1 Introduction

We discuss a topological curiosity discovered by Kuratowski (1922): the fact that the number of distinct operators on a topological space generated by compositions of closure and complement never exceeds 14, and is exactly 14 in the case of $\mathbb{R}$. In addition, we prove a theorem due to Chagrov (1982) that classifies topological spaces according to the number of such operators they support.

Kuratowski’s result, which is exposited in Whitty (2015) and Chapter 7 of Chamberland (2015), has already been treated in Mizar — see Bagińska and Grabowski (2003) and Grabowski (2004). To the best of our knowledge, we are the first to mechanize Chagrov’s result.

Our work is based on a presentation of Kuratowski’s and Chagrov’s results by Gardner and Jackson (2008).

We begin with some preliminary facts pertaining to the relationship between interiors of unions and unions of interiors (§2) and the relationship between $\mathbb{Q}$ and $\mathbb{R}$ (§3). We then prove Kuratowski’s result (§4) and the corollary that at most 7 distinct operators on a topological space can be generated by compositions of closure and interior (§5). Finally, we prove Chagrov’s result (§6).

2 Interiors and unions

definition

boundary :: 'a::topological_space set $\Rightarrow$ 'a set

where

boundary X = closure X - interior X
lemma boundary_empty:
  shows boundary {} = {}
⟨proof⟩

definition
  exterior :: 'a::topological_space set ⇒ 'a set
where
  exterior X = − (interior X ∪ boundary X)

lemma interior_union_boundary:
  shows interior (X ∪ Y) = interior X ∪ interior Y
  ←→ boundary X ∩ boundary Y ⊆ boundary (X ∪ Y) (is (?lhs1 = ?lhs2) ←→ ?rhs)
⟨proof⟩

lemma interior_union_closed_intervals:
  fixes a :: 'a::ordered_euclidean_space
  assumes b < c
  shows interior ({a..b} ∪ {c..d}) = interior {a..b} ∪ interior {c..d}
⟨proof⟩

3  Additional facts about the rationals and reals

lemma Rat_real_limpt:
  fixes x :: real
  shows x islimpt Q
⟨proof⟩

lemma Rat_closure:
  shows closure Q = (UNIV :: real set)
⟨proof⟩

lemma Rat_interval_closure:
  fixes x :: real
  assumes x < y
  shows closure ({x..<y} ∩ Q) = {x..y}
⟨proof⟩

lemma Rat_not_open:
  fixes T :: real set
  assumes open T
  assumes T ≠ {}
  shows ¬T ⊆ Q
⟨proof⟩

lemma Irrat_dense_in_real:
  fixes x :: real
  assumes x < y
  shows ∃ r ∈ −Q. x < r ∧ r < y
⟨proof⟩

lemma closed_interval_Int_compl:
  fixes x :: real
  assumes x < y
  assumes y < z
  shows − {x..y} ∩ − {y..z} = − {x..z}
⟨proof⟩
4 Kuratowski’s result

We prove that at most 14 distinct operators can be generated by compositions of closure and complement. For convenience, we give these operators short names and try to avoid pointwise reasoning. We treat the interior operator at the same time.

declare o_apply[simp del]

definition C :: 'a::topological_space set ⇒ 'a set where C X = − X

definition K :: 'a::topological_space set ⇒ 'a set where K X = closure X

definition I :: 'a::topological_space set ⇒ 'a set where I X = interior X

lemma C_C:
  shows C ◦ C = id
⟨proof⟩

lemma K_K:
  shows K ◦ K = K
⟨proof⟩

lemma I_I:
  shows I ◦ I = I
⟨proof⟩

lemma I_K:
  shows I = C ◦ K ◦ C
⟨proof⟩

lemma K_I:
  shows K = C ◦ I ◦ C
⟨proof⟩

lemma K_I_K_I:
  shows K ◦ I ◦ K ◦ I = K ◦ I
⟨proof⟩

lemma K_I_K_I:
  shows I ◦ K ◦ I ◦ K = I ◦ K
⟨proof⟩

lemma K_mono:
  assumes x ⊆ y
  shows K x ⊆ K y
⟨proof⟩

The following lemma embodies the crucial observation about compositions of C and K:

lemma KCKCKCKCK:
  shows K ◦ C ◦ K ◦ C ◦ K ◦ C ◦ K ◦ K = K ◦ C ◦ K (is ?lhs = ?rhs)
⟨proof⟩

The inductive set CK captures all operators that can be generated by compositions of C and K. We shallowly embed the operators; that is, we identify operators up to extensional equality.

inductive CK :: ('a::topological_space set ⇒ 'a set) ⇒ bool where
  CK C
| CK K
| [ CK f; CK g ] ⇒ CK (f ◦ g)
declare CK.intros[intro!]

lemma CK_id[intro!]:
  CK id
⟨proof⟩

The inductive set CK_nf captures the normal forms for the 14 distinct operators.

inductive CK_nf :: ('a::topological_space set ⇒ 'a set) ⇒ bool where
  CK_nf id
| CK_nf C
| CK_nf K
| CK_nf (C ◦ K)
| CK_nf (K ◦ C)
| CK_nf (C ◦ K ◦ C)
| CK_nf (K ◦ C ◦ K)
| CK_nf (C ◦ K ◦ C ◦ K)
| CK_nf (K ◦ C ◦ K ◦ C)
| CK_nf (C ◦ K ◦ C ◦ K ◦ C)
| CK_nf (K ◦ C ◦ K ◦ C ◦ K)
| CK_nf (C ◦ K ◦ C ◦ K ◦ C ◦ K)
| CK_nf (K ◦ C ◦ K ◦ C ◦ K ◦ C ◦ K)
| CK_nf (C ◦ K ◦ C ◦ K ◦ C ◦ K ◦ C ◦ K)
| CK_nf (K ◦ C ◦ K ◦ C ◦ K ◦ C ◦ K ◦ C ◦ K)
| CK_nf (C ◦ K ◦ C ◦ K ◦ C ◦ K ◦ C ◦ K ◦ C ◦ K)
| CK_nf (K ◦ C ◦ K ◦ C ◦ K ◦ C ◦ K ◦ C ◦ K ◦ C ◦ K)
| CK_nf (C ◦ K ◦ C ◦ K ◦ C ◦ K ◦ C ◦ K ◦ C ◦ K ◦ C ◦ K)
| CK_nf (K ◦ C ◦ K ◦ C ◦ K ◦ C ◦ K ◦ C ◦ K ◦ C ◦ K ◦ C ◦ K)
| CK_nf (C ◦ K ◦ C ◦ K ◦ C ◦ K ◦ C ◦ K ◦ C ◦ K ◦ C ◦ K ◦ C ◦ K)
| CK_nf (K ◦ C ◦ K ◦ C ◦ K ◦ C ◦ K ◦ C ◦ K ◦ C ◦ K ◦ C ◦ K ◦ C ◦ K)
| CK_nf (C ◦ K ◦ C ◦ K ◦ C ◦ K ◦ C ◦ K ◦ C ◦ K ◦ C ◦ K ◦ C ◦ K ◦ C ◦ K)
| CK_nf (K ◦ C ◦ K ◦ C ◦ K ◦ C ◦ K ◦ C ◦ K ◦ C ◦ K ◦ C ◦ K ◦ C ◦ K ◦ C ◦ K)

declare CK_nf.intros[intro!]

lemma CK_nf_set:
⟨proof⟩

That each operator generated by compositions of C and K is extensionally equivalent to one of the normal forms captured by CK_nf is demonstrated by means of an induction over the construction of CK_nf and an appeal to the facts proved above.

theorem CK_nf:
  CK f ↔ CK_nf f
⟨proof⟩

theorem CK_card:
  shows card \{ f . CK f \} ≤ 14
⟨proof⟩

We show, using the following subset of \( \mathbb{R} \) (an example taken from Rusin (2001)) as a witness, that there exist topological spaces on which all 14 operators are distinct.

definition
  RRR :: real set
where
  RRR = \{ 0<..<1 \} ∪ \{ 1<..<2 \} ∪ \{ 3 \} ∪ (\{ 5<..<7 \} ∩ \mathbb{Q} )

The following facts allow the required proofs to proceed by simp:

lemma RRR_closure:
  shows closure RRR = \{ 0..2 \} ∪ \{ 3 \} ∪ \{ 5..7 \}
⟨proof⟩

lemma RRR_interior:
  interior RRR = \{ 0<..<1 \} ∪ \{ 1<..<2 \} (is \_lhs = \_rhs)
⟨proof⟩
The operators can be distinguished by testing which of the points in \{1,2,3,4,6\} belong to their results.

**Definition**

\[ \text{test} :: (\text{real set} \Rightarrow \text{real set}) \Rightarrow \text{bool list} \]

where

\[ \text{test } f \equiv \text{map} \ (\lambda x. \ x \in f \text{ RRR}) \ [1,2,3,4,6] \]

**Lemma RRR_test:**

- **Assumes**: \( f \text{ RRR} = g \text{ RRR} \)
- **Shows**: \( \text{test } f = \text{test } g \)

**Lemma nf_RRR:**

**Shows**

\[
\begin{align*}
\text{test } \text{id} &= [\text{False}, \text{False}, \text{True}, \text{False}, \text{True}] \\
\text{test } C &= [\text{True}, \text{True}, \text{False}, \text{True}, \text{False}] \\
\text{test } K &= [\text{True}, \text{True}, \text{True}, \text{True}, \text{False}] \\
\text{test } (K \circ C) &= [\text{True}, \text{True}, \text{True}, \text{True}, \text{True}] \\
\text{test } (C \circ K) &= [\text{False}, \text{False}, \text{True}, \text{False}, \text{False}] \\
\text{test } (C \circ K \circ C) &= [\text{False}, \text{False}, \text{False}, \text{True}, \text{False}] \\
\text{test } (K \circ C \circ K) &= [\text{True}, \text{True}, \text{False}, \text{False}, \text{False}] \\
\text{test } (C \circ K \circ C \circ K) &= [\text{True}, \text{True}, \text{True}, \text{False}, \text{False}] \\
\text{test } (K \circ C \circ K \circ C \circ K \circ K) &= [\text{True}, \text{False}, \text{False}, \text{False}, \text{False}] \\
\text{test } (C \circ K \circ C \circ K \circ C \circ K) &= [\text{True}, \text{False}, \text{False}, \text{False}, \text{False}] \\
\text{test } (K \circ C \circ K \circ C \circ K \circ C \circ K \circ C) &= [\text{True}, \text{False}, \text{False}, \text{False}, \text{False}]
\end{align*}
\]

**Theorem CK_nf_real_card:**

**Shows**

\[
(\lambda f. f \text{ RRR} \ ' \ \{ f . \text{CK} \text{nf} \ f \}) \ = 14
\]

**Theorem CK_real_card:**

**Shows**

\[
\text{card} \ \{ f::\text{real set} \Rightarrow \text{real set. CK } f \} = 14 \ (\text{is } \text{lhs} = \text{rhs})
\]

5 A corollary of Kuratowski’s result

We show that it is a corollary of \( \text{CK} \text{real_card} \) that at most 7 distinct operators on a topological space can be generated by compositions of closure and interior. In the case of \( \mathbb{R} \), exactly 7 distinct operators can be so generated.

**Inductive IK :: (\text{a::topological_space set} \Rightarrow \text{a set}) \Rightarrow \text{bool where**

\[
\begin{align*}
\text{IK} \text{id} \\
\text{IK} \text{I} \\
\text{IK} \text{K} \\
[ [ \text{IK } f; \text{IK } g ] ] \Rightarrow \text{IK } (f \circ g)
\end{align*}
\]

**Inductive IK_nf :: (\text{a::topological_space set} \Rightarrow \text{a set}) \Rightarrow \text{bool where**

\[
\begin{align*}
\text{IK} \text{nf} \text{id} \\
\text{IK} \text{nf} \text{I} \\
\text{IK} \text{nf} \text{K} \\
[ [ \text{IK} \text{nf } (I \circ K) ] ] \\
[ [ \text{IK} \text{nf } (K \circ I) ] ]
\end{align*}
\]
6 Chagrov’s result

Chagrov’s theorem, which is discussed in Section 2.1 of Gardner and Jackson (2008), states that the number of distinct operators on a topological space that can be generated by compositions of closure and complement is one of 2, 6, 8, 10 or 14.

We begin by observing that the set of normal forms $CK_{nf}$ can be split into two disjoint sets, $CK_{nf_pos}$ and $CK_{nf_neg}$, which we define in terms of interior and closure.

inductive $CK_{nf_pos} :: \langle 'a::topological_space \Rightarrow 'a set \rangle \Rightarrow bool$
where
$CK_{nf_pos} id$
$| CK_{nf_pos} I$
$| CK_{nf_pos} K$
$| CK_{nf_pos} (I \circ K)$
$| CK_{nf_pos} (K \circ I)$
$| CK_{nf_pos} (I \circ K \circ I)$
$| CK_{nf_pos} (K \circ I \circ K)$

declare $CK_{nf_pos}.intros[intro!]

lemma $CK_{nf_pos}.set$: 
shows $\{ f \cdot CK_{nf_pos} f \} = \{ id, I, K, I \circ K, K \circ I, I \circ K \circ I, K \circ I \circ K \}$
⟨proof⟩

definition $CK_{nf_neg} :: \langle 'a::topological_space \Rightarrow 'a set \rangle \Rightarrow bool$
where
$CK_{nf_neg} f \leftarrow (\exists g. CK_{nf_pos} g \land f = C \circ g)$

lemma $CK_{nf_pos_neg_disjoint}$: 
assumes $CK_{nf_pos} f$
assumes $CK_{nf_neg} g$
shows \( f \neq g \)

\[
\langle \text{proof} \rangle
\]

\textbf{lemma} \( CK_{nf\_pos\_neg\_CK_{nf}} \):
\[
CK_{nf} f \iff CK_{nf\_pos} f \lor CK_{nf\_neg} f \quad (\text{is } \text{lhs} \iff \text{rhs})
\]

\[
\langle \text{proof} \rangle
\]

We now focus on \( CK_{nf\_pos} \). In particular, we show that its cardinality for any given topological space is one of 1, 3, 4, 5 or 7.

The proof consists of exhibiting normal forms for the operators supported by each of six classes of topological spaces. These are sublattices of the following lattice of \( CK_{nf\_pos} \) operators:

\textbf{lemmas} \( K\_I\_K\_\subseteq\_K = \text{closure\_mono[OF interior\_subset, of closure X, simplified]} \) for \( X \)

\textbf{lemma} \( CK_{nf\_pos\_lattice} \):

shows \[
\begin{align*}
I & \leq (id : 'a::topological\_space set \Rightarrow 'a set) \\
id & \leq (K : 'a::topological\_space set \Rightarrow 'a set) \\
I & \leq I \circ K \circ (I : 'a::topological\_space set \Rightarrow 'a set) \\
I \circ K \circ I & \leq I \circ (K : 'a::topological\_space set \Rightarrow 'a set) \\
I \circ K & \leq K \circ I \circ (K : 'a::topological\_space set \Rightarrow 'a set) \\
K \circ I & \leq K \circ I \circ (K : 'a::topological\_space set \Rightarrow 'a set) \\
K \circ I & \leq (K : 'a::topological\_space set \Rightarrow 'a set)
\end{align*}
\]

\[
\langle \text{proof} \rangle
\]

We define the six classes of topological spaces in question, and show that they are related by inclusion in the following way (as shown in Figure 2.3 of Gardner and Jackson (2008)):

\begin{center}
\begin{tikzpicture}
  \node (a) {Kuratowski spaces};
  \node (b) [below of=a] {Extremally disconnected spaces};
  \node (c) [below of=b] {Open unresolvable spaces};
  \node (d) [left of=c] {Partition spaces};
  \node (e) [right of=c] {Extremally disconnected and open unresolvable spaces};
  \node (f) [below of=e] {Discrete spaces};
  \draw (a) -- (b);
  \draw (b) -- (c);
  \draw (c) -- (d);
  \draw (c) -- (e);
  \draw (e) -- (f);
\end{tikzpicture}
\end{center}

\textbf{6.1 Discrete spaces}

\textbf{definition}
\[
discrete \ (X : 'a::topological\_space set) \iff I = (id::'a set \Rightarrow 'a set)
\]

\textbf{lemma} \( \text{discrete\_eqs} \):
\textbf{assumes} \( \text{discrete} \ (X : 'a::topological\_space set) \)
\textbf{shows} \[
\begin{align*}
I & = (id::'a set \Rightarrow 'a set) \\
K & = (id::'a set \Rightarrow 'a set)
\end{align*}
\]

\[
\langle \text{proof} \rangle
\]
lemma discrete_card:
  assumes discrete (X :: 'a::topological_space set)
  shows card {f. CK_nf_pos (f::'a set ⇒ 'a set)} = 1
  ⟨proof⟩

lemma discrete_discrete_topology:
  fixes X :: 'a::topological_space set
  assumes ∀Y::'a set. open Y
  shows discrete X
  ⟨proof⟩

6.2 Partition spaces

definition part (X :: 'a::topological_space set) ←→ K ◦ I = (I :: 'a set ⇒ 'a set)

lemma discrete_part:
  assumes discrete X
  shows part X
  ⟨proof⟩

lemma part_eqs:
  assumes part (X :: 'a::topological_space set)
  shows K ◦ I = (I :: 'a set ⇒ 'a set)
  I ◦ K = (K :: 'a set ⇒ 'a set)
  ⟨proof⟩

lemma part_not_discrete_card:
  assumes part (X :: 'a::topological_space set)
  assumes ¬discrete X
  shows card {f. CK_nf_pos (f::'a set ⇒ 'a set)} = 3
  ⟨proof⟩

A partition space is a topological space whose basis consists of the empty set and the equivalence classes of points of the space induced by some equivalence relation $R$ on the underlying set of the space. Equivalently, a partition space is one in which every open set is closed. Thus, for example, the class of partition spaces includes every topological space whose open sets form a boolean algebra.

datatype part_witness = a | b | c

lemma part_witness_UNIV:
  shows UNIV = set [a, b, c]
  ⟨proof⟩

lemmas part_witness_pow = subset_subseqs[OF subset_trans[OF subset_UNIV Set.equalityD1[OF part_witness_UNIV]]]

lemmas part_witness_Compl = Compl_eq.Diff_UNIV[where 'a=part_witness, unfolded part_witness_UNIV, simplified]

instantiation part_witness :: topological_space
begin

definition open_part_witness X ←→ X ∈ {{}, {a}, {b, c}, {a, b, c}}

lemma part_witness_ball:
  (∀ s ∈ S. s ∈ {{}, {a}, {b, c}, {a, b, c}}) ←→ S ⊆ set [{{}, {a}, {b, c}, {a, b, c}}]
  ⟨proof⟩
lemmas part_witness_subsets_pow = subset_subseqs[OF iffD1[OF part_witness_ball]]

instance (proof)

end

lemma part_witness_interior_simps:
shows
interior {a} = {a}
interior {b} = {}
interior {c} = {}
interior {a, b} = {a}
interior {a, c} = {a}
interior {b, c} = {b, c}
interior {a, b, c} = {a, b, c}
(proof)

lemma part_witness_part:
fixes X :: part_witness set
shows part X
(proof)

lemma part_witness_not_discrete:
fixes X :: part_witness set
shows ¬discrete X
(proof)

lemma part_witness_card:
shows card {f. CK_nf_pos (f::part_witness set ⇒ part_witness set)} = 3
(proof)

6.3 Extremally disconnected and open unresolvable spaces

definition
ed_ou (X :: 'a::topological_space set) = I ◦ K = K ◦ (I :: 'a set ⇒ 'a set)

lemma discrete_ed_ou:
assumes discrete X
shows ed_ou X
(proof)

lemma ed_ou_eqs:
assumes ed_ou (X :: 'a::topological_space set)
sows
I ◦ K ◦ I = K ◦ (I :: 'a set ⇒ 'a set)
K ◦ I ◦ K = K ◦ (I :: 'a set ⇒ 'a set)
I ◦ K = K ◦ (I :: 'a set ⇒ 'a set)
(proof)

lemma ed_ou_neqs:
assumes ed_ou (X :: 'a::topological_space set)
assumes ¬discrete X
shows
I ≠ (K :: 'a set ⇒ 'a set)
I ≠ K ◦ (I :: 'a set ⇒ 'a set)
K ≠ K ◦ (I :: 'a set ⇒ 'a set)
I ≠ (id :: 'a set ⇒ 'a set)
K ≠ (id :: 'a set ⇒ 'a set)
lemma ed_ou_not_discrete_card:
  assumes ed_ou (X :: 'a::topological_space set)
  assumes ¬discrete X
  shows card \{f. CK_nf_pos (f::'a set ⇒ 'a set)\} = 4
⟨proof⟩

We consider an example extremally disconnected and open unresolvable topological space.

datatype ed_ou_witness = a | b | c | d | e

lemma ed_ou_witness_UNIV:
  shows UNIV = set \{a, b, c, d, e\}
⟨proof⟩

lemmas ed_ou_witness_pow = subset_subseqs[OF subset_trans[OF subset_UNIV Set.equalityD1[OF ed_ou_witness_UNIV]]]

lemmas ed_ou_witness_Compl = Compl_eq_Diff_UNIV[where 'a=ed_ou_witness, unfolded ed_ou_witness_UNIV, simplified]

instance ed_ou_witness :: finite
⟨proof⟩

instantiation ed_ou_witness :: topological_space
begin

inductive open_ed_ou_witness :: ed_ou_witness set ⇒ bool where
  open_ed_ou_witness {} |
  open_ed_ou_witness \{a\} |
  open_ed_ou_witness \{b\} |
  open_ed_ou_witness \{e\} |
  open_ed_ou_witness \{a, c\} |
  open_ed_ou_witness \{b, d\} |
  open_ed_ou_witness \{a, c, e\} |
  open_ed_ou_witness \{a, b\} |
  open_ed_ou_witness \{a, e\} |
  open_ed_ou_witness \{b, e\} |
  open_ed_ou_witness \{a, b, c\} |
  open_ed_ou_witness \{a, b, d\} |
  open_ed_ou_witness \{a, b, e\} |
  open_ed_ou_witness \{b, d, e\} |
  open_ed_ou_witness \{a, b, c, d\} |
  open_ed_ou_witness \{a, b, c, e\} |
  open_ed_ou_witness \{a, b, d, e\} |
  open_ed_ou_witness \{a, b, c, d, e\}

declare open_ed_ou_witness.intros[intro!]

lemma ed_ou_witness_inter:
  fixes S :: ed_ou_witness set
  assumes open S |
  assumes open T
  shows open (S ∩ T)
⟨proof⟩

lemma ed_ou_witness_union:
  fixes X :: ed_ou_witness set set
assumes $\forall x \in X. \text{open } x$

shows open $(\bigcup X)$

⟨proof⟩

instance ⟨proof⟩

end

lemma ed_ou_witness_interior_simps:

shows

interior {a} = {a}
interior {b} = {b}
interior {c} = {}
interior {d} = {}
interior {e} = {e}
interior {a, b} = {a, b}
interior {a, c} = {a, c}
interior {a, d} = {a}
interior {a, e} = {a, e}
interior {b, c} = {b}
interior {b, d} = {b, d}
interior {b, e} = {b, e}
interior {c, d} = {}
interior {c, e} = {e}
interior {d, e} = {e}
interior {a, b, c} = {a, b, c}
interior {a, b, d} = {a, b, d}
interior {a, b, e} = {a, b, e}
interior {a, c, d} = {a, c}
interior {a, c, e} = {a, c, e}
interior {a, d, e} = {a, e}
interior {b, c, d} = {b, d}
interior {b, c, e} = {b, e}
interior {b, d, e} = {b, d, e}
interior {c, d, e} = {e}
interior {a, b, c, d} = {a, b, c, d}
interior {a, b, c, e} = {a, b, c, e}
interior {a, b, d, e} = {a, b, d, e}
interior {a, b, c, d, e} = {a, b, c, d, e}
interior {a, c, d, e} = {a, c, e}
interior {b, c, d, e} = {b, c, d, e}

⟨proof⟩

lemma ed_ou_witness_not_discrete:

fixes X :: ed_ou_witness set

shows $\neg\text{discrete } X$

⟨proof⟩

lemma ed_ou_witness_ed_ou:

fixes X :: ed_ou_witness set

shows ed_ou X

⟨proof⟩

lemma ed_ou_witness_card:

shows card \{f. CK_nf_pos (f :: ed_ou_witness set \Rightarrow ed_ou_witness set)} = 4

⟨proof⟩
6.4 Extremally disconnected spaces

definition

extremally_disconnected \( X :: \text{topological_space set} \) \( \iff \) \( K \circ I \circ K = I \circ (K :: \text{a set} \Rightarrow \text{a set}) \)

lemma ed_on_part_extremally_disconnected:

assumes ed_on \( X \)

assumes part \( X \)

shows extremally_disconnected \( X \)

⟨proof⟩

lemma extremally_disconnected_eqs:

fixes \( X :: \text{topological_space set} \)

assumes extremally_disconnected \( X \)

shows \( I \circ K \circ I = K \circ (I :: \text{a set} \Rightarrow \text{a set}) \)

\( K \circ I \circ K = I \circ (K :: \text{a set} \Rightarrow \text{a set}) \)

⟨proof⟩

lemma extremally_disconnected_not_part_not_ed_on_card:

fixes \( X :: \text{topological_space set} \)

assumes extremally_disconnected \( X \)

assumes ~part \( X \)

assumes ~ed_on \( X \)

shows card \( \{f. \text{CK_nf_pos (f::a set} \Rightarrow \text{a set)}\} = 5 \)

⟨proof⟩

Any topological space having an infinite underlying set and whose topology consists of the empty set and every cofinite subset of the underlying set is extremally disconnected. We consider an example such space having a countably infinite underlying set.

datatype \( \text{a cofinite} = \text{cofinite} \text{'}a \)

instantiation \( \text{cofinite :: (type) topological_space} \)

begin

definition open_cofinite = (\( \lambda X::\text{a cofinite set}. \text{finite (}\neg X\text{)} \lor X = \{\})\)

instance

⟨proof⟩

end

lemma cofinite_closure_finite:

fixes \( X :: \text{a cofinite set} \)

assumes finite \( X \)

shows closure \( X = X \)

⟨proof⟩

lemma cofinite_closure_infinite:

fixes \( X :: \text{a cofinite set} \)

assumes infinite \( X \)

shows closure \( X = \text{UNIV} \)

⟨proof⟩

lemma cofinite_interior_finite:

fixes \( X :: \text{a cofinite set} \)

assumes finite \( X \)

assumes infinite (\( \text{UNIV::a cofinite set}\))
shows interior $X = \{\}$

lemma cofinite_interior_infinite:
  fixes $X :: 'a cofinite set$
  assumes infinite $X$
  assumes infinite $(-X)$
  shows interior $X = \{\}$

abbreviation evens :: nat cofinite set ≡ \{cfinite $n$ | $n \cdot 3 i. n=2*i}\$

lemma evens_infinite:
  shows infinite evens

lemma cofinite_nat_infinite:
  shows infinite $(UNIV::nat cofinite set)$

lemma evens_Compl_infinite:
  shows infinite $(- evens)$

lemma evens_closure:
  shows closure evens = UNIV

lemma evens_interior:
  shows interior evens = \{\}

lemma cofinite_not_part:
  fixes $X :: nat cofinite set$
  shows $\neg$part $X$

lemma cofinite_not_ed_ou:
  fixes $X :: nat cofinite set$
  shows $\neg$ed_ou $X$

lemma cofinite_extremally_disconnected_aux:
  fixes $X :: nat cofinite set$
  shows closure (interior (closure $X$)) ⊆ interior (closure $X$)

lemma cofinite_extremally_disconnected:
  fixes $X :: nat cofinite set$
  shows extremally_disconnected $X$

lemma cofinite_card:
  shows card \{f. CK_nf_pos (f::nat cofinite set ⇒ nat cofinite set)\} = 5
6.5 Open unresolvable spaces

definition
open_unresolvable (X :: 'a::topological_space set) ←→ K o I o K = K o (I :: 'a set ⇒ 'a set)

lemma ed_ou_open_unresolvable:
  assumes ed_ou X
  shows open_unresolvable X
⟨proof⟩

lemma open_unresolvable_eqs:
  assumes open_unresolvable (X :: 'a::topological_space set)
  shows
    I o K o I = I o (K :: 'a set ⇒ 'a set)
    K o I o K = K o (I :: 'a set ⇒ 'a set)
⟨proof⟩

lemma not_ed_negs:
  assumes ¬ed_ou (X :: 'a::topological_space set)
  shows
    I ≠ I o (K :: 'a set ⇒ 'a set)
    K ≠ K o (I :: 'a set ⇒ 'a set)
⟨proof⟩

lemma open_unresolvable_not_ed_card:
  assumes open_unresolvable (X :: 'a::topological_space set)
  assumes ¬ed_ou X
  shows
    card {f. CK_nf_pos (f::'a set ⇒ 'a set)} = 5
⟨proof⟩

We show that the class of open unresolvable spaces is non-empty by exhibiting an example of such a space.

datatype ou_witness = a | b | c

lemma ou_witness_UNIV:
  shows UNIV = set [a, b, c]
⟨proof⟩

instantiation ou_witness :: topological_space
begin

definition open_ou_witness X ←→ a ∉ X ∨ X = UNIV

instance
⟨proof⟩

end

lemma ou_witness_closure_simps:
  shows
    closure {a} = {a}
    closure {b} = {a, b}
    closure {c} = {a, c}
    closure {a, b} = {a, b}
    closure {a, c} = {a, c}
    closure {a, b, c} = {a, b, c}
    closure {b, c} = {a, b, c}
⟨proof⟩
lemma ou_witness_open_unresolvable:
  fixes X :: ou_witness set
  shows open_unresolvable X
  ⟨proof⟩

lemma ou_witness_not_ed_ou:
  fixes X :: ou_witness set
  shows ¬ed_ou X
  ⟨proof⟩

lemma ou_witness_card:
  shows card {f. CK_nf_pos (f::ou_witness set ⇒ ou_witness set)} = 5
  ⟨proof⟩

6.6 Kuratowski spaces

definition kuratowski (X :: 'a::topological_space set) ⇔
  ¬extremally_disconnected X ∧ ¬open_unresolvable X

A Kuratowski space distinguishes all 7 positive operators.

lemma part_closed_open:
  fixes X :: 'a::topological_space set
  assumes I ◦ K ◦ I = (I::'a set ⇒ 'a set)
  assumes closed X
  shows open X
  ⟨proof⟩

lemma part_I_K_I:
  assumes I ◦ K ◦ I = (I::'a::topological_space set ⇒ 'a set)
  shows I ◦ K = (K::'a set ⇒ 'a set)
  ⟨proof⟩

lemma part_K_I_I:
  assumes I ◦ K ◦ I = (I::'a::topological_space set ⇒ 'a set)
  shows K ◦ I = (I::'a set ⇒ 'a set)
  ⟨proof⟩

lemma kuratowski_neqs:
  assumes kuratowski (X :: 'a::topological_space set)
  shows
    I ≠ I ◦ K ◦ I (I :: 'a set ⇒ 'a set)
    I ◦ K ◦ I ≠ K ◦ (I :: 'a set ⇒ 'a set)
    I ◦ K ◦ I ≠ I ◦ (K :: 'a set ⇒ 'a set)
    I ◦ K ≠ K ◦ I ◦ (K :: 'a set ⇒ 'a set)
    K ◦ I ≠ K ◦ I ◦ (K :: 'a set ⇒ 'a set)
    K ◦ I ≠ K (K :: 'a set ⇒ 'a set)
    I ◦ K ≠ K ◦ I (I :: 'a set ⇒ 'a set)
    I ≠ (id :: 'a set ⇒ 'a set)
    K ≠ (id :: 'a set ⇒ 'a set)
    I ◦ K ◦ I ≠ (id :: 'a set ⇒ 'a set)
    K ◦ I ◦ K ≠ (id :: 'a set ⇒ 'a set)
  ⟨proof⟩

lemma kuratowski_card:
  assumes kuratowski (X :: 'a::topological_space set)
  shows card {f. CK_nf_pos (f::'a set ⇒ 'a set)} = 7
  ⟨proof⟩
\( \mathbb{R} \) is a Kuratowski space.

**lemma kuratowski_reals:**
- shows kuratowski \((\mathbb{R} :: real set)\)

### 6.7 Chagrov’s theorem

**theorem chagrov:**
- fixes \( X :: 'a::topological_space set \)
- obtains discrete \( X \)
  - \( \neg \)discrete \( X \land \) part \( X \)
  - \( \neg \)discrete \( X \land \) ed_ou \( X \)
  - \( \neg \)ed_ou \( X \land \) open_unresolvable \( X \)
  - \( \neg \)ed_ou \( X \land \neg \)part \( X \land extremally_disconnected \) \( X \)
- kuratowski \( X \)

**corollary chagrov_card:**
- shows \( \text{card} \{ f. \ CK_{nf_pos} (f::'a::topological_space set \Rightarrow 'a set) \} \in \{1,3,4,5,7\} \)

**References**


