

The Kuratowski Closure-Complement Theorem

Peter Gammie and Gianpaolo Gioiosa

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1 Introduction

We discuss a topological curiosity discovered by [Kuratowski \(1922\)](#): the fact that the number of distinct operators on a topological space generated by compositions of closure and complement never exceeds 14, and is exactly 14 in the case of \mathbb{R} . In addition, we prove a theorem due to [Chagrov \(1982\)](#) that classifies topological spaces according to the number of such operators they support.

Kuratowski's result, which is exposted in [Whitty \(2015\)](#) and Chapter 7 of [Chamberland \(2015\)](#), has already been treated in Mizar — see [Bagińska and Grabowski \(2003\)](#) and [Grabowski \(2004\)](#). To the best of our knowledge, we are the first to mechanize Chagrov's result.

Our work is based on a presentation of Kuratowski's and Chagrov's results by [Gardner and Jackson \(2008\)](#).

We begin with some preliminary facts pertaining to the relationship between interiors of unions and unions of interiors (§2) and the relationship between \mathbb{Q} and \mathbb{R} (§3). We then prove Kuratowski's result (§4) and the corollary that at most 7 distinct operators on a topological space can be generated by compositions of closure and interior (§5). Finally, we prove Chagrov's result (§6).

2 Interiors and unions

definition

boundary :: 'a::topological_space set \Rightarrow 'a set

where

boundary $X = \text{closure } X - \text{interior } X$

lemma *boundary_empty*:
shows *boundary* $\{\} = \{\}$
unfolding *boundary_def* **by** *simp*

definition

exterior :: 'a::topological_space set \Rightarrow 'a set

where

exterior $X = - (interior\ X \cup boundary\ X)$

lemma *interior_union_boundary*:

shows *interior* $(X \cup Y) = interior\ X \cup interior\ Y$

$\longleftrightarrow boundary\ X \cap boundary\ Y \subseteq boundary\ (X \cup Y)$ (**is** $(?lhs1 = ?lhs2) \longleftrightarrow ?rhs$)

proof(*rule iffI[OF - subset_antisym[OF subsetI]]*)

assume $?lhs1 = ?lhs2$ **then show** $?rhs$ **by** (*force simp: boundary_def*)

next

fix x

assume $?rhs$ **and** $x \in ?lhs1$

have $x \in ?lhs2$ **if** $x \notin interior\ X$ $x \notin interior\ Y$

proof(*cases* $x \in boundary\ X \cap boundary\ Y$)

case *True* **with** $\langle ?rhs \rangle \langle x \in ?lhs1 \rangle$ **show** $?thesis$ **by** (*simp add: boundary_def subset_iff*)

next

case *False* **then consider** $(X) x \notin boundary\ X \mid (Y) x \notin boundary\ Y$ **by** *blast*

then show $?thesis$

proof *cases*

case X

from $X \langle x \notin interior\ X \rangle$ **have** $x \in exterior\ X$ **by** (*simp add: exterior_def*)

from $\langle x \notin boundary\ X \rangle \langle x \in exterior\ X \rangle \langle x \notin interior\ X \rangle$

obtain U **where** *open* U $U \subseteq - X$ $x \in U$

by (*metis ComplI DiffI boundary_def closure_interior interior_subset open_interior*)

from $\langle x \in interior\ (X \cup Y) \rangle$ **obtain** U' **where** *open* U' $U' \subseteq X \cup Y$ $x \in U'$

by (*meson interiorE*)

from $\langle U \subseteq - X \rangle \langle U' \subseteq X \cup Y \rangle$ **have** $U \cap U' \subseteq Y$ **by** *blast*

with $\langle x \notin interior\ Y \rangle \langle open\ U' \rangle \langle open\ U \rangle \langle x \in U' \rangle \langle x \in U \rangle$ **show** $?thesis$

by (*meson IntI interiorI open_Int*)

next

case Y

from $Y \langle x \notin interior\ Y \rangle$ **have** $x \in exterior\ Y$ **by** (*simp add: exterior_def*)

from $\langle x \notin boundary\ Y \rangle \langle x \in exterior\ Y \rangle \langle x \notin interior\ Y \rangle$

obtain U **where** *open* U $U \subseteq - Y$ $x \in U$

by (*metis ComplI DiffI boundary_def closure_interior interior_subset open_interior*)

from $\langle x \in interior\ (X \cup Y) \rangle$ **obtain** U' **where** *open* U' $U' \subseteq X \cup Y$ $x \in U'$

by (*meson interiorE*)

from $\langle U \subseteq - Y \rangle \langle U' \subseteq X \cup Y \rangle$ **have** $U \cap U' \subseteq X$ **by** *blast*

with $\langle x \notin interior\ X \rangle \langle open\ U' \rangle \langle open\ U \rangle \langle x \in U' \rangle \langle x \in U \rangle$ **show** $?thesis$

by (*meson IntI interiorI open_Int*)

qed

qed

with $\langle x \in ?lhs1 \rangle$ **show** $x \in ?lhs2$ **by** *blast*

next

show $?lhs2 \subseteq ?lhs1$ **by** (*simp add: interior_mono*)

qed

lemma *interior_union_closed_intervals*:

fixes $a :: 'a::ordered_euclidean_space$

assumes $b < c$

shows *interior* $(\{a..b\} \cup \{c..d\}) = interior\ \{a..b\} \cup interior\ \{c..d\}$

using *assms* **by** (*subst interior_union_boundary; auto simp: boundary_def*)

3 Additional facts about the rationals and reals

lemma *Rat_real_limpt*:

fixes $x :: \text{real}$

shows $x \text{ islimpt } \mathbb{Q}$

proof(*rule islimptI*)

fix T **assume** $x \in T$ **open** T

then obtain e **where** $0 < e$ **and** $\text{ball}: |x' - x| < e \longrightarrow x' \in T$ **for** x' **by** (*auto simp: open_real*)

from $(0 < e)$ **obtain** q **where** $x < \text{real.of_rat } q \wedge \text{real.of_rat } q < x + e$ **using** *of_rat_dense* **by force**

with ball **show** $\exists y \in \mathbb{Q}. y \in T \wedge y \neq x$ **by force**

qed

lemma *Rat_closure*:

shows $\text{closure } \mathbb{Q} = (\text{UNIV} :: \text{real set})$

unfolding *closure_def* **using** *Rat_real_limpt* **by blast**

lemma *Rat_interval_closure*:

fixes $x :: \text{real}$

assumes $x < y$

shows $\text{closure } (\{x <..<y\} \cap \mathbb{Q}) = \{x..y\}$

using *assms*

by (*metis (no_types, lifting) Rat_closure closure_closure closure_greaterThanLessThan closure_mono inf_le1 inf_top.right_ne open_Int_closure_subset open_real_greaterThanLessThan subset_antisym*)

lemma *Rat_not_open*:

fixes $T :: \text{real set}$

assumes *open* T

assumes $T \neq \{\}$

shows $\neg T \subseteq \mathbb{Q}$

using *assms* **by** (*simp add: countable_rat open_minus_countable subset_eq*)

lemma *Irrat_dense_in_real*:

fixes $x :: \text{real}$

assumes $x < y$

shows $\exists r \in -\mathbb{Q}. x < r \wedge r < y$

using *assms* *Rat_not_open*[**where** $T = \{x <..<y\}$] **by force**

lemma *closed_interval_Int_compl*:

fixes $x :: \text{real}$

assumes $x < y$

assumes $y < z$

shows $-\{x..y\} \cap -\{y..z\} = -\{x..z\}$

using *assms* **by auto**

4 Kuratowski's result

We prove that at most 14 distinct operators can be generated by compositions of *closure* and complement. For convenience, we give these operators short names and try to avoid pointwise reasoning. We treat the *interior* operator at the same time.

declare *o_apply*[*simp del*]

definition $C :: 'a :: \text{topological_space set} \Rightarrow 'a \text{ set}$ **where** $C X = - X$

definition $K :: 'a :: \text{topological_space set} \Rightarrow 'a \text{ set}$ **where** $K X = \text{closure } X$

definition $I :: 'a :: \text{topological_space set} \Rightarrow 'a \text{ set}$ **where** $I X = \text{interior } X$

lemma *C_C*:
shows $C \circ C = id$
by (*simp add: fun_eq_iff C_def o_apply*)

lemma *K_K*:
shows $K \circ K = K$
by (*simp add: fun_eq_iff K_def o_apply*)

lemma *I_I*:
shows $I \circ I = I$
unfolding *I_def* **by** (*simp add: o_def*)

lemma *I_K*:
shows $I = C \circ K \circ C$
unfolding *C_def I_def K_def* **by** (*simp add: o_def interior_closure*)

lemma *K_I*:
shows $K = C \circ I \circ C$
unfolding *C_def I_def K_def* **by** (*simp add: o_def interior_closure*)

lemma *K_I_K_I*:
shows $K \circ I \circ K \circ I = K \circ I$
unfolding *C_def I_def K_def*
by (*clarsimp simp: fun_eq_iff o_apply closure_minimal closure_mono closure_subset interior_maximal interior_subset subset_antisym*)

lemma *I_K_I_K*:
shows $I \circ K \circ I \circ K = I \circ K$
unfolding *C_def I_def K_def*
by (*simp add: fun_eq_iff o_apply*)
(*metis (no_types) closure_closure closure_mono closure_subset interior_maximal interior_mono interior_subset open_interior subset_antisym*)

lemma *K_mono*:
assumes $x \subseteq y$
shows $K x \subseteq K y$
using *assms* **unfolding** *K_def* **by** (*simp add: closure_mono*)

The following lemma embodies the crucial observation about compositions of C and K :

lemma *KCKCKCK_KCK*:
shows $K \circ C \circ K \circ C \circ K \circ C \circ K = K \circ C \circ K$ (**is** *?lhs = ?rhs*)
proof(*rule ext[OF equalityI]*)
fix x
have $(C \circ K \circ C \circ K \circ C \circ K) x \subseteq ?rhs x$ **by** (*simp add: C_def K_def closure_def o_apply*)
then have $(K \circ (C \circ K \circ C \circ K \circ C \circ K)) x \subseteq (K \circ ?rhs) x$ **by** (*simp add: K_mono o_apply*)
then show $?lhs x \subseteq ?rhs x$ **by** (*simp add: K_K o_assoc*)
next
fix $x :: 'a::topological_space\ set$
have $(C \circ K \circ C \circ K) x \subseteq K x$ **by** (*simp add: C_def K_def closure_def o_apply*)
then have $(K \circ (C \circ K \circ C \circ K)) x \subseteq (K \circ K) x$ **by** (*simp add: K_mono o_apply*)
then have $(C \circ (K \circ K)) x \subseteq (C \circ (K \circ (C \circ K \circ C \circ K))) x$ **by** (*simp add: C_def o_apply*)
then have $(K \circ (C \circ (K \circ K))) x \subseteq (K \circ (C \circ (K \circ (C \circ K \circ C \circ K)))) x$ **by** (*simp add: K_mono o_apply*)
then show $?rhs x \subseteq ?lhs x$ **by** (*simp add: K_K o_assoc*)
qed

The inductive set CK captures all operators that can be generated by compositions of C and K . We shallowly embed the operators; that is, we identify operators up to extensional equality.

inductive *CK* :: (*'a::topological_space set* \Rightarrow *'a set*) \Rightarrow *bool* **where**

```

  CK C
| CK K
| [ CK f; CK g ]  $\implies$  CK (f  $\circ$  g)

```

declare *CK.intros*[intro!]

lemma *CK_id*[intro!]:

```

  CK id
by (metis CK.intros(1) CK.intros(3) C_C)

```

The inductive set *CK_nf* captures the normal forms for the 14 distinct operators.

inductive *CK_nf* :: ('a::topological_space set \Rightarrow 'a set) \Rightarrow bool **where**

```

  CK_nf id
| CK_nf C
| CK_nf K
| CK_nf (C  $\circ$  K)
| CK_nf (K  $\circ$  C)
| CK_nf (C  $\circ$  K  $\circ$  C)
| CK_nf (K  $\circ$  C  $\circ$  K)
| CK_nf (C  $\circ$  K  $\circ$  C  $\circ$  K)
| CK_nf (K  $\circ$  C  $\circ$  K  $\circ$  C)
| CK_nf (C  $\circ$  K  $\circ$  C  $\circ$  K  $\circ$  C)
| CK_nf (K  $\circ$  C  $\circ$  K  $\circ$  C  $\circ$  K)
| CK_nf (C  $\circ$  K  $\circ$  C  $\circ$  K  $\circ$  C  $\circ$  K)
| CK_nf (K  $\circ$  C  $\circ$  K  $\circ$  C  $\circ$  K  $\circ$  C)
| CK_nf (C  $\circ$  K  $\circ$  C  $\circ$  K  $\circ$  C  $\circ$  K  $\circ$  C)

```

declare *CK_nf.intros*[intro!]

lemma *CK_nf_set*:

```

  shows {f . CK_nf f} = {id, C, K, C  $\circ$  K, K  $\circ$  C, C  $\circ$  K  $\circ$  C, K  $\circ$  C  $\circ$  K, C  $\circ$  K  $\circ$  C  $\circ$  K, K  $\circ$  C  $\circ$  K  $\circ$  C, C  $\circ$  K  $\circ$  C  $\circ$  K  $\circ$  C, K  $\circ$  C  $\circ$  K  $\circ$  C  $\circ$  K, C  $\circ$  K  $\circ$  C  $\circ$  K  $\circ$  C  $\circ$  K, K  $\circ$  C  $\circ$  K  $\circ$  C  $\circ$  K  $\circ$  C, C  $\circ$  K  $\circ$  C  $\circ$  K  $\circ$  C  $\circ$  K  $\circ$  C}
by (auto simp: CK_nf.simps)

```

That each operator generated by compositions of *C* and *K* is extensionally equivalent to one of the normal forms captured by *CK_nf* is demonstrated by means of an induction over the construction of *CK_nf* and an appeal to the facts proved above.

theorem *CK_nf*:

```

  CK f  $\longleftrightarrow$  CK_nf f
proof(rule iffI)
  assume CK f then show CK_nf f
  by induct
  (elim CK_nf.cases; clarsimp simp: id_def[symmetric] C_C K_K KCKCKCK_KCK o_assoc; simp add:
o_assoc[symmetric]; clarsimp simp: C_C K_K KCKCKCK_KCK o_assoc
| blast)+
next
  assume CK_nf f then show CK f by induct (auto simp: id_def[symmetric])
qed

```

theorem *CK_card*:

```

  shows card {f. CK f}  $\leq$  14
by (auto simp: CK_nf CK_nf_set card_insert intro!: le_trans[OF card_Diff1_le])

```

We show, using the following subset of \mathbb{R} (an example taken from Rusin (2001)) as a witness, that there exist topological spaces on which all 14 operators are distinct.

definition

```

RRR :: real set

```

where

$$RRR = \{0 < \cdot < 1\} \cup \{1 < \cdot < 2\} \cup \{3\} \cup (\{5 < \cdot < 7\} \cap \mathbb{Q})$$

The following facts allow the required proofs to proceed by *simp*:

lemma *RRR_closure*:

shows $\text{closure } RRR = \{0..2\} \cup \{3\} \cup \{5..7\}$

unfolding *RRR_def* **by** (*force simp: closure_insert Rat_interval_closure*)

lemma *RRR_interior*:

interior $RRR = \{0 < \cdot < 1\} \cup \{1 < \cdot < 2\}$ (**is** $?lhs = ?rhs$)

proof(*rule equalityI[OF subsetI subsetI]*)

fix x **assume** $x \in ?lhs$

then obtain T **where** *open* T **and** $x \in T$ **and** $T \subseteq RRR$ **by** (*blast elim: interiorE*)

then obtain e **where** $0 < e$ **and** $\text{ball } x \ e \subseteq T$ **by** (*blast elim!: openE*)

from $\langle x \in T \rangle \langle 0 < e \rangle \langle \text{ball } x \ e \subseteq T \rangle \langle T \subseteq RRR \rangle$

have *False* **if** $x = 3$

using *that* **unfolding** *RRR_def ball_def*

by (*auto dest!: subsetD[where c=min (3 + e/2) 4] simp: dist_real_def*)

moreover

from *Irrat_dense_in_real[where x=x and y=x + e/2] <0 < e>*

obtain r **where** $r \in -\mathbb{Q} \wedge x < r \wedge r < x + e / 2$ **by** *auto*

with $\langle x \in T \rangle \langle \text{ball } x \ e \subseteq T \rangle \langle T \subseteq RRR \rangle$

have *False* **if** $x \in \{5 < \cdot < 7\} \cap \mathbb{Q}$

using *that* **unfolding** *RRR_def ball_def*

by (*force simp: dist_real_def dest: subsetD[where c=r]*)

moreover note $\langle x \in \text{interior } RRR \rangle$

ultimately show $x \in ?rhs$

unfolding *RRR_def* **by** (*auto dest: subsetD[OF interior_subset]*)

next

fix x **assume** $x \in ?rhs$

then show $x \in ?lhs$

unfolding *RRR_def interior_def* **by** (*auto intro: open_real_greaterThanLessThan*)

qed

lemma *RRR_interior_closure[simplified]*:

shows $\text{interior } (\{0::\text{real}..2\} \cup \{3\} \cup \{5..7\}) = \{0 < \cdot < 2\} \cup \{5 < \cdot < 7\}$ (**is** $?lhs = ?rhs$)

proof –

have $?lhs = \text{interior } (\{0..2\} \cup \{5..7\})$

by (*metis (no_types, lifting) Un_assoc Un_commute closed_Un closed_eucl_atLeastAtMost interior_closed_Un_empty_interior_singleton*)

also have $\dots = ?rhs$

by (*simp add: interior_union_closed_intervals*)

finally show *?thesis* .

qed

The operators can be distinguished by testing which of the points in $\{1,2,3,4,6\}$ belong to their results.

definition

$\text{test} :: (\text{real set} \Rightarrow \text{real set}) \Rightarrow \text{bool list}$

where

$\text{test } f \equiv \text{map } (\lambda x. x \in f \text{ } RRR) [1,2,3,4,6]$

lemma *RRR_test*:

assumes $f \text{ } RRR = g \text{ } RRR$

shows $\text{test } f = \text{test } g$

unfolding *test_def* **using** *assms* **by** *simp*

lemma *nf_RRR*:

shows

```

test id = [False, False, True, False, True]
test C = [True, True, False, True, False]
test K = [True, True, True, False, True]
test (K ∘ C) = [True, True, True, True, True]
test (C ∘ K) = [False, False, False, True, False]
test (C ∘ K ∘ C) = [False, False, False, False, False]
test (K ∘ C ∘ K) = [False, True, True, True, False]
test (C ∘ K ∘ C ∘ K) = [True, False, False, False, True]
test (K ∘ C ∘ K ∘ C) = [True, True, False, False, False]
test (C ∘ K ∘ C ∘ K ∘ C) = [False, False, True, True, True]
test (K ∘ C ∘ K ∘ C ∘ K) = [True, True, False, False, True]
test (C ∘ K ∘ C ∘ K ∘ C ∘ K) = [False, False, True, True, False]
test (K ∘ C ∘ K ∘ C ∘ K ∘ C) = [False, True, True, True, True]
test (C ∘ K ∘ C ∘ K ∘ C ∘ K ∘ C) = [True, False, False, False, False]

```

unfolding *test_def C_def K_def*

by (*simp_all add: RRR_closure RRR_interior RRR_interior_closure closure_complement closed_interval_Int_compl o_apply*)
(*simp_all add: RRR_def*)

theorem *CK_nf_real_card*:

shows *card ((λ f. f RRR) ‘ {f . CK_nf f}) = 14*

by (*simp add: CK_nf_set*) (*(subst card_insert_disjoint; auto dest!: RRR_test simp: nf_RRR id_def[symmetric])[1])+*

theorem *CK_real_card*:

shows *card {f::real set ⇒ real set. CK f} = 14 (is ?lhs = ?rhs)*

proof(*rule antisym[OF CK_card]*)

show *?rhs ≤ ?lhs*

unfolding *CK_nf*

by (*rule le_trans[OF eq_imp_le[OF CK_nf_real_card[symmetric]] card_image_le]*)

(*simp add: CK_nf_set*)

qed

5 A corollary of Kuratowski’s result

We show that it is a corollary of *CK_real_card* that at most 7 distinct operators on a topological space can be generated by compositions of closure and interior. In the case of \mathbb{R} , exactly 7 distinct operators can be so generated.

inductive *IK* :: (*'a::topological_space set ⇒ 'a set*) ⇒ *bool* **where**

```

  IK id
| IK I
| IK K
| [ IK f; IK g ] ⇒ IK (f ∘ g)

```

inductive *IK_nf* :: (*'a::topological_space set ⇒ 'a set*) ⇒ *bool* **where**

```

  IK_nf id
| IK_nf I
| IK_nf K
| IK_nf (I ∘ K)
| IK_nf (K ∘ I)
| IK_nf (I ∘ K ∘ I)
| IK_nf (K ∘ I ∘ K)

```

declare *IK.intros[intro!]*

declare *IK_nf.intros[intro!]*

lemma *IK_nf_set*:

{f . IK_nf f} = {id, I, K, I ∘ K, K ∘ I, I ∘ K ∘ I, K ∘ I ∘ K}

by (auto simp: *IK_nf_simps*)

theorem *IK_nf*:

$IK\ f \longleftrightarrow IK_nf\ f$

proof(rule *iffI*)

assume $IK\ f$ then show $IK_nf\ f$

by induct

(elim *IK_nf.cases*; clarsimp simp: *id_def[symmetric]* *o_assoc*; simp add: *I_I K_K o_assoc[symmetric]*; clarsimp simp: *K_I K_I I_K I_K o_assoc*

| *blast*)+

next

assume $IK_nf\ f$ then show $IK\ f$ by induct *blast*+

qed

theorem *IK_card*:

shows $card\ \{f.\ IK\ f\} \leq 7$

by (auto simp: *IK_nf IK_nf_set card_insert intro!*: *le_trans[OF card_Diff1_le]*)

theorem *IK_nf_real_card*:

shows $card\ ((\lambda f.\ f\ RRR)\ \{\ f.\ IK_nf\ f\}) = 7$

by (simp add: *IK_nf_set*) ((subst *card_insert_disjoint*; auto dest!: *RRR_test simp: nf_RRR I_K id_def[symmetric] o_assoc*)[1])+

theorem *IK_real_card*:

shows $card\ \{f::real\ set \Rightarrow real\ set.\ IK\ f\} = 7$ (is ?lhs = ?rhs)

proof(rule *antisym[OF IK_card]*)

show ?rhs \leq ?lhs

unfolding *IK_nf*

by (rule *le_trans[OF eq_refl[OF IK_nf_real_card[symmetric]] card_image_le]*)

(simp add: *IK_nf_set*)

qed

6 Chagrov's result

Chagrov's theorem, which is discussed in Section 2.1 of Gardner and Jackson (2008), states that the number of distinct operators on a topological space that can be generated by compositions of closure and complement is one of 2, 6, 8, 10 or 14.

We begin by observing that the set of normal forms *CK_nf* can be split into two disjoint sets, *CK_nf_pos* and *CK_nf_neg*, which we define in terms of interior and closure.

inductive *CK_nf_pos* :: ('a::topological_space set \Rightarrow 'a set) \Rightarrow bool **where**

CK_nf_pos id
| *CK_nf_pos I*
| *CK_nf_pos K*
| *CK_nf_pos (I \circ K)*
| *CK_nf_pos (K \circ I)*
| *CK_nf_pos (I \circ K \circ I)*
| *CK_nf_pos (K \circ I \circ K)*

declare *CK_nf_pos.intros*[*intro!*]

lemma *CK_nf_pos_set*:

shows $\{f.\ CK_nf_pos\ f\} = \{id,\ I,\ K,\ I \circ K,\ K \circ I,\ I \circ K \circ I,\ K \circ I \circ K\}$

by (auto simp: *CK_nf_pos_simps*)

definition

CK_nf_neg :: ('a::topological_space set \Rightarrow 'a set) \Rightarrow bool

where

$$CK_nf_neg\ f \longleftrightarrow (\exists g. CK_nf_pos\ g \wedge f = C \circ g)$$

lemma *CK_nf_pos_neg_disjoint*:

assumes *CK_nf_pos* *f*

assumes *CK_nf_neg* *g*

shows $f \neq g$

using *assms unfolding CK_nf_neg_def*

by (*clarsimp simp: CK_nf_pos.simps; elim disjE; metis comp_def C_def I_def K_def Compl_iff closure_UNIV interior_UNIV id_apply*)

lemma *CK_nf_pos_neg_CK_nf*:

$$CK_nf\ f \longleftrightarrow CK_nf_pos\ f \vee CK_nf_neg\ f \text{ (is } ?lhs \longleftrightarrow ?rhs)$$

proof(*rule iffI*)

assume *?lhs* **then show** *?rhs*

unfolding *CK_nf_neg_def*

by (*rule CK_nf.cases; metis (no_types, lifting) CK_nf_pos.simps C_C I_K K_I comp_id o_assoc*)

next

assume *?rhs* **then show** *?lhs*

unfolding *CK_nf_neg_def*

by (*auto elim!: CK_nf_pos.cases simp: I_K C_C o_assoc*)

qed

We now focus on *CK_nf_pos*. In particular, we show that its cardinality for any given topological space is one of 1, 3, 4, 5 or 7.

The proof consists of exhibiting normal forms for the operators supported by each of six classes of topological spaces. These are sublattices of the following lattice of *CK_nf_pos* operators:

lemmas *K_I_K_subseteq_K = closure_mono[OF interior_subset, of closure X, simplified]* **for** *X*

lemma *CK_nf_pos_lattice*:

shows

$$I \leq (id :: 'a::topological_space\ set \Rightarrow 'a\ set)$$

$$id \leq (K :: 'a::topological_space\ set \Rightarrow 'a\ set)$$

$$I \leq I \circ K \circ (I :: 'a::topological_space\ set \Rightarrow 'a\ set)$$

$$I \circ K \circ I \leq I \circ (K :: 'a::topological_space\ set \Rightarrow 'a\ set)$$

$$I \circ K \circ I \leq K \circ (I :: 'a::topological_space\ set \Rightarrow 'a\ set)$$

$$I \circ K \leq K \circ I \circ (K :: 'a::topological_space\ set \Rightarrow 'a\ set)$$

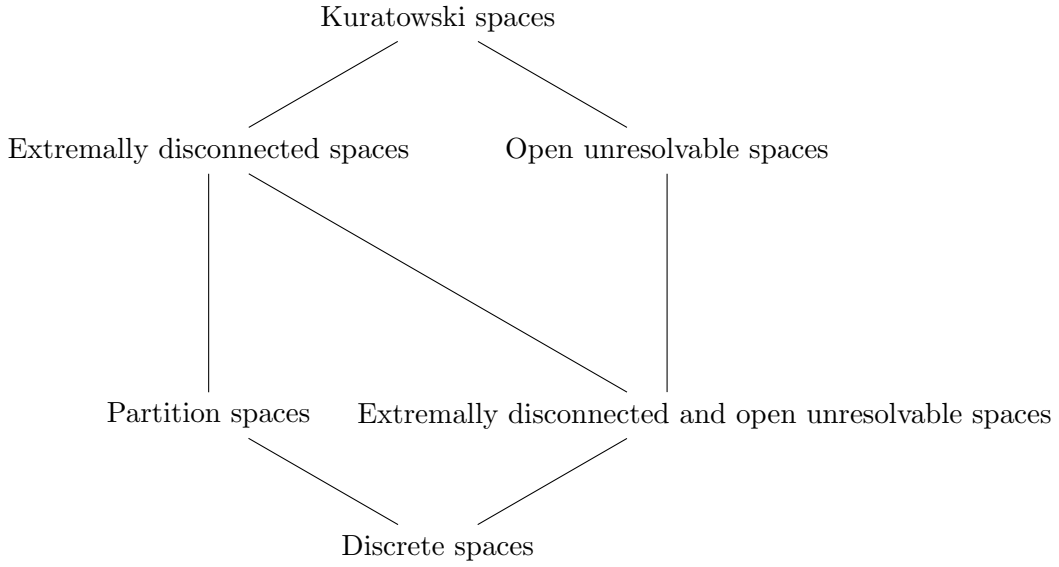
$$K \circ I \leq K \circ I \circ (K :: 'a::topological_space\ set \Rightarrow 'a\ set)$$

$$K \circ I \circ K \leq (K :: 'a::topological_space\ set \Rightarrow 'a\ set)$$

unfolding *I_def K_def*

by (*simp_all add: interior_subset closure_subset interior_maximal closure_mono o_apply interior_mono K_I_K_subseteq_K le_funI*)

We define the six classes of topological spaces in question, and show that they are related by inclusion in the following way (as shown in Figure 2.3 of [Gardner and Jackson \(2008\)](#)):



6.1 Discrete spaces

definition

$discrete (X :: 'a::topological_space\ set) \longleftrightarrow I = (id::'a\ set \Rightarrow 'a\ set)$

lemma *discrete_eqs*:

assumes $discrete (X :: 'a::topological_space\ set)$

shows

$I = (id::'a\ set \Rightarrow 'a\ set)$

$K = (id::'a\ set \Rightarrow 'a\ set)$

using *assms* **unfolding** *discrete_def* **by** (*auto simp: C_C K_I*)

lemma *discrete_card*:

assumes $discrete (X :: 'a::topological_space\ set)$

shows $card \{f. CK_nf_pos (f::'a\ set \Rightarrow 'a\ set)\} = 1$

using *discrete_eqs*[*OF assms*] *CK_nf_pos_lattice*[**where** $'a='a$] **by** (*simp add: CK_nf_pos_set*)

lemma *discrete_discrete_topology*:

fixes $X :: 'a::topological_space\ set$

assumes $\bigwedge Y::'a\ set. open\ Y$

shows $discrete\ X$

using *assms* **unfolding** *discrete_def I_def interior_def islimpt_def* **by** (*auto simp: fun_eq_iff*)

6.2 Partition spaces

definition

$part (X :: 'a::topological_space\ set) \longleftrightarrow K \circ I = (I :: 'a\ set \Rightarrow 'a\ set)$

lemma *discrete_part*:

assumes $discrete\ X$

shows $part\ X$

using *assms* **unfolding** *discrete_def part_def* **by** (*simp add: C_C K_I*)

lemma *part_eqs*:

assumes $part (X :: 'a::topological_space\ set)$

shows

$K \circ I = (I :: 'a\ set \Rightarrow 'a\ set)$

$I \circ K = (K :: 'a\ set \Rightarrow 'a\ set)$

using *assms* **unfolding** *part_def* **by** (*assumption, metis (no_types, hide_lams) I_I K_I o_assoc*)

```

lemma part_not_discrete_card:
  assumes part ( $X :: 'a::\text{topological\_space}$  set)
  assumes  $\neg \text{discrete } X$ 
  shows  $\text{card } \{f. \text{CK\_nf\_pos } (f::'a \text{ set} \Rightarrow 'a \text{ set})\} = 3$ 
using part_eqs[OF  $\langle \text{part } X \rangle$ ]  $\langle \neg \text{discrete } X \rangle$  CK\_nf\_pos\_lattice[where  $'a='a$ ]
unfolding discrete_def
by (simp add: CK\_nf\_pos\_set card_insert_if C_C I_K K_K o\_assoc;metis comp_id)

```

A partition space is a topological space whose basis consists of the empty set and the equivalence classes of points of the space induced by some equivalence relation R on the underlying set of the space. Equivalently, a partition space is one in which every open set is closed. Thus, for example, the class of partition spaces includes every topological space whose open sets form a boolean algebra.

```

datatype part_witness =  $a \mid b \mid c$ 

```

```

lemma part_witness_UNIV:
  shows  $\text{UNIV} = \text{set } [a, b, c]$ 
using part_witness.exhaust by auto

```

```

lemmas part_witness_pow = subset_subseqs[OF subset_trans[OF subset_UNIV Set.equalityD1[OF part_witness_UNIV]]]

```

```

lemmas part_witness_Compl = Compl_eq_Diff_UNIV[where  $'a=\text{part\_witness}$ , unfolded part_witness_UNIV, simplified]

```

```

instantiation part_witness :: topological_space
begin

```

```

definition open_part_witness  $X \longleftrightarrow X \in \{\{\}, \{a\}, \{b, c\}, \{a, b, c\}\}$ 

```

```

lemma part_witness_ball:
   $(\forall s \in S. s \in \{\{\}, \{a\}, \{b, c\}, \{a, b, c\}\}) \longleftrightarrow S \subseteq \text{set } [\{\}, \{a\}, \{b, c\}, \{a, b, c\}]$ 
by simp blast

```

```

lemmas part_witness_subsets_pow = subset_subseqs[OF iffD1[OF part_witness_ball]]

```

```

instance proof standard
  fix  $K :: \text{part\_witness}$  set set
  assume  $\forall S \in K. \text{open } S$  then show  $\text{open } (\bigcup K)$ 
  unfolding open_part_witness_def
  by  $-$  (drule part_witness_subsets_pow; clarsimp; elim disjE; simp add: insert_commute)
qed (auto simp: open_part_witness_def part_witness_UNIV)

```

```

end

```

```

lemma part_witness_interior_simps:
  shows
    interior  $\{a\} = \{a\}$ 
    interior  $\{b\} = \{\}$ 
    interior  $\{c\} = \{\}$ 
    interior  $\{a, b\} = \{a\}$ 
    interior  $\{a, c\} = \{a\}$ 
    interior  $\{b, c\} = \{b, c\}$ 
    interior  $\{a, b, c\} = \{a, b, c\}$ 
unfolding interior_def open_part_witness_def by auto

```

```

lemma part_witness_part:
  fixes  $X :: \text{part\_witness}$  set
  shows part  $X$ 
proof  $-$ 

```

```

have closure (interior Y) = interior Y for Y :: part_witness set
  using part_witness_pow[where X=Y]
  by (auto simp: closure_interior part_witness_interior_simps part_witness_Compl insert_Diff_if)
then show ?thesis
  unfolding part_def I_def K_def by (simp add: o_def)
qed

```

```

lemma part_witness_not_discrete:
  fixes X :: part_witness set
  shows ¬discrete X
unfolding discrete_def I_def
by (clarsimp simp: o_apply fun_eq_iff exI[where x={b}] part_witness_interior_simps)

```

```

lemma part_witness_card:
  shows card {f. CK_nf_pos (f::part_witness set ⇒ part_witness set)} = 3
by (rule part_not_discrete_card[OF part_witness_part part_witness_not_discrete])

```

6.3 Extremally disconnected and open unresolvable spaces

definition

```

ed_ou (X :: 'a::topological_space set) ⟷ I ∘ K = K ∘ (I :: 'a set ⇒ 'a set)

```

```

lemma discrete_ed_ou:
  assumes discrete X
  shows ed_ou X
using assms unfolding discrete_def ed_ou_def by simp

```

```

lemma ed_ou_eqs:
  assumes ed_ou (X :: 'a::topological_space set)
  shows
    I ∘ K ∘ I = K ∘ (I :: 'a set ⇒ 'a set)
    K ∘ I ∘ K = K ∘ (I :: 'a set ⇒ 'a set)
    I ∘ K = K ∘ (I :: 'a set ⇒ 'a set)
using assms unfolding ed_ou_def by (metis I I K K o_assoc)+

```

```

lemma ed_ou_neqs:
  assumes ed_ou (X :: 'a::topological_space set)
  assumes ¬discrete X
  shows
    I ≠ (K :: 'a set ⇒ 'a set)
    I ≠ K ∘ (I :: 'a set ⇒ 'a set)
    K ≠ K ∘ (I :: 'a set ⇒ 'a set)
    I ≠ (id :: 'a set ⇒ 'a set)
    K ≠ (id :: 'a set ⇒ 'a set)
using assms CK_nf_pos_lattice[where 'a='a]
unfolding ed_ou_def discrete_def
by (metis (no_types, lifting) C_C I K K I comp_id o_assoc antisym)+

```

```

lemma ed_ou_not_discrete_card:
  assumes ed_ou (X :: 'a::topological_space set)
  assumes ¬discrete X
  shows card {f. CK_nf_pos (f::'a set ⇒ 'a set)} = 4
using ed_ou_eqs[OF ed_ou X] ed_ou_neqs[OF assms]
by (subst CK_nf_pos_set) (subst card_insert_disjoint; (auto)[1])+

```

We consider an example extremally disconnected and open unresolvable topological space.

```

datatype ed_ou_witness = a | b | c | d | e

```

```

lemma ed_ou_witness_UNIV:
  shows  $UNIV = set [a, b, c, d, e]$ 
using ed_ou_witness.exhaust by auto

lemmas ed_ou_witness_pow = subset_subseqs[OF subset_trans[OF subset_UNIV Set.equalityD1 [OF ed_ou_witness_UNIV]]]

lemmas ed_ou_witness_Compl = Compl_eq_Diff_UNIV[where 'a=ed_ou_witness, unfolded ed_ou_witness_UNIV, simplified]

instance ed_ou_witness :: finite
by standard (simp add: ed_ou_witness_UNIV)

instantiation ed_ou_witness :: topological_space
begin

inductive open_ed_ou_witness :: ed_ou_witness set  $\Rightarrow$  bool where
  open_ed_ou_witness {}
| open_ed_ou_witness {a}
| open_ed_ou_witness {b}
| open_ed_ou_witness {c}
| open_ed_ou_witness {a, c}
| open_ed_ou_witness {b, d}
| open_ed_ou_witness {a, c, e}

| open_ed_ou_witness {a, b}
| open_ed_ou_witness {a, e}
| open_ed_ou_witness {b, e}
| open_ed_ou_witness {a, b, c}
| open_ed_ou_witness {a, b, d}
| open_ed_ou_witness {a, b, e}
| open_ed_ou_witness {b, d, e}
| open_ed_ou_witness {a, b, c, d}
| open_ed_ou_witness {a, b, c, e}
| open_ed_ou_witness {a, b, d, e}
| open_ed_ou_witness {a, b, c, d, e}

declare open_ed_ou_witness.intros[intro!]

lemma ed_ou_witness_inter:
  fixes S :: ed_ou_witness set
  assumes open S
  assumes open T
  shows open (S  $\cap$  T)
using assms by (auto elim!: open_ed_ou_witness.cases)

lemma ed_ou_witness_union:
  fixes X :: ed_ou_witness set set
  assumes  $\forall x \in X. open x$ 
  shows open ( $\bigcup X$ )
using finite[of X] assms
by (induct, force)
  (clarsimp; elim open_ed_ou_witness.cases; simp add: open_ed_ou_witness.simps subset_insertI2 insert_commute; metis Union_empty_conv)

instance
by standard (auto simp: ed_ou_witness_UNIV intro: ed_ou_witness_inter ed_ou_witness_union)

end

```

lemma *ed_ou_witness_interior_simps*:

shows

interior {*a*} = {*a*}
interior {*b*} = {*b*}
interior {*c*} = {}
interior {*d*} = {}
interior {*e*} = {*e*}
interior {*a*, *b*} = {*a*, *b*}
interior {*a*, *c*} = {*a*, *c*}
interior {*a*, *d*} = {*a*}
interior {*a*, *e*} = {*a*, *e*}
interior {*b*, *c*} = {*b*}
interior {*b*, *d*} = {*b*, *d*}
interior {*b*, *e*} = {*b*, *e*}
interior {*c*, *d*} = {}
interior {*c*, *e*} = {*e*}
interior {*d*, *e*} = {*e*}
interior {*a*, *b*, *c*} = {*a*, *b*, *c*}
interior {*a*, *b*, *d*} = {*a*, *b*, *d*}
interior {*a*, *b*, *e*} = {*a*, *b*, *e*}
interior {*a*, *c*, *d*} = {*a*, *c*}
interior {*a*, *c*, *e*} = {*a*, *c*, *e*}
interior {*a*, *d*, *e*} = {*a*, *e*}
interior {*b*, *c*, *d*} = {*b*, *d*}
interior {*b*, *c*, *e*} = {*b*, *e*}
interior {*b*, *d*, *e*} = {*b*, *d*, *e*}
interior {*c*, *d*, *e*} = {*e*}
interior {*a*, *b*, *c*, *d*} = {*a*, *b*, *c*, *d*}
interior {*a*, *b*, *c*, *e*} = {*a*, *b*, *c*, *e*}
interior {*a*, *b*, *d*, *e*} = {*a*, *b*, *d*, *e*}
interior {*a*, *b*, *c*, *d*, *e*} = {*a*, *b*, *c*, *d*, *e*}
interior {*a*, *c*, *d*, *e*} = {*a*, *c*, *e*}
interior {*b*, *c*, *d*, *e*} = {*b*, *d*, *e*}

unfolding *interior_def* **by** *safe (clarsimp simp: open_ed_ou_witness_simps; blast)+*

lemma *ed_ou_witness_not_discrete*:

fixes *X* :: *ed_ou_witness set*

shows \neg *discrete* *X*

unfolding *discrete_def I_def* **using** *ed_ou_witness_interior_simps* **by** (*force simp: fun_eq_iff*)

lemma *ed_ou_witness_ed_ou*:

fixes *X* :: *ed_ou_witness set*

shows *ed_ou* *X*

unfolding *ed_ou_def I_def K_def*

proof(*clarsimp simp: o_apply fun_eq_iff*)

fix *x* :: *ed_ou_witness set*

from *ed_ou_witness_pow*[*of x*]

show *interior* (*closure* *x*) = *closure* (*interior* *x*)

by $-$ (*simp; elim disjE; simp add: closure_interior ed_ou_witness_interior_simps ed_ou_witness_Cmpl insert_Diff_if*)

qed

lemma *ed_ou_witness_card*:

shows *card* {*f*. *CK_nf_pos* (*f*::*ed_ou_witness set* \Rightarrow *ed_ou_witness set*)} = 4

by (*rule ed_ou_not_discrete_card[OF ed_ou_witness_ed_ou ed_ou_witness_not_discrete]*)

6.4 Extremely disconnected spaces

definition

$extremally_disconnected (X :: 'a::topological_space\ set) \iff K \circ I \circ K = I \circ (K :: 'a\ set \Rightarrow 'a\ set)$

lemma *ed_ou_part_extremally_disconnected*:

assumes *ed_ou* *X*

assumes *part* *X*

shows *extremally_disconnected* *X*

using *assms* **unfolding** *extremally_disconnected_def ed_ou_def part_def* **by** *simp*

lemma *extremally_disconnected_eqs*:

fixes *X* :: *'a::topological_space set*

assumes *extremally_disconnected* *X*

shows

$I \circ K \circ I = K \circ (I :: 'a\ set \Rightarrow 'a\ set)$

$K \circ I \circ K = I \circ (K :: 'a\ set \Rightarrow 'a\ set)$

using *assms* **unfolding** *extremally_disconnected_def* **by** (*metis K_I_K_I*)**+**

lemma *extremally_disconnected_not_part_not_ed_ou_card*:

fixes *X* :: *'a::topological_space set*

assumes *extremally_disconnected* *X*

assumes \neg *part* *X*

assumes \neg *ed_ou* *X*

shows $card \{f. CK_nf_pos (f::'a\ set \Rightarrow 'a\ set)\} = 5$

using *extremally_disconnected_eqs*[*OF* \langle *extremally_disconnected* *X* \rangle] *CK_nf_pos_lattice*[**where** *'a='a*] *assms*(2,3)

unfolding *part_def ed_ou_def*

by (*simp add: CK_nf_pos_set C_C I_K K_K o_assoc card_insert_if; metis (no_types) C_C K_I id_comp o_assoc*)

Any topological space having an infinite underlying set and whose topology consists of the empty set and every cofinite subset of the underlying set is extremely disconnected. We consider an example such space having a countably infinite underlying set.

datatype *'a cofinite = cofinite 'a*

instantiation *cofinite* :: (*type*) *topological_space*

begin

definition *open_cofinite* = ($\lambda X::'a\ cofinite\ set. finite (-X) \vee X = \{\}$)

instance

by *standard (auto simp: open_cofinite_def uminus_Sup)*

end

lemma *cofinite_closure_finite*:

fixes *X* :: *'a cofinite set*

assumes *finite* *X*

shows *closure* *X* = *X*

using *assms* **by** (*simp add: closed_open open_cofinite_def*)

lemma *cofinite_closure_infinite*:

fixes *X* :: *'a cofinite set*

assumes *infinite* *X*

shows *closure* *X* = *UNIV*

using *assms* **by** (*metis Compl_empty_eq closure_subset double_compl finite_subset interior_complement open_cofinite_def open_interior*)

lemma *cofinite_interior_finite*:

fixes $X :: 'a$ cofinite set
assumes finite X
assumes infinite ($UNIV :: 'a$ cofinite set)
shows interior $X = \{\}$
using *assms* cofinite_closure_infinite[**where** $X = -X$] **by** (*simp* add: interior_closure)

lemma cofinite_interior_infinite:
fixes $X :: 'a$ cofinite set
assumes infinite X
assumes infinite ($-X$)
shows interior $X = \{\}$
using *assms* cofinite_closure_infinite[**where** $X = -X$] **by** (*simp* add: interior_closure)

abbreviation evens :: nat cofinite set $\equiv \{\text{cofinite } n \mid n. \exists i. n = 2 * i\}$

lemma evens_infinite:
shows infinite evens
proof(rule iffD2[OF infinite_iff_countable_subset], rule exI, rule conjI)
let $?f = \lambda n :: \text{nat}. \text{cofinite } (2 * n)$
show inj $?f$ **by** (*auto* intro: inj_onI)
show range $?f \subseteq \text{evens}$ **by** *auto*
qed

lemma cofinite_nat_infinite:
shows infinite ($UNIV :: \text{nat}$ cofinite set)
using evens_infinite finite_Diff2 **by** fastforce

lemma evens_Coimpl_infinite:
shows infinite ($- \text{evens}$)
proof(rule iffD2[OF infinite_iff_countable_subset], rule exI, rule conjI)
let $?f = \lambda n :: \text{nat}. \text{cofinite } (2 * n + 1)$
show inj $?f$ **by** (*auto* intro: inj_onI)
show range $?f \subseteq -\text{evens}$ **by** *clarsimp presburger*
qed

lemma evens_closure:
shows closure evens = $UNIV$
using evens_infinite **by** (rule cofinite_closure_infinite)

lemma evens_interior:
shows interior evens = $\{\}$
using evens_infinite evens_Coimpl_infinite **by** (rule cofinite_interior_infinite)

lemma cofinite_not_part:
fixes $X :: \text{nat}$ cofinite set
shows $\neg \text{part } X$
unfolding part_def I_def K_def
using cofinite_nat_infinite
by (*clarsimp simp: fun_eq_iff o_apply*)
(*metis* (*no_types*) cofinite_closure_finite cofinite_interior_finite double_coimpl finite.emptyI finite.insertI insert_not_empty interior_closure)

lemma cofinite_not_ed_ou:
fixes $X :: \text{nat}$ cofinite set
shows $\neg \text{ed_ou } X$
unfolding ed_ou_def I_def K_def
by (*clarsimp simp: fun_eq_iff o_apply evens_closure evens_interior exI*[**where** $x = \text{evens}$])

lemma *cofinite_extremally_disconnected_aux*:

fixes $X :: \text{nat cofinite set}$

shows $\text{closure } (\text{interior } (\text{closure } X)) \subseteq \text{interior } (\text{closure } X)$

by (*metis subsetI closure_closure closure_complement closure_def closure_empty finite_Un interior_eq open_cofinite_def open_interior*)

lemma *cofinite_extremally_disconnected*:

fixes $X :: \text{nat cofinite set}$

shows *extremally_disconnected* X

unfolding *extremally_disconnected_def I_def K_def*

by (*auto simp: fun_eq_iff o_apply dest: subsetD[OF closure_subset] subsetD[OF interior_subset] subsetD[OF cofinite_extremally_disconnected_aux]*)

lemma *cofinite_card*:

shows $\text{card } \{f. \text{CK_nf_pos } (f :: \text{nat cofinite set} \Rightarrow \text{nat cofinite set})\} = 5$

by (*rule extremally_disconnected_not_part_not_ed_ou_card[OF cofinite_extremally_disconnected cofinite_not_part cofinite_not_ed_ou]*)

6.5 Open unresolvable spaces

definition

open_unresolvable ($X :: 'a :: \text{topological_space set}$) $\longleftrightarrow K \circ I \circ K = K \circ (I :: 'a \text{ set} \Rightarrow 'a \text{ set})$

lemma *ed_ou_open_unresolvable*:

assumes *ed_ou* X

shows *open_unresolvable* X

using *assms unfolding open_unresolvable_def* **by** (*simp add: ed_ou_eqs*)

lemma *open_unresolvable_eqs*:

assumes *open_unresolvable* ($X :: 'a :: \text{topological_space set}$)

shows

$I \circ K \circ I = I \circ (K :: 'a \text{ set} \Rightarrow 'a \text{ set})$

$K \circ I \circ K = K \circ (I :: 'a \text{ set} \Rightarrow 'a \text{ set})$

using *assms unfolding open_unresolvable_def* **by** $-$ (*metis I_K_I_K o_assoc; simp*)

lemma *not_ed_ou_neqs*:

assumes $\neg \text{ed_ou } (X :: 'a :: \text{topological_space set})$

shows

$I \neq I \circ (K :: 'a \text{ set} \Rightarrow 'a \text{ set})$

$K \neq K \circ (I :: 'a \text{ set} \Rightarrow 'a \text{ set})$

using *assms unfolding ed_ou_def*

by (*simp_all add: fun_eq_iff I_K K_def C_def o_apply*)

(*metis (no_types, hide_lams) closure_eq_empty disjoint_eq_subset_Compl double_complement interior_Int interior_complement set_eq_subset*) $+$

lemma *open_unresolvable_not_ed_ou_card*:

assumes *open_unresolvable* ($X :: 'a :: \text{topological_space set}$)

assumes $\neg \text{ed_ou } X$

shows $\text{card } \{f. \text{CK_nf_pos } (f :: 'a \text{ set} \Rightarrow 'a \text{ set})\} = 5$

using *open_unresolvable_eqs[OF <open_unresolvable X>] not_ed_ou_neqs[OF <\neg ed_ou X>] <\neg ed_ou X>*

unfolding *ed_ou_def* **by** (*auto simp: CK_nf_pos_set card_insert_if*)

We show that the class of open unresolvable spaces is non-empty by exhibiting an example of such a space.

datatype *ou_witness* = $a \mid b \mid c$

lemma *ou_witness_UNIV*:

shows $\text{UNIV} = \text{set } [a, b, c]$

using *ou_witness.exhaust* **by** *auto*

instantiation *ou_witness* :: *topological_space*
begin

definition *open_ou_witness* $X \longleftrightarrow a \notin X \vee X = UNIV$

instance
by *standard* (*auto simp: open_ou_witness_def*)

end

lemma *ou_witness_closure_simps*:

shows

closure $\{a\} = \{a\}$
closure $\{b\} = \{a, b\}$
closure $\{c\} = \{a, c\}$
closure $\{a, b\} = \{a, b\}$
closure $\{a, c\} = \{a, c\}$
closure $\{a, b, c\} = \{a, b, c\}$
closure $\{b, c\} = \{a, b, c\}$

unfolding *closure_def islimpt_def open_ou_witness_def* **by** *force+*

lemma *ou_witness_open_unresolvable*:

fixes $X :: \text{ou_witness set}$

shows *open_unresolvable* X

unfolding *open_unresolvable_def I_def K_def*

by (*clarsimp simp: o_apply fun_eq_iff*)

(*metis (no_types, lifting) Compl_iff K_I_K_subseteq_K closure_complement closure_interior closure_mono closure_subset interior_eq interior_maximal open_ou_witness_def subset_antisym*)

lemma *ou_witness_not_ed_ou*:

fixes $X :: \text{ou_witness set}$

shows $\neg \text{ed_ou } X$

unfolding *ed_ou_def I_def K_def*

by (*clarsimp simp: o_apply fun_eq_iff*)

(*metis UNIV_I insert_iff interior_eq open_ou_witness_def singletonD*
ou_witness.distinct(4,5) ou_witness.simps(2) ou_witness_closure_simps(2))

lemma *ou_witness_card*:

shows $\text{card } \{f. \text{CK_nf_pos } (f :: \text{ou_witness set} \Rightarrow \text{ou_witness set})\} = 5$

by (*rule open_unresolvable_not_ed_ou_card[OF ou_witness_open_unresolvable ou_witness_not_ed_ou]*)

6.6 Kuratowski spaces

definition

kuratowski ($X :: 'a :: \text{topological_space set}$) \longleftrightarrow
 $\neg \text{extremally_disconnected } X \wedge \neg \text{open_unresolvable } X$

A Kuratowski space distinguishes all 7 positive operators.

lemma *part_closed_open*:

fixes $X :: 'a :: \text{topological_space set}$

assumes $I \circ K \circ I = (I :: 'a \text{ set} \Rightarrow 'a \text{ set})$

assumes *closed* X

shows *open* X

proof(*rule Topological_Spaces.openI*)

fix x **assume** $x \in X$

let $?S = I (-\{x\})$

let $?G = -K ?S$

have $x \in ?G$
proof –
from $\langle I \circ K \circ I = I \rangle$ **have** $I (K (I ?S)) = ?S I ?S = ?S$
unfolding $I_def K_def$ **by** (*simp_all add: o_def fun_eq_iff*)
then have $K (I ?S) \neq UNIV$
unfolding $I_def K_def$ **using** *interior_subset* **by** *fastforce*
moreover have $G \subseteq ?S \vee x \in G$ **if open** G **for** G
using that unfolding I_def **by** (*meson interior_maximal_subset_Compl_singleton*)
ultimately show *?thesis*
unfolding $I_def K_def$
by *clarsimp (metis (no_types, lifting) ComplD Compl_empty_eq closure_interior closure_subset ex_in_conv open_interior_subset_eq)*
qed
moreover from $\langle I \circ K \circ I = I \rangle$ **have** *open ?G*
unfolding $I_def K_def$ **by** (*auto simp: fun_eq_iff o_apply*)
moreover have $?G \subseteq X$
proof –
have $?G \subseteq K ?G$ **unfolding** K_def **using** *closure_subset* **by** *fastforce*
also from $\langle I \circ K \circ I = I \rangle$ **have** $\dots = K \{x\}$
unfolding $I_def K_def$ **by** (*metis closure_interior comp_def double_complement*)
also from $\langle \text{closed } X \rangle \langle x \in X \rangle$ **have** $\dots \subseteq X$
unfolding K_def **by** *clarsimp (meson closure_minimal contra_subsetD empty_subsetI insert_subset)*
finally show *?thesis* .
qed
ultimately show $\exists T. \text{open } T \wedge x \in T \wedge T \subseteq X$ **by** *blast*
qed

lemma *part_I_K_I*:

assumes $I \circ K \circ I = (I :: 'a :: \text{topological_space set} \Rightarrow 'a \text{ set})$
shows $I \circ K = (K :: 'a \text{ set} \Rightarrow 'a \text{ set})$
using *interior_open[OF part_closed_open[OF assms closed_closure]]* **unfolding** $I_def K_def o_def$ **by** *simp*

lemma *part_K_I_I*:

assumes $I \circ K \circ I = (I :: 'a :: \text{topological_space set} \Rightarrow 'a \text{ set})$
shows $K \circ I = (I :: 'a \text{ set} \Rightarrow 'a \text{ set})$
using *part_I_K_I[OF assms]* **assms** **by** *simp*

lemma *kuratowski_neqs*:

assumes *kuratowski* ($X :: 'a :: \text{topological_space set}$)
shows
 $I \neq I \circ K \circ (I :: 'a \text{ set} \Rightarrow 'a \text{ set})$
 $I \circ K \circ I \neq K \circ (I :: 'a \text{ set} \Rightarrow 'a \text{ set})$
 $I \circ K \circ I \neq I \circ (K :: 'a \text{ set} \Rightarrow 'a \text{ set})$
 $I \circ K \neq K \circ I \circ (K :: 'a \text{ set} \Rightarrow 'a \text{ set})$
 $K \circ I \neq K \circ I \circ (K :: 'a \text{ set} \Rightarrow 'a \text{ set})$
 $K \circ I \circ K \neq (K :: 'a \text{ set} \Rightarrow 'a \text{ set})$
 $I \circ K \neq K \circ (I :: 'a \text{ set} \Rightarrow 'a \text{ set})$
 $I \neq (id :: 'a \text{ set} \Rightarrow 'a \text{ set})$
 $K \neq (id :: 'a \text{ set} \Rightarrow 'a \text{ set})$
 $I \circ K \circ I \neq (id :: 'a \text{ set} \Rightarrow 'a \text{ set})$
 $K \circ I \circ K \neq (id :: 'a \text{ set} \Rightarrow 'a \text{ set})$
using *assms* **unfolding** *kuratowski_def extremally_disconnected_def open_unresolvable_def*
by (*metis (no_types, lifting) I_K K_K I_K_I_K K_I_K_I part_I_K_I part_K_I_I o_assoc comp_id*)**+**

lemma *kuratowski_card*:

assumes *kuratowski* ($X :: 'a :: \text{topological_space set}$)
shows $\text{card } \{f. CK_nf_pos (f :: 'a \text{ set} \Rightarrow 'a \text{ set})\} = 7$
using *CK_nf_pos_lattice* **[where** $'a = 'a$ **]** *kuratowski_neqs* **[OF** *assms***]** *assms*

unfolding *kuratowski_def extremally_disconnected_def open_unresolvable_def*
by (*subst CK_nf_pos_set*) (*subst card_insert_disjoint*; (*auto*)[1])+

\mathbb{R} is a Kuratowski space.

lemma *kuratowski_reals*:

shows *kuratowski* (\mathbb{R} :: *real set*)

unfolding *kuratowski_def extremally_disconnected_def open_unresolvable_def*

by (*rule conjI*)

(*metis* (*no_types*, *lifting*) *LK list.inject nf_RRR(11) nf_RRR(8) o_assoc*,

metis (*no_types*, *lifting*) *LK fun.map_comp list.inject nf_RRR(11) nf_RRR(9)*)

6.7 Chagrov's theorem

theorem *chagrov*:

fixes *X* :: '*a*::*topological_space set*

obtains *discrete X*

| \neg *discrete X* \wedge *part X*

| \neg *discrete X* \wedge *ed_ou X*

| \neg *ed_ou X* \wedge *open_unresolvable X*

| \neg *ed_ou X* \wedge \neg *part X* \wedge *extremally_disconnected X*

| *kuratowski X*

unfolding *kuratowski_def* **by** *metis*

corollary *chagrov_card*:

shows *card* {*f*. *CK_nf_pos* (*f*::'*a*::*topological_space set* \Rightarrow '*a set*)} \in {1,3,4,5,7}

using *discrete_card part_not_discrete_card ed_ou_not_discrete_card open_unresolvable_not_ed_ou_card*

extremally_disconnected_not_part_not_ed_ou_card kuratowski_card

by (*cases rule: chagrov*) *blast*+

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