The Kuratowski Closure-Complement Theorem

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Contents

1 Introduction

We discuss a topological curiosity discovered by Kuratowski (1922): the fact that the number of distinct operators on a topological space generated by compositions of closure and complement never exceeds 14, and is exactly 14 in the case of $\mathbb{R}$. In addition, we prove a theorem due to Chagrov (1982) that classifies topological spaces according to the number of such operators they support.

Kuratowski's result, which is exposited in Whitty (2015) and Chapter 7 of Chamberland (2015), has already been treated in Mizar — see Bagińska and Grabowski (2003) and Grabowski (2004). To the best of our knowledge, we are the first to mechanize Chagrov's result.

Our work is based on a presentation of Kuratowski's and Chagrov's results by Gardner and Jackson (2008).

We begin with some preliminary facts pertaining to the relationship between interiors of unions and unions of interiors (§2) and the relationship between $\mathbb{Q}$ and $\mathbb{R}$ (§3). We then prove Kuratowski's result (§4) and the corollary that at most 7 distinct operators on a topological space can be generated by compositions of closure and interior (§5). Finally, we prove Chagrov's result (§6).

2 Interiors and unions

definition
  boundary :: *'a::topological_space set* ⇒ *'a set*
where
  boundary X = *closure X* − *interior X*

lemma *boundary_empty*:
  shows *boundary* {} = {}
⟨*proof*⟩

definition
  exterior :: *'a::topological_space set* ⇒ *'a set*
where
  exterior X = − (*interior X* ∪ *boundary X*)

lemma *interior_union_boundary*:
  shows *interior* (*X* ∪ *Y*) = *interior X* ∪ *interior Y*
    ⟷ *boundary X* ∩ *boundary Y* ⊆ *boundary* (*X* ∪ *Y*) (**is** (*?lhs1* = *?lhs2*) ⟷ *?rhs*)
⟨*proof*⟩

lemma *interior_union_closed_intervals*:
  fixes *a* :: *'a::ordered_euclidean_space*
  assumes *b* < *c*
  shows *interior* ({*a..b*} ∪ {*c..d*}) = *interior* {*a..b*} ∪ *interior* {*c..d*}
⟨*proof*⟩

3 Additional facts about the rationals and reals

lemma *Rat_real_limpt*:
  fixes *x* :: *real*
  shows *x islimpt* ℚ
⟨*proof*⟩

lemma *Rat_closure*:
  shows *closure* ℚ = (*UNIV* :: *real set*)
⟨*proof*⟩

lemma *Rat_interval_closure*:
  fixes *x* :: *real*
  assumes *x* < *y*
  shows *closure* ({*x*<..<*y*} ∩ ℚ) = {*x..y*}
⟨*proof*⟩

lemma *Rat_not_open*:
  fixes *T* :: *real set*
  assumes *open T*
  assumes *T* ≠ {}
  shows ¬*T* ⊆ ℚ
⟨*proof*⟩

lemma *Irrat_dense_in_real*:
  fixes *x* :: *real*
  assumes *x* < *y*
  shows ∃ *r*∈−ℚ. *x* < *r* ∧ *r* < *y*
⟨*proof*⟩

lemma *closed_interval_Int_compl*:
  fixes *x* :: *real*
  assumes *x* < *y*
  assumes *y* < *z*
  shows − {*x..y*} ∩ − {*y..z*} = − {*x..z*}
⟨*proof*⟩

4 Kuratowski's result

We prove that at most 14 distinct operators can be generated by compositions of *closure* and complement. For convenience, we give these operators short names and try to avoid pointwise reasoning. We treat the *interior* operator at the same time.

declare *o_apply*[*simp del*]

definition C :: $'a$::*topological_space set* $\Rightarrow$ $'a$ *set* **where** $C\ X = -\ X$

definition K :: $'a$::*topological_space set* $\Rightarrow$ $'a$ *set* **where** $K\ X$ = *closure* X

definition I :: $'a$::*topological_space set* $\Rightarrow$ $'a$ *set* **where** $I\ X$ = *interior* X

lemma *C_C*:
  shows $C \circ C = id$
⟨*proof*⟩

lemma *K_K*:
  shows $K \circ K = K$
⟨*proof*⟩

lemma *I_I*:
  shows $I \circ I = I$
⟨*proof*⟩

lemma *I_K*:
  shows $I = C \circ K \circ C$
⟨*proof*⟩

lemma *K_I*:
  shows $K = C \circ I \circ C$
⟨*proof*⟩

lemma *K_I_K_I*:
  shows $K \circ I \circ K \circ I = K \circ I$
⟨*proof*⟩

lemma *I_K_I_K*:
  shows $I \circ K \circ I \circ K = I \circ K$
⟨*proof*⟩

lemma *K_mono*:
  assumes $x \subseteq y$
  shows $K\ x \subseteq K\ y$
⟨*proof*⟩

The following lemma embodies the crucial observation about compositions of C and K:

lemma *KCKCKCK_KCK*:
  shows $K \circ C \circ K \circ C \circ K \circ C \circ K = K \circ C \circ K$ (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

The inductive set CK captures all operators that can be generated by compositions of C and K. We shallowly embed the operators; that is, we identify operators up to extensional equality.

inductive CK :: ($'a$::*topological_space set* $\Rightarrow$ $'a$ *set*) $\Rightarrow$ *bool* **where**
  $CK\ C$
| $CK\ K$
| $[\![\ CK\ f;\ CK\ g\ ]\!] \Longrightarrow CK\ (f \circ g)$

declare $CK.intros[intro!]$

lemma $CK\_id[intro!]$:
  $CK\ id$
$\langle proof \rangle$

The inductive set CK_nf captures the normal forms for the 14 distinct operators.

inductive $CK\_nf :: ('a::topological\_space\ set \Rightarrow 'a\ set) \Rightarrow bool$ **where**
  $CK\_nf\ id$
| $CK\_nf\ C$
| $CK\_nf\ K$
| $CK\_nf\ (C \circ K)$
| $CK\_nf\ (K \circ C)$
| $CK\_nf\ (C \circ K \circ C)$
| $CK\_nf\ (K \circ C \circ K)$
| $CK\_nf\ (C \circ K \circ C \circ K)$
| $CK\_nf\ (K \circ C \circ K \circ C)$
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| $CK\_nf\ (K \circ C \circ K \circ C \circ K \circ C)$
| $CK\_nf\ (C \circ K \circ C \circ K \circ C \circ K \circ C)$

declare $CK\_nf.intros[intro!]$

lemma CK_nf_set:
  shows $\{f\ .\ CK\_nf\ f\} = \{id, C, K, C \circ K, K \circ C, C \circ K \circ C, K \circ C \circ K, C \circ K \circ C \circ K, K \circ C \circ K \circ C, C \circ K \circ C \circ K \circ C, K \circ C \circ K \circ C \circ K, C \circ K \circ C \circ K \circ C \circ K, K \circ C \circ K \circ C \circ K \circ C, C \circ K \circ C \circ K \circ C \circ K \circ C\}$
$\langle proof \rangle$

That each operator generated by compositions of C and K is extensionally equivalent to one of the normal forms captured by CK_nf is demonstrated by means of an induction over the construction of CK_nf and an appeal to the facts proved above.

theorem CK_nf:
  $CK\ f \longleftrightarrow CK\_nf\ f$
$\langle proof \rangle$

theorem CK_card:
  shows $card\ \{f.\ CK\ f\} \leq 14$
$\langle proof \rangle$

We show, using the following subset of $\mathbb{R}$ (an example taken from Rusin (2001)) as a witness, that there exist topological spaces on which all 14 operators are distinct.

definition
  $RRR :: real\ set$
where
  $RRR = \{0<..<1\} \cup \{1<..<2\} \cup \{3\} \cup (\{5<..<7\} \cap \mathbb{Q})$

The following facts allow the required proofs to proceed by *simp*:

lemma $RRR\_closure$:
  shows $closure\ RRR = \{0..2\} \cup \{3\} \cup \{5..7\}$
$\langle proof \rangle$

lemma $RRR\_interior$:
  $interior\ RRR = \{0<..<1\} \cup \{1<..<2\}$ (**is** $?lhs = ?rhs$)
$\langle proof \rangle$

lemma *RRR_interior_closure[simplified]*:
shows *interior* ({*0::real..2*} ∪ {*3*} ∪ {*5..7*}) = {*0<..<2*} ∪ {*5<..<7*} (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

The operators can be distinguished by testing which of the points in {*1*,*2*,*3*,*4*,*6*} belong to their results.

definition
test :: (*real set* ⇒ *real set*) ⇒ *bool list*
where
test f ≡ *map* (λ*x*. *x* ∈ *f RRR*) [*1*,*2*,*3*,*4*,*6*]

lemma *RRR_test*:
assumes *f RRR* = *g RRR*
shows *test f* = *test g*
⟨*proof*⟩

lemma *nf_RRR*:
shows
test id = [*False*, *False*, *True*, *False*, *True*]
test C = [*True*, *True*, *False*, *True*, *False*]
test K = [*True*, *True*, *True*, *False*, *True*]
test (*K* ∘ *C*) = [*True*, *True*, *True*, *True*, *True*]
test (*C* ∘ *K*) = [*False*, *False*, *False*, *True*, *False*]
test (*C* ∘ *K* ∘ *C*) = [*False*, *False*, *False*, *False*, *False*]
test (*K* ∘ *C* ∘ *K*) = [*False*, *True*, *True*, *True*, *False*]
test (*C* ∘ *K* ∘ *C* ∘ *K*) = [*True*, *False*, *False*, *False*, *True*]
test (*K* ∘ *C* ∘ *K* ∘ *C*) = [*True*, *True*, *False*, *False*, *False*]
test (*C* ∘ *K* ∘ *C* ∘ *K* ∘ *C*) = [*False*, *False*, *True*, *True*, *True*]
test (*K* ∘ *C* ∘ *K* ∘ *C* ∘ *K*) = [*True*, *True*, *False*, *False*, *True*]
test (*C* ∘ *K* ∘ *C* ∘ *K* ∘ *C* ∘ *K*) = [*False*, *False*, *True*, *True*, *False*]
test (*K* ∘ *C* ∘ *K* ∘ *C* ∘ *K* ∘ *C*) = [*False*, *True*, *True*, *True*, *True*]
test (*C* ∘ *K* ∘ *C* ∘ *K* ∘ *C* ∘ *K* ∘ *C*) = [*True*, *False*, *False*, *False*, *False*]
⟨*proof*⟩

theorem *CK_nf_real_card*:
shows *card* ((λ *f*. *f RRR*) ‘ {*f* . *CK_nf f*}) = *14*
⟨*proof*⟩

theorem *CK_real_card*:
shows *card* {*f::real set* ⇒ *real set*. *CK f*} = *14* (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

5 A corollary of Kuratowski's result

We show that it is a corollary of *CK_real_card* that at most 7 distinct operators on a topological space can be generated by compositions of closure and interior. In the case of ℝ, exactly 7 distinct operators can be so generated.

inductive *IK* :: (*'a::topological_space set* ⇒ *'a set*) ⇒ *bool* **where**
IK id
| *IK I*
| *IK K*
| ⟦ *IK f*; *IK g* ⟧ ⟹ *IK* (*f* ∘ *g*)

inductive *IK_nf* :: (*'a::topological_space set* ⇒ *'a set*) ⇒ *bool* **where**
IK_nf id
| *IK_nf I*
| *IK_nf K*
| *IK_nf* (*I* ∘ *K*)

| *IK_nf* $(K \circ I)$
| *IK_nf* $(I \circ K \circ I)$
| *IK_nf* $(K \circ I \circ K)$

declare *IK.intros*[*intro*!]

declare *IK_nf.intros*[*intro*!]

lemma *IK_nf_set*:
  $\{f\ .\ IK\_nf\ f\} = \{id, I, K, I \circ K, K \circ I, I \circ K \circ I, K \circ I \circ K\}$
⟨*proof*⟩

theorem *IK_nf*:
  $IK\ f \longleftrightarrow IK\_nf\ f$
⟨*proof*⟩

theorem *IK_card*:
  shows $card\ \{f.\ IK\ f\} \leq 7$
⟨*proof*⟩

theorem *IK_nf_real_card*:
  shows $card\ ((\lambda\ f.\ f\ RRR)\ `\ \{f\ .\ IK\_nf\ f\}) = 7$
⟨*proof*⟩

theorem *IK_real_card*:
  shows $card\ \{f{::}real\ set \Rightarrow real\ set.\ IK\ f\} = 7$ (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

6 Chagrov's result

Chagrov's theorem, which is discussed in Section 2.1 of Gardner and Jackson (2008), states that the number of distinct operators on a topological space that can be generated by compositions of closure and complement is one of 2, 6, 8, 10 or 14.

We begin by observing that the set of normal forms *CK_nf* can be split into two disjoint sets, *CK_nf_pos* and *CK_nf_neg*, which we define in terms of interior and closure.

inductive *CK_nf_pos* :: ($'a$::*topological_space set* ⇒ $'a$ *set*) ⇒ *bool* **where**
  CK_nf_pos id
| *CK_nf_pos I*
| *CK_nf_pos K*
| *CK_nf_pos* $(I \circ K)$
| *CK_nf_pos* $(K \circ I)$
| *CK_nf_pos* $(I \circ K \circ I)$
| *CK_nf_pos* $(K \circ I \circ K)$

declare *CK_nf_pos.intros*[*intro*!]

lemma *CK_nf_pos_set*:
  shows $\{f\ .\ CK\_nf\_pos\ f\} = \{id, I, K, I \circ K, K \circ I, I \circ K \circ I, K \circ I \circ K\}$
⟨*proof*⟩

definition
  CK_nf_neg :: ($'a$::*topological_space set* ⇒ $'a$ *set*) ⇒ *bool*
where
  $CK\_nf\_neg\ f \longleftrightarrow (\exists\, g.\ CK\_nf\_pos\ g \wedge f = C \circ g)$

lemma *CK_nf_pos_neg_disjoint*:
  assumes *CK_nf_pos f*

assumes *CK_nf_neg g*
shows $f \neq g$
⟨*proof*⟩

lemma *CK_nf_pos_neg_CK_nf*:
CK_nf f ⟷ *CK_nf_pos f* ∨ *CK_nf_neg f* (**is** *?lhs* ⟷ *?rhs*)
⟨*proof*⟩

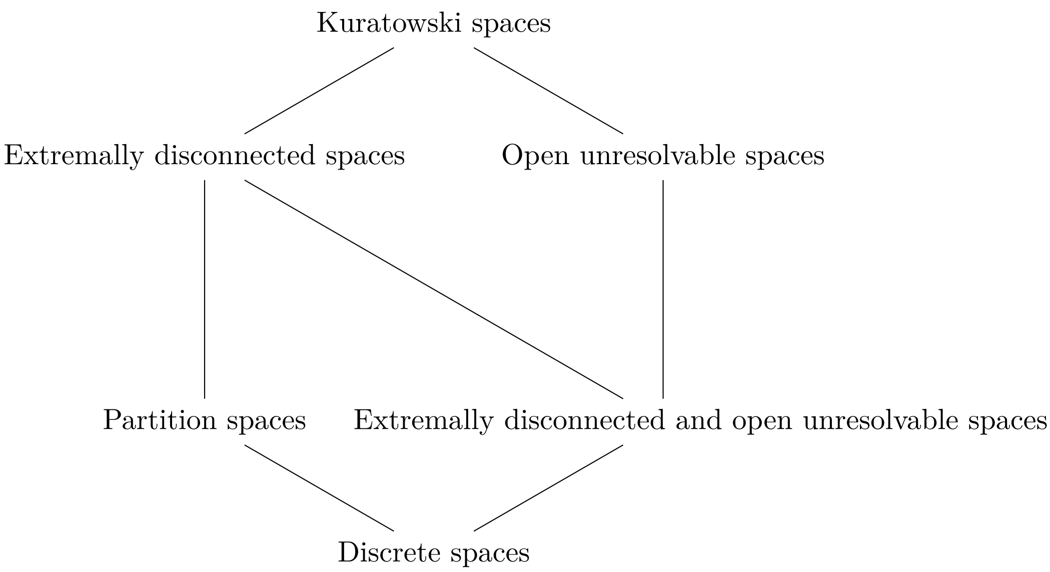
We now focus on *CK_nf_pos*. In particular, we show that its cardinality for any given topological space is one of 1, 3, 4, 5 or 7.

The proof consists of exhibiting normal forms for the operators supported by each of six classes of topological spaces. These are sublattices of the following lattice of *CK_nf_pos* operators:

lemmas *K_I_K_subseteq_K* = *closure_mono*[*OF interior_subset*, *of closure X*, *simplified*] **for** *X*

lemma *CK_nf_pos_lattice*:
shows
$I \le (id :: 'a::topological\_space\ set \Rightarrow 'a\ set)$
$id \le (K :: 'a::topological\_space\ set \Rightarrow 'a\ set)$
$I \le I \circ K \circ (I :: 'a::topological\_space\ set \Rightarrow 'a\ set)$
$I \circ K \circ I \le I \circ (K :: 'a::topological\_space\ set \Rightarrow 'a\ set)$
$I \circ K \circ I \le K \circ (I :: 'a::topological\_space\ set \Rightarrow 'a\ set)$
$I \circ K \le K \circ I \circ (K :: 'a::topological\_space\ set \Rightarrow 'a\ set)$
$K \circ I \le K \circ I \circ (K :: 'a::topological\_space\ set \Rightarrow 'a\ set)$
$K \circ I \circ K \le (K :: 'a::topological\_space\ set \Rightarrow 'a\ set)$
⟨*proof*⟩

We define the six classes of topological spaces in question, and show that they are related by inclusion in the following way (as shown in Figure 2.3 of Gardner and Jackson (2008)):



6.1 Discrete spaces

definition
discrete $(X :: 'a::topological\_space\ set) \longleftrightarrow I = (id::'a\ set \Rightarrow 'a\ set)$

lemma *discrete_eqs*:
assumes *discrete* $(X :: 'a::topological\_space\ set)$
shows
$I = (id::'a\ set \Rightarrow 'a\ set)$
$K = (id::'a\ set \Rightarrow 'a\ set)$
⟨*proof*⟩

lemma *discrete_card*:
  assumes *discrete* (*X* :: ′*a*::*topological_space set*)
  shows *card* {*f*. *CK_nf_pos* (*f*::′*a set* ⇒ ′*a set*)} = *1*
⟨*proof*⟩

lemma *discrete_discrete_topology*:
  fixes *X* :: ′*a*::*topological_space set*
  assumes ⋀*Y*::′*a set*. *open Y*
  shows *discrete X*
⟨*proof*⟩

6.2 Partition spaces

definition
  part (*X* :: ′*a*::*topological_space set*) ⟷ *K* ∘ *I* = (*I* :: ′*a set* ⇒ ′*a set*)

lemma *discrete_part*:
  assumes *discrete X*
  shows *part X*
⟨*proof*⟩

lemma *part_eqs*:
  assumes *part* (*X* :: ′*a*::*topological_space set*)
  shows
    K ∘ *I* = (*I* :: ′*a set* ⇒ ′*a set*)
    I ∘ *K* = (*K* :: ′*a set* ⇒ ′*a set*)
⟨*proof*⟩

lemma *part_not_discrete_card*:
  assumes *part* (*X* :: ′*a*::*topological_space set*)
  assumes ¬*discrete X*
  shows *card* {*f*. *CK_nf_pos* (*f*::′*a set* ⇒ ′*a set*)} = *3*
⟨*proof*⟩

A partition space is a topological space whose basis consists of the empty set and the equivalence classes of points of the space induced by some equivalence relation R on the underlying set of the space. Equivalently, a partition space is one in which every open set is closed. Thus, for example, the class of partition spaces includes every topological space whose open sets form a boolean algebra.

datatype *part_witness* = *a* | *b* | *c*

lemma *part_witness_UNIV*:
  shows *UNIV* = *set* [*a*, *b*, *c*]
⟨*proof*⟩

lemmas *part_witness_pow* = *subset_subseqs*[*OF subset_trans*[*OF subset_UNIV Set.equalityD1* [*OF part_witness_UNIV*

lemmas *part_witness_Compl* = *Compl_eq_Diff_UNIV*[**where** ′*a*=*part_witness*, *unfolded part_witness_UNIV*, *simplified*]

instantiation *part_witness* :: *topological_space*
begin

definition *open_part_witness X* ⟷ *X* ∈ {{}, {*a*}, {*b*, *c*}, {*a*, *b*, *c*}}

lemma *part_witness_ball*:
  (∀ *s*∈*S*. *s* ∈ {{}, {*a*}, {*b*, *c*}, {*a*, *b*, *c*}}) ⟷ *S* ⊆ *set* [{}, {*a*}, {*b*, *c*}, {*a*, *b*, *c*}]
⟨*proof*⟩

lemmas *part_witness_subsets_pow* = *subset_subseqs*[*OF iffD1*[*OF part_witness_ball*]]

instance ⟨*proof*⟩

end

lemma *part_witness_interior_simps*:
  shows
    interior {*a*} = {*a*}
    interior {*b*} = {}
    interior {*c*} = {}
    interior {*a*, *b*} = {*a*}
    interior {*a*, *c*} = {*a*}
    interior {*b*, *c*} = {*b*, *c*}
    interior {*a*, *b*, *c*} = {*a*, *b*, *c*}
⟨*proof*⟩

lemma *part_witness_part*:
  fixes *X* :: *part_witness set*
  shows *part X*
⟨*proof*⟩

lemma *part_witness_not_discrete*:
  fixes *X* :: *part_witness set*
  shows ¬*discrete X*
⟨*proof*⟩

lemma *part_witness_card*:
  shows *card* {*f*. *CK_nf_pos* (*f*::*part_witness set* ⇒ *part_witness set*)} = *3*
⟨*proof*⟩

6.3 Extremally disconnected and open unresolvable spaces

definition
  ed_ou (*X* :: ′*a*::*topological_space set*) ⟷ *I* ∘ *K* = *K* ∘ (*I* :: ′*a set* ⇒ ′*a set*)

lemma *discrete_ed_ou*:
  assumes *discrete X*
  shows *ed_ou X*
⟨*proof*⟩

lemma *ed_ou_eqs*:
  assumes *ed_ou* (*X* :: ′*a*::*topological_space set*)
  shows
    I ∘ *K* ∘ *I* = *K* ∘ (*I* :: ′*a set* ⇒ ′*a set*)
    K ∘ *I* ∘ *K* = *K* ∘ (*I* :: ′*a set* ⇒ ′*a set*)
    I ∘ *K* = *K* ∘ (*I* :: ′*a set* ⇒ ′*a set*)
⟨*proof*⟩

lemma *ed_ou_neqs*:
  assumes *ed_ou* (*X* :: ′*a*::*topological_space set*)
  assumes ¬*discrete X*
  shows
    I ≠ (*K* :: ′*a set* ⇒ ′*a set*)
    I ≠ *K* ∘ (*I* :: ′*a set* ⇒ ′*a set*)
    K ≠ *K* ∘ (*I* :: ′*a set* ⇒ ′*a set*)
    I ≠ (*id* :: ′*a set* ⇒ ′*a set*)

$K \neq (id :: 'a\ set \Rightarrow 'a\ set)$
⟨*proof*⟩

lemma *ed_ou_not_discrete_card*:
assumes *ed_ou* (*X* :: *'a::topological_space set*)
assumes ¬*discrete X*
shows *card* {*f*. *CK_nf_pos* (*f*::*'a set* ⇒ *'a set*)} = *4*
⟨*proof*⟩

We consider an example extremally disconnected and open unresolvable topological space.

datatype *ed_ou_witness* = *a* | *b* | *c* | *d* | *e*

lemma *ed_ou_witness_UNIV*:
shows *UNIV* = *set* [*a*, *b*, *c*, *d*, *e*]
⟨*proof*⟩

lemmas *ed_ou_witness_pow* = *subset_subseqs*[*OF subset_trans*[*OF subset_UNIV Set.equalityD1*[*OF ed_ou_witness_U*

lemmas *ed_ou_witness_Compl* = *Compl_eq_Diff_UNIV*[**where** *'a=ed_ou_witness*, *unfolded ed_ou_witness_UNIV*, *simplified*]

instance *ed_ou_witness* :: *finite*
⟨*proof*⟩

instantiation *ed_ou_witness* :: *topological_space*
begin

inductive *open_ed_ou_witness* :: *ed_ou_witness set* ⇒ *bool* **where**
open_ed_ou_witness {}
| *open_ed_ou_witness* {*a*}
| *open_ed_ou_witness* {*b*}
| *open_ed_ou_witness* {*e*}
| *open_ed_ou_witness* {*a*, *c*}
| *open_ed_ou_witness* {*b*, *d*}
| *open_ed_ou_witness* {*a*, *c*, *e*}

| *open_ed_ou_witness* {*a*, *b*}
| *open_ed_ou_witness* {*a*, *e*}
| *open_ed_ou_witness* {*b*, *e*}
| *open_ed_ou_witness* {*a*, *b*, *c*}
| *open_ed_ou_witness* {*a*, *b*, *d*}
| *open_ed_ou_witness* {*a*, *b*, *e*}
| *open_ed_ou_witness* {*b*, *d*, *e*}
| *open_ed_ou_witness* {*a*, *b*, *c*, *d*}
| *open_ed_ou_witness* {*a*, *b*, *c*, *e*}
| *open_ed_ou_witness* {*a*, *b*, *d*, *e*}
| *open_ed_ou_witness* {*a*, *b*, *c*, *d*, *e*}

declare *open_ed_ou_witness.intros*[*intro*!]

lemma *ed_ou_witness_inter*:
fixes *S* :: *ed_ou_witness set*
assumes *open S*
assumes *open T*
shows *open* (*S* ∩ *T*)
⟨*proof*⟩

lemma *ed_ou_witness_union*:

fixes X :: *ed_ou_witness set set*
assumes $\forall x \in X.$ *open* x
shows *open* $(\bigcup X)$
⟨*proof*⟩

instance
⟨*proof*⟩

end

lemma *ed_ou_witness_interior_simps*:
shows
interior $\{a\} = \{a\}$
interior $\{b\} = \{b\}$
interior $\{c\} = \{\}$
interior $\{d\} = \{\}$
interior $\{e\} = \{e\}$
interior $\{a, b\} = \{a, b\}$
interior $\{a, c\} = \{a, c\}$
interior $\{a, d\} = \{a\}$
interior $\{a, e\} = \{a, e\}$
interior $\{b, c\} = \{b\}$
interior $\{b, d\} = \{b, d\}$
interior $\{b, e\} = \{b, e\}$
interior $\{c, d\} = \{\}$
interior $\{c, e\} = \{e\}$
interior $\{d, e\} = \{e\}$
interior $\{a, b, c\} = \{a, b, c\}$
interior $\{a, b, d\} = \{a, b, d\}$
interior $\{a, b, e\} = \{a, b, e\}$
interior $\{a, c, d\} = \{a, c\}$
interior $\{a, c, e\} = \{a, c, e\}$
interior $\{a, d, e\} = \{a, e\}$
interior $\{b, c, d\} = \{b, d\}$
interior $\{b, c, e\} = \{b, e\}$
interior $\{b, d, e\} = \{b, d, e\}$
interior $\{c, d, e\} = \{e\}$
interior $\{a, b, c, d\} = \{a, b, c, d\}$
interior $\{a, b, c, e\} = \{a, b, c, e\}$
interior $\{a, b, d, e\} = \{a, b, d, e\}$
interior $\{a, b, c, d, e\} = \{a, b, c, d, e\}$
interior $\{a, c, d, e\} = \{a, c, e\}$
interior $\{b, c, d, e\} = \{b, d, e\}$
⟨*proof*⟩

lemma *ed_ou_witness_not_discrete*:
fixes X :: *ed_ou_witness set*
shows ¬*discrete* X
⟨*proof*⟩

lemma *ed_ou_witness_ed_ou*:
fixes X :: *ed_ou_witness set*
shows *ed_ou* X
⟨*proof*⟩

lemma *ed_ou_witness_card*:
shows *card* {*f*. *CK_nf_pos* (*f*::*ed_ou_witness set* ⇒ *ed_ou_witness set*)} = *4*
⟨*proof*⟩

6.4 Extremally disconnected spaces

definition
extremally_disconnected (X :: $'a$::*topological_space set*) $\longleftrightarrow$ $K \circ I \circ K = I \circ (K :: 'a\ set \Rightarrow 'a\ set)$

lemma *ed_ou_part_extremally_disconnected*:
assumes *ed_ou X*
assumes *part X*
shows *extremally_disconnected X*
⟨*proof*⟩

lemma *extremally_disconnected_eqs*:
fixes X :: $'a$::*topological_space set*
assumes *extremally_disconnected X*
shows
$I \circ K \circ I = K \circ (I :: 'a\ set \Rightarrow 'a\ set)$
$K \circ I \circ K = I \circ (K :: 'a\ set \Rightarrow 'a\ set)$
⟨*proof*⟩

lemma *extremally_disconnected_not_part_not_ed_ou_card*:
fixes X :: $'a$::*topological_space set*
assumes *extremally_disconnected X*
assumes ¬*part X*
assumes ¬*ed_ou X*
shows *card* {*f*. *CK_nf_pos* (*f*::$'a\ set \Rightarrow 'a\ set$)} = *5*
⟨*proof*⟩

Any topological space having an infinite underlying set and whose topology consists of the empty set and every cofinite subset of the underlying set is extremally disconnected. We consider an example such space having a countably infinite underlying set.

datatype $'a$ *cofinite* = *cofinite* $'a$

instantiation *cofinite* :: (*type*) *topological_space*
begin

definition *open_cofinite* = (λX::$'a$ *cofinite set*. *finite* ($-X$) ∨ X = {})

instance
⟨*proof*⟩

end

lemma *cofinite_closure_finite*:
fixes X :: $'a$ *cofinite set*
assumes *finite X*
shows *closure X* = X
⟨*proof*⟩

lemma *cofinite_closure_infinite*:
fixes X :: $'a$ *cofinite set*
assumes *infinite X*
shows *closure X* = *UNIV*
⟨*proof*⟩

lemma *cofinite_interior_finite*:
fixes X :: $'a$ *cofinite set*
assumes *finite X*
assumes *infinite* (*UNIV*::$'a$ *cofinite set*)

shows *interior X* = {}
⟨*proof*⟩

lemma *cofinite_interior_infinite*:
fixes *X* :: *'a cofinite set*
assumes *infinite X*
assumes *infinite* (−*X*)
shows *interior X* = {}
⟨*proof*⟩

abbreviation *evens* :: *nat cofinite set* ≡ {*cofinite n* | *n*. ∃ *i*. *n*=*2*∗*i*}

lemma *evens_infinite*:
shows *infinite evens*
⟨*proof*⟩

lemma *cofinite_nat_infinite*:
shows *infinite* (*UNIV*::*nat cofinite set*)
⟨*proof*⟩

lemma *evens_Compl_infinite*:
shows *infinite* (− *evens*)
⟨*proof*⟩

lemma *evens_closure*:
shows *closure evens* = *UNIV*
⟨*proof*⟩

lemma *evens_interior*:
shows *interior evens* = {}
⟨*proof*⟩

lemma *cofinite_not_part*:
fixes *X* :: *nat cofinite set*
shows ¬*part X*
⟨*proof*⟩

lemma *cofinite_not_ed_ou*:
fixes *X* :: *nat cofinite set*
shows ¬*ed_ou X*
⟨*proof*⟩

lemma *cofinite_extremally_disconnected_aux*:
fixes *X* :: *nat cofinite set*
shows *closure* (*interior* (*closure X*)) ⊆ *interior* (*closure X*)
⟨*proof*⟩

lemma *cofinite_extremally_disconnected*:
fixes *X* :: *nat cofinite set*
shows *extremally_disconnected X*
⟨*proof*⟩

lemma *cofinite_card*:
shows *card* {*f*. *CK_nf_pos* (*f*::*nat cofinite set* ⇒ *nat cofinite set*)} = *5*
⟨*proof*⟩

6.5 Open unresolvable spaces

definition
  open_unresolvable $(X :: 'a::topological\_space\ set) \longleftrightarrow K \circ I \circ K = K \circ (I :: 'a\ set \Rightarrow 'a\ set)$

lemma *ed_ou_open_unresolvable*:
  assumes *ed_ou X*
  shows *open_unresolvable X*
⟨*proof*⟩

lemma *open_unresolvable_eqs*:
  assumes *open_unresolvable* $(X :: 'a::topological\_space\ set)$
  shows
    $I \circ K \circ I = I \circ (K :: 'a\ set \Rightarrow 'a\ set)$
    $K \circ I \circ K = K \circ (I :: 'a\ set \Rightarrow 'a\ set)$
⟨*proof*⟩

lemma *not_ed_ou_neqs*:
  assumes $\neg$*ed_ou* $(X :: 'a::topological\_space\ set)$
  shows
    $I \neq I \circ (K :: 'a\ set \Rightarrow 'a\ set)$
    $K \neq K \circ (I :: 'a\ set \Rightarrow 'a\ set)$
⟨*proof*⟩

lemma *open_unresolvable_not_ed_ou_card*:
  assumes *open_unresolvable* $(X :: 'a::topological\_space\ set)$
  assumes $\neg$*ed_ou X*
  shows *card* $\{f.\ CK\_nf\_pos\ (f::'a\ set \Rightarrow 'a\ set)\} = 5$
⟨*proof*⟩

We show that the class of open unresolvable spaces is non-empty by exhibiting an example of such a space.

datatype *ou_witness* $= a \mid b \mid c$

lemma *ou_witness_UNIV*:
  shows $UNIV = set\ [a,\ b,\ c]$
⟨*proof*⟩

instantiation *ou_witness* :: *topological_space*
begin

definition *open_ou_witness* $X \longleftrightarrow a \notin X \lor X = UNIV$

instance
⟨*proof*⟩

end

lemma *ou_witness_closure_simps*:
  shows
    $closure\ \{a\} = \{a\}$
    $closure\ \{b\} = \{a,\ b\}$
    $closure\ \{c\} = \{a,\ c\}$
    $closure\ \{a,\ b\} = \{a,\ b\}$
    $closure\ \{a,\ c\} = \{a,\ c\}$
    $closure\ \{a,\ b,\ c\} = \{a,\ b,\ c\}$
    $closure\ \{b,\ c\} = \{a,\ b,\ c\}$
⟨*proof*⟩

lemma *ou_witness_open_unresolvable*:
  fixes X :: *ou_witness set*
  shows *open_unresolvable X*
⟨*proof*⟩

lemma *ou_witness_not_ed_ou*:
  fixes X :: *ou_witness set*
  shows ¬*ed_ou X*
⟨*proof*⟩

lemma *ou_witness_card*:
  shows *card* {*f*. *CK_nf_pos* (*f*::*ou_witness set* ⇒ *ou_witness set*)} = *5*
⟨*proof*⟩

6.6 Kuratowski spaces

definition
  kuratowski (X :: ′*a*::*topological_space set*) ⟷
    ¬*extremally_disconnected X* ∧ ¬*open_unresolvable X*

A Kuratowski space distinguishes all 7 positive operators.

lemma *part_closed_open*:
  fixes X :: ′*a*::*topological_space set*
  assumes $I \circ K \circ I$ = (I::′*a set* ⇒ ′*a set*)
  assumes *closed X*
  shows *open X*
⟨*proof*⟩

lemma *part_I_K_I*:
  assumes $I \circ K \circ I$ = (I::′*a*::*topological_space set* ⇒ ′*a set*)
  shows $I \circ K$ = (K::′*a set* ⇒ ′*a set*)
⟨*proof*⟩

lemma *part_K_I_I*:
  assumes $I \circ K \circ I$ = (I::′*a*::*topological_space set* ⇒ ′*a set*)
  shows $K \circ I$ = (I::′*a set* ⇒ ′*a set*)
⟨*proof*⟩

lemma *kuratowski_neqs*:
  assumes *kuratowski* (X :: ′*a*::*topological_space set*)
  shows
    $I \neq I \circ K \circ$ (I :: ′*a set* ⇒ ′*a set*)
    $I \circ K \circ I \neq K \circ$ (I :: ′*a set* ⇒ ′*a set*)
    $I \circ K \circ I \neq I \circ$ (K :: ′*a set* ⇒ ′*a set*)
    $I \circ K \neq K \circ I \circ$ (K :: ′*a set* ⇒ ′*a set*)
    $K \circ I \neq K \circ I \circ$ (K :: ′*a set* ⇒ ′*a set*)
    $K \circ I \circ K \neq$ (K :: ′*a set* ⇒ ′*a set*)
    $I \circ K \neq K \circ$ (I :: ′*a set* ⇒ ′*a set*)
    $I \neq$ (*id* :: ′*a set* ⇒ ′*a set*)
    $K \neq$ (*id* :: ′*a set* ⇒ ′*a set*)
    $I \circ K \circ I \neq$ (*id* :: ′*a set* ⇒ ′*a set*)
    $K \circ I \circ K \neq$ (*id* :: ′*a set* ⇒ ′*a set*)
⟨*proof*⟩

lemma *kuratowski_card*:
  assumes *kuratowski* (X :: ′*a*::*topological_space set*)
  shows *card* {*f*. *CK_nf_pos* (*f*::′*a set* ⇒ ′*a set*)} = *7*
⟨*proof*⟩

$\mathbb{R}$ is a Kuratowski space.

lemma *kuratowski_reals*:
  shows *kuratowski* ($\mathbb{R}$ :: *real set*)
⟨*proof*⟩

6.7 Chagrov's theorem

theorem *chagrov*:
  fixes *X* :: *'a::topological_space set*
  obtains *discrete X*
    | ¬*discrete X* ∧ *part X*
    | ¬*discrete X* ∧ *ed_ou X*
    | ¬*ed_ou X* ∧ *open_unresolvable X*
    | ¬*ed_ou X* ∧ ¬*part X* ∧ *extremally_disconnected X*
    | *kuratowski X*
⟨*proof*⟩

corollary *chagrov_card*:
  shows *card* {*f*. *CK_nf_pos* (*f*::*'a::topological_space set* ⇒ *'a set*)} ∈ {*1*,*3*,*4*,*5*,*7*}
⟨*proof*⟩

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