Laplace Transform

Fabian Immler

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Abstract

This entry formalizes the Laplace transform and concrete Laplace transforms for arithmetic functions, frequency shift, integration and (higher) differentiation in the time domain. It proves Lerch's lemma and uniqueness of the Laplace transform for continuous functions. In order to formalize the foundational assumptions, this entry contains a formalization of piecewise continuous functions and functions of exponential order.

Contents

theory *Laplace-Transform-Library*
  imports
    HOL−Analysis.Analysis
begin

1 References

Much of this formalization is based on Schiff's textbook [3]. Parts of this formalization are inspired by the HOL-Light formalization ([4], [1], [2]), but stated more generally for piecewise continuous (instead of piecewise continuously differentiable) functions.

2 Library Additions

2.1 Derivatives

lemma *DERIV-compose-FDERIV*:— TODO: generalize and move from HOL-ODE
  assumes $DERIV\ f\ (g\ x) :> f'$
  assumes $(g\ \mathit{has\text{-}derivative}\ g')\ (\mathit{at}\ x\ \mathit{within}\ s)$
  shows $((\lambda x.\ f\ (g\ x))\ \mathit{has\text{-}derivative}\ (\lambda x.\ g'\ x * f'))\ (\mathit{at}\ x\ \mathit{within}\ s)$
  ⟨*proof*⟩

lemmas *has-derivative-sin*[*derivative-intros*] = *DERIV-sin*[*THEN DERIV-compose-FDERIV*]
 and *has-derivative-cos*[*derivative-intros*] = *DERIV-cos*[*THEN DERIV-compose-FDERIV*]
 and *has-derivative-exp*[*derivative-intros*] = *DERIV-exp*[*THEN DERIV-compose-FDERIV*]

2.2 Integrals

lemma *negligible-real-ivlI*:
  fixes $a\ b::\mathit{real}$
  assumes $a \geq b$
  shows $\mathit{negligible}\ \{a\ ..\ b\}$
⟨*proof*⟩

lemma *absolutely-integrable-on-combine*:
  fixes $f :: \mathit{real} \Rightarrow {'a}::\mathit{euclidean\text{-}space}$
  assumes $f\ \mathit{absolutely\text{-}integrable\text{-}on}\ \{a..c\}$
    and $f\ \mathit{absolutely\text{-}integrable\text{-}on}\ \{c..b\}$
    and $a \leq c$
    and $c \leq b$
  shows $f\ \mathit{absolutely\text{-}integrable\text{-}on}\ \{a..b\}$
  ⟨*proof*⟩

lemma *dominated-convergence-at-top*:
  fixes $f :: \mathit{real} \Rightarrow {'n}::\mathit{euclidean\text{-}space} \Rightarrow {'m}::\mathit{euclidean\text{-}space}$
  assumes f: $\bigwedge k.\ (f\ k)\ \mathit{integrable\text{-}on}\ s$ **and** h: $h\ \mathit{integrable\text{-}on}\ s$
    and le: $\bigwedge k\ x.\ x \in s \Longrightarrow \mathit{norm}\ (f\ k\ x) \leq h\ x$
    and conv: $\forall x \in s.\ ((\lambda k.\ f\ k\ x) \longrightarrow g\ x)\ \mathit{at\text{-}top}$
  shows $g\ \mathit{integrable\text{-}on}\ s\ ((\lambda k.\ \mathit{integral}\ s\ (f\ k)) \longrightarrow \mathit{integral}\ s\ g)\ \mathit{at\text{-}top}$
⟨*proof*⟩

lemma *has-integral-dominated-convergence-at-top*:

fixes f :: *real* ⇒ *'n*::*euclidean-space* ⇒ *'m*::*euclidean-space*
assumes ⋀*k*. (f k *has-integral* y k) s h *integrable-on* s
⋀*k* *x*. *x*∈*s* ⟹ *norm* (f k x) ≤ h x ∀ *x*∈*s*. ((λ*k*. f k x) ⟶ g x) *at-top*
and x: (y ⟶ x) *at-top*
shows (g *has-integral* x) s
⟨*proof*⟩

lemma *integral-indicator-eq-restriction*:
fixes *f*::*'a*::*euclidean-space* ⇒ *'b*::*banach*
assumes f: f *integrable-on* R
and $R \subseteq S$
shows *integral* S (λ*x*. *indicator* R x $*_R$ f x) = *integral* R f
⟨*proof*⟩

lemma
improper-integral-at-top:
fixes *f*::*real* ⇒ *'a*::*euclidean-space*
assumes f *absolutely-integrable-on* {*a*..}
shows ((λ*x*. *integral* {*a*..*x*} f) ⟶ *integral* {*a*..} f) *at-top*
⟨*proof*⟩

lemma *norm-integrable-onI*: (λ*x*. *norm* (f x)) *integrable-on* S
if f *absolutely-integrable-on* S
for *f*::*'a*::*euclidean-space*⇒*'b*::*euclidean-space*
⟨*proof*⟩

lemma
has-integral-improper-at-topI:
fixes *f*::*real* ⇒ *'a*::*banach*
assumes I: $\forall_F$ k *in at-top*. (f *has-integral* I k) {*a*..*k*}
assumes J: (I ⟶ J) *at-top*
shows (f *has-integral* J) {*a*..}
⟨*proof*⟩

lemma *has-integral-improperE*:
fixes *f*::*real* ⇒ *'a*::*euclidean-space*
assumes I: (f *has-integral* I) {*a*..}
assumes *ai*: f *absolutely-integrable-on* {*a*..}
obtains J **where**
⋀*k*. (f *has-integral* J k) {*a*..*k*}
(J ⟶ I) *at-top*
⟨*proof*⟩

2.3 Miscellaneous

lemma *AE-BallI*: *AE* *x*∈*X* *in* F. P x **if** $\forall x \in X.\ P\ x$
⟨*proof*⟩

lemma *bounded-le-Sup*:

assumes *bounded* $(f \text{ ‘ } S)$
shows $\forall x \in S.\ norm\ (f\ x) \leq Sup\ (norm \text{ ‘ } f \text{ ‘ } S)$
⟨*proof*⟩

end

3 Piecewise Continous Functions

theory *Piecewise-Continuous*
imports
Laplace-Transform-Library
begin

3.1 at within filters

lemma *at-within-self-singleton*[*simp*]: $at\ i\ within\ \{i\} = bot$
⟨*proof*⟩

lemma *at-within-t1-space-avoid*:
$(at\ x\ within\ X - \{i\}) = (at\ x\ within\ X)$ **if** $x \neq i$ **for** $x\ i{::}'a{::}t1\text{-}space$
⟨*proof*⟩

lemma *at-within-t1-space-avoid-finite*:
$(at\ x\ within\ X - I) = (at\ x\ within\ X)$ **if** *finite* I $x \notin I$ **for** $x{::}'a{::}t1\text{-}space$
⟨*proof*⟩

lemma *at-within-interior*:
$NO\text{-}MATCH\ (UNIV{::}'a\ set)\ (S{::}'a{::}topological\text{-}space\ set) \Longrightarrow x \in interior\ S \Longrightarrow at\ x\ within\ S = at\ x$
⟨*proof*⟩

3.2 intervals

lemma *Compl-Icc*: $-\ \{a\ ..\ b\} = \{..{<}a\} \cup \{b{<}..\}$ **for** $a\ b{::}'a{::}linorder$
⟨*proof*⟩

lemma *interior-Icc*[*simp*]: $interior\ \{a..b\} = \{a{<}..{<}b\}$
for $a\ b{::}'a{::}\{linorder\text{-}topology,\ dense\text{-}order,\ no\text{-}bot,\ no\text{-}top\}$
— TODO: is *no-bot* and *no-top* really required?
⟨*proof*⟩

lemma *closure-finite*[*simp*]: $closure\ X = X$ **if** *finite* X **for** $X{::}'a{::}t1\text{-}space\ set$
⟨*proof*⟩

definition $piecewise\text{-}continuous\text{-}on :: 'a{::}linorder\text{-}topology \Rightarrow 'a \Rightarrow 'a\ set \Rightarrow ('a \Rightarrow 'b{::}topological\text{-}space) \Rightarrow bool$
where $piecewise\text{-}continuous\text{-}on\ a\ b\ I\ f \longleftrightarrow$
$(continuous\text{-}on\ (\{a\ ..\ b\} - I)\ f \wedge finite\ I \wedge$
$(\forall i \in I.\ (i \in \{a{<}..b\} \longrightarrow (\exists l.\ (f \longrightarrow l)\ (at\text{-}left\ i))) \wedge$

$(i \in \{a..<b\} \longrightarrow (\exists u.\ (f \longrightarrow u)\ (\textit{at-right}\ i)))))$

lemma *piecewise-continuous-on-subset*:
piecewise-continuous-on a b I f $\Longrightarrow \{c\ ..\ d\} \subseteq \{a\ ..\ b\} \Longrightarrow$ *piecewise-continuous-on c d I f*
⟨*proof*⟩

lemma *piecewise-continuous-onE*:
assumes *piecewise-continuous-on a b I f*
obtains *l u*
where *finite I*
and *continuous-on* $(\{a..b\} - I)\ f$
and $(\bigwedge i.\ i \in I \Longrightarrow a < i \Longrightarrow i \leq b \Longrightarrow (f \longrightarrow l\ i)\ (\textit{at-left}\ i))$
and $(\bigwedge i.\ i \in I \Longrightarrow a \leq i \Longrightarrow i < b \Longrightarrow (f \longrightarrow u\ i)\ (\textit{at-right}\ i))$
⟨*proof*⟩

lemma *piecewise-continuous-onI*:
assumes *finite I continuous-on* $(\{a..b\} - I)\ f$
and $(\bigwedge i.\ i \in I \Longrightarrow a < i \Longrightarrow i \leq b \Longrightarrow (f \longrightarrow l\ i)\ (\textit{at-left}\ i))$
and $(\bigwedge i.\ i \in I \Longrightarrow a \leq i \Longrightarrow i < b \Longrightarrow (f \longrightarrow u\ i)\ (\textit{at-right}\ i))$
shows *piecewise-continuous-on a b I f*
⟨*proof*⟩

lemma *piecewise-continuous-onI'*:
fixes *a b*::'*a*::{*linorder-topology*, *dense-order*, *no-bot*, *no-top*}
assumes *finite I* $\bigwedge x.\ a < x \Longrightarrow x < b \Longrightarrow$ *isCont f x*
and $a \notin I \Longrightarrow$ *continuous* (*at-right a*) *f*
and $b \notin I \Longrightarrow$ *continuous* (*at-left b*) *f*
and $(\bigwedge i.\ i \in I \Longrightarrow a < i \Longrightarrow i \leq b \Longrightarrow (f \longrightarrow l\ i)\ (\textit{at-left}\ i))$
and $(\bigwedge i.\ i \in I \Longrightarrow a \leq i \Longrightarrow i < b \Longrightarrow (f \longrightarrow u\ i)\ (\textit{at-right}\ i))$
shows *piecewise-continuous-on a b I f*
⟨*proof*⟩

lemma *piecewise-continuous-onE'*:
fixes *a b*::'*a*::{*linorder-topology*, *dense-order*, *no-bot*, *no-top*}
assumes *piecewise-continuous-on a b I f*
obtains *l u*
where *finite I*
and $\bigwedge x.\ a < x \Longrightarrow x < b \Longrightarrow x \notin I \Longrightarrow$ *isCont f x*
and $(\bigwedge x.\ a < x \Longrightarrow x \leq b \Longrightarrow (f \longrightarrow l\ x)\ (\textit{at-left}\ x))$
and $(\bigwedge x.\ a \leq x \Longrightarrow x < b \Longrightarrow (f \longrightarrow u\ x)\ (\textit{at-right}\ x))$
and $\bigwedge x.\ a \leq x \Longrightarrow x \leq b \Longrightarrow x \notin I \Longrightarrow f\ x = l\ x$
and $\bigwedge x.\ a \leq x \Longrightarrow x \leq b \Longrightarrow x \notin I \Longrightarrow f\ x = u\ x$
⟨*proof*⟩

lemma *tendsto-avoid-at-within*:
$(f \longrightarrow l)\ (\textit{at}\ x\ \textit{within}\ X)$
if $(f \longrightarrow l)\ (\textit{at}\ x\ \textit{within}\ X - \{x\})$
⟨*proof*⟩

lemma *tendsto-within-subset-eventuallyI*:
  $(f \longrightarrow fx)$ $(at\ x\ within\ X)$
  if *g*: $(g \longrightarrow gy)$ $(at\ y\ within\ Y)$
    and *ev*: $\forall_F\ x\ in\ (at\ y\ within\ Y).\ f\ x = g\ x$
    and *xy*: $x = y$
    and *fxgy*: $fx = gy$
    and *XY*: $X - \{x\} \subseteq Y$
  ⟨*proof*⟩

lemma *piecewise-continuous-on-insertE*:
  assumes *piecewise-continuous-on a b* (*insert i I*) *f*
  assumes $i \in \{a\ ..\ b\}$
  obtains *g h* **where**
    piecewise-continuous-on a i I g
    piecewise-continuous-on i b I h
    $\bigwedge x.\ a \le x \Longrightarrow x < i \Longrightarrow g\ x = f\ x$
    $\bigwedge x.\ i < x \Longrightarrow x \le b \Longrightarrow h\ x = f\ x$
⟨*proof*⟩

lemma *eventually-avoid-finite*:
  $\forall_F\ x\ in\ at\ y\ within\ Y.\ x \notin I$ **if** *finite I* **for** *y*::′*a*::*t1-space*
  ⟨*proof*⟩

lemma *eventually-at-left-linorder*:— TODO: generalize $?b < ?a \Longrightarrow \forall_F\ x\ in\ at\text{-}left\ ?a.\ x \in \{?b<..<?a\}$
  $a > (b :: 'a :: linorder\text{-}topology) \Longrightarrow eventually\ (\lambda x.\ x \in \{b<..<a\})\ (at\text{-}left\ a)$
  ⟨*proof*⟩

lemma *eventually-at-right-linorder*:— TODO: generalize $?a < ?b \Longrightarrow \forall_F\ x\ in\ at\text{-}right\ ?a.\ x \in \{?a<..<?b\}$
  $a > (b :: 'a :: linorder\text{-}topology) \Longrightarrow eventually\ (\lambda x.\ x \in \{b<..<a\})\ (at\text{-}right\ b)$
  ⟨*proof*⟩

lemma *piecewise-continuous-on-congI*:
  piecewise-continuous-on a b I g
  if *piecewise-continuous-on a b I f*
    and *eq*: $\bigwedge x.\ x \in \{a\ ..\ b\} - I \Longrightarrow g\ x = f\ x$
⟨*proof*⟩

lemma *piecewise-continuous-on-cong*[*cong*]:
  piecewise-continuous-on a b I f ⟷ *piecewise-continuous-on c d J g*
  if $a = c$
    $b = d$
    $I = J$
    $\bigwedge x.\ c \le x \Longrightarrow x \le d \Longrightarrow x \notin J \Longrightarrow f\ x = g\ x$
  ⟨*proof*⟩

lemma *tendsto-at-left-continuous-on-avoidI*: $(f \longrightarrow g\ i)$ $(at\text{-}left\ i)$

if g: *continuous-on* $(\{a..i\} - I)$ g
and gf: $\bigwedge x.\ a < x \Longrightarrow x < i \Longrightarrow g\ x = f\ x$
$i \notin I$ *finite* I $a < i$
for i::$'a$::*linorder-topology*
⟨*proof*⟩

lemma *tendsto-at-right-continuous-on-avoidI*: $(f \longrightarrow g\ i)$ $(\textit{at-right}\ i)$
if g: *continuous-on* $(\{i..b\} - I)$ g
and gf: $\bigwedge x.\ i < x \Longrightarrow x < b \Longrightarrow g\ x = f\ x$
$i \notin I$ *finite* I $i < b$
for i::$'a$::*linorder-topology*
⟨*proof*⟩

lemma *piecewise-continuous-on-insert-leftI*:
piecewise-continuous-on a b (*insert* a I) f **if** *piecewise-continuous-on* a b I f
⟨*proof*⟩

lemma *piecewise-continuous-on-insert-rightI*:
piecewise-continuous-on a b (*insert* b I) f **if** *piecewise-continuous-on* a b I f
⟨*proof*⟩

theorem *piecewise-continuous-on-induct*[*consumes 1*, *case-names empty combine weaken*]:
assumes *pc*: *piecewise-continuous-on* a b I f
assumes *1*: $\bigwedge a\ b\ f.$ *continuous-on* $\{a\ ..\ b\}$ $f \Longrightarrow P\ a\ b\ \{\}\ f$
assumes *2*: $\bigwedge a\ i\ b\ I\ f1\ f2\ f.\ a \le i \Longrightarrow i \le b \Longrightarrow i \notin I \Longrightarrow P\ a\ i\ I\ f1 \Longrightarrow P\ i\ b\ I\ f2 \Longrightarrow$
piecewise-continuous-on a i I $f1 \Longrightarrow$
piecewise-continuous-on i b I $f2 \Longrightarrow$
$(\bigwedge x.\ a \le x \Longrightarrow x < i \Longrightarrow f1\ x = f\ x) \Longrightarrow$
$(\bigwedge x.\ i < x \Longrightarrow x \le b \Longrightarrow f2\ x = f\ x) \Longrightarrow$
$(i > a \Longrightarrow (f \longrightarrow f1\ i)\ (\textit{at-left}\ i)) \Longrightarrow$
$(i < b \Longrightarrow (f \longrightarrow f2\ i)\ (\textit{at-right}\ i)) \Longrightarrow$
$P\ a\ b$ (*insert* i I) f
assumes *3*: $\bigwedge a\ b\ i\ I\ f.\ P\ a\ b\ I\ f \Longrightarrow$ *finite* $I \Longrightarrow i \notin I \Longrightarrow P\ a\ b$ (*insert* i I) f
shows $P\ a\ b\ I\ f$
⟨*proof*⟩

lemma *continuous-on-imp-piecewise-continuous-on*:
continuous-on $\{a\ ..\ b\}$ $f \Longrightarrow$ *piecewise-continuous-on* a b $\{\}$ f
⟨*proof*⟩

lemma *piecewise-continuous-on-imp-absolutely-integrable*:
fixes a b::*real* **and** f::*real* $\Rightarrow$ $'a$::*euclidean-space*
assumes *piecewise-continuous-on* a b I f
shows f *absolutely-integrable-on* $\{a..b\}$
⟨*proof*⟩

lemma *piecewise-continuous-on-integrable*:

fixes $a\ b$::*real* **and** f::*real* ⇒ *'a*::*euclidean-space*
assumes *piecewise-continuous-on a b I f*
shows *f integrable-on* $\{a..b\}$
⟨*proof*⟩

lemma *piecewise-continuous-on-comp*:
assumes *p*: *piecewise-continuous-on a b I f*
assumes *c*: $\bigwedge x.$ *isCont* $(\lambda(x, y).\ g\ x\ y)\ x$
shows *piecewise-continuous-on a b I* $(\lambda x.\ g\ x\ (f\ x))$
⟨*proof*⟩

lemma *bounded-piecewise-continuous-image*:
bounded $(f\ `\ \{a\ ..\ b\})$
if *piecewise-continuous-on a b I f* **for** $a\ b$::*real*
⟨*proof*⟩

lemma *tendsto-within-eventually*:
$(f \longrightarrow l)$ (*at x within X*)
if
$(f \longrightarrow l)$ (*at x within Y*)
$\forall_F$ *y in at x within X*. $y \in Y$
⟨*proof*⟩

lemma *at-within-eq-bot-lemma*:
at x within $\{b..c\}$ = (*if* $x < b \lor b > c$ *then bot else at x within* $\{b..c\}$)
for $x\ b\ c$::*'a*::*linorder-topology*
⟨*proof*⟩

lemma *at-within-eq-bot-lemma2*:
at x within $\{a..b\}$ = (*if* $x > b \lor a > b$ *then bot else at x within* $\{a..b\}$)
for $x\ a\ b$::*'a*::*linorder-topology*
⟨*proof*⟩

lemma *piecewise-continuous-on-combine*:
piecewise-continuous-on a c J f
if *piecewise-continuous-on a b J f piecewise-continuous-on b c J f*
⟨*proof*⟩

lemma *piecewise-continuous-on-finite-superset*:
piecewise-continuous-on a b I f ⟹ $I \subseteq J$ ⟹ *finite J* ⟹ *piecewise-continuous-on a b J f*
for $a\ b$::*'a*::{*linorder-topology*, *dense-order*, *no-bot*, *no-top*}
⟨*proof*⟩

lemma *piecewise-continuous-on-splitI*:
piecewise-continuous-on a c K f
if
piecewise-continuous-on a b I f
piecewise-continuous-on b c J f

$I \subseteq K$ $J \subseteq K$ *finite* K
for a b::$'a$::{*linorder-topology*, *dense-order*, *no-bot*, *no-top*}
⟨*proof*⟩

end

4 Existence

theory *Existence* **imports**
Piecewise-Continuous
begin

4.1 Definition

definition *has-laplace* :: (*real* ⇒ *complex*) ⇒ *complex* ⇒ *complex* ⇒ *bool*
(**infixr** ‹*has′-laplace*› *46*)
where (*f has-laplace L*) *s* ⟷ ((λ*t*. *exp* ($t *_R - s$) ∗ *f t*) *has-integral L*) {*0*..}

lemma *has-laplaceI*:
assumes ((λ*t*. *exp* ($t *_R - s$) ∗ *f t*) *has-integral L*) {*0*..}
shows (*f has-laplace L*) *s*
⟨*proof*⟩

lemma *has-laplaceD*:
assumes (*f has-laplace L*) *s*
shows ((λ*t*. *exp* ($t *_R - s$) ∗ *f t*) *has-integral L*) {*0*..}
⟨*proof*⟩

lemma *has-laplace-unique*:
$L = M$ **if**
(*f has-laplace L*) *s*
(*f has-laplace M*) *s*
⟨*proof*⟩

4.2 Condition for Existence: Exponential Order

definition *exponential-order M c f* ⟷ $0 < M$ ∧ ($\forall_F$ *t in at-top*. *norm* (*f t*) ≤ $M * exp$ ($c * t$))

lemma *exponential-orderI*:
assumes $0 < M$ **and** *eo*: $\forall_F$ *t in at-top*. *norm* (*f t*) ≤ $M * exp$ ($c * t$)
shows *exponential-order M c f*
⟨*proof*⟩

lemma *exponential-orderD*:
assumes *exponential-order M c f*
shows $0 < M$ $\forall_F$ *t in at-top*. *norm* (*f t*) ≤ $M * exp$ ($c * t$)
⟨*proof*⟩

context
  fixes *f*::*real* ⇒ *complex*
begin

definition *laplace-integrand*::*complex* ⇒ *real* ⇒ *complex*
  where *laplace-integrand s t* = *exp* ($t *_R - s$) * *f t*

lemma *laplace-integrand-absolutely-integrable-on-Icc*:
  laplace-integrand s absolutely-integrable-on {*a*..*b*}
  if *AE x*∈{*a*..*b*} *in lebesgue. cmod* (*f x*) ≤ *B f integrable-on* {*a*..*b*}
  ⟨*proof*⟩

lemma *laplace-integrand-integrable-on-Icc*:
  laplace-integrand s integrable-on {*a*..*b*}
  if *AE x*∈{*a*..*b*} *in lebesgue. cmod* (*f x*) ≤ *B f integrable-on* {*a*..*b*}
  ⟨*proof*⟩

lemma *eventually-laplace-integrand-le*:
  $\forall_F$ *t in at-top. cmod* (*laplace-integrand s t*) ≤ *M* * *exp* (− (*Re s* − *c*) * *t*)
  if *exponential-order M c f*
  ⟨*proof*⟩

lemma
  assumes *eo*: *exponential-order M c f*
    and *cs*: *c* < *Re s*
  shows *laplace-integrand-integrable-on-Ici-iff*:
    laplace-integrand s integrable-on {*a*..} ⟷
      (∀ *k*>*a*. *laplace-integrand s integrable-on* {*a*..*k*})
    (**is** *?th1*)
  and *laplace-integrand-absolutely-integrable-on-Ici-iff*:
    laplace-integrand s absolutely-integrable-on {*a*..} ⟷
      (∀ *k*>*a*. *laplace-integrand s absolutely-integrable-on* {*a*..*k*})
    (**is** *?th2*)
⟨*proof*⟩

theorem *laplace-exists-laplace-integrandI*:
  assumes *laplace-integrand s integrable-on* {*0*..}
  obtains *F* **where** (*f has-laplace F*) *s*
⟨*proof*⟩

lemma
  assumes *eo*: *exponential-order M c f*
    and *pc*: ⋀*k*. *AE x*∈{*0*..*k*} *in lebesgue. cmod* (*f x*) ≤ *B k* ⋀*k*. *f integrable-on* {*0*..*k*}
    and *s*: *Re s* > *c*
  shows *laplace-integrand-integrable*: *laplace-integrand s integrable-on* {*0*..} (**is** *?th1*)
    and *laplace-integrand-absolutely-integrable*:
      laplace-integrand s absolutely-integrable-on {*0*..} (**is** *?th2*)

⟨*proof*⟩

lemma *piecewise-continuous-on-AE-boundedE*:
assumes *pc*: $\bigwedge k$. *piecewise-continuous-on a k (I k) f*
obtains *B* **where** $\bigwedge k$. *AE x*∈{*a..k*} *in lebesgue. cmod (f x)* ≤ *B k*
⟨*proof*⟩

theorem *piecewise-continuous-on-has-laplace*:
assumes *eo*: *exponential-order M c f*
and *pc*: $\bigwedge k$. *piecewise-continuous-on 0 k (I k) f*
and *s*: *Re s* > *c*
obtains *F* **where** (*f has-laplace F*) *s*
⟨*proof*⟩

end

4.3 Concrete Laplace Transforms

lemma *exp-scaleR-has-vector-derivative-left′*[*derivative-intros*]:
((λ*t*. *exp* (*t* $*_R$ *A*)) *has-vector-derivative A* ∗ *exp* (*t* $*_R$ *A*)) (*at t within S*)
⟨*proof*⟩

lemma
fixes *a*::*complex*— TODO: generalize
assumes *a*: *0* < *Re a*
shows *integrable-on-cexp-minus-to-infinity*: (λ*x*. *exp* (*x* $*_R$ − *a*)) *integrable-on* {*c..*}
and *integral-cexp-minus-to-infinity*: *integral* {*c..*} (λ*x*. *exp* (*x* $*_R$ − *a*)) = *exp* (*c* $*_R$ − *a*) / *a*
⟨*proof*⟩

lemma *has-integral-cexp-minus-to-infinity*:
fixes *a*::*complex*— TODO: generalize
assumes *a*: *0* < *Re a*
shows ((λ*x*. *exp* (*x* $*_R$ − *a*)) *has-integral exp* (*c* $*_R$ − *a*) / *a*) {*c..*}
⟨*proof*⟩

lemma *has-laplace-one*:
((λ-. *1*) *has-laplace inverse s*) *s* **if** *Re s* > *0*
⟨*proof*⟩

lemma *has-laplace-add*:
assumes *f*: (*f has-laplace F*) *S*
assumes *g*: (*g has-laplace G*) *S*
shows ((λ*x*. *f x* + *g x*) *has-laplace F* + *G*) *S*
⟨*proof*⟩

lemma *has-laplace-cmul*:
assumes (*f has-laplace F*) *S*

shows (($\lambda x.\ r *_R f\ x$) *has-laplace* $r *_R F$) S
⟨*proof*⟩

lemma *has-laplace-uminus*:
assumes (f *has-laplace* F) S
shows (($\lambda x.\ - f\ x$) *has-laplace* $- F$) S
⟨*proof*⟩

lemma *has-laplace-minus*:
assumes f: (f *has-laplace* F) S
assumes g: (g *has-laplace* G) S
shows (($\lambda x.\ f\ x - g\ x$) *has-laplace* $F - G$) S
⟨*proof*⟩

lemma *has-laplace-spike*:
(f *has-laplace* L) s
if L: (g *has-laplace* L) s
and *negligible* T
and $\bigwedge t.\ t \notin T \Longrightarrow t \geq 0 \Longrightarrow f\ t = g\ t$
⟨*proof*⟩

lemma *has-laplace-frequency-shift*:— First Translation Theorem in Schiff
(($\lambda t.\ exp\ (t *_R b) * f\ t$) *has-laplace* L) s
if (f *has-laplace* L) ($s - b$)
⟨*proof*⟩

theorem *has-laplace-derivative-time-domain*:
(f' *has-laplace* $s * L - f0$) s
if L: (f *has-laplace* L) s
and f': $\bigwedge t.\ t > 0 \Longrightarrow$ (f *has-vector-derivative* $f'\ t$) (*at* t)
and $f0$: ($f \longrightarrow f0$) (*at-right* 0)
and *eo*: *exponential-order* $M\ c\ f$
and *cs*: $c <$ *Re* s
— Proof and statement follow "The Laplace Transform: Theory and Applications" by Joel L. Schiff.
⟨*proof*⟩

lemma *exp-times-has-integral*:
(($\lambda t.\ exp\ (c * t)$) *has-integral* (*if* $c = 0$ *then* t *else* $exp\ (c * t)\ /\ c$) $-$ (*if* $c = 0$ *then* $t0$ *else* $exp\ (c * t0)\ /\ c$)) $\{t0\ ..\ t\}$
if $t0 \leq t$
for $c\ t$::*real*
⟨*proof*⟩

lemma *integral-exp-times*:
integral $\{t0\ ..\ t\}$ ($\lambda t.\ exp\ (c * t)$) = (*if* $c = 0$ *then* $t - t0$ *else* $exp\ (c * t)\ /\ c - exp\ (c * t0)\ /\ c$)
if $t0 \leq t$

for *c t*::*real*
⟨*proof*⟩

lemma *filtermap-times-pos-at-top*: *filtermap* ((∗) *e*) *at-top* = *at-top*
if *e* > *0*
for *e*::*real*
⟨*proof*⟩

lemma *exponential-order-additiveI*:
assumes *0* < *M* **and** *eo*: $\forall_F$ *t in at-top. norm* (*f t*) ≤ *K* + *M* ∗ *exp* (*c* ∗ *t*) **and** *c* ≥ *0*
obtains *M′* **where** *exponential-order M′ c f*
⟨*proof*⟩

lemma *exponential-order-integral*:
fixes *f*::*real* ⇒ *′a*::*banach*
assumes *I*: ⋀*t*. *t* ≥ *a* ⟹ (*f has-integral I t*) {*a* .. *t*}
and *eo*: *exponential-order M c f*
and *c* > *0*
obtains *M′* **where** *exponential-order M′ c I*
⟨*proof*⟩

lemma *integral-has-vector-derivative-piecewise-continuous*:
fixes *f* :: *real* ⇒ *′a*::*euclidean-space*— TODO: generalize?
assumes *piecewise-continuous-on a b D f*
shows ⋀*x*. *x* ∈ {*a* .. *b*} − *D* ⟹
((λ*u*. *integral* {*a*..*u*} *f*) *has-vector-derivative f*(*x*)) (*at x within* {*a*..*b*} − *D*)
⟨*proof*⟩

lemma *has-derivative-at-split*:
(*f has-derivative f′*) (*at x*) ⟷ (*f has-derivative f′*) (*at-left x*) ∧ (*f has-derivative f′*) (*at-right x*)
for *x*::*′a*::{*linorder-topology*, *real-normed-vector*}
⟨*proof*⟩

lemma *has-vector-derivative-at-split*:
(*f has-vector-derivative f′*) (*at x*) ⟷
(*f has-vector-derivative f′*) (*at-left x*) ∧
(*f has-vector-derivative f′*) (*at-right x*)
⟨*proof*⟩

lemmas *differentiableI-vector*[*intro*]

lemma *differentiable-at-splitD*:
f differentiable at-left x
f differentiable at-right x
if *f differentiable* (*at x*)
for *x*::*real*
⟨*proof*⟩

lemma *integral-differentiable*:
fixes $f :: real \Rightarrow 'a::banach$
assumes *continuous-on* $\{a..b\}$ f
and $x \in \{a..b\}$
shows $(\lambda u.\ integral\ \{a..u\}\ f)$ *differentiable at x within* $\{a..b\}$
⟨*proof*⟩

theorem *integral-has-vector-derivative-piecewise-continuous'*:
fixes $f :: real \Rightarrow 'a::euclidean\text{-}space$— TODO: generalize?
assumes *piecewise-continuous-on a b D f* $a < b$
shows
$(\forall x.\ a < x \longrightarrow x < b \longrightarrow x \notin D \longrightarrow (\lambda u.\ integral\ \{a..u\}\ f)$ *differentiable at* $x) \wedge$
$(\forall x.\ a \leq x \longrightarrow x < b \longrightarrow (\lambda t.\ integral\ \{a..t\}\ f)$ *differentiable at-right* $x) \wedge$
$(\forall x.\ a < x \longrightarrow x \leq b \longrightarrow (\lambda t.\ integral\ \{a..t\}\ f)$ *differentiable at-left* $x)$
⟨*proof*⟩

lemma $closure\ (-S) \cap closure\ S = frontier\ S$
⟨*proof*⟩

theorem *integral-time-domain-has-laplace*:
$((\lambda t.\ integral\ \{0\ ..\ t\}\ f)\ has\text{-}laplace\ L\ /\ s)\ s$
if *pc*: $\bigwedge k.$ *piecewise-continuous-on 0 k D f*
and *eo*: *exponential-order M c f*
and *L*: $(f\ has\text{-}laplace\ L)\ s$
and *s*: $Re\ s > c$
and *c*: $c > 0$
and *TODO*: $D = \{\}$ — TODO: generalize to actual *piecewise-continuous-on*
for $f::real \Rightarrow complex$
⟨*proof*⟩

4.4 higher derivatives

definition $nderiv\ i\ f\ X = ((\lambda f.\ (\lambda x.\ vector\text{-}derivative\ f\ (at\ x\ within\ X)))^{\frown\frown}i)\ f$

definition $ndiff\ n\ f\ X \longleftrightarrow (\forall i<n.\ \forall x \in X.\ nderiv\ i\ f\ X$ *differentiable at x within* $X)$

lemma *nderiv-zero*[*simp*]: $nderiv\ 0\ f\ X = f$
⟨*proof*⟩

lemma *nderiv-Suc*[*simp*]:
$nderiv\ (Suc\ i)\ f\ X\ x = vector\text{-}derivative\ (nderiv\ i\ f\ X)\ (at\ x\ within\ X)$
⟨*proof*⟩

lemma *ndiff-zero*[*simp*]: $ndiff\ 0\ f\ X$
⟨*proof*⟩

lemma *ndiff-Sucs*[*simp*]:
  $ndiff\ (Suc\ i)\ f\ X \longleftrightarrow$
    $(ndiff\ i\ f\ X) \wedge$
    $(\forall x \in X.\ (nderiv\ i\ f\ X)\ differentiable\ (at\ x\ within\ X))$
  ⟨*proof*⟩

theorem *has-laplace-vector-derivative*:
  $((\lambda t.\ vector\text{-}derivative\ f\ (at\ t))\ has\text{-}laplace\ s * L - f0)\ s$
  if L: $(f\ has\text{-}laplace\ L)\ s$
    and f': $\bigwedge t.\ t > 0 \Longrightarrow f\ differentiable\ (at\ t)$
    and $f0$: $(f \longrightarrow f0)\ (at\text{-}right\ 0)$
    and eo: $exponential\text{-}order\ M\ c\ f$
    and cs: $c < Re\ s$
⟨*proof*⟩

lemma *has-laplace-nderiv*:
  $(nderiv\ n\ f\ \{0<..\}\ has\text{-}laplace\ s\,\hat{}\,n * L - (\sum i<n.\ s\,\hat{}\,(n - Suc\ i) * f0\ i))\ s$
  if L: $(f\ has\text{-}laplace\ L)\ s$
    and f': $ndiff\ n\ f\ \{0<..\}$
    and $f0$: $\bigwedge i.\ i < n \Longrightarrow (nderiv\ i\ f\ \{0<..\} \longrightarrow f0\ i)\ (at\text{-}right\ 0)$
    and eo: $\bigwedge i.\ i < n \Longrightarrow exponential\text{-}order\ M\ c\ (nderiv\ i\ f\ \{0<..\})$
    and cs: $c < Re\ s$
  ⟨*proof*⟩

end

5 Lerch Lemma

theory *Lerch-Lemma*
  imports
    HOL−Analysis.Analysis
begin

The main tool to prove uniqueness of the Laplace transform.

lemma *lerch-lemma-real*:
  fixes $h::real \Rightarrow real$
  assumes *h-cont*[*continuous-intros*]: $continuous\text{-}on\ \{0\ ..\ 1\}\ h$
  assumes *int-0*: $\bigwedge n.\ ((\lambda u.\ u\ \hat{}\ n * h\ u)\ has\text{-}integral\ 0)\ \{0\ ..\ 1\}$
  assumes u: $0 \le u\ u \le 1$
  shows $h\ u = 0$
⟨*proof*⟩

lemma *lerch-lemma*:
  fixes $h::real \Rightarrow {'a}::euclidean\text{-}space$
  assumes [*continuous-intros*]: $continuous\text{-}on\ \{0\ ..\ 1\}\ h$
  assumes *int-0*: $\bigwedge n.\ ((\lambda u.\ u\ \hat{}\ n *_R h\ u)\ has\text{-}integral\ 0)\ \{0\ ..\ 1\}$
  assumes u: $0 \le u\ u \le 1$
  shows $h\ u = 0$
⟨*proof*⟩

end

6 Uniqueness of Laplace Transform

theory *Uniqueness*
  imports
    Existence
    Lerch-Lemma
begin

We show uniqueness of the Laplace transform for continuous functions.

lemma *laplace-transform-zero*:— should also work for piecewise continuous
  assumes *cont-f*: *continuous-on* $\{0..\}$ *f*
  assumes *eo*: *exponential-order M a f*
  assumes *laplace*: $\bigwedge s.\ Re\ s > a \Longrightarrow (f\ has\text{-}laplace\ 0)\ s$
  assumes $t \geq 0$
  shows $f\ t = 0$
⟨*proof*⟩

lemma *exponential-order-eventually-eq*: *exponential-order M a f*
  if *exponential-order M a g* $\bigwedge t.\ t \geq k \Longrightarrow f\ t = g\ t$
⟨*proof*⟩

lemma *exponential-order-mono*:
  assumes *eo*: *exponential-order M a f*
  assumes $a \leq b$ $M \leq N$
  shows *exponential-order N b f*
⟨*proof*⟩

lemma *exponential-order-uminus-iff*:
  exponential-order M a $(\lambda x.\ -\ f\ x)$ = *exponential-order M a f*
  ⟨*proof*⟩

lemma *exponential-order-add*:
  assumes *exponential-order M a f exponential-order M a g*
  shows *exponential-order* $(2 * M)$ *a* $(\lambda x.\ f\ x + g\ x)$
  ⟨*proof*⟩

theorem *laplace-transform-unique*:
  assumes *f*: $\bigwedge s.\ Re\ s > a \Longrightarrow (f\ has\text{-}laplace\ F)\ s$
  assumes *g*: $\bigwedge s.\ Re\ s > b \Longrightarrow (g\ has\text{-}laplace\ F)\ s$
  assumes [*continuous-intros*]: *continuous-on* $\{0..\}$ *f*
  assumes [*continuous-intros*]: *continuous-on* $\{0..\}$ *g*
  assumes *eof*: *exponential-order M a f*
  assumes *eog*: *exponential-order N b g*
  assumes $t \geq 0$
  shows $f\ t = g\ t$
⟨*proof*⟩

end
theory *Laplace-Transform*
  imports
    Existence
    Uniqueness
begin

end

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