Laplace Transform

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Abstract

This entry formalizes the Laplace transform and concrete Laplace transforms for arithmetic functions, frequency shift, integration and (higher) differentiation in the time domain. It proves Lerch's lemma and uniqueness of the Laplace transform for continuous functions. In order to formalize the foundational assumptions, this entry contains a formalization of piecewise continuous functions and functions of exponential order.

Contents

theory *Laplace-Transform-Library*
  imports
    HOL−Analysis.Analysis
begin

1 References

Much of this formalization is based on Schiff's textbook [3]. Parts of this formalization are inspired by the HOL-Light formalization ([4], [1], [2]), but stated more generally for piecewise continuous (instead of piecewise continuously differentiable) functions.

2 Library Additions

2.1 Derivatives

lemma *DERIV-compose-FDERIV*:— TODO: generalize and move from HOL-ODE
  assumes $DERIV\ f\ (g\ x) :> f'$
  assumes $(g\ \mathit{has\text{-}derivative}\ g')\ (\mathit{at}\ x\ \mathit{within}\ s)$
  shows $((\lambda x.\ f\ (g\ x))\ \mathit{has\text{-}derivative}\ (\lambda x.\ g'\ x * f'))\ (\mathit{at}\ x\ \mathit{within}\ s)$
  ⟨*proof*⟩

lemmas *has-derivative-sin*[*derivative-intros*] = *DERIV-sin*[*THEN DERIV-compose-FDERIV*]
 and *has-derivative-cos*[*derivative-intros*] = *DERIV-cos*[*THEN DERIV-compose-FDERIV*]
 and *has-derivative-exp*[*derivative-intros*] = *DERIV-exp*[*THEN DERIV-compose-FDERIV*]

2.2 Integrals

lemma *negligible-real-ivlI*:
  fixes $a\ b::\mathit{real}$
  assumes $a \geq b$
  shows $\mathit{negligible}\ \{a\ ..\ b\}$
⟨*proof*⟩

lemma *absolutely-integrable-on-combine*:
  fixes $f :: \mathit{real} \Rightarrow {'a}::\mathit{euclidean\text{-}space}$
  assumes $f\ \mathit{absolutely\text{-}integrable\text{-}on}\ \{a..c\}$
    and $f\ \mathit{absolutely\text{-}integrable\text{-}on}\ \{c..b\}$
    and $a \leq c$
    and $c \leq b$
  shows $f\ \mathit{absolutely\text{-}integrable\text{-}on}\ \{a..b\}$
  ⟨*proof*⟩

lemma *dominated-convergence-at-top*:
  fixes $f :: \mathit{real} \Rightarrow {'n}::\mathit{euclidean\text{-}space} \Rightarrow {'m}::\mathit{euclidean\text{-}space}$
  assumes f: $\bigwedge k.\ (f\ k)\ \mathit{integrable\text{-}on}\ s$ **and** h: $h\ \mathit{integrable\text{-}on}\ s$
    and le: $\bigwedge k\ x.\ x \in s \Longrightarrow \mathit{norm}\ (f\ k\ x) \leq h\ x$
    and $conv$: $\forall x \in s.\ ((\lambda k.\ f\ k\ x) \longrightarrow g\ x)\ \mathit{at\text{-}top}$
  shows $g\ \mathit{integrable\text{-}on}\ s\ ((\lambda k.\ \mathit{integral}\ s\ (f\ k)) \longrightarrow \mathit{integral}\ s\ g)\ \mathit{at\text{-}top}$
⟨*proof*⟩

lemma *has-integral-dominated-convergence-at-top*:

fixes f :: *real* $\Rightarrow$ $'n$::*euclidean-space* $\Rightarrow$ $'m$::*euclidean-space*
assumes $\bigwedge k$. (f k *has-integral* y k) s h *integrable-on* s
$\bigwedge k\ x.\ x{\in}s \Longrightarrow$ *norm* (f k x) $\leq$ h x $\forall\, x{\in}s$. ((λk. f k x) $\longrightarrow$ g x) *at-top*
and x: (y $\longrightarrow$ x) *at-top*
shows (g *has-integral* x) s
⟨*proof*⟩

lemma *integral-indicator-eq-restriction*:
fixes f::$'a$::*euclidean-space* $\Rightarrow$ $'b$::*banach*
assumes f: f *integrable-on* R
and $R \subseteq S$
shows *integral* S (λx. *indicator* R x $*_R$ f x) = *integral* R f
⟨*proof*⟩

lemma
improper-integral-at-top:
fixes f::*real* $\Rightarrow$ $'a$::*euclidean-space*
assumes f *absolutely-integrable-on* $\{a..\}$
shows ((λx. *integral* $\{a..x\}$ f) $\longrightarrow$ *integral* $\{a..\}$ f) *at-top*
⟨*proof*⟩

lemma *norm-integrable-onI*: (λx. *norm* (f x)) *integrable-on* S
if f *absolutely-integrable-on* S
for f::$'a$::*euclidean-space*$\Rightarrow$$'b$::*euclidean-space*
⟨*proof*⟩

lemma
has-integral-improper-at-topI:
fixes f::*real* $\Rightarrow$ $'a$::*banach*
assumes I: $\forall_F$ k *in at-top*. (f *has-integral* I k) $\{a..k\}$
assumes J: (I $\longrightarrow$ J) *at-top*
shows (f *has-integral* J) $\{a..\}$
⟨*proof*⟩

lemma *has-integral-improperE*:
fixes f::*real* $\Rightarrow$ $'a$::*euclidean-space*
assumes I: (f *has-integral* I) $\{a..\}$
assumes *ai*: f *absolutely-integrable-on* $\{a..\}$
obtains J **where**
$\bigwedge k$. (f *has-integral* J k) $\{a..k\}$
(J $\longrightarrow$ I) *at-top*
⟨*proof*⟩

2.3 Miscellaneous

lemma *AE-BallI*: *AE* $x{\in}X$ *in* F. P x **if** $\forall\, x \in X$. P x
⟨*proof*⟩

lemma *bounded-le-Sup*:

assumes *bounded* $(f \text{ ` } S)$
shows $\forall\, x{\in}S.\ norm\ (f\ x) \leq Sup\ (norm \text{ ` } f \text{ ` } S)$
$\langle proof \rangle$

end

3 Piecewise Continous Functions

theory *Piecewise-Continuous*
imports
Laplace-Transform-Library
begin

3.1 at within filters

lemma *at-within-self-singleton*[*simp*]: $at\ i\ within\ \{i\} = bot$
$\langle proof \rangle$

lemma *at-within-t1-space-avoid*:
$(at\ x\ within\ X - \{i\}) = (at\ x\ within\ X)$ **if** $x \neq i$ **for** $x\ i{::}'a{::}t1\text{-}space$
$\langle proof \rangle$

lemma *at-within-t1-space-avoid-finite*:
$(at\ x\ within\ X - I) = (at\ x\ within\ X)$ **if** $finite\ I\ x \notin I$ **for** $x{::}'a{::}t1\text{-}space$
$\langle proof \rangle$

lemma *at-within-interior*:
$NO\text{-}MATCH\ (UNIV{::}'a\ set)\ (S{::}'a{::}topological\text{-}space\ set) \Longrightarrow x \in interior\ S \Longrightarrow at\ x\ within\ S = at\ x$
$\langle proof \rangle$

3.2 intervals

lemma *Compl-Icc*: $-\ \{a\ ..\ b\} = \{..{<}a\} \cup \{b{<}..\}$ **for** $a\ b{::}'a{::}linorder$
$\langle proof \rangle$

lemma *interior-Icc*[*simp*]: $interior\ \{a..b\} = \{a{<}..{<}b\}$
for $a\ b{::}'a{::}\{linorder\text{-}topology,\ dense\text{-}order,\ no\text{-}bot,\ no\text{-}top\}$
— TODO: is *no-bot* and *no-top* really required?
$\langle proof \rangle$

lemma *closure-finite*[*simp*]: $closure\ X = X$ **if** $finite\ X$ **for** $X{::}'a{::}t1\text{-}space\ set$
$\langle proof \rangle$

definition $piecewise\text{-}continuous\text{-}on :: 'a{::}linorder\text{-}topology \Rightarrow 'a \Rightarrow 'a\ set \Rightarrow ('a \Rightarrow 'b{::}topological\text{-}space) \Rightarrow bool$
where $piecewise\text{-}continuous\text{-}on\ a\ b\ I\ f \longleftrightarrow$
$(continuous\text{-}on\ (\{a\ ..\ b\} - I)\ f \wedge finite\ I \wedge$
$(\forall\, i{\in}I.\ (i \in \{a{<}..b\} \longrightarrow (\exists\, l.\ (f \longrightarrow l)\ (at\text{-}left\ i))) \wedge$

$(i \in \{a..<b\} \longrightarrow (\exists u.\ (f \longrightarrow u)\ (\textit{at-right}\ i)))))$

lemma *piecewise-continuous-on-subset*:
$\textit{piecewise-continuous-on}\ a\ b\ I\ f \Longrightarrow \{c\ ..\ d\} \subseteq \{a\ ..\ b\} \Longrightarrow \textit{piecewise-continuous-on}\ c\ d\ I\ f$
⟨*proof*⟩

lemma *piecewise-continuous-onE*:
assumes $\textit{piecewise-continuous-on}\ a\ b\ I\ f$
obtains $l\ u$
where $\textit{finite}\ I$
and $\textit{continuous-on}\ (\{a..b\} - I)\ f$
and $(\bigwedge i.\ i \in I \Longrightarrow a < i \Longrightarrow i \le b \Longrightarrow (f \longrightarrow l\ i)\ (\textit{at-left}\ i))$
and $(\bigwedge i.\ i \in I \Longrightarrow a \le i \Longrightarrow i < b \Longrightarrow (f \longrightarrow u\ i)\ (\textit{at-right}\ i))$
⟨*proof*⟩

lemma *piecewise-continuous-onI*:
assumes $\textit{finite}\ I\ \textit{continuous-on}\ (\{a..b\} - I)\ f$
and $(\bigwedge i.\ i \in I \Longrightarrow a < i \Longrightarrow i \le b \Longrightarrow (f \longrightarrow l\ i)\ (\textit{at-left}\ i))$
and $(\bigwedge i.\ i \in I \Longrightarrow a \le i \Longrightarrow i < b \Longrightarrow (f \longrightarrow u\ i)\ (\textit{at-right}\ i))$
shows $\textit{piecewise-continuous-on}\ a\ b\ I\ f$
⟨*proof*⟩

lemma *piecewise-continuous-onI'*:
fixes $a\ b{::}'a{::}\{\textit{linorder-topology},\ \textit{dense-order},\ \textit{no-bot},\ \textit{no-top}\}$
assumes $\textit{finite}\ I\ \bigwedge x.\ a < x \Longrightarrow x < b \Longrightarrow \textit{isCont}\ f\ x$
and $a \notin I \Longrightarrow \textit{continuous}\ (\textit{at-right}\ a)\ f$
and $b \notin I \Longrightarrow \textit{continuous}\ (\textit{at-left}\ b)\ f$
and $(\bigwedge i.\ i \in I \Longrightarrow a < i \Longrightarrow i \le b \Longrightarrow (f \longrightarrow l\ i)\ (\textit{at-left}\ i))$
and $(\bigwedge i.\ i \in I \Longrightarrow a \le i \Longrightarrow i < b \Longrightarrow (f \longrightarrow u\ i)\ (\textit{at-right}\ i))$
shows $\textit{piecewise-continuous-on}\ a\ b\ I\ f$
⟨*proof*⟩

lemma *piecewise-continuous-onE'*:
fixes $a\ b{::}'a{::}\{\textit{linorder-topology},\ \textit{dense-order},\ \textit{no-bot},\ \textit{no-top}\}$
assumes $\textit{piecewise-continuous-on}\ a\ b\ I\ f$
obtains $l\ u$
where $\textit{finite}\ I$
and $\bigwedge x.\ a < x \Longrightarrow x < b \Longrightarrow x \notin I \Longrightarrow \textit{isCont}\ f\ x$
and $(\bigwedge x.\ a < x \Longrightarrow x \le b \Longrightarrow (f \longrightarrow l\ x)\ (\textit{at-left}\ x))$
and $(\bigwedge x.\ a \le x \Longrightarrow x < b \Longrightarrow (f \longrightarrow u\ x)\ (\textit{at-right}\ x))$
and $\bigwedge x.\ a \le x \Longrightarrow x \le b \Longrightarrow x \notin I \Longrightarrow f\ x = l\ x$
and $\bigwedge x.\ a \le x \Longrightarrow x \le b \Longrightarrow x \notin I \Longrightarrow f\ x = u\ x$
⟨*proof*⟩

lemma *tendsto-avoid-at-within*:
$(f \longrightarrow l)\ (\textit{at}\ x\ \textit{within}\ X)$
if $(f \longrightarrow l)\ (\textit{at}\ x\ \textit{within}\ X - \{x\})$
⟨*proof*⟩

lemma *tendsto-within-subset-eventuallyI*:
$(f \longrightarrow fx)$ $(at\ x\ within\ X)$
if *g*: $(g \longrightarrow gy)$ $(at\ y\ within\ Y)$
and *ev*: $\forall_F\ x\ in\ (at\ y\ within\ Y).\ f\ x = g\ x$
and *xy*: $x = y$
and *fxgy*: $fx = gy$
and *XY*: $X - \{x\} \subseteq Y$
⟨*proof*⟩

lemma *piecewise-continuous-on-insertE*:
assumes *piecewise-continuous-on a b* (*insert i I*) *f*
assumes $i \in \{a\ ..\ b\}$
obtains *g h* **where**
piecewise-continuous-on a i I g
piecewise-continuous-on i b I h
$\bigwedge x.\ a \leq x \Longrightarrow x < i \Longrightarrow g\ x = f\ x$
$\bigwedge x.\ i < x \Longrightarrow x \leq b \Longrightarrow h\ x = f\ x$
⟨*proof*⟩

lemma *eventually-avoid-finite*:
$\forall_F\ x\ in\ at\ y\ within\ Y.\ x \notin I$ **if** *finite I* **for** *y*::*'a*::*t1-space*
⟨*proof*⟩

lemma *eventually-at-left-linorder*:— TODO: generalize $?b < ?a \Longrightarrow \forall_F\ x$ *in at-left* $?a.\ x \in \{?b{<}..{<}?a\}$
$a > (b :: 'a :: linorder\text{-}topology) \Longrightarrow eventually\ (\lambda x.\ x \in \{b{<}..{<}a\})\ (at\text{-}left\ a)$
⟨*proof*⟩

lemma *eventually-at-right-linorder*:— TODO: generalize $?a < ?b \Longrightarrow \forall_F\ x$ *in at-right* $?a.\ x \in \{?a{<}..{<}?b\}$
$a > (b :: 'a :: linorder\text{-}topology) \Longrightarrow eventually\ (\lambda x.\ x \in \{b{<}..{<}a\})\ (at\text{-}right\ b)$
⟨*proof*⟩

lemma *piecewise-continuous-on-congI*:
piecewise-continuous-on a b I g
if *piecewise-continuous-on a b I f*
and *eq*: $\bigwedge x.\ x \in \{a\ ..\ b\} - I \Longrightarrow g\ x = f\ x$
⟨*proof*⟩

lemma *piecewise-continuous-on-cong*[*cong*]:
piecewise-continuous-on a b I f ⟷ *piecewise-continuous-on c d J g*
if $a = c$
$b = d$
$I = J$
$\bigwedge x.\ c \leq x \Longrightarrow x \leq d \Longrightarrow x \notin J \Longrightarrow f\ x = g\ x$
⟨*proof*⟩

lemma *tendsto-at-left-continuous-on-avoidI*: $(f \longrightarrow g\ i)$ $(at\text{-}left\ i)$

if g: *continuous-on* $(\{a..i\} - I)$ g
and gf: $\bigwedge x.\ a < x \Longrightarrow x < i \Longrightarrow g\ x = f\ x$
$i \notin I$ *finite* I $a < i$
for i::$'a$::*linorder-topology*
⟨*proof*⟩

lemma *tendsto-at-right-continuous-on-avoidI*: $(f \longrightarrow g\ i)$ (*at-right* i)
if g: *continuous-on* $(\{i..b\} - I)$ g
and gf: $\bigwedge x.\ i < x \Longrightarrow x < b \Longrightarrow g\ x = f\ x$
$i \notin I$ *finite* I $i < b$
for i::$'a$::*linorder-topology*
⟨*proof*⟩

lemma *piecewise-continuous-on-insert-leftI*:
piecewise-continuous-on a b (*insert* a I) f **if** *piecewise-continuous-on* a b I f
⟨*proof*⟩

lemma *piecewise-continuous-on-insert-rightI*:
piecewise-continuous-on a b (*insert* b I) f **if** *piecewise-continuous-on* a b I f
⟨*proof*⟩

theorem *piecewise-continuous-on-induct*[*consumes 1*, *case-names empty combine weaken*]:
assumes *pc*: *piecewise-continuous-on* a b I f
assumes *1*: $\bigwedge a\ b\ f.$ *continuous-on* $\{a\ ..\ b\}$ $f \Longrightarrow P\ a\ b\ \{\}\ f$
assumes *2*: $\bigwedge a\ i\ b\ I\ f1\ f2\ f.\ a \le i \Longrightarrow i \le b \Longrightarrow i \notin I \Longrightarrow P\ a\ i\ I\ f1 \Longrightarrow P\ i\ b\ I\ f2 \Longrightarrow$
piecewise-continuous-on a i I $f1 \Longrightarrow$
piecewise-continuous-on i b I $f2 \Longrightarrow$
$(\bigwedge x.\ a \le x \Longrightarrow x < i \Longrightarrow f1\ x = f\ x) \Longrightarrow$
$(\bigwedge x.\ i < x \Longrightarrow x \le b \Longrightarrow f2\ x = f\ x) \Longrightarrow$
$(i > a \Longrightarrow (f \longrightarrow f1\ i)$ (*at-left* i)$) \Longrightarrow$
$(i < b \Longrightarrow (f \longrightarrow f2\ i)$ (*at-right* i)$) \Longrightarrow$
$P\ a\ b$ (*insert* i I) f
assumes *3*: $\bigwedge a\ b\ i\ I\ f.\ P\ a\ b\ I\ f \Longrightarrow$ *finite* $I \Longrightarrow i \notin I \Longrightarrow P\ a\ b$ (*insert* i I) f
shows $P\ a\ b\ I\ f$
⟨*proof*⟩

lemma *continuous-on-imp-piecewise-continuous-on*:
continuous-on $\{a\ ..\ b\}$ $f \Longrightarrow$ *piecewise-continuous-on* a b $\{\}$ f
⟨*proof*⟩

lemma *piecewise-continuous-on-imp-absolutely-integrable*:
fixes a b::*real* **and** f::*real* $\Rightarrow$ $'a$::*euclidean-space*
assumes *piecewise-continuous-on* a b I f
shows f *absolutely-integrable-on* $\{a..b\}$
⟨*proof*⟩

lemma *piecewise-continuous-on-integrable*:

fixes $a\ b$::*real* **and** f::*real* $\Rightarrow$ $'a$::*euclidean-space*
assumes *piecewise-continuous-on* $a\ b\ I\ f$
shows f *integrable-on* $\{a..b\}$
$\langle proof \rangle$

lemma *piecewise-continuous-on-comp*:
assumes p: *piecewise-continuous-on* $a\ b\ I\ f$
assumes c: $\bigwedge x.\ isCont\ (\lambda(x,\ y).\ g\ x\ y)\ x$
shows *piecewise-continuous-on* $a\ b\ I\ (\lambda x.\ g\ x\ (f\ x))$
$\langle proof \rangle$

lemma *bounded-piecewise-continuous-image*:
bounded $(f\ `\ \{a\ ..\ b\})$
if *piecewise-continuous-on* $a\ b\ I\ f$ **for** $a\ b$::*real*
$\langle proof \rangle$

lemma *tendsto-within-eventually*:
$(f \longrightarrow l)\ (at\ x\ within\ X)$
if
$(f \longrightarrow l)\ (at\ x\ within\ Y)$
$\forall_F\ y\ in\ at\ x\ within\ X.\ y \in Y$
$\langle proof \rangle$

lemma *at-within-eq-bot-lemma*:
$at\ x\ within\ \{b..c\} = (if\ x < b \lor b > c\ then\ bot\ else\ at\ x\ within\ \{b..c\})$
for $x\ b\ c$::$'a$::*linorder-topology*
$\langle proof \rangle$

lemma *at-within-eq-bot-lemma2*:
$at\ x\ within\ \{a..b\} = (if\ x > b \lor a > b\ then\ bot\ else\ at\ x\ within\ \{a..b\})$
for $x\ a\ b$::$'a$::*linorder-topology*
$\langle proof \rangle$

lemma *piecewise-continuous-on-combine*:
piecewise-continuous-on $a\ c\ J\ f$
if *piecewise-continuous-on* $a\ b\ J\ f$ *piecewise-continuous-on* $b\ c\ J\ f$
$\langle proof \rangle$

lemma *piecewise-continuous-on-finite-superset*:
piecewise-continuous-on $a\ b\ I\ f \Longrightarrow I \subseteq J \Longrightarrow$ *finite* $J \Longrightarrow$ *piecewise-continuous-on* $a\ b\ J\ f$
for $a\ b$::$'a$::$\{$*linorder-topology*, *dense-order*, *no-bot*, *no-top*$\}$
$\langle proof \rangle$

lemma *piecewise-continuous-on-splitI*:
piecewise-continuous-on $a\ c\ K\ f$
if
piecewise-continuous-on $a\ b\ I\ f$
piecewise-continuous-on $b\ c\ J\ f$

$I \subseteq K$ $J \subseteq K$ *finite* K
for a b::$'a$::{*linorder-topology*, *dense-order*, *no-bot*, *no-top*}
⟨*proof*⟩

end

4 Existence

theory *Existence* **imports**
Piecewise-Continuous
begin

4.1 Definition

definition *has-laplace* :: (*real* ⇒ *complex*) ⇒ *complex* ⇒ *complex* ⇒ *bool*
(**infixr** *has′-laplace* *46*)
where (*f* *has-laplace* L) s ⟷ ((λt. *exp* ($t *_R - s$) $*$ f t) *has-integral* L) {*0*..}

lemma *has-laplaceI*:
assumes ((λt. *exp* ($t *_R - s$) $*$ f t) *has-integral* L) {*0*..}
shows (*f* *has-laplace* L) s
⟨*proof*⟩

lemma *has-laplaceD*:
assumes (*f* *has-laplace* L) s
shows ((λt. *exp* ($t *_R - s$) $*$ f t) *has-integral* L) {*0*..}
⟨*proof*⟩

lemma *has-laplace-unique*:
$L = M$ **if**
(*f* *has-laplace* L) s
(*f* *has-laplace* M) s
⟨*proof*⟩

4.2 Condition for Existence: Exponential Order

definition *exponential-order* M c f ⟷ $0 < M$ ∧ ($\forall_F$ t *in* *at-top*. *norm* (f t) $\leq$ $M * exp$ ($c * t$))

lemma *exponential-orderI*:
assumes $0 < M$ **and** *eo*: $\forall_F$ t *in* *at-top*. *norm* (f t) $\leq$ $M * exp$ ($c * t$)
shows *exponential-order* M c f
⟨*proof*⟩

lemma *exponential-orderD*:
assumes *exponential-order* M c f
shows $0 < M$ $\forall_F$ t *in* *at-top*. *norm* (f t) $\leq$ $M * exp$ ($c * t$)
⟨*proof*⟩

context
fixes *f*::*real* ⇒ *complex*
begin

definition *laplace-integrand*::*complex* ⇒ *real* ⇒ *complex*
where *laplace-integrand s t* = *exp* ($t *_R - s$) ∗ *f t*

lemma *laplace-integrand-absolutely-integrable-on-Icc*:
laplace-integrand s absolutely-integrable-on {*a*..*b*}
if *AE x*∈{*a*..*b*} *in lebesgue. cmod* (*f x*) ≤ *B f integrable-on* {*a*..*b*}
⟨*proof*⟩

lemma *laplace-integrand-integrable-on-Icc*:
laplace-integrand s integrable-on {*a*..*b*}
if *AE x*∈{*a*..*b*} *in lebesgue. cmod* (*f x*) ≤ *B f integrable-on* {*a*..*b*}
⟨*proof*⟩

lemma *eventually-laplace-integrand-le*:
$\forall_F$ *t in at-top. cmod* (*laplace-integrand s t*) ≤ *M* ∗ *exp* (− (*Re s* − *c*) ∗ *t*)
if *exponential-order M c f*
⟨*proof*⟩

lemma
assumes *eo*: *exponential-order M c f*
and *cs*: *c* < *Re s*
shows *laplace-integrand-integrable-on-Ici-iff*:
laplace-integrand s integrable-on {*a*..} ⟷
(∀ *k*>*a*. *laplace-integrand s integrable-on* {*a*..*k*})
(**is** *?th1*)
and *laplace-integrand-absolutely-integrable-on-Ici-iff*:
laplace-integrand s absolutely-integrable-on {*a*..} ⟷
(∀ *k*>*a*. *laplace-integrand s absolutely-integrable-on* {*a*..*k*})
(**is** *?th2*)
⟨*proof*⟩

theorem *laplace-exists-laplace-integrandI*:
assumes *laplace-integrand s integrable-on* {*0*..}
obtains *F* **where** (*f has-laplace F*) *s*
⟨*proof*⟩

lemma
assumes *eo*: *exponential-order M c f*
and *pc*: ⋀*k*. *AE x*∈{*0*..*k*} *in lebesgue. cmod* (*f x*) ≤ *B k* ⋀*k*. *f integrable-on* {*0*..*k*}
and *s*: *Re s* > *c*
shows *laplace-integrand-integrable*: *laplace-integrand s integrable-on* {*0*..} (**is** *?th1*)
and *laplace-integrand-absolutely-integrable*:
laplace-integrand s absolutely-integrable-on {*0*..} (**is** *?th2*)

⟨*proof*⟩

lemma *piecewise-continuous-on-AE-boundedE*:
assumes *pc*: $\bigwedge k$. *piecewise-continuous-on a k (I k) f*
obtains *B* **where** $\bigwedge k$. *AE x*∈{*a..k*} *in lebesgue. cmod (f x)* ≤ *B k*
⟨*proof*⟩

theorem *piecewise-continuous-on-has-laplace*:
assumes *eo*: *exponential-order M c f*
and *pc*: $\bigwedge k$. *piecewise-continuous-on 0 k (I k) f*
and *s*: *Re s* > *c*
obtains *F* **where** (*f has-laplace F*) *s*
⟨*proof*⟩

end

4.3 Concrete Laplace Transforms

lemma *exp-scaleR-has-vector-derivative-left'*[*derivative-intros*]:
((λt. *exp* ($t *_R A$)) *has-vector-derivative A* ∗ *exp* ($t *_R A$)) (*at t within S*)
⟨*proof*⟩

lemma
fixes *a*::*complex*— TODO: generalize
assumes *a*: *0* < *Re a*
shows *integrable-on-cexp-minus-to-infinity*: (λx. *exp* ($x *_R - a$)) *integrable-on* {*c..*}
and *integral-cexp-minus-to-infinity*: *integral* {*c..*} (λx. *exp* ($x *_R - a$)) = *exp* ($c *_R - a$) / *a*
⟨*proof*⟩

lemma *has-integral-cexp-minus-to-infinity*:
fixes *a*::*complex*— TODO: generalize
assumes *a*: *0* < *Re a*
shows ((λx. *exp* ($x *_R - a$)) *has-integral exp* ($c *_R - a$) / *a*) {*c..*}
⟨*proof*⟩

lemma *has-laplace-one*:
((λ-. *1*) *has-laplace inverse s*) *s* **if** *Re s* > *0*
⟨*proof*⟩

lemma *has-laplace-add*:
assumes *f*: (*f has-laplace F*) *S*
assumes *g*: (*g has-laplace G*) *S*
shows ((λx. *f x* + *g x*) *has-laplace F* + *G*) *S*
⟨*proof*⟩

lemma *has-laplace-cmul*:
assumes (*f has-laplace F*) *S*

shows $((\lambda x.\ r *_R f\ x)$ *has-laplace* $r *_R F)\ S$
⟨*proof*⟩

lemma *has-laplace-uminus*:
assumes $(f$ *has-laplace* $F)\ S$
shows $((\lambda x.\ - f\ x)$ *has-laplace* $- F)\ S$
⟨*proof*⟩

lemma *has-laplace-minus*:
assumes f: $(f$ *has-laplace* $F)\ S$
assumes g: $(g$ *has-laplace* $G)\ S$
shows $((\lambda x.\ f\ x - g\ x)$ *has-laplace* $F - G)\ S$
⟨*proof*⟩

lemma *has-laplace-spike*:
$(f$ *has-laplace* $L)\ s$
if L: $(g$ *has-laplace* $L)\ s$
and *negligible* T
and $\bigwedge t.\ t \notin T \Longrightarrow t \geq 0 \Longrightarrow f\ t = g\ t$
⟨*proof*⟩

lemma *has-laplace-frequency-shift*:— First Translation Theorem in Schiff
$((\lambda t.\ exp\ (t *_R b) * f\ t)$ *has-laplace* $L)\ s$
if $(f$ *has-laplace* $L)\ (s - b)$
⟨*proof*⟩

theorem *has-laplace-derivative-time-domain*:
$(f'$ *has-laplace* $s * L - f0)\ s$
if L: $(f$ *has-laplace* $L)\ s$
and f': $\bigwedge t.\ t > 0 \Longrightarrow (f$ *has-vector-derivative* $f'\ t)\ (at\ t)$
and $f0$: $(f \longrightarrow f0)\ (at\text{-}right\ 0)$
and *eo*: *exponential-order* $M\ c\ f$
and *cs*: $c < Re\ s$
— Proof and statement follow "The Laplace Transform: Theory and Applications" by Joel L. Schiff.
⟨*proof*⟩

lemma *exp-times-has-integral*:
$((\lambda t.\ exp\ (c * t))$ *has-integral* $(if\ c = 0\ then\ t\ else\ exp\ (c * t)\ /\ c) - (if\ c = 0\ then\ t0\ else\ exp\ (c * t0)\ /\ c))\ \{t0\ ..\ t\}$
if $t0 \leq t$
for $c\ t::real$
⟨*proof*⟩

lemma *integral-exp-times*:
$integral\ \{t0\ ..\ t\}\ (\lambda t.\ exp\ (c * t)) = (if\ c = 0\ then\ t - t0\ else\ exp\ (c * t)\ /\ c - exp\ (c * t0)\ /\ c)$
if $t0 \leq t$

for *c t*::*real*
⟨*proof*⟩

lemma *filtermap-times-pos-at-top*: *filtermap* $((*)\ e)$ *at-top* = *at-top*
if $e > 0$
for *e*::*real*
⟨*proof*⟩

lemma *exponential-order-additiveI*:
assumes $0 < M$ **and** *eo*: $\forall_F$ *t in at-top. norm* $(f\ t) \leq K + M * exp\ (c * t)$ **and** $c \geq 0$
obtains M' **where** *exponential-order* M' *c f*
⟨*proof*⟩

lemma *exponential-order-integral*:
fixes *f*::*real* ⇒ *'a*::*banach*
assumes *I*: $\bigwedge t.\ t \geq a \Longrightarrow$ (*f has-integral I t*) $\{a\ ..\ t\}$
and *eo*: *exponential-order M c f*
and $c > 0$
obtains M' **where** *exponential-order* M' *c I*
⟨*proof*⟩

lemma *integral-has-vector-derivative-piecewise-continuous*:
fixes *f* :: *real* ⇒ *'a*::*euclidean-space*— TODO: generalize?
assumes *piecewise-continuous-on a b D f*
shows $\bigwedge x.\ x \in \{a\ ..\ b\} - D \Longrightarrow$
((λ*u. integral* {*a..u*} *f*) *has-vector-derivative f*(*x*)) (*at x within* {*a..b*} − *D*)
⟨*proof*⟩

lemma *has-derivative-at-split*:
(*f has-derivative f'*) (*at x*) ⟷ (*f has-derivative f'*) (*at-left x*) ∧ (*f has-derivative f'*) (*at-right x*)
for *x*::*'a*::{*linorder-topology*, *real-normed-vector*}
⟨*proof*⟩

lemma *has-vector-derivative-at-split*:
(*f has-vector-derivative f'*) (*at x*) ⟷
(*f has-vector-derivative f'*) (*at-left x*) ∧
(*f has-vector-derivative f'*) (*at-right x*)
⟨*proof*⟩

lemmas *differentiableI-vector*[*intro*]

lemma *differentiable-at-splitD*:
f differentiable at-left x
f differentiable at-right x
if *f differentiable* (*at x*)
for *x*::*real*
⟨*proof*⟩

lemma *integral-differentiable*:
fixes f :: *real* $\Rightarrow$ $'a$::*banach*
assumes *continuous-on* $\{a..b\}$ f
and $x \in \{a..b\}$
shows $(\lambda u.$ *integral* $\{a..u\}$ $f)$ *differentiable at* x *within* $\{a..b\}$
$\langle proof \rangle$

theorem *integral-has-vector-derivative-piecewise-continuous'*:
fixes f :: *real* $\Rightarrow$ $'a$::*euclidean-space*— TODO: generalize?
assumes *piecewise-continuous-on* a b D f $a < b$
shows
$(\forall x.\ a < x \longrightarrow x < b \longrightarrow x \notin D \longrightarrow (\lambda u.$ *integral* $\{a..u\}$ $f)$ *differentiable at* $x) \wedge$
$(\forall x.\ a \leq x \longrightarrow x < b \longrightarrow (\lambda t.$ *integral* $\{a..t\}$ $f)$ *differentiable at-right* $x) \wedge$
$(\forall x.\ a < x \longrightarrow x \leq b \longrightarrow (\lambda t.$ *integral* $\{a..t\}$ $f)$ *differentiable at-left* $x)$
$\langle proof \rangle$

lemma *closure* $(-S) \cap$ *closure* S = *frontier* S
$\langle proof \rangle$

theorem *integral-time-domain-has-laplace*:
$((\lambda t.$ *integral* $\{0\ ..\ t\}$ $f)$ *has-laplace* L / $s)$ s
if *pc*: $\bigwedge k.$ *piecewise-continuous-on* 0 k D f
and *eo*: *exponential-order* M c f
and L: (f *has-laplace* L) s
and s: *Re* $s > c$
and c: $c > 0$
and *TODO*: $D = \{\}$ — TODO: generalize to actual *piecewise-continuous-on*
for f::*real* $\Rightarrow$ *complex*
$\langle proof \rangle$

4.4 higher derivatives

definition *nderiv* i f X = $((\lambda f.\ (\lambda x.$ *vector-derivative* f (*at* x *within* $X)))$ ^^ $i)$ f

definition *ndiff* n f $X \longleftrightarrow (\forall i<n.\ \forall x \in X.$ *nderiv* i f X *differentiable at* x *within* $X)$

lemma *nderiv-zero*[*simp*]: *nderiv* 0 f X = f
$\langle proof \rangle$

lemma *nderiv-Suc*[*simp*]:
nderiv (*Suc* i) f X x = *vector-derivative* (*nderiv* i f X) (*at* x *within* X)
$\langle proof \rangle$

lemma *ndiff-zero*[*simp*]: *ndiff* 0 f X
$\langle proof \rangle$

lemma *ndiff-Sucs*[*simp*]:
  ndiff (*Suc i*) *f X* $\longleftrightarrow$
    (*ndiff i f X*) $\wedge$
    ($\forall x \in X$. (*nderiv i f X*) *differentiable* (*at x within X*))
  ⟨*proof*⟩

theorem *has-laplace-vector-derivative*:
  ((λt. *vector-derivative f* (*at t*)) *has-laplace s* $*$ *L* $-$ *f0*) *s*
  if *L*: (*f has-laplace L*) *s*
    and f': $\bigwedge t.\ t > 0 \Longrightarrow$ *f differentiable* (*at t*)
    and *f0*: (*f* $\longrightarrow$ *f0*) (*at-right 0*)
    and *eo*: *exponential-order M c f*
    and *cs*: *c* $<$ *Re s*
⟨*proof*⟩

lemma *has-laplace-nderiv*:
  (*nderiv n f* {*0<..*} *has-laplace s*^*n* $*$ *L* $-$ ($\sum$ *i<n*. *s*^(*n* $-$ *Suc i*) $*$ *f0 i*)) *s*
  if *L*: (*f has-laplace L*) *s*
    and f': *ndiff n f* {*0<..*}
    and *f0*: $\bigwedge i.\ i < n \Longrightarrow$ (*nderiv i f* {*0<..*} $\longrightarrow$ *f0 i*) (*at-right 0*)
    and *eo*: $\bigwedge i.\ i < n \Longrightarrow$ *exponential-order M c* (*nderiv i f* {*0<..*})
    and *cs*: *c* $<$ *Re s*
  ⟨*proof*⟩

end

5 Lerch Lemma

theory *Lerch-Lemma*
  imports
    HOL−Analysis.Analysis
begin

The main tool to prove uniqueness of the Laplace transform.

lemma *lerch-lemma-real*:
  fixes *h*::*real* $\Rightarrow$ *real*
  assumes *h-cont*[*continuous-intros*]: *continuous-on* {*0 .. 1*} *h*
  assumes *int-0*: $\bigwedge n$. ((λu. *u* ^ *n* $*$ *h u*) *has-integral 0*) {*0 .. 1*}
  assumes *u*: $0 \le u$ $u \le 1$
  shows $h\ u = 0$
⟨*proof*⟩

lemma *lerch-lemma*:
  fixes *h*::*real* $\Rightarrow$ *'a*::*euclidean-space*
  assumes [*continuous-intros*]: *continuous-on* {*0 .. 1*} *h*
  assumes *int-0*: $\bigwedge n$. ((λu. *u* ^ *n* $*_R$ *h u*) *has-integral 0*) {*0 .. 1*}
  assumes *u*: $0 \le u$ $u \le 1$
  shows $h\ u = 0$
⟨*proof*⟩

end

6 Uniqueness of Laplace Transform

theory *Uniqueness*
  imports
    Existence
    Lerch-Lemma
begin

We show uniqueness of the Laplace transform for continuous functions.

lemma *laplace-transform-zero*:— should also work for piecewise continuous
  assumes *cont-f*: *continuous-on* $\{0..\}$ *f*
  assumes *eo*: *exponential-order M a f*
  assumes *laplace*: $\bigwedge s.\ Re\ s > a \Longrightarrow (f\ has\text{-}laplace\ 0)\ s$
  assumes $t \geq 0$
  shows $f\ t = 0$
$\langle proof \rangle$

lemma *exponential-order-eventually-eq*: *exponential-order M a f*
  if *exponential-order M a g* $\bigwedge t.\ t \geq k \Longrightarrow f\ t = g\ t$
$\langle proof \rangle$

lemma *exponential-order-mono*:
  assumes *eo*: *exponential-order M a f*
  assumes $a \leq b$ $M \leq N$
  shows *exponential-order N b f*
$\langle proof \rangle$

lemma *exponential-order-uminus-iff*:
  exponential-order M a $(\lambda x.\ -\ f\ x)$ = *exponential-order M a f*
  $\langle proof \rangle$

lemma *exponential-order-add*:
  assumes *exponential-order M a f exponential-order M a g*
  shows *exponential-order* $(2 * M)$ *a* $(\lambda x.\ f\ x + g\ x)$
  $\langle proof \rangle$

theorem *laplace-transform-unique*:
  assumes *f*: $\bigwedge s.\ Re\ s > a \Longrightarrow (f\ has\text{-}laplace\ F)\ s$
  assumes *g*: $\bigwedge s.\ Re\ s > b \Longrightarrow (g\ has\text{-}laplace\ F)\ s$
  assumes [*continuous-intros*]: *continuous-on* $\{0..\}$ *f*
  assumes [*continuous-intros*]: *continuous-on* $\{0..\}$ *g*
  assumes *eof*: *exponential-order M a f*
  assumes *eog*: *exponential-order N b g*
  assumes $t \geq 0$
  shows $f\ t = g\ t$
$\langle proof \rangle$

end
theory *Laplace-Transform*
  imports
    Existence
    Uniqueness
begin

end

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