

Labeled Transition Systems

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Abstract

Labeled transition systems are ubiquitous in computer science. They are used e.g. for automata and for program graphs in program analysis. We formalize labeled transition systems with and without epsilon transitions. The main difference between formalizations of labeled transition systems is in their choice of how to represent the transition system. In the present formalization the set of nodes is a type, and a labeled transition system is represented as a locale fixing a set of transitions where each transition is a triple of respectively a start node, a label and an end node. Wimmer [Wim20] provides an overview of formalizations of graphs and transition systems.

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theory LTS imports Main "HOL-Library.Multiset_Order" begin

1 LTS

1.1 Transitions

type-synonym ('state, 'label) transition = "'state × 'label × 'state"

1.2 LTS functions

fun trans_hd :: "('state, 'label) transition ⇒ 'state" where
 "trans_hd (s1, γ, s2) = s1"

fun trans_tl :: "('state, 'label) transition ⇒ 'state" where
 "trans_tl (s1, γ, s2) = s2"

fun transitions_of :: "'state list * 'label list ⇒ ('state, 'label) transition multiset" where
 "transitions_of (s1#s2#ss, γ#w) = {# (s1, γ, s2) #} + transitions_of (s2#ss, w)"
 | "transitions_of ([s1], _) = {#}"
 | "transitions_of ([], _) = {#}"
 | "transitions_of (_, []) = {#}"

fun transition_list :: "'state list * 'label list ⇒ ('state, 'label) transition list" where
 "transition_list (s1#s2#ss, γ#w) = (s1, γ, s2) # (transition_list (s2#ss, w))"
 | "transition_list ([s1], _) = []"
 | "transition_list ([], _) = []"
 | "transition_list (_, []) = []"

fun transition_list' :: "'state * 'label list * 'state list * 'state ⇒ ('state, 'label) transition list" where
 "transition_list' (p, w, ss, q) = transition_list (ss, w)"

fun transitions_of' :: "'state * 'label list * 'state list * 'state ⇒ ('state, 'label) transition multiset" where
 "transitions_of' (p, w, ss, q) = transitions_of (ss, w)"

fun transition_list_of' where
 "transition_list_of' (p, γ#w, p'#p''#ss, q) = (p, γ, p'') # (transition_list_of' (p'', w, p''#ss, q))"
 | "transition_list_of' (p, [], _, p'') = []"
 | "transition_list_of' (p, _, [], p'') = []"
 | "transition_list_of' (v, va # vc, [vf], ve) = []"

fun append_path_with_word :: "('a list × 'b list) ⇒ ('a list × 'b list) ⇒ ('a list × 'b list)" (infix "@'" 65) where
 "(ss1, w1) @' (ss2, w2) = (ss1 @ (tl ss2), w1 @ w2)"

fun append_path_with_word_γ :: "((('a list × 'b list) * 'b) ⇒ ('a list × 'b list) ⇒ ('a list × 'b list))" (infix "@'" 65) where
 "((ss1, w1), γ) @' (ss2, w2) = (ss1 @ ss2, w1 @ [γ] @ w2)"

fun append_trans_star_states :: "('a × 'b list × 'a list × 'a) ⇒ ('a × 'b list × 'a list × 'a) ⇒ ('a × 'b list × 'a list × 'a)" (infix "@@" 65) where
 "(p1, w1, ss1, q1) @@" (p2, w2, ss2, q2) = (p1, w1 @ w2, ss1 @ (tl ss2), q2)"

fun append_trans_star_states_γ :: "((('a × 'b list × 'a list × 'a) * 'b) ⇒ ('a × 'b list × 'a list × 'a) ⇒ ('a × 'b list × 'a list × 'a list × 'a list × 'a list × 'a))" (infix "@@" 65) where
 "((p1, w1, ss1, q1), γ) @@" (p2, w2, ss2, q2) = (p1, w1 @ [γ] @ w2, ss1 @ ss2, q2)"

definition inters :: "('state, 'label) transition set ⇒ ('state, 'label) transition set ⇒ (('state * 'state), 'label) transition set" where
 "inters ts1 ts2 = {(p1, q1), α, (p2, q2)}. (p1, α, p2) ∈ ts1 ∧ (q1, α, q2) ∈ ts2}"

definition inters_finals :: "'state set ⇒ 'state set ⇒ ('state * 'state) set" where
 "inters_finals finals1 finals2 = finals1 × finals2"

lemma inters_code[code]:

“inters $ts1\ ts2 = (\bigcup (p1, \alpha, p2) \in ts1. \bigcup (q1, \alpha', q2) \in ts2. \text{if } \alpha = \alpha' \text{ then } \{(p1, q1), \alpha, (p2, q2)\} \text{ else } \{\})$ ”
 ⟨proof⟩

1.3 LTS locale

locale $LTS =$

fixes $transition_relation :: \text{“}('state, 'label) \text{ transition set”}$
 begin

More definitions.

definition $step_relp :: \text{“}('state \Rightarrow 'state \Rightarrow bool) \text{ (infix “} \Rightarrow \text{” 80) where}$
 $\text{“}c \Rightarrow c' \longleftrightarrow (\exists l. (c, l, c') \in transition_relation)\text{”}$

abbreviation $step_starp :: \text{“}('state \Rightarrow 'state \Rightarrow bool) \text{ (infix “} \Rightarrow^* \text{” 80) where}$
 $\text{“}c \Rightarrow^* c' \equiv step_relp^{**} c c'\text{”}$

definition $step_rel :: \text{“}('state \text{ rel}) \text{ where}$
 $\text{“}step_rel = \{(c, c'). step_relp c c'\}\text{”}$

definition $step_star :: \text{“}('state \text{ rel}) \text{ where}$
 $\text{“}step_star = \{(c, c'). step_starp c c'\}\text{”}$

definition $post_star :: \text{“}('state \text{ set} \Rightarrow 'state \text{ set}) \text{ where}$
 $\text{“}post_star C = \{c'. \exists c \in C. c \Rightarrow^* c'\}\text{”}$

definition $pre_star :: \text{“}('state \text{ set} \Rightarrow 'state \text{ set}) \text{ where}$
 $\text{“}pre_star C = \{c'. \exists c \in C. c' \Rightarrow^* c'\}\text{”}$

inductive-set $path :: \text{“}('state \text{ list set}) \text{ where}$
 $\text{“}[s] \in path\text{”}$
 $| \text{“}(s' \# ss) \in path \Longrightarrow (s, l, s') \in transition_relation \Longrightarrow s \# s' \# ss \in path\text{”}$

inductive-set $trans_star :: \text{“}('state * 'label \text{ list} * 'state) \text{ set}) \text{ where}$
 $trans_star_refl[iff]:$
 $\text{“}(p, [], p) \in trans_star\text{”}$
 $| trans_star_step:$
 $\text{“}(p, \gamma, q') \in transition_relation \Longrightarrow$
 $(q', w, q) \in trans_star \Longrightarrow$
 $(p, \gamma \# w, q) \in trans_star\text{”}$

inductive-cases $trans_star_empty [elim]: \text{“}(p, [], q) \in trans_star\text{”}$
inductive-cases $trans_star_cons: \text{“}(p, \gamma \# w, q) \in trans_star\text{”}$

inductive-set $trans_star_states :: \text{“}('state * 'label \text{ list} * 'state \text{ list} * 'state) \text{ set}) \text{ where}$
 $trans_star_states_refl[iff]:$
 $\text{“}(p, [], [p], p) \in trans_star_states\text{”}$
 $| trans_star_states_step:$
 $\text{“}(p, \gamma, q') \in transition_relation \Longrightarrow$
 $(q', w, ss, q) \in trans_star_states \Longrightarrow$
 $(p, \gamma \# w, p \# ss, q) \in trans_star_states\text{”}$

inductive-set $path_with_word :: \text{“}('state \text{ list} * 'label \text{ list}) \text{ set}) \text{ where}$
 $path_with_word_refl[iff]:$
 $\text{“}([s], []) \in path_with_word\text{”}$
 $| path_with_word_step:$
 $\text{“}(s' \# ss, w) \in path_with_word \Longrightarrow$
 $(s, l, s') \in transition_relation \Longrightarrow$
 $(s \# s' \# ss, l \# w) \in path_with_word\text{”}$

definition $start_of :: \text{“}('state \text{ list} \times 'label \text{ list}) \Rightarrow 'state\text{” where}$
 $\text{“}start_of \pi = hd (fst \pi)\text{”}$

definition *end_of* :: $(\text{'state list} \times \text{'label list}) \Rightarrow \text{'state}$ **where**
 $\text{"end_of } \pi = \text{last (fst } \pi \text{"}$

abbreviation *path_with_word_from* :: $\text{'state} \Rightarrow (\text{'state list} * \text{'label list}) \text{ set}$ **where**
 $\text{"path_with_word_from } q == \{\pi. \pi \in \text{path_with_word} \wedge \text{start_of } \pi = q\}$

abbreviation *path_with_word_to* :: $\text{'state} \Rightarrow (\text{'state list} * \text{'label list}) \text{ set}$ **where**
 $\text{"path_with_word_to } q == \{\pi. \pi \in \text{path_with_word} \wedge \text{end_of } \pi = q\}$

abbreviation *path_with_word_from_to* :: $\text{'state} \Rightarrow \text{'state} \Rightarrow (\text{'state list} * \text{'label list}) \text{ set}$ **where**
 $\text{"path_with_word_from_to } \text{start } \text{end} == \{\pi. \pi \in \text{path_with_word} \wedge \text{start_of } \pi = \text{start} \wedge \text{end_of } \pi = \text{end}\}$

inductive-set *transition_list_path* :: $(\text{'state}, \text{'label}) \text{ transition list set}$ **where**

$\text{"(q, l, q') \in transition_relation} \implies$
 $\text{[(q, l, q')] \in transition_list_path}$
 $| \text{"(q, l, q') \in transition_relation} \implies$
 $\text{(q', l', q'') \# ts \in transition_list_path} \implies$
 $\text{(q, l, q') \# (q', l', q'') \# ts \in transition_list_path}$

lemma *singleton_path_start_end*:
assumes $\text{"([s], []) \in LTS.path_with_word } pg$
shows $\text{"start_of } ([s], []) = \text{end_of } ([s], [])$
 $\langle \text{proof} \rangle$

lemma *path_with_word_length*:
assumes $\text{"(ss, w) \in path_with_word}$
shows $\text{"length } ss = \text{length } w + 1$
 $\langle \text{proof} \rangle$

lemma *path_with_word_lengths*:
assumes $\text{"(qs @ [qminus1], w) \in path_with_word}$
shows $\text{"length } qs = \text{length } w$
 $\langle \text{proof} \rangle$

lemma *path_with_word_butlast*:
assumes $\text{"(ss, w) \in path_with_word}$
assumes $\text{"length } ss \geq 2$
shows $\text{"(butlast } ss, \text{butlast } w) \in \text{path_with_word}$
 $\langle \text{proof} \rangle$

lemma *transition_butlast*:
assumes $\text{"(ss, w) \in path_with_word}$
assumes $\text{"length } ss \geq 2$
shows $\text{"(last (butlast } ss), \text{last } w, \text{last } ss) \in \text{transition_relation}$
 $\langle \text{proof} \rangle$

lemma *path_with_word_induct_reverse* [*consumes 1, case_names path_with_word_refl path_with_word_step_rev*]:
 $\text{"(ss, w) \in path_with_word} \implies$
 $\text{(\bigwedge s. } P \text{ [s] [])} \implies$
 $\text{(\bigwedge ss s w l s'. (ss @ [s], w) \in path_with_word} \implies$
 $\text{P (ss @ [s]) } w \implies$
 $\text{(s, l, s') \in transition_relation} \implies$
 $\text{P (ss @ [s, s']) (w @ [l])}$
 $\implies P \text{ ss } w$
 $\langle \text{proof} \rangle$

lemma *path_with_word_from_induct_reverse*:
 $\text{"(ss, w) \in path_with_word_from } \text{start} \implies$
 $\text{(\bigwedge s. } P \text{ [s] [])} \implies$
 $\text{(\bigwedge ss s w l s'. (ss @ [s], w) \in path_with_word_from } \text{start} \implies$
 $\text{P (ss @ [s]) } w \implies$
 $\text{(s, l, s') \in transition_relation} \implies$

$$\begin{aligned} & P (ss @ [s, s']) (w @ [l]) \\ \implies & P ss w \end{aligned}$$
 <proof>

inductive *transition_of* :: “('state, 'label) transition \Rightarrow 'state list * 'label list \Rightarrow bool” **where**
 “*transition_of* (s1, γ , s2) (s1 # s2 # ss, γ # w)”
 | “*transition_of* (s1, γ , s2) (ss, w) \implies
 transition_of (s1, γ , s2) (s # ss, μ # w)”

lemma *path_with_word_not_empty[simp]*: “ $\neg([\], w) \in \text{path_with_word}$ ”
 <proof>

lemma *trans_star_path_with_word*:
assumes “(p, w, q) \in *trans_star*”
shows “ $\exists ss. \text{hd } ss = p \wedge \text{last } ss = q \wedge (ss, w) \in \text{path_with_word}$ ”
 <proof>

lemma *trans_star_trans_star_states*:
assumes “(p, w, q) \in *trans_star*”
shows “ $\exists ss. (p, w, ss, q) \in \text{trans_star_states}$ ”
 <proof>

lemma *trans_star_states_trans_star*:
assumes “(p, w, ss, q) \in *trans_star_states*”
shows “(p, w, q) \in *trans_star*”
 <proof>

lemma *path_with_word_trans_star*:
assumes “(ss, w) \in *path_with_word*”
assumes “*length* ss \neq 0”
shows “(hd ss, w, last ss) \in *trans_star*”
 <proof>

lemma *path_with_word_trans_star_Cons*:
assumes “(s1 # ss @ [s2], w) \in *path_with_word*”
shows “(s1, w, s2) \in *trans_star*”
 <proof>

lemma *path_with_word_trans_star_Singleton*:
assumes “([s2], w) \in *path_with_word*”
shows “(s2, [], s2) \in *trans_star*”
 <proof>

lemma *trans_star_split*:
assumes “(p'', u1 @ w1, q) \in *trans_star*”
shows “ $\exists q1. (p'', u1, q1) \in \text{trans_star} \wedge (q1, w1, q) \in \text{trans_star}$ ”
 <proof>

lemma *trans_star_states_append*:
assumes “(p2, w2, w2_ss, q') \in *trans_star_states*”
assumes “(q', v, v_ss, q) \in *trans_star_states*”
shows “(p2, w2 @ v, w2_ss @ tl v_ss, q) \in *trans_star_states*”
 <proof>

lemma *trans_star_states_length*:
assumes “(p, u, u_ss, p1) \in *trans_star_states*”
shows “*length* u_ss = *Suc* (*length* u)”
 <proof>

lemma *trans_star_states_last*:
assumes “(p, u, u_ss, p1) \in *trans_star_states*”
shows “p1 = last u_ss”
 <proof>

lemma *trans_star_states_hd*:

assumes “ $(q', v, v_ss, q) \in \text{trans_star_states}$ ”
shows “ $q' = \text{hd } v_ss$ ”
<proof>

lemma *trans_star_states_transition_relation*:

assumes “ $(p, \gamma \# w_rest, ss, q) \in \text{trans_star_states}$ ”
shows “ $\exists s \gamma'. (s, \gamma', q) \in \text{transition_relation}$ ”
<proof>

lemma *trans_star_states_path_with_word*:

assumes “ $(p, w, ss, q) \in \text{trans_star_states}$ ”
shows “ $(ss, w) \in \text{path_with_word}$ ”
<proof>

lemma *path_with_word_trans_star_states*:

assumes “ $(ss, w) \in \text{path_with_word}$ ”
assumes “ $p = \text{hd } ss$ ”
assumes “ $q = \text{last } ss$ ”
shows “ $(p, w, ss, q) \in \text{trans_star_states}$ ”
<proof>

lemma *append_path_with_word_path_with_word*:

assumes “ $\text{last } \gamma 2ss = \text{hd } v_ss$ ”
assumes “ $(\gamma 2ss, \gamma 2\varepsilon) \in \text{path_with_word}$ ”
assumes “ $(v_ss, v) \in \text{path_with_word}$ ”
shows “ $(\gamma 2ss, \gamma 2\varepsilon) @' (v_ss, v) \in \text{path_with_word}$ ”
<proof>

lemma *hd_is_hd*:

assumes “ $(p, w, ss, q) \in \text{trans_star_states}$ ”
assumes “ $(p1, \gamma, q1) = \text{hd } (\text{transition_list}' (p, w, ss, q))$ ”
assumes “ $\text{transition_list}' (p, w, ss, q) \neq []$ ”
shows “ $p = p1$ ”
<proof>

definition *srcs* :: “'state set” **where**

“ $\text{srcs} = \{p. \exists q \gamma. (q, \gamma, p) \in \text{transition_relation}\}$ ”

definition *sinks* :: “'state set” **where**

“ $\text{sinks} = \{p. \exists q \gamma. (p, \gamma, q) \in \text{transition_relation}\}$ ”

definition *isolated* :: “'state set” **where**

“ $\text{isolated} = \text{srcs} \cap \text{sinks}$ ”

lemma *srcs_def2*:

“ $q \in \text{srcs} \iff (\exists q' \gamma. (q', \gamma, q) \in \text{transition_relation})$ ”
<proof>

lemma *sinks_def2*:

“ $q \in \text{sinks} \iff (\exists q' \gamma. (q, \gamma, q') \in \text{transition_relation})$ ”
<proof>

lemma *isolated_no_edges*:

assumes “ $(p, \gamma, q) \in \text{transition_relation}$ ”
shows “ $p \notin \text{isolated} \wedge q \notin \text{isolated}$ ”
<proof>

lemma *source_never_or_hd*:

assumes “ $(ss, w) \in \text{path_with_word}$ ”
assumes “ $p1 \in \text{srcs}$ ”
assumes “ $t = (p1, \gamma, q1)$ ”

shows “ $\text{count}(\text{transitions_of}(ss, w)) t = 0 \vee$
 $((\text{hd}(\text{transition_list}(ss, w)) = t \wedge \text{count}(\text{transitions_of}(ss, w)) t = 1))$ ”
 ⟨proof⟩

lemma *source_only_hd*:
assumes “ $(ss, w) \in \text{path_with_word}$ ”
assumes “ $p1 \in \text{srcs}$ ”
assumes “ $\text{count}(\text{transitions_of}(ss, w)) t > 0$ ”
assumes “ $t = (p1, \gamma, q1)$ ”
shows “ $\text{hd}(\text{transition_list}(ss, w)) = t \wedge \text{count}(\text{transitions_of}(ss, w)) t = 1$ ”
 ⟨proof⟩

lemma *no_end_in_source*:
assumes “ $(p, w, qq) \in \text{trans_star}$ ”
assumes “ $w \neq []$ ”
shows “ $qq \notin \text{srcs}$ ”
 ⟨proof⟩

lemma *transition_list_length_Cons*:
assumes “ $\text{length } ss = \text{Suc}(\text{length } w)$ ”
assumes “ $\text{hd}(\text{transition_list}(ss, w)) = (p, \gamma, q)$ ”
assumes “ $\text{transition_list}(ss, w) \neq []$ ”
shows “ $\exists w' ss'. w = \gamma \# w' \wedge ss = p \# q \# ss'$ ”
 ⟨proof⟩

lemma *transition_list_Cons*:
assumes “ $(p, w, ss, q) \in \text{trans_star_states}$ ”
assumes “ $\text{hd}(\text{transition_list}(ss, w)) = (p, \gamma, q1)$ ”
assumes “ $\text{transition_list}(ss, w) \neq []$ ”
shows “ $\exists w' ss'. w = \gamma \# w' \wedge ss = p \# q1 \# ss'$ ”
 ⟨proof⟩

lemma *nothing_after_sink*:
assumes “ $([q, q'] @ ss, \gamma1 \# w) \in \text{path_with_word}$ ”
assumes “ $q' \in \text{sinks}$ ”
shows “ $ss = [] \wedge w = []$ ”
 ⟨proof⟩

lemma *count_transitions_of'_tails*:
assumes “ $(p, \gamma', q'_add) \neq (p1, \gamma, q')$ ”
shows “ $\text{count}(\text{transitions_of}'(p, \gamma' \# w, p \# q'_add \# ss_rest, q)) (p1, \gamma, q') =$
 $\text{count}(\text{transitions_of}'(q'_add, w, q'_add \# ss_rest, q)) (p1, \gamma, q')$ ”
 ⟨proof⟩

lemma *avoid_count_zero*:
assumes “ $(p, w, ss, q) \in \text{trans_star_states}$ ”
assumes “ $(p1, \gamma, q') \notin \text{transition_relation}$ ”
shows “ $\text{count}(\text{transitions_of}'(p, w, ss, q)) (p1, \gamma, q') = 0$ ”
 ⟨proof⟩

lemma *transition_list_append*:
assumes “ $(ss, w) \in \text{path_with_word}$ ”
assumes “ $(ss', w') \in \text{path_with_word}$ ”
assumes “ $\text{last } ss = \text{hd } ss'$ ”
shows “ $\text{transition_list}((ss, w) @' (ss', w')) = \text{transition_list}(ss, w) @ \text{transition_list}(ss', w')$ ”
 ⟨proof⟩

lemma *split_path_with_word_beginning'*:
assumes “ $(SS, WW) \in \text{path_with_word}$ ”
assumes “ $SS = (ss @ ss')$ ”
assumes “ $\text{length } ss = \text{Suc}(\text{length } w)$ ”
assumes “ $WW = w @ w'$ ”
shows “ $(ss, w) \in \text{path_with_word}$ ”

<proof>

lemma *split_path_with_word_end'*:
 assumes “ $(SS, WW) \in \text{path_with_word}$ ”
 assumes “ $SS = (ss @ ss')$ ”
 assumes “ $\text{length } ss' = \text{Suc } (\text{length } w')$ ”
 assumes “ $WW = w @ w'$ ”
 shows “ $(ss', w') \in \text{path_with_word}$ ”
<proof>

lemma *split_path_with_word_end*:
 assumes “ $(ss @ ss', w @ w') \in \text{path_with_word}$ ”
 assumes “ $\text{length } ss' = \text{Suc } (\text{length } w')$ ”
 shows “ $(ss', w') \in \text{path_with_word}$ ”
<proof>

lemma *split_path_with_word_beginning'*:
 assumes “ $(ss @ ss', w @ w') \in \text{path_with_word}$ ”
 assumes “ $\text{length } ss = \text{Suc } (\text{length } w)$ ”
 shows “ $(ss, w) \in \text{path_with_word}$ ”
<proof>

lemma *split_path_with_word_beginning*:
 assumes “ $(ss, w) @' (ss', w') \in \text{path_with_word}$ ”
 assumes “ $\text{length } ss = \text{Suc } (\text{length } w)$ ”
 shows “ $(ss, w) \in \text{path_with_word}$ ”
<proof>

lemma *path_with_word_remove_last'*:
 assumes “ $(SS, W) \in \text{path_with_word}$ ”
 assumes “ $SS = ss @ [s, s']$ ”
 assumes “ $W = w @ [l]$ ”
 shows “ $(ss @ [s], w) \in \text{path_with_word}$ ”
<proof>

lemma *path_with_word_remove_last*:
 assumes “ $(ss @ [s, s'], w @ [l]) \in \text{path_with_word}$ ”
 shows “ $(ss @ [s], w) \in \text{path_with_word}$ ”
<proof>

lemma *transition_list_append_edge*:
 assumes “ $(ss @ [s, s'], w @ [l]) \in \text{path_with_word}$ ”
 shows “ $\text{transition_list } (ss @ [s, s'], w @ [l]) = \text{transition_list } (ss @ [s], w) @ [(s, l, s')]$ ”
<proof>

end

1.4 More LTS lemmas

lemma *hd_transition_list_append_path_with_word*:
 assumes “ $\text{hd } (\text{transition_list } (ss, w)) = (p1, \gamma, q1)$ ”
 assumes “ $\text{transition_list } (ss, w) \neq []$ ”
 shows “ $([p1, q1], [\gamma]) @' (tl ss, tl w) = (ss, w)$ ”
<proof>

lemma *counting*:
 “ $\text{count } (\text{transitions_of } ((hdss1, ww1, ss1, lastss1))) (s1, \gamma, s2) =$
 $\text{count } (\text{transitions_of } ((ss1, ww1))) (s1, \gamma, s2)$ ”
<proof>

lemma *count_append_path_with_word_γ*:
 assumes “ $\text{length } ss1 = \text{Suc } (\text{length } ww1)$ ”
 assumes “ $ss2 \neq []$ ”
 shows “ $\text{count } (\text{transitions_of } (((ss1, ww1), \gamma') @^\gamma (ss2, ww2))) (s1, \gamma, s2) =$

count (transitions_of (ss1,ww1)) (s1, γ, s2) +
 (if s1 = last ss1 ∧ s2 = hd ss2 ∧ γ = γ' then 1 else 0) +
 count (transitions_of (ss2,ww2)) (s1, γ, s2)”

⟨proof⟩

lemma count_append_path_with_word:

assumes “length ss1 = Suc (length ww1)”

assumes “ss2 ≠ []”

assumes “last ss1 = hd ss2”

shows “count (transitions_of (((ss1, ww1) @' (ss2, ww2))) (s1, γ, s2) =
 count (transitions_of (ss1, ww1)) (s1, γ, s2) +
 count (transitions_of (ss2, ww2)) (s1, γ, s2)”

⟨proof⟩

lemma count_append_trans_star_states_γ_length:

assumes “length (ss1) = Suc (length (ww1))”

assumes “ss2 ≠ []”

shows “count (transitions_of' (((hdss1,ww1,ss1,lastss1),γ') @@^γ (hdss2,ww2,ss2,lastss2))) (s1, γ, s2) =
 count (transitions_of' (hdss1,ww1,ss1,lastss1)) (s1, γ, s2) +
 (if s1 = last ss1 ∧ s2 = hd ss2 ∧ γ = γ' then 1 else 0) +
 count (transitions_of' (hdss2,ww2,ss2,lastss2)) (s1, γ, s2)”

⟨proof⟩

lemma count_append_trans_star_states_γ:

assumes “(hdss1,ww1,ss1,lastss1) ∈ LTS.trans_star_states A”

assumes “(hdss2,ww2,ss2,lastss2) ∈ LTS.trans_star_states A”

shows “count (transitions_of' (((hdss1,ww1,ss1,lastss1),γ') @@^γ (hdss2,ww2,ss2,lastss2))) (s1, γ, s2) =
 count (transitions_of' (hdss1,ww1,ss1,lastss1)) (s1, γ, s2) +
 (if s1 = last ss1 ∧ s2 = hd ss2 ∧ γ = γ' then 1 else 0) +
 count (transitions_of' (hdss2,ww2,ss2,lastss2)) (s1, γ, s2)”

⟨proof⟩

lemma count_append_trans_star_states_length:

assumes “length (ss1) = Suc (length (ww1))”

assumes “ss2 ≠ []”

assumes “last ss1 = hd ss2”

shows “count (transitions_of' (((hdss1,ww1,ss1,lastss1)) @' (hdss2,ww2,ss2,lastss2))) (s1, γ, s2) =
 count (transitions_of' (hdss1,ww1,ss1,lastss1)) (s1, γ, s2) +
 count (transitions_of' (hdss2,ww2,ss2,lastss2)) (s1, γ, s2)”

⟨proof⟩

lemma count_append_trans_star_states:

assumes “(hdss1,ww1,ss1,lastss1) ∈ LTS.trans_star_states A”

assumes “(lastss1,ww2,ss2,lastss2) ∈ LTS.trans_star_states A”

shows “count (transitions_of' (((hdss1,ww1,ss1,lastss1)) @@' (lastss1,ww2,ss2,lastss2))) (s1, γ, s2) =
 count (transitions_of' (hdss1,ww1,ss1,lastss1)) (s1, γ, s2) +
 count (transitions_of' (lastss1,ww2,ss2,lastss2)) (s1, γ, s2)”

⟨proof⟩

context fixes Δ :: “('state, 'label) transition set” **begin**

fun reach **where**

“reach p [] = {p}”

| “reach p (γ#w) =

($\bigcup q' \in (\bigcup (p',\gamma',q') \in \Delta. \text{if } p' = p \wedge \gamma' = \gamma \text{ then } \{q'\} \text{ else } \{\}).$
 reach q' w)”

end

lemma trans_star_imp_exec: “(p,w,q) ∈ LTS.trans_star Δ ⇒ q ∈ reach Δ p w”

⟨proof⟩

lemma reach_imp: “q ∈ reach Δ p w ⇒ (p,w,q) ∈ LTS.trans_star Δ”

⟨proof⟩

lemma trans_star_code[code_unfold]: “(p,w,q) ∈ LTS.trans_star Δ ⇔ q ∈ reach Δ p w”

⟨proof⟩

lemma *subset_srcs_code[code_unfold]*:

“ $X \subseteq LTS.srcs\ A \iff (\forall q \in X. q \notin snd\ 'A)$ ”
(proof)

lemma *LTS_trans_star_mono*:

“mono $LTS.trans_star$ ”

(proof)

lemma *count_next_0*:

assumes “count (transitions_of (s # s' # ss, l # w)) (p1, γ , q') = 0”

shows “count (transitions_of (s' # ss, w)) (p1, γ , q') = 0”

(proof)

lemma *count_next_hd*:

assumes “count (transitions_of (s # s' # ss, l # w)) (p1, γ , q') = 0”

shows “(s, l, s') \neq (p1, γ , q')”

(proof)

lemma *count_empty_zero*: “count (transitions_of' (p, [], [p_add], p_add)) (p1, γ , q') = 0”

(proof)

lemma *count_zero_remove_path_with_word*:

assumes “(ss, w) $\in LTS.path_with_word\ A_i$ ”

assumes “0 = count (transitions_of (ss, w)) (p1, γ , q')”

assumes “ $A_i = A_{i\minus 1} \cup \{(p1, \gamma, q')\}$ ”

shows “(ss, w) $\in LTS.path_with_word\ A_{i\minus 1}$ ”

(proof)

lemma *count_zero_remove_path_with_word_trans_star_states*:

assumes “(p, w, ss, q) $\in LTS.trans_star_states\ A_i$ ”

assumes “0 = count (transitions_of' (p, w, ss, q)) (p1, γ , q')”

assumes “ $A_i = A_{i\minus 1} \cup \{(p1, \gamma, q')\}$ ”

shows “(p, w, ss, q) $\in LTS.trans_star_states\ A_{i\minus 1}$ ”

(proof)

lemma *count_zero_remove_trans_star_states_trans_star*:

assumes “(p, w, ss, q) $\in LTS.trans_star_states\ A_i$ ”

assumes “0 = count (transitions_of' (p, w, ss, q)) (p1, γ , q')”

assumes “ $A_i = A_{i\minus 1} \cup \{(p1, \gamma, q')\}$ ”

shows “(p, w, q) $\in LTS.trans_star\ A_{i\minus 1}$ ”

(proof)

lemma *split_at_first_t*:

assumes “(p, w, ss, q) $\in LTS.trans_star_states\ A_i$ ”

assumes “Suc j' = count (transitions_of' (p, w, ss, q)) (p1, γ , q')”

assumes “(p1, γ , q') $\notin A_{i\minus 1}$ ”

assumes “ $A_i = A_{i\minus 1} \cup \{(p1, \gamma, q')\}$ ”

shows “ $\exists u\ v\ u_ss\ v_ss.$

ss = u_ss @ v_ss \wedge

w = u @ [γ] @ v \wedge

(p, u, u_ss, p1) $\in LTS.trans_star_states\ A_{i\minus 1} \wedge$

(p1, [γ], q') $\in LTS.trans_star\ A_i \wedge$

(q', v, v_ss, q) $\in LTS.trans_star_states\ A_i \wedge$

(p, w, ss, q) = ((p, u, u_ss, p1), γ) @ ^{γ} (q', v, v_ss, q)”

(proof)

lemma *trans_star_states_mono*:

assumes “(p, w, ss, q) $\in LTS.trans_star_states\ A_1$ ”

assumes “ $A_1 \subseteq A_2$ ”

shows “(p, w, ss, q) $\in LTS.trans_star_states\ A_2$ ”

(proof)

lemma *count_combine_trans_star_states_append*:
assumes “ $ss = u_ss @ v_ss \wedge w = u @ [\gamma] @ v$ ”
assumes “ $t = (p1, \gamma, q')$ ”
assumes “ $(p, u, u_ss, p1) \in LTS.trans_star_states A$ ”
assumes “ $(q', v, v_ss, q) \in LTS.trans_star_states B$ ”
shows “ $count (transitions_of' (p, w, ss, q)) t =$
 $count (transitions_of' (p, u, u_ss, p1)) t +$
 $1 +$
 $count (transitions_of' (q', v, v_ss, q)) t$ ”
<proof>

lemma *count_combine_trans_star_states*:
assumes “ $t = (p1, \gamma, q')$ ”
assumes “ $(p, u, u_ss, p1) \in LTS.trans_star_states A$ ”
assumes “ $(q', v, v_ss, q) \in LTS.trans_star_states B$ ”
shows “ $count (transitions_of' (((p, u, u_ss, p1), \gamma) @@^\gamma (q', v, v_ss, q))) t =$
 $count (transitions_of' (p, u, u_ss, p1)) t + 1 + count (transitions_of' (q', v, v_ss, q)) t$ ”
<proof>

lemma *transition_list_reversed_simp*:
assumes “ $length ss = length w$ ”
shows “ $transition_list (ss @ [s, s'], w @ [l]) = (transition_list (ss@[s], w)) @ [(s, l, s')]$ ”
<proof>

lemma *LTS_trans_star_mono'*:
“*mono LTS.trans_star_states*”
<proof>

lemma *path_with_word_mono'*:
assumes “ $(ss, w) \in LTS.path_with_word A1$ ”
assumes “ $A1 \subseteq A2$ ”
shows “ $(ss, w) \in LTS.path_with_word A2$ ”
<proof>

lemma *LTS_path_with_word_mono*:
“*mono LTS.path_with_word*”
<proof>

1.5 Reverse transition system

fun *rev_edge* :: “ $('n, 'v) transition \Rightarrow ('n, 'v) transition$ ” **where**
“ $rev_edge (q_s, \alpha, q_o) = (q_o, \alpha, q_s)$ ”

lemma *rev_edge_rev_edge_id[simp]*: “ $rev_edge (rev_edge x) = x$ ”
<proof>

fun *rev_path_with_word* :: “ $'n list * 'v list \Rightarrow 'n list * 'v list$ ” **where**
“ $rev_path_with_word (es, ls) = (rev es, rev ls)$ ”

definition *rev_edge_list* :: “ $('n, 'v) transition list \Rightarrow ('n, 'v) transition list$ ” **where**
“ $rev_edge_list ts = rev (map rev_edge ts)$ ”

context *LTS begin*

interpretation *rev_LTS*: *LTS* “ $(rev_edge \text{ ' transition_relation})$ ”
<proof>

lemma *rev_path_in_rev_pg*:
assumes “ $(ss, w) \in path_with_word$ ”
shows “ $(rev ss, rev w) \in rev_LTS.path_with_word$ ”
<proof>

lemma *transition_list_rev_edge_list*:

assumes “ $(ss, w) \in \text{path_with_word}$ ”
shows “ $\text{transition_list}(\text{rev } ss, \text{rev } w) = \text{rev_edge_list}(\text{transition_list}(ss, w))$ ”
 ⟨proof⟩

end

2 LTS with epsilon

2.1 LTS functions

context begin

private abbreviation ε :: “label option” where
 “ $\varepsilon == \text{None}$ ”

definition $\text{inters_}\varepsilon$:: “(state, label option) transition set \Rightarrow (state, label option) transition set \Rightarrow ((state * state), label option) transition set” where

“ $\text{inters_}\varepsilon \text{ ts1 ts2} =$
 $\{((p1, q1), \alpha, (p2, q2)) \mid p1 \ q1 \ \alpha \ p2 \ q2. (p1, \alpha, p2) \in \text{ts1} \wedge (q1, \alpha, q2) \in \text{ts2}\} \cup$
 $\{((p1, q1), \varepsilon, (p2, q1)) \mid p1 \ p2 \ q1. (p1, \varepsilon, p2) \in \text{ts1}\} \cup$
 $\{((p1, q1), \varepsilon, (p1, q2)) \mid p1 \ q1 \ q2. (q1, \varepsilon, q2) \in \text{ts2}\}$ ”

end

2.2 LTS with epsilon locale

locale $\text{LTS_}\varepsilon = \text{LTS transition_relation for transition_relation}$:: “(state, label option) transition set”
 begin

abbreviation ε :: “label option” where
 “ $\varepsilon == \text{None}$ ”

inductive-set $\text{trans_star_}\varepsilon$:: “(state * label list * state) set” where

$\text{trans_star_}\varepsilon \text{ refl[iff]}$: “ $(p, [], p) \in \text{trans_star_}\varepsilon$ ”
 | $\text{trans_star_}\varepsilon \text{ step_}\gamma$: “ $(p, \text{Some } \gamma, q) \in \text{transition_relation} \implies (q', w, q) \in \text{trans_star_}\varepsilon$
 $\implies (p, \gamma \# w, q) \in \text{trans_star_}\varepsilon$ ”
 | $\text{trans_star_}\varepsilon \text{ step_}\varepsilon$: “ $(p, \varepsilon, q) \in \text{transition_relation} \implies (q', w, q) \in \text{trans_star_}\varepsilon$
 $\implies (p, w, q) \in \text{trans_star_}\varepsilon$ ”

inductive-cases $\text{trans_star_}\varepsilon \text{ empty}$ [elim]: “ $(p, [], q) \in \text{trans_star_}\varepsilon$ ”

inductive-cases $\text{trans_star_}\varepsilon \text{ cons_}\varepsilon$: “ $(p, \gamma \# w, q) \in \text{trans_star}$ ”

definition $\text{remove_}\varepsilon$:: “label option list \Rightarrow label list” where
 “ $\text{remove_}\varepsilon w = \text{map the}(\text{removeAll } \varepsilon w)$ ”

definition $\varepsilon \text{ exp}$:: “label option list \Rightarrow label list \Rightarrow bool” where
 “ $\varepsilon \text{ exp } w' w \iff \text{map the}(\text{removeAll } \varepsilon w') = w$ ”

lemma $\text{trans_star_}\varepsilon \text{ trans_star_}\varepsilon$:

assumes “ $(p, w, q) \in \text{trans_star}$ ”
shows “ $(p, \text{map the}(\text{removeAll } \varepsilon w), q) \in \text{trans_star_}\varepsilon$ ”
 ⟨proof⟩

lemma $\text{trans_star_}\varepsilon \text{ exp_}\varepsilon \text{ trans_star}$:

assumes “ $(p, w, q) \in \text{trans_star_}\varepsilon$ ”
shows “ $\exists w'. \varepsilon \text{ exp } w' w \wedge (p, w', q) \in \text{trans_star}$ ”
 ⟨proof⟩

lemma $\text{trans_star_}\varepsilon \text{ iff_}\varepsilon \text{ exp_}\varepsilon \text{ trans_star}$:

“ $(p, w, q) \in \text{trans_star_}\varepsilon \iff (\exists w'. \varepsilon \text{ exp } w' w \wedge (p, w', q) \in \text{trans_star})$ ”
 ⟨proof⟩

lemma $\varepsilon \text{ exp_split'}$:

assumes “ $\varepsilon_exp\ u_ \varepsilon\ (\gamma 1\ \# \ u 1)$ ”
shows “ $\exists \gamma 1_ \varepsilon\ u 1_ \varepsilon. \varepsilon_exp\ \gamma 1_ \varepsilon\ [\gamma 1] \wedge \varepsilon_exp\ u 1_ \varepsilon\ u 1 \wedge u_ \varepsilon = \gamma 1_ \varepsilon\ @\ u 1_ \varepsilon$ ”
 ⟨proof⟩

lemma *remove_ε_append_dist*:
 “ $remove_ \varepsilon\ (w\ @\ w') = remove_ \varepsilon\ w\ @\ remove_ \varepsilon\ w'$ ”
 ⟨proof⟩

lemma *remove_ε_Cons_tl*:
assumes “ $remove_ \varepsilon\ w = remove_ \varepsilon\ (Some\ \gamma' \# \ tl\ w)$ ”
shows “ $\gamma' \# \ remove_ \varepsilon\ (tl\ w) = remove_ \varepsilon\ w$ ”
 ⟨proof⟩

lemma *trans_star_states_trans_star_ε*:
assumes “ $(p, w, ss, q) \in trans_star_states$ ”
shows “ $(p, LTS_ \varepsilon.remove_ \varepsilon\ w, q) \in trans_star_ \varepsilon$ ”
 ⟨proof⟩

lemma *no_edge_to_source_ε*:
assumes “ $(p, [\gamma], qq) \in trans_star_ \varepsilon$ ”
shows “ $qq \notin srcs$ ”
 ⟨proof⟩

lemma *trans_star_not_to_source_ε*:
assumes “ $(p''', w, q) \in trans_star_ \varepsilon$ ”
assumes “ $p''' \neq q$ ”
assumes “ $q' \in srcs$ ”
shows “ $q' \neq q$ ”
 ⟨proof⟩

lemma *append_edge_edge_trans_star_ε*:
assumes “ $(p1, Some\ \gamma', p2) \in transition_relation$ ”
assumes “ $(p2, Some\ \gamma'', q1) \in transition_relation$ ”
assumes “ $(q1, u1, q) \in trans_star_ \varepsilon$ ”
shows “ $(p1, [\gamma', \gamma''] @\ u1, q) \in trans_star_ \varepsilon$ ”
 ⟨proof⟩

inductive-set *trans_star_states_ε* :: “ $(state * label\ list * state\ list * state)\ set$ ” **where**
trans_star_states_ε_refl[iff]:
 “ $(p, [], [p], p) \in trans_star_states_ \varepsilon$ ”
 | *trans_star_states_ε_step_γ*:
 “ $(p, Some\ \gamma, q') \in transition_relation \implies$
 $(q', w, ss, q) \in trans_star_states_ \varepsilon \implies$
 $(p, \gamma \# w, p \# ss, q) \in trans_star_states_ \varepsilon$ ”
 | *trans_star_states_ε_step_ε*:
 “ $(p, \varepsilon, q') \in transition_relation \implies$
 $(q', w, ss, q) \in trans_star_states_ \varepsilon \implies$
 $(p, w, p \# ss, q) \in trans_star_states_ \varepsilon$ ”

inductive-set *path_with_word_ε* :: “ $(state\ list * label\ list)\ set$ ” **where**
path_with_word_ε_refl[iff]:
 “ $([s], []) \in path_with_word_ \varepsilon$ ”
 | *path_with_word_ε_step_γ*:
 “ $(s' \# ss, w) \in path_with_word_ \varepsilon \implies$
 $(s, Some\ l, s') \in transition_relation \implies$
 $(s \# s' \# ss, l \# w) \in path_with_word_ \varepsilon$ ”
 | *path_with_word_ε_step_ε*:
 “ $(s' \# ss, w) \in path_with_word_ \varepsilon \implies$
 $(s, \varepsilon, s') \in transition_relation \implies$
 $(s \# s' \# ss, w) \in path_with_word_ \varepsilon$ ”

lemma *ε_exp_Some_length*:

assumes “ $\varepsilon_exp (Some\ \alpha \# w1\ ^) w$ ”
shows “ $0 < length\ w$ ”
<proof>

lemma $\varepsilon_exp_Some_hd$:
assumes “ $\varepsilon_exp (Some\ \alpha \# w1\ ^) w$ ”
shows “ $hd\ w = \alpha$ ”
<proof>

lemma exp_empty_empty :
assumes “ $\varepsilon_exp []\ w$ ”
shows “ $w = []$ ”
<proof>

end

2.3 More LTS lemmas

lemma $LTS_ \varepsilon_trans_star_ \varepsilon_mono$:
“ $mono\ LTS_ \varepsilon.trans_star_ \varepsilon$ ”
<proof>

definition $\varepsilon_edge_of_edge$ **where**
“ $\varepsilon_edge_of_edge = (\lambda(a, l, b). (a, Some\ l, b))$ ”

definition $LTS_ \varepsilon_of_LTS$ **where**
“ $LTS_ \varepsilon_of_LTS\ transition_relation = \varepsilon_edge_of_edge\ 'transition_relation$ ”

end

References

[Wim20] Simon Wimmer. Archive of graph formalizations. 2020. <https://github.com/wimmers/archive-of-graph-formalizations>.