# The Kuratowski Closure-Complement Theorem 

Peter Gammie and Gianpaolo Gioiosa

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## 1 Introduction

We discuss a topological curiosity discovered by Kuratowski (1922): the fact that the number of distinct operators on a topological space generated by compositions of closure and complement never exceeds 14 , and is exactly 14 in the case of $\mathbb{R}$. In addition, we prove a theorem due to Chagrov (1982) that classifies topological spaces according to the number of such operators they support.
Kuratowski's result, which is exposited in Whitty (2015) and Chapter 7 of Chamberland (2015), has already been treated in Mizar - see Bagińska and Grabowski (2003) and Grabowski (2004). To the best of our knowledge, we are the first to mechanize Chagrov's result.
Our work is based on a presentation of Kuratowski's and Chagrov's results by Gardner and Jackson (2008).
We begin with some preliminary facts pertaining to the relationship between interiors of unions and unions of interiors ( $\S 2$ ) and the relationship between $\mathbb{Q}$ and $\mathbb{R}(\S 3)$. We then prove Kuratowski's result ( $\S 4$ ) and the corollary that at most 7 distinct operators on a topological space can be generated by compositions of closure and interior (§5). Finally, we prove Chagrov's result (§6).

## 2 Interiors and unions

```
definition
    boundary :: 'a::topological_space set = 'a set
where
    boundary X = closure X - interior }
```

lemma boundary＿empty：
shows boundary $\}=\{ \}$
unfolding boundary＿def by simp

## definition

exterior ：：＇a：：topological＿space set $\Rightarrow$＇$a$ set
where
exterior $X=-($ interior $X \cup$ boundary $X)$
lemma interior＿union＿boundary：
shows interior $(X \cup Y)=$ interior $X \cup$ interior $Y$
$\longleftrightarrow$ boundary $X \cap$ boundary $Y \subseteq$ boundary $(X \cup Y)($ is $($ ？lhs $1=$ ？lhs2 $) \longleftrightarrow$ ？rhs $)$
$\operatorname{proof}\left(r u l e ~ i f f I\left[O F ~ \_~ s u b s e t \_a n t i s y m[O F ~ s u b s e t I]\right]\right)$
assume ？lhs1＝？lhs2 then show ？rhs by（force simp：boundary＿def）
next
fix $x$
assume ？rhs and $x \in$ ？lhs1
have $x \in$ ？lhs 2 if $x \notin$ interior $X x \notin$ interior $Y$
proof（cases $x \in$ boundary $X \cap$ boundary $Y$ ）
case True with 〈？rhs〉 〈x ？？hs1〉 show ？thesis by（simp add：boundary＿def subset＿iff）
next
case False then consider $(X) x \notin$ boundary $X \mid(Y) x \notin$ boundary $Y$ by blast
then show？thesis
proof cases
case $X$
from $X\langle x \notin$ interior $X\rangle$ have $x \in$ exterior $X$ by（simp add：exterior＿def）
from $\langle x \notin$ boundary $X\rangle\langle x \in$ exterior $X\rangle\langle x \notin$ interior $X\rangle$
obtain $U$ where open $U U \subseteq-X x \in U$
by（metis ComplI DiffI boundary＿def closure＿interior interior＿subset open＿interior）
from $\left\langle x \in\right.$ interior $(X \cup Y)$ ）obtain $U^{\prime}$ where open $U^{\prime} U^{\prime} \subseteq X \cup Y x \in U^{\prime}$
by（meson interiorE）
from $\langle U \subseteq-X\rangle\left\langle U^{\prime} \subseteq X \cup Y\right\rangle$ have $U \cap U^{\prime} \subseteq Y$ by blast
with $\langle x \notin$ interior $Y\rangle\left\langle\right.$ open $\left.U^{\prime}\right\rangle\langle$ open $U\rangle\left\langle x \in U^{\prime}\right\rangle\langle x \in U\rangle$ show ？thesis
by（meson IntI interiorI open＿Int）
next
case $Y$
from $Y\langle x \notin$ interior $Y\rangle$ have $x \in$ exterior $Y$ by（simp add：exterior＿def）
from $\langle x \notin$ boundary $Y\rangle\langle x \in$ exterior $Y\rangle\langle x \notin$ interior $Y\rangle$
obtain $U$ where open $U U \subseteq-Y x \in U$
by（metis ComplI DiffI boundary＿def closure＿interior interior＿subset open＿interior）
from $\left\langle x \in\right.$ interior $(X \cup Y)$ ）obtain $U^{\prime}$ where open $U^{\prime} U^{\prime} \subseteq X \cup Y x \in U^{\prime}$
by（meson interiorE）
from $\langle U \subseteq-Y\rangle\left\langle U^{\prime} \subseteq X \cup Y\right\rangle$ have $U \cap U^{\prime} \subseteq X$ by blast
with $\langle x \notin$ interior $X\rangle\left\langle\right.$ open $\left.U^{\prime}\right\rangle\langle$ open $U\rangle\left\langle x \in U^{\prime}\right\rangle\langle x \in U\rangle$ show ？thesis
by（meson IntI interiorI open＿Int）
qed
qed
with $\langle x \in$ ？lhs1 $\rangle$ show $x \in$ ？lhs2 by blast
next
show ？lhs $2 \subseteq$ ？lhs1 by（simp add：interior＿mono）
qed
lemma interior＿union＿closed＿intervals：
fixes $a$ ：：＇a：：ordered＿euclidean＿space
assumes $b<c$
shows interior $(\{a . . b\} \cup\{c . . d\})=$ interior $\{a . . b\} \cup$ interior $\{c . . d\}$
using assms by（subst interior＿union＿boundary；auto simp：boundary＿def）

## 3 Additional facts about the rationals and reals

lemma Rat_real_limpt:
fixes $x$ :: real
shows $x$ islimpt $\mathbb{Q}$
proof (rule islimptI)
fix $T$ assume $x \in T$ open $T$
then obtain $e$ where $0<e$ and ball: $\left|x^{\prime}-x\right|<e \longrightarrow x^{\prime} \in T$ for $x^{\prime}$ by (auto simp: open_real)
from $\langle 0<e\rangle$ obtain $q$ where $x<$ real_of_rat $q \wedge$ real_of_rat $q<x+e$ using of_rat_dense by force
with ball show $\exists y \in \mathbb{Q} . y \in T \wedge y \neq x$ by force
qed
lemma Rat_closure:
shows closure $\mathbb{Q}=(U N I V$ :: real set $)$
unfolding closure_def using Rat_real_limpt by blast
lemma Rat_interval_closure:
fixes $x$ :: real
assumes $x<y$
shows closure $(\{x<. .<y\} \cap \mathbb{Q})=\{x . . y\}$
using assms
by (metis (no_types, lifting) Rat_closure closure_closure closure_greaterThanLessThan closure_mono inf_le1
inf_top.right_neutral open_Int_closure_subset open_real_greaterThanLessThan subset_antisym)
lemma Rat_not_open:
fixes $T$ :: real set
assumes open $T$
assumes $T \neq\{ \}$
shows $\neg T \subseteq \mathbb{Q}$
using assms by (simp add: countable_rat open_minus_countable subset_eq)
lemma Irrat__dense_in_real:
fixes $x$ :: real
assumes $x<y$
shows $\exists r \in-\mathbb{Q} . x<r \wedge r<y$
using assms Rat_not_open[where $T=\{x<. .<y\}]$ by force
lemma closed_interval_Int_compl:
fixes $x$ :: real
assumes $x<y$
assumes $y<z$
shows $-\{x . . y\} \cap-\{y . . z\}=-\{x . . z\}$
using assms by auto

## 4 Kuratowski's result

We prove that at most 14 distinct operators can be generated by compositions of closure and complement. For convenience, we give these operators short names and try to avoid pointwise reasoning. We treat the interior operator at the same time.
declare o_apply[simp del]
definition $C::{ }^{\prime} a::$ topological_space set $\Rightarrow$ 'a set where $C X=-X$
definition $K::$ ' $a::$ topological_space set $\Rightarrow$ 'a set where $K X=$ closure $X$
definition $I::$ 'a::topological_space set $\Rightarrow$ 'a set where $I X=$ interior $X$
lemma $C \_C$ :
shows $C \circ C=i d$
by (simp add: fun_eq_iff C_def o_apply)
lemma $K \_K$ :
shows $K \circ K=K$
by (simp add: fun_eq_iff $\left.K \_d e f o \_a p p l y\right)$
lemma $I \_I$ :
shows $I \circ I=I$
unfolding $I \quad$ def by ( $\operatorname{simp}$ add: o__def)
lemma $I \_K$ :
shows $I=C \circ K \circ C$
unfolding $C$ __def I_def $K \_d e f$ by (simp add: o_def interior_closure)
lemma $K_{\_} I$ :
shows $K=C \circ I \circ C$
unfolding $C \_d e f I \_d e f ~ K \_d e f$ by ( $\operatorname{simp}$ add: o_def interior_closure)
lemma $K \_I \_K \_I$ :
shows $K \circ I \circ K \circ I=K \circ I$
unfolding $C \_$def I_def $K \_d e f$
by (clarsimp simp: fun_eq_iff o__apply closure_minimal closure_mono closure_subset interior_maximal inte-
rior_subset subset_antisym)
lemma $I \_K \_I \_K$ :
shows $I \circ K \circ I \circ K=I \circ K$
unfolding $C$ _def $I \_d e f$ K_def
by (simp add: fun_eq_iff o_apply)
(metis (no_types) closure_closure closure_mono closure_subset interior_maximal interior_mono interior_subset open_interior subset__antisym)
lemma K_mono:
assumes $x \subseteq y$
shows $K x \subseteq K y$
using assms unfolding $K \_d e f$ by (simp add: closure_mono)
The following lemma embodies the crucial observation about compositions of $C$ and $K$ :
lemma $K C K C K C K \_K C K$ :
shows $K \circ C \circ K \circ C \circ K \circ C \circ K=K \circ C \circ K($ is ?lhs $=$ ? $r$ rhs $)$
proof (rule ext [OF equalityI])
fix $x$
have $(C \circ K \circ C \circ K \circ C \circ K) x \subseteq$ ?rhs $x$ by (simp add: C_def $K \_d e f$ closure_def o_apply)
then have $(K \circ(C \circ K \circ C \circ K \circ C \circ K)) x \subseteq(K \circ$ ?rhs $) x$ by (simp add: K_mono o_apply)
then show ?lhs $x \subseteq$ ? rhs $x$ by (simp add: K_K o_assoc)
next
fix $x$ :: ' $a$ ::topological_space set
have $(C \circ K \circ C \circ K) x \subseteq K x$ by (simp add: C_def K_def closure_def o_apply)
then have $(K \circ(C \circ K \circ C \circ K)) x \subseteq(K \circ K) x$ by (simp add: K_mono o_apply)
then have $(C \circ(K \circ K)) x \subseteq(C \circ(K \circ(C \circ K \circ C \circ K))) x$ by $(\operatorname{simp}$ add: $C$ _def o_apply $)$
then have $(K \circ(C \circ(K \circ K))) x \subseteq(K \circ(C \circ(K \circ(C \circ K \circ C \circ K)))) x$ by $\left(\operatorname{simp} a d d: K \_m o n o o \_a p p l y\right)$
then show ? rhs $x \subseteq$ ?lhs $x$ by (simp add: K_K o_assoc)
qed
The inductive set $C K$ captures all operators that can be generated by compositions of $C$ and $K$. We shallowly embed the operators; that is, we identify operators up to extensional equality.
inductive $C K$ :: ('a::topological_space set $\Rightarrow{ }^{\prime}$ a set) $\Rightarrow$ bool where
| $\llbracket C K f ; C K g \rrbracket \Longrightarrow C K(f \circ g)$
declare CK.intros[intro!]
lemma CK_id[intro!]:
CK id
by (metis CK.intros(1) CK.intros(3) C_C)

The inductive set $C K \_n f$ captures the normal forms for the 14 distinct operators.

```
inductive \(C K \_n f\) :: ('a::topological_space set \(\Rightarrow{ }^{\prime}\) 'a set) \(\Rightarrow\) bool where
    \(C K \_n f\) id
| \(C K \_n f C\)
| CK_nf K
\(C K \_n f(C \circ K)\)
\(C K \_n f(K \circ C)\)
\(C K \_n f(C \circ K \circ C)\)
| CK_nf \((K \circ C \circ K)\)
| CK_nf \((C \circ K \circ C \circ K)\)
\(C K \_n f(K \circ C \circ K \circ C)\)
| \(C K \_n f(C \circ K \circ C \circ K \circ C)\)
| \(C K \_n f(K \circ C \circ K \circ C \circ K)\)
| CK_nf \((C \circ K \circ C \circ K \circ C \circ K)\)
| CK_nf \((K \circ C \circ K \circ C \circ K \circ C)\)
\(\mid C K \_n f(C \circ K \circ C \circ K \circ C \circ K \circ C)\)
```

declare CK_nf.intros[intro!]
lemma $C K \_n f \_s e t$ :
shows $\left\{f . C K \_n f f\right\}=\{i d, C, K, C \circ K, K \circ C, C \circ K \circ C, K \circ C \circ K, C \circ K \circ C \circ K, K \circ C \circ K \circ$ $C, C \circ K \circ C \circ K \circ C, K \circ C \circ K \circ C \circ K, C \circ K \circ C \circ K \circ C \circ K, K \circ C \circ K \circ C \circ K \circ C, C \circ K \circ C$ $\circ K \circ C \circ K \circ C\}$
by (auto simp: CK_nf.simps)
That each operator generated by compositions of $C$ and $K$ is extensionally equivalent to one of the normal forms captured by $C K \_n f$ is demonstrated by means of an induction over the construction of $C K \_n f$ and an appeal to the facts proved above.
theorem CK_nf:
$C K f \longleftrightarrow C K \_n f f$
proof (rule iffI)
assume $C K f$ then show $C K \_n f f$
by induct
(elim CK_nf.cases; clarsimp simp: id_def[symmetric] $C \_C K \_K ~ K C K C K C K \_K C K ~ o \_a s s o c ; ~ s i m p ~ a d d: ~$
o_assoc[symmetric]; clarsimp simp: $C \_C K \_K K C K C K C K \_K C K ~ o \_a s s o c ~$
| blast)+
next
assume $C K \_n f f$ then show $C K f$ by induct (auto simp: id_def[symmetric])
qed
theorem CK_card:
shows card $\{f$. CK f $\} \leq 14$
by (auto simp: CK_nf CK_nf_set card.insert_remove intro!: le_trans[OF card_Diff1_le])
We show, using the following subset of $\mathbb{R}$ (an example taken from Rusin (2001)) as a witness, that there exist topological spaces on which all 14 operators are distinct.

## definition

$R R R$ :: real set
where

$$
R R R=\{0<. .<1\} \cup\{1<. .<2\} \cup\{3\} \cup(\{5<. .<7\} \cap \mathbb{Q})
$$

The following facts allow the required proofs to proceed by simp:
lemma RRR_closure:
shows closure $R R R=\{0 . .2\} \cup\{3\} \cup\{5 . .7\}$
unfolding $R R R \_$def by (force simp: closure_insert Rat_interval_closure)
lemma $R R R$ _interior:
interior $R R R=\{0<. .<1\} \cup\{1<. .<2\}$ (is ? 1 hs $=$ ? rhs )
proof(rule equality $[$ [OF subsetI subsetI])
fix $x$ assume $x \in$ ? $/ \mathrm{hs}$
then obtain $T$ where open $T$ and $x \in T$ and $T \subseteq R R R$ by (blast elim: interiorE)
then obtain $e$ where $0<e$ and ball $x e \subseteq T$ by (blast elim!: openE)
from $\langle x \in T\rangle\langle 0\langle e\rangle\langle b a l l x e \subseteq T\rangle\langle T \subseteq R R R\rangle$
have False if $x=3$
using that unfolding $R R R \_$def ball_def
by (auto dest!: subsetD[where $c=\min (3+e / 2)$ 4] simp: dist_real_def)

## moreover

from Irrat_dense_in_real[where $x=x$ and $y=x+e / 2]\langle 0<e\rangle$
obtain $r$ where $r \in-\mathbb{Q} \wedge x<r \wedge r<x+e / 2$ by auto
with $\langle x \in T\rangle\langle b a l l x e \subseteq T\rangle\langle T \subseteq R R R\rangle$
have False if $x \in\{5<. .<7\} \cap \mathbb{Q}$
using that unfolding $R R R \_d e f$ ball_def
by (force simp: dist_real_def dest: subsetD[where $c=r]$ )
moreover note $\langle x \in$ interior $R R R\rangle$
ultimately show $x \in$ ? rhs
unfolding $R R R \_$def by (auto dest: subset $D[O F$ interior_subset])
next
fix $x$ assume $x \in$ ? $r h s$
then show $x \in$ ? lhs
unfolding RRR_def interior_def by (auto intro: open_real_greaterThanLessThan)
qed
lemma $R$ RR_interior_closure[simplified]:
shows interior $(\{0::$ real.. 2$\} \cup\{3\} \cup\{5 . .7\})=\{0<. .<2\} \cup\{5<. .<7\}$ (is ?lhs $=$ ? $r$ rhs)
proof -
have ?lhs $=$ interior $(\{0 . .2\} \cup\{5 . .7\})$
by (metis (no_types, lifting) Un_assoc Un_commute closed_Un closed_eucl_atLeastAtMost interior_closed_Un_emp interior_singleton)
also have ... = ? rhs
by (simp add: interior_union_closed_intervals)
finally show ?thesis .
qed
The operators can be distinguished by testing which of the points in $\{1,2,3,4,6\}$ belong to their results.

## definition

test $::($ real set $\Rightarrow$ real set $) \Rightarrow$ bool list
where
test $f \equiv \operatorname{map}(\lambda x . x \in f R R R)[1,2,3,4,6]$
lemma RRR_test:
assumes $f R R R=g R R R$
shows test $f=$ test $g$
unfolding test_def using assms by simp
lemmanf_RRR:
shows
test id $=$ [False, False, True, False, True $]$
test $C=[$ True, True, False, True, False $]$
test $K=$ [True, True, True, False, True $]$
test $(K \circ C)=[$ True, True, True, True, True $]$
test $(C \circ K)=[$ False, False, False, True, False $]$
test $(C \circ K \circ C)=[$ False, False, False, False, False $]$
test $(K \circ C \circ K)=[$ False, True, True, True, False $]$
test $(C \circ K \circ C \circ K)=[$ True, False, False, False, True $]$
test $(K \circ C \circ K \circ C)=[$ True, True, False, False, False $]$
test $(C \circ K \circ C \circ K \circ C)=[$ False, False, True, True, True $]$
test $(K \circ C \circ K \circ C \circ K)=[$ True, True, False, False, True $]$
test $(C \circ K \circ C \circ K \circ C \circ K)=[$ False, False, True, True, False $]$
test $(K \circ C \circ K \circ C \circ K \circ C)=[$ False, True, True, True, True $]$
test $(C \circ K \circ C \circ K \circ C \circ K \circ C)=[$ True, False, False, False, False]
unfolding test_def $C$ __def $K \_d e f$
 o_apply)
(simp_all add: RRR_def)
theorem $C K \_n f \_r e a l \_c a r d$ :
shows $\operatorname{card}\left((\lambda f . f R R R) \cdot\left\{f . C K \_n f f\right\}\right)=14$
by (simp add: CK_nf_set) ((subst card_insert_disjoint; auto dest!: RRR_test simp: $n f \_R R R$ id__def[symmetric])[1])+

```
theorem CK_real_card:
    shows card {f::real set }=>\mathrm{ real set. CK f} = 14 (is ?lhs = ?rhs)
proof(rule antisym[OF CK_card])
    show ?rhs \leq?lhs
        unfolding CK_nf
        by (rule le_trans[OF eq_imp_le[OF CK_nf_real_card[symmetric]] card__image_le])
            (simp add:CK_nf_set)
qed
```


## 5 A corollary of Kuratowski's result

We show that it is a corollary of $C K \_r e a l \_c a r d$ that at most 7 distinct operators on a topological space can be generated by compositions of closure and interior. In the case of $\mathbb{R}$, exactly 7 distinct operators can be so generated.
inductive $I K::\left(' a:: t o p o l o g i c a l \_s p a c e ~ s e t ~ \Rightarrow ' a ~ s e t\right) ~ \Rightarrow b o o l$ where
IK id
| IK I
| IK K
$\mid \llbracket I K f ; I K g \rrbracket \Longrightarrow I K(f \circ g)$
inductive $I K \_n f::\left({ }^{\prime} a::\right.$ topological_space set $\Rightarrow{ }^{\prime} a$ set $) \Rightarrow$ bool where
IK_nfid
| IK_nf I
| IK_nf $K$
| IK_nf $(I \circ K)$
| IK_nf $(K \circ I)$
$\mid I K \_n f(I \circ K \circ I)$
$\mid I K \_n f(K \circ I \circ K)$
declare IK.intros[intro!]
declare IK_nf.intros[intro!]
lemma $I K \_n f \_s e t:$
$\left\{f . I K \_n f f\right\}=\{i d, I, K, I \circ K, K \circ I, I \circ K \circ I, K \circ I \circ K\}$
by (auto simp: IK_nf.simps)
theorem $I K \_n f$ :
$I K f \longleftrightarrow I K \_n f f$
proof(rule iffI)
assume $I K f$ then show $I K \_n f f$
by induct
(elim IK_nf.cases; clarsimp simp: id_def[symmetric] o_assoc; simp add: I_I K_K o_assoc[symmetric];
clarsimp simp: $K \_I \_K \_I I \_K \_I \_K o \_a s s o c$
| blast)+
next
assume $I K \_n f f$ then show $I K f$ by induct blast+
qed
theorem $I K$ card:
shows card $\{f$. IK $f\} \leq 7$
by (auto simp: IK_nf IK_nf_set card.insert_remove intro!: le_trans[OF card_Diff1_le])
theorem $I K \_n f \_r e a l \_c a r d:$
shows $\operatorname{card}\left((\lambda f . f R R R)\right.$ ' $\left.\left\{f . I K \_n f f\right\}\right)=7$
by (simp add: IK_nf_set) ((subst card_insert_disjoint; auto dest!: RRR_test simp: nf_RRR I_Kid_def[symmetric] o_assoc)[1])+

```
theorem IK_real_card:
    shows card \(\{f::\) real set \(\Rightarrow\) real set. IK \(f\}=7\) (is ?lhs \(=\) ?rhs)
proof(rule antisym[OF IK_card])
    show ?rhs \(\leq\) ? lhs
        unfolding \(I K \_n f\)
        by (rule le_trans[OF eq_refl[OF IK_nf_real_card[symmetric]] card_image_le])
            ( simp add: IK_nf_set)
    qed
```


## 6 Chagrov's result

Chagrov's theorem, which is discussed in Section 2.1 of Gardner and Jackson (2008), states that the number of distinct operators on a topological space that can be generated by compositions of closure and complement is one of $2,6,8,10$ or 14 .
We begin by observing that the set of normal forms $C K \_n f$ can be split into two disjoint sets, $C K \_n f \_p o s$ and $C K \_n f \_n e g$, which we define in terms of interior and closure.

```
inductive CK_nf_pos :: ('a::topological_space set \(\Rightarrow{ }^{\prime}\) ' \(a\) set \() \Rightarrow\) bool where
    CK_nf_pos id
    | CK_nf_pos I
    | CK_nf_pos \(K\)
    | CK_nf_pos \((I \circ K)\)
    |CK_nf_pos \((K \circ I)\)
    | CK_nf_pos \((I \circ K \circ I)\)
    | CK_nf_pos \((K \circ I \circ K)\)
```

declare CK_nf_pos.intros[intro!]
lemma $C K \_n f \_p o s \_s e t:$
shows $\left\{f . C K \_n f \_p o s f\right\}=\{i d, I, K, I \circ K, K \circ I, I \circ K \circ I, K \circ I \circ K\}$
by (auto simp: CK_nf_pos.simps)

## definition

$C K \_n f \_n e g ~::\left(' a:: t o p o l o g i c a l \_s p a c e ~ s e t ~ \Rightarrow ' a ~ s e t\right) ~ \Rightarrow b o o l$
where

$$
C K \_n f \_n e g ~ f \longleftrightarrow\left(\exists g . C K \_n f \_p o s g \wedge f=C \circ g\right)
$$

lemma $C K \_n f \_p o s \_n e g \_d i s j o i n t:$
assumes $C K \_n f \_p o s f$
assumes $C K \_n f \_n e g ~ g$
shows $f \neq g$
using assms unfolding $C K \_n f \_n e g \_d e f$
by (clarsimp simp: CK_nf_pos.simps; elim disjE; metis comp_def C_def I_def K_def Compl_iff closure_UNIV interior_UNIV id_apply)
lemma $C K \_n f \_p o s \_n e g \_C K \_n f$ :
$C K \_n f f \longleftrightarrow C K \_n f \_p o s f \vee C K \_n f \_n e g f$ (is ?lhs $\longleftrightarrow$ ?rhs)
proof(rule iffI)
assume ?lhs then show ?rhs
unfolding $C K \_n f \_n e g \_d e f$ by (rule CK_nf.cases; metis (no_types, lifting) CK_nf_pos.simps C_C I_K K_I comp_id o_assoc)
next
assume ?rhs then show ?lhs
unfolding $C K \_n f \_n e g \_d e f$
by (auto elim!: CK_nf_pos.cases simp: I_K C_Co_assoc)
qed
We now focus on $C K \_n f \_p o s$. In particular, we show that its cardinality for any given topological space is one of $1,3,4,5$ or 7 .
The proof consists of exhibiting normal forms for the operators supported by each of six classes of topological spaces. These are sublattices of the following lattice of $C K \_n f \_p o s$ operators:
lemmas $K \_I \_K \_s u b s e t e q \_K=$ closure_mono[OF interior_subset, of closure $X$, simplified $]$ for $X$
lemma $C K \_n f \_p o s \_l a t t i c e:$
shows
$I \leq(i d::$ 'a::topological_space set $\Rightarrow$ 'a set $)$
$i d \leq\left(K::\right.$ 'a::topological_space set $\Rightarrow{ }^{\prime} a$ set $)$
$I \leq I \circ K \circ(I:: ' a::$ topological_space set $\Rightarrow$ 'a set $)$
$I \circ K \circ I \leq I \circ\left(K::{ }^{\prime} a::\right.$ topological_space set $\Rightarrow$ 'a set $)$
$I \circ K \circ I \leq K \circ(I::$ 'a::topological_space set $\Rightarrow$ 'a set $)$
$I \circ K \leq K \circ I \circ\left(K:: ' a:: t o p o l o g i c a l \_\right.$space set $\Rightarrow$ 'a set $)$
$K \circ I \leq K \circ I \circ(K:: ' a::$ topological_space set $\Rightarrow$ 'a set $)$
$K \circ I \circ K \leq\left(K:: ' a:\right.$ :topological_space set $\Rightarrow{ }^{\prime} a$ set $)$
unfolding $I$ _def $K \_d e f$
by (simp_all add: interior_subset closure_subset interior_maximal closure_mono o_apply interior_mono K_I_K_subseteq_K le_funI)

We define the six classes of topological spaces in question, and show that they are related by inclusion in the following way (as shown in Figure 2.3 of Gardner and Jackson (2008)):


### 6.1 Discrete spaces

## definition

discrete ( $X$ :: 'a::topological_space set) $\longleftrightarrow I=\left(i d:: ' a\right.$ set $\Rightarrow{ }^{\prime} a$ set $)$
lemma discrete_eqs:
assumes discrete ( $X$ :: 'a::topological_space set)
shows
$I=(i d:: ' a$ set $\Rightarrow$ ' $a$ set $)$
$K=(i d:: ' a$ set $\Rightarrow$ 'a set $)$
using assms unfolding discrete_def by (auto simp: $C \_C K \_I$ )
lemma discrete_card:
assumes discrete ( $X$ :: 'a::topological_space set)
shows card $\{f$. CK_nf_pos $(f:: ' a$ set $\Rightarrow$ 'a set $)\}=1$
using discrete_eqs[OF assms] CK_nf_pos_lattice[where 'a='a] by (simp add: CK_nf_pos_set)
lemma discrete_discrete_topology:
fixes $X$ :: 'a::topological_space set
assumes $\wedge Y$ ::'a set. open $Y$
shows discrete $X$
using assms unfolding discrete_def I_def interior_def islimpt_def by (auto simp: fun_eq_iff)

### 6.2 Partition spaces

## definition

part ( $X$ :: 'a::topological_space set $) \longleftrightarrow K \circ I=(I:: ' a$ set $\Rightarrow$ 'a set $)$
lemma discrete_part:
assumes discrete $X$
shows part $X$
using assms unfolding discrete_def part_def by (simp add: C_C K_I)
lemma part_eqs:
assumes part (X :: 'a::topological_space set)
shows
$K \circ I=(I::$ 'a set $\Rightarrow$ 'a set $)$
$I \circ K=(K:: ' a$ set $\Rightarrow$ 'a set $)$
using assms unfolding part_def by (assumption, metis (no_types, opaque_lifting) I_I K_I o_assoc)
lemma part_not_discrete_card:
assumes part ( $X$ :: 'a::topological_space set)
assumes $\neg$ discrete $X$
shows card $\left\{f\right.$. CK_nf_pos ( $f::^{\prime} a$ set $\Rightarrow$ 'a set $\left.)\right\}=3$
using part_eqs $[O F\langle p a r t X\rangle]$ 〔discrete $X$ 〉 $C K \_n f \_p o s \_l a t t i c e\left[\right.$ where $\left.{ }^{\prime} a={ }^{\prime} a\right]$
unfolding discrete_def
by (simp add: CK_nf_pos_set card_insert_if C_C I_K K_K o_assoc; metis comp_id)
A partition space is a topological space whose basis consists of the empty set and the equivalence classes of points of the space induced by some equivalence relation $R$ on the underlying set of the space. Equivalently, a partition space is one in which every open set is closed. Thus, for example, the class of partition spaces includes every topological space whose open sets form a boolean algebra.
datatype part_witness $=a|b| c$
lemma part_witness_UNIV:
shows $U N I V=$ set $[a, b, c]$
using part_witness.exhaust by auto
lemmas part_witness_pow = subset_subseqs[OF subset_trans[OF subset_UNIV Set.equalityD1 [OF part_witness_UNIV
lemmas part_witness_Compl $=$ Compl_eq_Diff_UNIV[where ' $a=$ part_witness, unfolded part_witness_UNIV, simplified]
instantiation part_witness :: topological_space
begin
definition open_part_witness $X \longleftrightarrow X \in\{\},\{a\},\{b, c\},\{a, b, c\}\}$
lemma part_witness_ball:
$(\forall s \in S . s \in\{\{ \},\{a\},\{b, c\},\{a, b, c\}\}) \longleftrightarrow S \subseteq \operatorname{set}[\},\{a\},\{b, c\},\{a, b, c\}]$
by simp blast
lemmas part_witness_subsets_pow $=$ subset_subseqs[OF iffD1 $[$ OF part_witness_ball]]
instance proof standard
fix $K$ :: part_witness set set
assume $\forall S \in K$. open $S$ then show open $(\cup K)$
unfolding open_part_witness_def
by - (drule part_witness_subsets_pow; clarsimp; elim disjE; simp add: insert_commute)
qed (auto simp: open_part_witness_def part_witness_UNIV)
end
lemma part_witness_interior_simps:
shows
interior $\{a\}=\{a\}$
interior $\{b\}=\{ \}$
interior $\{c\}=\{ \}$
interior $\{a, b\}=\{a\}$
interior $\{a, c\}=\{a\}$
interior $\{b, c\}=\{b, c\}$
interior $\{a, b, c\}=\{a, b, c\}$
unfolding interior_def open_part_witness_def by auto
lemma part_witness_part:
fixes $X$ :: part_witness set
shows part $X$
proof -
have closure (interior $Y$ ) $=$ interior $Y$ for $Y::$ part_witness set
using part_witness_pow[where $X=Y]$
by (auto simp: closure_interior part_witness_interior_simps part_witness_Compl insert_Diff_if)
then show?thesis

qed
lemma part_witness_not_discrete:
fixes $X$ :: part_witness set
shows $\neg$ discrete $X$
unfolding discrete_def I_def
by (clarsimp simp: o_apply fun_eq_iff exI $[\mathbf{w h e r e} x=\{b\}]$ part_witness_interior_simps)
lemma part_witness_card:
shows card $\{f$. CK_nf_pos ( $f::$ part_witness set $\Rightarrow$ part_witness set $)\}=3$
by (rule part_not_discrete_card[OF part_witness_part part_witness_not_discrete])

### 6.3 Extremally disconnected and open unresolvable spaces

## definition

```
ed_ou (X :: 'a::topological_space set) \longleftrightarrowI \circ K=K ○(I :: 'a set # ' 'a set)
```

lemma discrete_ed_ou:
assumes discrete $X$
shows ed ou $X$
using assms unfolding discrete_def ed_ou_def by simp
lemma ed_ou_eqs:
assumes ed_ou ( $X$ :: 'a::topological_space set)
shows
$I \circ K \circ I=K \circ\left(I::{ }^{\prime} a\right.$ set $\Rightarrow{ }^{\prime} a$ set $)$
$K \circ I \circ K=K \circ\left(I::\right.$ 'a set $\Rightarrow{ }^{\prime}$ ' set $)$
$I \circ K=K \circ\left(I::\right.$ 'a set $\Rightarrow{ }^{\prime}$ 'a set $)$
using assms unfolding ed_ou_def by (metis I_I K_Ko_assoc)+
lemma ed_ou_neqs:
assumes ed_ou ( $X$ :: 'a::topological_space set)
assumes $\neg$ discrete $X$
shows
$I \neq(K::$ 'a set $\Rightarrow$ ' $a$ set $)$
$I \neq K \circ(I:: ' a$ set $\Rightarrow$ 'a set $)$
$K \neq K \circ(I:: ' a$ set $\Rightarrow$ 'a set $)$
$I \neq($ id $::$ 'a set $\Rightarrow$ ' $a$ set $)$
$K \neq(i d::$ 'a set $\Rightarrow$ 'a set $)$
using assms $C K \_n f \_p o s \_l a t t i c e\left[\right.$ where $\left.{ }^{\prime} a={ }^{\prime} a\right]$
unfolding ed_ou_def discrete_def
by (metis (no_types, lifting) C_C I_K K_I comp_id o_assoc antisym)+
lemma ed_ou_not_discrete_card:
assumes ed_ou ( $X$ :: 'a::topological_space set)
assumes $\neg$ discrete $X$
shows card $\left\{f\right.$. CK_nf_pos $\left(f::^{\prime} a\right.$ set $\Rightarrow{ }^{\prime} a$ set $\left.)\right\}=4$
using ed_ou_eqs[OF 〈ed_ou $X$ 〉] ed_ou_neqs $[O F$ assms]
by (subst CK_nf_pos_set) (subst card_insert_disjoint; (auto)[1])+
We consider an example extremally disconnected and open unresolvable topological space.
datatype ed_ou_witness $=a|b| c|d| e$
lemma ed_ou__witness__UNIV:
shows $U N I V=\operatorname{set}[a, b, c, d, e]$
using ed_ou_witness.exhaust by auto
lemmas ed__ou_witness_pow $=$ subset_subseqs[OF subset_trans[OF subset__UNIV Set.equalityD1[OF ed__ou_witness__
lemmas ed__ou_witness_Compl = Compl__eq_Diff__UNIV[where ' $a=e d \_$ou__witness, unfolded ed_ou_witness__UNIV, simplified]
instance ed_ou_witness :: finite
by standard (simp add: ed_ou__witness_UNIV)
instantiation ed_ou_witness :: topological_space
begin
inductive open_ed_oou_witness :: ed__ou_witness set $\Rightarrow$ bool where
open_ed_ou_witness $\}$
| open_ed_ou_witness $\{a\}$
|open_ed_ou_witness $\{b\}$
| open_ed_ou_witness $\{e\}$
| open_ed_ou_witness $\{a, c\}$
| open_ed_ou_witness $\{b, d\}$
|open_ed_ou_witness $\{a, c, e\}$
| open_ed_ou_witness $\{a, b\}$
| open_ed_ou_witness $\{a, e\}$
|open_ed_ou_witness $\{b, e\}$
|open_ed_ou_witness $\{a, b, c\}$
|open_ed_ou_witness $\{a, b, d\}$
|open_ed_ou_witness $\{a, b, e\}$
|open_ed_ou_witness $\{b, d, e\}$
| open_ed_ou_witness $\{a, b, c, d\}$
| open_ed_ou_witness $\{a, b, c, e\}$
$\mid$ open_ed_ou_witness $\{a, b, d, e\}$
$\mid$ open_ed_ou_witness $\{a, b, c, d, e\}$
declare open_ed_ou__witness.intros[intro!]
lemma ed__ou_witness_inter:
fixes $S$ :: ed_ou__witness set
assumes open $S$
assumes open $T$
shows open $(S \cap T)$
using assms by (auto elim!: open_ed_ou__witness.cases)
lemma ed_ou_witness_union:
fixes $X$ :: ed_ou_witness set set
assumes $\forall x \in X$. open $x$
shows open $(\bigcup X)$
using finite $[o f ~ X]$ assms
by (induct, force)
(clarsimp; elim open_ed_ou_witness.cases; simp add: open_ed_ou_witness.simps subset_insertI2 insert_commute;
metis Union_empty_conv)
instance
by standard (auto simp: ed_ou_witness_UNIV intro: ed_ou__witness_inter ed_ou_witness_union)
end

```
lemma ed_ou_witness_interior_simps:
    shows
    interior \(\{a\}=\{a\}\)
    interior \(\{b\}=\{b\}\)
    interior \(\{c\}=\{ \}\)
    interior \(\{d\}=\{ \}\)
    interior \(\{e\}=\{e\}\)
    interior \(\{a, b\}=\{a, b\}\)
    interior \(\{a, c\}=\{a, c\}\)
    interior \(\{a, d\}=\{a\}\)
    interior \(\{a, e\}=\{a, e\}\)
    interior \(\{b, c\}=\{b\}\)
    interior \(\{b, d\}=\{b, d\}\)
    interior \(\{b, e\}=\{b, e\}\)
    interior \(\{c, d\}=\{ \}\)
    interior \(\{c, e\}=\{e\}\)
    interior \(\{d, e\}=\{e\}\)
    interior \(\{a, b, c\}=\{a, b, c\}\)
    interior \(\{a, b, d\}=\{a, b, d\}\)
    interior \(\{a, b, e\}=\{a, b, e\}\)
    interior \(\{a, c, d\}=\{a, c\}\)
    interior \(\{a, c, e\}=\{a, c, e\}\)
    interior \(\{a, d, e\}=\{a, e\}\)
    interior \(\{b, c, d\}=\{b, d\}\)
    interior \(\{b, c, e\}=\{b, e\}\)
    interior \(\{b, d, e\}=\{b, d, e\}\)
    interior \(\{c, d, e\}=\{e\}\)
    interior \(\{a, b, c, d\}=\{a, b, c, d\}\)
    interior \(\{a, b, c, e\}=\{a, b, c, e\}\)
    interior \(\{a, b, d, e\}=\{a, b, d, e\}\)
    interior \(\{a, b, c, d, e\}=\{a, b, c, d, e\}\)
    interior \(\{a, c, d, e\}=\{a, c, e\}\)
    interior \(\{b, c, d, e\}=\{b, d, e\}\)
unfolding interior_def by safe (clarsimp simp: open_ed_ou_witness.simps; blast)+
lemma ed_ou_witness_not_discrete:
    fixes \(X\) :: ed_ou__witness set
    shows \(\neg\) discrete \(X\)
unfolding discrete_def I_def using ed_ou_witness_interior_simps by (force simp: fun_eq_iff)
lemma ed_ou__witness_ed_ou:
    fixes \(X\) :: ed_ou_witness set
    shows ed ou \(X\)
unfolding ed_ou_def I_def K_def
proof (clarsimp simp: o__apply fun_eq_iff)
    fix \(x\) :: ed_ou_witness set
    from ed_ou_witness_pow[of \(x\) ]
    show interior \((\) closure \(x)=\) closure \((\) interior \(x)\)
        by - (simp; elim disjE; simp add: closure_interior ed_ou_witness_interior_simps ed_ou_witness_Compl
insert_Diff_if)
qed
lemma ed_ou__witness_card:
    shows card \(\left\{f\right.\). CK__nf_pos \(\left.\left(f:: e d \_o u \_w i t n e s s ~ s e t ~ \Rightarrow e d \_o u \_w i t n e s s ~ s e t\right)\right\}=4\)
by (rule ed__ou_not_discrete_card[OF ed__ou_witness_ed_ou ed__ou_witness_not_discrete])
```


### 6.4 Extremally disconnected spaces

## definition

```
extremally_disconnected \((X::\) ' \(a::\) topological_space set \() \longleftrightarrow K \circ I \circ K=I \circ(K\) :: 'a set \(\Rightarrow\) ' \(a\) set \()\)
```

lemma ed_ou_part_extremally_disconnected:
assumes ed_ou $X$
assumes part $X$
shows extremally_disconnected $X$
using assms unfolding extremally_disconnected_def ed_ou_def part_def by simp
lemma extremally_disconnected_eqs:
fixes $X$ :: ' $a:$ :topological_space set
assumes extremally_disconnected $X$
shows
$I \circ K \circ I=K \circ(I:: ' a$ set $\Rightarrow$ 'a set $)$ $K \circ I \circ K=I \circ(K:: ' a$ set $\Rightarrow$ 'a set $)$
using assms unfolding extremally_disconnected_def by (metis K_I_K_I)+
lemma extremally_disconnected_not_part_not_ed_ou_card:
fixes $X$ :: 'a::topological_space set
assumes extremally_disconnected $X$
assumes $\neg$ part $X$
assumes $\neg e d \_o u X$
shows card $\left\{f\right.$. CK_nf_pos ( $f::^{\prime} a$ set $\Rightarrow{ }^{\prime} a$ set $\left.)\right\}=5$
using extremally_disconnected_eqs[OF 〈extremally_disconnected $X$ 〉] CK_nf_pos_lattice $[\mathbf{w h e r e}$ ' $a=$ ' $a$ ] assms $(2,3)$
unfolding part_def ed_ou_def
by (simp add: CK_nf_pos_set $C \_C I \_K K \_K o \_a s s o c ~ c a r d \_i n s e r t \_i f ;$ metis (no_types) $C \_C K \_I i d \_c o m p$ o_assoc)

Any topological space having an infinite underlying set and whose topology consists of the empty set and every cofinite subset of the underlying set is extremally disconnected. We consider an example such space having a countably infinite underlying set.
datatype ' $a$ cofinite $=$ cofinite ' $a$
instantiation cofinite :: (type) topological_space
begin
definition open_cofinite $=\left(\lambda X:::^{\prime} a\right.$ cofinite set. finite $\left.(-X) \vee X=\{ \}\right)$

## instance

by standard (auto simp: open_cofinite_defuminus_Sup)
end
lemma cofinite_closure_finite:
fixes $X$ :: 'a cofinite set
assumes finite $X$
shows closure $X=X$
using assms by (simp add: closed_open open_cofinite_def)
lemma cofinite_closure_infinite:
fixes $X$ :: ' $a$ cofinite set
assumes infinite $X$
shows closure $X=U N I V$
using assms by (metis Compl_empty_eq closure_subset double_compl finite_subset interior_complement open_cofinite open_interior)
lemma cofinite__interior_finite:
fixes $X$ :: 'a cofinite set
assumes finite $X$
assumes infinite (UNIV ::'a cofinite set)
shows interior $X=\{ \}$
using assms cofinite_closure_infinite $[$ where $X=-X]$ by (simp add: interior_cclosure)
lemma cofinite_interior_infinite:
fixes $X$ :: 'a cofinite set
assumes infinite $X$
assumes infinite $(-X)$
shows interior $X=\{ \}$
using assms cofinite_closure_infinite[where $X=-X]$ by (simp add: interior_closure)
abbreviation evens $::$ nat cofinite set $\equiv\{$ cofinite $n \mid n . \exists i . n=2 * i\}$
lemma evens_infinite:
shows infinite evens
proof $($ rule iffD2[OF infinite_iff_countable_subset], rule exI, rule conjI)
let ?f $=\lambda n$ ::nat. cofinite $(2 * n)$
show inj ?f by (auto intro: inj_onI)
show range ?f $\subseteq$ evens by auto
qed
lemma cofinite_nat_infinite:
shows infinite (UNIV::nat cofinite set)
using evens_infinite finite_Diff2 by fastforce
lemma evens_Compl_infinite:
shows infinite (- evens)
proof $($ rule iffD2[OF infinite_iff_countable_subset], rule exI, rule conjI)
let ?f $=\lambda n$ ::nat. cofinite $(2 * n+1)$
show inj ?f by (auto intro: inj_onI)
show range ?f $\subseteq-$ evens by clarsimp presburger
qed
lemma evens_closure:
shows closure evens $=U N I V$
using evens_infinite by (rule cofinite_closure_infinite)
lemma evens_interior:
shows interior evens $=\{ \}$
using evens_infinite evens_Compl_infinite by (rule cofinite_interior_infinite)
lemma cofinite_not_part:
fixes $X$ :: nat cofinite set
shows $\neg$ part $X$
unfolding part_def I_def K__def
using cofinite_nat_infinite
by (clarsimp simp: fun_eq_iff o_apply)
(metis (no_types) cofinite_closure_finite cofinite_interior_finite double_compl finite.emptyI finite.insertI
insert_not_empty interior_closure)
lemma cofinite_not_ed_ou:
fixes $X$ :: nat cofinite set
shows $\neg e d$ __ou $X$
unfolding ed_ou_def I_def K__def
by (clarsimp simp: fun_eq_iff o_apply evens_closure evens_interior exI [where $x=$ evens $]$ )
lemma cofinite_extremally_disconnected_aux:
fixes $X$ :: nat cofinite set
shows closure $($ interior $($ closure $X)) \subseteq$ interior (closure X)
by (metis subsetI closure_closure closure_complement closure_def closure_empty finite_Un interior_eq open_cofinite_de open_interior)
lemma cofinite_extremally_disconnected:
fixes $X$ :: nat cofinite set
shows extremally_disconnected $X$
unfolding extremally_disconnected_def I_def K_def
by (auto simp: fun_eq_iff o_apply dest: subsetD[OF closure_subset] subsetD[OF interior_subset] subsetD[OF
cofinite_extremally_disconnected_aux])
lemma cofinite_card:
shows card $\{f$. CK_nf_pos ( $f::$ nat cofinite set $\Rightarrow$ nat cofinite set) $\}=5$
by (rule extremally_disconnected_not_part_not_ed_ou_card[OF cofinite_extremally_disconnected cofinite_not_part cofinite_not_ed_ou])

### 6.5 Open unresolvable spaces

## definition

open_unresolvable $(X::$ 'a::topological_space set $) \longleftrightarrow K \circ I \circ K=K \circ(I:: ' a$ set $\Rightarrow$ ' $a$ set $)$
lemma ed_ou_open_unresolvable:
assumes ed_ou $X$
shows open_unresolvable $X$
using assms unfolding open_unresolvable_def by (simp add: ed_ou_eqs)
lemma open_unresolvable_eqs:
assumes open_unresolvable ( $X$ :: 'a::topological_space set)

## shows

$$
\begin{aligned}
& I \circ K \circ I=I \circ\left(K::{ }^{\prime} a \text { set } \Rightarrow \text { 'a set }\right) \\
& K \circ I \circ K=K \circ\left(I::{ }^{\prime} \text { a set } \Rightarrow \text { 'a set }\right)
\end{aligned}
$$

using assms unfolding open_unresolvable_def by $-\left(\right.$ metis $\left.I \_K \_I \_K o \_a s s o c ; ~ s i m p\right) ~$
lemma not_ed_ou_neqs:
assumes ᄀed_ou ( $X$ :: 'a::topological_space set)
shows
$I \neq I \circ\left(K::{ }^{\prime} a\right.$ set $\Rightarrow$ 'a set $)$
$K \neq K \circ(I:: ' a$ set $\Rightarrow$ 'a set $)$
using assms unfolding ed_ou_def
by (simp_all add: fun_eq_iff $\left.I \_K K \_d e f C \_d e f o \_a p p l y\right)$
(metis (no_types, opaque_lifting) closure_eq_empty disjoint_eq_subset_Compl double_complement interior_Int interior_complement set_eq_subset)+
lemma open_unresolvable_not_ed_ou_card:
assumes open_unresolvable ( $X$ :: 'a::topological_space set)
assumes $\neg e d \_o u ~ X$
shows card $\left\{f\right.$. CK_nf_pos $\left(f::^{\prime} a\right.$ set $\Rightarrow{ }^{\prime} a$ set $\left.)\right\}=5$
using open_unresolvable_eqs $[O F$ «open_unresolvable $X\rangle]$ not_ed_ou_neqs $\left[O F\left\langle\neg e d \_o u \quad X\right\rangle\right]\left\langle\neg e d \_o u \quad X\right\rangle$
unfolding ed_ou_def by (auto simp: CK_nf_pos_set card_insert_if)
We show that the class of open unresolvable spaces is non-empty by exhibiting an example of such a space.
datatype ou_witness $=a|b| c$
lemma ou_witness_UNIV:
shows UNIV $=\operatorname{set}[a, b, c]$
using ou witness.exhaust by auto
instantiation ou_witness :: topological_space
begin
definition open_ou_witness $X \longleftrightarrow a \notin X \vee X=U N I V$

## instance

by standard (auto simp: open_ou__witness_def)
end
lemma ou_witness_closure_simps:
shows

$$
\text { closure }\{a\}=\{a\}
$$

closure $\{b\}=\{a, b\}$
closure $\{c\}=\{a, c\}$
closure $\{a, b\}=\{a, b\}$
closure $\{a, c\}=\{a, c\}$
closure $\{a, b, c\}=\{a, b, c\}$
closure $\{b, c\}=\{a, b, c\}$
unfolding closure_def islimpt_def open_ou__witness_def by force+
lemma ou_witness_open_unresolvable:
fixes $X$ :: ou_witness set
shows open_unresolvable $X$
unfolding open_unresolvable_def I__def K_def
by (clarsimp simp: o_apply fun_eq_iff)
(metis (no_types, lifting) Compl_iff K_I_K_subseteq_K closure_complement closure_interior closure_mono
closure_subset interior_eq interior_maximal open_ou_witness_def subset_antisym)
lemma ou_witness_not_ed__ou:
fixes $X$ :: ou_witness set
shows $\neg e d \_$ou $X$
unfolding ed_ou_def I__def K__def
by (clarsimp simp: o_apply fun_eq_iff)
(metis UNIV_I insert_iff interior_eq open_ou_witness_def singletonD ou_witness.distinct $(4,5)$ ou_witness.simps(2) ou_witness_closure_simps(2))
lemma ou_witness_card:
shows card $\left\{f\right.$. CK_nf_pos ( $f:: o u \_w i t n e s s$ set $\Rightarrow$ ou_witness set $\left.)\right\}=5$
by (rule open_unresolvable_not_ed_ou_card[OF ou_witness_open_unresolvable ou_witness_not_ed_ou])

### 6.6 Kuratowski spaces

## definition

kuratowski ( $X$ :: 'a::topological_space set) $\longleftrightarrow$ $\neg$ extremally_disconnected $X \wedge \neg$ open_unresolvable $X$

A Kuratowski space distinguishes all 7 positive operators.
lemma part_closed__open:
fixes $X$ :: 'a::topological_space set
assumes $I \circ K \circ I=(I:: ' a$ set $\Rightarrow$ 'a set $)$
assumes closed $X$
shows open $X$
proof (rule Topological_Spaces.openI)
fix $x$ assume $x \in X$
let ? $S=I(-\{x\})$
let ? $G=-K$ ? $S$
have $x \in ? G$
proof -
from $\langle I \circ K \circ I=I\rangle$ have $I(K(I ? S))=? S I ? S=? S$
unfolding $I$ _def $K \_d e f$ by (simp_all add: o_def fun_eq_iff)
then have $K(I ? S) \neq U N I V$
unfolding $I \_d e f ~ K \_d e f$ using interior_subset by fastforce
moreover have $G \subseteq$ ? $S \vee x \in G$ if open $G$ for $G$
using that unfolding $I \_d e f$ by (meson interior_maximal subset_Compl_singleton)
ultimately show ?thesis
unfolding $I \_d e f$ K_def
by clarsimp (metis (no_types, lifting) ComplD Compl_empty_eq closure_interior closure_subset ex_in_conv open_interior subset_eq)
qed
moreover from $\langle I \circ K \circ I=I\rangle$ have open ? $G$
unfolding $I \_d e f K_{\_}$def by (auto simp: fun_eq_iff o_apply)
moreover have ? $G \subseteq X$
proof -
have ? $G \subseteq K$ ? $G$ unfolding $K \_d e f$ using closure_subset by fastforce
also from $\langle I \circ K \circ I=I\rangle$ have $\ldots=K\{x\}$
unfolding $I \_d e f$ K_def by (metis closure_interior comp_def double_complement)
also from $\langle$ closed $X\rangle\langle x \in X\rangle$ have $\ldots \subseteq X$
unfolding $K \_d e f$ by clarsimp (meson closure_minimal contra_subsetD empty_subsetI insert_subset)
finally show? ?thesis .
qed
ultimately show $\exists T$. open $T \wedge x \in T \wedge T \subseteq X$ by blast
qed
lemma part_I_K_I:
assumes $I \circ K \circ I=\left(I::{ }^{\prime} a::\right.$ topological_space set $\Rightarrow{ }^{\prime}$ 'a set $)$
shows $I \circ K=\left(K::{ }^{\prime} a\right.$ set $\Rightarrow$ 'a set $)$
using interior_open[OF part_closed_open[OF assms closed_closure]] unfolding $I$ _def $K \_d e f o \_d e f$ by simp
lemma part_K_I_I:
assumes $I \circ K \circ I=\left(I:: ' a::\right.$ topological_space set $\Rightarrow{ }^{\prime} a$ set $)$
shows $K \circ I=(I:: ' a$ set $\Rightarrow$ 'a set $)$
using part_I_K_I[OF assms] assms by simp
lemma kuratowski_neqs:
assumes kuratowski (X :: 'a::topological_space set)
shows
$I \neq I \circ K \circ\left(I::\right.$ 'a set $\Rightarrow{ }^{\prime}$ ' set $)$
$I \circ K \circ I \neq K \circ\left(I::{ }^{\prime} a\right.$ set $\Rightarrow{ }^{\prime} a$ set $)$
$I \circ K \circ I \neq I \circ\left(K::{ }^{\prime} a\right.$ set $\Rightarrow{ }^{\prime} a$ set $)$
$I \circ K \neq K \circ I \circ(K:: ' a$ set $\Rightarrow$ 'a set $)$
$K \circ I \neq K \circ I \circ(K:: ' a$ set $\Rightarrow$ 'a set $)$
$K \circ I \circ K \neq(K:: ' a$ set $\Rightarrow$ 'a set $)$
$I \circ K \neq K \circ(I::$ 'a set $\Rightarrow$ 'a set $)$
$I \neq(i d::$ 'a set $\Rightarrow$ ' $a$ set $)$
$K \neq(i d::$ 'a set $\Rightarrow$ 'a set $)$
$I \circ K \circ I \neq(i d::$ 'a set $\Rightarrow$ 'a set $)$
$K \circ I \circ K \neq(i d::$ 'a set $\Rightarrow$ 'a set $)$
using assms unfolding kuratowski_def extremally_disconnected_def open_unresolvable_def
by (metis (no_types, lifting) $\left.I \_K K \_K I \_K \_I \_K K \_I \_K \_I p a r t \_I \_K \_I p a r t \_K \_I \_I o \_a s s o c c o m p \_i d\right)+$
lemma kuratowski_card:
assumes kuratowski ( $X$ :: 'a::topological_space set)
shows card $\left\{f\right.$. CK_nf_pos $\left(f::^{\prime} a\right.$ set $\Rightarrow{ }^{\prime} a$ set $\left.)\right\}=7$
using $C K \_n f \_p o s \_l a t t i c e\left[\right.$ where $\left.{ }^{\prime} a=' a\right]$ kuratowski_neqs[OF assms] assms
unfolding kuratowski_def extremally_disconnected_def open_unresolvable_def
by (subst CK_nf_pos_set) (subst card_insert_disjoint; (auto)[1])+
$\mathbb{R}$ is a Kuratowski space.
lemma kuratowski_reals:
shows kuratowski ( $\mathbb{R}$ :: real set)
unfolding kuratowski_def extremally_disconnected_def open_unresolvable_def
by (rule conjI)
(metis (no_types, lifting) I_K list.inject $n f \_R R R(11) n f \_R R R(8) o \_a s s o c$,
metis (no_types, lifting) I_K fun.map_comp list.inject $\left.n f \_R R R(11) n f \_R R R(9)\right)$

### 6.7 Chagrov's theorem

theorem chagrov:
fixes $X$ :: 'a::topological_space set
obtains discrete $X$
$\mid \neg$ discrete $X \wedge$ part $X$
$\mid \neg$ discrete $X \wedge$ ed_ou $X$
$\mid \neg e d \_o u \quad X \wedge$ open_unresolvable $X$
$\mid \neg e d \_o u X \wedge \neg$ part $X \wedge$ extremally_disconnected $X$
| kuratowski X
unfolding kuratowski_def by metis
corollary chagrov_card:
shows card $\left\{f\right.$. CK_nf_pos ( $f::^{\prime}$ 'a::topological_space set $\Rightarrow{ }^{\prime}$ 'a set $\left.)\right\} \in\{1,3,4,5,7\}$
using discrete_card part_not_discrete_card ed_ou_not_discrete_card open_unresolvable_not_ed_ou_card extremally_disconnected_not_part_not_ed_ou_card kuratowski_card
by (cases rule: chagrov) blast+

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