

# Lattice Properties

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## Abstract

This formalization introduces and collects some algebraic structures based on lattices and complete lattices for use in other developments. The structures introduced are modular, and lattice ordered groups. In addition to the results proved for the new lattices, this formalization also introduces theorems about lattices and complete lattices in general.

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## 1 Overview

Section 2 introduces well founded and transitive relations. Section 3 introduces some properties about fixpoints of monotonic application which maps monotonic functions to monotonic functions. The most important property is that such a monotonic application has the least fixpoint monotonic. Section 4 introduces conjunctive, disjunctive, universally conjunctive, and universally disjunctive functions. In section 5 some simplification lemmas for lattices are proved. Section 6 introduces modular lattices and proves some

properties about them and about distributive lattices. The main result of this section is that a lattice is distributive if and only if it satisfies

$$\forall x y z : x \sqcap z = y \sqcap z \wedge x \sqcup z = y \sqcup z \longrightarrow x = y$$

Section 7 introduces lattice ordered groups and some of their properties. The most important is that they are distributive lattices, and this property is proved using the results from Section 5.

## 2 Well founded and transitive relations

**theory** *WellFoundedTransitive*

**imports** *Main*

**begin**

**class** *transitive* = *ord* +

**assumes** *order-trans1*:  $x < y \implies y < z \implies x < z$

**and** *less-eq-def*:  $x \leq y \iff x = y \vee x < y$

**begin**

**lemma** *eq-less-eq* [*simp*]:

$x = y \implies x \leq y$

*<proof>*

**lemma** *order-trans2* [*simp*]:

$x \leq y \implies y < z \implies x < z$

*<proof>*

**lemma** *order-trans3*:

$x < y \implies y \leq z \implies x < z$

*<proof>*

**end**

**class** *well-founded* = *ord* +

**assumes** *less-induct1* [*case-names less*]:  $(!!x . (!!y . y < x \implies P y) \implies P x) \implies P a$

**class** *well-founded-transitive* = *transitive* + *well-founded*

**instantiation** *prod*:: (*ord*, *ord*) *ord*

**begin**

**definition**

*less-pair-def*:  $a < b \iff \text{fst } a < \text{fst } b \vee (\text{fst } a = \text{fst } b \wedge \text{snd } a < \text{snd } b)$

**definition**

*less-eq-pair-def*:  $(a::('a::ord * 'b::ord)) \leq b \iff a = b \vee a < b$

**instance** *<proof>*

**end**

```

instantiation prod:: (transitive, transitive) transitive
begin
instance ⟨proof⟩
end

```

```

instantiation prod:: (well-founded, well-founded) well-founded
begin
instance ⟨proof⟩
end

```

```

instantiation prod:: (well-founded-transitive, well-founded-transitive) well-founded-transitive
begin
instance ⟨proof⟩
end

```

```

instantiation nat :: transitive
begin

```

```

instance ⟨proof⟩
end

```

```

instantiation nat:: well-founded
begin
instance ⟨proof⟩
end

```

```

instantiation nat:: well-founded-transitive
begin
instance ⟨proof⟩
end

```

```

end

```

### 3 Fixpoints and Complete Lattices

```

theory Complete-Lattice-Prop
imports WellFoundedTransitive
begin

```

This theory introduces some results about fixpoints of functions on complete lattices. The main result is that a monotonic function mapping monotonic functions to monotonic functions has the least fixpoint monotonic.

```

context complete-lattice begin

```

```

lemma inf-Inf: assumes nonempty:  $A \neq \{\}$ 
shows  $\text{inf } x \ (\text{Inf } A) = \text{Inf } ((\text{inf } x) \text{ ` } A)$ 
  ⟨proof⟩

```

**end**

**definition**

$mono\text{-}mono\ F = (mono\ F \wedge (\forall f . mono\ f \longrightarrow mono\ (F\ f)))$

**theorem** *lfp-mono* [*simp*]:

$mono\text{-}mono\ F \Longrightarrow mono\ (lfp\ F)$   
*<proof>*

**lemma** *gfp-ordinal-induct*:

**fixes**  $f :: 'a::complete\text{-}lattice \Rightarrow 'a$

**assumes**  $mono: mono\ f$

**and**  $P\text{-}f: !!S. P\ S \Longrightarrow P\ (f\ S)$

**and**  $P\text{-}Union: !!M. \forall S \in M. P\ S \Longrightarrow P\ (Inf\ M)$

**shows**  $P\ (gfp\ f)$

*<proof>*

**theorem** *gfp-mono* [*simp*]:

$mono\text{-}mono\ F \Longrightarrow mono\ (gfp\ F)$   
*<proof>*

**context** *complete-lattice* **begin**

**definition**

$Sup\text{-}less\ x\ (w::'b::well\text{-}founded) = Sup\ \{y :: 'a . \exists v < w . y = x\ v\}$

**lemma** *Sup-less-upper*:

$v < w \Longrightarrow P\ v \leq Sup\text{-}less\ P\ w$   
*<proof>*

**lemma** *Sup-less-least*:

$(!!v . v < w \Longrightarrow P\ v \leq Q) \Longrightarrow Sup\text{-}less\ P\ w \leq Q$   
*<proof>*

**end**

**lemma** *Sup-less-fun-eq*:

$((Sup\text{-}less\ P\ w)\ i) = (Sup\text{-}less\ (\lambda v . P\ v\ i))\ w$   
*<proof>*

**theorem** *fp-wf-induction*:

$f\ x = x \Longrightarrow mono\ f \Longrightarrow (\forall w . (y\ w) \leq f\ (Sup\text{-}less\ y\ w)) \Longrightarrow Sup\ (range\ y) \leq x$   
*<proof>*

end

## 4 Conjunctive and Disjunctive Functions

**theory** *Conj-Disj*  
**imports** *Main*  
**begin**

This theory introduces the definitions and some properties for conjunctive, disjunctive, universally conjunctive, and universally disjunctive functions.

**locale** *conjunctive* =  
  **fixes** *inf-b* :: 'b  $\Rightarrow$  'b  $\Rightarrow$  'b  
  **and** *inf-c* :: 'c  $\Rightarrow$  'c  $\Rightarrow$  'c  
  **and** *times-abc* :: 'a  $\Rightarrow$  'b  $\Rightarrow$  'c  
**begin**

**definition**

*conjunctive* =  $\{x . (\forall y z . \text{times-abc } x (\text{inf-b } y z) = \text{inf-c } (\text{times-abc } x y) (\text{times-abc } x z))\}$

**lemma** *conjunctiveI*:

**assumes**  $(\bigwedge b c . \text{times-abc } a (\text{inf-b } b c) = \text{inf-c } (\text{times-abc } a b) (\text{times-abc } a c))$   
  **shows**  $a \in \text{conjunctive}$   
   $\langle \text{proof} \rangle$

**lemma** *conjunctiveD*:  $x \in \text{conjunctive} \Longrightarrow \text{times-abc } x (\text{inf-b } y z) = \text{inf-c } (\text{times-abc } x y) (\text{times-abc } x z)$   
   $\langle \text{proof} \rangle$

end

**interpretation** *Apply*: *conjunctive inf::'a::semilattice-inf  $\Rightarrow$  'a  $\Rightarrow$  'a*  
*inf::'b::semilattice-inf  $\Rightarrow$  'b  $\Rightarrow$  'b  $\lambda f . f$*   
   $\langle \text{proof} \rangle$

**interpretation** *Comp*: *conjunctive inf::('a::lattice  $\Rightarrow$  'a)  $\Rightarrow$  ('a  $\Rightarrow$  'a)  $\Rightarrow$  ('a  $\Rightarrow$  'a)*  
*inf::('a::lattice  $\Rightarrow$  'a)  $\Rightarrow$  ('a  $\Rightarrow$  'a)  $\Rightarrow$  ('a  $\Rightarrow$  'a) (o)*  
   $\langle \text{proof} \rangle$

**lemma** *Apply.conjunctive = Comp.conjunctive*  
   $\langle \text{proof} \rangle$

**locale** *disjunctive* =  
  **fixes** *sup-b* :: 'b  $\Rightarrow$  'b  $\Rightarrow$  'b  
  **and** *sup-c* :: 'c  $\Rightarrow$  'c  $\Rightarrow$  'c  
  **and** *times-abc* :: 'a  $\Rightarrow$  'b  $\Rightarrow$  'c  
**begin**

**definition**

$disjunctive = \{x . (\forall y z . times-abc x (sup-b y z) = sup-c (times-abc x y) (times-abc x z))\}$

**lemma** *disjunctiveI*:

**assumes**  $(\bigwedge b c . times-abc a (sup-b b c) = sup-c (times-abc a b) (times-abc a c))$   
**shows**  $a \in disjunctive$   
 $\langle proof \rangle$

**lemma** *disjunctiveD*:  $x \in disjunctive \implies times-abc x (sup-b y z) = sup-c (times-abc x y) (times-abc x z)$

$\langle proof \rangle$

**end**

**interpretation** *Apply*:  $disjunctive\ sup::'a::semilattice-sup \Rightarrow 'a \Rightarrow 'a$

$sup::'b::semilattice-sup \Rightarrow 'b \Rightarrow 'b \lambda f . f$   
 $\langle proof \rangle$

**interpretation** *Comp*:  $disjunctive\ sup::('a::lattice \Rightarrow 'a) \Rightarrow ('a \Rightarrow 'a) \Rightarrow ('a \Rightarrow 'a)$

$sup::('a::lattice \Rightarrow 'a) \Rightarrow ('a \Rightarrow 'a) \Rightarrow ('a \Rightarrow 'a) (o)$   
 $\langle proof \rangle$

**lemma** *apply-comp-disjunctive*:  $Apply.disjunctive = Comp.disjunctive$

$\langle proof \rangle$

**locale** *Conjunctive* =

**fixes**  $Inf-b :: 'b\ set \Rightarrow 'b$   
**and**  $Inf-c :: 'c\ set \Rightarrow 'c$   
**and**  $times-abc :: 'a \Rightarrow 'b \Rightarrow 'c$

**begin**

**definition**

$Conjunctive = \{x . (\forall X . times-abc x (Inf-b X) = Inf-c ((times-abc x) ' X))\}$

**lemma** *ConjunctiveI*:

**assumes**  $\bigwedge A . times-abc a (Inf-b A) = Inf-c ((times-abc a) ' A)$   
**shows**  $a \in Conjunctive$   
 $\langle proof \rangle$

**lemma** *ConjunctiveD*:

**assumes**  $a \in Conjunctive$   
**shows**  $times-abc a (Inf-b A) = Inf-c ((times-abc a) ' A)$   
 $\langle proof \rangle$

**end**

**interpretation** *Apply: Conjunctive Inf*  $\text{Inf } \lambda f . f$   
*<proof>*

**interpretation** *Comp: Conjunctive Inf*::( $'a::\text{complete-lattice} \Rightarrow 'a$ ) *set*)  $\Rightarrow ('a \Rightarrow 'a)$   
*Inf*::( $'a::\text{complete-lattice} \Rightarrow 'a$ ) *set*)  $\Rightarrow ('a \Rightarrow 'a)$  (*o*)  
*<proof>*

**lemma** *Apply.Conjunctive = Comp.Conjunctive*  
*<proof>*

**locale** *Disjunctive =*  
  **fixes** *Sup-b* ::  $'b$  *set*  $\Rightarrow 'b$   
  **and** *Sup-c* ::  $'c$  *set*  $\Rightarrow 'c$   
  **and** *times-abc* ::  $'a \Rightarrow 'b \Rightarrow 'c$   
**begin**

**definition**

*Disjunctive =*  $\{x . (\forall X . \text{times-abc } x (\text{Sup-b } X) = \text{Sup-c } ((\text{times-abc } x) ' X))\}$

**lemma** *DisjunctiveI*:

**assumes**  $\bigwedge A . \text{times-abc } a (\text{Sup-b } A) = \text{Sup-c } ((\text{times-abc } a) ' A)$   
  **shows**  $a \in \text{Disjunctive}$   
  *<proof>*

**lemma** *DisjunctiveD*:  $x \in \text{Disjunctive} \Longrightarrow \text{times-abc } x (\text{Sup-b } X) = \text{Sup-c } ((\text{times-abc } x) ' X)$   
*<proof>*

**end**

**interpretation** *Apply: Disjunctive Sup*  $\text{Sup } \lambda f . f$   
*<proof>*

**interpretation** *Comp: Disjunctive Sup*::( $'a::\text{complete-lattice} \Rightarrow 'a$ ) *set*)  $\Rightarrow ('a \Rightarrow 'a)$   
*Sup*::( $'a::\text{complete-lattice} \Rightarrow 'a$ ) *set*)  $\Rightarrow ('a \Rightarrow 'a)$  (*o*)  
*<proof>*

**lemma** *Apply.Disjunctive = Comp.Disjunctive*  
*<proof>*

**lemma** [*simp*]:  $(F::'a::\text{complete-lattice} \Rightarrow 'b::\text{complete-lattice}) \in \text{Apply.Conjunctive} \Longrightarrow F \in \text{Apply.conjunctive}$   
*<proof>*

**lemma** [*simp*]:  $F \in \text{Apply.conjunctive} \Longrightarrow \text{mono } F$   
*<proof>*

**lemma** [simp]:  $(F::'a::\text{complete-lattice} \Rightarrow 'b::\text{complete-lattice}) \in \text{Apply.Conjunctive}$   
 $\Longrightarrow F \text{ top} = \text{top}$   
 ⟨proof⟩

**lemma** [simp]:  $(F::'a::\text{complete-lattice} \Rightarrow 'b::\text{complete-lattice}) \in \text{Apply.Disjunctive}$   
 $\Longrightarrow F \in \text{Apply.disjunctive}$   
 ⟨proof⟩

**lemma** [simp]:  $F \in \text{Apply.disjunctive} \Longrightarrow \text{mono } F$   
 ⟨proof⟩

**lemma** [simp]:  $(F::'a::\text{complete-lattice} \Rightarrow 'b::\text{complete-lattice}) \in \text{Apply.Disjunctive}$   
 $\Longrightarrow F \text{ bot} = \text{bot}$   
 ⟨proof⟩

**lemma** weak-fusion:  $h \in \text{Apply.Disjunctive} \Longrightarrow \text{mono } f \Longrightarrow \text{mono } g \Longrightarrow$   
 $h \circ f \leq g \circ h \Longrightarrow h (\text{lfp } f) \leq \text{lfp } g$   
 ⟨proof⟩

**lemma** inf-Disj:  $(\lambda (x::'a::\text{complete-distrib-lattice}) . \text{inf } x \ y) \in \text{Apply.Disjunctive}$   
 ⟨proof⟩

end

## 5 Simplification Lemmas for Lattices

**theory** Lattice-Prop  
**imports** Main  
**begin**

This theory introduces some simplification lemmas for semilattices and lattices

**notation**

$\text{inf}$  (infixl  $\sqcap$  70) and  
 $\text{sup}$  (infixl  $\sqcup$  65)

**context** semilattice-inf **begin**

**lemma** [simp]:  $(x \sqcap y) \sqcap z \leq x$   
 ⟨proof⟩

**lemma** [simp]:  $x \sqcap y \sqcap z \leq y$   
 ⟨proof⟩

**lemma** [simp]:  $x \sqcap (y \sqcap z) \leq y$   
 ⟨proof⟩

**lemma** [simp]:  $x \sqcap (y \sqcap z) \leq z$   
 ⟨proof⟩

**end**



**context** *semilattice-sup* **begin**

**lemma** [*simp*]:  $x \leq x \sqcup y \sqcup z$   
*<proof>*

**lemma** [*simp*]:  $y \leq x \sqcup y \sqcup z$   
*<proof>*

**lemma** [*simp*]:  $y \leq x \sqcup (y \sqcup z)$   
*<proof>*

**lemma** [*simp*]:  $z \leq x \sqcup (y \sqcup z)$   
*<proof>*

**end**

**context** *lattice* **begin**

**lemma** [*simp*]:  $x \sqcap y \leq x \sqcup z$   
*<proof>*

**lemma** [*simp*]:  $y \sqcap x \leq x \sqcup z$   
*<proof>*

**lemma** [*simp*]:  $x \sqcap y \leq z \sqcup x$   
*<proof>*

**lemma** [*simp*]:  $y \sqcap x \leq z \sqcup x$   
*<proof>*

**end**

**end**

## 6 Modular and Distributive Lattices

**theory** *Modular-Distrib-Lattice*

**imports** *Lattice-Prop*

**begin**

The main result of this theory is the fact that a lattice is distributive if and only if it satisfies the following property:

**term**  $(\forall x y z . x \sqcap z = y \sqcap z \wedge x \sqcup z = y \sqcup z \implies x = y)$

This result was proved by Bergmann in [1]. The formalization presented here is based on [3, 4].

**class** *modular-lattice* = *lattice* +

**assumes** *modular*:  $x \leq y \implies x \sqcup (y \sqcap z) = y \sqcap (x \sqcup z)$

**context** *distrib-lattice* **begin**

**subclass** *modular-lattice*

*<proof>*

**end**

**context** *lattice* **begin**

**definition**

$$d\text{-aux } a \ b \ c = (a \sqcap b) \sqcup (b \sqcap c) \sqcup (c \sqcap a)$$

**lemma** *d-b-c-a*:  $d\text{-aux } b \ c \ a = d\text{-aux } a \ b \ c$

*<proof>*

**lemma** *d-c-a-b*:  $d\text{-aux } c \ a \ b = d\text{-aux } a \ b \ c$

*<proof>*

**definition**

$$e\text{-aux } a \ b \ c = (a \sqcup b) \sqcap (b \sqcup c) \sqcap (c \sqcup a)$$

**lemma** *e-b-c-a*:  $e\text{-aux } b \ c \ a = e\text{-aux } a \ b \ c$

*<proof>*

**lemma** *e-c-a-b*:  $e\text{-aux } c \ a \ b = e\text{-aux } a \ b \ c$

*<proof>*

**definition**

$$a\text{-aux } a \ b \ c = (a \sqcap (e\text{-aux } a \ b \ c)) \sqcup (d\text{-aux } a \ b \ c)$$

**definition**

$$b\text{-aux } a \ b \ c = (b \sqcap (e\text{-aux } a \ b \ c)) \sqcup (d\text{-aux } a \ b \ c)$$

**definition**

$$c\text{-aux } a \ b \ c = (c \sqcap (e\text{-aux } a \ b \ c)) \sqcup (d\text{-aux } a \ b \ c)$$

**lemma** *b-a*:  $b\text{-aux } a \ b \ c = a\text{-aux } b \ c \ a$

*<proof>*

**lemma** *c-a*:  $c\text{-aux } a \ b \ c = a\text{-aux } c \ a \ b$

*<proof>*

**lemma** [*simp*]:  $a\text{-aux } a \ b \ c \leq e\text{-aux } a \ b \ c$

*<proof>*

**lemma** [*simp*]:  $b\text{-aux } a \ b \ c \leq e\text{-aux } a \ b \ c$

*<proof>*

**lemma** [*simp*]:  $c\text{-aux } a \ b \ c \leq e\text{-aux } a \ b \ c$

*<proof>*

**lemma** [*simp*]:  $d\text{-aux } a \ b \ c \leq a\text{-aux } a \ b \ c$   
*<proof>*

**lemma** [*simp*]:  $d\text{-aux } a \ b \ c \leq b\text{-aux } a \ b \ c$   
*<proof>*

**lemma** [*simp*]:  $d\text{-aux } a \ b \ c \leq c\text{-aux } a \ b \ c$   
*<proof>*

**lemma** *a-meet-e*:  $a \sqcap (e\text{-aux } a \ b \ c) = a \sqcap (b \sqcup c)$   
*<proof>*

**lemma** *b-meet-e*:  $b \sqcap (e\text{-aux } a \ b \ c) = b \sqcap (c \sqcup a)$   
*<proof>*

**lemma** *c-meet-e*:  $c \sqcap (e\text{-aux } a \ b \ c) = c \sqcap (a \sqcup b)$   
*<proof>*

**lemma** *a-join-d*:  $a \sqcup d\text{-aux } a \ b \ c = a \sqcup (b \sqcap c)$   
*<proof>*

**lemma** *b-join-d*:  $b \sqcup d\text{-aux } a \ b \ c = b \sqcup (c \sqcap a)$   
*<proof>*

**end**

**context** *lattice* **begin**

**definition**

*no-distrib*  $a \ b \ c = (a \sqcap b \sqcup c \sqcap a < a \sqcap (b \sqcup c))$

**definition**

*incomp*  $x \ y = (\neg x \leq y \wedge \neg y \leq x)$

**definition**

*N5-lattice*  $a \ b \ c = (a \sqcap c = b \sqcap c \wedge a < b \wedge a \sqcup c = b \sqcup c)$

**definition**

*M5-lattice*  $a \ b \ c = (a \sqcap b = b \sqcap c \wedge c \sqcap a = b \sqcap c \wedge a \sqcup b = b \sqcup c \wedge c \sqcup a = b \sqcup c \wedge a \sqcap b < a \sqcup b)$

**lemma** *M5-lattice-incomp*:  $M5\text{-lattice } a \ b \ c \implies \text{incomp } a \ b$   
*<proof>*

**end**

**context** *modular-lattice* **begin**

**lemma** *a-meet-d*:  $a \sqcap (d\text{-aux } a \ b \ c) = (a \sqcap b) \sqcup (c \sqcap a)$   
*<proof>*

**lemma** *b-meet-d*:  $b \sqcap (d\text{-aux } a \ b \ c) = (b \sqcap c) \sqcup (a \sqcap b)$   
 ⟨proof⟩

**lemma** *c-meet-d*:  $c \sqcap (d\text{-aux } a \ b \ c) = (c \sqcap a) \sqcup (b \sqcap c)$   
 ⟨proof⟩

**lemma** *d-less-e*: *no-distrib*  $a \ b \ c \implies d\text{-aux } a \ b \ c < e\text{-aux } a \ b \ c$   
 ⟨proof⟩

**lemma** *a-meet-b-eq-d*:  $d\text{-aux } a \ b \ c \leq e\text{-aux } a \ b \ c \implies a\text{-aux } a \ b \ c \sqcap b\text{-aux } a \ b \ c = d\text{-aux } a \ b \ c$   
 ⟨proof⟩

**lemma** *b-meet-c-eq-d*:  $d\text{-aux } a \ b \ c \leq e\text{-aux } a \ b \ c \implies b\text{-aux } a \ b \ c \sqcap c\text{-aux } a \ b \ c = d\text{-aux } a \ b \ c$   
 ⟨proof⟩

**lemma** *c-meet-a-eq-d*:  $d\text{-aux } a \ b \ c \leq e\text{-aux } a \ b \ c \implies c\text{-aux } a \ b \ c \sqcap a\text{-aux } a \ b \ c = d\text{-aux } a \ b \ c$   
 ⟨proof⟩

**lemma** *a-def-equiv*:  $d\text{-aux } a \ b \ c \leq e\text{-aux } a \ b \ c \implies a\text{-aux } a \ b \ c = (a \sqcup d\text{-aux } a \ b \ c) \sqcap e\text{-aux } a \ b \ c$   
 ⟨proof⟩

**lemma** *b-def-equiv*:  $d\text{-aux } a \ b \ c \leq e\text{-aux } a \ b \ c \implies b\text{-aux } a \ b \ c = (b \sqcup d\text{-aux } a \ b \ c) \sqcap e\text{-aux } a \ b \ c$   
 ⟨proof⟩

**lemma** *c-def-equiv*:  $d\text{-aux } a \ b \ c \leq e\text{-aux } a \ b \ c \implies c\text{-aux } a \ b \ c = (c \sqcup d\text{-aux } a \ b \ c) \sqcap e\text{-aux } a \ b \ c$   
 ⟨proof⟩

**lemma** *a-join-b-eq-e*:  $d\text{-aux } a \ b \ c \leq e\text{-aux } a \ b \ c \implies a\text{-aux } a \ b \ c \sqcup b\text{-aux } a \ b \ c = e\text{-aux } a \ b \ c$   
 ⟨proof⟩

**lemma** *b-join-c-eq-e*:  $d\text{-aux } a \ b \ c \leq e\text{-aux } a \ b \ c \implies b\text{-aux } a \ b \ c \sqcup c\text{-aux } a \ b \ c = e\text{-aux } a \ b \ c$   
 ⟨proof⟩

**lemma** *c-join-a-eq-e*:  $d\text{-aux } a \ b \ c \leq e\text{-aux } a \ b \ c \implies c\text{-aux } a \ b \ c \sqcup a\text{-aux } a \ b \ c = e\text{-aux } a \ b \ c$   
 ⟨proof⟩

**lemma** *no-distrib*  $a \ b \ c \implies \text{incomp } a \ b$   
 ⟨proof⟩

**lemma** *M5-modular: no-distrib*  $a b c \implies M5\text{-lattice } (a\text{-aux } a b c) (b\text{-aux } a b c)$   
 $(c\text{-aux } a b c)$   
 $\langle proof \rangle$

**lemma** *M5-modular-def*:  $M5\text{-lattice } a b c = (a \sqcap b = b \sqcap c \wedge c \sqcap a = b \sqcap c \wedge a \sqcup b = b \sqcup c \wedge c \sqcup a = b \sqcup c \wedge a \sqcap b < a \sqcup b)$   
 $\langle proof \rangle$

**end**

**context** *lattice* **begin**

**lemma** *not-modular-N5*:  $(\neg \text{class.modular-lattice inf } ((\leq)::'a \Rightarrow 'a \Rightarrow \text{bool}) (<) \text{sup}) =$   
 $(\exists a b c::'a . N5\text{-lattice } a b c)$   
 $\langle proof \rangle$

**lemma** *not-distrib-N5-M5*:  $(\neg \text{class.distrib-lattice } (\sqcap) ((\leq)::'a \Rightarrow 'a \Rightarrow \text{bool}) (<) (\sqcup)) =$   
 $(\exists a b c::'a . N5\text{-lattice } a b c) \vee (\exists a b c::'a . M5\text{-lattice } a b c)$   
 $\langle proof \rangle$

**lemma** *distrib-not-N5-M5*:  $(\text{class.distrib-lattice } (\sqcap) ((\leq)::'a \Rightarrow 'a \Rightarrow \text{bool}) (<) (\sqcup)) =$   
 $(\forall a b c::'a . \neg N5\text{-lattice } a b c) \wedge (\forall a b c::'a . \neg M5\text{-lattice } a b c)$   
 $\langle proof \rangle$

**lemma** *distrib-inf-sup-eq*:  
 $(\text{class.distrib-lattice } (\sqcap) ((\leq)::'a \Rightarrow 'a \Rightarrow \text{bool}) (<) (\sqcup)) =$   
 $(\forall x y z::'a . x \sqcap z = y \sqcap z \wedge x \sqcup z = y \sqcup z \longrightarrow x = y)$   
 $\langle proof \rangle$

**end**

**class** *inf-sup-eq-lattice* = *lattice* +  
**assumes** *inf-sup-eq*:  $x \sqcap z = y \sqcap z \implies x \sqcup z = y \sqcup z \implies x = y$   
**begin**  
**subclass** *distrib-lattice*  
 $\langle proof \rangle$

**end**

**end**

## 7 Lattice Orderd Groups

**theory** *Lattice-Ordered-Group*  
**imports** *Modular-Distrib-Lattice*  
**begin**

This theory introduces lattice ordered groups [2] and proves some results about them. The most important result is that a lattice ordered group is also a distributive lattice.

```
class lgroup = group-add + lattice +
assumes add-order-preserving:  $a \leq b \implies u + a + v \leq u + b + v$ 
begin
```

```
lemma add-order-preserving-left:  $a \leq b \implies u + a \leq u + b$ 
  <proof>
```

```
lemma add-order-preserving-right:  $a \leq b \implies a + v \leq b + v$ 
  <proof>
```

```
lemma minus-order:  $-a \leq -b \implies b \leq a$ 
  <proof>
```

```
lemma right-move-to-left:  $a + -c \leq b \implies a \leq b + c$ 
  <proof>
```

```
lemma right-move-to-right:  $a \leq b + -c \implies a + c \leq b$ 
  <proof>
```

```
lemma [simp]:  $(a \sqcap b) + c = (a + c) \sqcap (b + c)$ 
  <proof>
```

```
lemma [simp]:  $(a \sqcap b) - c = (a - c) \sqcap (b - c)$ 
  <proof>
```

```
lemma left-move-to-left:  $-c + a \leq b \implies a \leq c + b$ 
  <proof>
```

```
lemma left-move-to-right:  $a \leq -c + b \implies c + a \leq b$ 
  <proof>
```

```
lemma [simp]:  $c + (a \sqcap b) = (c + a) \sqcap (c + b)$ 
  <proof>
```

```
lemma [simp]:  $-(a \sqcap b) = (-a) \sqcup (-b)$ 
  <proof>
```

```
lemma [simp]:  $(a \sqcup b) + c = (a + c) \sqcup (b + c)$ 
  <proof>
```

```
lemma [simp]:  $c + (a \sqcup b) = (c + a) \sqcup (c + b)$ 
  <proof>
```

**lemma** [*simp*]:  $c - (a \sqcap b) = (c - a) \sqcup (c - b)$   
*<proof>*

**lemma** [*simp*]:  $(a \sqcup b) - c = (a - c) \sqcup (b - c)$   
*<proof>*

**lemma** [*simp*]:  $-(a \sqcup b) = (-a) \sqcap (-b)$   
*<proof>*

**lemma** [*simp*]:  $c - (a \sqcup b) = (c - a) \sqcap (c - b)$   
*<proof>*

**lemma** *add-pos*:  $0 \leq a \implies b \leq b + a$   
*<proof>*

**lemma** *add-pos-left*:  $0 \leq a \implies b \leq a + b$   
*<proof>*

**lemma** *inf-sup*:  $a - (a \sqcap b) + b = a \sqcup b$   
*<proof>*

**lemma** *inf-sup-2*:  $b = (a \sqcap b) - a + (a \sqcup b)$   
*<proof>*

**subclass** *inf-sup-eq-lattice*  
*<proof>*

**end**

**end**

## References

- [1] G. Bergmann. Zur axiomatik der elementargeometrie. *Monatshefte für Mathematik*, 36:269–284, 1929. 10.1007/BF02307616.
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- [4] S. Burris and H. P. Sankappanavar. *A course in universal algebra*, volume 78 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1981.