Duality of Linear Programming

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Abstract

We formalize the weak and strong duality theorems of linear programming. For the strong duality theorem we provide three sufficient preconditions: both the primal problem and the dual problem are satisfiable, the primal problem is satisfiable and bounded, or the dual problem is satisfiable and bounded. The proofs are based on an existing formalization of Farkas' Lemma.

Contents

1 Introduction

The proofs are taken from a textbook on linear programming [3]. There clearly is already an related AFP entry on linear programming [2] and we briefly explain the relationship between that entry and this one.

- The other AFP entry provides an algorithm for solving linear programs based on an existing simplex implementation. Since the simplex implementation is formulated only for rational numbers, several results are only available for rational numbers. Moreover, the simplex algorithm internally works on sets of inequalities that are represented by linear polynomials, and there are conversions between matrix-vector inequalities and linear polynomial inequalities. Finally, that AFP entry does not contain the strong duality theorem, which is the essential result in this AFP entry.
- This AFP entry has completely been formalized in the matrix-vector representation. It mainly consists of the strong duality theorems without any algorithms. The proof of these theorems are based on Farkas'

Lemma which is provided in [1] for arbitrary linearly ordered fields. Therefore, also the duality theorems are proven in that generality without the restriction to rational numbers.

2 Minimum and Maximum of Potentially Infinite Sets

theory *Minimum-Maximum*
  imports *Main*
begin

We define minima and maxima of sets. In contrast to the existing *Min* and *Max* operators, these operators are not restricted to finite sets

definition *Maximum* :: $'a$ :: *linorder set* $\Rightarrow$ $'a$ **where**
  $\mathit{Maximum}\ S = (\mathit{THE}\ x.\ x \in S \wedge (\forall\ y \in S.\ y \leq x))$
definition *Minimum* :: $'a$ :: *linorder set* $\Rightarrow$ $'a$ **where**
  $\mathit{Minimum}\ S = (\mathit{THE}\ x.\ x \in S \wedge (\forall\ y \in S.\ x \leq y))$

definition *has-Maximum* **where** $\mathit{has\text{-}Maximum}\ S = (\exists\ x.\ x \in S \wedge (\forall\ y \in S.\ y \leq x))$
definition *has-Minimum* **where** $\mathit{has\text{-}Minimum}\ S = (\exists\ x.\ x \in S \wedge (\forall\ y \in S.\ x \leq y))$

lemma *eqMaximumI*:
  assumes $x \in S$
  and $\bigwedge y.\ y \in S \Longrightarrow y \leq x$
shows $\mathit{Maximum}\ S = x$
  ⟨*proof*⟩

lemma *eqMinimumI*:
  assumes $x \in S$
  and $\bigwedge y.\ y \in S \Longrightarrow x \leq y$
shows $\mathit{Minimum}\ S = x$
  ⟨*proof*⟩

lemma *has-MaximumD*:
  assumes *has-Maximum S*
  shows $\mathit{Maximum}\ S \in S$
    $x \in S \Longrightarrow x \leq \mathit{Maximum}\ S$
⟨*proof*⟩

lemma *has-MinimumD*:
  assumes *has-Minimum S*
  shows $\mathit{Minimum}\ S \in S$
    $x \in S \Longrightarrow \mathit{Minimum}\ S \leq x$
⟨*proof*⟩

On non-empty finite sets, *Minimum* and *Min* coincide, and similarly *Maxi-*

mum and *Max*.

lemma *Minimum-Min*: **assumes** *finite S S* $\neq$ {}
  shows *Minimum S* = *Min S*
  ⟨*proof*⟩

lemma *Maximum-Max*: **assumes** *finite S S* $\neq$ {}
  shows *Maximum S* = *Max S*
  ⟨*proof*⟩

end

3 Weak and Strong Duality of Linear Programming

theory *LP-Duality*
  imports
    Linear-Inequalities.Farkas-Lemma
    Minimum-Maximum
begin

lemma *weak-duality-theorem*:
  fixes *A* :: *'a* :: *linordered-comm-semiring-strict mat*
  assumes *A*: $A \in$ *carrier-mat nr nc*
    and *b*: $b \in$ *carrier-vec nr*
    and *c*: $c \in$ *carrier-vec nc*
    and *x*: $x \in$ *carrier-vec nc*
    and *Axb*: $A *_v x \leq b$
    and *y0*: $y \geq 0_v\ nr$
    and *yA*: $A^T *_v y = c$
  shows $c \cdot x \leq b \cdot y$
⟨*proof*⟩

corollary *unbounded-primal-solutions*:
  fixes *A* :: *'a* :: *linordered-idom mat*
  assumes *A*: $A \in$ *carrier-mat nr nc*
    and *b*: $b \in$ *carrier-vec nr*
    and *c*: $c \in$ *carrier-vec nc*
    and *unbounded*: $\forall\ v.\ \exists\ x \in$ *carrier-vec nc*. $A *_v x \leq b \wedge c \cdot x \geq v$
  shows $\neg$ ($\exists\ y.\ y \geq 0_v\ nr \wedge A^T *_v y = c$)
⟨*proof*⟩

corollary *unbounded-dual-solutions*:
  fixes *A* :: *'a* :: *linordered-idom mat*
  assumes *A*: $A \in$ *carrier-mat nr nc*
    and *b*: $b \in$ *carrier-vec nr*
    and *c*: $c \in$ *carrier-vec nc*
    and *unbounded*: $\forall\ v.\ \exists\ y.\ y \geq 0_v\ nr \wedge A^T *_v y = c \wedge b \cdot y \leq v$
  shows $\neg$ ($\exists\ x \in$ *carrier-vec nc*. $A *_v x \leq b$)

⟨*proof*⟩

A version of the strong duality theorem which demands that both primal and dual problem are solvable. At this point we do not use min- or max-operations

theorem *strong-duality-theorem-both-sat*:
  fixes A :: $'a$ :: *trivial-conjugatable-linordered-field mat*
  assumes A: $A \in$ *carrier-mat nr nc*
    and b: $b \in$ *carrier-vec nr*
    and c: $c \in$ *carrier-vec nc*
    and *primal*: $\exists\ x \in$ *carrier-vec nc*. $A *_v x \leq b$
    and *dual*: $\exists\ y.\ y \geq 0_v\ nr \wedge A^T *_v y = c$
  shows $\exists\ x\ y.$
      $x \in$ *carrier-vec nc* $\wedge A *_v x \leq b \wedge$
      $y \geq 0_v\ nr \wedge A^T *_v y = c \wedge$
      $c \cdot x = b \cdot y$
⟨*proof*⟩

A version of the strong duality theorem which demands that the primal problem is solvable and the objective function is bounded.

theorem *strong-duality-theorem-primal-sat-bounded*:
  fixes *bound* :: $'a$ :: *trivial-conjugatable-linordered-field*
  assumes A: $A \in$ *carrier-mat nr nc*
    and b: $b \in$ *carrier-vec nr*
    and c: $c \in$ *carrier-vec nc*
    and *sat*: $\exists\ x \in$ *carrier-vec nc*. $A *_v x \leq b$
    and *bounded*: $\forall\ x \in$ *carrier-vec nc*. $A *_v x \leq b \longrightarrow c \cdot x \leq$ *bound*
  shows $\exists\ x\ y.$
      $x \in$ *carrier-vec nc* $\wedge A *_v x \leq b \wedge$
      $y \geq 0_v\ nr \wedge A^T *_v y = c \wedge$
      $c \cdot x = b \cdot y$
⟨*proof*⟩

A version of the strong duality theorem which demands that the dual problem is solvable and the objective function is bounded.

theorem *strong-duality-theorem-dual-sat-bounded*:
  fixes *bound* :: $'a$ :: *trivial-conjugatable-linordered-field*
  assumes A: $A \in$ *carrier-mat nr nc*
    and b: $b \in$ *carrier-vec nr*
    and c: $c \in$ *carrier-vec nc*
    and *sat*: $\exists\ y.\ y \geq 0_v\ nr \wedge A^T *_v y = c$
    and *bounded*: $\forall\ y.\ y \geq 0_v\ nr \wedge A^T *_v y = c \longrightarrow$ *bound* $\leq b \cdot y$
  shows $\exists\ x\ y.$
      $x \in$ *carrier-vec nc* $\wedge A *_v x \leq b \wedge$
      $y \geq 0_v\ nr \wedge A^T *_v y = c \wedge$
      $c \cdot x = b \cdot y$
⟨*proof*⟩

Now the previous three duality theorems are formulated via min/max.

corollary *strong-duality-theorem-min-max*:
fixes $A :: 'a :: trivial\text{-}conjugatable\text{-}linordered\text{-}field\ mat$
assumes A: $A \in carrier\text{-}mat\ nr\ nc$
and b: $b \in carrier\text{-}vec\ nr$
and c: $c \in carrier\text{-}vec\ nc$
and *primal*: $\exists\ x \in carrier\text{-}vec\ nc.\ A *_v x \leq b$
and *dual*: $\exists\ y.\ y \geq 0_v\ nr \wedge A^T *_v y = c$
shows $Maximum\ \{c \cdot x \mid x.\ x \in carrier\text{-}vec\ nc \wedge A *_v x \leq b\}$
$= Minimum\ \{b \cdot y \mid y.\ y \geq 0_v\ nr \wedge A^T *_v y = c\}$
and $has\text{-}Maximum\ \{c \cdot x \mid x.\ x \in carrier\text{-}vec\ nc \wedge A *_v x \leq b\}$
and $has\text{-}Minimum\ \{b \cdot y \mid y.\ y \geq 0_v\ nr \wedge A^T *_v y = c\}$
⟨*proof*⟩

corollary *strong-duality-theorem-primal-sat-bounded-min-max*:
fixes $bound :: 'a :: trivial\text{-}conjugatable\text{-}linordered\text{-}field$
assumes A: $A \in carrier\text{-}mat\ nr\ nc$
and b: $b \in carrier\text{-}vec\ nr$
and c: $c \in carrier\text{-}vec\ nc$
and *sat*: $\exists\ x \in carrier\text{-}vec\ nc.\ A *_v x \leq b$
and *bounded*: $\forall\ x \in carrier\text{-}vec\ nc.\ A *_v x \leq b \longrightarrow c \cdot x \leq bound$
shows $Maximum\ \{c \cdot x \mid x.\ x \in carrier\text{-}vec\ nc \wedge A *_v x \leq b\}$
$= Minimum\ \{b \cdot y \mid y.\ y \geq 0_v\ nr \wedge A^T *_v y = c\}$
and $has\text{-}Maximum\ \{c \cdot x \mid x.\ x \in carrier\text{-}vec\ nc \wedge A *_v x \leq b\}$
and $has\text{-}Minimum\ \{b \cdot y \mid y.\ y \geq 0_v\ nr \wedge A^T *_v y = c\}$
⟨*proof*⟩

corollary *strong-duality-theorem-dual-sat-bounded-min-max*:
fixes $bound :: 'a :: trivial\text{-}conjugatable\text{-}linordered\text{-}field$
assumes A: $A \in carrier\text{-}mat\ nr\ nc$
and b: $b \in carrier\text{-}vec\ nr$
and c: $c \in carrier\text{-}vec\ nc$
and *sat*: $\exists\ y.\ y \geq 0_v\ nr \wedge A^T *_v y = c$
and *bounded*: $\forall\ y.\ y \geq 0_v\ nr \wedge A^T *_v y = c \longrightarrow bound \leq b \cdot y$
shows $Maximum\ \{c \cdot x \mid x.\ x \in carrier\text{-}vec\ nc \wedge A *_v x \leq b\}$
$= Minimum\ \{b \cdot y \mid y.\ y \geq 0_v\ nr \wedge A^T *_v y = c\}$
and $has\text{-}Maximum\ \{c \cdot x \mid x.\ x \in carrier\text{-}vec\ nc \wedge A *_v x \leq b\}$
and $has\text{-}Minimum\ \{b \cdot y \mid y.\ y \geq 0_v\ nr \wedge A^T *_v y = c\}$
⟨*proof*⟩

end

References

[1] R. Bottesch, A. Reynaud, and R. Thiemann. Linear inequalities. *Archive of Formal Proofs*, June 2019. https://isa-afp.org/entries/Linear_Inequalities.html, Formal proof development.

[2] J. Parsert and C. Kaliszyk. Linear programming. *Archive of Formal*

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[3] A. Schrijver. *Theory of linear and integer programming.* John Wiley & Sons, 1998.