\text{var} := v)) y$ 
    using Lam(1,2) by (simp add: fresh-at-base)
  hence  $\bigwedge v. \llbracket \text{exp} \rrbracket_{\rho(\text{var} := v)} = \llbracket \text{exp}[x::=y] \rrbracket_{\rho(\text{var} := v)}$ 
    by (rule Lam)
  thus ?case using Lam(1,2) by simp
next
case IfThenElse
  from IfThenElse(1)[OF IfThenElse(4)] IfThenElse(2)[OF IfThenElse(4)] IfThenElse(3)[OF IfThenElse(4)]
  show ?case
    by simp
next
case Nil thus ?case by auto
next
case Cons
  from Cons(1,2)[OF Cons(3)] Cons(3)
  show ?case by auto
qed auto

lemma ESem-subst:
  shows  $\llbracket e \rrbracket_{\sigma(x := \sigma y)} = \llbracket e[x::=y] \rrbracket_{\sigma}$ 
proof (cases x = y)
  case False
    hence  $[simp]: x \notin \text{fv } e[x::=y]$  by (auto simp add: fv-def supp-subst supp-at-base dest: subsetD[OF supp-subst])

    have  $\llbracket e \rrbracket_{\sigma(x := \sigma y)} = \llbracket e[x::=y] \rrbracket_{\sigma(x := \sigma y)}$ 
      by (rule ESem-subst-same) simp
    also have  $\dots = \llbracket e[x::=y] \rrbracket_{\sigma}$ 
      by (rule ESem-fresh-cong) simp
    finally
      show ?thesis.
  next
  case True
    thus ?thesis by simp

```

qed

end

end

5.6 Denotational

theory *Denotational*

imports *Abstract-Denotational-Props Value-Nominal*

begin

This is the actual denotational semantics as found in [Lau93].

interpretation *semantic-domain Fn Fn-project B B-project* ($\Lambda x. x$).

notation *ESem-syn* ($\llbracket - \rrbracket_{[60,60] 60}$)

notation *EvalHeapSem-syn* ($\llbracket - \rrbracket_{[0,0] 110}$)

notation *HSem-syn* ($\{\!\!-\!\!\}_{[60,60] 60}$)

notation *AHSem-bot* ($\{\!\!-\!\!\}_{[60] 60}$)

lemma *ESem-simps-as-defined:*

$\llbracket \text{Lam } [x]. e \rrbracket_{\varrho} = \text{Fn} \cdot (\Lambda v. \llbracket e \rrbracket_{(\varrho f | \cdot (fv (\text{Lam } [x]. e)))} (x := v))$

$\llbracket \text{App } e x \rrbracket_{\varrho} = \llbracket e \rrbracket_{\varrho} \downarrow \text{Fn } \varrho x$

$\llbracket \text{Var } x \rrbracket_{\varrho} = \varrho x$

$\llbracket \text{Bool } b \rrbracket_{\varrho} = B \cdot (\text{Discr } b)$

$\llbracket (\text{scrut } ? e_1 : e_2) \rrbracket_{\varrho} = B\text{-project} \cdot (\llbracket \text{scrut} \rrbracket_{\varrho}) \cdot (\llbracket e_1 \rrbracket_{\varrho}) \cdot (\llbracket e_2 \rrbracket_{\varrho})$

$\llbracket \text{Let } \Gamma \text{ body} \rrbracket_{\varrho} = \llbracket \text{body} \rrbracket_{\{\!\!-\!\!\}_{\Gamma}(\varrho f | \cdot fv (\text{Let } \Gamma \text{ body}))}$

by (*simp-all del: ESem-Lam ESem-Let add: ESem.simps(1,4)*)

lemma *ESem-simps:*

$\llbracket \text{Lam } [x]. e \rrbracket_{\varrho} = \text{Fn} \cdot (\Lambda v. \llbracket e \rrbracket_{\varrho(x := v)})$

$\llbracket \text{App } e x \rrbracket_{\varrho} = \llbracket e \rrbracket_{\varrho} \downarrow \text{Fn } \varrho x$

$\llbracket \text{Var } x \rrbracket_{\varrho} = \varrho x$

$\llbracket \text{Bool } b \rrbracket_{\varrho} = B \cdot (\text{Discr } b)$

$\llbracket (\text{scrut } ? e_1 : e_2) \rrbracket_{\varrho} = B\text{-project} \cdot (\llbracket \text{scrut} \rrbracket_{\varrho}) \cdot (\llbracket e_1 \rrbracket_{\varrho}) \cdot (\llbracket e_2 \rrbracket_{\varrho})$

$\llbracket \text{Let } \Gamma \text{ body} \rrbracket_{\varrho} = \llbracket \text{body} \rrbracket_{\{\!\!-\!\!\}_{\Gamma} \varrho}$

by *simp-all*

end

6 Resourced denotational domain

6.1 C

```
theory C
imports HOLCF Mono-Nat-Fun
begin
```

```
default-sort cpo
```

The initial solution to the domain equation $C = C_{\perp}$, i.e. the completion of the natural numbers.

```
domain C = C (lazy C)
```

```
lemma below-C:  $x \sqsubseteq C \cdot x$ 
  by (induct x) auto
```

```
definition Cinf (C∞) where C∞ = fix C
```

```
lemma C-Cinf[simp]:  $C \cdot C^{\infty} = C^{\infty}$  unfolding Cinf-def by (rule fix-eq[symmetric])
```

```
abbreviation Cpow (Cn) where Cn ≡ iterate n C · ⊥
```

```
lemma C-below-C[simp]:  $(C^i \sqsubseteq C^j) \longleftrightarrow i \leq j$ 
  apply (induction i arbitrary: j)
  apply simp
  apply (case-tac j, auto)
  done
```

```
lemma below-Cinf[simp]:  $r \sqsubseteq C^{\infty}$ 
  apply (induct r)
  apply simp-all[2]
  apply (metis (full-types) C-Cinf monofun-cfun-arg)
  done
```

```
lemma C-eq-Cinf[simp]:  $C^i \neq C^{\infty}$ 
  by (metis C-below-C Suc-n-not-le-n below-Cinf)
```

```
lemma Cinf-eq-C[simp]:  $C^{\infty} = C \cdot r \longleftrightarrow C^{\infty} = r$ 
  by (metis C.injects C-Cinf)
```

```
lemma C-eq-C[simp]:  $(C^i = C^j) \longleftrightarrow i = j$ 
  by (metis C-below-C le-antisym le-refl)
```

```
lemma case-of-C-below: (case r of C · y ⇒ x)  $\sqsubseteq x$ 
  by (cases r) auto
```

```
lemma C-case-below: C-case · f  $\sqsubseteq f$ 
```

by (metis cfun-belowI C.case-rews(2) below-C monofun-cfun-arg)

lemma *C-case-bot[simp]*: $C\text{-case} \cdot \perp = \perp$
apply (subst eq-bottom-iff)
apply (rule C-case-below)
done

lemma *C-case-cong*:
assumes $\bigwedge r'. r = C \cdot r' \implies f \cdot r' = g \cdot r'$
shows $C\text{-case} \cdot f \cdot r = C\text{-case} \cdot g \cdot r$
using *assms* **by** (cases r) auto

lemma *C-cases*:

obtains n **where** $r = C^n \mid r = C^\infty$

proof–

{ **fix** m
have $\exists n. C\text{-take } m \cdot r = C^n$
proof (rule C.finite-induct)
have $\perp = C^0$ **by** simp
thus $\exists n. \perp = C^n$..
next
fix r
show $\exists n. r = C^n \implies \exists n. C \cdot r = C^n$
by (auto simp del: iterate-Suc simp add: iterate-Suc[symmetric])
qed

}

then obtain f **where** $\text{take}: \bigwedge m. C\text{-take } m \cdot r = C^f m$ **by** metis

have $\text{chain } (\lambda m. C^f m)$ **using** ch2ch-Rep-cfunL[OF C.chain-take, **where** $x=r$, unfolded take].

hence $\text{mono } f$ **by** (auto simp add: mono-iff-le-Suc chain-def elim!:chainE)

have $r: r = (\bigsqcup m. C^f m)$ **by** (metis (lifting) take C.reach lub-eq)

from $\langle \text{mono } f \rangle$

show *thesis*

proof(rule nat-mono-characterization)

fix n
assume $n: \bigwedge m. n \leq m \implies f n = f m$
have $\text{max-in-chain } n (\lambda m. C^f m)$
apply (rule max-in-chainI)
apply simp
apply (erule n)
done

hence $(\bigsqcup m. C^f m) = C^f n$ **unfolding** *maxinch-is-thelub*[OF $\langle \text{chain } \rightarrow \rangle$].

thus *?thesis* **using** *that* **unfolding** r **by** blast

next

assume $\bigwedge m. \exists n. m \leq f n$

hence $\bigwedge n. C^n \sqsubseteq r$ **unfolding** r **by** (fastforce intro: below-lub[OF $\langle \text{chain } \rightarrow \rangle$])

hence $(\bigsqcup n. C^n) \sqsubseteq r$

by (rule lub-below[OF chain-iterate])

```

  hence  $C^\infty \sqsubseteq r$  unfolding Cinf-def fix-def2.
  hence  $C^\infty = r$  using below-Cinf by (metis below-antisym)
  thus thesis using that by blast
qed
qed

```

```

lemma C-case-Cinf[simp]:  $C\text{-case} \cdot f \cdot C^\infty = f \cdot C^\infty$ 
  unfolding Cinf-def
  by (subst fix-eq) simp

```

end

6.2 C-Meet

```

theory C-Meet
imports C HOLCF-Meet
begin

```

```

instantiation C :: Finite-Meet-cpo begin
  fixrec C-meet ::  $C \rightarrow C \rightarrow C$ 
    where  $C\text{-meet} \cdot (C \cdot a) \cdot (C \cdot b) = C \cdot (C\text{-meet} \cdot a \cdot b)$ 

```

```

lemma[simp]:  $C\text{-meet} \cdot \perp \cdot y = \perp$   $C\text{-meet} \cdot x \cdot \perp = \perp$  by (fixrec-simp, cases x, fixrec-simp+)

```

```

instance
  apply standard
  proof(intro exI conjI strip)
    fix  $x y$ 
    {
      fix  $t$ 
      have  $(t \sqsubseteq C\text{-meet} \cdot x \cdot y) = (t \sqsubseteq x \wedge t \sqsubseteq y)$ 
      proof (induct t rule:C.take-induct)
        fix  $n$ 
        show  $(C\text{-take } n \cdot t \sqsubseteq C\text{-meet} \cdot x \cdot y) = (C\text{-take } n \cdot t \sqsubseteq x \wedge C\text{-take } n \cdot t \sqsubseteq y)$ 
        proof (induct n arbitrary: t x y rule:nat-induct)
          case 0 thus ?case by auto
          next
          case (Suc n t x y)
            with  $C.nchotomy[of t]$   $C.nchotomy[of x]$   $C.nchotomy[of y]$ 
            show ?case by fastforce
        qed
      qed auto
    } note * = this
    show  $C\text{-meet} \cdot x \cdot y \sqsubseteq x$  using * by auto
    show  $C\text{-meet} \cdot x \cdot y \sqsubseteq y$  using * by auto
    fix  $z$ 
    assume  $z \sqsubseteq x$  and  $z \sqsubseteq y$ 
    thus  $z \sqsubseteq C\text{-meet} \cdot x \cdot y$  using * by auto

```

qed
end

lemma *C-meet-is-meet*: $(z \sqsubseteq C\text{-meet}\cdot x\cdot y) = (z \sqsubseteq x \wedge z \sqsubseteq y)$

proof (*induct z rule:C.take-induct*)

fix *n*

show $(C\text{-take } n\cdot z \sqsubseteq C\text{-meet}\cdot x\cdot y) = (C\text{-take } n\cdot z \sqsubseteq x \wedge C\text{-take } n\cdot z \sqsubseteq y)$

proof (*induct n arbitrary: z x y rule:nat-induct*)

case 0 **thus** ?*case* **by** *auto*

next

case (*Suc n z x y*) **thus** ?*case*

apply –

apply (*cases z, simp*)

apply (*cases x, simp*)

apply (*cases y, simp*)

apply (*fastforce simp add: cfun-below-iff*)

done

qed

qed *auto*

instance *C* :: *cont-binary-meet*

proof (*standard, goal-cases*)

have [*simp*]: $\bigwedge x y. x \sqcap y = C\text{-meet}\cdot x\cdot y$

using *C-meet-is-meet*

by (*blast intro: is-meetI*)

case 1 **thus** ?*case*

by (*simp add: ch2ch-Rep-cfunR contlub-cfun-arg contlub-cfun-fun*)

qed

lemma [*simp*]: $C\cdot r \sqcap r = r$

by (*auto intro: is-meetI simp add: below-C*)

lemma [*simp*]: $r \sqcap C\cdot r = r$

by (*auto intro: is-meetI simp add: below-C*)

lemma [*simp*]: $C\cdot r \sqcap C\cdot r' = C\cdot (r \sqcap r')$

apply (*rule is-meetI*)

apply (*metis below-refl meet-below1 monofun-cfun-arg*)

apply (*metis below-refl meet-below2 monofun-cfun-arg*)

apply (*case-tac a*)

apply *auto*

by (*metis meet-above-iff*)

end

6.3 C-restr

theory *C-restr*

imports *C C-Meet HOLCF-Utills*

begin

6.3.1 The demand of a C -function

The demand is the least amount of resources required to produce a non-bottom element, if at all.

definition $demand :: (C \rightarrow 'a::pcpo) \Rightarrow C$ **where**
 $demand\ f = (if\ f \cdot C^\infty \neq \perp\ then\ C^{(LEAST\ n.\ f \cdot C^n \neq \perp)}\ else\ C^\infty)$

Because of continuity, a non-bottom value can always be obtained with finite resources.

lemma *finite-resources-suffice*:

assumes $f \cdot C^\infty \neq \perp$

obtains n **where** $f \cdot C^n \neq \perp$

proof–

{

assume $\forall n.\ f \cdot (C^n) = \perp$

hence $f \cdot C^\infty \sqsubseteq \perp$

by (*auto intro: lub-below[OF ch2ch-Rep-cfunR[OF chain-iterate]]*
simp add: Cinf-def fix-def2 contlub-cfun-arg[OF chain-iterate])

with *assms* **have** *False* **by** *simp*

}

thus *?thesis* **using** *that* **by** *blast*

qed

Because of monotonicity, a non-bottom value can always be obtained with more resources.

lemma *more-resources-suffice*:

assumes $f \cdot r \neq \perp$ **and** $r \sqsubseteq r'$

shows $f \cdot r' \neq \perp$

using *assms(1) monofun-cfun-arg[OF assms(2)]*, **where** $f = f$

by *auto*

lemma *infinite-resources-suffice*:

shows $f \cdot r \neq \perp \implies f \cdot C^\infty \neq \perp$

by (*erule more-resources-suffice[OF - below-Cinf]*)

lemma *demand-suffices*:

assumes $f \cdot C^\infty \neq \perp$

shows $f \cdot (demand\ f) \neq \perp$

apply (*simp add: assms demand-def*)

apply (*rule finite-resources-suffice[OF assms]*)

apply (*rule LeastI*)

apply *assumption*

done

lemma *not-bot-demand*:

$f \cdot r \neq \perp \iff demand\ f \neq C^\infty \wedge demand\ f \sqsubseteq r$

proof(*intro iffI*)
assume $f \cdot r \neq \perp$
thus $\text{demand } f \neq C^\infty \wedge \text{demand } f \sqsubseteq r$
apply (*cases r rule:C-cases*)
apply (*auto intro: Least-le simp add: demand-def dest: infinite-resources-suffice*)
done
next
assume $*$: $\text{demand } f \neq C^\infty \wedge \text{demand } f \sqsubseteq r$
then have $f \cdot C^\infty \neq \perp$ **by** (*auto intro: Least-le simp add: demand-def dest: infinite-resources-suffice*)
hence $f \cdot (\text{demand } f) \neq \perp$ **by** (*rule demand-suffices*)
moreover from $*$ **have** $\text{demand } f \sqsubseteq r..$
ultimately
show $f \cdot r \neq \perp$ **by** (*rule more-resources-suffice*)
qed

lemma *infinity-bot-demand*:
 $f \cdot C^\infty = \perp \iff \text{demand } f = C^\infty$
by (*metis below-Cinf not-bot-demand*)

lemma *demand-suffices'*:
assumes $\text{demand } f = C^n$
shows $f \cdot (\text{demand } f) \neq \perp$
by (*metis C-eq-Cinf assms demand-suffices infinity-bot-demand*)

lemma *demand-Suc-Least*:
assumes [*simp*]: $f \cdot \perp = \perp$
assumes $\text{demand } f \neq C^\infty$
shows $\text{demand } f = C^{(\text{Suc } (\text{LEAST } n. f \cdot C^{\text{Suc } n} \neq \perp))}$

proof –
from *assms*
have $\text{demand } f = C^{(\text{LEAST } n. f \cdot C^n \neq \perp)}$ **by** (*auto simp add: demand-def*)
also
then obtain n **where** $f \cdot C^n \neq \perp$ **by** (*metis demand-suffices'*)
hence $(\text{LEAST } n. f \cdot C^n \neq \perp) = \text{Suc } (\text{LEAST } n. f \cdot C^{\text{Suc } n} \neq \perp)$
apply (*rule Least-Suc*) **by** *simp*
finally show *?thesis*.
qed

lemma *demand-C-case[simp]*: $\text{demand } (C\text{-case}\cdot f) = C \cdot (\text{demand } f)$
proof(*cases demand (C-case\cdot f) = C^\infty*)

case *True*
then have $C\text{-case}\cdot f \cdot C^\infty = \perp$
by (*metis infinity-bot-demand*)
with *True*
show *?thesis* **apply** *auto* **by** (*metis infinity-bot-demand*)

next
case *False*
hence $\text{demand } (C\text{-case}\cdot f) = C^{\text{Suc } (\text{LEAST } n. (C\text{-case}\cdot f) \cdot C^{\text{Suc } n} \neq \perp)}$
by (*rule demand-Suc-Least[OF C.case-rews(1)]*)

also have $\dots = C.C^{LEAST\ n.\ f.C^n} \neq \perp$ **by** *simp*
 also have $\dots = C.(demand\ f)$
 using *False unfolding demand-def* **by** *auto*
 finally show *?thesis*.
qed

lemma *demand-contravariant*:
 assumes $f \sqsubseteq g$
 shows $demand\ g \sqsubseteq demand\ f$
proof(*cases demand f rule:C-cases*)
 fix n
 assume $demand\ f = C^n$
 hence $f.(demand\ f) \neq \perp$ **by** (*metis demand-suffices'*)
 also note *monofun-cfun-fun[OF assms]*
 finally have $g.(demand\ f) \neq \perp$ **by** *this (intro cont2cont)*
 thus $demand\ g \sqsubseteq demand\ f$ **unfolding not-bot-demand** **by** *auto*
qed *auto*

6.3.2 Restricting functions with domain C

fixrec $C-restr :: C \rightarrow (C \rightarrow 'a::pcpo) \rightarrow (C \rightarrow 'a)$
 where $C-restr.r.f.r' = (f.(r \sqcap r'))$

abbreviation $C-restr-syn :: (C \rightarrow 'a::pcpo) \Rightarrow C \Rightarrow (C \rightarrow 'a)$ (*-|_ [111,110] 110*)
 where $f|_r \equiv C-restr.r.f$

lemma [*simp*]: $\perp|_r = \perp$ **by** *fixrec-simp*
lemma [*simp*]: $f.\perp = \perp \implies f|_{\perp} = \perp$ **by** *fixrec-simp*

lemma $C-restr-C-restr$ [*simp*]: $(v|_{r'})|_r = v|_{(r' \sqcap r)}$
by (*rule cfun-eqI*) *simp*

lemma $C-restr-eqD$:
 assumes $f|_r = g|_r$
 assumes $r' \sqsubseteq r$
 shows $f.r' = g.r'$
by (*metis C-restr.simps assms below-refl is-meetI*)

lemma $C-restr-eq-lower$:
 assumes $f|_r = g|_r$
 assumes $r' \sqsubseteq r$
 shows $f|_{r'} = g|_{r'}$
by (*metis C-restr-C-restr assms below-refl is-meetI*)

lemma $C-restr-below$ [*intro, simp*]:
 $x|_r \sqsubseteq x$
apply (*rule cfun-belowI*)
apply *simp*
by (*metis below-refl meet-below2 monofun-cfun-arg*)

lemma *C-restr-below-cong*:

$(\bigwedge r'. r' \sqsubseteq r \implies f \cdot r' \sqsubseteq g \cdot r') \implies f|_r \sqsubseteq g|_r$
apply (*rule cfun-belowI*)
apply *simp*
by (*metis below-refl meet-below1*)

lemma *C-restr-cong*:

$(\bigwedge r'. r' \sqsubseteq r \implies f \cdot r' = g \cdot r') \implies f|_r = g|_r$
apply (*intro below-antisym C-restr-below-cong*)
by (*metis below-refl*)⁺

lemma *C-restr-C-cong*:

$(\bigwedge r'. r' \sqsubseteq r \implies f \cdot (C \cdot r') = g \cdot (C \cdot r')) \implies f \cdot \perp = g \cdot \perp \implies f|_{C \cdot r} = g|_{C \cdot r}$
apply (*rule C-restr-cong*)
by (*case-tac r'*, *auto*)

lemma *C-restr-C-case[simp]*:

$(C \cdot \text{case} \cdot f)|_{C \cdot r} = C \cdot \text{case} \cdot (f|_r)$
apply (*rule cfun-eqI*)
apply *simp*
apply (*case-tac x*)
apply *simp*
apply *simp*
done

lemma *C-restr-bot-demand*:

assumes $C \cdot r \sqsubseteq \text{demand } f$
shows $f|_r = \perp$
proof (*rule cfun-eqI*)
fix r'
have $f \cdot (r \sqcap r') = \perp$
proof (*rule classical*)
have $r \sqsubseteq C \cdot r$ **by** (*rule below-C*)
also
note *assms*
also
assume $*$: $f \cdot (r \sqcap r') \neq \perp$
hence $\text{demand } f \sqsubseteq (r \sqcap r')$ **unfolding** *not-bot-demand* **by** *auto*
hence $\text{demand } f \sqsubseteq r$ **by** (*metis below-refl meet-below1 below-trans*)
finally (*below-antisym*) **have** $r = \text{demand } f$ **by** *this* (*intro cont2cont*)
with *assms*
have $\text{demand } f = C^\infty$ **by** (*cases demand f rule: C-cases*) (*auto simp add: iterate-Suc[symmetric]*)
simp del: iterate-Suc
thus $f \cdot (r \sqcap r') = \perp$ **by** (*metis not-bot-demand*)
qed
thus $(f|_r) \cdot r' = \perp \cdot r'$ **by** *simp*
qed

6.3.3 Restricting maps of C-ranged functions

definition $env\text{-}C\text{-restr} :: C \rightarrow ('var::type \Rightarrow (C \rightarrow 'a::pcpo)) \rightarrow ('var \Rightarrow (C \rightarrow 'a))$ **where**
 $env\text{-}C\text{-restr} = (\Lambda r f. cfun\text{-}comp.(C\text{-restr}.r).f)$

abbreviation $env\text{-}C\text{-restr}\text{-}syn :: ('var::type \Rightarrow (C \rightarrow 'a::pcpo)) \Rightarrow C \Rightarrow ('var \Rightarrow (C \rightarrow 'a))$ ($\cdot|^\circ$ - [111,110] 110)
where $f|^\circ_r \equiv env\text{-}C\text{-restr}.r.f$

lemma $env\text{-}C\text{-restr}\text{-}upd[simp]: (\varrho(x := v))|^\circ_r = (\varrho|^\circ_r)(x := v|_r)$
unfolding $env\text{-}C\text{-restr}\text{-}def$ **by** $simp$

lemma $env\text{-}C\text{-restr}\text{-}lookup[simp]: (\varrho|^\circ_r) v = \varrho v|_r$
unfolding $env\text{-}C\text{-restr}\text{-}def$ **by** $simp$

lemma $env\text{-}C\text{-restr}\text{-}bot[simp]: \perp|^\circ_r = \perp$
unfolding $env\text{-}C\text{-restr}\text{-}def$ **by** $auto$

lemma $env\text{-}C\text{-restr}\text{-}restr\text{-}below[intro]: \varrho|^\circ_r \sqsubseteq \varrho$
by ($auto$ $intro: fun\text{-}belowI$)

lemma $env\text{-}C\text{-restr}\text{-}env\text{-}C\text{-restr}[simp]: (v|^\circ_{r'})|^\circ_r = v|^\circ_{(r' \sqcap r)}$
unfolding $env\text{-}C\text{-restr}\text{-}def$ **by** $auto$

lemma $env\text{-}C\text{-restr}\text{-}cong:$
 $(\Lambda x r'. r' \sqsubseteq r \implies f x \cdot r' = g x \cdot r') \implies f|^\circ_r = g|^\circ_r$
unfolding $env\text{-}C\text{-restr}\text{-}def$
by ($rule ext$) ($auto$ $intro: C\text{-restr}\text{-}cong$)

end

6.4 CValue

theory $CValue$
imports C
begin

domain $CValue$
 $= CFn$ (**lazy** $(C \rightarrow CValue) \rightarrow (C \rightarrow CValue)$)
 $| CB$ (**lazy** $bool$ $discr$)

fixrec $CFn\text{-}project :: CValue \rightarrow (C \rightarrow CValue) \rightarrow (C \rightarrow CValue)$
where $CFn\text{-}project.(CFn.f).v = f \cdot v$

abbreviation $CFn\text{-}project\text{-}abbr$ (**infix** $\downarrow CFn$ 55)
where $f \downarrow CFn v \equiv CFn\text{-}project.f.v$

lemma $CFn\text{-}project\text{-}strict[simp]:$

$\perp \downarrow CFn\ v = \perp$
 $CB \cdot b \downarrow CFn\ v = \perp$
by (*fixrec-simp*)⁺

lemma *CB-below[simp]*: $CB \cdot b \sqsubseteq v \longleftrightarrow v = CB \cdot b$
by (*cases v*) *auto*

fixrec *CB-project* :: $CValue \rightarrow CValue \rightarrow CValue \rightarrow CValue$ **where**
 $CB\text{-project} \cdot (CB \cdot db) \cdot v_1 \cdot v_2 = (\text{if } \text{undiscr } db \text{ then } v_1 \text{ else } v_2)$

lemma [*simp*]:
 $CB\text{-project} \cdot (CB \cdot (Discr\ b)) \cdot v_1 \cdot v_2 = (\text{if } b \text{ then } v_1 \text{ else } v_2)$
 $CB\text{-project} \cdot \perp \cdot v_1 \cdot v_2 = \perp$
 $CB\text{-project} \cdot (CFn \cdot f) \cdot v_1 \cdot v_2 = \perp$
by *fixrec-simp*⁺

lemma *CB-project-not-bot*:
 $CB\text{-project} \cdot scrut \cdot v_1 \cdot v_2 \neq \perp \longleftrightarrow (\exists b. scrut = CB \cdot (Discr\ b) \wedge (\text{if } b \text{ then } v_1 \text{ else } v_2) \neq \perp)$
apply (*cases scrut*)
apply *simp*
apply *simp*
by (*metis (poly-guards-query) CB-project.simps CValue.injects(2) discr.exhaust undiscr-Discr*)

HOLCF provides us $CValue\text{-take}::nat \Rightarrow CValue \rightarrow CValue$; we want a similar function for $C \rightarrow CValue$.

abbreviation *C-to-CValue-take* :: $nat \Rightarrow (C \rightarrow CValue) \rightarrow (C \rightarrow CValue)$
where $C\text{-to-}CValue\text{-take } n \equiv cfun\text{-map}\cdot ID \cdot (CValue\text{-take } n)$

lemma *C-to-CValue-chain-take*: *chain C-to-CValue-take*
by (*auto intro: chainI cfun-belowI chainE[OF CValue.chain-take] monofun-cfun-fun*)

lemma *C-to-CValue-reach*: $(\bigsqcup n. C\text{-to-}CValue\text{-take } n \cdot x) = x$
by (*auto intro: cfun-eqI simp add: contlub-cfun-fun[OF ch2ch-Rep-cfunL[OF C-to-CValue-chain-take]] CValue.reach*)

end

6.5 CValue-Nominal

theory *CValue-Nominal*
imports *CValue Nominal-Utills Nominal-HOLCF*
begin

instantiation *C* :: *pure*

begin

definition $p \cdot (c::C) = c$

instance **by** *standard (auto simp add: permute-C-def)*

end
instance $C :: \text{pcpo-pt}$
 by *standard* (*simp add: pure-permute-id*)

instantiation $CValue :: \text{pure}$
begin
definition $p \cdot (v :: CValue) = v$
instance
apply *standard*
apply (*auto simp add: permute-CValue-def*)
done
end

instance $CValue :: \text{pcpo-pt}$
 by *standard* (*simp add: pure-permute-id*)

end

6.6 ResourcedDenotational

theory *ResourcedDenotational*
imports *Abstract-Denotational-Props CValue-Nominal C-restr*
begin

type-synonym $CEnv = \text{var} \Rightarrow (C \rightarrow CValue)$

interpretation *semantic-domain*
 $\Lambda f . \Lambda r . CFn \cdot (\Lambda v . (f \cdot (v)) | r)$
 $\Lambda x y . (\Lambda r . (x \cdot r \downarrow CFn y | r) \cdot r)$
 $\Lambda b r . CB \cdot b$
 $\Lambda scrut v1 v2 r . CB\text{-project} \cdot (scrut \cdot r) \cdot (v1 \cdot r) \cdot (v2 \cdot r)$
C-case.

notation *ESem-syn* ($\mathcal{N} \llbracket - \rrbracket_{\rho} \cdot [60, 60] 60$)
notation *EvalHeapSem-syn* ($\mathcal{N} \llbracket - \rrbracket_{\rho} \cdot [0, 0] 110$)
notation *HSem-syn* ($\mathcal{N} \{\!-\!\} \cdot [60, 60] 60$)
notation *AHSem-bot* ($\mathcal{N} \{\!-\!\} \cdot [60] 60$)

Here we re-state the simplification rules, cleaned up by beta-reducing the locale parameters.

lemma *CESem-simps:*

$\mathcal{N} \llbracket \text{Lam } [x]. e \rrbracket_{\rho} = (\Lambda (C \cdot r) . CFn \cdot (\Lambda v . (\mathcal{N} \llbracket e \rrbracket_{\rho} (x := v)) | r))$
 $\mathcal{N} \llbracket \text{App } e x \rrbracket_{\rho} = (\Lambda (C \cdot r) . ((\mathcal{N} \llbracket e \rrbracket_{\rho}) \cdot r \downarrow CFn \rho x | r) \cdot r)$
 $\mathcal{N} \llbracket \text{Var } x \rrbracket_{\rho} = (\Lambda (C \cdot r) . (\rho x) \cdot r)$
 $\mathcal{N} \llbracket \text{Bool } b \rrbracket_{\rho} = (\Lambda (C \cdot r) . CB \cdot (\text{Discr } b))$
 $\mathcal{N} \llbracket (\text{scrut } ? e_1 : e_2) \rrbracket_{\rho} = (\Lambda (C \cdot r) . CB\text{-project} \cdot ((\mathcal{N} \llbracket \text{scrut} \rrbracket_{\rho}) \cdot r) \cdot ((\mathcal{N} \llbracket e_1 \rrbracket_{\rho}) \cdot r) \cdot ((\mathcal{N} \llbracket e_2 \rrbracket_{\rho}) \cdot r))$

$\mathcal{N} \llbracket \text{Let as body} \rrbracket_{\varrho} = (\Lambda (C \cdot r) \cdot (\mathcal{N} \llbracket \text{body} \rrbracket_{\mathcal{N} \{as\}_{\varrho}}) \cdot r)$
by (*auto simp add: eta-cfun*)

lemma *CESem-bot[simp]*: $(\mathcal{N} \llbracket e \rrbracket_{\sigma}) \cdot \perp = \perp$
by (*nominal-induct e arbitrary: σ rule: exp-strong-induct*) *auto*

lemma *CHSem-bot[simp]*: $(\mathcal{N} \{ \Gamma \} x) \cdot \perp = \perp$
by (*cases $x \in \text{dom} A \Gamma$*) (*auto simp add: lookup-HSem-heap lookup-HSem-other*)

Sometimes we do not care much about the resource usage and just want a simpler formula.

lemma *CESem-simps-no-tick*:

$(\mathcal{N} \llbracket \text{Lam } [x]. e \rrbracket_{\varrho}) \cdot r \sqsubseteq \text{CFn} \cdot (\Lambda v. (\mathcal{N} \llbracket e \rrbracket_{\varrho}(x := v)) | r)$
 $(\mathcal{N} \llbracket \text{App } e x \rrbracket_{\varrho}) \cdot r \sqsubseteq ((\mathcal{N} \llbracket e \rrbracket_{\varrho}) \cdot r \downarrow \text{CFn } \varrho x | r) \cdot r$
 $\mathcal{N} \llbracket \text{Var } x \rrbracket_{\varrho} \sqsubseteq \varrho x$
 $(\mathcal{N} \llbracket \text{scrut } ? e_1 : e_2 \rrbracket_{\varrho}) \cdot r \sqsubseteq \text{CB-project} \cdot ((\mathcal{N} \llbracket \text{scrut} \rrbracket_{\varrho}) \cdot r) \cdot ((\mathcal{N} \llbracket e_1 \rrbracket_{\varrho}) \cdot r) \cdot ((\mathcal{N} \llbracket e_2 \rrbracket_{\varrho}) \cdot r)$
 $\mathcal{N} \llbracket \text{Let as body} \rrbracket_{\varrho} \sqsubseteq \mathcal{N} \llbracket \text{body} \rrbracket_{\mathcal{N} \{as\}_{\varrho}}$

apply –

apply (*rule below-trans[OF monofun-cfun-arg[OF below-C]], simp*)

apply (*rule below-trans[OF monofun-cfun-arg[OF below-C]], simp*)

apply (*rule cfun-belowI, rule below-trans[OF monofun-cfun-arg[OF below-C]], simp*)

apply (*rule below-trans[OF monofun-cfun-arg[OF below-C]], simp*)

apply (*rule cfun-belowI, rule below-trans[OF monofun-cfun-arg[OF below-C]], simp*)

done

lemma *CELam-no-restr*: $(\mathcal{N} \llbracket \text{Lam } [x]. e \rrbracket_{\varrho}) \cdot r \sqsubseteq \text{CFn} \cdot (\Lambda v. (\mathcal{N} \llbracket e \rrbracket_{\varrho}(x := v)))$

proof –

have $(\mathcal{N} \llbracket \text{Lam } [x]. e \rrbracket_{\varrho}) \cdot r \sqsubseteq \text{CFn} \cdot (\Lambda v. (\mathcal{N} \llbracket e \rrbracket_{\varrho}(x := v)) | r)$ **by** (*rule CESem-simps-no-tick*)

also have $\dots \sqsubseteq \text{CFn} \cdot (\Lambda v. (\mathcal{N} \llbracket e \rrbracket_{\varrho}(x := v)))$

by (*intro cont2cont monofun-LAM below-trans[OF C-restr-below] monofun-cfun-arg below-refl fun-upd-mono*)

finally show *?thesis by this* (*intro cont2cont*)

qed

lemma *CEApp-no-restr*: $(\mathcal{N} \llbracket \text{App } e x \rrbracket_{\varrho}) \cdot r \sqsubseteq ((\mathcal{N} \llbracket e \rrbracket_{\varrho}) \cdot r \downarrow \text{CFn } \varrho x) \cdot r$

proof –

have $(\mathcal{N} \llbracket \text{App } e x \rrbracket_{\varrho}) \cdot r \sqsubseteq ((\mathcal{N} \llbracket e \rrbracket_{\varrho}) \cdot r \downarrow \text{CFn } \varrho x | r) \cdot r$ **by** (*rule CESem-simps-no-tick*)

also have $\varrho x | r \sqsubseteq \varrho x$ **by** (*rule C-restr-below*)

finally show *?thesis by this* (*intro cont2cont*)

qed

end

7 Correctness of the natural semantics

7.1 CorrectnessOriginal

```
theory CorrectnessOriginal
imports Denotational Launchbury
begin
```

This is the main correctness theorem, Theorem 2 from [Lau93].

```
theorem correctness:
  assumes  $\Gamma : e \Downarrow_L \Delta : v$ 
  and  $fv(\Gamma, e) \subseteq set\ L \cup domA\ \Gamma$ 
  shows  $\llbracket e \rrbracket_{\Gamma}^{\varrho} = \llbracket v \rrbracket_{\Delta}^{\varrho}$ 
  and  $(\llbracket \Gamma \rrbracket^{\varrho}) f|^{\cdot} domA\ \Gamma = (\llbracket \Delta \rrbracket^{\varrho}) f|^{\cdot} domA\ \Gamma$ 
  using assms
proof(nominal-induct arbitrary:  $\varrho$  rule:reds.strong-induct)
case Lambda
  case 1 show ?case..
  case 2 show ?case..
next
case (Application  $y\ \Gamma\ e\ x\ L\ \Delta\ \Theta\ v\ e'$ )
  have Gamma-subset:  $domA\ \Gamma \subseteq domA\ \Delta$ 
  by (rule reds-doesnt-forget[OF Application.hyps(8)])

  case 1
  hence prem1:  $fv(\Gamma, e) \subseteq set\ L \cup domA\ \Gamma$  and  $x \in set\ L \cup domA\ \Gamma$  by auto
  moreover
  note reds-pres-closed[OF Application.hyps(8) prem1]
  moreover
  note reds-doesnt-forget[OF Application.hyps(8)]
  moreover
  have  $fv(e'[y::=x]) \subseteq fv(Lam\ [y].\ e') \cup \{x\}$ 
  by (auto simp add: fv-subst-eq)
  ultimately
  have prem2:  $fv(\Delta, e'[y::=x]) \subseteq set\ L \cup domA\ \Delta$  by auto

  have *:  $(\llbracket \Gamma \rrbracket^{\varrho})\ x = (\llbracket \Delta \rrbracket^{\varrho})\ x$ 
  proof(cases  $x \in domA\ \Gamma$ )
  case True
    from Application.hyps(10)[OF prem1, where  $\varrho = \varrho$ ]
    have  $((\llbracket \Gamma \rrbracket^{\varrho}) f|^{\cdot} domA\ \Gamma)\ x = ((\llbracket \Delta \rrbracket^{\varrho}) f|^{\cdot} domA\ \Gamma)\ x$  by simp
    with True show ?thesis by simp
  case False
    from False  $\langle x \in set\ L \cup domA\ \Gamma \rangle$  reds-avoids-live[OF Application.hyps(8)]
    show ?thesis by (auto simp add: lookup-HSem-other)
  qed
  have  $\llbracket App\ e\ x \rrbracket_{\Gamma}^{\varrho} = (\llbracket e \rrbracket_{\Gamma}^{\varrho}) \Downarrow_{Fn\ (\llbracket \Gamma \rrbracket^{\varrho})\ x}$ 
  by simp
```

also have $\dots = (\llbracket \text{Lam } [y]. e' \rrbracket_{\{\Delta\}\varrho} \downarrow \text{Fn } (\{\Gamma\}\varrho)) x$
using *Application.hyps(9)*[*OF prem1*] **by** *simp*
also have $\dots = (\llbracket \text{Lam } [y]. e' \rrbracket_{\{\Delta\}\varrho} \downarrow \text{Fn } (\{\Delta\}\varrho)) x$
unfolding $*$..
also have $\dots = (\text{Fn}(\Lambda z. \llbracket e' \rrbracket_{(\{\Delta\}\varrho)(y := z)})) \downarrow \text{Fn } (\{\Delta\}\varrho) x$
by *simp*
also have $\dots = \llbracket e' \rrbracket_{(\{\Delta\}\varrho)(y := (\{\Delta\}\varrho) x)}$
by *simp*
also have $\dots = \llbracket e'[y ::= x] \rrbracket_{\{\Delta\}\varrho}$
unfolding *ESem-subst.*
also have $\dots = \llbracket v \rrbracket_{\{\Theta\}\varrho}$
by (*rule Application.hyps(12)*[*OF prem2*])
finally
show $\llbracket \text{App } e x \rrbracket_{\{\Gamma\}\varrho} = \llbracket v \rrbracket_{\{\Theta\}\varrho}$. **show** $(\{\Gamma\}\varrho) f|' \text{domA } \Gamma = (\{\Theta\}\varrho) f|' \text{domA } \Gamma$
using *Application.hyps(10)*[*OF prem1*]
env-restr-eq-subset[*OF Gamma-subset Application.hyps(13)*[*OF prem2*]]
by (*rule trans*)
next
case (*Variable* $\Gamma x e L \Delta v$)
hence [*simp*]: $x \in \text{domA } \Gamma$ **by** (*metis domA-from-set map-of-SomeD*)

let $?\Gamma = \text{delete } x \Gamma$

case 2
have $x \notin \text{domA } \Delta$
by (*rule reds-avoids-live*[*OF Variable.hyps(2)*], *simp-all*)

have *subset*: $\text{domA } ?\Gamma \subseteq \text{domA } \Delta$
by (*rule reds-doesnt-forget*[*OF Variable.hyps(2)*])

let $?new = \text{domA } \Delta - \text{domA } \Gamma$
have $\text{fv } (?\Gamma, e) \cup \{x\} \subseteq \text{fv } (\Gamma, \text{Var } x)$
by (*rule fv-delete-heap*[*OF map-of* $\Gamma x = \text{Some } e$])
hence *prem*: $\text{fv } (?\Gamma, e) \subseteq \text{set } (x \# L) \cup \text{domA } ?\Gamma$ **using** 2 **by** *auto*
hence *fv-subset*: $\text{fv } (?\Gamma, e) - \text{domA } ?\Gamma \subseteq - ?new$
using *reds-avoids-live'*[*OF Variable.hyps(2)*] **by** *auto*

have $\text{domA } \Gamma \subseteq (- ?new)$ **by** *auto*

have $\{\Gamma\}\varrho = \{(x, e) \# ?\Gamma\}\varrho$
by (*rule HSem-reorder*[*OF map-of-delete-insert*[*symmetric*, *OF Variable(1)*]])
also have $\dots = (\mu \varrho'. (\varrho ++_{\text{domA } ?\Gamma} (\{\?\Gamma\}\varrho')) (x := \llbracket e \rrbracket_{\varrho'}))$
by (*rule iterative-HSem*, *simp*)
also have $\dots = (\mu \varrho'. (\varrho ++_{\text{domA } ?\Gamma} (\{\?\Gamma\}\varrho')) (x := \llbracket e \rrbracket_{\{\?\Gamma\}\varrho'}))$
by (*rule iterative-HSem'*, *simp*)
finally
have $(\{\Gamma\}\varrho) f|' (- ?new) = (\dots) f|' (- ?new)$ **by** *simp*
also have $\dots = (\mu \varrho'. (\varrho ++_{\text{domA } \Delta} (\{\Delta\}\varrho')) (x := \llbracket v \rrbracket_{\{\Delta\}\varrho'})) f|' (- ?new)$

proof (*induction rule: parallel-fix-ind*[**where** $P = \lambda x y. x f |' (- ?new) = y f |' (- ?new)$])
case 1 show $?case$ **by** *simp*
next
case 2 show $?case ..$
next
case ($\exists \sigma \sigma'$)
hence $\llbracket e \rrbracket_{\{\Gamma\}\sigma} = \llbracket e \rrbracket_{\{\Gamma\}\sigma'}$
and $(\{\{\Gamma\}\sigma\} f |' \text{dom}A \ ?\Gamma = (\{\{\Gamma\}\sigma'\} f |' \text{dom}A \ ?\Gamma$
using *fv-subset* **by** (*auto intro: ESem-fresh-cong HSem-fresh-cong env-restr-eq-subset*[*OF*
- 3])
from *trans*[*OF this*(1) *Variable*(3)[*OF prem*]] *trans*[*OF this*(2) *Variable*(4)[*OF prem*]]
have $\llbracket e \rrbracket_{\{\Gamma\}\sigma} = \llbracket v \rrbracket_{\{\Delta\}\sigma'}$
and $(\{\{\Gamma\}\sigma\} f |' \text{dom}A \ ?\Gamma = (\{\{\Delta\}\sigma'\} f |' \text{dom}A \ ?\Gamma.$
thus $?case$
using *subset*
by (*fastforce simp add: lookup-override-on-eq lookup-env-restr-eq dest: env-restr-eqD*)
qed
also have $\dots = (\mu \varrho'. (\varrho ++_{\text{dom}A \ \Delta} (\{\{\Delta\}\varrho'\})) (x := \llbracket v \rrbracket_{\varrho'})) f |' (- ?new)$
by (*rule arg-cong*[*OF iterative-HSem*[*symmetric*], *OF* $\langle x \notin \text{dom}A \ \Delta \rangle$])
also have $\dots = (\{\{(x,v) \# \Delta\}\varrho\} f |' (- ?new)$
by (*rule arg-cong*[*OF iterative-HSem*[*symmetric*], *OF* $\langle x \notin \text{dom}A \ \Delta \rangle$])
finally
show *le*: $?case$ **by** (*rule env-restr-eq-subset*[*OF* $\langle \text{dom}A \ \Gamma \subseteq (- ?new) \rangle$])

have $\llbracket \text{Var } x \rrbracket_{\{\Gamma\}\varrho} = \llbracket \text{Var } x \rrbracket_{\{(x,v) \# \Delta\}\varrho}$
using *env-restr-eqD*[*OF le*, **where** $x = x$]
by *simp*
also have $\dots = \llbracket v \rrbracket_{\{(x,v) \# \Delta\}\varrho}$
by (*auto simp add: lookup-HSem-heap*)
finally
show $\llbracket \text{Var } x \rrbracket_{\{\Gamma\}\varrho} = \llbracket v \rrbracket_{\{(x,v) \# \Delta\}\varrho}.$
next
case (*Bool* b)
case 1
show $?case$ **by** *simp*
case 2
show $?case$ **by** *simp*
next
case (*IfThenElse* $\Gamma \text{ scrut } L \ \Delta \ b \ e_1 \ e_2 \ \Theta \ v$)
have *Gamma-subset*: $\text{dom}A \ \Gamma \subseteq \text{dom}A \ \Delta$
by (*rule reds-doesnt-forget*[*OF IfThenElse.hyps*(1)])

let $?e = \text{if } b \text{ then } e_1 \text{ else } e_2$

case 1

hence *prem1*: $\text{fv}(\Gamma, \text{scrut}) \subseteq \text{set } L \cup \text{dom}A \ \Gamma$
and *prem2*: $\text{fv}(\Delta, ?e) \subseteq \text{set } L \cup \text{dom}A \ \Delta$

```

and  $fv\ ?e \subseteq domA\ \Gamma \cup set\ L$ 
using new-free-vars-on-heap[OF IfThenElse.hyps(1)] Gamma-subset by auto

have  $\llbracket (scrut\ ?\ e_1 : e_2) \rrbracket_{\{\Gamma\}\varrho} = B\text{-project} \cdot (\llbracket scrut \rrbracket_{\{\Gamma\}\varrho}) \cdot (\llbracket e_1 \rrbracket_{\{\Gamma\}\varrho}) \cdot (\llbracket e_2 \rrbracket_{\{\Gamma\}\varrho})$  by simp
also have  $\dots = B\text{-project} \cdot (\llbracket Bool\ b \rrbracket_{\{\Delta\}\varrho}) \cdot (\llbracket e_1 \rrbracket_{\{\Gamma\}\varrho}) \cdot (\llbracket e_2 \rrbracket_{\{\Gamma\}\varrho})$ 
  unfolding IfThenElse.hyps(2)[OF prem1]..
also have  $\dots = \llbracket ?e \rrbracket_{\{\Gamma\}\varrho}$  by simp
also have  $\dots = \llbracket ?e \rrbracket_{\{\Delta\}\varrho}$ 
proof(rule ESem-fresh-cong-subset[OF <fv ?e ⊆ domA Γ ∪ set L> env-restr-eqI])
  fix  $x$ 
  assume  $x \in domA\ \Gamma \cup set\ L$ 
  thus  $(\llbracket \Gamma \rrbracket_{\varrho})\ x = (\llbracket \Delta \rrbracket_{\varrho})\ x$ 
  proof(cases  $x \in domA\ \Gamma$ )
    assume  $x \in domA\ \Gamma$ 
    from IfThenElse.hyps(3)[OF prem1]
    have  $((\llbracket \Gamma \rrbracket_{\varrho})\ f|' domA\ \Gamma)\ x = ((\llbracket \Delta \rrbracket_{\varrho})\ f|' domA\ \Gamma)\ x$  by simp
    with  $\langle x \in domA\ \Gamma \rangle$  show ?thesis by simp
  next
  assume  $x \notin domA\ \Gamma$ 
  from this  $\langle x \in domA\ \Gamma \cup set\ L \rangle$  reds-avoids-live[OF IfThenElse.hyps(1)]
  show ?thesis
  by (simp add: lookup-HSem-other)
  qed
qed
also have  $\dots = \llbracket v \rrbracket_{\{\Theta\}\varrho}$ 
  unfolding IfThenElse.hyps(5)[OF prem2]..
finally
show ?case.
thm env-restr-eq-subset
show  $(\llbracket \Gamma \rrbracket_{\varrho})\ f|' domA\ \Gamma = (\llbracket \Theta \rrbracket_{\varrho})\ f|' domA\ \Gamma$ 
  using IfThenElse.hyps(3)[OF prem1]
  env-restr-eq-subset[OF Gamma-subset IfThenElse.hyps(6)][OF prem2]
  by (rule trans)
next
case (Let as Γ L body Δ v)
  case 1
  { fix  $a$ 
    assume  $a: a \in domA\ as$ 
    have  $atom\ a \# \Gamma$ 
      by (rule Let(1)[unfolded fresh-star-def, rule-format, OF imageI][OF a])
    hence  $a \notin domA\ \Gamma$ 
      by (metis domA-not-fresh)
  }
note  $* = this$ 

have  $fv\ (as\ @\ \Gamma,\ body) - domA\ (as\ @\ \Gamma) \subseteq fv\ (\Gamma,\ Let\ as\ body) - domA\ \Gamma$ 
by auto

```

```

with 1 have prem: fv (as @ Γ, body) ⊆ set L ∪ domA (as @ Γ) by auto

have f1: atom ' domA as #* Γ
  using Let(1) by (simp add: set-bn-to-atom-domA)

have [ Let as body ]_{Γ} ρ = [ body ]_{as} _{Γ} ρ
  by (simp)
also have ... = [ body ]_{as @ Γ} ρ
  by (rule arg-cong[OF HSem-merge[OF f1]])
also have ... = [ v ]_{Δ} ρ
  by (rule Let.hyps(4)[OF prem])
finally
show ?case.

have (_{Γ} ρ) f|' (domA Γ) = (_{as} (_{Γ} ρ)) f|' (domA Γ)
  apply (rule ext)
  apply (case-tac x ∈ domA as)
  apply (auto simp add: lookup-HSem-other lookup-env-restr-eq *)
  done
also have ... = (_{as @ Γ} ρ) f|' (domA Γ)
  by (rule arg-cong[OF HSem-merge[OF f1]])
also have ... = (_{Δ} ρ) f|' (domA Γ)
  by (rule env-restr-eq-subset[OF - Let.hyps(5)[OF prem]]) simp
finally
show (_{Γ} ρ) f|' domA Γ = (_{Δ} ρ) f|' domA Γ.
qed

end



## 7.2 CorrectnessResourced


theory CorrectnessResourced
  imports ResourcedDenotational Launchbury
begin

theorem correctness:
  assumes Γ : e ↓L Δ : z
  and fv (Γ, e) ⊆ set L ∪ domA Γ
  shows N[e]_{N}_{Γ} ρ ⊆ N[z]_{N}_{Δ} ρ and (N_{Γ} ρ) f|' domA Γ ⊆ (N_{Δ} ρ) f|' domA Γ
  using assms
proof (nominal-induct arbitrary: ρ rule:reds.strong-induct)
case Lambda
  case 1 show ?case..
  case 2 show ?case..
next
case (Application y Γ e x L Δ Θ z e')
  have Gamma-subset: domA Γ ⊆ domA Δ
  by (rule reds-doesnt-forget[OF Application.hyps(8)])

```

case 1
hence $\text{prem1}: \text{fv}(\Gamma, e) \subseteq \text{set } L \cup \text{domA } \Gamma$ **and** $x \in \text{set } L \cup \text{domA } \Gamma$ **by** *auto*
moreover
note $\text{reds-pres-closed}[OF \text{ Application.hyps}(8) \text{ prem1}]$
moreover
note $\text{reds-doesnt-forget}[OF \text{ Application.hyps}(8)]$
moreover
have $\text{fv}(e'[y ::= x]) \subseteq \text{fv}(\text{Lam } [y]. e') \cup \{x\}$
by *(auto simp add: fv-subst-eq)*
ultimately
have $\text{prem2}: \text{fv}(\Delta, e'[y ::= x]) \subseteq \text{set } L \cup \text{domA } \Delta$ **by** *auto*

have $*$: $(\mathcal{N}\{\Gamma\}\varrho) x \sqsubseteq (\mathcal{N}\{\Delta\}\varrho) x$
proof(*cases* $x \in \text{domA } \Gamma$)
case *True*
thus *?thesis*
using $\text{fun-belowD}[OF \text{ Application.hyps}(10)[OF \text{ prem1}], \text{ where } \varrho1 = \varrho \text{ and } x = x]$
by *simp*
next
case *False*
from $\text{False} \langle x \in \text{set } L \cup \text{domA } \Gamma \rangle$ $\text{reds-avoids-live}[OF \text{ Application.hyps}(8)]$
show *?thesis* **by** *(auto simp add: lookup-HSem-other)*
qed

{
fix r
have $(\mathcal{N}[\text{App } e \ x]_{\mathcal{N}\{\Gamma\}\varrho}) \cdot r \sqsubseteq ((\mathcal{N}[e]_{\mathcal{N}\{\Gamma\}\varrho}) \cdot r \downarrow \text{CFn } (\mathcal{N}\{\Gamma\}\varrho) x) \cdot r$
by *(rule CEApp-no-restr)*
also have $((\mathcal{N}[e]_{\mathcal{N}\{\Gamma\}\varrho})) \sqsubseteq ((\mathcal{N}[\text{Lam } [y]. e']_{\mathcal{N}\{\Delta\}\varrho}))$
using $\text{Application.hyps}(9)[OF \text{ prem1}]$.
also note $\langle (\mathcal{N}\{\Gamma\}\varrho) x \sqsubseteq (\mathcal{N}\{\Delta\}\varrho) x \rangle$
also have $(\mathcal{N}[\text{Lam } [y]. e']_{\mathcal{N}\{\Delta\}\varrho}) \cdot r \sqsubseteq (\text{CFn} \cdot (\Lambda v. (\mathcal{N}[e']_{(\mathcal{N}\{\Delta\}\varrho)(y := v)})))$
by *(rule CELam-no-restr)*
also have $\text{CFn} \cdot (\Lambda v. (\mathcal{N}[e']_{(\mathcal{N}\{\Delta\}\varrho)(y := v)})) \downarrow \text{CFn } ((\mathcal{N}\{\Delta\}\varrho) x) = (\mathcal{N}[e']_{(\mathcal{N}\{\Delta\}\varrho)(y := (\mathcal{N}\{\Delta\}\varrho) x)})$
by *simp*
also have $\dots = (\mathcal{N}[e'[y ::= x]]_{(\mathcal{N}\{\Delta\}\varrho)})$
unfolding *ESem-subst..*
also have $\dots \sqsubseteq \mathcal{N}[z]_{\mathcal{N}\{\Theta\}\varrho}$
using $\text{Application.hyps}(12)[OF \text{ prem2}]$.
finally
have $(\mathcal{N}[\text{App } e \ x]_{\mathcal{N}\{\Gamma\}\varrho}) \cdot r \sqsubseteq (\mathcal{N}[z]_{\mathcal{N}\{\Theta\}\varrho}) \cdot r$ **by** *this (intro cont2cont)+*
}
thus *?case* **by** *(rule cfun-belowI)*

show $(\mathcal{N}\{\Gamma\}\varrho) f|'(\text{domA } \Gamma) \sqsubseteq (\mathcal{N}\{\Theta\}\varrho) f|'(\text{domA } \Gamma)$
using $\text{Application.hyps}(10)[OF \text{ prem1}]$

```

      env-restr-below-subset[OF Gamma-subset Application.hyps(13)][OF prem2]]
    by (rule below-trans)
next
case (Variable  $\Gamma$   $x \in L$   $\Delta$   $z$ )
  hence [simp]:  $x \in \text{dom} A \ \Gamma$ 
    by (metis domA-from-set map-of-SomeD)

  case 2

  have  $x \notin \text{dom} A \ \Delta$ 
    by (rule reds-avoids-live[OF Variable.hyps(2)], simp-all)

  have subset:  $\text{dom} A \ (\text{delete } x \ \Gamma) \subseteq \text{dom} A \ \Delta$ 
    by (rule reds-doesnt-forget[OF Variable.hyps(2)])

  let ?new =  $\text{dom} A \ \Delta - \text{dom} A \ \Gamma$ 
  have fv (delete  $x \ \Gamma$ ,  $e$ )  $\cup \{x\} \subseteq \text{fv} \ (\Gamma, \text{Var } x)$ 
    by (rule fv-delete-heap[OF map-of  $\Gamma$   $x = \text{Some } e$ ])
  hence prem:  $\text{fv} \ (\text{delete } x \ \Gamma, e) \subseteq \text{set} \ (x \# L) \cup \text{dom} A \ (\text{delete } x \ \Gamma)$  using 2 by auto
  hence fv-subset:  $\text{fv} \ (\text{delete } x \ \Gamma, e) - \text{dom} A \ (\text{delete } x \ \Gamma) \subseteq - \ ?new$ 
    using reds-avoids-live'[OF Variable.hyps(2)] by auto

  have  $\text{dom} A \ \Gamma \subseteq (- \ ?new)$  by auto

  have  $\mathcal{N}\{\Gamma\}\varrho = \mathcal{N}\{(x,e) \# \text{delete } x \ \Gamma\}\varrho$ 
    by (rule HSem-reorder[OF map-of-delete-insert[symmetric, OF Variable(1)]])
  also have ... =  $(\mu \varrho'. (\varrho \ ++_{\text{dom} A \ (\text{delete } x \ \Gamma)} (\mathcal{N}\{\text{delete } x \ \Gamma\}\varrho')))(x := \mathcal{N}\llbracket e \rrbracket_{\varrho'})$ 
    by (rule iterative-HSem, simp)
  also have ... =  $(\mu \varrho'. (\varrho \ ++_{\text{dom} A \ (\text{delete } x \ \Gamma)} (\mathcal{N}\{\text{delete } x \ \Gamma\}\varrho')))(x := \mathcal{N}\llbracket e \rrbracket_{\mathcal{N}\{\text{delete } x \ \Gamma\}\varrho'})$ 
    by (rule iterative-HSem', simp)
  finally
  have  $(\mathcal{N}\{\Gamma\}\varrho) f |' (- \ ?new) \sqsubseteq (\dots) f |' (- \ ?new)$  by (rule ssubst) (rule below-refl)
  also have ...  $\sqsubseteq (\mu \varrho'. (\varrho \ ++_{\text{dom} A \ \Delta} (\mathcal{N}\{\Delta\}\varrho')))(x := \mathcal{N}\llbracket z \rrbracket_{\mathcal{N}\{\Delta\}\varrho'}) f |' (- \ ?new)$ 
  proof (induction rule: parallel-fix-ind[where  $P = \lambda x y. x f |' (- \ ?new) \sqsubseteq y f |' (- \ ?new)$ ])
    case 1 show ?case by simp
  next
    case 2 show ?case ..
  next
    case (3  $\sigma \ \sigma'$ )
    hence  $\mathcal{N}\llbracket e \rrbracket_{\mathcal{N}\{\text{delete } x \ \Gamma\}\sigma} \sqsubseteq \mathcal{N}\llbracket e \rrbracket_{\mathcal{N}\{\text{delete } x \ \Gamma\}\sigma'}$ 
      and  $(\mathcal{N}\{\text{delete } x \ \Gamma\}\sigma) f |' \text{dom} A \ (\text{delete } x \ \Gamma) \sqsubseteq (\mathcal{N}\{\text{delete } x \ \Gamma\}\sigma') f |' \text{dom} A \ (\text{delete } x \ \Gamma)$ 
      using fv-subset by (auto intro: ESem-fresh-cong-below HSem-fresh-cong-below env-restr-below-subset[OF
- 3])
    from below-trans[OF this(1) Variable(3)][OF prem] below-trans[OF this(2) Variable(4)][OF
premi]]

```

have $\mathcal{N}\llbracket e \rrbracket_{\mathcal{N}\{\text{delete } x \Gamma\}\sigma} \sqsubseteq \mathcal{N}\llbracket z \rrbracket_{\mathcal{N}\{\Delta\}\sigma'}$
and $(\mathcal{N}\{\text{delete } x \Gamma\}\sigma) f|' \text{dom}A (\text{delete } x \Gamma) \sqsubseteq (\mathcal{N}\{\Delta\}\sigma') f|' \text{dom}A (\text{delete } x \Gamma)$.
thus *?case*
using *subset*
by (*auto intro!*: *fun-belowI simp add: lookup-override-on-eq lookup-env-restr-eq elim:*
env-restr-belowD)
qed
also have $\dots = (\mu \varrho'. (\varrho ++_{\text{dom}A} \Delta (\mathcal{N}\{\Delta\}\varrho')) (x := \mathcal{N}\llbracket z \rrbracket_{\varrho'})) f|' (-?new)$
by (*rule arg-cong[OF iterative-HSem[symmetric], OF ‹x ∉ domA Δ›]*)
also have $\dots = (\mathcal{N}\{(x,z) \# \Delta\}\varrho) f|' (-?new)$
by (*rule arg-cong[OF iterative-HSem[symmetric], OF ‹x ∉ domA Δ›]*)
finally
show *le*: *?case* **by** (*rule env-restr-below-subset[OF ‹domA Γ ⊆ (-?new)›]*) (*intro cont2cont*)+

have $\mathcal{N}\llbracket \text{Var } x \rrbracket_{\mathcal{N}\{\Gamma\}\varrho} \sqsubseteq (\mathcal{N}\{\Gamma\}\varrho) x$ **by** (*rule CESem-simps-no-tick*)
also have $\dots \sqsubseteq (\mathcal{N}\{(x,z) \# \Delta\}\varrho) x$
using *fun-belowD[OF le, where x = x]* **by** *simp*
also have $\dots = \mathcal{N}\llbracket z \rrbracket_{\mathcal{N}\{(x,z) \# \Delta\}\varrho}$
by (*simp add: lookup-HSem-heap*)
finally
show $\mathcal{N}\llbracket \text{Var } x \rrbracket_{\mathcal{N}\{\Gamma\}\varrho} \sqsubseteq \mathcal{N}\llbracket z \rrbracket_{\mathcal{N}\{(x,z) \# \Delta\}\varrho}$ **by** *this* (*intro cont2cont*)+
next
case (*Bool b*)
case 1
show *?case* **by** *simp*
case 2
show *?case* **by** *simp*
next
case (*IfThenElse Γ scrut L Δ b e₁ e₂ Θ z*)
have *Gamma-subset*: $\text{dom}A \Gamma \subseteq \text{dom}A \Delta$
by (*rule reds-doesnt-forget[OF IfThenElse.hyps(1)]*)

let *?e* = *if b then e₁ else e₂*

case 1

hence *prem1*: $\text{fv}(\Gamma, \text{scrut}) \subseteq \text{set } L \cup \text{dom}A \Gamma$
and *prem2*: $\text{fv}(\Delta, ?e) \subseteq \text{set } L \cup \text{dom}A \Delta$
and $\text{fv } ?e \subseteq \text{dom}A \Gamma \cup \text{set } L$
using *new-free-vars-on-heap[OF IfThenElse.hyps(1)] Gamma-subset* **by** *auto*

{
fix *r*
have $(\mathcal{N}\llbracket (\text{scrut } ?e_1 : e_2) \rrbracket_{\mathcal{N}\{\Gamma\}\varrho}) \cdot r \sqsubseteq \text{CB-project} \cdot ((\mathcal{N}\llbracket \text{scrut} \rrbracket_{\mathcal{N}\{\Gamma\}\varrho}) \cdot r) \cdot ((\mathcal{N}\llbracket e_1 \rrbracket_{\mathcal{N}\{\Gamma\}\varrho}) \cdot r) \cdot ((\mathcal{N}\llbracket e_2 \rrbracket_{\mathcal{N}\{\Gamma\}\varrho}) \cdot r)$
by (*rule CESem-simps-no-tick*)
also have $\dots \sqsubseteq \text{CB-project} \cdot ((\mathcal{N}\llbracket \text{Bool } b \rrbracket_{\mathcal{N}\{\Delta\}\varrho}) \cdot r) \cdot ((\mathcal{N}\llbracket e_1 \rrbracket_{\mathcal{N}\{\Gamma\}\varrho}) \cdot r) \cdot ((\mathcal{N}\llbracket e_2 \rrbracket_{\mathcal{N}\{\Gamma\}\varrho}) \cdot r)$
by (*intro monofun-cfun-fun monofun-cfun-arg IfThenElse.hyps(2)[OF prem1]*)
}

also have ... = $(\mathcal{N}[\ ?e \]_{\mathcal{N}\{\Gamma\}\varrho}) \cdot r$ **by** (*cases r simp-all*)
also have ... $\sqsubseteq (\mathcal{N}[\ ?e \]_{\mathcal{N}\{\Delta\}\varrho}) \cdot r$
proof(*rule monofun-cfun-fun[OF ESem-fresh-cong-below-subset[OF $\langle fv \ ?e \subseteq domA \ \Gamma \cup set \ L \rangle Env.env-restr-belowI$]]*)
 fix x
 assume $x \in domA \ \Gamma \cup set \ L$
 thus $(\mathcal{N}\{\Gamma\}\varrho) \ x \sqsubseteq (\mathcal{N}\{\Delta\}\varrho) \ x$
 proof(*cases $x \in domA \ \Gamma$*)
 assume $x \in domA \ \Gamma$
 from *IfThenElse.hyps(3)[OF prem1]*
 have $((\mathcal{N}\{\Gamma\}\varrho) \ f) \ ' \ domA \ \Gamma \ x \sqsubseteq ((\mathcal{N}\{\Delta\}\varrho) \ f) \ ' \ domA \ \Gamma \ x$ **by** (*rule fun-belowD*)
 with $\langle x \in domA \ \Gamma \rangle$ **show** *?thesis* **by** *simp*
 next
 assume $x \notin domA \ \Gamma$
 from *this $\langle x \in domA \ \Gamma \cup set \ L \rangle reds-avoids-live[OF IfThenElse.hyps(1)]$*
 show *?thesis*
 by (*simp add: lookup-HSem-other*)
 qed
 qed
also have ... $\sqsubseteq (\mathcal{N}[\ z \]_{\mathcal{N}\{\Theta\}\varrho}) \cdot r$
 by (*intro monofun-cfun-fun monofun-cfun-arg IfThenElse.hyps(5)[OF prem2]*)
finally
have $(\mathcal{N}[\ (scrut \ ? \ e_1 : e_2) \]_{\mathcal{N}\{\Gamma\}\varrho}) \cdot r \sqsubseteq (\mathcal{N}[\ z \]_{\mathcal{N}\{\Theta\}\varrho}) \cdot r$ **by** *this (intro cont2cont)+*
}
thus *?case* **by** (*rule cfun-belowI*)

show $(\mathcal{N}\{\Gamma\}\varrho) \ f \ ' \ (domA \ \Gamma) \sqsubseteq (\mathcal{N}\{\Theta\}\varrho) \ f \ ' \ (domA \ \Gamma)$
 using *IfThenElse.hyps(3)[OF prem1]*
 env-restr-below-subset[OF Gamma-subset IfThenElse.hyps(6)[OF prem2]]
 by (*rule below-trans*)
next
case (*Let as $\Gamma \ L \ body \ \Delta \ z$*)
 case *1*
 have $*$: $domA \ as \cap domA \ \Gamma = \{\}$ **by** (*metis Let.hyps(1) fresh-distinct*)

 have $fv \ (as \ @ \ \Gamma, \ body) - domA \ (as \ @ \ \Gamma) \subseteq fv \ (\Gamma, \ Let \ as \ body) - domA \ \Gamma$
 by *auto*
 with *1* **have** *prem*: $fv \ (as \ @ \ \Gamma, \ body) \subseteq set \ L \cup domA \ (as \ @ \ \Gamma)$ **by** *auto*

 have $f1$: $atom \ ' \ domA \ as \ \# \ * \ \Gamma$
 using *Let(1)* **by** (*simp add: set-bn-to-atom-domA*)

 have $\mathcal{N}[\ Let \ as \ body \]_{\mathcal{N}\{\Gamma\}\varrho} \sqsubseteq \mathcal{N}[\ body \]_{\mathcal{N}\{as\}\mathcal{N}\{\Gamma\}\varrho}$
 by (*rule CESem-simps-no-tick*)
 also have ... = $\mathcal{N}[\ body \]_{\mathcal{N}\{as \ @ \ \Gamma\}\varrho}$
 by (*rule arg-cong[OF HSem-merge[OF f1]]*)
 also have ... $\sqsubseteq \mathcal{N}[\ z \]_{\mathcal{N}\{\Delta\}\varrho}$
 by (*rule Let.hyps(4)[OF prem]*)

finally
show $?case$ **by** $this (intro cont2cont)+$

have $(\mathcal{N}\{\Gamma\}\varrho) f|' (domA \Gamma) = (\mathcal{N}\{as\}(\mathcal{N}\{\Gamma\}\varrho)) f|' (domA \Gamma)$
unfolding $env-restr-HSem[OF *]..$
also have $\mathcal{N}\{as\}(\mathcal{N}\{\Gamma\}\varrho) = (\mathcal{N}\{as @ \Gamma\}\varrho)$
by $(rule HSem-merge[OF f1])$
also have $\dots f|' domA \Gamma \sqsubseteq (\mathcal{N}\{\Delta\}\varrho) f|' domA \Gamma$
by $(rule env-restr-below-subset[OF - Let.hyps(5)[OF prem]]) simp$
finally
show $(\mathcal{N}\{\Gamma\}\varrho) f|' domA \Gamma \sqsubseteq (\mathcal{N}\{\Delta\}\varrho) f|' domA \Gamma.$
qed

corollary $correctness-empty-env:$

assumes $\Gamma : e \Downarrow_L \Delta : z$
and $fv(\Gamma, e) \subseteq set L$
shows $\mathcal{N}[e]_{\mathcal{N}\{\Gamma\}} \sqsubseteq \mathcal{N}[z]_{\mathcal{N}\{\Delta\}}$ **and** $\mathcal{N}\{\Gamma\} \sqsubseteq \mathcal{N}\{\Delta\}$

proof–

from $assms(2)$ **have** $fv(\Gamma, e) \subseteq set L \cup domA \Gamma$ **by** $auto$
note $corr = correctness[OF assms(1) this, where \varrho = \perp]$

show $\mathcal{N}[e]_{\mathcal{N}\{\Gamma\}} \sqsubseteq \mathcal{N}[z]_{\mathcal{N}\{\Delta\}}$ **using** $corr(1).$

have $\mathcal{N}\{\Gamma\} = (\mathcal{N}\{\Gamma\}) f|' domA \Gamma$
using $env-restr-useless[OF HSem-edom-subset, where \varrho1 = \perp]$ **by** $simp$
also have $\dots \sqsubseteq (\mathcal{N}\{\Delta\}) f|' domA \Gamma$ **using** $corr(2).$
also have $\dots \sqsubseteq \mathcal{N}\{\Delta\}$ **by** $(rule env-restr-below-itself)$
finally show $\mathcal{N}\{\Gamma\} \sqsubseteq \mathcal{N}\{\Delta\}$ **by** $this (intro cont2cont)+$

qed

end

8 Equivalence of the denotational semantics

8.1 ValueSimilarity

```
theory ValueSimilarity
imports Value CValue Pointwise
begin
```

This theory formalizes Section 3 of [SGHHOM11]. Their domain D is our type $Value$, their domain E is our type $CValue$ and A corresponds to $C \rightarrow CValue$.

In our case, the construction of the domains was taken care of by the HOLCF package ([Huf12]), so where [SGHHOM11] refers to elements of the domain approximations D_n resp. E_n , these are just elements of $Value$ resp. $CValue$ here. Therefore the n -injection $\phi_n^E: E_n \rightarrow E$ is the identity here.

The projections correspond to the take-functions generated by the HOLCF package:

$$\begin{aligned} \psi_n^E: E \rightarrow E_n & \text{ corresponds to } & CValue\text{-take}::nat \Rightarrow CValue \rightarrow CValue \\ \psi_n^A: A \rightarrow A_n & \text{ corresponds to } & C\text{-to-}CValue\text{-take}::nat \Rightarrow (C \rightarrow CValue) \rightarrow C \rightarrow CValue \\ \psi_n^D: D \rightarrow D_n & \text{ corresponds to } & Value\text{-take}::nat \Rightarrow Value \rightarrow Value. \end{aligned}$$

The syntactic overloading of $e(a)(c)$ to mean either $Ap_{E_n}^\perp$ or AP_E^\perp turns into our non-overloaded $\downarrow CFn \text{ } :: CValue \Rightarrow (C \rightarrow CValue) \Rightarrow C \rightarrow CValue$.

To have our presentation closer to [SGHHOM11], we introduce some notation:

```
notation Value-take ( $\psi^D \cdot$ )
notation C-to-CValue-take ( $\psi^A \cdot$ )
notation CValue-take ( $\psi^E \cdot$ )
```

8.1.1 A note about section 2.3

Section 2.3 of [SGHHOM11] contains equations (2) and (3) which do not hold in general. We demonstrate that fact here using our corresponding definition, but the counter-example carries over to the original formulation. Lemma (2) is a generalisation of (3) to the resourced semantics, so the counter-example for (3) is the simpler and more educating:

lemma counter-example:

```
assumes Equation (3):  $\bigwedge n d d'. \psi_n^D.(d \downarrow Fn d') = \psi_{Suc\ n}^D.d \downarrow Fn \psi_n^D.d'$ 
shows False
```

proof—

```
define n :: nat where n = 1
define d where d = Fn.( $\Lambda e. (e \downarrow Fn \perp)$ )
define d' where d' = Fn.( $\Lambda -. Fn.( \Lambda -. \perp)$ )
have Fn.( $\Lambda -. \perp$ ) =  $\psi_n^D.(d \downarrow Fn d')$ 
```

```

    by (simp add: d-def d'-def n-def cfun-map-def)
  also
  have ... =  $\psi^D_{Suc\ n} \cdot d \downarrow Fn \psi^D_n \cdot d'$ 
    using Equation (3).
  also have ... =  $\perp$ 
    by (simp add: d-def d'-def n-def)
  finally show False by simp
qed

```

For completeness, and to avoid making false assertions, the counter-example to equation (2):

lemma *counter-example2*:

```

  assumes Equation (2):  $\bigwedge n\ e\ a\ c. \psi^E_n \cdot ((e \downarrow CFn\ a) \cdot c) = (\psi^E_{Suc\ n} \cdot e \downarrow CFn\ \psi^A_n \cdot a) \cdot c$ 
  shows False

```

proof –

```

  define n :: nat where n = 1
  define e where e = CFn (λ e r. (e · r ↓ CFn ⊥) · r)
  define a :: C → CValue where a = (λ -. CFn (λ -. CFn (λ -. ⊥)))
  fix c :: C
  have CFn (λ -. ⊥) =  $\psi^E_n \cdot ((e \downarrow CFn\ a) \cdot c)$ 
    by (simp add: e-def a-def n-def cfun-map-def)
  also
  have ... =  $(\psi^E_{Suc\ n} \cdot e \downarrow CFn\ \psi^A_n \cdot a) \cdot c$ 
    using Equation (2).
  also have ... =  $\perp$ 
    by (simp add: e-def a-def n-def)
  finally show False by simp
qed

```

A suitable substitute for the lemma can be found in 4.3.5 (1) in [AO93], which in our setting becomes the following (note the extra invocation of ψ^D_n on the left hand side):

lemma *Abramsky 4,3,5 (1)*:

```

 $\psi^D_n \cdot (d \downarrow Fn \psi^D_n \cdot d') = \psi^D_{Suc\ n} \cdot d \downarrow Fn \psi^D_n \cdot d'$ 
  by (cases d) (auto simp add: Value.take-take)

```

The problematic equations are used in the proof of the only-if direction of proposition 9 in [SGHHOM11]. It can be fixed by applying take-induction, which inserts the extra call to ψ^D_n in the right spot.

8.1.2 Working with Value and CValue

Combined case distinguishing and induction rules.

lemma *value-CValue-cases*:

```

  obtains
  x =  $\perp$  y =  $\perp$  |
  f where x = Fn · f y =  $\perp$  |

```

g where $x = \perp$ $y = CFn.g$ |
 $f g$ where $x = Fn.f y = CFn \cdot g$ |
 b_1 where $x = B.(Discr b_1)$ $y = \perp$ |
 $b_1 g$ where $x = B.(Discr b_1)$ $y = CFn.g$ |
 $b_1 b_2$ where $x = B.(Discr b_1)$ $y = CB.(Discr b_2)$ |
 $f b_2$ where $x = Fn.f y = CB.(Discr b_2)$ |
 b_2 where $x = \perp$ $y = CB.(Discr b_2)$
by (*metis CValue.exhaust Discr-undiscr Value.exhaust*)

lemma *Value-CValue-take-induct*:

assumes *adm* (*case-prod P*)
assumes $\bigwedge n. P (\psi^D n.x) (\psi^A n.y)$
shows $P x y$

proof–

have *case-prod P* ($\bigsqcup n. (\psi^D n.x, \psi^A n.y)$)

by (*rule admD[OF ‹adm (case-prod P)› ch2ch-Pair[OF ch2ch-Rep-cfunL[OF Value.chain-take] ch2ch-Rep-cfunL[OF C-to-CValue-chain-take]]]*)

(*simp add: assms(2)*)

hence *case-prod P* (x, y)

by (*simp add: lub-Pair[OF ch2ch-Rep-cfunL[OF Value.chain-take] ch2ch-Rep-cfunL[OF C-to-CValue-chain-take]]*)

Value.reach C-to-CValue-reach)

thus *?thesis* **by** *simp*

qed

8.1.3 Restricted similarity is defined recursively

The base case

inductive *similar'-base* :: *Value* \Rightarrow *CValue* \Rightarrow *bool* **where**
bot-similar'-base[*simp,intro*]: *similar'-base* $\perp \perp$

inductive-cases [*elim!*]:
similar'-base $x y$

The inductive case

inductive *similar'-step* :: (*Value* \Rightarrow *CValue* \Rightarrow *bool*) \Rightarrow *Value* \Rightarrow *CValue* \Rightarrow *bool* **for** s **where**

bot-similar'-step[*intro!*]: *similar'-step* $s \perp \perp$ |

bool-similar'-step[*intro*]: *similar'-step* $s (B \cdot b) (CB \cdot b)$ |

Fun-similar'-step[*intro*]: ($\bigwedge x y . s x (y \cdot C^\infty) \Longrightarrow s (f \cdot x) (g \cdot y \cdot C^\infty)$) \Longrightarrow *similar'-step* $s (Fn \cdot f) (CFn \cdot g)$

inductive-cases [*elim!*]:

similar'-step $s x \perp$

similar'-step $s \perp y$

similar'-step $s (B \cdot f) (CB \cdot g)$

similar'-step $s (Fn \cdot f) (CFn \cdot g)$

We now create the restricted similarity relation, by primitive recursion over n .

This cannot be done using an inductive definition, as it would not be monotone.

```

fun similar' where
  similar' 0 = similar'-base |
  similar' (Suc n) = similar'-step (similar' n)
declare similar'.simps[simp del]

abbreviation similar'-syn (-  $\triangleleft$  - [50,50,50] 50)
  where similar'-syn x n y  $\equiv$  similar' n x y

lemma similar'-botI[intro!,simp]:  $\perp \triangleleft_n \perp$ 
  by (cases n) (auto simp add: similar'.simps)

lemma similar'-FnI[intro!]:
  assumes  $\bigwedge x y. x \triangleleft_n y \cdot C^\infty \implies f \cdot x \triangleleft_n g \cdot y \cdot C^\infty$ 
  shows  $Fn \cdot f \triangleleft_{Suc\ n} CFn \cdot g$ 
using assms by (auto simp add: similar'.simps)

lemma similar'-FnE[elim!]:
  assumes  $Fn \cdot f \triangleleft_{Suc\ n} CFn \cdot g$ 
  assumes  $(\bigwedge x y. x \triangleleft_n y \cdot C^\infty \implies f \cdot x \triangleleft_n g \cdot y \cdot C^\infty) \implies P$ 
  shows P
using assms by (auto simp add: similar'.simps)

lemma bot-or-not-bot':
   $x \triangleleft_n y \implies (x = \perp \longleftrightarrow y = \perp)$ 
  by (cases n) (auto simp add: similar'.simps elim: similar'-base.cases similar'-step.cases)

lemma similar'-bot[elim-format, elim!]:
   $\perp \triangleleft_n x \implies x = \perp$ 
   $y \triangleleft_n \perp \implies y = \perp$ 
by (metis bot-or-not-bot')+

lemma similar'-typed[simp]:
   $\neg B \cdot b \triangleleft_n CFn \cdot g$ 
   $\neg Fn \cdot f \triangleleft_n CB \cdot b$ 
  by (cases n, auto simp add: similar'.simps elim: similar'-base.cases similar'-step.cases)+

lemma similar'-bool[simp]:
   $B \cdot b_1 \triangleleft_{Suc\ n} CB \cdot b_2 \longleftrightarrow b_1 = b_2$ 
  by (auto simp add: similar'.simps elim: similar'-base.cases similar'-step.cases)

```

8.1.4 Moving up and down the similarity relations

These correspond to Lemma 7 in [SGHHOM11].

```

lemma similar'-down:  $d \triangleleft_{Suc\ n} e \implies \psi^D_n \cdot d \triangleleft_n \psi^E_n \cdot e$ 
  and similar'-up:  $d \triangleleft_n e \implies \psi^D_n \cdot d \triangleleft_{Suc\ n} \psi^E_n \cdot e$ 
proof (induction n arbitrary: d e)
  case (Suc n) case 1 with Suc

```

```

show ?case
  by (cases d e rule:value-CValue-cases) auto
next
  case (Suc n) case 2 with Suc
  show ?case
    by (cases d e rule:value-CValue-cases) auto
qed auto

```

A generalisation of the above, doing multiple steps at once.

```

lemma similar'-up-le:  $n \leq m \implies \psi^D_n \cdot d \triangleleft_n \psi^E_n \cdot e \implies \psi^D_n \cdot d \triangleleft_m \psi^E_n \cdot e$ 
  by (induction rule: dec-induct )
  (auto dest: similar'-up simp add: Value.take-take CValue.take-take min-absorb2)

```

```

lemma similar'-down-le:  $n \leq m \implies \psi^D_m \cdot d \triangleleft_m \psi^E_m \cdot e \implies \psi^D_n \cdot d \triangleleft_n \psi^E_n \cdot e$ 
  by (induction rule: inc-induct )
  (auto dest: similar'-down simp add: Value.take-take CValue.take-take min-absorb1)

```

```

lemma similar'-take:  $d \triangleleft_n e \implies \psi^D_n \cdot d \triangleleft_n \psi^E_n \cdot e$ 
  apply (drule similar'-up)
  apply (drule similar'-down)
  apply (simp add: Value.take-take CValue.take-take)
done

```

8.1.5 Admissibility

A technical prerequisite for induction is admissibility of the predicate, i.e. that the predicate holds for the limit of a chain, given that it holds for all elements.

```

lemma similar'-base-adm: adm ( $\lambda x. \text{similar}'\text{-base } (fst\ x) (snd\ x)$ )
proof (rule admI, goal-cases)
  case (1 Y)
  then have  $Y = (\lambda \cdot . \perp)$  by (metis prod.exhaust fst-eqD inst-prod-pcpo similar'-base.simps snd-eqD)
  thus ?case by auto
qed

```

```

lemma similar'-step-adm:
  assumes adm ( $\lambda x. s\ (fst\ x)\ (snd\ x)$ )
  shows adm ( $\lambda x. \text{similar}'\text{-step } s\ (fst\ x)\ (snd\ x)$ )
proof (rule admI, goal-cases)
  case prems: (1 Y)
  from ⟨chain Y⟩
  have chain ( $\lambda i. \text{fst } (Y\ i)$ ) by (rule ch2ch-fst)
  thus ?case
  proof(cases rule: Value-chainE)
  case bot
  hence *:  $\bigwedge i. \text{fst } (Y\ i) = \perp$  by metis
  with prems(2)[unfolded split-beta]
  have  $\bigwedge i. \text{snd } (Y\ i) = \perp$  by auto

```

```

hence  $Y = (\lambda i. (\perp, \perp))$  using * by (metis surjective-pairing)
thus ?thesis by auto
next
case (B n b)
hence  $\forall i. \text{fst } (Y (i + n)) = B \cdot b$  by (metis add.commute not-add-less1)
with prems(2)
have  $\forall i. Y (i + n) = (B \cdot b, CB \cdot b)$ 
  apply auto
  apply (erule-tac x = i + n in allE)
  apply (erule-tac x = i in allE)
  apply (erule similar'-step.cases)
  apply auto
  by (metis fst-conv old.prod.exhaust snd-conv)
hence similar'-step s ( $\text{fst } (\bigsqcup i. Y (i + n))$ ) ( $\text{snd } (\bigsqcup i. Y (i + n))$ ) by auto
thus ?thesis
  by (simp add: lub-range-shift[OF ‹chain Y›])
next
fix n
fix Y'
assume chain Y' and  $(\lambda i. \text{fst } (Y i)) = (\lambda m. (\text{if } m < n \text{ then } \perp \text{ else } \text{Fn} \cdot (Y' (m - n))))$ 
hence  $Y': \bigwedge i. \text{fst } (Y (i + n)) = \text{Fn} \cdot (Y' i)$  by (metis add-diff-cancel-right' not-add-less2)
with prems(2)[unfolded split-beta]
have  $\bigwedge i. \exists g'. \text{snd } (Y (i + n)) = \text{CFn} \cdot g'$ 
  by  $\text{--}(\text{erule-tac } x = i + n \text{ in allE, auto elim!: similar'-step.cases})$ 
then obtain  $Y''$  where  $Y'': \bigwedge i. \text{snd } (Y (i + n)) = \text{CFn} \cdot (Y'' i)$  by metis
from prems(1) have  $\bigwedge i. Y i \sqsubseteq Y (\text{Suc } i)$ 
  by (simp add: po-class.chain-def)
then have *:  $\bigwedge i. Y (i + n) \sqsubseteq Y (\text{Suc } i + n)$ 
  by simp
have chain Y''
  apply (rule chainI)
  apply (rule iffD1[OF CValue.inverts(1)])
  apply (subst (1 2) Y''[symmetric])
  apply (rule snd-monofun)
  apply (rule *)
  done

have similar'-step s ( $\text{Fn} \cdot (\bigsqcup i. (Y' i))$ ) ( $\text{CFn} \cdot (\bigsqcup i. Y'' i)$ )
proof (rule Fun-similar'-step)
  fix x y
  from prems(2)  $Y' Y''$ 
  have  $\bigwedge i. \text{similar'-step s } (\text{Fn} \cdot (Y' i)) (\text{CFn} \cdot (Y'' i))$  by metis
  moreover
  assume  $s x (y \cdot C^\infty)$ 
  ultimately
  have  $\bigwedge i. s (Y' i \cdot x) (Y'' i \cdot y \cdot C^\infty)$  by auto
  hence case-prod s ( $\bigsqcup i. ((Y' i) \cdot x, (Y'' i) \cdot y \cdot C^\infty)$ )
  apply  $\text{--}$ 
  apply (rule admD[OF adm-case-prod[where P =  $\lambda \cdot . s$ , OF assms]])

```

```

apply (simp add: ‹chain Y'› ‹chain Y''›)
apply simp
done
thus s (( $\sqcup$  i. Y' i)·x) (( $\sqcup$  i. Y'' i)·y·C∞)
  by (simp add: lub-Pair ch2ch-Rep-cfunL contlub-cfun-fun ‹chain Y'› ‹chain Y''›)
qed
hence similar'-step s (fst ( $\sqcup$  i. Y (i+n))) (snd ( $\sqcup$  i. Y (i+n)))
  by (simp add: Y' Y''
    cont2contlubE[OF cont-fst chain-shift[OF prems(1)]] cont2contlubE[OF cont-snd
chain-shift[OF prems(1)]]
    contlub-cfun-arg[OF ‹chain Y''›] contlub-cfun-arg[OF ‹chain Y'›])
thus similar'-step s (fst ( $\sqcup$  i. Y i)) (snd ( $\sqcup$  i. Y i))
  by (simp add: lub-range-shift[OF ‹chain Y'›])
qed
qed

```

lemma similar'-adm: adm (λx . fst x \triangleleft_n snd x)
by (induct n) (auto simp add: similar'.simps intro: similar'-base-adm similar'-step-adm)

lemma similar'-admI: cont f \implies cont g \implies adm (λx . f x \triangleleft_n g x)
by (rule adm-subst[OF similar'-adm, where t = λx . (f x, g x), simplified]) auto

8.1.6 The real similarity relation

This is the goal of the theory: A relation between *Value* and *CValue*.

definition similar :: Value \Rightarrow CValue \Rightarrow bool (**infix** \triangleleft 50) **where**
 $x \triangleleft y \iff (\forall n. \psi^D_n \cdot x \triangleleft_n \psi^E_n \cdot y)$

lemma similarI:
 $(\bigwedge n. \psi^D_n \cdot x \triangleleft_n \psi^E_n \cdot y) \implies x \triangleleft y$
unfolding similar-def **by** blast

lemma similarE:
 $x \triangleleft y \implies \psi^D_n \cdot x \triangleleft_n \psi^E_n \cdot y$
unfolding similar-def **by** blast

lemma similar-bot[simp]: $\perp \triangleleft \perp$ **by** (auto intro: similarI)

lemma similar-bool[simp]: $B \cdot b \triangleleft CB \cdot b$
by (rule similarI, case-tac n, auto)

lemma [elim-format, elim!]: $x \triangleleft \perp \implies x = \perp$
unfolding similar-def
apply (cases x)
apply auto
apply (erule-tac x = Suc 0 in allE, auto)+
done

lemma [*elim-format, elim!*]: $x \triangleleft CB \cdot b \implies x = B \cdot b$
unfolding *similar-def*
apply (*cases x*)
apply *auto*
apply (*erule-tac x = Suc 0 in allE, auto*)
done

lemma [*elim-format, elim!*]: $\perp \triangleleft y \implies y = \perp$
unfolding *similar-def*
apply (*cases y*)
apply *auto*
apply (*erule-tac x = Suc 0 in allE, auto*)
done

lemma [*elim-format, elim!*]: $B \cdot b \triangleleft y \implies y = CB \cdot b$
unfolding *similar-def*
apply (*cases y*)
apply *auto*
apply (*erule-tac x = Suc 0 in allE, auto*)
done

lemma *take-similar'-similar*:

assumes $x \triangleleft_n y$
shows $\psi^D_n \cdot x \triangleleft \psi^E_n \cdot y$
proof(*rule similarI*)
fix m
from *assms*
have $\psi^D_n \cdot x \triangleleft_n \psi^E_n \cdot y$ **by** (*rule similar'-take*)
moreover
have $n \leq m \vee m \leq n$ **by** *auto*
ultimately
show $\psi^D_m \cdot (\psi^D_n \cdot x) \triangleleft_m \psi^E_m \cdot (\psi^E_n \cdot y)$
by (*auto elim: similar'-up-le similar'-down-le dest: similar'-take*
simp add: min-absorb2 min-absorb1 Value.take-take CValue.take-take)
qed

lemma *bot-or-not-bot*:

$x \triangleleft y \implies (x = \perp \longleftrightarrow y = \perp)$
by (*cases x y rule:value-CValue-cases*) *auto*

lemma *bool-or-not-bool*:

$x \triangleleft y \implies (x = B \cdot b \longleftrightarrow (y = CB \cdot b))$
by (*cases x y rule:value-CValue-cases*) *auto*

lemma *similar-bot-cases*[*consumes 1, case-names bot bool Fn*]:

assumes $x \triangleleft y$
obtains $x = \perp \ y = \perp$ |
b where $x = B \cdot (Discr\ b) \ y = CB \cdot (Discr\ b)$ |

f g where $x = Fn.f y = CFn \cdot g$
using *assms*
by (*metis CValue.exhaust Value.exhaust bool-or-not-bool bot-or-not-bot discr.exhaust*)

lemma *similar-adm*: $adm (\lambda x. fst x \triangleleft snd x)$
unfolding *similar-def*
by (*intro adm-lemmas similar'-admI cont2cont*)

lemma *similar-admI*: $cont f \implies cont g \implies adm (\lambda x. f x \triangleleft g x)$
by (*rule adm-subst[OF - similar-adm, where t = $\lambda x. (f x, g x)$, simplified]*) *auto*

Having constructed the relation we can now show that it indeed is the desired relation, relating \perp with \perp and functions with functions, if they take related arguments to related values. This corresponds to Proposition 9 in [SGHHOM11].

lemma *similar-nice-def*: $x \triangleleft y \iff (x = \perp \wedge y = \perp \vee (\exists b. x = B \cdot (Discr b) \wedge y = CB \cdot (Discr b)) \vee (\exists f g. x = Fn.f \wedge y = CFn.g \wedge (\forall a b. a \triangleleft b \cdot C^\infty \implies f \cdot a \triangleleft g \cdot b \cdot C^\infty)))$
(is ?L \iff ?R)

proof

assume *?L*

thus *?R*

proof (*cases x y rule:similar-bot-cases*)

case *bot* **thus** *?thesis* **by** *simp*

next

case *bool* **thus** *?thesis* **by** *simp*

next

case (*Fn f g*)

note $\langle ?L \rangle$ [*unfolded Fn*]

have $\forall a b. a \triangleleft b \cdot C^\infty \implies f \cdot a \triangleleft g \cdot b \cdot C^\infty$

proof (*intro impI allI*)

fix *a b*

assume $a \triangleleft b \cdot C^\infty$

show $f \cdot a \triangleleft g \cdot b \cdot C^\infty$

proof (*rule similarI*)

fix *n*

have $adm (\lambda (b, a). \psi^D_n \cdot (f \cdot b) \triangleleft_n \psi^E_n \cdot (g \cdot a \cdot C^\infty))$

by (*intro adm-case-prod similar'-admI cont2cont*)

thus $\psi^D_n \cdot (f \cdot a) \triangleleft_n \psi^E_n \cdot (g \cdot b \cdot C^\infty)$

proof (*induct a b rule: Value-CValue-take-induct[consumes 1]*)

This take induction is required to avoid the wrong equation shown above.

fix *m*

from $\langle a \triangleleft b \cdot C^\infty \rangle$

have $\psi^D_m \cdot a \triangleleft_m \psi^E_m \cdot (b \cdot C^\infty)$ **by** (*rule similarE*)

hence $\psi^D_m \cdot a \triangleleft_{\max m n} \psi^E_m \cdot (b \cdot C^\infty)$ **by** (*rule similar'-up-le[rotated]*) *auto*

moreover

from $\langle Fn.f \triangleleft CFn.g \rangle$

```

      have  $\psi^D_{\text{Suc } (max\ m\ n)} \cdot (Fn \cdot f) \triangleleft_{\text{Suc } (max\ m\ n)} \psi^E_{\text{Suc } (max\ m\ n)} \cdot (CFn \cdot g)$  by (rule
similarE)
    ultimately
      have  $\psi^D_{max\ m\ n} \cdot (f \cdot (\psi^D_{max\ m\ n} \cdot (\psi^D_m \cdot a))) \triangleleft_{max\ m\ n} \psi^E_{max\ m\ n} \cdot (g \cdot (\psi^A_{max\ m\ n} \cdot (\psi^A_m \cdot b))) \cdot C^\infty$ 
      by auto
      hence  $\psi^D_{max\ m\ n} \cdot (f \cdot (\psi^D_m \cdot a)) \triangleleft_{max\ m\ n} \psi^E_{max\ m\ n} \cdot (g \cdot (\psi^A_m \cdot b)) \cdot C^\infty$ 
      by (simp add: Value.take-take cfun-map-map CValue.take-take ID-def eta-cfun
min-absorb2 min-absorb1)
      thus  $\psi^D_n \cdot (f \cdot (\psi^D_m \cdot a)) \triangleleft_n \psi^E_n \cdot (g \cdot (\psi^A_m \cdot b)) \cdot C^\infty$ 
      by (rule similar'-down-le[rotated]) auto
    qed
  qed
  qed
  thus ?thesis unfolding Fn by simp
qed
next
assume ?R
thus ?L
proof(elim conjE disjE exE ssubst)
  show  $\perp \triangleleft \perp$  by simp
next
  fix b
  show  $B \cdot (Discr\ b) \triangleleft CB \cdot (Discr\ b)$  by simp
next
  fix f g
  assume imp:  $\forall a\ b. a \triangleleft b \cdot C^\infty \longrightarrow f \cdot a \triangleleft g \cdot b \cdot C^\infty$ 
  show  $Fn \cdot f \triangleleft CFn \cdot g$ 
  proof (rule similarI)
    fix n
    show  $\psi^D_n \cdot (Fn \cdot f) \triangleleft_n \psi^E_n \cdot (CFn \cdot g)$ 
    proof(cases n)
      case 0 thus ?thesis by simp
    next
      case (Suc n)
      { fix x y
        assume  $x \triangleleft_n y \cdot C^\infty$ 
        hence  $\psi^D_n \cdot x \triangleleft \psi^E_n \cdot (y \cdot C^\infty)$  by (rule take-similar'-similar)
        hence  $f \cdot (\psi^D_n \cdot x) \triangleleft g \cdot (\psi^A_n \cdot y) \cdot C^\infty$  using imp by auto
        hence  $\psi^D_n \cdot (f \cdot (\psi^D_n \cdot x)) \triangleleft_n \psi^E_n \cdot (g \cdot (\psi^A_n \cdot y)) \cdot C^\infty$ 
        by (rule similarE)
      }
    with Suc
    show ?thesis by auto
  qed
  qed
  qed
  qed

```

lemma similar-FnI[intro]:

assumes $\bigwedge x y. x \triangleleft y \cdot C^\infty \implies f \cdot x \triangleleft g \cdot y \cdot C^\infty$
shows $Fn \cdot f \triangleleft CFn \cdot g$
by (*metis assms similar-nice-def*)

lemma *similar-FnD[elim!]*:
assumes $Fn \cdot f \triangleleft CFn \cdot g$
assumes $x \triangleleft y \cdot C^\infty$
shows $f \cdot x \triangleleft g \cdot y \cdot C^\infty$
using *assms*
by (*subst (asm) similar-nice-def*) *auto*

lemma *similar-FnE[elim!]*:
assumes $Fn \cdot f \triangleleft CFn \cdot g$
assumes $(\bigwedge x y. x \triangleleft y \cdot C^\infty \implies f \cdot x \triangleleft g \cdot y \cdot C^\infty) \implies P$
shows P
by (*metis assms similar-FnD*)

8.1.7 The similarity relation lifted pointwise to functions.

abbreviation *fun-similar* :: $('a::type \Rightarrow Value) \Rightarrow ('a \Rightarrow (C \rightarrow CValue)) \Rightarrow bool$ (**infix** \triangleleft^*
 50) **where**
fun-similar \equiv *pointwise* $(\lambda x y. x \triangleleft y \cdot C^\infty)$

lemma *fun-similar-fmap-bottom[simp]*: $\perp \triangleleft^* \perp$
by *auto*

lemma *fun-similarE[elim]*:
assumes $m \triangleleft^* m'$
assumes $(\bigwedge x. (m \ x) \triangleleft (m' \ x) \cdot C^\infty) \implies Q$
shows Q
using *assms unfolding pointwise-def* **by** *blast*

end

8.2 Denotational-Related

theory *Denotational-Related*
imports *Denotational ResourcedDenotational ValueSimilarity*
begin

Given the similarity relation it is straight-forward to prove that the standard and the re-sourced denotational semantics produce similar results. (Theorem 10 in [SGHHOM11]).

theorem *denotational-semantics-similar*:
assumes $\varrho \triangleleft^* \sigma$
shows $\llbracket e \rrbracket_\varrho \triangleleft (\mathcal{N} \llbracket e \rrbracket_\sigma) \cdot C^\infty$
using *assms*
proof (*induct e arbitrary: $\varrho \ \sigma$ rule:exp-induct*)
case (*Var v*)

```

from Var have  $\varrho v \triangleleft (\sigma v) \cdot C^\infty$  by cases auto
thus ?case by simp
next
case (Lam v e)
{ fix x y
  assume  $x \triangleleft y \cdot C^\infty$ 
  with  $\langle \varrho \triangleleft^* \sigma \rangle$ 
  have  $\varrho(v := x) \triangleleft^* \sigma(v := y)$ 
    by (auto 1 4)
  hence  $\llbracket e \rrbracket_{\varrho(v := x)} \triangleleft (\mathcal{N} \llbracket e \rrbracket_{\sigma(v := y)}) \cdot C^\infty$ 
    by (rule Lam.hyps)
}
thus ?case by auto
next
case (App e v ρ σ)
hence App':  $\llbracket e \rrbracket_\varrho \triangleleft (\mathcal{N} \llbracket e \rrbracket_\sigma) \cdot C^\infty$  by auto
thus ?case
proof (cases rule: similar-bot-cases)
  case (Fn f g)
  from  $\langle \varrho \triangleleft^* \sigma \rangle$ 
  have  $\varrho v \triangleleft (\sigma v) \cdot C^\infty$  by auto
  thus ?thesis using Fn App' by auto
qed auto
next
case (Bool b)
thus  $\llbracket \text{Bool } b \rrbracket_\varrho \triangleleft (\mathcal{N} \llbracket \text{Bool } b \rrbracket_\sigma) \cdot C^\infty$  by auto
next
case (IfThenElse scrut e1 e2)
hence IfThenElse':
   $\llbracket \text{scrut} \rrbracket_\varrho \triangleleft (\mathcal{N} \llbracket \text{scrut} \rrbracket_\sigma) \cdot C^\infty$ 
   $\llbracket e_1 \rrbracket_\varrho \triangleleft (\mathcal{N} \llbracket e_1 \rrbracket_\sigma) \cdot C^\infty$ 
   $\llbracket e_2 \rrbracket_\varrho \triangleleft (\mathcal{N} \llbracket e_2 \rrbracket_\sigma) \cdot C^\infty$  by auto
from IfThenElse'(1)
show ?case
proof (cases rule: similar-bot-cases)
  case (bool b)
  thus ?thesis using IfThenElse' by auto
qed auto
next
case (Let as e ρ σ)
have  $\{as\}_\varrho \triangleleft^* \mathcal{N} \{as\}_\sigma$ 
proof (rule parallel-HSem-ind-different-ESem[OF pointwise-adm[OF similar-admI] fun-similar-fmap-bottom])
  fix  $\varrho' :: \text{var} \Rightarrow \text{Value}$  and  $\sigma' :: \text{var} \Rightarrow C \rightarrow C\text{Value}$ 
  assume  $\varrho' \triangleleft^* \sigma'$ 
  show  $\varrho ++ \text{domA } as \llbracket as \rrbracket_{\varrho'} \triangleleft^* \sigma ++ \text{domA } as \text{evalHeap } as (\lambda e. \mathcal{N} \llbracket e \rrbracket_{\sigma'})$ 
  proof(rule pointwiseI, goal-cases)
  case (1 x)
  show ?case using  $\langle \varrho \triangleleft^* \sigma \rangle$ 
    by (auto simp add: lookup-override-on-eq lookupEvalHeap elim: Let(1)[OF -  $\langle \varrho' \triangleleft^* \sigma' \rangle$ ])

```

)
qed
qed auto
hence $\llbracket e \rrbracket_{\{\text{as}\}_\rho} \triangleleft (\mathcal{N}[\llbracket e \rrbracket_{\mathcal{N}\{\text{as}\}_\sigma}] \cdot C^\infty)$ **by** (*rule Let(2)*)
thus *?case by simp*
qed

corollary *evalHeap-similar*:

$\bigwedge y z. y \triangleleft^* z \implies \llbracket \Gamma \rrbracket_y \triangleleft^* \mathcal{N}[\llbracket \Gamma \rrbracket_z]$

by (*rule pointwiseI*)

(*case-tac x ∈ dom A Γ, auto simp add: lookupEvalHeap denotational-semantics-similar*)

theorem *heaps-similar*: $\{\Gamma\} \triangleleft^* \mathcal{N}\{\Gamma\}$

by (*rule parallel-HSem-ind-different-ESem[OF pointwise-adm[OF similar-admI]]*)

(*auto simp add: evalHeap-similar*)

end

9 Adequacy

9.1 ResourcedAdequacy

```
theory ResourcedAdequacy
imports ResourcedDenotational Launchbury AList–Utils CorrectnessResourced
begin
```

```
lemma demand-not-0: demand (N[e]ρ) ≠ ⊥
proof
  assume demand (N[e]ρ) = ⊥
  with demand-suffices'[where n = 0, simplified, OF this]
  have (N[e]ρ)·⊥ ≠ ⊥ by simp
  thus False by simp
qed
```

The semantics of an expression, given only r resources, will only use values from the environment with less resources.

```
lemma restr-can-restrict-env: (N[e]ρ)|C·r = (N[e]ρ|or})|C·r
proof(induction e arbitrary: ρ r rule: exp-induct)
  case (Var x)
  show ?case
  proof(rule C-restr-C-cong)
    fix r'
    assume r' ⊆ r
    have (N[Var x]ρ)·(C·r') = ρ x·r' by simp
    also have ... = ((ρ x)|r)·r' using ⟨r' ⊆ r⟩ by simp
    also have ... = (N[Var x]ρ|or})·(C·r') by simp
    finally show (N[Var x]ρ)·(C·r') = (N[Var x]ρ|or})·(C·r').
  qed simp
next
  case (Lam x e)
  show ?case
  proof(rule C-restr-C-cong)
    fix r'
    assume r' ⊆ r
    hence r' ⊆ C·r by (metis below-C below-trans)
    {
      fix v
      have ρ(x := v)|r = (ρ|or)(x := v)|r
      by simp
      hence (N[e]ρ(x := v)})|r' = (N[e](ρ|or)(x := v)})|r'
      by (subst (1 2) C-restr-eq-lower[OF Lam ⟨r' ⊆ C·r⟩]) simp
    }
    thus (N[Lam [x]. e]ρ)·(C·r') = (N[Lam [x]. e]ρ|or})·(C·r')
    by simp
  qed simp
next
```

case (*App e x*)
show *?case*
proof (*rule C-restr-C-cong*)
 fix r'
 assume $r' \sqsubseteq r$
 hence $r' \sqsubseteq C \cdot r$ **by** (*metis below-C below-trans*)
 hence $(\mathcal{N} \llbracket e \rrbracket_{\varrho}) \cdot r' = (\mathcal{N} \llbracket e \rrbracket_{\varrho|_r^\circ}) \cdot r'$
 by (*rule C-restr-eqD[OF App]*)
 thus $(\mathcal{N} \llbracket \text{App } e \ x \rrbracket_{\varrho}) \cdot (C \cdot r') = (\mathcal{N} \llbracket \text{App } e \ x \rrbracket_{\varrho|_r^\circ}) \cdot (C \cdot r')$
 using $\langle r' \sqsubseteq r \rangle$ **by** *simp*
 qed *simp*
next
case (*Bool b*)
show *?case* **by** *simp*
next
case (*IfThenElse scrut e₁ e₂*)
show *?case*
proof (*rule C-restr-C-cong*)
 fix r'
 assume $r' \sqsubseteq r$
 hence $r' \sqsubseteq C \cdot r$ **by** (*metis below-C below-trans*)

 have $(\mathcal{N} \llbracket \text{scrut} \rrbracket_{\varrho}) \cdot r' = (\mathcal{N} \llbracket \text{scrut} \rrbracket_{\varrho|_r^\circ}) \cdot r'$
 using $\langle r' \sqsubseteq C \cdot r \rangle$ **by** (*rule C-restr-eqD[OF IfThenElse(1)]*)
 moreover
 have $(\mathcal{N} \llbracket e_1 \rrbracket_{\varrho}) \cdot r' = (\mathcal{N} \llbracket e_1 \rrbracket_{\varrho|_r^\circ}) \cdot r'$
 using $\langle r' \sqsubseteq C \cdot r \rangle$ **by** (*rule C-restr-eqD[OF IfThenElse(2)]*)
 moreover
 have $(\mathcal{N} \llbracket e_2 \rrbracket_{\varrho}) \cdot r' = (\mathcal{N} \llbracket e_2 \rrbracket_{\varrho|_r^\circ}) \cdot r'$
 using $\langle r' \sqsubseteq C \cdot r \rangle$ **by** (*rule C-restr-eqD[OF IfThenElse(3)]*)
 ultimately
 show $(\mathcal{N} \llbracket (\text{scrut } ? e_1 : e_2) \rrbracket_{\varrho}) \cdot (C \cdot r') = (\mathcal{N} \llbracket (\text{scrut } ? e_1 : e_2) \rrbracket_{\varrho|_r^\circ}) \cdot (C \cdot r')$
 using $\langle r' \sqsubseteq r \rangle$ **by** *simp*
 qed *simp*
next
case (*Let Γ e*)

The lemma, lifted to heaps

have *restr-can-restrict-env-heap* : $\bigwedge r. (\mathcal{N} \llbracket \Gamma \rrbracket_{\varrho})|_r^\circ = (\mathcal{N} \llbracket \Gamma \rrbracket_{\varrho|_r^\circ})|_r^\circ$
proof(*rule has-ESem.parallel-HSem-ind*)
 fix $\varrho_1 \ \varrho_2 :: CEnv$ **and** $r :: C$
 assume $\varrho_1|_r^\circ = \varrho_2|_r^\circ$

 show $(\varrho \ ++_{domA \ \Gamma} \mathcal{N} \llbracket \Gamma \rrbracket_{\varrho_1})|_r^\circ = (\varrho|_r^\circ \ ++_{domA \ \Gamma} \mathcal{N} \llbracket \Gamma \rrbracket_{\varrho_2})|_r^\circ$
proof(*rule env-C-restr-cong*)
 fix x **and** r'
 assume $r' \sqsubseteq r$
 hence $r' \sqsubseteq C \cdot r$ **by** (*metis below-C below-trans*)

show $(\varrho \text{ ++ } \text{dom}A \Gamma \mathcal{N}[\Gamma]_{\varrho_1}) x \cdot r' = (\varrho|^\circ_r \text{ ++ } \text{dom}A \Gamma \mathcal{N}[\Gamma]_{\varrho_2}) x \cdot r'$
proof (*cases* $x \in \text{dom}A \Gamma$)
 case *True*
 have $(\mathcal{N}[\text{the } (\text{map-of } \Gamma \ x) \]_{\varrho_1}) \cdot r' = (\mathcal{N}[\text{the } (\text{map-of } \Gamma \ x) \]_{\varrho_1|^\circ_r}) \cdot r'$
 by (*rule C-restr-eqD[OF Let(1)[OF True] ⟨r' ⊆ C·r⟩*)
 also have $\dots = (\mathcal{N}[\text{the } (\text{map-of } \Gamma \ x) \]_{\varrho_2|^\circ_r}) \cdot r'$
 unfolding $\langle \varrho_1|^\circ_r = \varrho_2|^\circ_r \rangle \dots$
 also have $\dots = (\mathcal{N}[\text{the } (\text{map-of } \Gamma \ x) \]_{\varrho_2}) \cdot r'$
 by (*rule C-restr-eqD[OF Let(1)[OF True] ⟨r' ⊆ C·r⟩, symmetric*)
 finally
 show *?thesis using True by (simp add: lookupEvalHeap)*
next
 case *False*
 with $\langle r' \subseteq r \rangle$
 show *?thesis by simp*
qed
qed
qed *simp-all*

show *?case*
proof (*rule C-restr-C-cong*)
 fix r'
 assume $r' \subseteq r$
 hence $r' \subseteq C \cdot r$ **by** (*metis below-C below-trans*)

have $(\mathcal{N}\{\Gamma\}\varrho)|^\circ_r = (\mathcal{N}\{\Gamma\}(\varrho|^\circ_r))|^\circ_r$
 by (*rule restr-can-restrict-env-heap*)
 hence $(\mathcal{N}[e]_{\mathcal{N}\{\Gamma\}\varrho}) \cdot r' = (\mathcal{N}[e]_{\mathcal{N}\{\Gamma\}\varrho|^\circ_r}) \cdot r'$
 by (*subst (1 2) C-restr-eqD[OF Let(2) ⟨r' ⊆ C·r⟩] simp*)

thus $(\mathcal{N}[\text{Let } \Gamma \ e \]_{\varrho}) \cdot (C \cdot r') = (\mathcal{N}[\text{Let } \Gamma \ e \]_{\varrho|^\circ_r}) \cdot (C \cdot r')$
 using $\langle r' \subseteq r \rangle$ **by** *simp*
qed *simp*
qed

lemma *can-restrict-env:*

$(\mathcal{N}[e]_{\varrho}) \cdot (C \cdot r) = (\mathcal{N}[e]_{\varrho|^\circ_r}) \cdot (C \cdot r)$
by (*rule C-restr-eqD[OF restr-can-restrict-env below-refl]*)

When an expression e terminates, then we can remove such an expression from the heap and it still terminates. This is the crucial trick to handle black-holing in the resourced semantics.

lemma *add-BH:*

assumes *map-of* $\Gamma \ x = \text{Some } e$
assumes $(\mathcal{N}[e]_{\mathcal{N}\{\Gamma\}}) \cdot r' \neq \perp$
shows $(\mathcal{N}[e]_{\mathcal{N}\{\text{delete } x \ \Gamma\}}) \cdot r' \neq \perp$

proof-

obtain r **where** $r: C \cdot r = \text{demand } (\mathcal{N} \llbracket e \rrbracket_{\mathcal{N}\{\Gamma\}})$
using *demand-not-0* **by** (*cases demand* $(\mathcal{N} \llbracket e \rrbracket_{\mathcal{N}\{\Gamma\}})$) *auto*

from *assms(2)*
have $C \cdot r \sqsubseteq r'$ **unfolding** r *not-bot-demand* **by** *simp*

from *assms(1)*
have [*simp*]: *the* $(\text{map-of } \Gamma \ x) = e$ **by** (*metis option.sel*)

from *assms(1)*
have [*simp*]: $x \in \text{domA } \Gamma$ **by** (*metis domIff dom-map-of-conv-domA not-Some-eq*)

define ub **where** $ub = \mathcal{N}\{\Gamma\}$ — An upper bound for the induction

have *heaps*: $(\mathcal{N}\{\Gamma\})|_r^\circ \sqsubseteq \mathcal{N}\{\text{delete } x \ \Gamma\}$ **and** $\mathcal{N}\{\Gamma\} \sqsubseteq ub$

proof (*induction rule: HSem-bot-ind*)

fix ϱ
assume $\varrho|_r^\circ \sqsubseteq \mathcal{N}\{\text{delete } x \ \Gamma\}$
assume $\varrho \sqsubseteq ub$

show $(\mathcal{N} \llbracket \Gamma \rrbracket_{\varrho})|_r^\circ \sqsubseteq \mathcal{N}\{\text{delete } x \ \Gamma\}$

proof (*rule fun-belowI*)

fix y
show $((\mathcal{N} \llbracket \Gamma \rrbracket_{\varrho})|_r^\circ) \ y \sqsubseteq (\mathcal{N}\{\text{delete } x \ \Gamma\}) \ y$

proof (*cases y = x*)

case *True*
have $((\mathcal{N} \llbracket \Gamma \rrbracket_{\varrho})|_r^\circ) \ x = (\mathcal{N} \llbracket e \rrbracket_{\varrho})|_r$
by (*simp add: lookupEvalHeap*)
also have $\dots \sqsubseteq (\mathcal{N} \llbracket e \rrbracket_{ub})|_r$
using $\langle \varrho \sqsubseteq ub \rangle$ **by** (*intro monofun-cfun-arg*)
also have $\dots = (\mathcal{N} \llbracket e \rrbracket_{\mathcal{N}\{\Gamma\}})|_r$
unfolding *ub-def..*
also have $\dots = \perp$
using r **by** (*rule C-restr-bot-demand[OF eq-imp-below]*)
also have $\dots = (\mathcal{N}\{\text{delete } x \ \Gamma\}) \ x$
by (*simp add: lookup-HSem-other*)
finally
show *?thesis* **unfolding** *True.*

next

case *False*
show *?thesis*

proof (*cases y \in domA \Gamma*)

case *True*
have $(\mathcal{N} \llbracket \text{the } (\text{map-of } \Gamma \ y) \rrbracket_{\varrho})|_r = (\mathcal{N} \llbracket \text{the } (\text{map-of } \Gamma \ y) \rrbracket_{\varrho|_r^\circ})|_r$
by (*rule C-restr-eq-lower[OF restr-can-restrict-env below-C]*)
also have $\dots \sqsubseteq \mathcal{N} \llbracket \text{the } (\text{map-of } \Gamma \ y) \rrbracket_{\varrho|_r^\circ}$
by (*rule C-restr-below*)
also note $\langle \varrho|_r^\circ \sqsubseteq \mathcal{N}\{\text{delete } x \ \Gamma\} \rangle$

```

finally
show ?thesis
  using ⟨y ∈ domA Γ⟩ ⟨y ≠ x⟩
  by (simp add: lookupEvalHeap lookup-HSem-heap)
next
  case False
  thus ?thesis by simp
qed
qed
qed

from ⟨ρ ⊆ ub⟩
have (N[[Γ]]_ρ) ⊆ (N[[Γ]]_ub)
  by (rule cont2monofunE[rotated]) simp
also have ... = ub
  unfolding ub-def HSem-bot-eq[symmetric]..
finally
show (N[[Γ]]_ρ) ⊆ ub.
qed simp-all

from assms(2)
have (N[[e]]_N[[Γ]])·(C·r) ≠ ⊥
  unfolding r
  by (rule demand-suffices[OF infinite-resources-suffice])
also
have (N[[e]]_N[[Γ]])·(C·r) = (N[[e]]_(N[[Γ]]|°_r))·(C·r)
  by (rule can-restrict-env)
also
have ... ⊆ (N[[e]]_N[[delete x Γ]])·(C·r)
  by (intro monofun-cfun-arg monofun-cfun-fun heaps )
also
have ... ⊆ (N[[e]]_N[[delete x Γ]])·r'
  using ⟨C·r ⊆ r'⟩ by (rule monofun-cfun-arg)
finally
show ?thesis by this (intro cont2cont)+
qed

```

The semantics is continuous, so we can apply induction here:

```

lemma resourced-adequacy:
  assumes (N[[e]]_N[[Γ]])·r ≠ ⊥
  shows ∃ Δ v. Γ : e ↓S Δ : v
using assms
proof (induction r arbitrary: Γ e S rule: C.induct[case-names adm bot step])
  case adm show ?case by simp
next
  case bot
  hence False by auto
  thus ?case..

```

next
case (*step r*)
show *?case*
proof(*cases e rule:exp-strong-exhaust(1)*)[**where** $c = (\Gamma, S)$, *case-names Var App Let Lam Bool IfThenElse*]
case (*Var x*)
let *?e = the (map-of Γ x)*
from *step.premis[unfolded Var]*
have $x \in \text{domA } \Gamma$
by (*auto intro: ccontr simp add: lookup-HSem-other*)
hence *map-of Γ x = Some ?e* **by** (*rule domA-map-of-Some-the*)
moreover
from *step.premis[unfolded Var] $\langle \text{map-of } \Gamma \text{ x} = \text{Some } ?e \rangle \langle x \in \text{domA } \Gamma \rangle$*
have $(\mathcal{N}[\![?e]\!]_{\mathcal{N}\{\Gamma\}}) \cdot r \neq \perp$ **by** (*auto simp add: lookup-HSem-heap simp del: app-strict*)
hence $(\mathcal{N}[\![?e]\!]_{\mathcal{N}\{\text{delete } x \Gamma\}}) \cdot r \neq \perp$ **by** (*rule add-BH[OF $\langle \text{map-of } \Gamma \text{ x} = \text{Some } ?e \rangle$]*)
from *step.IH[OF this]*
obtain Δ *v* **where** *delete x Γ : ?e $\Downarrow_x \# S \Delta : v$* **by** *blast*
ultimately
have $\Gamma : (\text{Var } x) \Downarrow_S (x, v) \# \Delta : v$ **by** (*rule Variable*)
thus *?thesis using Var by auto*
next
case (*App e' x*)
have *finite (set S \cup fv (Γ , e'))* **by** *simp*
from *finite-list[OF this]*
obtain S' **where** *S': set S' = set S \cup fv (Γ , e')..*

from *step.premis[unfolded App]*
have *prem: (($\mathcal{N}[\![e']\!]_{\mathcal{N}\{\Gamma\}}) \cdot r \Downarrow_{CFn (\mathcal{N}\{\Gamma\}) x|_r}) \cdot r \neq \perp$* **by** (*auto simp del: app-strict*)
hence $(\mathcal{N}[\![e']\!]_{\mathcal{N}\{\Gamma\}}) \cdot r \neq \perp$ **by** *auto*
from *step.IH[OF this]*
obtain Δ *v* **where** *lhs': $\Gamma : e' \Downarrow_{S'} \Delta : v$* **by** *blast*

have *fv (Γ , e') \subseteq set S'* **using** *S'* **by** *auto*
from *correctness-empty-env[OF lhs' this]*
have *correct1: $\mathcal{N}[\![e']\!]_{\mathcal{N}\{\Gamma\}} \sqsubseteq \mathcal{N}[\![v]\!]_{\mathcal{N}\{\Delta\}}$ and $\mathcal{N}\{\Gamma\} \sqsubseteq \mathcal{N}\{\Delta\}$* **by** *auto*

from *prem*
have $((\mathcal{N}[\![v]\!]_{\mathcal{N}\{\Delta\}}) \cdot r \Downarrow_{CFn (\mathcal{N}\{\Gamma\}) x|_r}) \cdot r \neq \perp$
by (*rule not-bot-below-trans*)(*intro correct1 monofun-cfun-fun monofun-cfun-arg*)
with *result-evaluated[OF lhs']*
have *isLam v* **by** (*cases r, auto, cases v rule: isVal.cases, auto*)
then obtain y e'' **where** $n': v = (\text{Lam } [y]. e'')$ **by** (*rule isLam-obtain-fresh*)
with *lhs'*
have *lhs: $\Gamma : e' \Downarrow_{S'} \Delta : \text{Lam } [y]. e''$* **by** *simp*

have $((\mathcal{N}[\![v]\!]_{\mathcal{N}\{\Delta\}}) \cdot r \Downarrow_{CFn (\mathcal{N}\{\Gamma\}) x|_r}) \cdot r \neq \perp$ **by** *fact*
also have $(\mathcal{N}\{\Gamma\}) x|_r \sqsubseteq (\mathcal{N}\{\Gamma\}) x$ **by** (*rule C-restr-below*)
also note $\langle v = \rightarrow$

also note $\langle \mathcal{N}\{\Gamma\} \sqsubseteq \mathcal{N}\{\Delta\} \rangle$
also have $(\mathcal{N}\llbracket \text{Lam } [y]. e'' \rrbracket_{\mathcal{N}\{\Delta\}} \cdot r \sqsubseteq \text{CFn} \cdot (\Lambda v. \mathcal{N}\llbracket e'' \rrbracket_{\mathcal{N}\{\Delta\}}(y := v))$
by (rule *CELam-no-restr*)
also have $(\dots \downarrow \text{CFn } (\mathcal{N}\{\Delta\}) x) \cdot r = (\mathcal{N}\llbracket e'' \rrbracket_{\mathcal{N}\{\Delta\}}(y := ((\mathcal{N}\{\Delta\}) x))) \cdot r$ **by** *simp*
also have $\dots = (\mathcal{N}\llbracket e''[y::=x] \rrbracket_{\mathcal{N}\{\Delta\}} \cdot r$
unfolding *ESem-subst..*
finally
have $\dots \neq \perp$ **by** *this* (intro *cont2cont cont-fun*) +
then
obtain $\Theta v'$ **where** *rhs*: $\Delta : e''[y::=x] \downarrow_{S'} \Theta : v'$ **using** *step.IH* **by** *blast*

have $\Gamma : \text{App } e' x \downarrow_{S'} \Theta : v'$
by (rule *reds-ApplicationI[OF lhs rhs]*)
hence $\Gamma : \text{App } e' x \downarrow_{S'} \Theta : v'$
apply (rule *reds-smaller-L*) **using** *S'* **by** *auto*
thus *?thesis* **using** *App* **by** *auto*

next
case (*Lam v e'*)
have $\Gamma : \text{Lam } [v]. e' \downarrow_{S'} \Gamma : \text{Lam } [v]. e' ..$
thus *?thesis* **using** *Lam* **by** *blast*

next
case (*Bool b*)
have $\Gamma : \text{Bool } b \downarrow_{S'} \Gamma : \text{Bool } b$ **by** *rule*
thus *?thesis* **using** *Bool* **by** *blast*

next
case (*IfThenElse scrut e1 e2*)

from *step.prem**s[unfolded IfThenElse]*
have *prem*: $\text{CB-project} \cdot ((\mathcal{N}\llbracket \text{scrut} \rrbracket_{\mathcal{N}\{\Gamma\}} \cdot r) \cdot ((\mathcal{N}\llbracket e_1 \rrbracket_{\mathcal{N}\{\Gamma\}} \cdot r) \cdot ((\mathcal{N}\llbracket e_2 \rrbracket_{\mathcal{N}\{\Gamma\}} \cdot r) \neq \perp$ **by**
(auto simp del: app-strict)

then obtain *b* **where**
is-CB: $(\mathcal{N}\llbracket \text{scrut} \rrbracket_{\mathcal{N}\{\Gamma\}} \cdot r = \text{CB} \cdot (\text{Discr } b)$
and *not-bot2*: $((\mathcal{N}\llbracket (\text{if } b \text{ then } e_1 \text{ else } e_2) \rrbracket_{\mathcal{N}\{\Gamma\}} \cdot r) \neq \perp$
unfolding *CB-project-not-bot* **by** (*auto split: if-splits*)

have *finite* (*set S* \cup *fv* (Γ , *scrut*)) **by** *simp*
from *finite-list[OF this]*
obtain *S'* **where** *S'*: *set S'* = *set S* \cup *fv* (Γ , *scrut*)..

from *is-CB* **have** $(\mathcal{N}\llbracket \text{scrut} \rrbracket_{\mathcal{N}\{\Gamma\}} \cdot r \neq \perp$ **by** *simp*
from *step.IH[OF this]*
obtain Δv **where** *lhs'*: $\Gamma : \text{scrut} \downarrow_{S'} \Delta : v$ **by** *blast*
then have *isVal v* **by** (rule *result-evaluated*)

have *fv* (Γ , *scrut*) \subseteq *set S'* **using** *S'* **by** *simp*
from *correctness-empty-env[OF lhs' this]*
have *correct1*: $\mathcal{N}\llbracket \text{scrut} \rrbracket_{\mathcal{N}\{\Gamma\}} \sqsubseteq \mathcal{N}\llbracket v \rrbracket_{\mathcal{N}\{\Delta\}}$ **and** *correct2*: $\mathcal{N}\{\Gamma\} \sqsubseteq \mathcal{N}\{\Delta\}$ **by** *auto*

```

from correct1
have  $(\mathcal{N}[\llbracket \text{scrut } \llbracket \mathcal{N}\{\Gamma\} \rrbracket \cdot r \sqsubseteq (\mathcal{N}[\llbracket v \llbracket \mathcal{N}\{\Delta\} \rrbracket \cdot r \text{ by (rule monofun-cfun-fun)}$ 
with is-CB
have  $(\mathcal{N}[\llbracket v \llbracket \mathcal{N}\{\Delta\} \rrbracket \cdot r = \text{CB} \cdot (\text{Discr } b) \text{ by simp}$ 
with  $\langle \text{isVal } v \rangle$ 
have  $v = \text{Bool } b \text{ by (cases v rule: isVal.cases) (case-tac r, auto)+}$ 

from not-bot2  $\langle \mathcal{N}\{\Gamma\} \sqsubseteq \mathcal{N}\{\Delta\} \rangle$ 
have  $(\mathcal{N}[\llbracket (\text{if } b \text{ then } e_1 \text{ else } e_2) \llbracket \mathcal{N}\{\Delta\} \rrbracket \cdot r \neq \perp$ 
  by  $(\text{rule not-bot-below-trans}[\text{OF - monofun-cfun-fun}[\text{OF monofun-cfun-arg}]])$ 
from step.IH  $[\text{OF this}]$ 
obtain  $\Theta \ v' \text{ where rhs: } \Delta : (\text{if } b \text{ then } e_1 \text{ else } e_2) \Downarrow_{S'} \Theta : v' \text{ by blast}$ 

from lhs'  $[\text{unfolded } \langle v = - \rangle] \text{ rhs}$ 
have  $\Gamma : (\text{scrut } ? e_1 : e_2) \Downarrow_{S'} \Theta : v' \text{ by rule}$ 
hence  $\Gamma : (\text{scrut } ? e_1 : e_2) \Downarrow_S \Theta : v'$ 
  apply  $(\text{rule reds-smaller-L}) \text{ using } S' \text{ by auto}$ 
thus ?thesis unfolding IfThenElse by blast
next
case  $(\text{Let } \Delta \ e')$ 
from step.prem  $[\text{unfolded } \text{Let}(2)]$ 
have prem:  $(\mathcal{N}[\llbracket e' \llbracket \mathcal{N}\{\Delta\} \llbracket \mathcal{N}\{\Gamma\} \rrbracket \cdot r \neq \perp$ 
  by  $(\text{simp del: app-strict})$ 
also
  have atom  $\langle \text{domA } \Delta \ \#* \ \Gamma \text{ using } \text{Let}(1) \text{ by (simp add: fresh-star-Pair)}$ 
  hence  $\mathcal{N}\{\Delta\} \llbracket \mathcal{N}\{\Gamma\} = \mathcal{N}\{\Delta \ @ \ \Gamma\} \text{ by (rule HSem-merge)}$ 
finally
have  $(\mathcal{N}[\llbracket e' \llbracket \mathcal{N}\{\Delta \ @ \ \Gamma\} \rrbracket \cdot r \neq \perp.$ 
then
obtain  $\Theta \ v \text{ where } \Delta \ @ \ \Gamma : e' \Downarrow_S \Theta : v \text{ using step.IH by blast}$ 
hence  $\Gamma : \text{Let } \Delta \ e' \Downarrow_S \Theta : v$ 
  by  $(\text{rule reds.Let}[\text{OF } \text{Let}(1)])$ 
thus ?thesis using Let by auto
qed
qed
end

```

9.2 Adequacy

```

theory Adequacy
imports ResourcedAdequacy Denotational-Related
begin

```

```

theorem adequacy:
  assumes  $\llbracket e \rrbracket_{\Gamma} \neq \perp$ 
  shows  $\exists \Delta \ v. \Gamma : e \Downarrow_S \Delta : v$ 

```

proof–
have $\{\Gamma\} \triangleleft^* \mathcal{N}\{\Gamma\}$ **by** (*rule heaps-similar*)
hence $\llbracket e \rrbracket_{\{\Gamma\}} \triangleleft (\mathcal{N}\llbracket e \rrbracket_{\mathcal{N}\{\Gamma\}}) \cdot C^\infty$ **by** (*rule denotational-semantics-similar*)
from *bot-or-not-bot[OF this] assms*
have $(\mathcal{N}\llbracket e \rrbracket_{\mathcal{N}\{\Gamma\}}) \cdot C^\infty \neq \perp$ **by** *metis*
thus *?thesis* **by** (*rule resourced-adequacy*)
qed

end

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