# The Kuratowski Closure-Complement Theorem 

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## 1 Introduction

We discuss a topological curiosity discovered by Kuratowski (1922): the fact that the number of distinct operators on a topological space generated by compositions of closure and complement never exceeds 14 , and is exactly 14 in the case of $\mathbb{R}$. In addition, we prove a theorem due to Chagrov (1982) that classifies topological spaces according to the number of such operators they support.
Kuratowski's result, which is exposited in Whitty (2015) and Chapter 7 of Chamberland (2015), has already been treated in Mizar - see Bagińska and Grabowski (2003) and Grabowski (2004). To the best of our knowledge, we are the first to mechanize Chagrov's result.
Our work is based on a presentation of Kuratowski's and Chagrov's results by Gardner and Jackson (2008).
We begin with some preliminary facts pertaining to the relationship between interiors of unions and unions of interiors (§2) and the relationship between $\mathbb{Q}$ and $\mathbb{R}(\S 3)$. We then prove Kuratowski's result (§4) and the corollary that at most 7 distinct operators on a topological space can be generated by compositions of closure and interior (§5). Finally, we prove Chagrov's result (§6).

## 2 Interiors and unions

```
definition
    boundary :: 'a::topological_space set }=>\mathrm{ ''a set
where
    boundary X = closure X - interior X
```

lemma boundary_empty:
shows boundary $\}=\{ \}$
$\langle$ proof $\rangle$

## definition

exterior :: 'a::topological_space set $\Rightarrow$ ' $a$ set
where
exterior $X=-($ interior $X \cup$ boundary $X)$
lemma interior_union_boundary:
shows interior $(X \cup Y)=$ interior $X \cup$ interior $Y$
$\longleftrightarrow$ boundary $X \cap$ boundary $Y \subseteq$ boundary $(X \cup Y)($ is $($ ?lhs $1=$ ?lhs2) $\longleftrightarrow$ ?rhs)
$\langle p r o o f\rangle$
lemma interior_union_closed_intervals:
fixes $a::$ 'a::ordered_euclidean_space
assumes $b<c$
shows interior $(\{a . . b\} \cup\{c . . d\})=$ interior $\{a . . b\} \cup$ interior $\{c . . d\}$
$\langle$ proof $\rangle$

## 3 Additional facts about the rationals and reals

lemma Rat_real_limpt:
fixes $x$ :: real
shows $x$ islimpt $\mathbb{Q}$
$\langle$ proof $\rangle$
lemma Rat_closure:
shows closure $\mathbb{Q}=($ UNIV :: real set $)$
$\langle p r o o f\rangle$
lemma Rat_interval_closure:
fixes $x:$ real
assumes $x<y$
shows closure $(\{x<. .<y\} \cap \mathbb{Q})=\{x . . y\}$
$\langle p r o o f\rangle$
lemma Rat_not_open:
fixes $T$ :: real set
assumes open $T$
assumes $T \neq\{ \}$
shows $\neg T \subseteq \mathbb{Q}$
$\langle$ proof $\rangle$
lemma Irrat_dense_in_real:
fixes $x:$ real
assumes $x<y$
shows $\exists r \in-\mathbb{Q}$. $x<r \wedge r<y$
〈proof〉
lemma closed_interval_Int_compl:
fixes $x:$ real
assumes $x<y$
assumes $y<z$
shows $-\{x . . y\} \cap-\{y . . z\}=-\{x . . z\}$
$\langle$ proof $\rangle$

## 4 Kuratowski＇s result

We prove that at most 14 distinct operators can be generated by compositions of closure and complement．For convenience，we give these operators short names and try to avoid pointwise reasoning．We treat the interior operator at the same time．
declare o＿apply［simp del］
definition $C$ ：：＇a：：topological＿space set $\Rightarrow$＇a set where $C X=-X$
definition $K$ ：：＇a：：topological＿space set $\Rightarrow$＇$a$ set where $K X=$ closure $X$
definition $I$ ：：＇$a:$ ：topological＿space set $\Rightarrow$＇$a$ set where $I X=$ interior $X$
lemma $C \_C$ ：
shows $C \circ C=i d$
〈proof〉
lemma $K \_K$ ：
shows $K \circ K=K$
〈proof〉
lemma $I \_I$ ：
shows $I \circ I=I$
〈proof〉
lemma $I \_K$ ：
shows $I=C \circ K \circ C$
〈proof〉
lemma $K \_I$ ：
shows $K=C \circ I \circ C$
〈proof〉
lemma $K \_I \_K \_I$ ：
shows $K \circ I \circ K \circ I=K \circ I$
〈proof〉
lemma $I \_K \_I \_K$ ：
shows $I \circ K \circ I \circ K=I \circ K$
$\langle p r o o f\rangle$
lemma $K \_m o n o$ ：
assumes $x \subseteq y$
shows $K x \subseteq K y$
〈proof〉
The following lemma embodies the crucial observation about compositions of $C$ and $K$ ：
lemma KCKCKCK＿KCK：
shows $K \circ C \circ K \circ C \circ K \circ C \circ K=K \circ C \circ K$（is ？lhs $=$ ？？$r$ ss $)$
$\langle p r o o f\rangle$
The inductive set $C K$ captures all operators that can be generated by compositions of $C$ and $K$ ．We shallowly embed the operators；that is，we identify operators up to extensional equality．
inductive $C K::\left(' a::\right.$ topological＿space set $\Rightarrow{ }^{\prime} a$ set）$\Rightarrow$ bool where
CK C
｜CK K
｜【CKf；CKg】 $\Longrightarrow C K(f \circ g)$
declare CK．intros［intro！］
lemma $C K \_i d[$ intro！$]:$
CK id
$\langle p r o o f\rangle$
The inductive set $C K \_n f$ captures the normal forms for the 14 distinct operators．

```
inductive CK_nf :: ('a::topological_space set \(\Rightarrow\) 'a set) \(\Rightarrow\) bool where
    CK_nf id
| \(C K \_n f C\)
| \(C K \_n f K\)
| \(C K \_n f(C \circ K)\)
| \(C K \_n f(K \circ C)\)
| \(C K \_n f(C \circ K \circ C)\)
\(C K \_n f(K \circ C \circ K)\)
| CK_nf \((C \circ K \circ C \circ K)\)
| \(C K \_n f(K \circ C \circ K \circ C)\)
| CK_nf \((C \circ K \circ C \circ K \circ C)\)
| CK_nf \((K \circ C \circ K \circ C \circ K)\)
| CK_nf \((C \circ K \circ C \circ K \circ C \circ K)\)
| CK_nf \((K \circ C \circ K \circ C \circ K \circ C)\)
\(\mid C K \_n f(C \circ K \circ C \circ K \circ C \circ K \circ C)\)
declare \(C K \_n f . i n t r o s[i n t r o!]\)
lemma \(C K \_n f \_\)set:
    shows \(\left\{f . C K \_n f f\right\}=\{i d, C, K, C \circ K, K \circ C, C \circ K \circ C, K \circ C \circ K, C \circ K \circ C \circ K, K \circ C \circ K \circ\)
\(C, C \circ K \circ C \circ K \circ C, K \circ C \circ K \circ C \circ K, C \circ K \circ C \circ K \circ C \circ K, K \circ C \circ K \circ C \circ K \circ C, C \circ K \circ C\)
\(\circ K \circ C \circ K \circ C\}\)
〈proof〉
```

That each operator generated by compositions of $C$ and $K$ is extensionally equivalent to one of the normal forms captured by $C K \_n f$ is demonstrated by means of an induction over the construction of $C K \_n f$ and an appeal to the facts proved above．
theorem CK＿nf：
$C K f \longleftrightarrow C K \_n f f$
$\langle$ proof〉
theorem CK＿card：
shows card $\{f$ ．CK f $\} \leq 14$
＜proof〉
We show，using the following subset of $\mathbb{R}$（an example taken from Rusin（2001））as a witness，that there exist topological spaces on which all 14 operators are distinct．

## definition

$R R R$ ：：real set

## where

```
RRR={0<..<1}\cup{1<..<2}\cup{3}\cup({5<..<7}\cap\mathbb{Q})
```

The following facts allow the required proofs to proceed by simp：

```
lemma RRR_closure:
    shows closure \(R R R=\{0 . .2\} \cup\{3\} \cup\{5 . .7\}\)
\(\langle\) proof〉
```

lemma $R$ RR＿interior：
interior $R R R=\{0<. .<1\} \cup\{1<. .<2\}$ (is ?lhs $=$ ? rhs)
〈proof〉
lemma RRR_interior_closure[simplified]:
shows interior $(\{0::$ real.. 2$\} \cup\{3\} \cup\{5 . .7\})=\{0<. .<2\} \cup\{5<. .<7\}$ (is ?lhs $=$ ? rhs )
$\langle p r o o f\rangle$
The operators can be distinguished by testing which of the points in $\{1,2,3,4,6\}$ belong to their results.

## definition

test $::($ real set $\Rightarrow$ real set $) \Rightarrow$ bool list
where
test $f \equiv \operatorname{map}(\lambda x . x \in f R R R)[1,2,3,4,6]$
lemma $R R R \_t e s t$ :
assumes $f R R R=g R R R$
shows test $f=$ test $g$
$\langle$ proof $\rangle$
lemma $n f \_R R R$ :
shows
test id $=[$ False, False, True, False, True $]$
test $C=[$ True, True, False, True, False $]$
test $K=[$ True, True, True, False, True $]$
test $(K \circ C)=[$ True, True, True, True, True $]$
test $(C \circ K)=[$ False, False, False, True, False $]$
test $(C \circ K \circ C)=[$ False, False, False, False, False $]$
test $(K \circ C \circ K)=[$ False, True, True, True, False $]$
test $(C \circ K \circ C \circ K)=[$ True, False, False, False, True $]$
test $(K \circ C \circ K \circ C)=[$ True, True, False, False, False $]$
test $(C \circ K \circ C \circ K \circ C)=[$ False, False, True, True, True $]$
test $(K \circ C \circ K \circ C \circ K)=[$ True, True, False, False, True $]$
test $(C \circ K \circ C \circ K \circ C \circ K)=[$ False, False, True, True, False $]$
test $(K \circ C \circ K \circ C \circ K \circ C)=[$ False, True, True, True, True $]$
test $(C \circ K \circ C \circ K \circ C \circ K \circ C)=[$ True, False, False, False, False $]$
$\langle$ proof $\rangle$
theorem $C K \_n f \_r e a l \_c a r d$ :
shows $\operatorname{card}\left((\lambda f . f \bar{R} R R) \cdot\left\{f . C K \_n f f\right\}\right)=14$
$\langle p r o o f\rangle$
theorem CK_real_card:
shows card $\{f::$ real set $\Rightarrow$ real set. $C K f\}=14$ (is ? $/$ hs $=$ ?rhs)
$\langle p r o o f\rangle$

## 5 A corollary of Kuratowski's result

We show that it is a corollary of $C K \_$real_card that at most 7 distinct operators on a topological space can be generated by compositions of closure and interior. In the case of $\mathbb{R}$, exactly 7 distinct operators can be so generated.
inductive $I K::\left({ }^{\prime} a::\right.$ topological_space set $\Rightarrow{ }^{\prime} a$ set $) \Rightarrow$ bool where
IK id
| IK I
| IK K
$\mid \llbracket I K f ; I K g \rrbracket \Longrightarrow I K(f \circ g)$
inductive $I K \_n f::\left({ }^{\prime} a::\right.$ topological_space set $\Rightarrow{ }^{\prime} a$ set $) \Rightarrow$ bool where
IK_nf id
| IK_nf I
| $I K \_n f K$
$\mid I K \_n f(I \circ K)$

```
| IK_nf \((K \circ I)\)
| IK_nf \((I \circ K \circ I)\)
| \(I K \_n f(K \circ I \circ K)\)
```

declare IK.intros[intro!]
declare $I K \_n f . i n t r o s[i n t r o!]$
lemma $I K \_n f \_s e t:$
$\left\{f . I K \_n f f\right\}=\{i d, I, K, I \circ K, K \circ I, I \circ K \circ I, K \circ I \circ K\}$
〈proof〉
theorem $I K \_n f$ :
$I K f \longleftrightarrow I K \_n f f$
$\langle p r o o f\rangle$
theorem $I K \_c a r d$ :
shows card $\{f$. IK $f\} \leq 7$
〈proof〉
theorem IK_nf_real_card:
shows $\operatorname{card}\left((\lambda f . f R R R) \cdot\left\{f . I K \_n f f\right\}\right)=7$
〈proof〉
theorem IK_real_card:
shows card $\{f::$ real set $\Rightarrow$ real set. IK $f\}=7$ (is ?lhs $=$ ?rhs)
〈proof〉

## 6 Chagrov＇s result

Chagrov＇s theorem，which is discussed in Section 2.1 of Gardner and Jackson（2008），states that the number of distinct operators on a topological space that can be generated by compositions of closure and complement is one of $2,6,8,10$ or 14 ．
We begin by observing that the set of normal forms $C K \_n f$ can be split into two disjoint sets，$C K \_n f \_p o s$ and $C K \_n f \_n e g$ ，which we define in terms of interior and closure．

```
inductive CK_nf_pos :: ('a::topological_space set \(\Rightarrow{ }^{\prime} a\) set) \(\Rightarrow\) bool where
    CK_nf_posid
    | CK_nf_pos I
    | CK_nf_pos \(K\)
    |CK_nf_pos \((I \circ K)\)
    | CK_nf_pos \((K \circ I)\)
    |CK_nf_pos \((I \circ K \circ I)\)
    | \(C K \_n f \_p o s(K \circ I \circ K)\)
    declare CK_nf_pos.intros[intro!]
    lemma \(C K \_n f \_p o s \_s e t:\)
    shows \(\left\{f . C K \_n f \_p o s f\right\}=\{i d, I, K, I \circ K, K \circ I, I \circ K \circ I, K \circ I \circ K\}\)
    <proof〉
definition
    \(C K \_n f \_n e g ~::\left(' a:: t o p o l o g i c a l \_s p a c e ~ s e t ~ \Rightarrow ' a ~ s e t\right) \Rightarrow b o o l\)
where
    \(C K \_n f \_n e g f \longleftrightarrow\left(\exists g . C K \_n f \_p o s g \wedge f=C \circ g\right)\)
```

lemma $C K \_n f \_p o s \_n e g \_d i s j o i n t:$
assumes $C K \_n f \_p o s f$
assumes $C K \_n f \_n e g g$
shows $f \neq g$
$\langle p r o o f\rangle$
lemma $C K \_n f \_p o s \_n e g \_C K \_n f$ :
$C K \_n f f \longleftrightarrow C K \_n f \_p o s f \vee C K \_n f \_n e g f($ is ?lhs $\longleftrightarrow$ ? $\mathrm{rh} s$ )
$\langle p r o o f\rangle$
We now focus on $C K \_n f \_p o s$. In particular, we show that its cardinality for any given topological space is one of $1,3,4,5$ or 7 .
The proof consists of exhibiting normal forms for the operators supported by each of six classes of topological spaces. These are sublattices of the following lattice of $C K \_n f \_p o s$ operators:
lemmas $K \_I \_K \_s u b s e t e q \_K=$ closure_mono[OF interior_subset, of closure $X$, simplified $]$ for $X$
lemma $C K \_n f \_p o s \_l a t t i c e:$
shows
$I \leq(i d::$ 'a::topological_space set $\Rightarrow$ 'a set $)$
$i d \leq\left(K::\right.$ 'a::topological_space set $\Rightarrow{ }^{\prime} a$ set $)$
$I \leq I \circ K \circ(I::$ 'a::topological_space set $\Rightarrow$ 'a set $)$
$I \circ K \circ I \leq I \circ\left(K::{ }^{\prime} a::\right.$ topological_space set $\Rightarrow{ }^{\prime}$ a set $)$
$I \circ K \circ I \leq K \circ\left(I::{ }^{\prime} a::\right.$ topological_space set $\Rightarrow$ 'a set $)$
$I \circ K \leq K \circ I \circ\left(K:: ' a::\right.$ topological_space set $\Rightarrow{ }^{\prime} a$ set $)$
$K \circ I \leq K \circ I \circ(K:: ' a::$ topological_space set $\Rightarrow$ 'a set $)$
$K \circ I \circ K \leq\left(K::{ }^{\prime} a::\right.$ topological_space set $\Rightarrow{ }^{\prime} a$ set $)$
$\langle p r o o f\rangle$
We define the six classes of topological spaces in question, and show that they are related by inclusion in the following way (as shown in Figure 2.3 of Gardner and Jackson (2008)):


### 6.1 Discrete spaces

## definition

discrete $(X::$ 'a::topological__space set $) \longleftrightarrow I=\left(i d::^{\prime} a\right.$ set $\Rightarrow$ 'a set $)$
lemma discrete_eqs:
assumes discrete ( $X$ :: 'a::topological_space set)
shows
$I=\left(i d::^{\prime} a\right.$ set $\Rightarrow{ }^{\prime}$ a set $)$
$K=\left(i d::^{\prime}\right.$ a set $\Rightarrow$ 'a set $)$
$\langle p r o o f\rangle$
lemma discrete＿card：
assumes discrete（ $X$ ：：＇a：：topological＿space set）
shows card $\left\{f\right.$ ．CK＿nf＿pos $\left(f::^{\prime} a\right.$ set $\Rightarrow{ }^{\prime} a$ set $\left.)\right\}=1$
$\langle p r o o f\rangle$
lemma discrete＿discrete＿topology：
fixes $X$ ：：＇a：：topological＿space set
assumes $\wedge Y$ ：：＇a set．open $Y$
shows discrete $X$
〈proof〉

## 6．2 Partition spaces

## definition

```
part (X ::'a::topological_space set) \longleftrightarrowK ○I=(I ::'a set }=>\mp@subsup{}{}{\prime}'a set
```

lemma discrete＿part：
assumes discrete $X$
shows part $X$
$\langle p r o o f\rangle$
lemma part＿eqs：
assumes part（ $X$ ：：＇a：：topological＿space set）
shows
$K \circ I=(I::$＇a set $\Rightarrow$＇a set $)$
$I \circ K=\left(K::{ }^{\prime}\right.$ a set $\Rightarrow{ }^{\prime}$ a set $)$
$\langle p r o o f\rangle$
lemma part＿not＿discrete＿card：
assumes part（ $X$ ：：＇a：：topological＿space set）
assumes $\neg$ discrete $X$
shows card $\left\{f\right.$ ．CK＿nf＿pos $\left(f::^{\prime} a\right.$ set $\Rightarrow{ }^{\prime} a$ set $\left.)\right\}=3$
〈proof〉
A partition space is a topological space whose basis consists of the empty set and the equivalence classes of points of the space induced by some equivalence relation $R$ on the underlying set of the space．Equivalently，a partition space is one in which every open set is closed．Thus，for example，the class of partition spaces includes every topological space whose open sets form a boolean algebra．
datatype part＿witness $=a|b| c$
lemma part＿＿witness＿＿UNIV：
shows $U N I V=\operatorname{set}[a, b, c]$
$\langle p r o o f\rangle$
lemmas part＿witness＿pow＝subset＿subseqs［OF subset＿trans［OF subset＿UNIV Set．equalityD1［OF part＿witness＿UNIV
lemmas part＿witness＿Compl $=$ Compl＿eq＿Diff＿＿UNIV［where＇$a=$ part＿＿witness，unfolded part＿witness＿＿UNIV， simplified］
instantiation part＿＿witness ：：topological＿＿space
begin
definition open＿part＿＿witness $X \longleftrightarrow X \in\{\},\{a\},\{b, c\},\{a, b, c\}\}$
lemma part＿witness＿ball：
$(\forall s \in S . s \in\{\{ \},\{a\},\{b, c\},\{a, b, c\}\}) \longleftrightarrow S \subseteq \operatorname{set}[\},\{a\},\{b, c\},\{a, b, c\}]$
$\langle p r o o f\rangle$

```
lemmas part_witness_subsets_pow = subset_subseqs[OF iffD1[OF part_witness_ball]]
instance \(\langle p r o o f\rangle\)
end
lemma part_witness_interior_simps:
    shows
        interior \(\{a\}=\{a\}\)
        interior \(\{b\}=\{ \}\)
        interior \(\{c\}=\{ \}\)
        interior \(\{a, b\}=\{a\}\)
        interior \(\{a, c\}=\{a\}\)
        interior \(\{b, c\}=\{b, c\}\)
        interior \(\{a, b, c\}=\{a, b, c\}\)
〈proof〉
lemma part_witness_part:
    fixes \(X\) :: part_witness set
    shows part \(X\)
〈proof〉
lemma part_witness_not_discrete:
    fixes \(X\) :: part_witness set
    shows \(\neg\) discrete \(X\)
〈proof〉
lemma part_witness_card:
    shows card \(\{f\). CK_nf_pos ( \(f::\) part_witness set \(\Rightarrow\) part_witness set \()\}=3\)
〈proof〉
```


## 6．3 Extremally disconnected and open unresolvable spaces

## definition

```
ed_ou (X :: 'a::topological_space set) \longleftrightarrowI\circK=K \circ(I :: 'a set = 'a set)
```

ed_ou (X :: 'a::topological_space set) \longleftrightarrowI\circK=K \circ(I :: 'a set = 'a set)
lemma discrete＿ed＿ou：
assumes discrete $X$
shows ed＿ou $X$
$\langle p r o o f\rangle$
lemma ed＿ou＿eqs：
assumes ed＿ou（ $X$ ：：＇a：：topological＿space set）
shows
$I \circ K \circ I=K \circ(I:: ' a$ set $\Rightarrow$＇a set $)$
$K \circ I \circ K=K \circ\left(I::{ }^{\prime} a\right.$ set $\Rightarrow{ }^{\prime} a$ set $)$
$I \circ K=K \circ\left(I::\right.$＇a set $\Rightarrow{ }^{\prime}$＇a set $)$
$\langle p r o o f\rangle$
lemma ed＿ou＿neqs：
assumes ed＿ou（ $X$ ：：＇a：：topological＿space set）
assumes $\neg$ discrete $X$
shows
$I \neq\left(K::\right.$＇a set $\Rightarrow{ }^{\prime}$＇ set $)$
$I \neq K \circ(I:: ' a$ set $\Rightarrow ' a$ set $)$
$K \neq K \circ(I:: ' a$ set $\Rightarrow$＇a set $)$
$I \neq($ id $::$＇a set $\Rightarrow$＇a set $)$

```
\(K \neq(i d::\) 'a set \(\Rightarrow\) 'a set \()\)
\(\langle p r o o f\rangle\)
lemma ed_ou_not_discrete_card:
assumes ed_ou ( \(X\) :: 'a::topological_space set)
assumes \(\neg\) discrete \(X\)
shows card \(\left\{f\right.\). CK_nf_pos \(\left(f::^{\prime} a\right.\) set \(\Rightarrow ' a\) set \(\left.)\right\}=4\)
\(\langle p r o o f\rangle\)
We consider an example extremally disconnected and open unresolvable topological space.
datatype ed_ou_witness \(=a|b| c|d| e\)
lemma ed_ou_witness_UNIV:
shows \(U N I V=\operatorname{set}[a, b, c, d, e]\)
\(\langle p r o o f\rangle\)
lemmas ed_ou__witness_pow \(=\) subset__subseqs \([O F\) subset__trans \([O F\) subset__UNIV Set.equalityD1[OF ed__ou_witness__
lemmas ed_ou_witness_Compl \(=\) Compl_eq_Diff_UNIV[where ' \(a=e d \_o u \_w i t n e s s, u n f o l d e d ~ e d \_o u \_w i t n e s s \_U N I V\), simplified]
instance ed_ou_witness :: finite
\(\langle p r o o f\rangle\)
instantiation ed_ou_witness :: topological_space
begin
inductive open_ed__ou_witness :: ed_ou_witness set \(\Rightarrow\) bool where
open_ed_ou_witness \{\}
| open_ed_ou_witness \(\{a\}\)
| open_ed_ou_witness \(\{b\}\)
| open_ed_ou_witness \(\{e\}\)
|open_ed_ou_witness \(\{a, c\}\)
| open_ed_ou_witness \(\{b, d\}\)
|open_ed_ou_witness \(\{a, c, e\}\)
| open_ed_ou_witness \(\{a, b\}\)
| open_ed_ou_witness \(\{a, e\}\)
| open_ed_ou_witness \(\{b, e\}\)
|open_ed_ou_witness \(\{a, b, c\}\)
| open_ed_ou_witness \(\{a, b, d\}\)
| open_ed_ou_witness \(\{a, b, e\}\)
|open_ed_ou_witness \(\{b, d, e\}\)
| open_ed_ou_witness \(\{a, b, c, d\}\)
| open_ed_ou_witness \(\{a, b, c, e\}\)
| open_ed_ou_witness \(\{a, b, d, e\}\)
| open_ed_ou_witness \(\{a, b, c, d, e\}\)
declare open_ed_ou_witness.intros[intro!]
lemma ed_ou_witness_inter:
fixes \(S\) :: ed_ou__witness set
assumes open \(S\)
assumes open \(T\)
shows open \((S \cap T)\)
\(\langle p r o o f\rangle\)
lemma ed_ou_witness_union:
fixes \(X\) ：：ed ou witness set set
assumes \(\forall x \in X\) ．open \(x\)
shows open \((\bigcup X)\)
〈proof〉

\section*{instance}

〈proof〉
end
lemma ed＿ou＿witness＿interior＿simps：
shows
interior \(\{a\}=\{a\}\)
interior \(\{b\}=\{b\}\)
interior \(\{c\}=\{ \}\)
interior \(\{d\}=\{ \}\)
interior \(\{e\}=\{e\}\)
interior \(\{a, b\}=\{a, b\}\)
interior \(\{a, c\}=\{a, c\}\)
interior \(\{a, d\}=\{a\}\)
interior \(\{a, e\}=\{a, e\}\)
interior \(\{b, c\}=\{b\}\)
interior \(\{b, d\}=\{b, d\}\)
interior \(\{b, e\}=\{b, e\}\)
interior \(\{c, d\}=\{ \}\)
interior \(\{c, e\}=\{e\}\)
interior \(\{d, e\}=\{e\}\)
interior \(\{a, b, c\}=\{a, b, c\}\)
interior \(\{a, b, d\}=\{a, b, d\}\)
interior \(\{a, b, e\}=\{a, b, e\}\)
interior \(\{a, c, d\}=\{a, c\}\)
interior \(\{a, c, e\}=\{a, c, e\}\)
interior \(\{a, d, e\}=\{a, e\}\)
interior \(\{b, c, d\}=\{b, d\}\)
interior \(\{b, c, e\}=\{b, e\}\)
interior \(\{b, d, e\}=\{b, d, e\}\)
interior \(\{c, d, e\}=\{e\}\)
interior \(\{a, b, c, d\}=\{a, b, c, d\}\)
interior \(\{a, b, c, e\}=\{a, b, c, e\}\)
interior \(\{a, b, d, e\}=\{a, b, d, e\}\)
interior \(\{a, b, c, d, e\}=\{a, b, c, d, e\}\)
interior \(\{a, c, d, e\}=\{a, c, e\}\)
interior \(\{b, c, d, e\}=\{b, d, e\}\)
〈proof〉
lemma ed＿ou＿＿witness＿not＿＿discrete：
fixes \(X\) ：：ed＿ou＿witness set
shows \(\neg\) discrete \(X\)
\(\langle p r o o f\rangle\)
lemma ed＿ou＿＿witness＿ed＿ou：
fixes \(X\) ：：ed＿ou＿witness set
shows ed＿ou \(X\)
\(\langle p r o o f\rangle\)
lemma ed＿ou＿witness＿card：
shows card \(\left\{f . C K \_n f \_p o s\left(f:: e d \_o u \_w i t n e s s\right.\right.\) set \(\Rightarrow\) ed＿＿ou＿witness set \(\left.)\right\}=4\)
\(\langle p r o o f\rangle\)

\section*{6．4 Extremally disconnected spaces}

\section*{definition}
extremally＿disconnected \(\left(X::{ }^{\prime} a::\right.\) topological＿space set \() \longleftrightarrow K \circ I \circ K=I \circ\left(K::{ }^{\prime} a\right.\) set \(\Rightarrow\)＇a set）
lemma ed＿ou＿part＿extremally＿disconnected：
assumes ed＿＿ou \(X\)
assumes part \(X\)
shows extremally＿disconnected \(X\)
〈proof〉
lemma extremally＿disconnected＿eqs：
fixes \(X\) ：：＇a：：topological＿space set
assumes extremally＿disconnected \(X\)
shows
\(I \circ K \circ I=K \circ\left(I::{ }^{\prime} a\right.\) set \(\Rightarrow{ }^{\prime} a\) set \()\)
\(K \circ I \circ K=I \circ\left(K::{ }^{\prime} a\right.\) set \(\Rightarrow{ }^{\prime} a\) set \()\)
\(\langle p r o o f\rangle\)
lemma extremally＿disconnected＿＿not＿part＿＿not＿ed＿ou＿card：
fixes \(X\) ：：＇a：：topological＿space set
assumes extremally＿disconnected \(X\)
assumes \(\neg\) part \(X\)
assumes \(\neg e d \_\)ou \(X\)
shows card \(\left\{f\right.\) ．CK＿nf＿pos \(\left(f::^{\prime} a\right.\) set \(\Rightarrow ' a\) set \(\left.)\right\}=5\)
\(\langle p r o o f\rangle\)
Any topological space having an infinite underlying set and whose topology consists of the empty set and every cofinite subset of the underlying set is extremally disconnected．We consider an example such space having a countably infinite underlying set．
datatype＇\(a\) cofinite \(=\) cofinite \({ }^{\prime} a\)
instantiation cofinite ：：（type）topological＿space
begin
definition open＿cofinite \(=\left(\lambda X::^{\prime} a\right.\) cofinite set．finite \(\left.(-X) \vee X=\{ \}\right)\)

\section*{instance}

〈proof〉
end
lemma cofinite＿closure＿finite：
fixes \(X\) ：：＇a cofinite set
assumes finite \(X\)
shows closure \(X=X\)
\(\langle p r o o f\rangle\)
lemma cofinite＿closure＿infinite：
fixes \(X\) ：：＇a cofinite set
assumes infinite \(X\)
shows closure \(X=U N I V\)
\(\langle p r o o f\rangle\)
lemma cofinite＿interior＿finite：
fixes \(X\) ：：＇a cofinite set
assumes finite \(X\)
assumes infinite（UNIV：：＇a cofinite set）
shows interior \(X=\{ \}\)
\(\langle\) proof \(\rangle\)
lemma cofinite＿interior＿infinite：
fixes \(X\) ：：＇\(a\) cofinite set
assumes infinite \(X\)
assumes infinite（ \(-X\) ）
shows interior \(X=\{ \}\)
\(\langle p r o o f\rangle\)
abbreviation evens ：：nat cofinite set \(\equiv\{\) cofinite \(n \mid n . \exists i . n=2 * i\}\)
lemma evens＿infinite：
shows infinite evens
\(\langle\) proof \(\rangle\)
lemma cofinite＿nat＿infinite：
shows infinite（UNIV：：nat cofinite set）
\(\langle\) proof \(\rangle\)
lemma evens＿Compl＿infinite：
shows infinite（－evens）
\(\langle\) proof \(\rangle\)
lemma evens＿closure：
shows closure evens \(=\) UNIV
＜proof〉
lemma evens＿interior：
shows interior evens \(=\{ \}\)
\(\langle\) proof \(\rangle\)
lemma cofinite＿not＿part：
fixes \(X\) ：：nat cofinite set
shows \(\neg\) part \(X\)
〈proof〉
lemma cofinite＿not＿ed＿ou：
fixes \(X\) ：：nat cofinite set
shows \(\neg e d \_\)ou \(X\)
\(\langle p r o o f\rangle\)
lemma cofinite＿extremally＿disconnected＿aux：
fixes \(X\) ：：nat cofinite set
shows closure \((\) interior \((\) closure \(X)) \subseteq\) interior（closure \(X\) ）
〈proof〉
lemma cofinite＿extremally＿disconnected：
fixes \(X\) ：：nat cofinite set
shows extremally＿disconnected \(X\)
\(\langle p r o o f\rangle\)
lemma cofinite＿card：
shows card \(\{f\) ．CK＿nf＿pos（ \(f::\) nat cofinite set \(\Rightarrow\) nat cofinite set \()\}=5\)
\(\langle p r o o f\rangle\)

\section*{6．5 Open unresolvable spaces}

\section*{definition}
open＿unresolvable \((X::\)＇\(a::\) topological＿space set \() \longleftrightarrow K \circ I \circ K=K \circ\left(I::{ }^{\prime} a\right.\) set \(\Rightarrow{ }^{\prime} a\) set \()\)
lemma ed＿ou＿open＿unresolvable：
assumes ed＿ou \(X\)
shows open＿unresolvable \(X\)
\(\langle p r o o f\rangle\)
lemma open＿unresolvable＿eqs：
assumes open＿unresolvable（ \(X\) ：：＇a：：topological＿space set）
shows
\(I \circ K \circ I=I \circ\left(K::{ }^{\prime} a\right.\) set \(\Rightarrow{ }^{\prime} a\) set \()\)
\(K \circ I \circ K=K \circ\left(I::{ }^{\prime} a\right.\) set \(\Rightarrow{ }^{\prime}\) a set \()\)
\(\langle p r o o f\rangle\)
lemma not＿ed＿ou＿neqs：
assumes \(\neg e d \_\)ou（ \(X::\)＇a：：topological＿space set）
shows
\(I \neq I \circ(K:: ' a\) set \(\Rightarrow\)＇a set \()\)
\(K \neq K \circ(I::\)＇a set \(\Rightarrow\)＇a set \()\)
\(\langle p r o o f\rangle\)
lemma open＿unresolvable＿not＿ed＿＿ou＿card：
assumes open＿unresolvable（ \(X\) ：：＇a：：topological＿space set）
assumes \(\neg e d \_\)ou \(X\)
shows card \(\left\{f\right.\) ．CK＿nf＿pos \(\left(f::^{\prime}\right.\) a set \(\Rightarrow\)＇a set \(\left.)\right\}=5\)
〈proof〉
We show that the class of open unresolvable spaces is non－empty by exhibiting an example of such a space．
datatype ou＿witness \(=a|b| c\)
lemma ou＿witness＿UNIV：
shows \(U N I V=\operatorname{set}[a, b, c]\)
\(\langle p r o o f\rangle\)
instantiation ou＿witness ：：topological＿space
begin
definition open＿ou＿＿witness \(X \longleftrightarrow a \notin X \vee X=U N I V\)

\section*{instance}

〈proof〉
end
lemma ou＿witness＿closure＿simps：
shows
\[
\text { closure }\{a\}=\{a\}
\]
closure \(\{b\}=\{a, b\}\)
closure \(\{c\}=\{a, c\}\)
closure \(\{a, b\}=\{a, b\}\)
closure \(\{a, c\}=\{a, c\}\)
closure \(\{a, b, c\}=\{a, b, c\}\)
closure \(\{b, c\}=\{a, b, c\}\)
\(\langle p r o o f\rangle\)
lemma ou＿witness＿open＿unresolvable：
fixes \(X\) ：：ou＿witness set
shows open＿unresolvable \(X\)
\(\langle p r o o f\rangle\)
lemma ou＿witness＿not＿ed＿ou：
fixes \(X\) ：：ou＿witness set
shows \(\neg e d \_\)ou \(X\)
\(\langle p r o o f\rangle\)
lemma ou＿witness＿card：
shows card \(\left\{f\right.\) ．CK＿nf＿pos（ \(f:: o u \_\)witness set \(\Rightarrow\) ou＿witness set \(\left.)\right\}=5\)
\(\langle\) proof \(\rangle\)

\section*{6．6 Kuratowski spaces}

\section*{definition}
kuratowski（ \(X\) ：：＇a：：topological＿space set）\(\longleftrightarrow\)
\(\neg\) extremally＿disconnected \(X \wedge \neg\) open＿unresolvable \(X\)
A Kuratowski space distinguishes all 7 positive operators．
```

lemma part_closed_open:
fixes }X\mathrm{ :: 'a::topological_space set
assumes }I\circK\circI=(I::'a set => 'a set
assumes closed X
shows open X
<proof>

```
lemma part_I_K_I:
    assumes \(I \circ K \circ I=\left(I::{ }^{\prime} a::\right.\) topological_space set \(\Rightarrow{ }^{\prime} a\) set \()\)
    shows \(I \circ K=(K:: ' a\) set \(\Rightarrow\) 'a set \()\)
〈proof〉
lemma part_K_I_I:
    assumes \(I \circ K \circ I=\left(I:: ' a::\right.\) topological_space set \(\Rightarrow{ }^{\prime} a\) set \()\)
    shows \(K \circ I=(I:: ' a\) set \(\Rightarrow ' a\) set \()\)
〈proof〉
lemma kuratowski_neqs:
    assumes kuratowski (X :: 'a::topological_space set)
    shows
        \(I \neq I \circ K \circ\left(I::{ }^{\prime}\right.\) a set \(\Rightarrow\) 'a set \()\)
        \(I \circ K \circ I \neq K \circ\left(I::{ }^{\prime} a\right.\) set \(\Rightarrow\) 'a set \()\)
        \(I \circ K \circ I \neq I \circ\left(K::\right.\) 'a set \(\Rightarrow^{\prime}\) 'a set \()\)
        \(I \circ K \neq K \circ I \circ\left(K:: ' a\right.\) set \(\Rightarrow{ }^{\prime} a\) set \()\)
        \(K \circ I \neq K \circ I \circ\left(K:: ' a\right.\) set \(\Rightarrow{ }^{\prime} a\) set \()\)
        \(K \circ I \circ K \neq(K::\) 'a set \(\Rightarrow\) 'a set \()\)
        \(I \circ K \neq K \circ(I::\) 'a set \(\Rightarrow\) ' \(a\) set \()\)
        \(I \neq(i d::\) 'a set \(\Rightarrow\) ' \(a\) set \()\)
        \(K \neq(i d::\) 'a set \(\Rightarrow\) 'a set \()\)
        \(I \circ K \circ I \neq(i d::\) 'a set \(\Rightarrow\) 'a set \()\)
        \(K \circ I \circ K \neq(i d::\) 'a set \(\Rightarrow\) 'a set \()\)
    \(\langle p r o o f\rangle\)
lemma kuratowski_card:
    assumes kuratowski (X :: 'a::topological_space set)
    shows card \(\left\{f\right.\). CK_nf_pos \(\left(f::^{\prime} a\right.\) set \(\Rightarrow{ }^{\prime} a\) set \(\left.)\right\}=7\)
〈proof〉
\(\mathbb{R}\) is a Kuratowski space.
lemma kuratowski_reals:
shows kuratowski \((\mathbb{R}\) :: real set)
〈proof〉

\subsection*{6.7 Chagrov's theorem}
theorem chagrov:
fixes \(X\) :: 'a::topological_space set
obtains discrete \(X\)
\(\mid \neg\) discrete \(X \wedge\) part \(X\)
\(\mid \neg\) discrete \(X \wedge\) ed_ou \(X\)
\(\mid \neg e d \_o u \quad X \wedge\) open_unresolvable \(X\)
\(\mid \neg e d \_o u \quad X \wedge \neg\) part \(X \wedge\) extremally_disconnected \(X\)
| kuratowski X
\(\langle p r o o f\rangle\)
corollary chagrov_card:
shows card \(\left\{f\right.\). CK_nf_pos ( \(f::^{\prime} a::\) topological_space set \(\Rightarrow{ }^{\prime} a\) set \(\left.)\right\} \in\{1,3,4,5,7\}\)
\(\langle p r o o f\rangle\)

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