Laplace Transform

Fabian Immler

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Abstract

This entry formalizes the Laplace transform and concrete Laplace transforms for arithmetic functions, frequency shift, integration and (higher) differentiation in the time domain. It proves Lerch's lemma and uniqueness of the Laplace transform for continuous functions. In order to formalize the foundational assumptions, this entry contains a formalization of piecewise continuous functions and functions of exponential order.

Contents

theory *Laplace-Transform-Library*
 imports
 HOL−Analysis.Analysis
begin

1 References

Much of this formalization is based on Schiff's textbook [3]. Parts of this formalization are inspired by the HOL-Light formalization ([4], [1], [2]), but stated more generally for piecewise continuous (instead of piecewise continuously differentiable) functions.

2 Library Additions

2.1 Derivatives

lemma *DERIV-compose-FDERIV*:— TODO: generalize and move from HOL-ODE
  assumes $DERIV\ f\ (g\ x) :> f'$
  assumes $(g\ \mathit{has\text{-}derivative}\ g')\ (\mathit{at}\ x\ \mathit{within}\ s)$
  shows $((\lambda x.\ f\ (g\ x))\ \mathit{has\text{-}derivative}\ (\lambda x.\ g'\ x * f'))\ (\mathit{at}\ x\ \mathit{within}\ s)$
  ⟨*proof*⟩

lemmas *has-derivative-sin*[*derivative-intros*] = *DERIV-sin*[*THEN DERIV-compose-FDERIV*]
 and *has-derivative-cos*[*derivative-intros*] = *DERIV-cos*[*THEN DERIV-compose-FDERIV*]
 and *has-derivative-exp*[*derivative-intros*] = *DERIV-exp*[*THEN DERIV-compose-FDERIV*]

2.2 Integrals

lemma *negligible-real-ivlI*:
  fixes $a\ b{::}\mathit{real}$
  assumes $a \geq b$
  shows $\mathit{negligible}\ \{a\ ..\ b\}$
⟨*proof*⟩

lemma *absolutely-integrable-on-combine*:
  fixes $f :: \mathit{real} \Rightarrow 'a{::}\mathit{euclidean\text{-}space}$
  assumes $f\ \mathit{absolutely\text{-}integrable\text{-}on}\ \{a..c\}$
    and $f\ \mathit{absolutely\text{-}integrable\text{-}on}\ \{c..b\}$
    and $a \leq c$
    and $c \leq b$
  shows $f\ \mathit{absolutely\text{-}integrable\text{-}on}\ \{a..b\}$
  ⟨*proof*⟩

lemma *dominated-convergence-at-top*:
  fixes $f :: \mathit{real} \Rightarrow 'n{::}\mathit{euclidean\text{-}space} \Rightarrow 'm{::}\mathit{euclidean\text{-}space}$
  assumes f: $\bigwedge k.\ (f\ k)\ \mathit{integrable\text{-}on}\ s$ **and** h: $h\ \mathit{integrable\text{-}on}\ s$
    and le: $\bigwedge k\ x.\ x \in s \Longrightarrow \mathit{norm}\ (f\ k\ x) \leq h\ x$
    and conv: $\forall x \in s.\ ((\lambda k.\ f\ k\ x) \longrightarrow g\ x)\ \mathit{at\text{-}top}$
  shows $g\ \mathit{integrable\text{-}on}\ s\ ((\lambda k.\ \mathit{integral}\ s\ (f\ k)) \longrightarrow \mathit{integral}\ s\ g)\ \mathit{at\text{-}top}$
⟨*proof*⟩

lemma *has-integral-dominated-convergence-at-top*:

fixes f :: *real* $\Rightarrow$ $'n$::*euclidean-space* $\Rightarrow$ $'m$::*euclidean-space*
assumes $\bigwedge k$. (f k *has-integral* y k) s h *integrable-on* s
$\bigwedge k$ x. $x{\in}s \Longrightarrow$ *norm* (f k x) $\le$ h x $\forall x{\in}s$. ((λk. f k x) $\longrightarrow$ g x) *at-top*
and x: (y $\longrightarrow$ x) *at-top*
shows (g *has-integral* x) s
⟨*proof*⟩

lemma *integral-indicator-eq-restriction*:
fixes f::$'a$::*euclidean-space* $\Rightarrow$ $'b$::*banach*
assumes f: f *integrable-on* R
and $R \subseteq S$
shows *integral* S (λx. *indicator* R x $*_R$ f x) = *integral* R f
⟨*proof*⟩

lemma
improper-integral-at-top:
fixes f::*real* $\Rightarrow$ $'a$::*euclidean-space*
assumes f *absolutely-integrable-on* $\{a..\}$
shows ((λx. *integral* $\{a..x\}$ f) $\longrightarrow$ *integral* $\{a..\}$ f) *at-top*
⟨*proof*⟩

lemma *norm-integrable-onI*: (λx. *norm* (f x)) *integrable-on* S
if f *absolutely-integrable-on* S
for f::$'a$::*euclidean-space*$\Rightarrow$$'b$::*euclidean-space*
⟨*proof*⟩

lemma
has-integral-improper-at-topI:
fixes f::*real* $\Rightarrow$ $'a$::*banach*
assumes I: $\forall_F$ k *in at-top*. (f *has-integral* I k) $\{a..k\}$
assumes J: (I $\longrightarrow$ J) *at-top*
shows (f *has-integral* J) $\{a..\}$
⟨*proof*⟩

lemma *has-integral-improperE*:
fixes f::*real* $\Rightarrow$ $'a$::*euclidean-space*
assumes I: (f *has-integral* I) $\{a..\}$
assumes *ai*: f *absolutely-integrable-on* $\{a..\}$
obtains J **where**
$\bigwedge k$. (f *has-integral* J k) $\{a..k\}$
(J $\longrightarrow$ I) *at-top*
⟨*proof*⟩

2.3 Miscellaneous

lemma *AE-BallI*: *AE* $x{\in}X$ *in* F. P x **if** $\forall x \in X$. P x
⟨*proof*⟩

lemma *bounded-le-Sup*:

assumes *bounded* $(f \text{ ` } S)$
shows $\forall\, x{\in}S.\ norm\ (f\ x) \le Sup\ (norm \text{ ` } f \text{ ` } S)$
$\langle proof \rangle$

end

3 Piecewise Continous Functions

theory *Piecewise-Continuous*
imports
Laplace-Transform-Library
begin

3.1 at within filters

lemma *at-within-self-singleton*[*simp*]: $at\ i\ within\ \{i\} = bot$
$\langle proof \rangle$

lemma *at-within-t1-space-avoid*:
$(at\ x\ within\ X - \{i\}) = (at\ x\ within\ X)$ **if** $x \neq i$ **for** $x\ i{::}'a{::}t1\text{-}space$
$\langle proof \rangle$

lemma *at-within-t1-space-avoid-finite*:
$(at\ x\ within\ X - I) = (at\ x\ within\ X)$ **if** $finite\ I\ x \notin I$ **for** $x{::}'a{::}t1\text{-}space$
$\langle proof \rangle$

lemma *at-within-interior*:
$NO\text{-}MATCH\ (UNIV{::}'a\ set)\ (S{::}'a{::}topological\text{-}space\ set) \Longrightarrow x \in interior\ S \Longrightarrow at\ x\ within\ S = at\ x$
$\langle proof \rangle$

3.2 intervals

lemma *Compl-Icc*: $-\ \{a\ ..\ b\} = \{..{<}a\} \cup \{b{<}..\}$ **for** $a\ b{::}'a{::}linorder$
$\langle proof \rangle$

lemma *interior-Icc*[*simp*]: $interior\ \{a..b\} = \{a{<}..{<}b\}$
for $a\ b{::}'a{::}\{linorder\text{-}topology,\ dense\text{-}order,\ no\text{-}bot,\ no\text{-}top\}$
— TODO: is *no-bot* and *no-top* really required?
$\langle proof \rangle$

lemma *closure-finite*[*simp*]: $closure\ X = X$ **if** $finite\ X$ **for** $X{::}'a{::}t1\text{-}space\ set$
$\langle proof \rangle$

definition $piecewise\text{-}continuous\text{-}on :: 'a{::}linorder\text{-}topology \Rightarrow 'a \Rightarrow 'a\ set \Rightarrow ('a \Rightarrow 'b{::}topological\text{-}space) \Rightarrow bool$
where $piecewise\text{-}continuous\text{-}on\ a\ b\ I\ f \longleftrightarrow$
$(continuous\text{-}on\ (\{a\ ..\ b\} - I)\ f \wedge finite\ I \wedge$
$(\forall\, i{\in}I.\ (i \in \{a{<}..b\} \longrightarrow (\exists\, l.\ (f \longrightarrow l)\ (at\text{-}left\ i))) \wedge$

$(i \in \{a..<b\} \longrightarrow (\exists u.\ (f \longrightarrow u)\ (\textit{at-right } i)))))$

lemma *piecewise-continuous-on-subset*:
piecewise-continuous-on $a\ b\ I\ f \Longrightarrow \{c\ ..\ d\} \subseteq \{a\ ..\ b\} \Longrightarrow$ *piecewise-continuous-on* $c\ d\ I\ f$
⟨*proof*⟩

lemma *piecewise-continuous-onE*:
assumes *piecewise-continuous-on* $a\ b\ I\ f$
obtains $l\ u$
where *finite* I
and *continuous-on* $(\{a..b\} - I)\ f$
and $(\bigwedge i.\ i \in I \Longrightarrow a < i \Longrightarrow i \leq b \Longrightarrow (f \longrightarrow l\ i)\ (\textit{at-left } i))$
and $(\bigwedge i.\ i \in I \Longrightarrow a \leq i \Longrightarrow i < b \Longrightarrow (f \longrightarrow u\ i)\ (\textit{at-right } i))$
⟨*proof*⟩

lemma *piecewise-continuous-onI*:
assumes *finite* I *continuous-on* $(\{a..b\} - I)\ f$
and $(\bigwedge i.\ i \in I \Longrightarrow a < i \Longrightarrow i \leq b \Longrightarrow (f \longrightarrow l\ i)\ (\textit{at-left } i))$
and $(\bigwedge i.\ i \in I \Longrightarrow a \leq i \Longrightarrow i < b \Longrightarrow (f \longrightarrow u\ i)\ (\textit{at-right } i))$
shows *piecewise-continuous-on* $a\ b\ I\ f$
⟨*proof*⟩

lemma *piecewise-continuous-onI'*:
fixes $a\ b$::'a::{*linorder-topology*, *dense-order*, *no-bot*, *no-top*}
assumes *finite* I $\bigwedge x.\ a < x \Longrightarrow x < b \Longrightarrow$ *isCont* $f\ x$
and $a \notin I \Longrightarrow$ *continuous* (*at-right* a) f
and $b \notin I \Longrightarrow$ *continuous* (*at-left* b) f
and $(\bigwedge i.\ i \in I \Longrightarrow a < i \Longrightarrow i \leq b \Longrightarrow (f \longrightarrow l\ i)\ (\textit{at-left } i))$
and $(\bigwedge i.\ i \in I \Longrightarrow a \leq i \Longrightarrow i < b \Longrightarrow (f \longrightarrow u\ i)\ (\textit{at-right } i))$
shows *piecewise-continuous-on* $a\ b\ I\ f$
⟨*proof*⟩

lemma *piecewise-continuous-onE'*:
fixes $a\ b$::'a::{*linorder-topology*, *dense-order*, *no-bot*, *no-top*}
assumes *piecewise-continuous-on* $a\ b\ I\ f$
obtains $l\ u$
where *finite* I
and $\bigwedge x.\ a < x \Longrightarrow x < b \Longrightarrow x \notin I \Longrightarrow$ *isCont* $f\ x$
and $(\bigwedge x.\ a < x \Longrightarrow x \leq b \Longrightarrow (f \longrightarrow l\ x)\ (\textit{at-left } x))$
and $(\bigwedge x.\ a \leq x \Longrightarrow x < b \Longrightarrow (f \longrightarrow u\ x)\ (\textit{at-right } x))$
and $\bigwedge x.\ a \leq x \Longrightarrow x \leq b \Longrightarrow x \notin I \Longrightarrow f\ x = l\ x$
and $\bigwedge x.\ a \leq x \Longrightarrow x \leq b \Longrightarrow x \notin I \Longrightarrow f\ x = u\ x$
⟨*proof*⟩

lemma *tendsto-avoid-at-within*:
$(f \longrightarrow l)\ (\textit{at } x \textit{ within } X)$
if $(f \longrightarrow l)\ (\textit{at } x \textit{ within } X - \{x\})$
⟨*proof*⟩

lemma *tendsto-within-subset-eventuallyI*:
$(f \longrightarrow fx)$ $(at\ x\ within\ X)$
if g: $(g \longrightarrow gy)$ $(at\ y\ within\ Y)$
and ev: $\forall_F\ x\ in\ (at\ y\ within\ Y).\ f\ x = g\ x$
and xy: $x = y$
and $fxgy$: $fx = gy$
and XY: $X - \{x\} \subseteq Y$
⟨*proof*⟩

lemma *piecewise-continuous-on-insertE*:
assumes *piecewise-continuous-on* $a\ b\ (insert\ i\ I)\ f$
assumes $i \in \{a\ ..\ b\}$
obtains $g\ h$ **where**
piecewise-continuous-on $a\ i\ I\ g$
piecewise-continuous-on $i\ b\ I\ h$
$\bigwedge x.\ a \le x \Longrightarrow x < i \Longrightarrow g\ x = f\ x$
$\bigwedge x.\ i < x \Longrightarrow x \le b \Longrightarrow h\ x = f\ x$
⟨*proof*⟩

lemma *eventually-avoid-finite*:
$\forall_F\ x\ in\ at\ y\ within\ Y.\ x \notin I$ **if** *finite* I **for** $y::'a::t1\text{-}space$
⟨*proof*⟩

lemma *eventually-at-left-linorder*:— TODO: generalize $?b < ?a \Longrightarrow \forall_F\ x\ in\ at\text{-}left\ ?a.\ x \in \{?b<..<?a\}$
$a > (b :: 'a :: linorder\text{-}topology) \Longrightarrow eventually\ (\lambda x.\ x \in \{b<..<a\})\ (at\text{-}left\ a)$
⟨*proof*⟩

lemma *eventually-at-right-linorder*:— TODO: generalize $?a < ?b \Longrightarrow \forall_F\ x\ in\ at\text{-}right\ ?a.\ x \in \{?a<..<?b\}$
$a > (b :: 'a :: linorder\text{-}topology) \Longrightarrow eventually\ (\lambda x.\ x \in \{b<..<a\})\ (at\text{-}right\ b)$
⟨*proof*⟩

lemma *piecewise-continuous-on-congI*:
piecewise-continuous-on $a\ b\ I\ g$
if *piecewise-continuous-on* $a\ b\ I\ f$
and eq: $\bigwedge x.\ x \in \{a\ ..\ b\} - I \Longrightarrow g\ x = f\ x$
⟨*proof*⟩

lemma *piecewise-continuous-on-cong*[*cong*]:
piecewise-continuous-on $a\ b\ I\ f \longleftrightarrow$ *piecewise-continuous-on* $c\ d\ J\ g$
if $a = c$
$b = d$
$I = J$
$\bigwedge x.\ c \le x \Longrightarrow x \le d \Longrightarrow x \notin J \Longrightarrow f\ x = g\ x$
⟨*proof*⟩

lemma *tendsto-at-left-continuous-on-avoidI*: $(f \longrightarrow g\ i)$ $(at\text{-}left\ i)$

if g: *continuous-on* $(\{a..i\} - I)$ g
and gf: $\bigwedge x.\ a < x \Longrightarrow x < i \Longrightarrow g\ x = f\ x$
$i \notin I$ *finite* I $a < i$
for i::$'a$::*linorder-topology*
⟨*proof*⟩

lemma *tendsto-at-right-continuous-on-avoidI*: $(f \longrightarrow g\ i)$ $(at\text{-}right\ i)$
if g: *continuous-on* $(\{i..b\} - I)$ g
and gf: $\bigwedge x.\ i < x \Longrightarrow x < b \Longrightarrow g\ x = f\ x$
$i \notin I$ *finite* I $i < b$
for i::$'a$::*linorder-topology*
⟨*proof*⟩

lemma *piecewise-continuous-on-insert-leftI*:
piecewise-continuous-on a b $(insert\ a\ I)$ f **if** *piecewise-continuous-on* a b I f
⟨*proof*⟩

lemma *piecewise-continuous-on-insert-rightI*:
piecewise-continuous-on a b $(insert\ b\ I)$ f **if** *piecewise-continuous-on* a b I f
⟨*proof*⟩

theorem *piecewise-continuous-on-induct*[*consumes 1*, *case-names empty combine weaken*]:
assumes pc: *piecewise-continuous-on* a b I f
assumes 1: $\bigwedge a\ b\ f.$ *continuous-on* $\{a\ ..\ b\}$ $f \Longrightarrow P\ a\ b\ \{\}\ f$
assumes 2: $\bigwedge a\ i\ b\ I\ f1\ f2\ f.\ a \le i \Longrightarrow i \le b \Longrightarrow i \notin I \Longrightarrow P\ a\ i\ I\ f1 \Longrightarrow P\ i\ b\ I\ f2 \Longrightarrow$
piecewise-continuous-on a i I $f1$ $\Longrightarrow$
piecewise-continuous-on i b I $f2$ $\Longrightarrow$
$(\bigwedge x.\ a \le x \Longrightarrow x < i \Longrightarrow f1\ x = f\ x) \Longrightarrow$
$(\bigwedge x.\ i < x \Longrightarrow x \le b \Longrightarrow f2\ x = f\ x) \Longrightarrow$
$(i > a \Longrightarrow (f \longrightarrow f1\ i)\ (at\text{-}left\ i)) \Longrightarrow$
$(i < b \Longrightarrow (f \longrightarrow f2\ i)\ (at\text{-}right\ i)) \Longrightarrow$
$P\ a\ b\ (insert\ i\ I)\ f$
assumes 3: $\bigwedge a\ b\ i\ I\ f.\ P\ a\ b\ I\ f \Longrightarrow finite\ I \Longrightarrow i \notin I \Longrightarrow P\ a\ b\ (insert\ i\ I)\ f$
shows $P\ a\ b\ I\ f$
⟨*proof*⟩

lemma *continuous-on-imp-piecewise-continuous-on*:
continuous-on $\{a\ ..\ b\}$ $f \Longrightarrow$ *piecewise-continuous-on* a b $\{\}$ f
⟨*proof*⟩

lemma *piecewise-continuous-on-imp-absolutely-integrable*:
fixes a b::*real* **and** f::*real* $\Rightarrow$ $'a$::*euclidean-space*
assumes *piecewise-continuous-on* a b I f
shows f *absolutely-integrable-on* $\{a..b\}$
⟨*proof*⟩

lemma *piecewise-continuous-on-integrable*:

fixes $a\ b$::*real* **and** f::*real* $\Rightarrow$ *'a*::*euclidean-space*
assumes *piecewise-continuous-on* $a\ b\ I\ f$
shows f *integrable-on* $\{a..b\}$
⟨*proof*⟩

lemma *piecewise-continuous-on-comp*:
assumes p: *piecewise-continuous-on* $a\ b\ I\ f$
assumes c: $\bigwedge x$. *isCont* $(\lambda(x, y).\ g\ x\ y)\ x$
shows *piecewise-continuous-on* $a\ b\ I\ (\lambda x.\ g\ x\ (f\ x))$
⟨*proof*⟩

lemma *bounded-piecewise-continuous-image*:
bounded $(f\ `\ \{a\ ..\ b\})$
if *piecewise-continuous-on* $a\ b\ I\ f$ **for** $a\ b$::*real*
⟨*proof*⟩

lemma *tendsto-within-eventually*:
$(f \longrightarrow l)$ (*at* x *within* X)
if
$(f \longrightarrow l)$ (*at* x *within* Y)
$\forall_F\ y$ *in at* x *within* $X.\ y \in Y$
⟨*proof*⟩

lemma *at-within-eq-bot-lemma*:
at x *within* $\{b..c\}$ = (*if* $x < b \vee b > c$ *then bot else at* x *within* $\{b..c\}$)
for $x\ b\ c$::*'a*::*linorder-topology*
⟨*proof*⟩

lemma *at-within-eq-bot-lemma2*:
at x *within* $\{a..b\}$ = (*if* $x > b \vee a > b$ *then bot else at* x *within* $\{a..b\}$)
for $x\ a\ b$::*'a*::*linorder-topology*
⟨*proof*⟩

lemma *piecewise-continuous-on-combine*:
piecewise-continuous-on $a\ c\ J\ f$
if *piecewise-continuous-on* $a\ b\ J\ f$ *piecewise-continuous-on* $b\ c\ J\ f$
⟨*proof*⟩

lemma *piecewise-continuous-on-finite-superset*:
piecewise-continuous-on $a\ b\ I\ f \Longrightarrow I \subseteq J \Longrightarrow$ *finite* $J \Longrightarrow$ *piecewise-continuous-on* $a\ b\ J\ f$
for $a\ b$::*'a*::{*linorder-topology*, *dense-order*, *no-bot*, *no-top*}
⟨*proof*⟩

lemma *piecewise-continuous-on-splitI*:
piecewise-continuous-on $a\ c\ K\ f$
if
piecewise-continuous-on $a\ b\ I\ f$
piecewise-continuous-on $b\ c\ J\ f$

$I \subseteq K$ $J \subseteq K$ *finite* K
for a b::$'a$::{*linorder-topology*, *dense-order*, *no-bot*, *no-top*}
⟨*proof*⟩

end

4 Existence

theory *Existence* **imports**
Piecewise-Continuous
begin

4.1 Definition

definition *has-laplace* :: (*real* ⇒ *complex*) ⇒ *complex* ⇒ *complex* ⇒ *bool*
(**infixr** *has′-laplace 46*)
where (f *has-laplace* L) s ⟷ (($\lambda t.$ *exp* ($t *_R - s$) $*$ f t) *has-integral* L) {*0*..}

lemma *has-laplaceI*:
assumes (($\lambda t.$ *exp* ($t *_R - s$) $*$ f t) *has-integral* L) {*0*..}
shows (f *has-laplace* L) s
⟨*proof*⟩

lemma *has-laplaceD*:
assumes (f *has-laplace* L) s
shows (($\lambda t.$ *exp* ($t *_R - s$) $*$ f t) *has-integral* L) {*0*..}
⟨*proof*⟩

lemma *has-laplace-unique*:
$L = M$ **if**
(f *has-laplace* L) s
(f *has-laplace* M) s
⟨*proof*⟩

4.2 Condition for Existence: Exponential Order

definition *exponential-order* M c f ⟷ $0 < M \wedge (\forall_F$ t *in at-top. norm* (f t) $\leq M * exp$ ($c * t$))

lemma *exponential-orderI*:
assumes $0 < M$ **and** *eo*: $\forall_F$ t *in at-top. norm* (f t) $\leq M * exp$ ($c * t$)
shows *exponential-order* M c f
⟨*proof*⟩

lemma *exponential-orderD*:
assumes *exponential-order* M c f
shows $0 < M$ $\forall_F$ t *in at-top. norm* (f t) $\leq M * exp$ ($c * t$)
⟨*proof*⟩

context
fixes *f*::*real* ⇒ *complex*
begin

definition *laplace-integrand*::*complex* ⇒ *real* ⇒ *complex*
where *laplace-integrand s t* = *exp* (*t* $*_R$ − *s*) ∗ *f t*

lemma *laplace-integrand-absolutely-integrable-on-Icc*:
laplace-integrand s absolutely-integrable-on {*a*..*b*}
if *AE x*∈{*a*..*b*} *in lebesgue*. *cmod* (*f x*) ≤ *B f integrable-on* {*a*..*b*}
⟨*proof*⟩

lemma *laplace-integrand-integrable-on-Icc*:
laplace-integrand s integrable-on {*a*..*b*}
if *AE x*∈{*a*..*b*} *in lebesgue*. *cmod* (*f x*) ≤ *B f integrable-on* {*a*..*b*}
⟨*proof*⟩

lemma *eventually-laplace-integrand-le*:
$\forall_F$ *t in at-top*. *cmod* (*laplace-integrand s t*) ≤ *M* ∗ *exp* (− (*Re s* − *c*) ∗ *t*)
if *exponential-order M c f*
⟨*proof*⟩

lemma
assumes *eo*: *exponential-order M c f*
and *cs*: *c* < *Re s*
shows *laplace-integrand-integrable-on-Ici-iff*:
laplace-integrand s integrable-on {*a*..} ⟷
(∀ *k*>*a*. *laplace-integrand s integrable-on* {*a*..*k*})
(**is** *?th1*)
and *laplace-integrand-absolutely-integrable-on-Ici-iff*:
laplace-integrand s absolutely-integrable-on {*a*..} ⟷
(∀ *k*>*a*. *laplace-integrand s absolutely-integrable-on* {*a*..*k*})
(**is** *?th2*)
⟨*proof*⟩

theorem *laplace-exists-laplace-integrandI*:
assumes *laplace-integrand s integrable-on* {*0*..}
obtains *F* **where** (*f has-laplace F*) *s*
⟨*proof*⟩

lemma
assumes *eo*: *exponential-order M c f*
and *pc*: ⋀*k*. *AE x*∈{*0*..*k*} *in lebesgue*. *cmod* (*f x*) ≤ *B k* ⋀*k*. *f integrable-on* {*0*..*k*}
and *s*: *Re s* > *c*
shows *laplace-integrand-integrable*: *laplace-integrand s integrable-on* {*0*..} (**is** *?th1*)
and *laplace-integrand-absolutely-integrable*:
laplace-integrand s absolutely-integrable-on {*0*..} (**is** *?th2*)

⟨proof⟩

lemma *piecewise-continuous-on-AE-boundedE*:
assumes *pc*: $\bigwedge k$. *piecewise-continuous-on a k (I k) f*
obtains *B* **where** $\bigwedge k$. *AE* $x \in \{a..k\}$ *in lebesgue. cmod (f x)* $\leq$ *B k*
⟨proof⟩

theorem *piecewise-continuous-on-has-laplace*:
assumes *eo*: *exponential-order M c f*
and *pc*: $\bigwedge k$. *piecewise-continuous-on 0 k (I k) f*
and *s*: *Re s* $>$ *c*
obtains *F* **where** *(f has-laplace F) s*
⟨proof⟩

end

4.3 Concrete Laplace Transforms

lemma *exp-scaleR-has-vector-derivative-left'*[*derivative-intros*]:
((λt. *exp* ($t *_R A$)) *has-vector-derivative* $A *$ *exp* ($t *_R A$)) (*at t within S*)
⟨proof⟩

lemma
fixes *a*::*complex*— TODO: generalize
assumes *a*: $0 < Re\ a$
shows *integrable-on-cexp-minus-to-infinity*: (λx. *exp* ($x *_R - a$)) *integrable-on* $\{c..\}$
and *integral-cexp-minus-to-infinity*: *integral* $\{c..\}$ (λx. *exp* ($x *_R - a$)) = *exp* ($c *_R - a$) / *a*
⟨proof⟩

lemma *has-integral-cexp-minus-to-infinity*:
fixes *a*::*complex*— TODO: generalize
assumes *a*: $0 < Re\ a$
shows ((λx. *exp* ($x *_R - a$)) *has-integral exp* ($c *_R - a$) / *a*) $\{c..\}$
⟨proof⟩

lemma *has-laplace-one*:
((λ-. *1*) *has-laplace inverse s*) *s* **if** *Re s* > 0
⟨proof⟩

lemma *has-laplace-add*:
assumes *f*: (*f has-laplace F*) *S*
assumes *g*: (*g has-laplace G*) *S*
shows ((λx. *f x* + *g x*) *has-laplace F* + *G*) *S*
⟨proof⟩

lemma *has-laplace-cmul*:
assumes (*f has-laplace F*) *S*

shows $((\lambda x.\ r *_R f\ x)$ *has-laplace* $r *_R F)\ S$
⟨*proof*⟩

lemma *has-laplace-uminus*:
assumes $(f$ *has-laplace* $F)\ S$
shows $((\lambda x.\ - f\ x)$ *has-laplace* $- F)\ S$
⟨*proof*⟩

lemma *has-laplace-minus*:
assumes f: $(f$ *has-laplace* $F)\ S$
assumes g: $(g$ *has-laplace* $G)\ S$
shows $((\lambda x.\ f\ x - g\ x)$ *has-laplace* $F - G)\ S$
⟨*proof*⟩

lemma *has-laplace-spike*:
$(f$ *has-laplace* $L)\ s$
if L: $(g$ *has-laplace* $L)\ s$
and *negligible* T
and $\bigwedge t.\ t \notin T \Longrightarrow t \geq 0 \Longrightarrow f\ t = g\ t$
⟨*proof*⟩

lemma *has-laplace-frequency-shift*:— First Translation Theorem in Schiff
$((\lambda t.\ exp\ (t *_R b) * f\ t)$ *has-laplace* $L)\ s$
if $(f$ *has-laplace* $L)\ (s - b)$
⟨*proof*⟩

theorem *has-laplace-derivative-time-domain*:
$(f'$ *has-laplace* $s * L - f0)\ s$
if L: $(f$ *has-laplace* $L)\ s$
and f': $\bigwedge t.\ t > 0 \Longrightarrow (f$ *has-vector-derivative* $f'\ t)\ (at\ t)$
and $f0$: $(f \longrightarrow f0)\ (at\text{-}right\ 0)$
and *eo*: *exponential-order* $M\ c\ f$
and *cs*: $c < Re\ s$
— Proof and statement follow "The Laplace Transform: Theory and Applications" by Joel L. Schiff.
⟨*proof*⟩

lemma *exp-times-has-integral*:
$((\lambda t.\ exp\ (c * t))$ *has-integral* $(if\ c = 0\ then\ t\ else\ exp\ (c * t)\ /\ c) - (if\ c = 0\ then\ t0\ else\ exp\ (c * t0)\ /\ c))\ \{t0\ ..\ t\}$
if $t0 \leq t$
for $c\ t{::}real$
⟨*proof*⟩

lemma *integral-exp-times*:
$integral\ \{t0\ ..\ t\}\ (\lambda t.\ exp\ (c * t)) = (if\ c = 0\ then\ t - t0\ else\ exp\ (c * t)\ /\ c - exp\ (c * t0)\ /\ c)$
if $t0 \leq t$

for *c t*::*real*
⟨*proof*⟩

lemma *filtermap-times-pos-at-top*: *filtermap* $((*)\ e)$ *at-top* = *at-top*
if $e > 0$
for *e*::*real*
⟨*proof*⟩

lemma *exponential-order-additiveI*:
assumes $0 < M$ **and** *eo*: $\forall_F$ *t in at-top. norm* $(f\ t) \leq K + M * exp\ (c * t)$ **and** $c \geq 0$
obtains M' **where** *exponential-order* M' *c f*
⟨*proof*⟩

lemma *exponential-order-integral*:
fixes *f*::*real* ⇒ *'a*::*banach*
assumes *I*: $\bigwedge t.\ t \geq a \Longrightarrow$ (*f has-integral I t*) $\{a\ ..\ t\}$
and *eo*: *exponential-order M c f*
and $c > 0$
obtains M' **where** *exponential-order* M' *c I*
⟨*proof*⟩

lemma *integral-has-vector-derivative-piecewise-continuous*:
fixes *f* :: *real* ⇒ *'a*::*euclidean-space*— TODO: generalize?
assumes *piecewise-continuous-on a b D f*
shows $\bigwedge x.\ x \in \{a\ ..\ b\} - D \Longrightarrow$
$((\lambda u.$ *integral* $\{a..u\}\ f)$ *has-vector-derivative* $f(x))$ (*at x within* $\{a..b\} - D$)
⟨*proof*⟩

lemma *has-derivative-at-split*:
(*f has-derivative f'*) (*at x*) ⟷ (*f has-derivative f'*) (*at-left x*) ∧ (*f has-derivative f'*) (*at-right x*)
for *x*::*'a*::{*linorder-topology*, *real-normed-vector*}
⟨*proof*⟩

lemma *has-vector-derivative-at-split*:
(*f has-vector-derivative f'*) (*at x*) ⟷
(*f has-vector-derivative f'*) (*at-left x*) ∧
(*f has-vector-derivative f'*) (*at-right x*)
⟨*proof*⟩

lemmas *differentiableI-vector*[*intro*]

lemma *differentiable-at-splitD*:
f differentiable at-left x
f differentiable at-right x
if *f differentiable* (*at x*)
for *x*::*real*
⟨*proof*⟩

lemma *integral-differentiable*:
  fixes $f :: real \Rightarrow 'a::banach$
  assumes *continuous-on* $\{a..b\}$ f
    and $x \in \{a..b\}$
  shows $(\lambda u.\ integral\ \{a..u\}\ f)$ *differentiable at* x *within* $\{a..b\}$
  ⟨*proof*⟩

theorem *integral-has-vector-derivative-piecewise-continuous'*:
  fixes $f :: real \Rightarrow 'a::euclidean\text{-}space$— TODO: generalize?
  assumes *piecewise-continuous-on* a b D f $a < b$
  shows
    $(\forall x.\ a < x \longrightarrow x < b \longrightarrow x \notin D \longrightarrow (\lambda u.\ integral\ \{a..u\}\ f)$ *differentiable at* $x) \wedge$
    $(\forall x.\ a \leq x \longrightarrow x < b \longrightarrow (\lambda t.\ integral\ \{a..t\}\ f)$ *differentiable at-right* $x) \wedge$
    $(\forall x.\ a < x \longrightarrow x \leq b \longrightarrow (\lambda t.\ integral\ \{a..t\}\ f)$ *differentiable at-left* $x)$
  ⟨*proof*⟩

lemma *closure* $(-S) \cap$ *closure* $S =$ *frontier* S
  ⟨*proof*⟩

theorem *integral-time-domain-has-laplace*:
  $((\lambda t.\ integral\ \{0\ ..\ t\}\ f)$ *has-laplace* $L\ /\ s)\ s$
  if *pc*: $\bigwedge k.$ *piecewise-continuous-on* 0 k D f
    and *eo*: *exponential-order* M c f
    and L: $(f$ *has-laplace* $L)$ s
    and s: $Re\ s > c$
    and c: $c > 0$
    and *TODO*: $D = \{\}$ — TODO: generalize to actual *piecewise-continuous-on*
  for $f::real \Rightarrow complex$
⟨*proof*⟩

4.4 higher derivatives

definition *nderiv* i f $X = ((\lambda f.\ (\lambda x.\ vector\text{-}derivative\ f\ (at\ x\ within\ X)))\widehat{\ }\widehat{\ }i)\ f$

definition *ndiff* n f $X \longleftrightarrow (\forall\, i{<}n.\ \forall\, x \in X.$ *nderiv* i f X *differentiable at* x *within* $X)$

lemma *nderiv-zero*[*simp*]: *nderiv* 0 f $X = f$
  ⟨*proof*⟩

lemma *nderiv-Suc*[*simp*]:
  nderiv $(Suc\ i)$ f X $x =$ *vector-derivative* $(nderiv\ i\ f\ X)$ $(at\ x\ within\ X)$
  ⟨*proof*⟩

lemma *ndiff-zero*[*simp*]: *ndiff* 0 f X
  ⟨*proof*⟩

lemma *ndiff-Sucs*[*simp*]:
  $ndiff\ (Suc\ i)\ f\ X \longleftrightarrow$
    $(ndiff\ i\ f\ X) \wedge$
    $(\forall x \in X.\ (nderiv\ i\ f\ X)\ differentiable\ (at\ x\ within\ X))$
  ⟨*proof*⟩

theorem *has-laplace-vector-derivative*:
  $((\lambda t.\ vector\text{-}derivative\ f\ (at\ t))\ has\text{-}laplace\ s * L - f0)\ s$
  if L: $(f\ has\text{-}laplace\ L)\ s$
    and f': $\bigwedge t.\ t > 0 \Longrightarrow f\ differentiable\ (at\ t)$
    and $f0$: $(f \longrightarrow f0)\ (at\text{-}right\ 0)$
    and *eo*: $exponential\text{-}order\ M\ c\ f$
    and *cs*: $c < Re\ s$
⟨*proof*⟩

lemma *has-laplace-nderiv*:
  $(nderiv\ n\ f\ \{0<..\}\ has\text{-}laplace\ s\hat{\ }n * L - (\sum i<n.\ s\hat{\ }(n - Suc\ i) * f0\ i))\ s$
  if L: $(f\ has\text{-}laplace\ L)\ s$
    and f': $ndiff\ n\ f\ \{0<..\}$
    and $f0$: $\bigwedge i.\ i < n \Longrightarrow (nderiv\ i\ f\ \{0<..\} \longrightarrow f0\ i)\ (at\text{-}right\ 0)$
    and *eo*: $\bigwedge i.\ i < n \Longrightarrow exponential\text{-}order\ M\ c\ (nderiv\ i\ f\ \{0<..\})$
    and *cs*: $c < Re\ s$
  ⟨*proof*⟩

end

5 Lerch Lemma

theory *Lerch-Lemma*
  imports
    HOL−Analysis.Analysis
begin

The main tool to prove uniqueness of the Laplace transform.

lemma *lerch-lemma-real*:
  fixes $h::real \Rightarrow real$
  assumes *h-cont*[*continuous-intros*]: $continuous\text{-}on\ \{0\ ..\ 1\}\ h$
  assumes *int-0*: $\bigwedge n.\ ((\lambda u.\ u\ \hat{}\ n * h\ u)\ has\text{-}integral\ 0)\ \{0\ ..\ 1\}$
  assumes u: $0 \leq u\ u \leq 1$
  shows $h\ u = 0$
⟨*proof*⟩

lemma *lerch-lemma*:
  fixes $h::real \Rightarrow 'a::euclidean\text{-}space$
  assumes [*continuous-intros*]: $continuous\text{-}on\ \{0\ ..\ 1\}\ h$
  assumes *int-0*: $\bigwedge n.\ ((\lambda u.\ u\ \hat{}\ n *_R h\ u)\ has\text{-}integral\ 0)\ \{0\ ..\ 1\}$
  assumes u: $0 \leq u\ u \leq 1$
  shows $h\ u = 0$
⟨*proof*⟩

end

6 Uniqueness of Laplace Transform

theory *Uniqueness*
imports
Existence
Lerch-Lemma
begin

We show uniqueness of the Laplace transform for continuous functions.

lemma *laplace-transform-zero*:— should also work for piecewise continuous
assumes *cont-f*: *continuous-on* $\{0..\}$ *f*
assumes *eo*: *exponential-order* M a f
assumes *laplace*: $\bigwedge s.$ *Re* $s > a \Longrightarrow$ (f *has-laplace* 0) s
assumes $t \geq 0$
shows $f\ t = 0$
⟨*proof*⟩

lemma *exponential-order-eventually-eq*: *exponential-order* M a f
if *exponential-order* M a g $\bigwedge t.\ t \geq k \Longrightarrow f\ t = g\ t$
⟨*proof*⟩

lemma *exponential-order-mono*:
assumes *eo*: *exponential-order* M a f
assumes $a \leq b$ $M \leq N$
shows *exponential-order* N b f
⟨*proof*⟩

lemma *exponential-order-uminus-iff*:
exponential-order M a $(\lambda x. - f\ x)$ = *exponential-order* M a f
⟨*proof*⟩

lemma *exponential-order-add*:
assumes *exponential-order* M a f *exponential-order* M a g
shows *exponential-order* $(2 * M)$ a $(\lambda x.\ f\ x + g\ x)$
⟨*proof*⟩

theorem *laplace-transform-unique*:
assumes *f*: $\bigwedge s.$ *Re* $s > a \Longrightarrow$ (f *has-laplace* F) s
assumes *g*: $\bigwedge s.$ *Re* $s > b \Longrightarrow$ (g *has-laplace* F) s
assumes [*continuous-intros*]: *continuous-on* $\{0..\}$ *f*
assumes [*continuous-intros*]: *continuous-on* $\{0..\}$ *g*
assumes *eof*: *exponential-order* M a f
assumes *eog*: *exponential-order* N b g
assumes $t \geq 0$
shows $f\ t = g\ t$
⟨*proof*⟩

end
theory *Laplace-Transform*
imports
Existence
Uniqueness
begin

end

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