The Lambert W Function on the Reals

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Abstract

The Lambert W function is a multi-valued function defined as the inverse function of $x \mapsto xe^x$. Besides numerous applications in combinatorics, physics, and engineering, it also frequently occurs when solving equations containing both e^x and x, or both x and $\log x$.

This article provides a definition of the two real-valued branches $W_0(x)$ and $W_{-1}(x)$ and proves various properties such as basic identities and inequalities, monotonicity, differentiability, asymptotic expansions, and the MacLaurin series of $W_0(x)$ at $x = 0$.

Contents

1 The Lambert W Function on the reals

theory *Lambert-W*
imports
Complex-Main
HOL−Library.FuncSet
HOL−Real-Asymp.Real-Asymp
begin
⟨*proof*⟩⟨*proof*⟩

1.1 Properties of the function $x \mapsto xe^x$

lemma *exp-times-self-gt*:
assumes $x \neq -1$
shows $x * exp\ x > -exp\ (-1::real)$
⟨*proof*⟩

lemma *exp-times-self-ge*: $x * exp\ x \geq -exp\ (-1::real)$
⟨*proof*⟩

lemma *exp-times-self-strict-mono*:
assumes $x \geq -1$ $x < (y :: real)$
shows $x * exp\ x < y * exp\ y$
⟨*proof*⟩

lemma *exp-times-self-strict-antimono*:
assumes $y \leq -1$ $x < (y :: real)$
shows $x * exp\ x > y * exp\ y$
⟨*proof*⟩

lemma *exp-times-self-mono*:
assumes $x \geq -1$ $x \leq (y :: real)$
shows $x * exp\ x \leq y * exp\ y$
⟨*proof*⟩

lemma *exp-times-self-antimono*:
assumes $y \leq -1$ $x \leq (y :: real)$
shows $x * exp\ x \geq y * exp\ y$
⟨*proof*⟩

lemma *exp-times-self-inj*: *inj-on* $(\lambda x::real.\ x * exp\ x)\ \{-1..\}$
⟨*proof*⟩

lemma *exp-times-self-inj'*: *inj-on* $(\lambda x::real.\ x * exp\ x)\ \{..-1\}$
⟨*proof*⟩

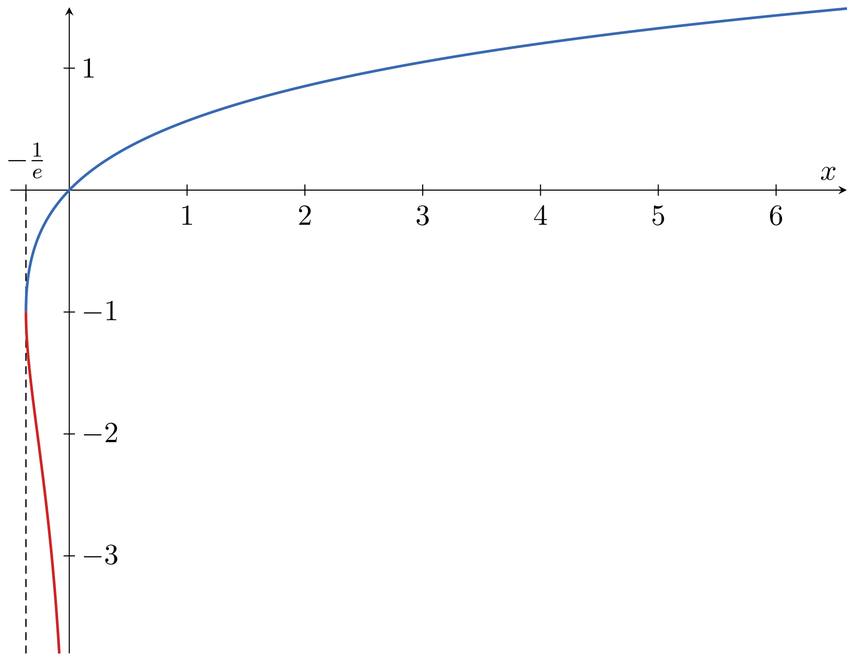

Figure 1: The two real branches of the Lambert W function: W_0 (blue) and W_{-1} (red).

1.2 Definition

The following are the two branches $W_0(x)$ and $W_{-1}(x)$ of the Lambert W function on the real numbers. These are the inverse functions of the function $x \mapsto xe^x$, i. e. we have $W(x)e^{W(x)} = x$ for both branches wherever they are defined. The two branches meet at the point $x = -\frac{1}{e}$.
$W_0(x)$ is the principal branch, whose domain is $[-\frac{1}{e}; \infty)$ and whose range is $[-1; \infty)$. $W_{-1}(x)$ has the domain $[-\frac{1}{e}; 0)$ and the range $(-\infty; -1]$. Figure 1 shows plots of these two branches for illustration.

definition *Lambert-W* :: *real* ⇒ *real* **where**
Lambert-W x = (*if x* < −*exp*(−*1*) *then* −*1 else* (*THE w. w* ≥ −*1* ∧ *w* ∗ *exp w* = *x*))

definition *Lambert-W′* :: *real* ⇒ *real* **where**
Lambert-W′ x = (*if x* ∈ {−*exp*(−*1*)..<*0*} *then* (*THE w. w* ≤ −*1* ∧ *w* ∗ *exp w* = *x*) *else* −*1*)

lemma *Lambert-W-ex1*:
assumes (*x*::*real*) ≥ −*exp* (−*1*)
shows ∃!*w*. *w* ≥ −*1* ∧ *w* ∗ *exp w* = *x*
⟨*proof*⟩

lemma *Lambert-W′-ex1*:
assumes $(x::real) \in \{-exp\ (-1)..<0\}$
shows $\exists!w.\ w \leq -1 \wedge w * exp\ w = x$
⟨*proof*⟩

lemma *Lambert-W-times-exp-self*:
assumes $x \geq -exp\ (-1)$
shows $Lambert\text{-}W\ x * exp\ (Lambert\text{-}W\ x) = x$
⟨*proof*⟩

lemma *Lambert-W-times-exp-self′*:
assumes $x \geq -exp\ (-1)$
shows $exp\ (Lambert\text{-}W\ x) * Lambert\text{-}W\ x = x$
⟨*proof*⟩

lemma *Lambert-W′-times-exp-self*:
assumes $x \in \{-exp\ (-1)..<0\}$
shows $Lambert\text{-}W'\ x * exp\ (Lambert\text{-}W'\ x) = x$
⟨*proof*⟩

lemma *Lambert-W′-times-exp-self′*:
assumes $x \in \{-exp\ (-1)..<0\}$
shows $exp\ (Lambert\text{-}W'\ x) * Lambert\text{-}W'\ x = x$
⟨*proof*⟩

lemma *Lambert-W-ge*: $Lambert\text{-}W\ x \geq -1$
⟨*proof*⟩

lemma *Lambert-W′-le*: $Lambert\text{-}W'\ x \leq -1$
⟨*proof*⟩

lemma *Lambert-W-eqI*:
assumes $w \geq -1$ $w * exp\ w = x$
shows $Lambert\text{-}W\ x = w$
⟨*proof*⟩

lemma *Lambert-W′-eqI*:
assumes $w \leq -1$ $w * exp\ w = x$
shows $Lambert\text{-}W'\ x = w$
⟨*proof*⟩

$W_0(x)$ and $W_{-1}(x)$ together fully cover all solutions of $we^w = x$:

lemma *exp-times-self-eqD*:
assumes $w * exp\ w = x$
shows $x \geq -exp\ (-1)$ **and** $w = Lambert\text{-}W\ x \vee x < 0 \wedge w = Lambert\text{-}W'\ x$
⟨*proof*⟩

theorem *exp-times-self-eq-iff*:

$w * exp\ w = x \longleftrightarrow x \geq -exp\ (-1) \wedge (w = Lambert\text{-}W\ x \vee x < 0 \wedge w = Lambert\text{-}W'\ x)$
⟨proof⟩

lemma *Lambert-W-exp-times-self* [*simp*]: $x \geq -1 \Longrightarrow Lambert\text{-}W\ (x * exp\ x) = x$
⟨proof⟩

lemma *Lambert-W-exp-times-self'* [*simp*]: $x \geq -1 \Longrightarrow Lambert\text{-}W\ (exp\ x * x) = x$
⟨proof⟩

lemma *Lambert-W'-exp-times-self* [*simp*]: $x \leq -1 \Longrightarrow Lambert\text{-}W'\ (x * exp\ x) = x$
⟨proof⟩

lemma *Lambert-W'-exp-times-self'* [*simp*]: $x \leq -1 \Longrightarrow Lambert\text{-}W'\ (exp\ x * x) = x$
⟨proof⟩

lemma *Lambert-W-times-ln-self*:
assumes $x \geq exp\ (-1)$
shows $Lambert\text{-}W\ (x * ln\ x) = ln\ x$
⟨proof⟩

lemma *Lambert-W-times-ln-self'*:
assumes $x \geq exp\ (-1)$
shows $Lambert\text{-}W\ (ln\ x\ * x) = ln\ x$
⟨proof⟩

lemma *Lambert-W-eq-minus-exp-minus1* [*simp*]: $Lambert\text{-}W\ (-exp\ (-1)) = -1$
⟨proof⟩

lemma *Lambert-W'-eq-minus-exp-minus1* [*simp*]: $Lambert\text{-}W'\ (-exp\ (-1)) = -1$
⟨proof⟩

lemma *Lambert-W-0* [*simp*]: $Lambert\text{-}W\ 0 = 0$
⟨proof⟩

1.3 Monotonicity properties

lemma *Lambert-W-strict-mono*:
assumes $x \geq -exp(-1)\ x < y$
shows $Lambert\text{-}W\ x < Lambert\text{-}W\ y$
⟨proof⟩

lemma *Lambert-W-mono*:
assumes $x \geq -exp(-1)\ x \leq y$
shows $Lambert\text{-}W\ x \leq Lambert\text{-}W\ y$

⟨*proof*⟩

lemma *Lambert-W-eq-iff* [*simp*]:
$x \geq -exp(-1) \Longrightarrow y \geq -exp(-1) \Longrightarrow Lambert\text{-}W\ x = Lambert\text{-}W\ y \longleftrightarrow x = y$
⟨*proof*⟩

lemma *Lambert-W-le-iff* [*simp*]:
$x \geq -exp(-1) \Longrightarrow y \geq -exp(-1) \Longrightarrow Lambert\text{-}W\ x \leq Lambert\text{-}W\ y \longleftrightarrow x \leq y$
⟨*proof*⟩

lemma *Lambert-W-less-iff* [*simp*]:
$x \geq -exp(-1) \Longrightarrow y \geq -exp(-1) \Longrightarrow Lambert\text{-}W\ x < Lambert\text{-}W\ y \longleftrightarrow x < y$
⟨*proof*⟩

lemma *Lambert-W-le-minus-one*:
assumes $x \leq -exp(-1)$
shows $Lambert\text{-}W\ x = -1$
⟨*proof*⟩

lemma *Lambert-W-pos-iff* [*simp*]: $Lambert\text{-}W\ x > 0 \longleftrightarrow x > 0$
⟨*proof*⟩

lemma *Lambert-W-eq-0-iff* [*simp*]: $Lambert\text{-}W\ x = 0 \longleftrightarrow x = 0$
⟨*proof*⟩

lemma *Lambert-W-nonneg-iff* [*simp*]: $Lambert\text{-}W\ x \geq 0 \longleftrightarrow x \geq 0$
⟨*proof*⟩

lemma *Lambert-W-neg-iff* [*simp*]: $Lambert\text{-}W\ x < 0 \longleftrightarrow x < 0$
⟨*proof*⟩

lemma *Lambert-W-nonpos-iff* [*simp*]: $Lambert\text{-}W\ x \leq 0 \longleftrightarrow x \leq 0$
⟨*proof*⟩

lemma *Lambert-W-geI*:
assumes $y * exp\ y \leq x$
shows $Lambert\text{-}W\ x \geq y$
⟨*proof*⟩

lemma *Lambert-W-gtI*:
assumes $y * exp\ y < x$
shows $Lambert\text{-}W\ x > y$
⟨*proof*⟩

lemma *Lambert-W-leI*:
assumes $y * exp\ y \geq x\ y \geq -1\ x \geq -exp\ (-1)$
shows $Lambert\text{-}W\ x \leq y$
⟨*proof*⟩

lemma *Lambert-W-lessI*:
assumes $y * exp\ y > x$ $y \geq -1$ $x \geq -exp\ (-1)$
shows $Lambert\text{-}W\ x < y$
⟨*proof*⟩

lemma *Lambert-W'-strict-antimono*:
assumes $-exp\ (-1) \leq x$ $x < y$ $y < 0$
shows $Lambert\text{-}W'\ x > Lambert\text{-}W'\ y$
⟨*proof*⟩

lemma *Lambert-W'-antimono*:
assumes $x \geq -exp(-1)$ $x \leq y$ $y < 0$
shows $Lambert\text{-}W'\ x \geq Lambert\text{-}W'\ y$
⟨*proof*⟩

lemma *Lambert-W'-eq-iff* [*simp*]:
$x \in \{-exp(-1)..<0\} \Longrightarrow y \in \{-exp(-1)..<0\} \Longrightarrow Lambert\text{-}W'\ x = Lambert\text{-}W'\ y \longleftrightarrow x = y$
⟨*proof*⟩

lemma *Lambert-W'-le-iff* [*simp*]:
$x \in \{-exp(-1)..<0\} \Longrightarrow y \in \{-exp(-1)..<0\} \Longrightarrow Lambert\text{-}W'\ x \leq Lambert\text{-}W'\ y \longleftrightarrow x \geq y$
⟨*proof*⟩

lemma *Lambert-W'-less-iff* [*simp*]:
$x \in \{-exp(-1)..<0\} \Longrightarrow y \in \{-exp(-1)..<0\} \Longrightarrow Lambert\text{-}W'\ x < Lambert\text{-}W'\ y \longleftrightarrow x > y$
⟨*proof*⟩

lemma *Lambert-W'-le-minus-one*:
assumes $x \leq -exp(-1)$
shows $Lambert\text{-}W'\ x = -1$
⟨*proof*⟩

lemma *Lambert-W'-ge-zero*: $x \geq 0 \Longrightarrow Lambert\text{-}W'\ x = -1$
⟨*proof*⟩

lemma *Lambert-W'-neg*: $Lambert\text{-}W'\ x < 0$
⟨*proof*⟩

lemma *Lambert-W'-nz* [*simp*]: $Lambert\text{-}W'\ x \neq 0$
⟨*proof*⟩

lemma *Lambert-W'-geI*:
assumes $y * exp\ y \geq x$ $y \leq -1$ $x \geq -exp(-1)$
shows $Lambert\text{-}W'\ x \geq y$

⟨*proof*⟩

lemma *Lambert-W′-gtI*:
assumes $y * exp\ y > x$ $y \leq -1$ $x \geq -exp(-1)$
shows $Lambert\text{-}W'\ x \geq y$
⟨*proof*⟩

lemma *Lambert-W′-leI*:
assumes $y * exp\ y \leq x$ $x < 0$
shows $Lambert\text{-}W'\ x \leq y$
⟨*proof*⟩

lemma *Lambert-W′-lessI*:
assumes $y * exp\ y < x$ $x < 0$
shows $Lambert\text{-}W'\ x < y$
⟨*proof*⟩

lemma *bij-betw-exp-times-self-atLeastAtMost*:
fixes $a\ b$ :: *real*
assumes $a \geq -1$ $a \leq b$
shows $bij\text{-}betw\ (\lambda x.\ x * exp\ x)\ \{a..b\}\ \{a * exp\ a..b * exp\ b\}$
⟨*proof*⟩

lemma *bij-betw-exp-times-self-atLeastAtMost′*:
fixes $a\ b$ :: *real*
assumes $a \leq b$ $b \leq -1$
shows $bij\text{-}betw\ (\lambda x.\ x * exp\ x)\ \{a..b\}\ \{b * exp\ b..a * exp\ a\}$
⟨*proof*⟩

lemma *bij-betw-exp-times-self-atLeast*:
fixes a :: *real*
assumes $a \geq -1$
shows $bij\text{-}betw\ (\lambda x.\ x * exp\ x)\ \{a..\}\ \{a * exp\ a..\}$
⟨*proof*⟩

1.4 Basic identities and bounds

lemma *Lambert-W-2-ln-2* [*simp*]: $Lambert\text{-}W\ (2 * ln\ 2) = ln\ 2$
⟨*proof*⟩

lemma *Lambert-W-exp-1* [*simp*]: $Lambert\text{-}W\ (exp\ 1) = 1$
⟨*proof*⟩

lemma *Lambert-W-neg-ln-over-self*:
assumes $x \in \{exp\ (-1)..exp\ 1\}$
shows $Lambert\text{-}W\ (-ln\ x\ /\ x) = -ln\ x$
⟨*proof*⟩

lemma *Lambert-W′-neg-ln-over-self*:
assumes $x \geq exp\ 1$
shows $Lambert\text{-}W'\ (-ln\ x\ /\ x) = -ln\ x$
⟨*proof*⟩

lemma *exp-Lambert-W*: $x \geq -exp\ (-1) \Longrightarrow x \neq 0 \Longrightarrow exp\ (Lambert\text{-}W\ x) = x\ /\ Lambert\text{-}W\ x$
⟨*proof*⟩

lemma *exp-Lambert-W′*: $x \in \{-exp\ (-1)..<0\} \Longrightarrow exp\ (Lambert\text{-}W'\ x) = x\ /\ Lambert\text{-}W'\ x$
⟨*proof*⟩

lemma *ln-Lambert-W*:
assumes $x > 0$
shows $ln\ (Lambert\text{-}W\ x) = ln\ x - Lambert\text{-}W\ x$
⟨*proof*⟩

lemma *ln-minus-Lambert-W′*:
assumes $x \in \{-exp\ (-1)..<0\}$
shows $ln\ (-Lambert\text{-}W'\ x) = ln\ (-x) - Lambert\text{-}W'\ x$
⟨*proof*⟩

lemma *Lambert-W-plus-Lambert-W-eq*:
assumes $x > 0$ $y > 0$
shows $Lambert\text{-}W\ x + Lambert\text{-}W\ y = Lambert\text{-}W\ (x * y * (1\ /\ Lambert\text{-}W\ x + 1\ /\ Lambert\text{-}W\ y))$
⟨*proof*⟩

lemma *Lambert-W′-plus-Lambert-W′-eq*:
assumes $x \in \{-exp(-1)..<0\}$ $y \in \{-exp(-1)..<0\}$
shows $Lambert\text{-}W'\ x + Lambert\text{-}W'\ y = Lambert\text{-}W'\ (x * y * (1\ /\ Lambert\text{-}W'\ x + 1\ /\ Lambert\text{-}W'\ y))$
⟨*proof*⟩

lemma *Lambert-W-gt-ln-minus-ln-ln*:
assumes $x > exp\ 1$
shows $Lambert\text{-}W\ x > ln\ x - ln\ (ln\ x)$
⟨*proof*⟩

lemma *Lambert-W-less-ln*:
assumes $x > exp\ 1$
shows $Lambert\text{-}W\ x < ln\ x$
⟨*proof*⟩

1.5 Limits, continuity, and differentiability

lemma *filterlim-Lambert-W-at-top* [*tendsto-intros*]: *filterlim Lambert-W at-top at-top*
⟨*proof*⟩

lemma *filterlim-Lambert-W-at-left-0* [*tendsto-intros*]:
filterlim Lambert-W′ at-bot (*at-left 0*)
⟨*proof*⟩

lemma *continuous-on-Lambert-W* [*continuous-intros*]: *continuous-on* {−*exp* (−*1*)..} *Lambert-W*
⟨*proof*⟩

lemma *continuous-on-Lambert-W-alt* [*continuous-intros*]:
assumes *continuous-on A f* $\bigwedge x.\ x \in A \Longrightarrow f\ x \geq -exp\ (-1)$
shows *continuous-on A* (λ*x*. *Lambert-W* (*f x*))
⟨*proof*⟩

lemma *continuous-on-Lambert-W′* [*continuous-intros*]: *continuous-on* {−*exp* (−*1*)..<*0*} *Lambert-W′*
⟨*proof*⟩

lemma *continuous-on-Lambert-W′-alt* [*continuous-intros*]:
assumes *continuous-on A f* $\bigwedge x.\ x \in A \Longrightarrow f\ x \in \{-exp\ (-1)..<0\}$
shows *continuous-on A* (λ*x*. *Lambert-W′* (*f x*))
⟨*proof*⟩

lemma *tendsto-Lambert-W-1*:
assumes $(f \longrightarrow L)\ F$ *eventually* $(\lambda x.\ f\ x \geq -exp\ (-1))\ F$
shows $((\lambda x.\ Lambert\text{-}W\ (f\ x)) \longrightarrow Lambert\text{-}W\ L)\ F$
⟨*proof*⟩

lemma *tendsto-Lambert-W-2*:
assumes $(f \longrightarrow L)\ F\ L > -exp\ (-1)$
shows $((\lambda x.\ Lambert\text{-}W\ (f\ x)) \longrightarrow Lambert\text{-}W\ L)\ F$
⟨*proof*⟩

lemma *tendsto-Lambert-W* [*tendsto-intros*]:
assumes $(f \longrightarrow L)\ F$ *eventually* $(\lambda x.\ f\ x \geq -exp\ (-1))\ F \lor L > -exp\ (-1)$
shows $((\lambda x.\ Lambert\text{-}W\ (f\ x)) \longrightarrow Lambert\text{-}W\ L)\ F$
⟨*proof*⟩

lemma *tendsto-Lambert-W′-1*:
assumes $(f \longrightarrow L)\ F$ *eventually* $(\lambda x.\ f\ x \geq -exp\ (-1))\ F\ L < 0$
shows $((\lambda x.\ Lambert\text{-}W'\ (f\ x)) \longrightarrow Lambert\text{-}W'\ L)\ F$
⟨*proof*⟩

lemma *tendsto-Lambert-W′-2*:
assumes $(f \longrightarrow L)\ F\ L > -exp\ (-1)\ L < 0$
shows $((\lambda x.\ Lambert\text{-}W'\ (f\ x)) \longrightarrow Lambert\text{-}W'\ L)\ F$
⟨*proof*⟩

lemma *tendsto-Lambert-W'* [*tendsto-intros*]:
assumes $(f \longrightarrow L)\ F$ *eventually* $(\lambda x.\ f\ x \geq -exp\ (-1))\ F \vee L > -exp\ (-1)$
$L < 0$
shows $((\lambda x.\ Lambert\text{-}W'\ (f\ x)) \longrightarrow Lambert\text{-}W'\ L)\ F$
⟨*proof*⟩

lemma *continuous-Lambert-W* [*continuous-intros*]:
assumes *continuous F f f* $(Lim\ F\ (\lambda x.\ x)) > -exp\ (-1) \vee$ *eventually* $(\lambda x.\ f\ x$
$\geq -exp\ (-1))\ F$
shows *continuous F* $(\lambda x.\ Lambert\text{-}W\ (f\ x))$
⟨*proof*⟩

lemma *continuous-Lambert-W'* [*continuous-intros*]:
assumes *continuous F f f* $(Lim\ F\ (\lambda x.\ x)) > -exp\ (-1) \vee$ *eventually* $(\lambda x.\ f\ x$
$\geq -exp\ (-1))\ F$
$f\ (Lim\ F\ (\lambda x.\ x)) < 0$
shows *continuous F* $(\lambda x.\ Lambert\text{-}W'\ (f\ x))$
⟨*proof*⟩

lemma *has-field-derivative-Lambert-W* [*derivative-intros*]:
assumes x: $x > -exp\ (-1)$
shows (*Lambert-W has-real-derivative inverse* $(x + exp\ (Lambert\text{-}W\ x)))$ (*at x within A*)
⟨*proof*⟩

lemma *has-field-derivative-Lambert-W-gen* [*derivative-intros*]:
assumes (*f has-real-derivative f'*) (*at x within A*) $f\ x > -exp\ (-1)$
shows $((\lambda x.\ Lambert\text{-}W\ (f\ x))$ *has-real-derivative*
$(f' / (f\ x + exp\ (Lambert\text{-}W\ (f\ x)))))$ (*at x within A*)
⟨*proof*⟩

lemma *has-field-derivative-Lambert-W'* [*derivative-intros*]:
assumes x: $x \in \{-exp\ (-1)<..<0\}$
shows (*Lambert-W' has-real-derivative inverse* $(x + exp\ (Lambert\text{-}W'\ x)))$ (*at x within A*)
⟨*proof*⟩

lemma *has-field-derivative-Lambert-W'-gen* [*derivative-intros*]:
assumes (*f has-real-derivative f'*) (*at x within A*) $f\ x \in \{-exp\ (-1)<..<0\}$
shows $((\lambda x.\ Lambert\text{-}W'\ (f\ x))$ *has-real-derivative*
$(f' / (f\ x + exp\ (Lambert\text{-}W'\ (f\ x)))))$ (*at x within A*)
⟨*proof*⟩

1.6 Asymptotic expansion

Lastly, we prove some more detailed asymptotic expansions of W and W' at their singularities. First, we show that:

$$W(x) = \log x - \log\log x + o(\log\log x) \qquad \text{for } x \to \infty$$
$$W'(x) = \log(-x) - \log(-\log(-x)) + o(\log(-\log(-x))) \qquad \text{for } x \to 0^-$$

theorem *Lambert-W-asymp-equiv-at-top*:
 $(\lambda x.\ \mathit{Lambert\text{-}W}\ x - \mathit{ln}\ x) \sim[\mathit{at\text{-}top}]\ (\lambda x.\ -\mathit{ln}\ (\mathit{ln}\ x))$
⟨*proof*⟩

lemma *Lambert-W-asymp-equiv-at-top′* [*asymp-equiv-intros*]:
 Lambert-W ∼[*at-top*] *ln*
⟨*proof*⟩

theorem *Lambert-W′-asymp-equiv-at-left-0*:
 $(\lambda x.\ \mathit{Lambert\text{-}W'}\ x - \mathit{ln}\ (-x)) \sim[\mathit{at\text{-}left}\ 0]\ (\lambda x.\ -\mathit{ln}\ (-\mathit{ln}\ (-x)))$
⟨*proof*⟩

lemma *Lambert-W′-asymp-equiv′-at-left-0* [*asymp-equiv-intros*]:
 Lambert-W′ ∼[*at-left 0*] $(\lambda x.\ \mathit{ln}\ (-x))$
⟨*proof*⟩

Next, we look at the branching point $a := \frac{1}{e}$. Here, the asymptotic behaviour is as follows:

$$W(x) = -1 + \sqrt{2e}(x-a)^{\frac{1}{2}} - \tfrac{2}{3}e(x-a) + o(x-a) \qquad \text{for} x \to a^+$$
$$W'(x) = -1 - \sqrt{2e}(x-a)^{\frac{1}{2}} - \tfrac{2}{3}e(x-a) + o(x-a) \qquad \text{for} x \to a^+$$

lemma *sqrt-sqrt-mult*:
 assumes $x \geq (0 :: \mathit{real})$
 shows $\mathit{sqrt}\ x * (\mathit{sqrt}\ x * y) = x * y$
 ⟨*proof*⟩

theorem *Lambert-W-asymp-equiv-at-right-minus-exp-minus1*:
 defines $e \equiv \mathit{exp}\ 1$
 defines $a \equiv -\mathit{exp}\ (-1)$
 defines $C1 \equiv \mathit{sqrt}\ (2 * \mathit{exp}\ 1)$
 defines $f \equiv (\lambda x.\ -1 + C1 * \mathit{sqrt}\ (x - a))$
 shows $(\lambda x.\ \mathit{Lambert\text{-}W}\ x - f\ x) \sim[\mathit{at\text{-}right}\ a]\ (\lambda x.\ -2/3 * e * (x - a))$
⟨*proof*⟩

theorem *Lambert-W′-asymp-equiv-at-right-minus-exp-minus1*:
 defines $e \equiv \mathit{exp}\ 1$
 defines $a \equiv -\mathit{exp}\ (-1)$
 defines $C1 \equiv \mathit{sqrt}\ (2 * \mathit{exp}\ 1)$
 defines $f \equiv (\lambda x.\ -1 - C1 * \mathit{sqrt}\ (x - a))$

shows $(\lambda x.\ \textit{Lambert-W}'\ x - f\ x) \sim[\textit{at-right}\ a]\ (\lambda x.\ -2/3 * e * (x - a))$
⟨*proof*⟩

Lastly, just for fun, we derive a slightly more accurate expansion of $W_0(x)$ for $x \to \infty$:

theorem *Lambert-W-asymp-equiv-at-top''*:
$(\lambda x.\ \textit{Lambert-W}\ x - \ln x + \ln (\ln x)) \sim[\textit{at-top}]\ (\lambda x.\ \ln (\ln x) / \ln x)$
⟨*proof*⟩

end

theory *Lambert-W-MacLaurin-Series*
imports
HOL−Computational-Algebra.Formal-Power-Series
Bernoulli.Bernoulli-FPS
Stirling-Formula.Stirling-Formula
Lambert-W
begin

1.7 The MacLaurin series of $W_0(x)$ at $x = 0$

In this section, we derive the MacLaurin series of $W_0(x)$ as a formal power series at $x = 0$ and prove that its radius of convergenge is e^{-1}.

We do not actually show that this series evaluates to 1 since Isabelle's library does not contain the required theorems about convergence of the composition of two power series yet. If it did, however, this last remaining step would be trivial since we did all the real work here.

lemma *Stirling-Suc-n-n*: *Stirling* (*Suc n*) *n* = (*Suc n choose 2*)
⟨*proof*⟩

lemma *Stirling-n-n-minus-1*: *n* > *0* ⟹ *Stirling n* (*n* − *1*) = (*n choose 2*)
⟨*proof*⟩

The following defines the power series $W(X)$ as the formal inverse of the formal power series Xe^X:

definition *fps-Lambert-W* :: *real fps* **where**
fps-Lambert-W = *fps-inv* (*fps-X* ∗ *fps-exp 1*)

The formal composition of $W(X)$ and Xe^X is, in fact, the identity (in both directions).

lemma *fps-compose-Lambert-W*: *fps-compose fps-Lambert-W* (*fps-X* ∗ *fps-exp 1*) = *fps-X*
⟨*proof*⟩

lemma *fps-compose-Lambert-W'*: *fps-compose* (*fps-X* ∗ *fps-exp 1*) *fps-Lambert-W* = *fps-X*
⟨*proof*⟩

We have $W(0) = 0$, which shows that $W(X)$ indeed represents the branch W_0.

lemma *fps-nth-Lambert-W-0* [*simp*]: *fps-nth fps-Lambert-W 0* = *0*
⟨*proof*⟩

lemma *fps-nth-Lambert-W-1* [*simp*]: *fps-nth fps-Lambert-W 1* = *1*
⟨*proof*⟩

All the equalities that hold for the analytic Lambert W function in a neighbourhood of 0 also hold formally for the formal power series, e.g. $W(X) = Xe^{-W(X)}$:

lemma *fps-Lambert-W-over-X*:
fps-Lambert-W = *fps-X* ∗ *fps-compose* (*fps-exp* (−*1*)) *fps-Lambert-W*
⟨*proof*⟩

We now derive the closed-form expression

$$W(X) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} X^n \ .$$

lemma *fps-nth-Lambert-W*: *fps-nth fps-Lambert-W n* = (*if n* = *0 then 0 else* ((−*n*)^(*n*−*1*) / *fact n*))
⟨*proof*⟩

Next, we need a few auxiliary lemmas about summability and convergence radii that should go into Isabelle's standard library at some point:

lemma *summable-comparison-test-bigo*:
fixes *f* :: *nat* ⇒ *real*
assumes *summable* (λ*n. norm* (*g n*)) *f* ∈ *O*(*g*)
shows *summable f*
⟨*proof*⟩

lemma *summable-comparison-test-bigo′*:
assumes *summable* (λ*n. norm* (*g n*))
assumes (λ*n. norm* (*f n* :: ′*a* :: *banach*)) ∈ *O*(λ*n. norm* (*g n*))
shows *summable f*
⟨*proof*⟩

lemma *conv-radius-conv-Sup′*:
fixes *f* :: *nat* ⇒ ′*a* :: {*banach*, *real-normed-div-algebra*}
shows *conv-radius f* = *Sup* {*r*. ∀ *z*. *ereal* (*norm z*) < *r* ⟶ *summable* (λ*n. norm* (*f n* ∗ *z* ^ *n*))}
⟨*proof*⟩

lemma *bigo-imp-conv-radius-ge*:
fixes *f g* :: *nat* ⇒ ′*a* :: {*banach*, *real-normed-field*}
assumes *f* ∈ *O*(*g*)

shows *conv-radius f* $\geq$ *conv-radius g*
⟨*proof*⟩

lemma *conv-radius-cong-bigtheta*:
assumes $f \in \Theta(g)$
shows *conv-radius f* = *conv-radius g*
⟨*proof*⟩

lemma *conv-radius-eqI-smallomega-smallo*:
fixes *f* :: *nat* ⇒ *'a* :: {*real-normed-div-algebra*, *banach*}
assumes $\bigwedge\varepsilon.\ \varepsilon > l \Longrightarrow \varepsilon < inverse\ C \Longrightarrow (\lambda n.\ norm\ (f\ n)) \in \omega(\lambda n.\ \varepsilon\ \widehat{}\ n)$
assumes $\bigwedge\varepsilon.\ \varepsilon < u \Longrightarrow \varepsilon > inverse\ C \Longrightarrow (\lambda n.\ norm\ (f\ n)) \in o(\lambda n.\ \varepsilon\ \widehat{}\ n)$
assumes *C*: *C* > *0* **and** *lu*: *l* > *0 l* < *inverse C u* > *inverse C*
shows *conv-radius f* = *ereal C*
⟨*proof*⟩

Finally, we show that the radius of convergence of $W(X)$ is e^{-1} by directly computing

$$\lim_{n\to\infty} \sqrt[n]{|[X^n]\, W(X)|} = e$$

using Stirling's formula for $n!$:

lemma *fps-conv-radius-Lambert-W*: *fps-conv-radius fps-Lambert-W* = *exp* (*−1*)
⟨*proof*⟩

end

References

[1] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, and D. E. Knuth. On the Lambert W function. *Advances in Computational Mathematics*, 5(1):329–359, Dec. 1996.