Linear Temporal Logic

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Abstract

This theory provides a formalisation of linear temporal logic (LTL) and unifies previous formalisations within the AFP. This entry establishes syntax and semantics for this logic and decouples it from existing entries, yielding a common environment for theories reasoning about LTL. Furthermore a parser written in SML and an executable simplifier are provided.

Contents

1 Linear Temporal Logic

theory *LTL*
imports
Main HOL−Library.Omega-Words-Fun

begin

This theory provides a formalisation of linear temporal logic. It provides three variants:

1. LTL with syntactic sugar. This variant is the semantic reference and the included parser generates ASTs of this datatype.
2. LTL in negation normal form without syntactic sugar. This variant is used by the included rewriting engine and is used for the translation to automata (implemented in other entries).
3. LTL in restricted negation normal form without the rather uncommon operators "weak until" and "strong release". It is used by the formalization of Gerth's algorithm.
4. PLTL. A variant with a reduced set of operators.

This theory subsumes (and partly reuses) the existing formalisation found in LTL_to_GBA and Stuttering_Equivalence and unifies them.

1.1 LTL with Syntactic Sugar

In this section, we provide a formulation of LTL with explicit syntactic sugar deeply embedded. This formalization serves as a reference semantics.

1.1.1 Syntax

datatype (*atoms-ltlc*: *'a*) *ltlc* =
  True-ltlc ($true_c$)
| *False-ltlc* ($false_c$)
| *Prop-ltlc 'a* ($prop_c$'(-'))
| *Not-ltlc 'a ltlc* (not_c - [*85*] *85*)
| *And-ltlc 'a ltlc 'a ltlc* (- and_c - [*82*,*82*] *81*)
| *Or-ltlc 'a ltlc 'a ltlc* (- or_c - [*81*,*81*] *80*)
| *Implies-ltlc 'a ltlc 'a ltlc* (- $implies_c$ - [*81*,*81*] *80*)
| *Next-ltlc 'a ltlc* (X_c - [*88*] *87*)
| *Final-ltlc 'a ltlc* (F_c - [*88*] *87*)
| *Global-ltlc 'a ltlc* (G_c - [*88*] *87*)
| *Until-ltlc 'a ltlc 'a ltlc* (- U_c - [*84*,*84*] *83*)
| *Release-ltlc 'a ltlc 'a ltlc* (- R_c - [*84*,*84*] *83*)
| *WeakUntil-ltlc 'a ltlc 'a ltlc* (- W_c - [*84*,*84*] *83*)
| *StrongRelease-ltlc 'a ltlc 'a ltlc* (- M_c - [*84*,*84*] *83*)

definition *Iff-ltlc* (- iff_c - [*81*,*81*] *80*)
where
  φ iff_c $\psi \equiv (\varphi$ $implies_c$ $\psi)$ and_c $(\psi$ $implies_c$ $\varphi)$

1.1.2 Semantics

primrec *semantics-ltlc* :: [*'a set word*, *'a ltlc*] ⇒ *bool* (- ⊨$_c$ - [*80*,*80*] *80*)
where
  $\xi \models_c true_c = True$
| $\xi \models_c false_c = False$
| $\xi \models_c prop_c(q) = (q{\in}\xi\ 0)$
| $\xi \models_c not_c\ \varphi = (\neg\ \xi \models_c \varphi)$
| $\xi \models_c \varphi\ and_c\ \psi = (\xi \models_c \varphi \wedge \xi \models_c \psi)$
| $\xi \models_c \varphi\ or_c\ \psi = (\xi \models_c \varphi \vee \xi \models_c \psi)$
| $\xi \models_c \varphi\ implies_c\ \psi = (\xi \models_c \varphi \longrightarrow \xi \models_c \psi)$
| $\xi \models_c X_c\ \varphi = (suffix\ 1\ \xi \models_c \varphi)$
| $\xi \models_c F_c\ \varphi = (\exists\, i.\ suffix\ i\ \xi \models_c \varphi)$
| $\xi \models_c G_c\ \varphi = (\forall\, i.\ suffix\ i\ \xi \models_c \varphi)$
| $\xi \models_c \varphi\ U_c\ \psi = (\exists\, i.\ suffix\ i\ \xi \models_c \psi \wedge (\forall\, j{<}i.\ suffix\ j\ \xi \models_c \varphi))$
| $\xi \models_c \varphi\ R_c\ \psi = (\forall\, i.\ suffix\ i\ \xi \models_c \psi \vee (\exists\, j{<}i.\ suffix\ j\ \xi \models_c \varphi))$
| $\xi \models_c \varphi\ W_c\ \psi = (\forall\, i.\ suffix\ i\ \xi \models_c \varphi \vee (\exists\, j{\leq}i.\ suffix\ j\ \xi \models_c \psi))$
| $\xi \models_c \varphi\ M_c\ \psi = (\exists\, i.\ suffix\ i\ \xi \models_c \varphi \wedge (\forall\, j{\leq}i.\ suffix\ j\ \xi \models_c \psi))$

lemma *semantics-ltlc-sugar* [*simp*]:
  $\xi \models_c \varphi\ iff_c\ \psi = (\xi \models_c \varphi \longleftrightarrow \xi \models_c \psi)$
  $\xi \models_c F_c\ \varphi = \xi \models_c (true_c\ U_c\ \varphi)$
  $\xi \models_c G_c\ \varphi = \xi \models_c (false_c\ R_c\ \varphi)$
  ⟨*proof*⟩

definition *language-ltlc* $\varphi \equiv \{\xi.\ \xi \models_c \varphi\}$

lemma *language-ltlc-negate*[*simp*]:
  language-ltlc $(not_c\ \varphi) = -$ *language-ltlc* φ
  ⟨*proof*⟩

lemma *ltl-true-or-con*[*simp*]:
  $\xi \models_c prop_c(p)\ or_c\ (not_c\ prop_c(p))$
  ⟨*proof*⟩

lemma *ltl-false-true-con*[*simp*]:
  $\xi \models_c not_c\ true_c \longleftrightarrow \xi \models_c false_c$
  ⟨*proof*⟩

lemma *ltl-Next-Neg-con*[*simp*]:
  $\xi \models_c X_c\ (not_c\ \varphi) \longleftrightarrow \xi \models_c not_c\ X_c\ \varphi$
  ⟨*proof*⟩

lemma *ltl-Until-Release-con*:

$\xi \models_c \varphi\ R_c\ \psi \longleftrightarrow (\neg\ \xi \models_c (not_c\ \varphi)\ U_c\ (not_c\ \psi))$
$\xi \models_c \varphi\ U_c\ \psi \longleftrightarrow (\neg\ \xi \models_c (not_c\ \varphi)\ R_c\ (not_c\ \psi))$
⟨*proof*⟩

lemma *ltl-WeakUntil-StrongRelease-con*:
$\xi \models_c \varphi\ W_c\ \psi \longleftrightarrow (\neg\ \xi \models_c (not_c\ \varphi)\ M_c\ (not_c\ \psi))$
$\xi \models_c \varphi\ M_c\ \psi \longleftrightarrow (\neg\ \xi \models_c (not_c\ \varphi)\ W_c\ (not_c\ \psi))$
⟨*proof*⟩

lemma *ltl-Release-StrongRelease-con*:
$\xi \models_c \varphi\ R_c\ \psi \longleftrightarrow \xi \models_c (\varphi\ M_c\ \psi)\ or_c\ (G_c\ \psi)$
$\xi \models_c \varphi\ M_c\ \psi \longleftrightarrow \xi \models_c (\varphi\ R_c\ \psi)\ and_c\ (F_c\ \varphi)$
⟨*proof*⟩

lemma *ltl-Until-WeakUntil-con*:
$\xi \models_c \varphi\ U_c\ \psi \longleftrightarrow \xi \models_c (\varphi\ W_c\ \psi)\ and_c\ (F_c\ \psi)$
$\xi \models_c \varphi\ W_c\ \psi \longleftrightarrow \xi \models_c (\varphi\ U_c\ \psi)\ or_c\ (G_c\ \varphi)$
⟨*proof*⟩

lemma *ltl-StrongRelease-Until-con*:
$\xi \models_c \varphi\ M_c\ \psi \longleftrightarrow \xi \models_c \psi\ U_c\ (\varphi\ and_c\ \psi)$
⟨*proof*⟩

lemma *ltl-WeakUntil-Release-con*:
$\xi \models_c \varphi\ R_c\ \psi \longleftrightarrow \xi \models_c \psi\ W_c\ (\varphi\ and_c\ \psi)$
⟨*proof*⟩

definition *pw-eq-on* $S\ w\ w' \equiv \forall\, i.\ w\ i \cap S = w'\ i \cap S$

lemma *pw-eq-on-refl*[*simp*]: *pw-eq-on* $S\ w\ w$
and *pw-eq-on-sym*: *pw-eq-on* $S\ w\ w' \Longrightarrow$ *pw-eq-on* $S\ w'\ w$
and *pw-eq-on-trans*[*trans*]: ⟦*pw-eq-on* $S\ w\ w'$; *pw-eq-on* $S\ w'\ w''$⟧ $\Longrightarrow$ *pw-eq-on* $S\ w\ w''$
⟨*proof*⟩

lemma *pw-eq-on-suffix*:
pw-eq-on $S\ w\ w' \Longrightarrow$ *pw-eq-on* S (*suffix* $k\ w$) (*suffix* $k\ w'$)
⟨*proof*⟩

lemma *pw-eq-on-subset*:
$S \subseteq S' \Longrightarrow$ *pw-eq-on* $S'\ w\ w' \Longrightarrow$ *pw-eq-on* $S\ w\ w'$
⟨*proof*⟩

lemma *ltlc-eq-on-aux*:
pw-eq-on (*atoms-ltlc* φ) w $w' \Longrightarrow w \models_c \varphi \Longrightarrow w' \models_c \varphi$
⟨*proof*⟩

lemma *ltlc-eq-on*:
pw-eq-on (*atoms-ltlc* φ) w $w' \Longrightarrow w \models_c \varphi \longleftrightarrow w' \models_c \varphi$
⟨*proof*⟩

lemma *suffix-comp*: $(\lambda i.\ f\ (\mathit{suffix}\ k\ w\ i)) = \mathit{suffix}\ k\ (f\ o\ w)$
⟨*proof*⟩

lemma *suffix-range*: $\bigcup(\mathit{range}\ \xi) \subseteq \mathit{APs} \Longrightarrow \bigcup(\mathit{range}\ (\mathit{suffix}\ k\ \xi)) \subseteq \mathit{APs}$
⟨*proof*⟩

lemma *map-semantics-ltlc-aux*:
assumes *inj-on f APs*
assumes $\bigcup(\mathit{range}\ w) \subseteq \mathit{APs}$
assumes *atoms-ltlc* $\varphi \subseteq \mathit{APs}$
shows $w \models_c \varphi \longleftrightarrow (\lambda i.\ f\ `\ w\ i) \models_c$ *map-ltlc f* φ
⟨*proof*⟩

definition *map-props f APs* $\equiv \{i.\ \exists\, p{\in}\mathit{APs}.\ f\ p = \mathit{Some}\ i\}$

lemma *map-semantics-ltlc*:
assumes *INJ*: *inj-on f* (*dom f*) **and** *DOM*: *atoms-ltlc* $\varphi \subseteq$ *dom f*
shows $\xi \models_c \varphi \longleftrightarrow$ (*map-props f o* ξ) $\models_c$ *map-ltlc* (*the o f*) φ
⟨*proof*⟩

lemma *map-semantics-ltlc-inv*:
assumes *INJ*: *inj-on f* (*dom f*) **and** *DOM*: *atoms-ltlc* $\varphi \subseteq$ *dom f*
shows $\xi \models_c$ *map-ltlc* (*the o f*) $\varphi \longleftrightarrow (\lambda i.\ (\mathit{the}\ o\ f) -`\ \xi\ i) \models_c \varphi$
⟨*proof*⟩

1.2 LTL in Negation Normal Form

We define a type of LTL formula in negation normal form (NNF).

1.2.1 Syntax

datatype (*atoms-ltln*: $'a$) *ltln* =

True-ltln	($true_n$)
| *False-ltln*	($false_n$)
| *Prop-ltln* $'a$	($prop_n$ $'$(-$'$))
| *Nprop-ltln* $'a$	($nprop_n$ $'$(-$'$))

| *And-ltln 'a ltln 'a ltln* (- and_n - [*82*,*82*] *81*)
| *Or-ltln 'a ltln 'a ltln* (- or_n - [*84*,*84*] *83*)
| *Next-ltln 'a ltln* (X_n - [*88*] *87*)
| *Until-ltln 'a ltln 'a ltln* (- U_n - [*84*,*84*] *83*)
| *Release-ltln 'a ltln 'a ltln* (- R_n - [*84*,*84*] *83*)
| *WeakUntil-ltln 'a ltln 'a ltln* (- W_n - [*84*,*84*] *83*)
| *StrongRelease-ltln 'a ltln 'a ltln* (- M_n - [*84*,*84*] *83*)

abbreviation *finally*$_n$ :: *'a ltln* ⇒ *'a ltln* (F_n - [*88*] *87*)
where
$F_n\ \varphi \equiv true_n\ U_n\ \varphi$

notation (*input*) *finally*$_n$ ($\Diamond_n$ - [*88*] *87*)

abbreviation *globally*$_n$ :: *'a ltln* ⇒ *'a ltln* (G_n - [*88*] *87*)
where
$G_n\ \varphi \equiv false_n\ R_n\ \varphi$

notation (*input*) *globally*$_n$ ($\Box_n$ - [*88*] *87*)

1.2.2 Semantics

primrec *semantics-ltln* :: [*'a set word*, *'a ltln*] ⇒ *bool* (- $\models_n$ - [*80*,*80*] *80*)
where
$\xi \models_n true_n = True$
| $\xi \models_n false_n = False$
| $\xi \models_n prop_n(q) = (q \in \xi\ 0)$
| $\xi \models_n nprop_n(q) = (q \notin \xi\ 0)$
| $\xi \models_n \varphi\ and_n\ \psi = (\xi \models_n \varphi \wedge \xi \models_n \psi)$
| $\xi \models_n \varphi\ or_n\ \psi = (\xi \models_n \varphi \vee \xi \models_n \psi)$
| $\xi \models_n X_n\ \varphi = (suffix\ 1\ \xi \models_n \varphi)$
| $\xi \models_n \varphi\ U_n\ \psi = (\exists\, i.\ suffix\ i\ \xi \models_n \psi \wedge (\forall\, j{<}i.\ suffix\ j\ \xi \models_n \varphi))$
| $\xi \models_n \varphi\ R_n\ \psi = (\forall\, i.\ suffix\ i\ \xi \models_n \psi \vee (\exists\, j{<}i.\ suffix\ j\ \xi \models_n \varphi))$
| $\xi \models_n \varphi\ W_n\ \psi = (\forall\, i.\ suffix\ i\ \xi \models_n \varphi \vee (\exists\, j{\leq}i.\ suffix\ j\ \xi \models_n \psi))$
| $\xi \models_n \varphi\ M_n\ \psi = (\exists\, i.\ suffix\ i\ \xi \models_n \varphi \wedge (\forall\, j{\leq}i.\ suffix\ j\ \xi \models_n \psi))$

definition *language-ltln* $\varphi \equiv \{\xi.\ \xi \models_n \varphi\}$

lemma *semantics-ltln-ite-simps*[*simp*]:
$w \models_n (if\ P\ then\ true_n\ else\ false_n) = P$
$w \models_n (if\ P\ then\ false_n\ else\ true_n) = (\neg P)$
⟨*proof*⟩

1.2.3 Conversion

fun *ltlc-to-ltln'* :: *bool* ⇒ *'a ltlc* ⇒ *'a ltln*
where
  ltlc-to-ltln' False $true_c$ = $true_n$
| *ltlc-to-ltln' False* $false_c$ = $false_n$
| *ltlc-to-ltln' False* $prop_c(q)$ = $prop_n(q)$
| *ltlc-to-ltln' False* (φ and_c ψ) = (*ltlc-to-ltln' False* φ) and_n (*ltlc-to-ltln' False* ψ)
| *ltlc-to-ltln' False* (φ or_c ψ) = (*ltlc-to-ltln' False* φ) or_n (*ltlc-to-ltln' False* ψ)
| *ltlc-to-ltln' False* (φ $implies_c$ ψ) = (*ltlc-to-ltln' True* φ) or_n (*ltlc-to-ltln' False* ψ)
| *ltlc-to-ltln' False* (F_c φ) = $true_n$ U_n (*ltlc-to-ltln' False* φ)
| *ltlc-to-ltln' False* (G_c φ) = $false_n$ R_n (*ltlc-to-ltln' False* φ)
| *ltlc-to-ltln' False* (φ U_c ψ) = (*ltlc-to-ltln' False* φ) U_n (*ltlc-to-ltln' False* ψ)
| *ltlc-to-ltln' False* (φ R_c ψ) = (*ltlc-to-ltln' False* φ) R_n (*ltlc-to-ltln' False* ψ)
| *ltlc-to-ltln' False* (φ W_c ψ) = (*ltlc-to-ltln' False* φ) W_n (*ltlc-to-ltln' False* ψ)
| *ltlc-to-ltln' False* (φ M_c ψ) = (*ltlc-to-ltln' False* φ) M_n (*ltlc-to-ltln' False* ψ)
| *ltlc-to-ltln' True* $true_c$ = $false_n$
| *ltlc-to-ltln' True* $false_c$ = $true_n$
| *ltlc-to-ltln' True* $prop_c(q)$ = $nprop_n(q)$
| *ltlc-to-ltln' True* (φ and_c ψ) = (*ltlc-to-ltln' True* φ) or_n (*ltlc-to-ltln' True* ψ)
| *ltlc-to-ltln' True* (φ or_c ψ) = (*ltlc-to-ltln' True* φ) and_n (*ltlc-to-ltln' True* ψ)
| *ltlc-to-ltln' True* (φ $implies_c$ ψ) = (*ltlc-to-ltln' False* φ) and_n (*ltlc-to-ltln' True* ψ)
| *ltlc-to-ltln' True* (F_c φ) = $false_n$ R_n (*ltlc-to-ltln' True* φ)
| *ltlc-to-ltln' True* (G_c φ) = $true_n$ U_n (*ltlc-to-ltln' True* φ)
| *ltlc-to-ltln' True* (φ U_c ψ) = (*ltlc-to-ltln' True* φ) R_n (*ltlc-to-ltln' True* ψ)
| *ltlc-to-ltln' True* (φ R_c ψ) = (*ltlc-to-ltln' True* φ) U_n (*ltlc-to-ltln' True* ψ)
| *ltlc-to-ltln' True* (φ W_c ψ) = (*ltlc-to-ltln' True* φ) M_n (*ltlc-to-ltln' True* ψ)
| *ltlc-to-ltln' True* (φ M_c ψ) = (*ltlc-to-ltln' True* φ) W_n (*ltlc-to-ltln' True* ψ)
| *ltlc-to-ltln' b* (not_c φ) = *ltlc-to-ltln'* ($\neg$ *b*) φ
| *ltlc-to-ltln' b* (X_c φ) = X_n (*ltlc-to-ltln' b* φ)

fun *ltlc-to-ltln* :: *'a ltlc* ⇒ *'a ltln*

where
ltlc-to-ltln φ = *ltlc-to-ltln'* *False* φ

fun *ltln-to-ltlc* :: *'a ltln* ⇒ *'a ltlc*
where
ltln-to-ltlc $true_n$ = $true_c$
| *ltln-to-ltlc* $false_n$ = $false_c$
| *ltln-to-ltlc* $prop_n(q)$ = $prop_c(q)$
| *ltln-to-ltlc* $nprop_n(q)$ = not_c ($prop_c(q)$)
| *ltln-to-ltlc* (φ and_n ψ) = (*ltln-to-ltlc* φ and_c *ltln-to-ltlc* ψ)
| *ltln-to-ltlc* (φ or_n ψ) = (*ltln-to-ltlc* φ or_c *ltln-to-ltlc* ψ)
| *ltln-to-ltlc* (X_n φ) = (X_c *ltln-to-ltlc* φ)
| *ltln-to-ltlc* (φ U_n ψ) = (*ltln-to-ltlc* φ U_c *ltln-to-ltlc* ψ)
| *ltln-to-ltlc* (φ R_n ψ) = (*ltln-to-ltlc* φ R_c *ltln-to-ltlc* ψ)
| *ltln-to-ltlc* (φ W_n ψ) = (*ltln-to-ltlc* φ W_c *ltln-to-ltlc* ψ)
| *ltln-to-ltlc* (φ M_n ψ) = (*ltln-to-ltlc* φ M_c *ltln-to-ltlc* ψ)

lemma *ltlc-to-ltln'-correct*:
$w \models_n$ (*ltlc-to-ltln'* *True* φ) ⟷ ¬ $w \models_c \varphi$
$w \models_n$ (*ltlc-to-ltln'* *False* φ) ⟷ $w \models_c \varphi$
size (*ltlc-to-ltln'* *True* φ) ≤ 2 ∗ *size* φ
size (*ltlc-to-ltln'* *False* φ) ≤ 2 ∗ *size* φ
⟨*proof*⟩

lemma *ltlc-to-ltln-semantics* [*simp*]:
$w \models_n$ *ltlc-to-ltln* φ ⟷ $w \models_c \varphi$
⟨*proof*⟩

lemma *ltlc-to-ltln-size*:
size (*ltlc-to-ltln* φ) ≤ 2 ∗ *size* φ
⟨*proof*⟩

lemma *ltln-to-ltlc-semantics* [*simp*]:
$w \models_c$ *ltln-to-ltlc* φ ⟷ $w \models_n \varphi$
⟨*proof*⟩

lemma *ltlc-to-ltln-atoms*:
atoms-ltln (*ltlc-to-ltln* φ) = *atoms-ltlc* φ
⟨*proof*⟩

1.2.4 Negation

fun not_n
where

$not_n\ true_n = false_n$
| $not_n\ false_n = true_n$
| $not_n\ prop_n(a) = nprop_n(a)$
| $not_n\ nprop_n(a) = prop_n(a)$
| $not_n\ (\varphi\ and_n\ \psi) = (not_n\ \varphi)\ or_n\ (not_n\ \psi)$
| $not_n\ (\varphi\ or_n\ \psi) = (not_n\ \varphi)\ and_n\ (not_n\ \psi)$
| $not_n\ (X_n\ \varphi) = X_n\ (not_n\ \varphi)$
| $not_n\ (\varphi\ U_n\ \psi) = (not_n\ \varphi)\ R_n\ (not_n\ \psi)$
| $not_n\ (\varphi\ R_n\ \psi) = (not_n\ \varphi)\ U_n\ (not_n\ \psi)$
| $not_n\ (\varphi\ W_n\ \psi) = (not_n\ \varphi)\ M_n\ (not_n\ \psi)$
| $not_n\ (\varphi\ M_n\ \psi) = (not_n\ \varphi)\ W_n\ (not_n\ \psi)$

lemma *not_n-semantics*[*simp*]:
$w \models_n not_n\ \varphi \longleftrightarrow \neg\ w \models_n \varphi$
⟨*proof*⟩

lemma *not_n-size*:
$size\ (not_n\ \varphi) = size\ \varphi$
⟨*proof*⟩

1.2.5 Subformulas

fun *subfrmlsn* :: *'a ltln* ⇒ *'a ltln set*
where
$subfrmlsn\ (\varphi\ and_n\ \psi) = \{\varphi\ and_n\ \psi\} \cup subfrmlsn\ \varphi \cup subfrmlsn\ \psi$
| $subfrmlsn\ (\varphi\ or_n\ \psi) = \{\varphi\ or_n\ \psi\} \cup subfrmlsn\ \varphi \cup subfrmlsn\ \psi$
| $subfrmlsn\ (X_n\ \varphi) = \{X_n\ \varphi\} \cup subfrmlsn\ \varphi$
| $subfrmlsn\ (\varphi\ U_n\ \psi) = \{\varphi\ U_n\ \psi\} \cup subfrmlsn\ \varphi \cup subfrmlsn\ \psi$
| $subfrmlsn\ (\varphi\ R_n\ \psi) = \{\varphi\ R_n\ \psi\} \cup subfrmlsn\ \varphi \cup subfrmlsn\ \psi$
| $subfrmlsn\ (\varphi\ W_n\ \psi) = \{\varphi\ W_n\ \psi\} \cup subfrmlsn\ \varphi \cup subfrmlsn\ \psi$
| $subfrmlsn\ (\varphi\ M_n\ \psi) = \{\varphi\ M_n\ \psi\} \cup subfrmlsn\ \varphi \cup subfrmlsn\ \psi$
| $subfrmlsn\ \varphi = \{\varphi\}$

lemma *subfrmlsn-id*[*simp*]:
$\varphi \in subfrmlsn\ \varphi$
⟨*proof*⟩

lemma *subfrmlsn-finite*:
$finite\ (subfrmlsn\ \varphi)$
⟨*proof*⟩

lemma *subfrmlsn-card*:
$card\ (subfrmlsn\ \varphi) \leq size\ \varphi$
⟨*proof*⟩

lemma *subfrmlsn-subset*:
$\psi \in \mathit{subfrmlsn}\ \varphi \Longrightarrow \mathit{subfrmlsn}\ \psi \subseteq \mathit{subfrmlsn}\ \varphi$
⟨*proof*⟩

lemma *subfrmlsn-size*:
$\psi \in \mathit{subfrmlsn}\ \varphi \Longrightarrow \mathit{size}\ \psi < \mathit{size}\ \varphi \vee \psi = \varphi$
⟨*proof*⟩

abbreviation
$\mathit{size\text{-}set}\ S \equiv \mathit{sum}\ (\lambda x.\ 2 * \mathit{size}\ x + 1)\ S$

lemma *size-set-diff*:
$\mathit{finite}\ S \Longrightarrow S' \subseteq S \Longrightarrow \mathit{size\text{-}set}\ (S - S') = \mathit{size\text{-}set}\ S - \mathit{size\text{-}set}\ S'$
⟨*proof*⟩

1.2.6 Constant Folding

lemma *U-consts*[*intro*, *simp*]:
$w \models_n \varphi\ U_n\ \mathit{true}_n$
$\neg\ (w \models_n \varphi\ U_n\ \mathit{false}_n)$
$(w \models_n \mathit{false}_n\ U_n\ \varphi) = (w \models_n \varphi)$
⟨*proof*⟩

lemma *R-consts*[*intro*, *simp*]:
$w \models_n \varphi\ R_n\ \mathit{true}_n$
$\neg\ (w \models_n \varphi\ R_n\ \mathit{false}_n)$
$(w \models_n \mathit{true}_n\ R_n\ \varphi) = (w \models_n \varphi)$
⟨*proof*⟩

lemma *W-consts*[*intro*, *simp*]:
$w \models_n \mathit{true}_n\ W_n\ \varphi$
$w \models_n \varphi\ W_n\ \mathit{true}_n$
$(w \models_n \mathit{false}_n\ W_n\ \varphi) = (w \models_n \varphi)$
$(w \models_n \varphi\ W_n\ \mathit{false}_n) = (w \models_n G_n\ \varphi)$
⟨*proof*⟩

lemma *M-consts*[*intro*, *simp*]:
$\neg\ (w \models_n \mathit{false}_n\ M_n\ \varphi)$
$\neg\ (w \models_n \varphi\ M_n\ \mathit{false}_n)$
$(w \models_n \mathit{true}_n\ M_n\ \varphi) = (w \models_n \varphi)$
$(w \models_n \varphi\ M_n\ \mathit{true}_n) = (w \models_n F_n\ \varphi)$
⟨*proof*⟩

1.2.7 Distributivity

lemma *until-and-left-distrib*:
$w \models_n (\varphi_1\ and_n\ \varphi_2)\ U_n\ \psi \longleftrightarrow w \models_n (\varphi_1\ U_n\ \psi)\ and_n\ (\varphi_2\ U_n\ \psi)$
⟨*proof*⟩

lemma *until-or-right-distrib*:
$w \models_n \varphi\ U_n\ (\psi_1\ or_n\ \psi_2) \longleftrightarrow w \models_n (\varphi\ U_n\ \psi_1)\ or_n\ (\varphi\ U_n\ \psi_2)$
⟨*proof*⟩

lemma *release-and-right-distrib*:
$w \models_n \varphi\ R_n\ (\psi_1\ and_n\ \psi_2) \longleftrightarrow w \models_n (\varphi\ R_n\ \psi_1)\ and_n\ (\varphi\ R_n\ \psi_2)$
⟨*proof*⟩

lemma *release-or-left-distrib*:
$w \models_n (\varphi_1\ or_n\ \varphi_2)\ R_n\ \psi \longleftrightarrow w \models_n (\varphi_1\ R_n\ \psi)\ or_n\ (\varphi_2\ R_n\ \psi)$
⟨*proof*⟩

lemma *strong-release-and-right-distrib*:
$w \models_n \varphi\ M_n\ (\psi_1\ and_n\ \psi_2) \longleftrightarrow w \models_n (\varphi\ M_n\ \psi_1)\ and_n\ (\varphi\ M_n\ \psi_2)$
⟨*proof*⟩

lemma *strong-release-or-left-distrib*:
$w \models_n (\varphi_1\ or_n\ \varphi_2)\ M_n\ \psi \longleftrightarrow w \models_n (\varphi_1\ M_n\ \psi)\ or_n\ (\varphi_2\ M_n\ \psi)$
⟨*proof*⟩

lemma *weak-until-and-left-distrib*:
$w \models_n (\varphi_1\ and_n\ \varphi_2)\ W_n\ \psi \longleftrightarrow w \models_n (\varphi_1\ W_n\ \psi)\ and_n\ (\varphi_2\ W_n\ \psi)$
⟨*proof*⟩

lemma *weak-until-or-right-distrib*:
$w \models_n \varphi\ W_n\ (\psi_1\ or_n\ \psi_2) \longleftrightarrow w \models_n (\varphi\ W_n\ \psi_1)\ or_n\ (\varphi\ W_n\ \psi_2)$
⟨*proof*⟩

lemma *next-until-distrib*:
$w \models_n X_n\ (\varphi\ U_n\ \psi) \longleftrightarrow w \models_n (X_n\ \varphi)\ U_n\ (X_n\ \psi)$
⟨*proof*⟩

lemma *next-release-distrib*:
$w \models_n X_n\ (\varphi\ R_n\ \psi) \longleftrightarrow w \models_n (X_n\ \varphi)\ R_n\ (X_n\ \psi)$
⟨*proof*⟩

lemma *next-weak-until-distrib*:

$w \models_n X_n\ (\varphi\ W_n\ \psi) \longleftrightarrow w \models_n (X_n\ \varphi)\ W_n\ (X_n\ \psi)$
⟨*proof*⟩

lemma *next-strong-release-distrib*:
$w \models_n X_n\ (\varphi\ M_n\ \psi) \longleftrightarrow w \models_n (X_n\ \varphi)\ M_n\ (X_n\ \psi)$
⟨*proof*⟩

1.2.8 Nested operators

lemma *finally-until*[*simp*]:
$w \models_n F_n\ (\varphi\ U_n\ \psi) \longleftrightarrow w \models_n F_n\ \psi$
⟨*proof*⟩

lemma *globally-release*[*simp*]:
$w \models_n G_n\ (\varphi\ R_n\ \psi) \longleftrightarrow w \models_n G_n\ \psi$
⟨*proof*⟩

lemma *globally-weak-until*[*simp*]:
$w \models_n G_n\ (\varphi\ W_n\ \psi) \longleftrightarrow w \models_n G_n\ (\varphi\ or_n\ \psi)$
⟨*proof*⟩

lemma *finally-strong-release*[*simp*]:
$w \models_n F_n\ (\varphi\ M_n\ \psi) \longleftrightarrow w \models_n F_n\ (\varphi\ and_n\ \psi)$
⟨*proof*⟩

1.2.9 Weak and strong operators

lemma *ltln-weak-strong*:
$w \models_n \varphi\ W_n\ \psi \longleftrightarrow w \models_n (G_n\ \varphi)\ or_n\ (\varphi\ U_n\ \psi)$
$w \models_n \varphi\ R_n\ \psi \longleftrightarrow w \models_n (G_n\ \psi)\ or_n\ (\varphi\ M_n\ \psi)$
⟨*proof*⟩

lemma *ltln-strong-weak*:
$w \models_n \varphi\ U_n\ \psi \longleftrightarrow w \models_n (F_n\ \psi)\ and_n\ (\varphi\ W_n\ \psi)$
$w \models_n \varphi\ M_n\ \psi \longleftrightarrow w \models_n (F_n\ \varphi)\ and_n\ (\varphi\ R_n\ \psi)$
⟨*proof*⟩

lemma *ltln-strong-to-weak*:
$w \models_n \varphi\ U_n\ \psi \Longrightarrow w \models_n \varphi\ W_n\ \psi$
$w \models_n \varphi\ M_n\ \psi \Longrightarrow w \models_n \varphi\ R_n\ \psi$
⟨*proof*⟩

lemma *ltln-weak-to-strong*:
$[\![w \models_n \varphi\ W_n\ \psi;\ \neg\ w \models_n G_n\ \varphi]\!] \Longrightarrow w \models_n \varphi\ U_n\ \psi$

$[\![w \models_n \varphi\ W_n\ \psi;\ w \models_n F_n\ \psi]\!] \Longrightarrow w \models_n \varphi\ U_n\ \psi$
$[\![w \models_n \varphi\ R_n\ \psi;\ \neg\ w \models_n G_n\ \psi]\!] \Longrightarrow w \models_n \varphi\ M_n\ \psi$
$[\![w \models_n \varphi\ R_n\ \psi;\ w \models_n F_n\ \varphi]\!] \Longrightarrow w \models_n \varphi\ M_n\ \psi$
⟨*proof*⟩

lemma *ltln-StrongRelease-to-Until*:
$w \models_n \varphi\ M_n\ \psi \longleftrightarrow w \models_n \psi\ U_n\ (\varphi\ and_n\ \psi)$
⟨*proof*⟩

lemma *ltln-Release-to-WeakUntil*:
$w \models_n \varphi\ R_n\ \psi \longleftrightarrow w \models_n \psi\ W_n\ (\varphi\ and_n\ \psi)$
⟨*proof*⟩

lemma *ltln-WeakUntil-to-Release*:
$w \models_n \varphi\ W_n\ \psi \longleftrightarrow w \models_n \psi\ R_n\ (\varphi\ or_n\ \psi)$
⟨*proof*⟩

lemma *ltln-Until-to-StrongRelease*:
$w \models_n \varphi\ U_n\ \psi \longleftrightarrow w \models_n \psi\ M_n\ (\varphi\ or_n\ \psi)$
⟨*proof*⟩

1.2.10 GF and FG semantics

lemma *GF-suffix*:
suffix $i\ w \models_n G_n\ (F_n\ \psi) \longleftrightarrow w \models_n G_n\ (F_n\ \psi)$
⟨*proof*⟩

lemma *FG-suffix*:
suffix $i\ w \models_n F_n\ (G_n\ \psi) \longleftrightarrow w \models_n F_n\ (G_n\ \psi)$
⟨*proof*⟩

lemma *GF-Inf-many*:
$w \models_n G_n\ (F_n\ \varphi) \longleftrightarrow (\exists_\infty\ i.\ suffix\ i\ w \models_n \varphi)$
⟨*proof*⟩

lemma *FG-Alm-all*:
$w \models_n F_n\ (G_n\ \varphi) \longleftrightarrow (\forall_\infty\ i.\ suffix\ i\ w \models_n \varphi)$
⟨*proof*⟩

lemma *MOST-nat-add*:
$(\forall_\infty i{::}nat.\ P\ i) \longleftrightarrow (\forall_\infty i.\ P\ (i + j))$

⟨*proof*⟩

lemma *INFM-nat-add*:
$(\exists_\infty i{::}nat.\ P\ i) \longleftrightarrow (\exists_\infty i.\ P\ (i + j))$
⟨*proof*⟩

lemma *FG-suffix-G*:
$w \models_n F_n\ (G_n\ \varphi) \Longrightarrow \forall_\infty i.\ suffix\ i\ w \models_n G_n\ \varphi$
⟨*proof*⟩

lemma *Alm-all-GF-F*:
$\forall_\infty i.\ suffix\ i\ w \models_n G_n\ (F_n\ \psi) \longleftrightarrow suffix\ i\ w \models_n F_n\ \psi$
⟨*proof*⟩

lemma *Alm-all-FG-G*:
$\forall_\infty i.\ suffix\ i\ w \models_n F_n\ (G_n\ \psi) \longleftrightarrow suffix\ i\ w \models_n G_n\ \psi$
⟨*proof*⟩

1.2.11 Expansion

lemma *ltln-expand-Until*:
$\xi \models_n \varphi\ U_n\ \psi \longleftrightarrow (\xi \models_n \psi\ or_n\ (\varphi\ and_n\ (X_n\ (\varphi\ U_n\ \psi))))$
(**is** *?lhs = ?rhs*)
⟨*proof*⟩

lemma *ltln-expand-Release*:
$\xi \models_n \varphi\ R_n\ \psi \longleftrightarrow (\xi \models_n \psi\ and_n\ (\varphi\ or_n\ (X_n\ (\varphi\ R_n\ \psi))))$
(**is** *?lhs = ?rhs*)
⟨*proof*⟩

lemma *ltln-expand-WeakUntil*:
$\xi \models_n \varphi\ W_n\ \psi \longleftrightarrow (\xi \models_n \psi\ or_n\ (\varphi\ and_n\ (X_n\ (\varphi\ W_n\ \psi))))$
(**is** *?lhs = ?rhs*)
⟨*proof*⟩

lemma *ltln-expand-StrongRelease*:
$\xi \models_n \varphi\ M_n\ \psi \longleftrightarrow (\xi \models_n \psi\ and_n\ (\varphi\ or_n\ (X_n\ (\varphi\ M_n\ \psi))))$
(**is** *?lhs = ?rhs*)
⟨*proof*⟩

lemma *ltln-Release-alterdef*:
$w \models_n \varphi\ R_n\ \psi \longleftrightarrow w \models_n (G_n\ \psi)\ or_n\ (\psi\ U_n\ (\varphi\ and_n\ \psi))$
⟨*proof*⟩

1.3 LTL in restricted Negation Normal Form

Some algorithms do not handle the operators W and M, hence we also provide a datatype without these two operators.

1.3.1 Syntax

datatype (*atoms-ltlr*: *'a*) *ltlr* =
  True-ltlr ($true_r$)
| *False-ltlr* ($false_r$)
| *Prop-ltlr 'a* ($prop_r$ *'*(-*'*))
| *Nprop-ltlr 'a* ($nprop_r$ *'*(-*'*))
| *And-ltlr 'a ltlr 'a ltlr* (- and_r - [*82*,*82*] *81*)
| *Or-ltlr 'a ltlr 'a ltlr* (- or_r - [*84*,*84*] *83*)
| *Next-ltlr 'a ltlr* (X_r - [*88*] *87*)
| *Until-ltlr 'a ltlr 'a ltlr* (- U_r - [*84*,*84*] *83*)
| *Release-ltlr 'a ltlr 'a ltlr* (- R_r - [*84*,*84*] *83*)

1.3.2 Semantics

primrec *semantics-ltlr* :: [*'a set word*, *'a ltlr*] ⇒ *bool* (- $\models_r$ - [*80*,*80*] *80*)
where
  $\xi \models_r true_r = True$
| $\xi \models_r false_r = False$
| $\xi \models_r prop_r(q) = (q \in \xi\ 0)$
| $\xi \models_r nprop_r(q) = (q \notin \xi\ 0)$
| $\xi \models_r \varphi\ and_r\ \psi = (\xi \models_r \varphi \wedge \xi \models_r \psi)$
| $\xi \models_r \varphi\ or_r\ \psi = (\xi \models_r \varphi \vee \xi \models_r \psi)$
| $\xi \models_r X_r\ \varphi = (suffix\ 1\ \xi \models_r \varphi)$
| $\xi \models_r \varphi\ U_r\ \psi = (\exists i.\ suffix\ i\ \xi \models_r \psi \wedge (\forall j{<}i.\ suffix\ j\ \xi \models_r \varphi))$
| $\xi \models_r \varphi\ R_r\ \psi = (\forall i.\ suffix\ i\ \xi \models_r \psi \vee (\exists j{<}i.\ suffix\ j\ \xi \models_r \varphi))$

1.3.3 Conversion

fun *ltln-to-ltlr* :: *'a ltln* ⇒ *'a ltlr*
where
  ltln-to-ltlr $true_n = true_r$
| *ltln-to-ltlr* $false_n = false_r$
| *ltln-to-ltlr* $prop_n(a) = prop_r(a)$
| *ltln-to-ltlr* $nprop_n(a) = nprop_r(a)$
| *ltln-to-ltlr* ($\varphi\ and_n\ \psi$) = (*ltln-to-ltlr* φ) and_r (*ltln-to-ltlr* ψ)
| *ltln-to-ltlr* ($\varphi\ or_n\ \psi$) = (*ltln-to-ltlr* φ) or_r (*ltln-to-ltlr* ψ)
| *ltln-to-ltlr* ($X_n\ \varphi$) = X_r (*ltln-to-ltlr* φ)
| *ltln-to-ltlr* ($\varphi\ U_n\ \psi$) = (*ltln-to-ltlr* φ) U_r (*ltln-to-ltlr* ψ)
| *ltln-to-ltlr* ($\varphi\ R_n\ \psi$) = (*ltln-to-ltlr* φ) R_r (*ltln-to-ltlr* ψ)

| *ltln-to-ltlr* (φ W_n ψ) = (*ltln-to-ltlr* ψ) R_r ((*ltln-to-ltlr* φ) or_r (*ltln-to-ltlr* ψ))
| *ltln-to-ltlr* (φ M_n ψ) = (*ltln-to-ltlr* ψ) U_r ((*ltln-to-ltlr* φ) and_r (*ltln-to-ltlr* ψ))

fun *ltlr-to-ltln* :: *'a ltlr* ⇒ *'a ltln*
where
ltlr-to-ltln $true_r$ = $true_n$
| *ltlr-to-ltln* $false_r$ = $false_n$
| *ltlr-to-ltln* $prop_r(a)$ = $prop_n(a)$
| *ltlr-to-ltln* $nprop_r(a)$ = $nprop_n(a)$
| *ltlr-to-ltln* (φ and_r ψ) = (*ltlr-to-ltln* φ) and_n (*ltlr-to-ltln* ψ)
| *ltlr-to-ltln* (φ or_r ψ) = (*ltlr-to-ltln* φ) or_n (*ltlr-to-ltln* ψ)
| *ltlr-to-ltln* (X_r φ) = X_n (*ltlr-to-ltln* φ)
| *ltlr-to-ltln* (φ U_r ψ) = (*ltlr-to-ltln* φ) U_n (*ltlr-to-ltln* ψ)
| *ltlr-to-ltln* (φ R_r ψ) = (*ltlr-to-ltln* φ) R_n (*ltlr-to-ltln* ψ)

lemma *ltln-to-ltlr-semantics*:
$w \models_r$ *ltln-to-ltlr* $\varphi \longleftrightarrow w \models_n \varphi$
⟨*proof*⟩

lemma *ltlr-to-ltln-semantics*:
$w \models_n$ *ltlr-to-ltln* $\varphi \longleftrightarrow w \models_r \varphi$
⟨*proof*⟩

1.3.4 Negation

fun not_r
where
not_r $true_r$ = $false_r$
| not_r $false_r$ = $true_r$
| not_r $prop_r(a)$ = $nprop_r(a)$
| not_r $nprop_r(a)$ = $prop_r(a)$
| not_r (φ and_r ψ) = (not_r φ) or_r (not_r ψ)
| not_r (φ or_r ψ) = (not_r φ) and_r (not_r ψ)
| not_r (X_r φ) = X_r (not_r φ)
| not_r (φ U_r ψ) = (not_r φ) R_r (not_r ψ)
| not_r (φ R_r ψ) = (not_r φ) U_r (not_r ψ)

lemma not_r-*semantics* [*simp*]:
$w \models_r$ not_r $\varphi \longleftrightarrow \neg$ $w \models_r \varphi$
⟨*proof*⟩

1.3.5 Subformulas

fun *subfrmlsr* :: *'a ltlr* ⇒ *'a ltlr set*
where
  $subfrmlsr\ (\varphi\ and_r\ \psi) = \{\varphi\ and_r\ \psi\} \cup subfrmlsr\ \varphi \cup subfrmlsr\ \psi$
| $subfrmlsr\ (\varphi\ or_r\ \psi) = \{\varphi\ or_r\ \psi\} \cup subfrmlsr\ \varphi \cup subfrmlsr\ \psi$
| $subfrmlsr\ (\varphi\ U_r\ \psi) = \{\varphi\ U_r\ \psi\} \cup subfrmlsr\ \varphi \cup subfrmlsr\ \psi$
| $subfrmlsr\ (\varphi\ R_r\ \psi) = \{\varphi\ R_r\ \psi\} \cup subfrmlsr\ \varphi \cup subfrmlsr\ \psi$
| $subfrmlsr\ (X_r\ \varphi) = \{X_r\ \varphi\} \cup subfrmlsr\ \varphi$
| $subfrmlsr\ x = \{x\}$

lemma *subfrmlsr-id*[*simp*]:
  $\varphi \in subfrmlsr\ \varphi$
  ⟨*proof*⟩

lemma *subfrmlsr-finite*:
  $finite\ (subfrmlsr\ \varphi)$
  ⟨*proof*⟩

lemma *subfrmlsr-subset*:
  $\psi \in subfrmlsr\ \varphi \Longrightarrow subfrmlsr\ \psi \subseteq subfrmlsr\ \varphi$
  ⟨*proof*⟩

lemma *subfrmlsr-size*:
  $\psi \in subfrmlsr\ \varphi \Longrightarrow size\ \psi < size\ \varphi \vee \psi = \varphi$
  ⟨*proof*⟩

1.3.6 Expansion lemmas

lemma *ltlr-expand-Until*:
  $\xi \models_r \varphi\ U_r\ \psi \longleftrightarrow (\xi \models_r \psi\ or_r\ (\varphi\ and_r\ (X_r\ (\varphi\ U_r\ \psi))))$
  ⟨*proof*⟩

lemma *ltlr-expand-Release*:
  $\xi \models_r \varphi\ R_r\ \psi \longleftrightarrow (\xi \models_r \psi\ and_r\ (\varphi\ or_r\ (X_r\ (\varphi\ R_r\ \psi))))$
  ⟨*proof*⟩

1.4 Propositional LTL

We define the syntax and semantics of propositional linear-time temporal logic PLTL. PLTL formulas are built from atomic formulas, propositional connectives, and the temporal operators "next" and "until". The following data type definition is parameterized by the type of states over which formulas are evaluated.

1.4.1 Syntax

datatype *'a pltl* =
  False-ltlp ($false_p$)
| *Atom-ltlp 'a ⇒ bool* ($atom_p$'(-'))
| *Implies-ltlp 'a pltl 'a pltl* (- $implies_p$ - [*81*,*81*] *80*)
| *Next-ltlp 'a pltl* (X_p - [*88*] *87*)
| *Until-ltlp 'a pltl 'a pltl* (- U_p - [*84*,*84*] *83*)

— Further connectives of PLTL can be defined in terms of the existing syntax.

definition *Not-ltlp* (not_p - [*85*] *85*)
where
  $not_p\ \varphi \equiv \varphi\ implies_p\ false_p$

definition *True-ltlp* ($true_p$)
where
  $true_p \equiv not_p\ false_p$

definition *Or-ltlp* (- or_p - [*81*,*81*] *80*)
where
  $\varphi\ or_p\ \psi \equiv (not_p\ \varphi)\ implies_p\ \psi$

definition *And-ltlp* (- and_p - [*82*,*82*] *81*)
where
  $\varphi\ and_p\ \psi \equiv not_p\ ((not_p\ \varphi)\ or_p\ (not_p\ \psi))$

definition *Eventually-ltlp* (F_p - [*88*] *87*)
where
  $F_p\ \varphi \equiv true_p\ U_p\ \varphi$

definition *Always-ltlp* (G_p - [*88*] *87*)
where
  $G_p\ \varphi \equiv not_p\ (F_p\ (not_p\ \varphi))$

definition *Release-ltlp* (- R_p - [*84*,*84*] *83*)
where
  $\varphi\ R_p\ \psi \equiv not_p\ ((not_p\ \varphi)\ U_p\ (not_p\ \psi))$

definition *WeakUntil-ltlp* (- W_p - [*84*,*84*] *83*)
where
  $\varphi\ W_p\ \psi \equiv \psi\ R_p\ (\varphi\ or_p\ \psi)$

definition *StrongRelease-ltlp* (- M_p - [*84*,*84*] *83*)
where
$\varphi\ M_p\ \psi \equiv \psi\ U_p\ (\varphi\ and_p\ \psi)$

1.4.2 Semantics

fun *semantics-pltl* :: ['*a word*, '*a pltl*] ⇒ *bool* (- $\models_p$ - [*80*,*80*] *80*)
where
$w \models_p false_p = False$
| $w \models_p atom_p(p) = (p\ (w\ 0))$
| $w \models_p \varphi\ implies_p\ \psi = (w \models_p \varphi \longrightarrow w \models_p \psi)$
| $w \models_p X_p\ \varphi = (suffix\ 1\ w \models_p \varphi)$
| $w \models_p \varphi\ U_p\ \psi = (\exists\, i.\ suffix\ i\ w \models_p \psi \wedge (\forall\, j{<}i.\ suffix\ j\ w \models_p \varphi))$

lemma *semantics-pltl-sugar* [*simp*]:
$w \models_p not_p\ \varphi = (\neg w \models_p \varphi)$
$w \models_p true_p = True$
$w \models_p \varphi\ or_p\ \psi = (w \models_p \varphi \vee w \models_p \psi)$
$w \models_p \varphi\ and_p\ \psi = (w \models_p \varphi \wedge w \models_p \psi)$
$w \models_p F_p\ \varphi = (\exists\, i.\ suffix\ i\ w \models_p \varphi)$
$w \models_p G_p\ \varphi = (\forall\, i.\ suffix\ i\ w \models_p \varphi)$
$w \models_p \varphi\ R_p\ \psi = (\forall\, i.\ suffix\ i\ w \models_p \psi \vee (\exists\, j{<}i.\ suffix\ j\ w \models_p \varphi))$
$w \models_p \varphi\ W_p\ \psi = (\forall\, i.\ suffix\ i\ w \models_p \varphi \vee (\exists\, j{\leq}i.\ suffix\ j\ w \models_p \psi))$
$w \models_p \varphi\ M_p\ \psi = (\exists\, i.\ suffix\ i\ w \models_p \varphi \wedge (\forall\, j{\leq}i.\ suffix\ j\ w \models_p \psi))$
⟨*proof*⟩

definition *language-ltlp* $\varphi \equiv \{\xi.\ \xi \models_p \varphi\}$

1.4.3 Conversion

fun *ltlc-to-pltl* :: '*a ltlc* ⇒ '*a set pltl*
where
ltlc-to-pltl $true_c = true_p$
| *ltlc-to-pltl* $false_c = false_p$
| *ltlc-to-pltl* $(prop_c(q)) = atom_p((\in)\ q)$
| *ltlc-to-pltl* $(not_c\ \varphi) = not_p\ (ltlc\text{-}to\text{-}pltl\ \varphi)$
| *ltlc-to-pltl* $(\varphi\ and_c\ \psi) = (ltlc\text{-}to\text{-}pltl\ \varphi)\ and_p\ (ltlc\text{-}to\text{-}pltl\ \psi)$
| *ltlc-to-pltl* $(\varphi\ or_c\ \psi) = (ltlc\text{-}to\text{-}pltl\ \varphi)\ or_p\ (ltlc\text{-}to\text{-}pltl\ \psi)$
| *ltlc-to-pltl* $(\varphi\ implies_c\ \psi) = (ltlc\text{-}to\text{-}pltl\ \varphi)\ implies_p\ (ltlc\text{-}to\text{-}pltl\ \psi)$
| *ltlc-to-pltl* $(X_c\ \varphi) = X_p\ (ltlc\text{-}to\text{-}pltl\ \varphi)$
| *ltlc-to-pltl* $(F_c\ \varphi) = F_p\ (ltlc\text{-}to\text{-}pltl\ \varphi)$
| *ltlc-to-pltl* $(G_c\ \varphi) = G_p\ (ltlc\text{-}to\text{-}pltl\ \varphi)$
| *ltlc-to-pltl* $(\varphi\ U_c\ \psi) = (ltlc\text{-}to\text{-}pltl\ \varphi)\ U_p\ (ltlc\text{-}to\text{-}pltl\ \psi)$
| *ltlc-to-pltl* $(\varphi\ R_c\ \psi) = (ltlc\text{-}to\text{-}pltl\ \varphi)\ R_p\ (ltlc\text{-}to\text{-}pltl\ \psi)$

| *ltlc-to-pltl* (φ W_c ψ) = (*ltlc-to-pltl* φ) W_p (*ltlc-to-pltl* ψ)
| *ltlc-to-pltl* (φ M_c ψ) = (*ltlc-to-pltl* φ) M_p (*ltlc-to-pltl* ψ)

lemma *ltlc-to-pltl-semantics* [*simp*]:
$w \models_p$ (*ltlc-to-pltl* φ) $\longleftrightarrow$ $w \models_c \varphi$
⟨*proof*⟩

1.4.4 Atoms

fun *atoms-pltl* :: $'a$ *pltl* $\Rightarrow$ ($'a$ $\Rightarrow$ *bool*) *set*
where
atoms-pltl $false_p$ = {}
| *atoms-pltl* $atom_p(p)$ = {p}
| *atoms-pltl* (φ $implies_p$ ψ) = *atoms-pltl* φ ∪ *atoms-pltl* ψ
| *atoms-pltl* (X_p φ) = *atoms-pltl* φ
| *atoms-pltl* (φ U_p ψ) = *atoms-pltl* φ ∪ *atoms-pltl* ψ

lemma *atoms-finite* [*iff*]:
finite (*atoms-pltl* φ)
⟨*proof*⟩

lemma *atoms-pltl-sugar* [*simp*]:
atoms-pltl (not_p φ) = *atoms-pltl* φ
atoms-pltl $true_p$ = {}
atoms-pltl (φ or_p ψ) = *atoms-pltl* φ ∪ *atoms-pltl* ψ
atoms-pltl (φ and_p ψ) = *atoms-pltl* φ ∪ *atoms-pltl* ψ
atoms-pltl (F_p φ) = *atoms-pltl* φ
atoms-pltl (G_p φ) = *atoms-pltl* φ
⟨*proof*⟩

end

2 Rewrite Rules for LTL Simplification

theory *Rewriting*
imports
LTL HOL−Library.Extended-Nat
begin

This theory provides rewrite rules for the simplification of LTL formulas. It supports:

- Constants Removal
- *Next-ltln*-Normalisation

- Modal Simplification (based on pure eventual, pure universal, or suspendable formulas)
- Syntactic Implication Checking

It reuses parts of LTL_Rewrite.thy (CAVA, LTL_TO_GBA). Furthermore, some rules were taken from [2] and [1]. All functions are defined for *ltln*.

2.1 Constant Eliminating Constructors

definition *mk-and*
where
mk-and x y ≡ *case x of false*$_n$ ⇒ *false*$_n$ | *true*$_n$ ⇒ *y* | - ⇒ (*case y of false*$_n$ ⇒ *false*$_n$ | *true*$_n$ ⇒ *x* | - ⇒ *x and*$_n$ *y*)

definition *mk-or*
where
mk-or x y ≡ *case x of false*$_n$ ⇒ *y* | *true*$_n$ ⇒ *true*$_n$ | - ⇒ (*case y of true*$_n$ ⇒ *true*$_n$ | *false*$_n$ ⇒ *x* | - ⇒ *x or*$_n$ *y*)

fun *remove-strong-ops*
where
remove-strong-ops (*x U*$_n$ *y*) = *remove-strong-ops y*
| *remove-strong-ops* (*x M*$_n$ *y*) = *x and*$_n$ *y*
| *remove-strong-ops* (*x or*$_n$ *y*) = *remove-strong-ops x or*$_n$ *remove-strong-ops y*
| *remove-strong-ops x* = *x*

fun *remove-weak-ops*
where
remove-weak-ops (*x R*$_n$ *y*) = *remove-weak-ops y*
| *remove-weak-ops* (*x W*$_n$ *y*) = *x or*$_n$ *y*
| *remove-weak-ops* (*x and*$_n$ *y*) = *remove-weak-ops x and*$_n$ *remove-weak-ops y*
| *remove-weak-ops x* = *x*

definition *mk-finally*
where
mk-finally x ≡ *case x of true*$_n$ ⇒ *true*$_n$ | *false*$_n$ ⇒ *false*$_n$ | - ⇒ *F*$_n$ (*remove-strong-ops x*)

definition *mk-globally*
where
mk-globally x ≡ *case x of true*$_n$ ⇒ *true*$_n$ | *false*$_n$ ⇒ *false*$_n$ | - ⇒ *G*$_n$ (*remove-weak-ops x*)

definition *mk-until*
where
mk-until x y ≡ *case x of false*$_n$ ⇒ *y*
| *true*$_n$ ⇒ *mk-finally y*
| - ⇒ (*case y of true*$_n$ ⇒ *true*$_n$ | *false*$_n$ ⇒ *false*$_n$ | - ⇒ *x* U_n *y*)

definition *mk-release*
where
mk-release x y ≡ *case x of true*$_n$ ⇒ *y*
| *false*$_n$ ⇒ *mk-globally y*
| - ⇒ (*case y of true*$_n$ ⇒ *true*$_n$ | *false*$_n$ ⇒ *false*$_n$ | - ⇒ *x* R_n *y*)

definition *mk-weak-until*
where
mk-weak-until x y ≡ *case y of true*$_n$ ⇒ *true*$_n$
| *false*$_n$ ⇒ *mk-globally x*
| - ⇒ (*case x of true*$_n$ ⇒ *true*$_n$ | *false*$_n$ ⇒ *y* | - ⇒ *x* W_n *y*)

definition *mk-strong-release*
where
mk-strong-release x y ≡ *case y of false*$_n$ ⇒ *false*$_n$
| *true*$_n$ ⇒ *mk-finally x*
| - ⇒ (*case x of true*$_n$ ⇒ *y* | *false*$_n$ ⇒ *false*$_n$ | - ⇒ *x* M_n *y*)

definition *mk-next*
where
mk-next x ≡ *case x of true*$_n$ ⇒ *true*$_n$ | *false*$_n$ ⇒ *false*$_n$ | - ⇒ X_n *x*

definition *mk-next-pow* ($X_n{}''$)
where
mk-next-pow n x ≡ *case x of true*$_n$ ⇒ *true*$_n$ | *false*$_n$ ⇒ *false*$_n$ | - ⇒ (*Next-ltln* $\widehat{\ }\widehat{\ }$ *n*) *x*

lemma *mk-and-semantics* [*simp*]:
$w \models_n$ *mk-and x y* ⟷ $w \models_n$ *x and*$_n$ *y*
⟨*proof*⟩

lemma *mk-or-semantics* [*simp*]:
$w \models_n$ *mk-or x y* ⟷ $w \models_n$ *x or*$_n$ *y*
⟨*proof*⟩

lemma *remove-strong-ops-sound* [*simp*]:
$w \models_n F_n$ (*remove-strong-ops* y) $\longleftrightarrow$ $w \models_n F_n\ y$
⟨*proof*⟩

lemma *remove-weak-ops-sound* [*simp*]:
$w \models_n G_n$ (*remove-weak-ops* y) $\longleftrightarrow$ $w \models_n G_n\ y$
⟨*proof*⟩

lemma *mk-finally-semantics* [*simp*]:
$w \models_n$ *mk-finally* $x \longleftrightarrow w \models_n F_n\ x$
⟨*proof*⟩

lemma *mk-globally-semantics* [*simp*]:
$w \models_n$ *mk-globally* $x \longleftrightarrow w \models_n G_n\ x$
⟨*proof*⟩

lemma *mk-until-semantics* [*simp*]:
$w \models_n$ *mk-until* $x\ y \longleftrightarrow w \models_n x\ U_n\ y$
⟨*proof*⟩

lemma *mk-release-semantics* [*simp*]:
$w \models_n$ *mk-release* $x\ y \longleftrightarrow w \models_n x\ R_n\ y$
⟨*proof*⟩

lemma *mk-weak-until-semantics* [*simp*]:
$w \models_n$ *mk-weak-until* $x\ y \longleftrightarrow w \models_n x\ W_n\ y$
⟨*proof*⟩

lemma *mk-strong-release-semantics* [*simp*]:
$w \models_n$ *mk-strong-release* $x\ y \longleftrightarrow w \models_n x\ M_n\ y$
⟨*proof*⟩

lemma *mk-next-semantics* [*simp*]:
$w \models_n$ *mk-next* $x \longleftrightarrow w \models_n X_n\ x$
⟨*proof*⟩

lemma *mk-next-pow-semantics* [*simp*]:
$w \models_n$ *mk-next-pow* $i\ x \longleftrightarrow$ *suffix* $i\ w \models_n x$
⟨*proof*⟩

lemma *mk-next-pow-simp* [*simp*, *code-unfold*]:
mk-next-pow $0\ x = x$
mk-next-pow $1\ x =$ *mk-next* x
⟨*proof*⟩

2.2 Constant Propagation

fun *is-constant* :: *'a ltln* ⇒ *bool*
where
is-constant $true_n$ = *True*
| *is-constant* $false_n$ = *True*
| *is-constant* - = *False*

lemma *is-constant-constructorsI*:
is-constant x ⟹ *is-constant y* ⟹ *is-constant* (*mk-and x y*)
¬*is-constant x* ⟹ ¬*is-constant y* ⟹ ¬*is-constant* (*mk-and x y*)
is-constant x ⟹ *is-constant y* ⟹ *is-constant* (*mk-or x y*)
¬*is-constant x* ⟹ ¬*is-constant y* ⟹ ¬*is-constant* (*mk-or x y*)
is-constant x ⟹ *is-constant* (*mk-finally x*)
¬*is-constant x* ⟹ ¬*is-constant* (*mk-finally x*)
is-constant x ⟹ *is-constant* (*mk-globally x*)
¬*is-constant x* ⟹ ¬*is-constant* (*mk-globally x*)
is-constant x ⟹ *is-constant* (*mk-until y x*)
¬*is-constant x* ⟹ ¬*is-constant* (*mk-until y x*)
is-constant x ⟹ *is-constant* (*mk-release y x*)
¬*is-constant x* ⟹ ¬*is-constant* (*mk-release y x*)
is-constant x ⟹ *is-constant y* ⟹ *is-constant* (*mk-weak-until x y*)
¬*is-constant x* ⟹ ¬*is-constant y* ⟹ ¬*is-constant* (*mk-weak-until x y*)
is-constant x ⟹ *is-constant y* ⟹ *is-constant* (*mk-strong-release x y*)
¬*is-constant x* ⟹ ¬*is-constant y* ⟹ ¬*is-constant* (*mk-strong-release x y*)
is-constant x ⟹ *is-constant* (*mk-next x*)
¬*is-constant x* ⟹ ¬*is-constant* (*mk-next x*)
is-constant x ⟹ *is-constant* (*mk-next-pow n x*)
⟨*proof*⟩

lemma *is-constant-constructors-simps*:
mk-next-pow n x = $false_n$ ⟷ *x* = $false_n$
mk-next-pow n x = $true_n$ ⟷ *x* = $true_n$
is-constant (*mk-next-pow n x*) ⟷ *is-constant x*
⟨*proof*⟩

lemma *is-constant-constructors-simps2*:
is-constant (*mk-and x y*) ⟷ (*x* = $true_n$ ∧ *y* = $true_n$ ∨ *x* = $false_n$ ∨ *y* = $false_n$)
is-constant (*mk-or x y*) ⟷ (*x* = $false_n$ ∧ *y* = $false_n$ ∨ *x* = $true_n$ ∨ *y* = $true_n$)
is-constant (*mk-finally x*) ⟷ *is-constant x*
is-constant (*mk-globally x*) ⟷ *is-constant x*

is-constant (*mk-until y x*) $\longleftrightarrow$ *is-constant x*
is-constant (*mk-release y x*) $\longleftrightarrow$ *is-constant x*
is-constant (*mk-next x*) $\longleftrightarrow$ *is-constant x*
⟨*proof*⟩

lemma *is-constant-constructors-simps3*:
is-constant (*mk-weak-until x y*) $\longleftrightarrow$ ($x = false_n \wedge y = false_n \vee x = true_n \vee y = true_n$)
is-constant (*mk-strong-release x y*) $\longleftrightarrow$ ($x = true_n \wedge y = true_n \vee x = false_n \vee y = false_n$)
⟨*proof*⟩

lemma *is-constant-semantics*:
is-constant $\varphi \Longrightarrow ((\forall\, w.\ w \models_n \varphi) \vee \neg(\exists\, w.\ w \models_n \varphi))$
⟨*proof*⟩

lemma *until-constant-simp*:
is-constant $\psi \Longrightarrow w \models_n \varphi\ U_n\ \psi \longleftrightarrow w \models_n \psi$
⟨*proof*⟩

lemma *release-constant-simp*:
is-constant $\psi \Longrightarrow w \models_n \varphi\ R_n\ \psi \longleftrightarrow w \models_n \psi$
⟨*proof*⟩

lemma *mk-next-pow-dist*:
mk-next-pow $(i + j)\ \varphi =$ *mk-next-pow i* (*mk-next-pow j* φ)
⟨*proof*⟩

lemma *mk-next-pow-until*:
suffix (*min i j*) $w \models_n$ (*mk-next-pow* $(i - j)\ \varphi)\ U_n$ (*mk-next-pow* $(j - i)\ \psi) \longleftrightarrow w \models_n$ (*mk-next-pow i* $\varphi)\ U_n$ (*mk-next-pow j* ψ)
⟨*proof*⟩

lemma *mk-next-pow-release*:
suffix (*min i j*) $w \models_n$ (*mk-next-pow* $(i - j)\ \varphi)\ R_n$ (*mk-next-pow* $(j - i)\ \psi) \longleftrightarrow w \models_n$ (*mk-next-pow i* $\varphi)\ R_n$ (*mk-next-pow j* ψ)
⟨*proof*⟩

2.3 X-Normalisation

The following rewrite functions pulls the X-operator up in the syntax tree. This preprocessing step enables the removal of X-operators in front of suspendable formulas. Furthermore constants are removed as far as possible.

fun *the-enat-0* :: *enat* ⇒ *nat*
where
  the-enat-0 i = *i*
| *the-enat-0* ∞ = *0*

lemma *the-enat-0-simp* [*simp*]:
  the-enat-0 0 = *0*
  the-enat-0 1 = *1*
  ⟨*proof*⟩

fun *combine* :: (′*a ltln* ⇒ ′*a ltln* ⇒ ′*a ltln*) ⇒ (′*a ltln* ∗ *enat*) ⇒ (′*a ltln* ∗ *enat*) ⇒ (′*a ltln* ∗ *enat*)
where
  combine binop (φ, *i*) (ψ, *j*) = (
    let
     χ = *binop* (*mk-next-pow* (*the-enat-0* (*i* − *j*)) φ) (*mk-next-pow* (*the-enat-0* (*j* − *i*)) ψ)
    in
      (χ, *if is-constant* χ *then* ∞ *else min i j*))

lemma *fst-combine*:
  fst (*combine binop* (φ, *i*) (ψ, *j*)) = *binop* (*mk-next-pow* (*the-enat-0* (*i* − *j*)) φ) (*mk-next-pow* (*the-enat-0* (*j* − *i*)) ψ)
  ⟨*proof*⟩

abbreviation *to-ltln* :: (′*a ltln* ∗ *enat*) ⇒ ′*a ltln*
where
  to-ltln x ≡ *mk-next-pow* (*the-enat-0* (*snd x*)) (*fst x*)

fun *rewrite-X-enat* :: ′*a ltln* ⇒ (′*a ltln* ∗ *enat*)
where
  rewrite-X-enat $true_n$ = ($true_n$, ∞)
| *rewrite-X-enat* $false_n$ = ($false_n$, ∞)
| *rewrite-X-enat* $prop_n(a)$ = ($prop_n(a)$, *0*)
| *rewrite-X-enat* $nprop_n(a)$ = ($nprop_n(a)$, *0*)
| *rewrite-X-enat* (φ and_n ψ) = *combine mk-and* (*rewrite-X-enat* φ) (*rewrite-X-enat* ψ)
| *rewrite-X-enat* (φ or_n ψ) = *combine mk-or* (*rewrite-X-enat* φ) (*rewrite-X-enat* ψ)
| *rewrite-X-enat* (φ U_n ψ) = *combine mk-until* (*rewrite-X-enat* φ) (*rewrite-X-enat* ψ)
| *rewrite-X-enat* (φ R_n ψ) = *combine mk-release* (*rewrite-X-enat* φ) (*rewrite-X-enat* ψ)
| *rewrite-X-enat* (φ W_n ψ) = *combine mk-weak-until* (*rewrite-X-enat* φ)

(*rewrite-X-enat* ψ)
| *rewrite-X-enat* (φ M_n ψ) = *combine mk-strong-release* (*rewrite-X-enat* φ) (*rewrite-X-enat* ψ)
| *rewrite-X-enat* (X_n φ) = ($\lambda(\varphi,\ n).\ (\varphi,\ eSuc\ n)$) (*rewrite-X-enat* φ)

definition
rewrite-X φ = *to-ltln* (*rewrite-X-enat* φ)

lemma *combine-infinity-invariant*:
assumes $i = \infty \longleftrightarrow$ *is-constant* x
assumes $j = \infty \longleftrightarrow$ *is-constant* y
shows *combine mk-and* (x, i) (y, j) = (z, k) $\Longrightarrow$ ($k = \infty \longleftrightarrow$ *is-constant* z)
and *combine mk-or* (x, i) (y, j) = (z, k) $\Longrightarrow$ ($k = \infty \longleftrightarrow$ *is-constant* z)
and *combine mk-until* (x, i) (y, j) = (z, k) $\Longrightarrow$ ($k = \infty \longleftrightarrow$ *is-constant* z)
and *combine mk-release* (x, i) (y, j) = (z, k) $\Longrightarrow$ ($k = \infty \longleftrightarrow$ *is-constant* z)
and *combine mk-weak-until* (x, i) (y, j) = (z, k) $\Longrightarrow$ ($k = \infty \longleftrightarrow$ *is-constant* z)
and *combine mk-strong-release* (x, i) (y, j) = (z, k) $\Longrightarrow$ ($k = \infty \longleftrightarrow$ *is-constant* z)
⟨*proof*⟩

lemma *combine-and-or-semantics*:
assumes $i = \infty \longleftrightarrow$ *is-constant* φ
assumes $j = \infty \longleftrightarrow$ *is-constant* ψ
shows $w \models_n$ *to-ltln* (*combine mk-and* (φ, i) (ψ, j)) $\longleftrightarrow$ $w \models_n$ *to-ltln* (φ, i) and_n *to-ltln* (ψ, j)
and $w \models_n$ *to-ltln* (*combine mk-or* (φ, i) (ψ, j)) $\longleftrightarrow$ $w \models_n$ *to-ltln* (φ, i) or_n *to-ltln* (ψ, j)
⟨*proof*⟩

lemma *combine-until-release-semantics*:
assumes $i = \infty \longleftrightarrow$ *is-constant* φ
assumes $j = \infty \longleftrightarrow$ *is-constant* ψ
shows $w \models_n$ *to-ltln* (*combine mk-until* (φ, i) (ψ, j)) $\longleftrightarrow$ $w \models_n$ *to-ltln* (φ, i) U_n *to-ltln* (ψ, j)
and $w \models_n$ *to-ltln* (*combine mk-release* (φ, i) (ψ, j)) $\longleftrightarrow$ $w \models_n$ *to-ltln* (φ, i) R_n *to-ltln* (ψ, j)
⟨*proof*⟩

lemma *combine-weak-until-strong-release-semantics*:
assumes $i = \infty \longleftrightarrow$ *is-constant* φ
assumes $j = \infty \longleftrightarrow$ *is-constant* ψ
shows $w \models_n$ *to-ltln* (*combine mk-weak-until* (φ, i) (ψ, j)) $\longleftrightarrow$ $w \models_n$ *to-ltln* (φ, i) W_n *to-ltln* (ψ, j)
and $w \models_n$ *to-ltln* (*combine mk-strong-release* (φ, i) (ψ, j)) $\longleftrightarrow$ $w \models_n$ *to-ltln* (φ, i) M_n *to-ltln* (ψ, j)
⟨*proof*⟩

lemma *rewrite-X-enat-infinity-invariant*:
snd (*rewrite-X-enat* φ) $= \infty \longleftrightarrow$ *is-constant* (*fst* (*rewrite-X-enat* φ))
⟨*proof*⟩

lemma *rewrite-X-enat-correct*:
$w \models_n \varphi \longleftrightarrow w \models_n$ *to-ltln* (*rewrite-X-enat* φ)
⟨*proof*⟩

lemma *rewrite-X-sound* [*simp*]:
$w \models_n$ *rewrite-X* $\varphi \longleftrightarrow w \models_n \varphi$
⟨*proof*⟩

2.4 Pure Eventual, Pure Universal, and Suspendable Formulas

fun *pure-eventual* :: $'a$ *ltln* $\Rightarrow$ *bool*
where
pure-eventual $true_n$ = *True*
| *pure-eventual* $false_n$ = *True*
| *pure-eventual* (μ and_n μ') = (*pure-eventual* $\mu \wedge$ *pure-eventual* μ')
| *pure-eventual* (μ or_n μ') = (*pure-eventual* $\mu \wedge$ *pure-eventual* μ')
| *pure-eventual* (μ U_n μ') = ($\mu = true_n \vee$ *pure-eventual* μ')
| *pure-eventual* (μ R_n μ') = (*pure-eventual* $\mu \wedge$ *pure-eventual* μ')
| *pure-eventual* (μ W_n μ') = (*pure-eventual* $\mu \wedge$ *pure-eventual* μ')
| *pure-eventual* (μ M_n μ') = (*pure-eventual* $\mu \wedge$ *pure-eventual* $\mu' \vee \mu' = true_n$)
| *pure-eventual* (X_n μ) = *pure-eventual* μ
| *pure-eventual* - = *False*

fun *pure-universal* :: $'a$ *ltln* $\Rightarrow$ *bool*
where
pure-universal $true_n$ = *True*
| *pure-universal* $false_n$ = *True*

| *pure-universal* (ν *and*$_n$ ν') = (*pure-universal* ν $\wedge$ *pure-universal* ν')
| *pure-universal* (ν *or*$_n$ ν') = (*pure-universal* ν $\wedge$ *pure-universal* ν')
| *pure-universal* (ν U_n ν') = (*pure-universal* ν $\wedge$ *pure-universal* ν')
| *pure-universal* (ν R_n ν') = (ν = *false*$_n$ $\vee$ *pure-universal* ν')
| *pure-universal* (ν W_n ν') = (*pure-universal* ν $\wedge$ *pure-universal* ν' $\vee$ ν' = *false*$_n$)
| *pure-universal* (ν M_n ν') = (*pure-universal* ν $\wedge$ *pure-universal* ν')
| *pure-universal* (X_n ν) = *pure-universal* ν
| *pure-universal* - = *False*

fun *suspendable* :: *'a ltln* $\Rightarrow$ *bool*
where
suspendable *true*$_n$ = *True*
| *suspendable* *false*$_n$ = *True*
| *suspendable* (ξ *and*$_n$ ξ') = (*suspendable* ξ $\wedge$ *suspendable* ξ')
| *suspendable* (ξ *or*$_n$ ξ') = (*suspendable* ξ $\wedge$ *suspendable* ξ')
| *suspendable* (φ U_n ξ) = (φ = *true*$_n$ $\wedge$ *pure-universal* ξ $\vee$ *suspendable* ξ)
| *suspendable* (φ R_n ξ) = (φ = *false*$_n$ $\wedge$ *pure-eventual* ξ $\vee$ *suspendable* ξ)
| *suspendable* (φ W_n ξ) = (*suspendable* φ $\wedge$ *suspendable* ξ $\vee$ *pure-eventual* φ $\wedge$ ξ = *false*$_n$)
| *suspendable* (φ M_n ξ) = (*suspendable* φ $\wedge$ *suspendable* ξ $\vee$ *pure-universal* φ $\wedge$ ξ = *true*$_n$)
| *suspendable* (X_n ξ) = *suspendable* ξ
| *suspendable* - = *False*

lemma *pure-eventual-left-append*:
pure-eventual μ $\Longrightarrow$ $w \models_n \mu$ $\Longrightarrow$ $(u \frown w) \models_n \mu$
⟨*proof*⟩

lemma *pure-universal-suffix-closed*:
pure-universal ν $\Longrightarrow$ $(u \frown w) \models_n \nu$ $\Longrightarrow$ $w \models_n \nu$
⟨*proof*⟩

lemma *suspendable-prefix-invariant*:
suspendable ξ $\Longrightarrow$ $(u \frown w) \models_n \xi \longleftrightarrow w \models_n \xi$
⟨*proof*⟩

theorem *pure-eventual-until-simp*:
assumes *pure-eventual* μ
shows $w \models_n \varphi\ U_n\ \mu \longleftrightarrow w \models_n \mu$
⟨*proof*⟩

theorem *pure-universal-release-simp*:
assumes *pure-universal* ν

shows $w \models_n \varphi\ R_n\ \nu \longleftrightarrow w \models_n \nu$
⟨*proof*⟩

theorem *pure-universal-weak-until-simp*:
assumes *pure-universal* φ **and** *pure-universal* ψ
shows $w \models_n \varphi\ W_n\ \psi \longleftrightarrow w \models_n \varphi\ or_n\ \psi$
⟨*proof*⟩

theorem *pure-eventual-strong-release-simp*:
assumes *pure-eventual* φ **and** *pure-eventual* ψ
shows $w \models_n \varphi\ M_n\ \psi \longleftrightarrow w \models_n \varphi\ and_n\ \psi$
⟨*proof*⟩

theorem *suspendable-formula-simp*:
assumes *suspendable* ξ
shows $w \models_n X_n\ \xi \longleftrightarrow w \models_n \xi$ (**is** *?t1*)
and $w \models_n \varphi\ U_n\ \xi \longleftrightarrow w \models_n \xi$ (**is** *?t2*)
and $w \models_n \varphi\ R_n\ \xi \longleftrightarrow w \models_n \xi$ (**is** *?t3*)
⟨*proof*⟩

theorem *suspendable-formula-simp2*:
assumes *suspendable* φ **and** *suspendable* ψ
shows $w \models_n \varphi\ W_n\ \psi \longleftrightarrow w \models_n \varphi\ or_n\ \psi$ (**is** *?t1*)
and $w \models_n \varphi\ M_n\ \psi \longleftrightarrow w \models_n \varphi\ and_n\ \psi$ (**is** *?t2*)
⟨*proof*⟩

fun *rewrite-modal* :: *'a ltln* ⇒ *'a ltln*
where
rewrite-modal $true_n$ = $true_n$
| *rewrite-modal* $false_n$ = $false_n$
| *rewrite-modal* (φ and_n ψ) = (*rewrite-modal* φ and_n *rewrite-modal* ψ)
| *rewrite-modal* (φ or_n ψ) = (*rewrite-modal* φ or_n *rewrite-modal* ψ)
| *rewrite-modal* (φ U_n ψ) = (*if pure-eventual* ψ ∨ *suspendable* ψ *then rewrite-modal* ψ *else* (*rewrite-modal* φ U_n *rewrite-modal* ψ))
| *rewrite-modal* (φ R_n ψ) = (*if pure-universal* ψ ∨ *suspendable* ψ *then rewrite-modal* ψ *else* (*rewrite-modal* φ R_n *rewrite-modal* ψ))
| *rewrite-modal* (φ W_n ψ) = (*if pure-universal* φ ∧ *pure-universal* ψ ∨ *suspendable* φ ∧ *suspendable* ψ *then* (*rewrite-modal* φ or_n *rewrite-modal* ψ) *else* (*rewrite-modal* φ W_n *rewrite-modal* ψ))
| *rewrite-modal* (φ M_n ψ) = (*if pure-eventual* φ ∧ *pure-eventual* ψ ∨ *suspendable* φ ∧ *suspendable* ψ *then* (*rewrite-modal* φ and_n *rewrite-modal* ψ) *else* (*rewrite-modal* φ M_n *rewrite-modal* ψ))
| *rewrite-modal* (X_n φ) = (*if suspendable* φ *then rewrite-modal* φ *else* X_n

(*rewrite-modal* φ))
| *rewrite-modal* $\varphi = \varphi$

lemma *rewrite-modal-sound* [*simp*]:
$w \models_n$ *rewrite-modal* $\varphi \longleftrightarrow w \models_n \varphi$
⟨*proof*⟩

lemma *rewrite-modal-size*:
size (*rewrite-modal* φ) $\leq$ *size* φ
⟨*proof*⟩

2.5 Syntactical Implication Based Simplification

inductive *syntactical-implies* :: $'a$ *ltln* $\Rightarrow$ $'a$ *ltln* $\Rightarrow$ *bool* (- $\vdash_s$ - [*80*, *80*])
where
- $\vdash_s$ $true_n$
| $false_n \vdash_s$ -
| $\varphi = \varphi \Longrightarrow \varphi \vdash_s \varphi$

| $\varphi \vdash_s \chi \Longrightarrow (\varphi\ and_n\ \psi) \vdash_s \chi$
| $\psi \vdash_s \chi \Longrightarrow (\varphi\ and_n\ \psi) \vdash_s \chi$
| $\varphi \vdash_s \psi \Longrightarrow \varphi \vdash_s \chi \Longrightarrow \varphi \vdash_s (\psi\ and_n\ \chi)$

| $\varphi \vdash_s \psi \Longrightarrow \varphi \vdash_s (\psi\ or_n\ \chi)$
| $\varphi \vdash_s \chi \Longrightarrow \varphi \vdash_s (\psi\ or_n\ \chi)$
| $\varphi \vdash_s \chi \Longrightarrow \psi \vdash_s \chi \Longrightarrow (\varphi\ or_n\ \psi) \vdash_s \chi$

| $\varphi \vdash_s \chi \Longrightarrow \varphi \vdash_s (\psi\ U_n\ \chi)$
| $\varphi \vdash_s \chi \Longrightarrow \psi \vdash_s \chi \Longrightarrow (\varphi\ U_n\ \psi) \vdash_s \chi$
| $\varphi \vdash_s \chi \Longrightarrow \psi \vdash_s \nu \Longrightarrow (\varphi\ U_n\ \psi) \vdash_s (\chi\ U_n\ \nu)$

| $\chi \vdash_s \varphi \Longrightarrow (\psi\ R_n\ \chi) \vdash_s \varphi$
| $\chi \vdash_s \varphi \Longrightarrow \chi \vdash_s \psi \Longrightarrow \chi \vdash_s (\varphi\ R_n\ \psi)$
| $\varphi \vdash_s \chi \Longrightarrow \psi \vdash_s \nu \Longrightarrow (\varphi\ R_n\ \psi) \vdash_s (\chi\ R_n\ \nu)$

| $(false_n\ R_n\ \varphi) \vdash_s \psi \Longrightarrow (false_n\ R_n\ \varphi) \vdash_s X_n\ \psi$
| $\varphi \vdash_s (true_n\ U_n\ \psi) \Longrightarrow (X_n\ \varphi) \vdash_s (true_n\ U_n\ \psi)$
| $\varphi \vdash_s \psi \Longrightarrow (X_n\ \varphi) \vdash_s (X_n\ \psi)$

lemma *syntactical-implies-correct*:
$\varphi \vdash_s \psi \Longrightarrow w \models_n \varphi \Longrightarrow w \models_n \psi$
⟨*proof*⟩

fun *rewrite-syn-imp*

where
rewrite-syn-imp (φ and_n ψ) = (
if $\varphi \vdash_s \psi$ *then*
rewrite-syn-imp φ
else if $\psi \vdash_s \varphi$ *then*
rewrite-syn-imp ψ
else if $\varphi \vdash_s (not_n\ \psi) \vee \psi \vdash_s (not_n\ \varphi)$ *then*
$false_n$
else
mk-and (*rewrite-syn-imp* φ) (*rewrite-syn-imp* ψ))
| *rewrite-syn-imp* (φ or_n ψ) = (
if $\varphi \vdash_s \psi$ *then*
rewrite-syn-imp ψ
else if $\psi \vdash_s \varphi$ *then*
rewrite-syn-imp φ
else if $(not_n\ \varphi) \vdash_s \psi \vee (not_n\ \psi) \vdash_s \varphi$ *then*
$true_n$
else
mk-or (*rewrite-syn-imp* φ) (*rewrite-syn-imp* ψ))
| *rewrite-syn-imp* (φ U_n ψ) = (
if $\varphi \vdash_s \psi$ *then*
rewrite-syn-imp ψ
else if $(not_n\ \varphi) \vdash_s \psi$ *then*
mk-finally (*rewrite-syn-imp* ψ)
else
mk-until (*rewrite-syn-imp* φ) (*rewrite-syn-imp* ψ))
| *rewrite-syn-imp* (φ R_n ψ) = (
if $\psi \vdash_s \varphi$ *then*
rewrite-syn-imp ψ
else if $\psi \vdash_s (not_n\ \varphi)$ *then*
mk-globally (*rewrite-syn-imp* ψ)
else
mk-release (*rewrite-syn-imp* φ) (*rewrite-syn-imp* ψ))
| *rewrite-syn-imp* (X_n φ) = *mk-next* (*rewrite-syn-imp* φ)
| *rewrite-syn-imp* φ = φ

lemma *rewrite-syn-imp-sound*:
$w \models_n$ *rewrite-syn-imp* $\varphi \longleftrightarrow w \models_n \varphi$
⟨*proof*⟩

2.6 Iterated Rewriting

fun *iterate*
where

iterate f x 0 = x
| *iterate f x (Suc n) = (let x′ = f x in if x = x′ then x else iterate f x′ n)*

definition
rewrite-iter-fast φ ≡ iterate (rewrite-modal o rewrite-X) φ (size φ)

definition
rewrite-iter-slow φ ≡ iterate (rewrite-syn-imp o rewrite-modal o rewrite-X) φ (size φ)

The rewriting functions defined in the previous subsections can be used as-is. However, in the most cases one wants to iterate these rules until the formula cannot be simplified further. *rewrite-iter-fast* pulls X operators up in the syntax tree and the uses the "modal" simplification rules. *rewrite-iter-slow* additionally tries to simplify the formula using syntactic implication checking.

lemma *iterate-sound*:
assumes $\bigwedge\varphi.\ w \models_n f\ \varphi \longleftrightarrow w \models_n \varphi$
shows $w \models_n$ *iterate f φ n* $\longleftrightarrow w \models_n \varphi$
⟨*proof*⟩

theorem *rewrite-iter-fast-sound* [*simp*]:
$w \models_n$ *rewrite-iter-fast φ* $\longleftrightarrow w \models_n \varphi$
⟨*proof*⟩

theorem *rewrite-iter-slow-sound* [*simp*]:
$w \models_n$ *rewrite-iter-slow φ* $\longleftrightarrow w \models_n \varphi$
⟨*proof*⟩

2.7 Preservation of atoms

lemma *iterate-atoms*:
assumes
$\bigwedge\varphi.$ *atoms-ltln (f φ) ⊆ atoms-ltln φ*
shows
atoms-ltln (iterate f φ n) ⊆ atoms-ltln φ
⟨*proof*⟩

lemma *rewrite-modal-atoms*:
atoms-ltln (rewrite-modal φ) ⊆ atoms-ltln φ
⟨*proof*⟩

lemma *mk-and-atoms*:
atoms-ltln (mk-and φ ψ) ⊆ atoms-ltln φ ∪ atoms-ltln ψ

⟨*proof*⟩

lemma *mk-or-atoms*:
atoms-ltln (*mk-or* φ ψ) $\subseteq$ *atoms-ltln* φ $\cup$ *atoms-ltln* ψ
⟨*proof*⟩

lemma *remove-strong-ops-atoms*:
atoms-ltln (*remove-strong-ops* φ) $\subseteq$ *atoms-ltln* φ
⟨*proof*⟩

lemma *remove-weak-ops-atoms*:
atoms-ltln (*remove-weak-ops* φ) $\subseteq$ *atoms-ltln* φ
⟨*proof*⟩

lemma *mk-finally-atoms*:
atoms-ltln (*mk-finally* φ) $\subseteq$ *atoms-ltln* φ
⟨*proof*⟩

lemma *mk-globally-atoms*:
atoms-ltln (*mk-globally* φ) $\subseteq$ *atoms-ltln* φ
⟨*proof*⟩

lemma *mk-until-atoms*:
atoms-ltln (*mk-until* φ ψ) $\subseteq$ *atoms-ltln* φ $\cup$ *atoms-ltln* ψ
⟨*proof*⟩

lemma *mk-release-atoms*:
atoms-ltln (*mk-release* φ ψ) $\subseteq$ *atoms-ltln* φ $\cup$ *atoms-ltln* ψ
⟨*proof*⟩

lemma *mk-weak-until-atoms*:
atoms-ltln (*mk-weak-until* φ ψ) $\subseteq$ *atoms-ltln* φ $\cup$ *atoms-ltln* ψ
⟨*proof*⟩

lemma *mk-strong-release-atoms*:
atoms-ltln (*mk-strong-release* φ ψ) $\subseteq$ *atoms-ltln* φ $\cup$ *atoms-ltln* ψ
⟨*proof*⟩

lemma *mk-next-atoms*:
atoms-ltln (*mk-next* φ) = *atoms-ltln* φ
⟨*proof*⟩

lemma *mk-next-pow-atoms*:
atoms-ltln (*mk-next-pow* n φ) = *atoms-ltln* φ

⟨*proof*⟩

lemma *combine-atoms*:
assumes
⋀φ ψ. *atoms-ltln* (*f* φ ψ) ⊆ *atoms-ltln* φ ∪ *atoms-ltln* ψ
shows
atoms-ltln (*fst* (*combine f x y*)) ⊆ *atoms-ltln* (*fst x*) ∪ *atoms-ltln* (*fst y*)
⟨*proof*⟩

lemmas *combine-mk-atoms* =
combine-atoms[*OF mk-and-atoms*]
combine-atoms[*OF mk-or-atoms*]
combine-atoms[*OF mk-until-atoms*]
combine-atoms[*OF mk-release-atoms*]
combine-atoms[*OF mk-weak-until-atoms*]
combine-atoms[*OF mk-strong-release-atoms*]

lemma *rewrite-X-enat-atoms*:
atoms-ltln (*fst* (*rewrite-X-enat* φ)) ⊆ *atoms-ltln* φ
⟨*proof*⟩

lemma *rewrite-X-atoms*:
atoms-ltln (*rewrite-X* φ) ⊆ *atoms-ltln* φ
⟨*proof*⟩

lemma *rewrite-syn-imp-atoms*:
atoms-ltln (*rewrite-syn-imp* φ) ⊆ *atoms-ltln* φ
⟨*proof*⟩

lemma *rewrite-iter-fast-atoms*:
atoms-ltln (*rewrite-iter-fast* φ) ⊆ *atoms-ltln* φ
⟨*proof*⟩

lemma *rewrite-iter-slow-atoms*:
atoms-ltln (*rewrite-iter-slow* φ) ⊆ *atoms-ltln* φ
⟨*proof*⟩

2.8 Simplifier

We now define a convenience wrapper for the rewriting engine

datatype *mode* = *Nop* | *Fast* | *Slow*

fun *simplify* :: *mode* ⇒ *'a ltln* ⇒ *'a ltln*

where
simplify Nop = *id*
| *simplify Fast* = *rewrite-iter-fast*
| *simplify Slow* = *rewrite-iter-slow*

theorem *simplify-correct*:
$w \models_n$ *simplify* $m\ \varphi \longleftrightarrow w \models_n \varphi$
⟨*proof*⟩

lemma *simplify-atoms*:
atoms-ltln (*simplify* $m\ \varphi$) $\subseteq$ *atoms-ltln* φ
⟨*proof*⟩

2.9 Code Generation

code-pred *syntactical-implies* ⟨*proof*⟩

export-code *simplify* **checking**

lemma *rewrite-iter-fast* (F_n (G_n (X_n $prop_n(''a'')$))) = (F_n (G_n $prop_n(''a'')$))
⟨*proof*⟩

lemma *rewrite-iter-fast* (X_n $prop_n(''a'')$ U_n (X_n $nprop_n(''a'')$)) = X_n ($prop_n(''a'')$ U_n $nprop_n(''a'')$)
⟨*proof*⟩

lemma *rewrite-iter-slow* (X_n $prop_n(''a'')$ U_n (X_n $nprop_n(''a'')$)) = X_n (F_n $nprop_n(''a'')$)
⟨*proof*⟩

end

3 Equivalence Relations for LTL formulas

theory *Equivalence-Relations*
imports
LTL
begin

3.1 Language Equivalence

definition *ltl-lang-equiv* :: *'a ltln* ⇒ *'a ltln* ⇒ *bool* (**infix** $\sim_L$ *75*)
where
$\varphi \sim_L \psi \equiv \forall\, w.\ w \models_n \varphi \longleftrightarrow w \models_n \psi$

lemma *ltl-lang-equiv-equivp*:
$\mathit{equivp}\ (\sim_L)$
⟨*proof*⟩

lemma *ltl-lang-equiv-and-true*[*simp*]:
$\varphi_1\ \mathit{and}_n\ \varphi_2 \sim_L \mathit{true}_n \longleftrightarrow \varphi_1 \sim_L \mathit{true}_n \wedge \varphi_2 \sim_L \mathit{true}_n$
⟨*proof*⟩

lemma *ltl-lang-equiv-and-false*[*intro*, *simp*]:
$\varphi_1 \sim_L \mathit{false}_n \Longrightarrow \varphi_1\ \mathit{and}_n\ \varphi_2 \sim_L \mathit{false}_n$
$\varphi_2 \sim_L \mathit{false}_n \Longrightarrow \varphi_1\ \mathit{and}_n\ \varphi_2 \sim_L \mathit{false}_n$
⟨*proof*⟩

lemma *ltl-lang-equiv-or-false*[*simp*]:
$\varphi_1\ \mathit{or}_n\ \varphi_2 \sim_L \mathit{false}_n \longleftrightarrow \varphi_1 \sim_L \mathit{false}_n \wedge \varphi_2 \sim_L \mathit{false}_n$
⟨*proof*⟩

lemma *ltl-lang-equiv-or-const*[*intro*, *simp*]:
$\varphi_1 \sim_L \mathit{true}_n \Longrightarrow \varphi_1\ \mathit{or}_n\ \varphi_2 \sim_L \mathit{true}_n$
$\varphi_2 \sim_L \mathit{true}_n \Longrightarrow \varphi_1\ \mathit{or}_n\ \varphi_2 \sim_L \mathit{true}_n$
⟨*proof*⟩

3.2 Propositional Equivalence

fun *ltl-prop-entailment* :: *'a ltln set* ⇒ *'a ltln* ⇒ *bool* (**infix** $\models_P$ *80*)
where
$\mathcal{A} \models_P \mathit{true}_n = \mathit{True}$
| $\mathcal{A} \models_P \mathit{false}_n = \mathit{False}$
| $\mathcal{A} \models_P \varphi\ \mathit{and}_n\ \psi = (\mathcal{A} \models_P \varphi \wedge \mathcal{A} \models_P \psi)$
| $\mathcal{A} \models_P \varphi\ \mathit{or}_n\ \psi = (\mathcal{A} \models_P \varphi \vee \mathcal{A} \models_P \psi)$
| $\mathcal{A} \models_P \varphi = (\varphi \in \mathcal{A})$

lemma *ltl-prop-entailment-monotonI*[*intro*]:
$S \models_P \varphi \Longrightarrow S \subseteq S' \Longrightarrow S' \models_P \varphi$
⟨*proof*⟩

lemma *ltl-models-equiv-prop-entailment*:
$w \models_n \varphi \longleftrightarrow \{\psi.\ w \models_n \psi\} \models_P \varphi$
⟨*proof*⟩

definition *ltl-prop-equiv* :: *'a ltln* ⇒ *'a ltln* ⇒ *bool* (**infix** $\sim_P$ *75*)
where
$\varphi \sim_P \psi \equiv \forall \mathcal{A}.\ \mathcal{A} \models_P \varphi \longleftrightarrow \mathcal{A} \models_P \psi$

definition *ltl-prop-implies* :: *'a ltln* ⇒ *'a ltln* ⇒ *bool* (**infix** $\longrightarrow_P$ *75*)
where
$\varphi \longrightarrow_P \psi \equiv \forall \mathcal{A}.\ \mathcal{A} \models_P \varphi \longrightarrow \mathcal{A} \models_P \psi$

lemma *ltl-prop-implies-equiv*:
$\varphi \sim_P \psi \longleftrightarrow (\varphi \longrightarrow_P \psi \wedge \psi \longrightarrow_P \varphi)$
⟨*proof*⟩

lemma *ltl-prop-equiv-equivp*:
equivp $(\sim_P)$
⟨*proof*⟩

lemma *ltl-prop-equiv-trans*[*trans*]:
$\varphi \sim_P \psi \Longrightarrow \psi \sim_P \chi \Longrightarrow \varphi \sim_P \chi$
⟨*proof*⟩

lemma *ltl-prop-equiv-true*:
$\varphi \sim_P true_n \longleftrightarrow \{\} \models_P \varphi$
⟨*proof*⟩

lemma *ltl-prop-equiv-false*:
$\varphi \sim_P false_n \longleftrightarrow \neg\ UNIV \models_P \varphi$
⟨*proof*⟩

lemma *ltl-prop-equiv-true-implies-true*:
$x \sim_P true_n \Longrightarrow x \longrightarrow_P y \Longrightarrow y \sim_P true_n$
⟨*proof*⟩

lemma *ltl-prop-equiv-false-implied-by-false*:
$y \sim_P false_n \Longrightarrow x \longrightarrow_P y \Longrightarrow x \sim_P false_n$
⟨*proof*⟩

lemma *ltl-prop-implication-implies-ltl-implication*:
$w \models_n \varphi \Longrightarrow \varphi \longrightarrow_P \psi \Longrightarrow w \models_n \psi$
⟨*proof*⟩

lemma *ltl-prop-equiv-implies-ltl-lang-equiv*:
$\varphi \sim_P \psi \Longrightarrow \varphi \sim_L \psi$
⟨*proof*⟩

lemma *ltl-prop-equiv-lt-ltl-lang-equiv*[*simp*]:
$(\sim_P) \leq (\sim_L)$
⟨*proof*⟩

3.3 Constants Equivalence

datatype *tvl* = *Yes* | *No* | *Maybe*

definition *eval-and* :: *tvl* ⇒ *tvl* ⇒ *tvl*
where
eval-and φ ψ =
(*case* (φ, ψ) *of*
(*Yes*, *Yes*) ⇒ *Yes*
| (*No*, -) ⇒ *No*
| (-, *No*) ⇒ *No*
| - ⇒ *Maybe*)

definition *eval-or* :: *tvl* ⇒ *tvl* ⇒ *tvl*
where
eval-or φ ψ =
(*case* (φ, ψ) *of*
(*No*, *No*) ⇒ *No*
| (*Yes*, -) ⇒ *Yes*
| (-, *Yes*) ⇒ *Yes*
| - ⇒ *Maybe*)

fun *eval* :: *'a ltln* ⇒ *tvl*
where
eval $true_n$ = *Yes*
| *eval* $false_n$ = *No*
| *eval* (φ and_n ψ) = *eval-and* (*eval* φ) (*eval* ψ)
| *eval* (φ or_n ψ) = *eval-or* (*eval* φ) (*eval* ψ)
| *eval* φ = *Maybe*

lemma *eval-and-const*[*simp*]:
eval-and φ ψ = *No* ⟷ φ = *No* ∨ ψ = *No*
eval-and φ ψ = *Yes* ⟷ φ = *Yes* ∧ ψ = *Yes*
⟨*proof*⟩

lemma *eval-or-const*[*simp*]:
eval-or φ ψ = *Yes* ⟷ φ = *Yes* ∨ ψ = *Yes*
eval-or φ ψ = *No* ⟷ φ = *No* ∧ ψ = *No*
⟨*proof*⟩

lemma *eval-prop-entailment*:
eval φ = *Yes* ⟷ {} $\models_P$ φ
eval φ = *No* ⟷ ¬ *UNIV* $\models_P$ φ
⟨*proof*⟩

definition *ltl-const-equiv* :: *'a ltln* ⇒ *'a ltln* ⇒ *bool* (**infix** $\sim_C$ *75*)
where
$\varphi \sim_C \psi \equiv \varphi = \psi \vee (eval\ \varphi = eval\ \psi \wedge eval\ \psi \neq Maybe)$

lemma *ltl-const-equiv-equivp*:
equivp $(\sim_C)$
⟨*proof*⟩

lemma *ltl-const-equiv-const*:
$\varphi \sim_C true_n \longleftrightarrow eval\ \varphi = Yes$
$\varphi \sim_C false_n \longleftrightarrow eval\ \varphi = No$
⟨*proof*⟩

lemma *ltl-const-equiv-and-const*[*simp*]:
$\varphi_1\ and_n\ \varphi_2 \sim_C true_n \longleftrightarrow \varphi_1 \sim_C true_n \wedge \varphi_2 \sim_C true_n$
$\varphi_1\ and_n\ \varphi_2 \sim_C false_n \longleftrightarrow \varphi_1 \sim_C false_n \vee \varphi_2 \sim_C false_n$
⟨*proof*⟩

lemma *ltl-const-equiv-or-const*[*simp*]:
$\varphi_1\ or_n\ \varphi_2 \sim_C true_n \longleftrightarrow \varphi_1 \sim_C true_n \vee \varphi_2 \sim_C true_n$
$\varphi_1\ or_n\ \varphi_2 \sim_C false_n \longleftrightarrow \varphi_1 \sim_C false_n \wedge \varphi_2 \sim_C false_n$
⟨*proof*⟩

lemma *ltl-const-equiv-other*[*simp*]:
$\varphi \sim_C prop_n(a) \longleftrightarrow \varphi = prop_n(a)$
$\varphi \sim_C nprop_n(a) \longleftrightarrow \varphi = nprop_n(a)$
$\varphi \sim_C X_n\ \psi \longleftrightarrow \varphi = X_n\ \psi$
$\varphi \sim_C \psi_1\ U_n\ \psi_2 \longleftrightarrow \varphi = \psi_1\ U_n\ \psi_2$
$\varphi \sim_C \psi_1\ R_n\ \psi_2 \longleftrightarrow \varphi = \psi_1\ R_n\ \psi_2$
$\varphi \sim_C \psi_1\ W_n\ \psi_2 \longleftrightarrow \varphi = \psi_1\ W_n\ \psi_2$
$\varphi \sim_C \psi_1\ M_n\ \psi_2 \longleftrightarrow \varphi = \psi_1\ M_n\ \psi_2$
⟨*proof*⟩

lemma *ltl-const-equiv-no-const-singleton*:
$eval\ \psi = Maybe \Longrightarrow \varphi \sim_C \psi \Longrightarrow \varphi = \psi$
⟨*proof*⟩

lemma *ltl-const-equiv-implies-prop-equiv*:
$\varphi \sim_C true_n \longleftrightarrow \varphi \sim_P true_n$
$\varphi \sim_C false_n \longleftrightarrow \varphi \sim_P false_n$
⟨*proof*⟩

lemma *ltl-const-equiv-no-const-prop-equiv*:

eval ψ = *Maybe* $\Longrightarrow$ φ $\sim_C$ ψ $\Longrightarrow$ φ $\sim_P$ ψ
⟨*proof*⟩

lemma *ltl-const-equiv-implies-ltl-prop-equiv*:
φ $\sim_C$ ψ $\Longrightarrow$ φ $\sim_P$ ψ
⟨*proof*⟩

lemma *ltl-const-equiv-lt-ltl-prop-equiv*[*simp*]:
($\sim_C$) $\leq$ ($\sim_P$)
⟨*proof*⟩

3.4 Quotient types

quotient-type $'a$ $ltln_L$ = $'a$ *ltln* / ($\sim_L$)
⟨*proof*⟩

instantiation $ltln_L$ :: (*type*) *equal*
begin

lift-definition $ltln_L$-*eq-test* :: $'a$ $ltln_L$ $\Rightarrow$ $'a$ $ltln_L$ $\Rightarrow$ *bool* **is** λx y. x $\sim_L$ y
⟨*proof*⟩

definition
eq_L: *equal-class.equal* $\equiv$ $ltln_L$-*eq-test*

instance
⟨*proof*⟩

end

quotient-type $'a$ $ltln_P$ = $'a$ *ltln* / ($\sim_P$)
⟨*proof*⟩

instantiation $ltln_P$ :: (*type*) *equal*
begin

lift-definition $ltln_P$-*eq-test* :: $'a$ $ltln_P$ $\Rightarrow$ $'a$ $ltln_P$ $\Rightarrow$ *bool* **is** λx y. x $\sim_P$ y
⟨*proof*⟩

definition
eq_P: *equal-class.equal* $\equiv$ $ltln_P$-*eq-test*

instance

⟨*proof*⟩

end

quotient-type $'a\ ltln_C = 'a\ ltln\ /\ (\sim_C)$
⟨*proof*⟩

instantiation $ltln_C :: (type)\ equal$
begin

lift-definition $ltln_C\text{-}eq\text{-}test :: 'a\ ltln_C \Rightarrow 'a\ ltln_C \Rightarrow bool$ **is** $\lambda x\ y.\ x \sim_C y$
⟨*proof*⟩

definition
eq_C: $equal\text{-}class.equal \equiv ltln_C\text{-}eq\text{-}test$

instance
⟨*proof*⟩

end

3.5 Cardinality of propositional quotient sets

definition $sat\text{-}models :: 'a\ ltln_P \Rightarrow 'a\ ltln\ set\ set$
where
$sat\text{-}models\ \varphi = \{\mathcal{A}.\ \mathcal{A} \models_P rep\text{-}ltln_P\ \varphi\}$

lemma *Rep-Abs-prop-entailment*[*simp*]:
$\mathcal{A} \models_P rep\text{-}ltln_P\ (abs\text{-}ltln_P\ \varphi) = \mathcal{A} \models_P \varphi$
⟨*proof*⟩

lemma *sat-models-Abs*:
$\mathcal{A} \in sat\text{-}models\ (abs\text{-}ltln_P\ \varphi) = \mathcal{A} \models_P \varphi$
⟨*proof*⟩

lemma *sat-models-inj*:
inj sat-models
⟨*proof*⟩

fun $prop\text{-}atoms :: 'a\ ltln \Rightarrow 'a\ ltln\ set$
where
$prop\text{-}atoms\ true_n = \{\}$

| *prop-atoms* $false_n = \{\}$
| *prop-atoms* $(\varphi\ and_n\ \psi) =$ *prop-atoms* $\varphi \cup$ *prop-atoms* ψ
| *prop-atoms* $(\varphi\ or_n\ \psi) =$ *prop-atoms* $\varphi \cup$ *prop-atoms* ψ
| *prop-atoms* $\varphi = \{\varphi\}$

fun *nested-prop-atoms* :: *'a ltln* ⇒ *'a ltln set*
where
nested-prop-atoms $true_n = \{\}$
| *nested-prop-atoms* $false_n = \{\}$
| *nested-prop-atoms* $(\varphi\ and_n\ \psi) =$ *nested-prop-atoms* $\varphi \cup$ *nested-prop-atoms* ψ
| *nested-prop-atoms* $(\varphi\ or_n\ \psi) =$ *nested-prop-atoms* $\varphi \cup$ *nested-prop-atoms* ψ
| *nested-prop-atoms* $(X_n\ \varphi) = \{X_n\ \varphi\} \cup$ *nested-prop-atoms* φ
| *nested-prop-atoms* $(\varphi\ U_n\ \psi) = \{\varphi\ U_n\ \psi\} \cup$ *nested-prop-atoms* $\varphi \cup$ *nested-prop-atoms* ψ
| *nested-prop-atoms* $(\varphi\ R_n\ \psi) = \{\varphi\ R_n\ \psi\} \cup$ *nested-prop-atoms* $\varphi \cup$ *nested-prop-atoms* ψ
| *nested-prop-atoms* $(\varphi\ W_n\ \psi) = \{\varphi\ W_n\ \psi\} \cup$ *nested-prop-atoms* $\varphi \cup$ *nested-prop-atoms* ψ
| *nested-prop-atoms* $(\varphi\ M_n\ \psi) = \{\varphi\ M_n\ \psi\} \cup$ *nested-prop-atoms* $\varphi \cup$ *nested-prop-atoms* ψ
| *nested-prop-atoms* $\varphi = \{\varphi\}$

lemma *prop-atoms-nested-prop-atoms*:
prop-atoms $\varphi \subseteq$ *nested-prop-atoms* φ
⟨*proof*⟩

lemma *prop-atoms-subfrmlsn*:
prop-atoms $\varphi \subseteq$ *subfrmlsn* φ
⟨*proof*⟩

lemma *nested-prop-atoms-subfrmlsn*:
nested-prop-atoms $\varphi \subseteq$ *subfrmlsn* φ
⟨*proof*⟩

lemma *prop-atoms-notin*[*simp*]:
$true_n \notin$ *prop-atoms* φ
$false_n \notin$ *prop-atoms* φ
$\varphi_1\ and_n\ \varphi_2 \notin$ *prop-atoms* φ
$\varphi_1\ or_n\ \varphi_2 \notin$ *prop-atoms* φ
⟨*proof*⟩

lemma *nested-prop-atoms-notin*[*simp*]:

$true_n \notin$ *nested-prop-atoms* φ
$false_n \notin$ *nested-prop-atoms* φ
φ_1 and_n $\varphi_2 \notin$ *nested-prop-atoms* φ
φ_1 or_n $\varphi_2 \notin$ *nested-prop-atoms* φ
⟨*proof*⟩

lemma *prop-atoms-finite*:
finite (*prop-atoms* φ)
⟨*proof*⟩

lemma *nested-prop-atoms-finite*:
finite (*nested-prop-atoms* φ)
⟨*proof*⟩

lemma *prop-atoms-entailment-iff*:
$\varphi \in$ *prop-atoms* $\psi \Longrightarrow \mathcal{A} \models_P \varphi \longleftrightarrow \varphi \in \mathcal{A}$
⟨*proof*⟩

lemma *prop-atoms-entailment-inter*:
prop-atoms $\varphi \subseteq P \Longrightarrow (\mathcal{A} \cap P) \models_P \varphi = \mathcal{A} \models_P \varphi$
⟨*proof*⟩

lemma *nested-prop-atoms-entailment-inter*:
nested-prop-atoms $\varphi \subseteq P \Longrightarrow (\mathcal{A} \cap P) \models_P \varphi = \mathcal{A} \models_P \varphi$
⟨*proof*⟩

lemma *sat-models-inter-inj-helper*:
assumes
prop-atoms $\varphi \subseteq P$
and
prop-atoms $\psi \subseteq P$
and
sat-models (*abs-ltln*$_P$ φ) $\cap$ *Pow* P = *sat-models* (*abs-ltln*$_P$ ψ) $\cap$ *Pow* P
shows
$\varphi \sim_P \psi$
⟨*proof*⟩

lemma *sat-models-inter-inj*:
inj-on ($\lambda\varphi$. *sat-models* $\varphi \cap$ *Pow* P) {*abs-ltln*$_P$ φ |φ. *prop-atoms* $\varphi \subseteq P$}
⟨*proof*⟩

lemma *sat-models-pow-pow*:
{*sat-models* (*abs-ltln*$_P$ φ) $\cap$ *Pow* P | φ. *prop-atoms* $\varphi \subseteq P$} $\subseteq$ *Pow* (*Pow* P)

⟨*proof*⟩

lemma *sat-models-finite*:
finite $P \Longrightarrow$ *finite* $\{$*sat-models* (*abs-ltln*$_P$ φ) $\cap$ *Pow* P | φ. *prop-atoms* $\varphi \subseteq P\}$
⟨*proof*⟩

lemma *sat-models-card*:
finite $P \Longrightarrow$ *card* $(\{$*sat-models* (*abs-ltln*$_P$ φ) $\cap$ *Pow* P | φ. *prop-atoms* $\varphi \subseteq P\}) \leq 2 \mathbin{\hat{}} 2 \mathbin{\hat{}}$ *card* P
⟨*proof*⟩

lemma *image-filter*:
f ‘ $\{g\ a \mid a.\ P\ a\} = \{f\ (g\ a) \mid a.\ P\ a\}$
⟨*proof*⟩

lemma *prop-equiv-finite*:
finite $P \Longrightarrow$ *finite* $\{$*abs-ltln*$_P$ ψ | ψ. *prop-atoms* $\psi \subseteq P\}$
⟨*proof*⟩

lemma *prop-equiv-card*:
finite $P \Longrightarrow$ *card* $\{$*abs-ltln*$_P$ ψ | ψ. *prop-atoms* $\psi \subseteq P\} \leq 2 \mathbin{\hat{}} 2 \mathbin{\hat{}}$ *card* P
⟨*proof*⟩

lemma *prop-equiv-subset*:
$\{$*abs-ltln*$_P$ ψ |ψ. *nested-prop-atoms* $\psi \subseteq P\} \subseteq \{$*abs-ltln*$_P$ ψ |ψ. *prop-atoms* $\psi \subseteq P\}$
⟨*proof*⟩

lemma *prop-equiv-finite′*:
finite $P \Longrightarrow$ *finite* $\{$*abs-ltln*$_P$ ψ | ψ. *nested-prop-atoms* $\psi \subseteq P\}$
⟨*proof*⟩

lemma *prop-equiv-card′*:
finite $P \Longrightarrow$ *card* $\{$*abs-ltln*$_P$ ψ | ψ. *nested-prop-atoms* $\psi \subseteq P\} \leq 2 \mathbin{\hat{}} 2 \mathbin{\hat{}}$ *card* P
⟨*proof*⟩

3.6 Substitution

fun *subst* :: $'a$ *ltln* $\Rightarrow$ ($'a$ *ltln* $\rightharpoonup$ $'a$ *ltln*) $\Rightarrow$ $'a$ *ltln*
where

subst $true_n$ m = $true_n$
| *subst* $false_n$ m = $false_n$
| *subst* (φ and_n ψ) m = *subst* φ m and_n *subst* ψ m
| *subst* (φ or_n ψ) m = *subst* φ m or_n *subst* ψ m
| *subst* φ m = (*case* m φ *of Some* ψ $\Rightarrow$ ψ | *None* $\Rightarrow$ φ)

Based on Uwe Schoening's Translation Lemma (Logic for CS, p. 54)

lemma *ltl-prop-equiv-subst-S*:
$S \models_P$ *subst* φ m = ((S − *dom* m) ∪ {χ | χ χ'. $\chi \in$ *dom* $m \wedge m\ \chi$ = *Some* $\chi' \wedge S \models_P \chi'$}) $\models_P$ φ
⟨*proof*⟩

lemma *subst-respects-ltl-prop-entailment*:
$\varphi \longrightarrow_P \psi \Longrightarrow$ *subst* φ $m \longrightarrow_P$ *subst* ψ m
$\varphi \sim_P \psi \Longrightarrow$ *subst* φ $m \sim_P$ *subst* ψ m
⟨*proof*⟩

lemma *eval-subst*:
eval φ = *Yes* $\Longrightarrow$ *eval* (*subst* φ m) = *Yes*
eval φ = *No* $\Longrightarrow$ *eval* (*subst* φ m) = *No*
⟨*proof*⟩

lemma *subst-respects-ltl-const-entailment*:
$\varphi \sim_C \psi \Longrightarrow$ *subst* φ $m \sim_C$ *subst* ψ m
⟨*proof*⟩

3.7 Order of Equivalence Relations

locale *ltl-equivalence* =
fixes
eq :: *'a ltln* ⇒ *'a ltln* ⇒ *bool* (**infix** ∼ *75*)
assumes
eq-equivp: *equivp* (∼)
and
ge-const-equiv: ($\sim_C$) ≤ (∼)
and
le-lang-equiv: (∼) ≤ ($\sim_L$)
begin

lemma *eq-implies-ltl-equiv*:
$\varphi \sim \psi \Longrightarrow w \models_n \varphi = w \models_n \psi$
⟨*proof*⟩

lemma *const-implies-eq*:
$\varphi \sim_C \psi \Longrightarrow \varphi \sim \psi$
⟨*proof*⟩

lemma *eq-implies-lang*:
$\varphi \sim \psi \Longrightarrow \varphi \sim_L \psi$
⟨*proof*⟩

lemma *eq-refl*[*simp*]:
$\varphi \sim \varphi$
⟨*proof*⟩

lemma *eq-sym*[*sym*]:
$\varphi \sim \psi \Longrightarrow \psi \sim \varphi$
⟨*proof*⟩

lemma *eq-trans*[*trans*]:
$\varphi \sim \psi \Longrightarrow \psi \sim \chi \Longrightarrow \varphi \sim \chi$
⟨*proof*⟩

end

interpretation *ltl-lang-equivalence*: *ltl-equivalence* $(\sim_L)$
⟨*proof*⟩

interpretation *ltl-prop-equivalence*: *ltl-equivalence* $(\sim_P)$
⟨*proof*⟩

interpretation *ltl-const-equivalence*: *ltl-equivalence* $(\sim_C)$
⟨*proof*⟩

end

4 Disjunctive Normal Form of LTL formulas

theory *Disjunctive-Normal-Form*
imports
LTL Equivalence-Relations HOL−Library.FSet
begin

We use the propositional representation of LTL formulas to define the minimal disjunctive normal form of our formulas. For this purpose we define the minimal product $\otimes_m$ and union $\cup_m$. In the end we show that for a set $\mathcal{A}$ of literals, $\mathcal{A} \models_P \varphi$ if, and only if, there exists a subset of $\mathcal{A}$ in the minimal

DNF of φ.

4.1 Definition of Minimum Sets

definition (**in** *ord*) *min-set* :: *'a set* ⇒ *'a set* **where**
min-set X = {*y* ∈ *X*. ∀ *x* ∈ *X*. *x* ≤ *y* ⟶ *x* = *y*}

lemma *min-set-iff*:
x ∈ *min-set X* ⟷ *x* ∈ *X* ∧ (∀ *y* ∈ *X*. *y* ≤ *x* ⟶ *y* = *x*)
⟨*proof*⟩

lemma *min-set-subset*:
min-set X ⊆ *X*
⟨*proof*⟩

lemma *min-set-idem*[*simp*]:
min-set (*min-set X*) = *min-set X*
⟨*proof*⟩

lemma *min-set-empty*[*simp*]:
min-set {} = {}
⟨*proof*⟩

lemma *min-set-singleton*[*simp*]:
min-set {*x*} = {*x*}
⟨*proof*⟩

lemma *min-set-finite*:
finite X ⟹ *finite* (*min-set X*)
⟨*proof*⟩

lemma *min-set-obtains-helper*:
A ∈ *B* ⟹ ∃ *C*. *C* |⊆| *A* ∧ *C* ∈ *min-set B*
⟨*proof*⟩

lemma *min-set-obtains*:
assumes *A* ∈ *B*
obtains *C* **where** *C* |⊆| *A* **and** *C* ∈ *min-set B*
⟨*proof*⟩

4.2 Minimal operators on sets

definition *product* :: *'a fset set* ⇒ *'a fset set* ⇒ *'a fset set* (**infixr** ⊗ *65*)
where *A* ⊗ *B* = {*a* |∪| *b* | *a b*. *a* ∈ *A* ∧ *b* ∈ *B*}

definition *min-product* :: *'a fset set* ⇒ *'a fset set* ⇒ *'a fset set* (**infixr** $\otimes_m$ *65*)
where $A \otimes_m B = \textit{min-set}\ (A \otimes B)$

definition *min-union* :: *'a fset set* ⇒ *'a fset set* ⇒ *'a fset set* (**infixr** $\cup_m$ *65*)
where $A \cup_m B = \textit{min-set}\ (A \cup B)$

definition *product-set* :: *'a fset set set* ⇒ *'a fset set* ($\bigotimes$)
where $\bigotimes\ X = \textit{Finite-Set.fold product}\ \{\{||\}\}\ X$

definition *min-product-set* :: *'a fset set set* ⇒ *'a fset set* ($\bigotimes_m$)
where $\bigotimes_m\ X = \textit{Finite-Set.fold min-product}\ \{\{||\}\}\ X$

lemma *min-product-idem*[*simp*]:
$A \otimes_m A = \textit{min-set}\ A$
⟨*proof*⟩

lemma *min-union-idem*[*simp*]:
$A \cup_m A = \textit{min-set}\ A$
⟨*proof*⟩

lemma *product-empty*[*simp*]:
$A \otimes \{\} = \{\}$
$\{\} \otimes A = \{\}$
⟨*proof*⟩

lemma *min-product-empty*[*simp*]:
$A \otimes_m \{\} = \{\}$
$\{\} \otimes_m A = \{\}$
⟨*proof*⟩

lemma *min-union-empty*[*simp*]:
$A \cup_m \{\} = \textit{min-set}\ A$
$\{\} \cup_m A = \textit{min-set}\ A$
⟨*proof*⟩

lemma *product-empty-singleton*[*simp*]:
$A \otimes \{\{||\}\} = A$
$\{\{||\}\} \otimes A = A$
⟨*proof*⟩

lemma *min-product-empty-singleton*[*simp*]:
$A \otimes_m \{\{||\}\} = \textit{min-set}\ A$
$\{\{||\}\} \otimes_m A = \textit{min-set}\ A$
⟨*proof*⟩

lemma *product-singleton-singleton*:
$A \otimes \{\{|x|\}\} = \textit{finsert}\ x\ `\ A$
$\{\{|x|\}\} \otimes A = \textit{finsert}\ x\ `\ A$
⟨*proof*⟩

lemma *product-mono*:
$A \subseteq B \Longrightarrow A \otimes C \subseteq B \otimes C$
$B \subseteq C \Longrightarrow A \otimes B \subseteq A \otimes C$
⟨*proof*⟩

lemma *product-finite*:
$\textit{finite}\ A \Longrightarrow \textit{finite}\ B \Longrightarrow \textit{finite}\ (A \otimes B)$
⟨*proof*⟩

lemma *min-product-finite*:
$\textit{finite}\ A \Longrightarrow \textit{finite}\ B \Longrightarrow \textit{finite}\ (A \otimes_m B)$
⟨*proof*⟩

lemma *min-union-finite*:
$\textit{finite}\ A \Longrightarrow \textit{finite}\ B \Longrightarrow \textit{finite}\ (A \cup_m B)$
⟨*proof*⟩

lemma *product-set-infinite*[*simp*]:
$\textit{infinite}\ X \Longrightarrow \bigotimes\ X = \{\{||\}\}$
⟨*proof*⟩

lemma *min-product-set-infinite*[*simp*]:
$\textit{infinite}\ X \Longrightarrow \bigotimes_m\ X = \{\{||\}\}$
⟨*proof*⟩

lemma *product-comm*:
$A \otimes B = B \otimes A$
⟨*proof*⟩

lemma *min-product-comm*:
$A \otimes_m B = B \otimes_m A$
⟨*proof*⟩

lemma *min-union-comm*:
$A \cup_m B = B \cup_m A$
⟨*proof*⟩

lemma *product-iff*:
$x \in A \otimes B \longleftrightarrow (\exists\, a \in A.\ \exists\, b \in B.\ x = a \mid\cup\mid b)$
⟨*proof*⟩

lemma *min-product-iff*:
$x \in A \otimes_m B \longleftrightarrow (\exists\, a \in A.\ \exists\, b \in B.\ x = a \mid\cup\mid b) \wedge (\forall\, a \in A.\ \forall\, b \in B.\ a \mid\cup\mid b \mid\subseteq\mid x \longrightarrow a \mid\cup\mid b = x)$
⟨*proof*⟩

lemma *min-union-iff*:
$x \in A \cup_m B \longleftrightarrow x \in A \cup B \wedge (\forall\, a \in A.\ a \mid\subseteq\mid x \longrightarrow a = x) \wedge (\forall\, b \in B.\ b \mid\subseteq\mid x \longrightarrow b = x)$
⟨*proof*⟩

lemma *min-set-min-product-helper*:
$x \in (\textit{min-set}\ A) \otimes_m B \longleftrightarrow x \in A \otimes_m B$
⟨*proof*⟩

lemma *min-set-min-product*[*simp*]:
$(\textit{min-set}\ A) \otimes_m B = A \otimes_m B$
$A \otimes_m (\textit{min-set}\ B) = A \otimes_m B$
⟨*proof*⟩

lemma *min-set-min-union*[*simp*]:
$(\textit{min-set}\ A) \cup_m B = A \cup_m B$
$A \cup_m (\textit{min-set}\ B) = A \cup_m B$
⟨*proof*⟩

lemma *product-assoc*[*simp*]:
$(A \otimes B) \otimes C = A \otimes (B \otimes C)$
⟨*proof*⟩

lemma *min-product-assoc*[*simp*]:

$(A \otimes_m B) \otimes_m C = A \otimes_m (B \otimes_m C)$
⟨*proof*⟩

lemma *min-union-assoc*[*simp*]:
$(A \cup_m B) \cup_m C = A \cup_m (B \cup_m C)$
⟨*proof*⟩

lemma *min-product-comp*:
$a \in A \Longrightarrow b \in B \Longrightarrow \exists\, c.\ c \mid\subseteq\mid (a \mid\cup\mid b) \wedge c \in A \otimes_m B$
⟨*proof*⟩

lemma *min-union-comp*:
$a \in A \Longrightarrow \exists\, c.\ c \mid\subseteq\mid a \wedge c \in A \cup_m B$
⟨*proof*⟩

interpretation *product-set-thms*: *Finite-Set.comp-fun-commute product*
⟨*proof*⟩

interpretation *min-product-set-thms*: *Finite-Set.comp-fun-idem min-product*
⟨*proof*⟩

interpretation *min-union-set-thms*: *Finite-Set.comp-fun-idem min-union*
⟨*proof*⟩

lemma *product-set-empty*[*simp*]:
$\bigotimes\ \{\} = \{\{||\}\}$
$\bigotimes\ \{\{\}\} = \{\}$
$\bigotimes\ \{\{\{||\}\}\} = \{\{||\}\}$
⟨*proof*⟩

lemma *min-product-set-empty*[*simp*]:
$\bigotimes_m\ \{\} = \{\{||\}\}$
$\bigotimes_m\ \{\{\}\} = \{\}$
$\bigotimes_m\ \{\{\{||\}\}\} = \{\{||\}\}$
⟨*proof*⟩

lemma *product-set-code*[*code*]:
$\bigotimes\ (\mathit{set}\ \mathit{xs}) = \mathit{fold}\ \mathit{product}\ (\mathit{remdups}\ \mathit{xs})\ \{\{||\}\}$
⟨*proof*⟩

lemma *min-product-set-code*[*code*]:
$\bigotimes_m$ (*set xs*) = *fold min-product* (*remdups xs*) {{||}}
⟨*proof*⟩

lemma *product-set-insert*[*simp*]:
finite X ⟹ $\bigotimes$ (*insert x X*) = $x \otimes (\bigotimes\ (X - \{x\}))$
⟨*proof*⟩

lemma *min-product-set-insert*[*simp*]:
finite X ⟹ $\bigotimes_m$ (*insert x X*) = $x \otimes_m (\bigotimes_m\ X)$
⟨*proof*⟩

lemma *min-product-subseteq*:
$x \in A \otimes_m B \Longrightarrow \exists\, a.\ a \mid\subseteq\mid x \wedge a \in A$
⟨*proof*⟩

lemma *min-product-set-subseteq*:
finite X ⟹ $x \in \bigotimes_m X \Longrightarrow A \in X \Longrightarrow \exists\, a \in A.\ a \mid\subseteq\mid x$
⟨*proof*⟩

lemma *min-set-product-set*:
$\bigotimes_m A$ = *min-set* ($\bigotimes\ A$)
⟨*proof*⟩

lemma *min-product-min-set*[*simp*]:
min-set ($A \otimes_m B$) = $A \otimes_m B$
⟨*proof*⟩

lemma *min-union-min-set*[*simp*]:
min-set ($A \cup_m B$) = $A \cup_m B$
⟨*proof*⟩

lemma *min-product-set-min-set*[*simp*]:
finite X ⟹ *min-set* ($\bigotimes_m X$) = $\bigotimes_m X$
⟨*proof*⟩

lemma *min-set-min-product-set*[*simp*]:
finite X ⟹ $\bigotimes_m$ (*min-set* ‘ *X*) = $\bigotimes_m X$
⟨*proof*⟩

lemma *min-product-set-union*[*simp*]:
finite X ⟹ *finite Y* ⟹ $\bigotimes_m (X \cup Y) = (\bigotimes_m X) \otimes_m (\bigotimes_m Y)$
⟨*proof*⟩

lemma *product-set-finite*:
$(\bigwedge x.\ x \in X \Longrightarrow \mathit{finite}\ x) \Longrightarrow \mathit{finite}\ (\bigotimes\ X)$
⟨*proof*⟩

lemma *min-product-set-finite*:
$(\bigwedge x.\ x \in X \Longrightarrow \mathit{finite}\ x) \Longrightarrow \mathit{finite}\ (\bigotimes_m\ X)$
⟨*proof*⟩

4.3 Disjunctive Normal Form

fun *dnf* :: *'a ltln* ⇒ *'a ltln fset set*
where
dnf true$_n$ = {{||}}
| *dnf false*$_n$ = {}
| *dnf* (φ *and*$_n$ ψ) = (*dnf* φ) ⊗ (*dnf* ψ)
| *dnf* (φ *or*$_n$ ψ) = (*dnf* φ) ∪ (*dnf* ψ)
| *dnf* φ = {{|φ|}}

fun *min-dnf* :: *'a ltln* ⇒ *'a ltln fset set*
where
min-dnf true$_n$ = {{||}}
| *min-dnf false*$_n$ = {}
| *min-dnf* (φ *and*$_n$ ψ) = (*min-dnf* φ) ⊗$_m$ (*min-dnf* ψ)
| *min-dnf* (φ *or*$_n$ ψ) = (*min-dnf* φ) ∪$_m$ (*min-dnf* ψ)
| *min-dnf* φ = {{|φ|}}

lemma *dnf-min-set*:
min-dnf φ = *min-set* (*dnf* φ)
⟨*proof*⟩

lemma *dnf-finite*:
finite (*dnf* φ)
⟨*proof*⟩

lemma *min-dnf-finite*:
finite (*min-dnf* φ)
⟨*proof*⟩

lemma *dnf-Abs-fset*[*simp*]:
fset (*Abs-fset* (*dnf* φ)) = *dnf* φ
⟨*proof*⟩

lemma *min-dnf-Abs-fset*[*simp*]:
fset (*Abs-fset* (*min-dnf* φ)) = *min-dnf* φ
⟨*proof*⟩

lemma *dnf-prop-atoms*:
$\Phi \in$ *dnf* $\varphi \Longrightarrow$ *fset* $\Phi \subseteq$ *prop-atoms* φ
⟨*proof*⟩

lemma *min-dnf-prop-atoms*:
$\Phi \in$ *min-dnf* $\varphi \Longrightarrow$ *fset* $\Phi \subseteq$ *prop-atoms* φ
⟨*proof*⟩

lemma *min-dnf-atoms-dnf*:
$\Phi \in$ *min-dnf* $\psi \Longrightarrow \varphi \in$ *fset* $\Phi \Longrightarrow$ *dnf* $\varphi = \{\{|\varphi|\}\}$
⟨*proof*⟩

lemma *min-dnf-min-set*[*simp*]:
min-set (*min-dnf* φ) = *min-dnf* φ
⟨*proof*⟩

lemma *min-dnf-iff-prop-assignment-subset*:
$\mathcal{A} \models_P \varphi \longleftrightarrow (\exists\, B.$ *fset* $B \subseteq \mathcal{A} \wedge B \in$ *min-dnf* $\varphi)$
⟨*proof*⟩

lemma *ltl-prop-implies-min-dnf*:
$\varphi \longrightarrow_P \psi = (\forall\, A \in$ *min-dnf* $\varphi.\ \exists\, B \in$ *min-dnf* $\psi.\ B \mid\subseteq\mid A)$
⟨*proof*⟩

lemma *ltl-prop-equiv-min-dnf*:
$\varphi \sim_P \psi =$ (*min-dnf* $\varphi =$ *min-dnf* ψ)
⟨*proof*⟩

lemma *min-dnf-rep-abs*[*simp*]:
min-dnf (*rep-ltln*$_P$ (*abs-ltln*$_P$ φ)) = *min-dnf* φ
⟨*proof*⟩

4.4 Folding of and_n and or_n over Finite Sets

definition And_n :: *'a ltln set* ⇒ *'a ltln*
where
$And_n\ \Phi \equiv$ *SOME* φ. *fold-graph And-ltln True-ltln* $\Phi\ \varphi$

definition $Or_n :: 'a\ ltln\ set \Rightarrow 'a\ ltln$
where
$Or_n\ \Phi \equiv SOME\ \varphi.\ fold\text{-}graph\ Or\text{-}ltln\ False\text{-}ltln\ \Phi\ \varphi$

lemma *fold-graph-*And_n:
finite $\Phi \Longrightarrow$ *fold-graph And-ltln True-ltln* Φ $(And_n\ \Phi)$
⟨*proof*⟩

lemma *fold-graph-*Or_n:
finite $\Phi \Longrightarrow$ *fold-graph Or-ltln False-ltln* Φ $(Or_n\ \Phi)$
⟨*proof*⟩

lemma Or_n*-empty*[*simp*]:
$Or_n\ \{\} = False\text{-}ltln$
⟨*proof*⟩

lemma And_n*-empty*[*simp*]:
$And_n\ \{\} = True\text{-}ltln$
⟨*proof*⟩

interpretation *dnf-union-thms*: *Finite-Set.comp-fun-commute* $\lambda\varphi.\ (\cup)\ (f\ \varphi)$
⟨*proof*⟩

interpretation *dnf-product-thms*: *Finite-Set.comp-fun-commute* $\lambda\varphi.\ (\otimes)\ (f\ \varphi)$
⟨*proof*⟩
lemma *fold-graph-finite*:
assumes *fold-graph f z A y*
shows *finite A*
⟨*proof*⟩

Taking the DNF of And_n and Or_n is the same as folding over the individual DNFs.

lemma And_n*-dnf*:
finite $\Phi \Longrightarrow$ *dnf* $(And_n\ \Phi) =$ *Finite-Set.fold* $(\lambda\varphi.\ (\otimes)\ (dnf\ \varphi))\ \{\{||\}\}\ \Phi$
⟨*proof*⟩

lemma Or_n*-dnf*:
finite $\Phi \Longrightarrow$ *dnf* $(Or_n\ \Phi) =$ *Finite-Set.fold* $(\lambda\varphi.\ (\cup)\ (dnf\ \varphi))\ \{\}\ \Phi$
⟨*proof*⟩

And_n and Or_n are injective on finite sets.

lemma And_n*-inj*:

inj-on And_n $\{s.$ *finite* $s\}$
⟨*proof*⟩

lemma Or_n*-inj*:
inj-on Or_n $\{s.$ *finite* $s\}$
⟨*proof*⟩

The semantics of And_n and Or_n can be expressed using quantifiers.

lemma And_n*-semantics*:
finite $\Phi \Longrightarrow w \models_n And_n\ \Phi \longleftrightarrow (\forall \varphi \in \Phi.\ w \models_n \varphi)$
⟨*proof*⟩

lemma Or_n*-semantics*:
finite $\Phi \Longrightarrow w \models_n Or_n\ \Phi \longleftrightarrow (\exists \varphi \in \Phi.\ w \models_n \varphi)$
⟨*proof*⟩

lemma And_n*-prop-semantics*:
finite $\Phi \Longrightarrow \mathcal{A} \models_P And_n\ \Phi \longleftrightarrow (\forall \varphi \in \Phi.\ \mathcal{A} \models_P \varphi)$
⟨*proof*⟩

lemma Or_n*-prop-semantics*:
finite $\Phi \Longrightarrow \mathcal{A} \models_P Or_n\ \Phi \longleftrightarrow (\exists \varphi \in \Phi.\ \mathcal{A} \models_P \varphi)$
⟨*proof*⟩

lemma Or_n*-*And_n*-image-semantics*:
assumes *finite* $\mathcal{A}$ **and** $\bigwedge \Phi.\ \Phi \in \mathcal{A} \Longrightarrow$ *finite* Φ
shows $w \models_n Or_n\ (And_n\ `\ \mathcal{A}) \longleftrightarrow (\exists \Phi \in \mathcal{A}.\ \forall \varphi \in \Phi.\ w \models_n \varphi)$
⟨*proof*⟩

lemma Or_n*-*And_n*-image-prop-semantics*:
assumes *finite* $\mathcal{A}$ **and** $\bigwedge \Phi.\ \Phi \in \mathcal{A} \Longrightarrow$ *finite* Φ
shows $\mathcal{I} \models_P Or_n\ (And_n\ `\ \mathcal{A}) \longleftrightarrow (\exists \Phi \in \mathcal{A}.\ \forall \varphi \in \Phi.\ \mathcal{I} \models_P \varphi)$
⟨*proof*⟩

4.5 DNF to LTL conversion

definition *ltln-of-dnf* :: *'a ltln fset set* ⇒ *'a ltln*
where
ltln-of-dnf $\mathcal{A} =$ $Or_n\ (And_n\ `$ *fset* $`\ \mathcal{A})$

lemma *ltln-of-dnf-semantics*:
assumes *finite* $\mathcal{A}$
shows $w \models_n$ *ltln-of-dnf* $\mathcal{A} \longleftrightarrow (\exists \Phi \in \mathcal{A}.\ \forall \varphi.\ \varphi\ |\in|\ \Phi \longrightarrow w \models_n \varphi)$
⟨*proof*⟩

lemma *ltln-of-dnf-prop-semantics*:
assumes *finite* $\mathcal{A}$
shows $\mathcal{I} \models_P$ *ltln-of-dnf* $\mathcal{A} \longleftrightarrow (\exists\, \Phi \in \mathcal{A}.\ \forall\, \varphi.\ \varphi \mid\in\mid \Phi \longrightarrow \mathcal{I} \models_P \varphi)$
⟨*proof*⟩

lemma *ltln-of-dnf-prop-equiv*:
ltln-of-dnf (*min-dnf* φ) $\sim_P \varphi$
⟨*proof*⟩

lemma *min-dnf-ltln-of-dnf*[*simp*]:
min-dnf (*ltln-of-dnf* (*min-dnf* φ)) = *min-dnf* φ
⟨*proof*⟩

4.6 Substitution in DNF formulas

definition *subst-clause* :: *'a ltln fset* ⇒ (*'a ltln* ⇀ *'a ltln*) ⇒ *'a ltln fset set*
where
subst-clause Φ m = $\bigotimes_m$ {*min-dnf* (*subst* φ m) | φ. $\varphi \in$ *fset* Φ}

definition *subst-dnf* :: *'a ltln fset set* ⇒ (*'a ltln* ⇀ *'a ltln*) ⇒ *'a ltln fset set*
where
subst-dnf $\mathcal{A}$ m = $(\bigcup \Phi \in \mathcal{A}.$ *subst-clause* Φ $m)$

lemma *subst-clause-empty*[*simp*]:
subst-clause {||} m = {{||}}
⟨*proof*⟩

lemma *subst-dnf-empty*[*simp*]:
subst-dnf {} m = {}
⟨*proof*⟩

lemma *subst-clause-inner-finite*:
finite {*min-dnf* (*subst* φ m) | φ. $\varphi \in \Phi$} **if** *finite* Φ
⟨*proof*⟩

lemma *subst-clause-finite*:
finite (*subst-clause* Φ m)
⟨*proof*⟩

lemma *subst-dnf-finite*:
finite $\mathcal{A}$ ⟹ *finite* (*subst-dnf* $\mathcal{A}$ m)
⟨*proof*⟩

lemma *subst-dnf-mono*:
$\mathcal{A} \subseteq \mathcal{B} \Longrightarrow$ *subst-dnf* $\mathcal{A}$ $m \subseteq$ *subst-dnf* $\mathcal{B}$ m
⟨*proof*⟩

lemma *subst-clause-min-set*[*simp*]:
min-set (*subst-clause* Φ m) = *subst-clause* Φ m
⟨*proof*⟩

lemma *subst-clause-finsert*[*simp*]:
subst-clause (*finsert* φ Φ) m = (*min-dnf* (*subst* φ m)) $\otimes_m$ (*subst-clause* Φ m)
⟨*proof*⟩

lemma *subst-clause-funion*[*simp*]:
subst-clause (Φ |∪| Ψ) m = (*subst-clause* Φ m) $\otimes_m$ (*subst-clause* Ψ m)
⟨*proof*⟩

For the proof of correctness, we redefine the ($\otimes$) operator on lists.

definition *list-product* :: *'a list set* ⇒ *'a list set* ⇒ *'a list set* (**infixl** $\otimes_l$ *65*)
where
$A \otimes_l B = \{a \mathbin{@} b \mid a\ b.\ a \in A \wedge b \in B\}$

lemma *list-product-fset-of-list*[*simp*]:
fset-of-list ‘ (*A* $\otimes_l$ *B*) = (*fset-of-list* ‘ *A*) ⊗ (*fset-of-list* ‘ *B*)
⟨*proof*⟩

lemma *list-product-finite*:
finite A ⟹ *finite B* ⟹ *finite* (*A* $\otimes_l$ *B*)
⟨*proof*⟩

lemma *list-product-iff*:
$x \in A \otimes_l B \longleftrightarrow (\exists\, a\ b.\ a \in A \wedge b \in B \wedge x = a \mathbin{@} b)$
⟨*proof*⟩

lemma *list-product-assoc*[*simp*]:
$A \otimes_l (B \otimes_l C) = A \otimes_l B \otimes_l C$
⟨*proof*⟩

Furthermore, we introduct DNFs where the clauses are represented as lists.

fun *list-dnf* :: *'a ltln* ⇒ *'a ltln list set*
where
list-dnf $true_n = \{[]\}$
| *list-dnf* $false_n = \{\}$
| *list-dnf* (φ and_n ψ) = (*list-dnf* φ) $\otimes_l$ (*list-dnf* ψ)

| *list-dnf* $(\varphi\ or_n\ \psi) = (\mathit{list\text{-}dnf}\ \varphi) \cup (\mathit{list\text{-}dnf}\ \psi)$
| *list-dnf* $\varphi = \{[\varphi]\}$

definition *list-dnf-to-dnf* :: $'a\ list\ set \Rightarrow 'a\ fset\ set$
where
list-dnf-to-dnf $X = \mathit{fset\text{-}of\text{-}list}\ `\ X$

lemma *list-dnf-to-dnf-list-dnf*[*simp*]:
list-dnf-to-dnf $(\mathit{list\text{-}dnf}\ \varphi) = \mathit{dnf}\ \varphi$
⟨*proof*⟩

lemma *list-dnf-finite*:
finite $(\mathit{list\text{-}dnf}\ \varphi)$
⟨*proof*⟩

We use this to redefine *subst-clause* and *subst-dnf* on list DNFs.

definition *subst-clause'* :: $'a\ ltln\ list \Rightarrow ('a\ ltln \rightharpoonup 'a\ ltln) \Rightarrow 'a\ ltln\ list\ set$
where
$\mathit{subst\text{-}clause'}\ \Phi\ m = \mathit{fold}\ (\lambda\varphi\ \mathit{acc}.\ \mathit{acc} \otimes_l \mathit{list\text{-}dnf}\ (\mathit{subst}\ \varphi\ m))\ \Phi\ \{[]\}$

definition *subst-dnf'* :: $'a\ ltln\ list\ set \Rightarrow ('a\ ltln \rightharpoonup 'a\ ltln) \Rightarrow 'a\ ltln\ list\ set$
where
$\mathit{subst\text{-}dnf'}\ \mathcal{A}\ m = (\bigcup \Phi \in \mathcal{A}.\ \mathit{subst\text{-}clause'}\ \Phi\ m)$

lemma *subst-clause'-finite*:
finite $(\mathit{subst\text{-}clause'}\ \Phi\ m)$
⟨*proof*⟩

lemma *subst-clause'-nil*[*simp*]:
$\mathit{subst\text{-}clause'}\ []\ m = \{[]\}$
⟨*proof*⟩

lemma *subst-clause'-cons*[*simp*]:
$\mathit{subst\text{-}clause'}\ (xs\ @\ [x])\ m = \mathit{subst\text{-}clause'}\ xs\ m \otimes_l \mathit{list\text{-}dnf}\ (\mathit{subst}\ x\ m)$
⟨*proof*⟩

lemma *subst-clause'-append*[*simp*]:
$\mathit{subst\text{-}clause'}\ (A\ @\ B)\ m = \mathit{subst\text{-}clause'}\ A\ m \otimes_l \mathit{subst\text{-}clause'}\ B\ m$
⟨*proof*⟩

lemma *subst-dnf'-iff*:
$x \in \mathit{subst\text{-}dnf'}\ A\ m \longleftrightarrow (\exists \Phi \in A.\ x \in \mathit{subst\text{-}clause'}\ \Phi\ m)$
⟨*proof*⟩

lemma *subst-dnf'-product*:
subst-dnf' (*A* $\otimes_l$ *B*) *m* = (*subst-dnf'* *A* *m*) $\otimes_l$ (*subst-dnf'* *B* *m*) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *subst-dnf'-list-dnf*:
subst-dnf' (*list-dnf* φ) *m* = *list-dnf* (*subst* φ *m*)
⟨*proof*⟩

lemma *min-set-Union*:
finite X $\Longrightarrow$ *min-set* ($\bigcup$ (*min-set* ‘ *X*)) = *min-set* ($\bigcup$ *X*) **for** *X* :: *'a fset set set*
⟨*proof*⟩

lemma *min-set-Union-image*:
finite X $\Longrightarrow$ *min-set* ($\bigcup x \in X$. *min-set* (*f x*)) = *min-set* ($\bigcup x \in X$. *f x*) **for** *f* :: *'b* $\Rightarrow$ *'a fset set*
⟨*proof*⟩

lemma *subst-clause-fset-of-list*:
subst-clause (*fset-of-list* Φ) *m* = *min-set* (*list-dnf-to-dnf* (*subst-clause'* Φ *m*))
⟨*proof*⟩

lemma *min-set-list-dnf-to-dnf-subst-dnf'*:
finite X $\Longrightarrow$ *min-set* (*list-dnf-to-dnf* (*subst-dnf'* *X m*)) = *min-set* (*subst-dnf* (*list-dnf-to-dnf X*) *m*)
⟨*proof*⟩

lemma *subst-dnf-dnf*:
min-set (*subst-dnf* (*dnf* φ) *m*) = *min-dnf* (*subst* φ *m*)
⟨*proof*⟩

This is almost the lemma we need. However, we need to show that the same holds for *min-dnf* φ, too.

lemma *fold-product*:
Finite-Set.fold (λx. ($\otimes$) {{|*x*|}}) {{||}} (*fset x*) = {*x*}
⟨*proof*⟩

lemma *fold-union*:
Finite-Set.fold (λx. ($\cup$) {*x*}) {} (*fset x*) = *fset x*
⟨*proof*⟩

lemma *fold-union-fold-product*:
assumes *finite* X **and** $\bigwedge \Psi\ \psi.\ \Psi \in X \Longrightarrow \psi \in fset\ \Psi \Longrightarrow dnf\ \psi = \{\{|\psi|\}\}$
shows *Finite-Set.fold* $(\lambda x.\ (\cup)\ (Finite\text{-}Set.fold\ (\lambda \varphi.\ (\otimes)\ (dnf\ \varphi))\ \{\{||\}\}\ (fset\ x)))\ \{\}\ X = X$ (**is** *?lhs* = *X*)
⟨*proof*⟩

lemma *dnf-ltln-of-dnf-min-dnf*:
dnf (*ltln-of-dnf* (*min-dnf* φ)) = *min-dnf* φ
⟨*proof*⟩

lemma *min-dnf-subst*:
min-set (*subst-dnf* (*min-dnf* φ) m) = *min-dnf* (*subst* φ m) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

end

5 Code lemmas for abstract definitions

theory *Code-Equations*
imports
LTL Equivalence-Relations
Boolean-Expression-Checkers.Boolean-Expression-Checkers
Boolean-Expression-Checkers.Boolean-Expression-Checkers-AList-Mapping
begin

5.1 Propositional Equivalence

fun *ifex-of-ltl* :: $'a$ *ltln* ⇒ $'a$ *ltln ifex*
where
ifex-of-ltl $true_n$ = *Trueif*
| *ifex-of-ltl* $false_n$ = *Falseif*
| *ifex-of-ltl* (φ and_n ψ) = *normif Mapping.empty* (*ifex-of-ltl* φ) (*ifex-of-ltl* ψ) *Falseif*
| *ifex-of-ltl* (φ or_n ψ) = *normif Mapping.empty* (*ifex-of-ltl* φ) *Trueif* (*ifex-of-ltl* ψ)
| *ifex-of-ltl* φ = *IF* φ *Trueif Falseif*

lemma *val-ifex*:
val-ifex (*ifex-of-ltl* b) $s = \{x.\ s\ x\} \models_P b$
⟨*proof*⟩

lemma *reduced-ifex*:

reduced (*ifex-of-ltl b*) {}
⟨*proof*⟩

lemma *ifex-of-ltl-reduced-bdt-checker*:
reduced-bdt-checkers ifex-of-ltl (λ*y s*. {*x*. *s x*} ⊨$_P$ *y*)
⟨*proof*⟩

lemma *ltl-prop-equiv-impl*[*code*]:
(*φ* ∼$_P$ *ψ*) = *equiv-test ifex-of-ltl φ ψ*
⟨*proof*⟩

lemma *ltl-prop-implies-impl*[*code*]:
(*φ* ⟶$_P$ *ψ*) = *impl-test ifex-of-ltl φ ψ*
⟨*proof*⟩
export-code (∼$_P$) (⟶$_P$) **checking**

end

6 Example

theory *Example*
imports
../LTL ../Rewriting HOL−Library.Code-Target-Numeral
begin

— The included parser always returns a *String.literal ltlc*. If a different labelling is needed one can use *map-ltlc* to relabel the leafs. In our example we prepend a string to each atom.

definition *rewrite* :: *String.literal ltlc* ⇒ *String.literal ltlc*
where
rewrite ≡ *ltln-to-ltlc o rewrite-iter-slow o ltlc-to-ltln o* (*map-ltlc* (λ*s*. *String.implode* ″*prop*(″ + *s* + *String.implode* ″)″))

— Export rewriting engine (and also constructors)

export-code *true$_c$ Iff-ltlc rewrite* **in** *SML* **file-prefix** ‹*rewrite-example*›

end

References

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[2] F. Somenzi and R. Bloem. Efficient büchi automata from LTL formulae. In E. A. Emerson and A. P. Sistla, editors, *Computer Aided Verification, 12th International Conference, CAV 2000, Chicago, IL, USA, July 15-19, 2000, Proceedings*, volume 1855 of *Lecture Notes in Computer Science*, pages 248–263. Springer, 2000.