

# Verified Synthesis of Knowledge-Based Programs in Finite Synchronous Environments

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## Abstract

Knowledge-based programs (KBPs) are a formalism for directly relating an agent’s knowledge and behaviour. Here we present a general scheme for compiling KBPs to executable automata with a proof of correctness in Isabelle/HOL. We develop the algorithm top-down, using Isabelle’s locale mechanism to structure these proofs, and show that two classic examples can be synthesised using Isabelle’s code generator.

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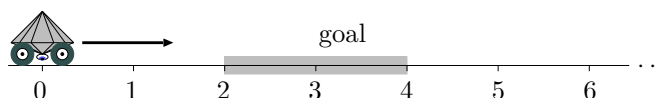
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## 1 Introduction

Imagine a robot stranded at zero on a discrete number line, hoping to reach and remain in the goal region  $\{2, 3, 4\}$ . The environment helpfully pushes the robot to the right, zero or one steps per unit time, and the robot can sense the current position with an error of plus or minus one. If the only action the robot can take is to halt at its current position, what program should it execute?



An intuitive way to specify the robot’s behaviour is with this *knowledge-based program* (KBP), using the syntax of Dijkstra’s guarded commands:

```

do
  []  $\mathbf{K}_{\text{robot goal}}$   $\rightarrow$  Halt
  []  $\neg\mathbf{K}_{\text{robot goal}}$   $\rightarrow$  Nothing
od

```

Here “ $\mathbf{K}_{\text{robot goal}}$ ” intuitively denotes “the robot knows it is in the goal region” (Fagin et al. 1995, Example 7.2.2). We will make this precise in §3, but for now note that what the robot knows depends on the rest of the scenario, which in general may involve other agents also running KBPs. In a sense a KBP is a very literal rendition of a venerable artificial intelligence trope, that what an agent does should depend on its knowledge, and what an agent knows depends on what it does. It has been argued elsewhere Bickford et al. (2004); Engelhardt et al. (2000); Fagin et al. (1995) that this is a useful level of abstraction at which to reason about distributed systems, and some kinds of multi-agent systems Shoham and Leyton-Brown (2008). The cost is that these specifications are not directly executable, and it may take significant effort to find a concrete program that has the required behaviour.

The robot does have a simple implementation however: it should halt iff the sensor reads at least 3. That this is correct can be shown by an epistemic model checker such as MCK Gammie and van der Meyden (2004) or pencil-and-paper refinement Engelhardt et al. (2000). In contrast the goal of this work is to

algorithmically discover such implementations, which is a step towards making the work of van der Meyden [van der Meyden \(1996\)](#) practical.

The contributions of this work are as follows: §2 develops enough of the theory of KBPs in Isabelle/HOL [Nipkow et al. \(2002\)](#) to support a formal proof of the possibility of their implementation by finite-state automata (§6). The later sections extend this development with a full top-down derivation of an original algorithm that constructs these implementations (§6.9) and two instances of it (§7.3 and §??), culminating in the mechanical synthesis of two standard examples from the literature: the aforementioned robot (§??) and the muddy children (§??).

We make judicious use of parametric polymorphism and Isabelle’s locale mechanism [Ballarin \(2006\)](#) to establish and instantiate this theory in a top-down style. Isabelle’s code generator [Haftmann and Nipkow \(2010\)](#) allows the algorithm developed here to be directly executed on the two examples, showing that the theory is both sound and usable. The complete development, available from the Archive of Formal Proofs [Gammie \(2011\)](#), includes the full formal details of all claims made in this paper.

In the following we adopt the Isabelle convention of using an apostrophe to prefix fixed but unknown types, such as  $'a$ , and postfix type constructors as in  $'a$  list. Other non-standard syntax will be explained as it arises.

## 2 A modal logic of knowledge

We begin with the standard syntax and semantics of the propositional logic of knowledge based on *Kripke structures*. More extensive treatments can be found in [Lenzen \(1978\)](#), [Chellas \(1980\)](#), [Hintikka \(1962\)](#) and [Fagin et al. \(1995, Chapter 2\)](#).

The syntax includes one knowledge modality per agent, and one for *common knowledge* amongst a set of agents. It is parameterised by the type  $'a$  of agents and  $'p$  of propositions.

```
datatype ('a, 'p) Kform
  = Kprop 'p
  | Knot ('a, 'p) Kform
  | Kand ('a, 'p) Kform ('a, 'p) Kform
  | Kknows 'a ('a, 'p) Kform (K- -)
  | Kcknows 'a list ('a, 'p) Kform (C- -)
```

A Kripke structure consists of a set of *worlds* of type  $'w$ , one *accessibility relation* between worlds for each agent and a *valuation function* that indicates the truth of a proposition at a world. This is a very general story that we will quickly specialise.

```
type-synonym 'w Relation = ('w × 'w) set
```

```
record ('a, 'p, 'w) KripkeStructure =
  worlds :: 'w set
```

$relations :: 'a \Rightarrow 'w \text{ Relation}$   
 $valuation :: 'w \Rightarrow 'p \Rightarrow bool$

**definition**  $kripke :: ('a, 'p, 'w) \text{ KripkeStructure} \Rightarrow bool$  **where**  
 $kripke M \equiv \forall a. relations M a \subseteq worlds M \times worlds M$

**definition**

$mkKripke :: 'w \text{ set} \Rightarrow ('a \Rightarrow 'w \text{ Relation}) \Rightarrow ('w \Rightarrow 'p \Rightarrow bool)$   
 $\Rightarrow ('a, 'p, 'w) \text{ KripkeStructure}$

**where**

$mkKripke \text{ ws rels val} \equiv$

$(\lambda worlds = ws, relations = \lambda a. rels a \cap ws \times ws, valuation = val) \langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle$

The standard semantics for knowledge is given by taking the accessibility relations to be equivalence relations, yielding the  $S5_n$  structures, so-called due to their axiomatisation.

**definition**  $S5n :: ('a, 'p, 'w) \text{ KripkeStructure} \Rightarrow bool$  **where**

$S5n M \equiv \forall a. equiv (worlds M) (relations M a) \langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle$

Intuitively an agent considers two worlds to be equivalent if it cannot distinguish between them.

## 2.1 Satisfaction

A formula  $\phi$  is satisfied at a world  $w$  in Kripke structure  $M$  in the following way:

**fun**  $models :: ('a, 'p, 'w) \text{ KripkeStructure} \Rightarrow 'w \Rightarrow ('a, 'p) \text{ Kform}$   
 $\Rightarrow bool ((-, - \models -) [80,0,80] 80)$  **where**

$M, w \models (Kprop p) = valuation M w p$

$| M, w \models (Knot \varphi) = (\neg M, w \models \varphi)$

$| M, w \models (Kand \varphi \psi) = (M, w \models \varphi \wedge M, w \models \psi)$

$| M, w \models (\mathbf{K}_a \varphi) = (\forall w' \in relations M a \text{ `` } \{w\}. M, w' \models \varphi)$

$| M, w \models (\mathbf{C}_{as} \varphi) = (\forall w' \in (\bigcup a \in set \text{ as. relations M a})^+ \text{ `` } \{w\}. M, w' \models \varphi)$

The first three clauses are standard.

The clause for  $\mathbf{K}_a \varphi$  expresses the idea that an agent knows  $\varphi$  at world  $w$  in structure  $M$  iff  $\varphi$  is true at all worlds it considers possible.

The clause for  $\mathbf{C}_{as} \varphi$  captures what it means for the set of agents  $as$  to commonly know  $\varphi$ ; roughly, everyone knows  $\varphi$  and knows that everyone knows it, and so forth. Note that the transitive closure and the reflexive-transitive closure generate the same relation due to the reflexivity of the agents' accessibility relations; we use the former as it has a more pleasant induction principle.

$\langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle$

The relation between knowledge and common knowledge can be understood as follows, following Fagin et al. (1995, §2.4). Firstly, that  $\phi$  is common knowledge to a set of agents  $as$  can be seen as asserting that everyone in  $as$  knows  $\phi$  and moreover knows that it is common knowledge amongst  $as$ .

**lemma** *S5n-common-knowledge-fixed-point:*

**assumes** *S5n*:  $S5n\ M$

**assumes** *w*:  $w \in worlds\ M$

**assumes** *a*:  $a \in set\ as$

**shows**  $M, w \models Kcknows\ as\ \varphi$

$\longleftrightarrow M, w \models Kand\ (Kknows\ a\ \varphi)\ (Kknows\ a\ (Kcknows\ as\ \varphi))\langle proof \rangle$

Secondly we can provide an induction schema for the introduction of common knowledge: from everyone in *as* knows that  $\phi$  implies  $\phi \wedge \psi$ , and that  $\phi$  is satisfied at world *w*, infer that  $\psi$  is common knowledge amongst *as* at *w*.

**lemma** *S5n-common-knowledge-induct:*

**assumes** *S5n*:  $S5n\ M$

**assumes** *w*:  $w \in worlds\ M$

**assumes** *E*:  $\forall a \in set\ as. \forall w \in worlds\ M.$

$M, w \models \varphi \longrightarrow M, w \models \mathbf{K}_a\ (Kand\ \varphi\ \psi)$

**assumes** *p*:  $M, w \models \varphi$

**shows**  $M, w \models \mathbf{C}_{as}\ \psi\langle proof \rangle\langle proof \rangle$

## 2.2 Generated models

The rest of this section introduces the technical machinery we use to relate Kripke structures.

Intuitively the truth of a formula at a world depends only on the worlds that are reachable from it in zero or more steps, using any of the accessibility relations at each step. Traditionally this result is called the *generated model property* (Chellas 1980, §3.4).

Given the model generated by *w* in *M*:

**definition**

*gen-model* ::  $(\ 'a, \ 'p, \ 'w) \ KripkeStructure \Rightarrow \ 'w \Rightarrow (\ 'a, \ 'p, \ 'w) \ KripkeStructure$

**where**

*gen-model* *M* *w*  $\equiv$

let  $ws' = worlds\ M \cap (\bigcup a. relations\ M\ a)^* \ \{w\}$

in  $(\ \{ \} \ world = ws',$

$relations = \lambda a. relations\ M\ a \cap (ws' \times ws'),$

$valuation = valuation\ M \ \rangle\langle proof \rangle\langle proof \rangle\langle proof \rangle\langle proof \rangle\langle proof \rangle\langle proof \rangle\langle proof \rangle\langle proof \rangle\langle proof \rangle\langle proof \rangle$

where we take the image of *w* under the reflexive transitive closure of the agents' relations, we can show that the satisfaction of a formula  $\varphi$  at a world *w'* is preserved, provided *w'* is relevant to the world *w* that the sub-model is based upon:

**lemma** *gen-model-semantic-equivalence:*

**assumes** *M*: *kripke* *M*

**assumes** *w'*:  $w' \in worlds\ (gen-model\ M\ w)$

**shows**  $M, w' \models \varphi \longleftrightarrow (gen-model\ M\ w), w' \models \varphi\langle proof \rangle$

This is shown by a straightforward structural induction over the formula  $\varphi$ .

$\langle proof \rangle\langle proof \rangle\langle proof \rangle\langle proof \rangle$

## 2.3 Simulations

A *simulation*, or *p-morphism*, is a mapping from the worlds of one Kripke structure to another that preserves the truth of all formulas at related worlds (Chellas 1980, §3.4, Ex. 3.60). Such a function  $f$  must satisfy four properties. Firstly, the image of the set of worlds of  $M$  under  $f$  should equal the set of worlds of  $M'$ .

**definition**

$$\begin{aligned} \text{sim-range} &:: ('a, 'p, 'w1) \text{ KripkeStructure} \\ &\Rightarrow ('a, 'p, 'w2) \text{ KripkeStructure} \Rightarrow ('w1 \Rightarrow 'w2) \Rightarrow \text{bool} \end{aligned}$$

**where**

$$\begin{aligned} \text{sim-range } M M' f &\equiv \text{worlds } M' = f \text{ ` worlds } M \\ &\wedge (\forall a. \text{relations } M' a \subseteq \text{worlds } M' \times \text{worlds } M) \end{aligned}$$

The value of a proposition should be the same at corresponding worlds:

**definition**

$$\begin{aligned} \text{sim-val} &:: ('a, 'p, 'w1) \text{ KripkeStructure} \\ &\Rightarrow ('a, 'p, 'w2) \text{ KripkeStructure} \Rightarrow ('w1 \Rightarrow 'w2) \Rightarrow \text{bool} \end{aligned}$$

**where**

$$\text{sim-val } M M' f \equiv \forall u \in \text{worlds } M. \text{valuation } M u = \text{valuation } M' (f u)$$

If two worlds are related in  $M$ , then the simulation maps them to related worlds in  $M'$ ; intuitively the simulation relates enough worlds. We term this the *forward* property.

**definition**

$$\begin{aligned} \text{sim-f} &:: ('a, 'p, 'w1) \text{ KripkeStructure} \\ &\Rightarrow ('a, 'p, 'w2) \text{ KripkeStructure} \Rightarrow ('w1 \Rightarrow 'w2) \Rightarrow \text{bool} \end{aligned}$$

**where**

$$\begin{aligned} \text{sim-f } M M' f &\equiv \\ &\forall a u v. (u, v) \in \text{relations } M a \longrightarrow (f u, f v) \in \text{relations } M' a \end{aligned}$$

Conversely, if two worlds  $f u$  and  $v'$  are related in  $M'$ , then there is a pair of related worlds  $u$  and  $v$  in  $M$  where  $f v = v'$ . Intuitively the simulation makes enough distinctions. We term this the *reverse* property.

**definition**

$$\begin{aligned} \text{sim-r} &:: ('a, 'p, 'w1) \text{ KripkeStructure} \\ &\Rightarrow ('a, 'p, 'w2) \text{ KripkeStructure} \Rightarrow ('w1 \Rightarrow 'w2) \Rightarrow \text{bool} \end{aligned}$$

**where**

$$\begin{aligned} \text{sim-r } M M' f &\equiv \forall a. \forall u \in \text{worlds } M. \forall v'. \\ &(f u, v') \in \text{relations } M' a \\ &\longrightarrow (\exists v. (u, v) \in \text{relations } M a \wedge f v = v') \end{aligned}$$

**definition**

$$\begin{aligned} \text{sim } M M' f &\equiv \text{sim-range } M M' f \wedge \text{sim-val } M M' f \\ &\wedge \text{sim-f } M M' f \wedge \text{sim-r } M M' f \langle \text{proof} \rangle \langle \text{proof} \rangle \langle \text{proof} \rangle \langle \text{proof} \rangle \langle \text{proof} \rangle \langle \text{proof} \rangle \langle \text{proof} \rangle \langle \text{proof} \rangle \end{aligned}$$

Due to the common knowledge modality, we need to show the simulation properties lift through the transitive closure. In particular we can show that forward simulation is preserved:

**lemma** *sim-f-tc*:

**assumes**  $s: \text{sim } M M' f$   
**assumes**  $uv': (u, v) \in (\bigcup a \in \text{as. relations } M a)^+$   
**shows**  $(f u, f v) \in (\bigcup a \in \text{as. relations } M' a)^+ \langle \text{proof} \rangle$

Reverse simulation also:

**lemma** *sim-r-tc*:

**assumes**  $M: \text{kripke } M$   
**assumes**  $s: \text{sim } M M' f$   
**assumes**  $u: u \in \text{worlds } M$   
**assumes**  $fuv': (f u, v') \in (\bigcup a \in \text{as. relations } M' a)^+$   
**obtains**  $v$  **where**  $f v = v'$  **and**  $(u, v) \in (\bigcup a \in \text{as. relations } M a)^+ \langle \text{proof} \rangle \langle \text{proof} \rangle \langle \text{proof} \rangle \langle \text{proof} \rangle \langle \text{proof} \rangle \langle \text{proof} \rangle$

Finally we establish the key property of simulations, that they preserve the satisfaction of all formulas in the following way:

**lemma** *sim-semantic-equivalence*:

**assumes**  $M: \text{kripke } M$   
**assumes**  $s: \text{sim } M M' f$   
**assumes**  $u: u \in \text{worlds } M$   
**shows**  $M, u \models \varphi \longleftrightarrow M', f u \models \varphi \langle \text{proof} \rangle$

The proof is by structural induction over the formula  $\varphi$ . The knowledge cases appeal to our two simulation preservation lemmas.

Sangiorgi (2009) surveys the history of p-morphisms and the related concept of *bisimulation*.

This is all we need to know about Kripke structures.

### 3 Knowledge-based Programs

A knowledge-based programs (KBPs) encodes the dependency of action on knowledge by a sequence of guarded commands, and a *joint knowledge-based program* (JKBP) assigns a KBP to each agent:

**record**  $(\prime a, \prime p, \prime aAct) GC =$   
 $\text{guard} :: (\prime a, \prime p) Kform$   
 $\text{action} :: \prime aAct$

**type-synonym**  $(\prime a, \prime p, \prime aAct) KBP = (\prime a, \prime p, \prime aAct) GC \text{ list}$   
**type-synonym**  $(\prime a, \prime p, \prime aAct) JKBP = \prime a \Rightarrow (\prime a, \prime p, \prime aAct) KBP$

We use a list of guarded commands just so we can reuse this definition and others in algorithmic contexts; we would otherwise use a set as there is no problem with infinite programs or actions, and we always ignore the sequential structure.

Intuitively a KBP for an agent cannot directly evaluate the truth of an arbitrary formula as it may depend on propositions that the agent has no certainty about. For example, a card-playing agent cannot determine which cards are in the deck, despite being sure that those in her hand are not. Conversely agent  $a$



can evaluate formulas of the form  $\mathbf{K}_a \varphi$  as these depend only on the worlds the agent thinks is possible.

Thus we restrict the guards of the JKBP to be boolean combinations of *subjective* formulas:

```
fun subjective :: 'a  $\Rightarrow$  ('a, 'p) Kform  $\Rightarrow$  bool where
  subjective a (Kprop p)      = False
| subjective a (Knot f)       = subjective a f
| subjective a (Kand f g)     = (subjective a f  $\wedge$  subjective a g)
| subjective a (Kknows a' f) = (a = a')
| subjective a (Kcknows as f) = (a  $\in$  set as)
```

All JKBP's in the following sections are assumed to be subjective.

This syntactic restriction implies the desired semantic property, that we can evaluate a guard at an arbitrary world that is compatible with a given observation (Fagin et al. 1997, §3).

**lemma** *S5n-subjective-eq*:

```
assumes S5n: S5n M
assumes subj: subjective a  $\varphi$ 
assumes ww': (w, w')  $\in$  relations M a
shows M, w  $\models$   $\varphi \iff$  M, w'  $\models$   $\varphi$ (proof)
```

The proof is by induction over the formula  $\varphi$ , using the properties of  $S5_n$  Kripke structures in the knowledge cases.

We capture the fixed but arbitrary JKBP using a locale, and work in this context for the rest of this section.

**locale** JKBP =

```
fixes jkbp :: ('a, 'p, 'aAct) JKBP
assumes subj:  $\forall a$  gc. gc  $\in$  set (jkbp a)  $\longrightarrow$  subjective a (guard gc)
```

**context** JKBP

**begin**

The action of the JKBP at a world is the list of all actions that are enabled at that world:

```
definition jAction :: ('a, 'p, 'w) KripkeStructure  $\Rightarrow$  'w  $\Rightarrow$  'a  $\Rightarrow$  'aAct list
where jAction  $\equiv$   $\lambda M w a$ . [ action gc. gc  $\leftarrow$  jkbp a, M, w  $\models$  guard gc ]
```

All of our machinery on Kripke structures lifts from the models relation of §2 through *jAction*, due to the subjectivity requirement. In particular, the KBP for agent  $a$  behaves the same at worlds that  $a$  cannot distinguish amongst:

**lemma** *S5n-jAction-eq*:

```
assumes S5n: S5n M
assumes ww': (w, w')  $\in$  relations M a
shows jAction M w a = jAction M w' a(proof)
```

Also the JKBP behaves the same on relevant generated models for all agents:

**lemma** *gen-model-jAction-eq*:  
**assumes**  $S$ : *gen-model*  $M$   $w = \text{gen-model } M' w$   
**assumes**  $w'$ :  $w' \in \text{worlds } (\text{gen-model } M' w)$   
**assumes**  $M$ : *kripke*  $M$   
**assumes**  $M'$ : *kripke*  $M'$   
**shows** *jAction*  $M$   $w' = \text{jAction } M' w'$  *(proof)*

Finally, *jAction* is invariant under simulations:

**lemma** *simulation-jAction-eq*:  
**assumes**  $M$ : *kripke*  $M$   
**assumes**  $\text{sim}$ : *sim*  $M$   $M' f$   
**assumes**  $w$ :  $w \in \text{worlds } M$   
**shows** *jAction*  $M$   $w = \text{jAction } M' (f w)$  *(proof)*  
**end**

## 4 Environments and Views

The previous section showed how a JKBP can be interpreted statically, with respect to a fixed Kripke structure. As we also wish to capture how agents interact, we adopt the *interpreted systems* and *contexts* of Fagin et al. (1995), which we term *environments* following van der Meyden (1996).

A *pre-environment* consists of the following:

- *envInit*, an arbitrary set of initial states;
- The protocol of the environment *envAction*, which depends on the current state;
- A transition function *envTrans*, which incorporates the environment's action and agents' behaviour into a state change; and
- A propositional evaluation function *envVal*.

We extend the *JKBP* locale with these constants:

**locale** *PreEnvironment* = *JKBP jkbp* **for**  $\text{jkbp} :: ('a, 'p, 'aAct)$  *JKBP*  
**+ fixes** *envInit* ::  $'s$  *list*  
**and** *envAction* ::  $'s \Rightarrow 'eAct$  *list*  
**and** *envTrans* ::  $'eAct \Rightarrow ('a \Rightarrow 'aAct) \Rightarrow 's \Rightarrow 's$   
**and** *envVal* ::  $'s \Rightarrow 'p \Rightarrow \text{bool}$

We represent the possible evolutions of the system as finite sequences of states, represented by a left-recursive type  $'s$  *Trace* with constructors *tInit*  $s$  and  $t \rightsquigarrow s$ , equipped with *tFirst*, *tLast*, *tLength* and *tMap* functions.

Constructing these traces requires us to determine the agents' actions at a given state. To do so we need to find an appropriate  $S5_n$  structure for interpreting *jkbp*.

Given that we want the agents to make optimal use of the information they have access to, we allow these structures to depend on the entire history of the

system, suitably conditioned by what the agents can observe. We capture this notion of observation with a *view* (van der Meyden 1996), which is an arbitrary function of a trace:

**type-synonym**  $( 's, 'tview ) \textit{View} = 's \textit{Trace} \Rightarrow 'tview$   
**type-synonym**  $( 'a, 's, 'tview ) \textit{JointView} = 'a \Rightarrow 's \textit{Trace} \Rightarrow 'tview$

We require views to be *synchronous*, i.e. that agents be able to tell the time using their view by distinguishing two traces of different lengths. As we will see in the next section, this guarantees that the JKBP has an essentially unique implementation.

We extend the *PreEnvironment* locale with a view:

**locale** *PreEnvironmentJView* =  
*PreEnvironment jkbp envInit envAction envTrans envVal*  
**for** *jkbp* ::  $( 'a, 'p, 'aAct ) \textit{JKBP}$   
**and** *envInit* ::  $'s \textit{list}$   
**and** *envAction* ::  $'s \Rightarrow 'eAct \textit{list}$   
**and** *envTrans* ::  $'eAct \Rightarrow ( 'a \Rightarrow 'aAct ) \Rightarrow 's \Rightarrow 's$   
**and** *envVal* ::  $'s \Rightarrow 'p \Rightarrow \textit{bool}$   
+ **fixes** *jview* ::  $( 'a, 's, 'tview ) \textit{JointView}$   
**assumes** *sync*:  $\forall a t t'. \textit{jview} a t = \textit{jview} a t' \longrightarrow \textit{tLength} t = \textit{tLength} t'$

The two principle synchronous views are the clock view and the perfect-recall view which we discuss further in §7. We will later derive an agent’s concrete view from an instantaneous observation of the global state in §6.1.

We build a Kripke structure from a set of traces by relating traces that yield the same view. To obtain an  $S5_n$  structure we also need a way to evaluate propositions: we apply *envVal* to the final state of a trace:

**definition** (in *PreEnvironmentJView*)  
*mkM* ::  $'s \textit{Trace set} \Rightarrow ( 'a, 'p, 's \textit{Trace} ) \textit{KripkeStructure}$   
**where**  
*mkM T*  $\equiv$   
 $\{ \textit{worlds} = T,$   
 $\textit{relations} = \lambda a. \{ (t, t') . \{t, t'\} \subseteq T \wedge \textit{jview} a t = \textit{jview} a t' \},$   
 $\textit{valuation} = \textit{envVal} \circ \textit{tLast} \} \langle \textit{proof} \rangle \langle \textit{proof} \rangle \langle \textit{proof} \rangle \langle \textit{proof} \rangle$

This construction supplants the role of the *local states* of Fagin et al. (1995).

The following section shows how we can canonically interpret the JKBP with respect to this structure.

## 5 Canonical Structures

Our goal in this section is to find the canonical set of traces for a given JKBP in a particular environment. As we will see, this always exists with respect to synchronous views.

We inductively define an *interpretation* of a JKBP with respect to an arbitrary set of traces  $T$  by constructing a sequence of sets of traces of increasing length:

**fun**  $jkbpT_n :: nat \Rightarrow 's \text{ Trace set} \Rightarrow 's \text{ Trace set}$  **where**  
 $jkbpT_0 T = \{ tInit\ s \mid s. s \in set\ envInit \}$   
 $| jkbpT_{Suc\ n} T = \{ t \rightsquigarrow envTrans\ eact\ aact\ (tLast\ t) \mid t\ eact\ aact.$   
 $\quad t \in jkbpT_n\ T \wedge eact \in set\ (envAction\ (tLast\ t))$   
 $\quad \wedge (\forall a. aact\ a \in set\ (jAction\ (mkM\ T)\ t\ a)) \}$

This model reflects the failure of any agent to provide an action as failure of the entire system. In general  $envTrans$  may incorporate a scheduler and communication failure models.

The union of this sequence gives us a closure property:

**definition**  $jkbpT :: 's \text{ Trace set} \Rightarrow 's \text{ Trace set}$  **where**  
 $jkbpT\ T \equiv \bigcup n. jkbpT_n\ T \langle proof \rangle \langle proof \rangle \langle proof \rangle$

We say that a set of traces  $T$  represents a JKBP if it is closed under  $jkbpT$ :

**definition**  $represents :: 's \text{ Trace set} \Rightarrow bool$  **where**  
 $represents\ T \equiv jkbpT\ T = T \langle proof \rangle \langle proof \rangle$

This is the vicious cycle that we break using our assumption that the view is synchronous. The key property of such views is that the satisfaction of an epistemic formula is determined by the set of traces in the model that have the same length. Lifted to  $jAction$ , we have:

$\langle proof \rangle \langle proof \rangle$  **lemma**  $sync\ jview\ jAction\ eq$ :  
**assumes**  $traces: \{ t \in T . tLength\ t = n \} = \{ t \in T' . tLength\ t = n \}$   
**assumes**  $tT: t \in \{ t \in T . tLength\ t = n \}$   
**shows**  $jAction\ (mkM\ T)\ t = jAction\ (mkM\ T')\ t \langle proof \rangle$

This implies that for a synchronous view we can inductively define the *canonical traces* of a JKBP. These are the traces that a JKBP generates when it is interpreted with respect to those very same traces. We do this by constructing the sequence  $jkbpC_n$  of (*canonical*) *temporal slices* similarly to  $jkbpT_n$ :

**fun**  $jkbpC_n :: nat \Rightarrow 's \text{ Trace set}$  **where**  
 $jkbpC_0 = \{ tInit\ s \mid s. s \in set\ envInit \}$   
 $| jkbpC_{Suc\ n} = \{ t \rightsquigarrow envTrans\ eact\ aact\ (tLast\ t) \mid t\ eact\ aact.$   
 $\quad t \in jkbpC_n \wedge eact \in set\ (envAction\ (tLast\ t))$   
 $\quad \wedge (\forall a. aact\ a \in set\ (jAction\ (mkM\ jkbpC_n)\ t\ a)) \}$

**abbreviation**  $MC_n :: nat \Rightarrow ('a, 'p, 's \text{ Trace}) \text{ KripkeStructure}$  **where**  
 $MC_n \equiv mkM\ jkbpC_n \langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle$

The canonical set of traces for a JKBP with respect to a joint view is the set of canonical traces of all lengths.

**definition**  $jkbpC :: 's \text{ Trace set}$  **where**  
 $jkbpC \equiv \bigcup n. jkbpC_n$

**abbreviation**  $MC :: ('a, 'p, 's \text{ Trace}) \text{ KripkeStructure}$  **where**  
 $MC \equiv mkM\ jkbpC \langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle$

We can show that  $jkbpC$  represents the joint knowledge-based program  $jkbp$  with respect to  $jview$ :

**lemma**  $jkbpC$ - $jkbpCn$ - $jAction$ - $eq$ :  
**assumes**  $tCn$ :  $t \in jkbpCn$   
**shows**  $jAction MC t = jAction MC_n t$   $\langle proof \rangle$

**lemma**  $jkbpTn$ - $jkbpCn$ - $represents$ :  $jkbpTn jkbpC = jkbpCn$   
 $\langle proof \rangle$

**theorem**  $jkbpC$ - $represents$ :  $represents jkbpC$   $\langle proof \rangle$

We can show uniqueness too, by a similar argument:

**theorem**  $jkbpC$ - $represents$ - $uniquely$ :  
**assumes**  $repT$ :  $represents T$   
**shows**  $T = jkbpC$   $\langle proof \rangle$   
**end**

Thus, at least with synchronous views, we are justified in talking about *the* representation of a JKBP in a given environment. More generally these results are also valid for the more general notion of *provides witnesses* as shown by [Fagin et al. \(1995, Lemma 7.2.4\)](#) and [Fagin et al. \(1997\)](#): it requires only that if a subjective knowledge formula is false on a trace then there is a trace of the same length or less that bears witness to that effect. This is a useful generalisation in asynchronous settings.

The next section shows how we can construct canonical representations of JKBP's using automata.

## 6 Automata Synthesis

Our attention now shifts to showing how we can synthesise standard automata that *implement* a JKBP under certain conditions. We proceed by defining *incremental views* following [van der Meyden \(1996\)](#), which provide the interface between the system and these automata. The algorithm itself is presented in §6.9.

### 6.1 Incremental views

Intuitively an agent instantaneously observes the system state, and so must maintain her view of the system *incrementally*: her new view must be a function of her current view and some new observation. We allow this observation to be an arbitrary projection  $envObs a$  of the system state for each agent  $a$ :

**locale**  $Environment =$   
 $PreEnvironment jkbp envInit envAction envTrans envVal$   
**for**  $jkbp :: ('a, 'p, 'aAct) JKBP$   
**and**  $envInit :: 's list$

```

and envAction :: 's ⇒ 'eAct list
and envTrans :: 'eAct ⇒ ('a ⇒ 'aAct) ⇒ 's ⇒ 's
and envVal :: 's ⇒ 'p ⇒ bool
+ fixes envObs :: 'a ⇒ 's ⇒ 'obs

```

An incremental view therefore consists of two functions with these types:

```

type-synonym ('a, 'obs, 'tv) InitialIncrJointView = 'a ⇒ 'obs ⇒ 'tv
type-synonym ('a, 'obs, 'tv) IncrJointView = 'a ⇒ 'obs ⇒ 'tv ⇒ 'tv

```

These functions are required to commute with their corresponding trace-based joint view in the obvious way:

```

locale IncrEnvironment =
  Environment jkbp envInit envAction envTrans envVal envObs
+ PreEnvironmentJView jkbp envInit envAction envTrans envVal jview
  for jkbp :: ('a, 'p, 'aAct) JKBP
  and envInit :: 's list
  and envAction :: 's ⇒ 'eAct list
  and envTrans :: 'eAct ⇒ ('a ⇒ 'aAct) ⇒ 's ⇒ 's
  and envVal :: 's ⇒ 'p ⇒ bool
  and jview :: ('a, 's, 'tv) JointView
  and envObs :: 'a ⇒ 's ⇒ 'obs
+ fixes jviewInit :: ('a, 'obs, 'tv) InitialIncrJointView
  fixes jviewIncr :: ('a, 'obs, 'tv) IncrJointView
  assumes jviewInit: ∀ a s. jviewInit a (envObs a s) = jview a (tInit s)
  assumes jviewIncr: ∀ a t s. jview a (t ↘ s)
                    = jviewIncr a (envObs a s) (jview a t)

```

Armed with these definitions, the following sections show that there are automata that implement a JKBP in a given environment.

## 6.2 Automata

Our implementations of JKBP's take the form of deterministic Moore automata, where transitions are labelled by observation and states with the action to be performed. We will use the term *protocols* interchangeably with automata, following the KBP literature, and adopt *joint protocols* for the assignment of one such to each agent:

```

record ('obs, 'aAct, 'ps) Protocol =
  pInit :: 'obs ⇒ 'ps
  pTrans :: 'obs ⇒ 'ps ⇒ 'ps
  pAct :: 'ps ⇒ 'aAct list

type-synonym ('a, 'obs, 'aAct, 'ps) JointProtocol
  = 'a ⇒ ('obs, 'aAct, 'ps) Protocol

```

```

context IncrEnvironment
begin

```

To ease composition with the system we adopt the function  $pInit$  which maps the initial observation to an initial automaton state.

van der Meyden (1996) shows that even non-deterministic JKBP's can be implemented with deterministic transition functions; intuitively all relevant uncertainty the agent has about the system must be encoded into each automaton state, so there is no benefit to doing this non-deterministically. In contrast we model the non-deterministic choice of action by making  $pAct$  a relation.

Running a protocol on a trace is entirely standard, as is running a joint protocol, and determining their actions:

```

fun runJP :: ('a, 'obs, 'aAct, 'ps) JointProtocol
           => 's Trace => 'a => 'ps
where
  runJP jp (tInit s) a = pInit (jp a) (envObs a s)
| runJP jp (t ~> s) a = pTrans (jp a) (envObs a s) (runJP jp t a)

```

```

abbreviation actJP :: ('a, 'obs, 'aAct, 'ps) JointProtocol
              => 's Trace => 'a => 'aAct list where
  actJP jp ≡ λt a. pAct (jp a) (runJP jp t a)

```

Similarly to §5 we will reason about the set of traces generated by a joint protocol in a fixed environment:

```

inductive-set
  jpTraces :: ('a, 'obs, 'aAct, 'ps) JointProtocol => 's Trace set
  for jp :: ('a, 'obs, 'aAct, 'ps) JointProtocol
where
  s ∈ set envInit ⟹ tInit s ∈ jpTraces jp
| [ [ t ∈ jpTraces jp; eact ∈ set (envAction (tLast t));
  ∧ a. aact a ∈ set (actJP jp t a); s = envTrans eact aact (tLast t) ] ]
  ⟹ t ~> s ∈ jpTraces jp⟨proof⟩⟨proof⟩⟨proof⟩⟨proof⟩⟨proof⟩
end

```

### 6.3 The Implementation Relation

With this machinery in hand, we now relate automata with JKBP's. We say a joint protocol  $jp$  implements a JKBP when they perform the same actions on the canonical of traces. Note that the behaviour of  $jp$  on other traces is arbitrary.

```

context IncrEnvironment
begin

```

```

definition
  implements :: ('a, 'obs, 'aAct, 'ps) JointProtocol => bool
where
  implements jp ≡ (∀ t ∈ jkbpC. set ∘ actJP jp t = set ∘ jAction MC t)

```

Clearly there are environments where the canonical trace set  $jkbpC$  can be generated by actions that differ from those prescribed by the JKBP. We can

show that the *implements* relation is a stronger requirement than the mere trace-inclusion required by the *represents* relation of §5.

$\langle \text{proof} \rangle \langle \text{proof} \rangle \langle \text{proof} \rangle$  **lemma** *implements-represents*:  
**assumes** *impl*: *implements* *jp*  
**shows** *represents* (*jpTraces* *jp*)  $\langle \text{proof} \rangle \langle \text{proof} \rangle$

The proof is by a straightforward induction over the lengths of traces generated by the joint protocol.

Our final piece of technical machinery allows us to refine automata definitions: we say that two joint protocols are *behaviourally equivalent* if the actions they propose coincide for each canonical trace. The implementation relation is preserved by this relation.

**definition**

*behaviourally-equiv* :: (*'a*, *'obs*, *'aAct*, *'ps*) *JointProtocol*  
 $\Rightarrow$  (*'a*, *'obs*, *'aAct*, *'ps*) *JointProtocol*  
 $\Rightarrow$  *bool*

**where**

*behaviourally-equiv* *jp* *jp'*  $\equiv \forall t \in \text{jkbp}C. \text{set} \circ \text{act}JP \text{ } jp \text{ } t = \text{set} \circ \text{act}JP \text{ } jp' \text{ } t$   
 $\langle \text{proof} \rangle$

**lemma** *behaviourally-equiv-implements*:

**assumes** *behaviourally-equiv* *jp* *jp'*  
**shows** *implements* *jp*  $\longleftrightarrow$  *implements* *jp'*  $\langle \text{proof} \rangle$

**end**

## 6.4 Automata using Equivalence Classes

We now show that there is an implementation of every JKBP with respect to every incremental synchronous view. Intuitively the states of the automaton for agent *a* represent the equivalence classes of traces that *a* considers possible, and the transitions update these sets according to her KBP.

**context** *IncrEnvironment*

**begin**

**definition**

*mkAutoEC* :: (*'a*, *'obs*, *'aAct*, *'s* *Trace* *set*) *JointProtocol*

**where**

*mkAutoEC*  $\equiv \lambda a.$   
 $\langle \text{ } pInit = \lambda obs. \{ t \in \text{jkbp}C. \text{jview}Init \text{ } a \text{ } obs = \text{jview} \text{ } a \text{ } t \},$   
 $\text{ } pTrans = \lambda obs \text{ } ps. \{ t \mid t'. t \in \text{jkbp}C \wedge t' \in ps$   
 $\quad \wedge \text{jview} \text{ } a \text{ } t = \text{jview}Incr \text{ } a \text{ } obs (\text{jview} \text{ } a \text{ } t') \},$   
 $\text{ } pAct = \lambda ps. \text{jAction} \text{ } MC (\text{SOME } t. t \in ps) \text{ } a \text{ } \rangle$

The function *SOME* is Hilbert's indefinite description operator  $\varepsilon$ , used here to choose an arbitrary trace from the protocol state.

That this automaton maintains the correct equivalence class on a trace *t* follows from an easy induction over *t*.



**lemma** *mkAutoEC-ec*:  
**assumes**  $t \in jkbpC$   
**shows**  $runJP\ mkAutoEC\ t\ a = \{ t' \in jkbpC . jview\ a\ t' = jview\ a\ t \}$ *<proof>*

We can show that the construction yields an implementation by appealing to the previous lemma and showing that the *pAct* functions coincide.

**lemma** *mkAutoEC-implements*: *implements mkAutoEC**<proof>*

This definition leans on the canonical trace set *jkbpC*, and is indeed effective: we can enumerate all canonical traces and are sure to find one that has the view we expect. Then it is sufficient to consider other traces of the same length due to synchrony. We would need to do this computation dynamically, as the automaton will (in general) have an infinite state space.

**end**

## 6.5 Simulations

Our goal now is to reduce the space required by the automaton constructed by *mkAutoEC* by *simulating* the equivalence classes (§2.3).

The following locale captures the framework of [van der Meyden \(1996\)](#):

**locale** *SimIncrEnvironment* =  
*IncrEnvironment jkbp envInit envAction envTrans envVal jview envObs*  
*jviewInit jviewIncr*  
**for**  $jkbp :: ('a, 'p, 'aAct)\ JKBP$   
  
**and**  $envInit :: 's\ list$   
**and**  $envAction :: 's \Rightarrow 'eAct\ list$   
**and**  $envTrans :: 'eAct \Rightarrow ('a \Rightarrow 'aAct) \Rightarrow 's \Rightarrow 's$   
**and**  $envVal :: 's \Rightarrow 'p \Rightarrow bool$   
**and**  $jview :: ('a, 's, 'tv)\ JointView$   
**and**  $envObs :: 'a \Rightarrow 's \Rightarrow 'obs$   
**and**  $jviewInit :: ('a, 'obs, 'tv)\ InitialIncrJointView$   
**and**  $jviewIncr :: ('a, 'obs, 'tv)\ IncrJointView$   
+ **fixes**  $simf :: 's\ Trace \Rightarrow 'ss$   
**fixes**  $simRels :: 'a \Rightarrow 'ss\ Relation$   
**fixes**  $simVal :: 'ss \Rightarrow 'p \Rightarrow bool$   
**assumes**  $simf: sim\ MC\ (mkKripke\ (simf\ 'jkbpC)\ simRels\ simVal)\ simf$

**context** *SimIncrEnvironment*  
**begin**

Note that the back tick ‘ is Isabelle/HOL’s relational image operator. In context it says that *simf* must be a simulation from *jkbpC* to its image under *simf*.

Firstly we lift our familiar canonical trace sets and Kripke structures through the simulation.

**abbreviation**  $jkbpCSn :: nat \Rightarrow 'ss\ set$  **where**  
 $jkbpCSn \equiv simf\ 'jkbpC_n$

**abbreviation**  $jkbpCS :: 'ss \text{ set where}$   
 $jkbpCS \equiv \text{simf } ' jkbpC$

**abbreviation**  $MCSn :: nat \Rightarrow ('a, 'p, 'ss) \text{ KripkeStructure where}$   
 $MCSn \equiv \text{mkKripke } jkbpCSn \text{ simRels simVal}$

**abbreviation**  $MCS :: ('a, 'p, 'ss) \text{ KripkeStructure where}$   
 $MCS \equiv \text{mkKripke } jkbpCS \text{ simRels simVal} \langle \text{proof} \rangle$

We will be often be concerned with the equivalence class of traces generated by agent  $a$ 's view:

**abbreviation**  $\text{sim-equiv-class} :: 'a \Rightarrow 's \text{ Trace} \Rightarrow 'ss \text{ set where}$   
 $\text{sim-equiv-class } a \ t \equiv \text{simf } ' \{ t' \in jkbpC . jview \ a \ t' = jview \ a \ t \}$

**abbreviation**  $jkbpSEC :: 'ss \text{ set set where}$   
 $jkbpSEC \equiv \bigcup a. \text{sim-equiv-class } a \ ' jkbpC$

With some effort we can show that the temporal slice of the simulated structure is adequate for determining the actions of the JKBP. The proof is tedious and routine, exploiting the sub-model property (§2.2).

$\langle \text{proof} \rangle$

**lemma**  $jkbpC\text{-}jkbpCSn\text{-}jAction\text{-}eq$ :

**assumes**  $tCn: t \in jkbpCn \ n$

**shows**  $jAction \ MC \ t = jAction \ (MCSn \ n) \ (\text{simf } t) \langle \text{proof} \rangle$

**end**

It can be shown that a suitable simulation into a finite structure is adequate to establish the existence of finite-state implementations (van der Meyden 1996, Theorem 2): essentially we apply the simulation to the states of  $mkAutoEC$ . However this result does not make it clear how the transition function can be incrementally constructed. One approach is to maintain  $jkbpC$  while extending the automaton, which is quite space inefficient.

Intuitively we would like to compute the possible  $\text{sim-equiv-class}$  successors of a given  $\text{sim-equiv-class}$  without reference to  $jkbpC$ , and this should be possible as the reachable simulated worlds must contain enough information to differentiate themselves from every other simulated world (reachable or not) that represents a trace that is observationally distinct to their own.

This leads us to asking for some extra functionality of our simulation, which we do in the following section.

## 6.6 Automata using simulations

The locale in Figure 1 captures our extra requirements of a simulation.

Firstly we relate the concrete representation  $'rep$  of equivalence classes under simulation to differ from the abstract representation  $'ss \text{ set}$  using the abstraction

```

locale AlgSimIncrEnvironment =
  SimIncrEnvironment jkbp envInit envAction envTrans envVal
    jview envObs jviewInit jviewIncr simf simRels simVal
  for jkbp :: ('a, 'p, 'aAct) JKBP
  and envInit :: 's list
  and envAction :: 's ⇒ 'eAct list
  and envTrans :: 'eAct ⇒ ('a ⇒ 'aAct) ⇒ 's ⇒ 's
  and envVal :: 's ⇒ 'p ⇒ bool

  and jview :: ('a, 's, 'tw) JointView
  and envObs :: 'a ⇒ 's ⇒ 'obs
  and jviewInit :: ('a, 'obs, 'tw) InitialIncrJointView
  and jviewIncr :: ('a, 'obs, 'tw) IncrJointView

  and simf :: 's Trace ⇒ 'ss
  and simRels :: 'a ⇒ 'ss Relation
  and simVal :: 'ss ⇒ 'p ⇒ bool

+ fixes simAbs :: 'rep ⇒ 'ss set

  and simObs :: 'a ⇒ 'rep ⇒ 'obs
  and simInit :: 'a ⇒ 'obs ⇒ 'rep
  and simTrans :: 'a ⇒ 'rep ⇒ 'rep list
  and simAction :: 'a ⇒ 'rep ⇒ 'aAct list

assumes simInit:
  ∀ a iobs. iobs ∈ envObs a ' set envInit
    → simAbs (simInit a iobs)
    = simf ' { t' ∈ jkbpC. jview a t' = jviewInit a iobs }

and simObs:
  ∀ a ec t. t ∈ jkbpC ∧ simAbs ec = sim-equiv-class a t
    → simObs a ec = envObs a (tLast t)

and simAction:
  ∀ a ec t. t ∈ jkbpC ∧ simAbs ec = sim-equiv-class a t
    → set (simAction a ec) = set (jAction MC t a)

and simTrans:
  ∀ a ec t. t ∈ jkbpC ∧ simAbs ec = sim-equiv-class a t
    → simAbs ' set (simTrans a ec)
    = { sim-equiv-class a (t' ∼ s)
      | t' s. t' ∼ s ∈ jkbpC ∧ jview a t' = jview a t }

```

Figure 1: The *SimEnvironment* locale extends the *Environment* locale with simulation and algorithmic operations. The backtick ' is Isabelle/HOL's image-of-a-set-under-a-function operator.

function  $simAbs$  (de Roever and Engelhardt 1998); there is no one-size-fits-all concrete representation, as we will see.

Secondly we ask for a function  $simInit$   $a$   $iobs$  that faithfully generates a representation of the equivalence class of simulated initial states that are possible for agent  $a$  given the valid initial observation  $iobs$ .

Thirdly the  $simObs$  function allows us to partition the results of  $simTrans$  according to the recurrent observation that agent  $a$  makes of the equivalence class.

Fourthly, the function  $simAction$  computes a list of actions enabled by the JKBP on a state that concretely represents a canonical equivalence class.

Finally we expect to compute the list of represented *sim-equiv-class* successors of a given *sim-equiv-class* using  $simTrans$ .

Note that these definitions are stated relative to the environment and the JKBP, allowing us to treat specialised cases such as broadcast (§7.4 and §7.5).

With these functions in hand, we can define our desired automaton:

**definition** (in *AlgSimIncrEnvironment*)

$mkAutoSim :: ('a, 'obs, 'aAct, 'rep) JointProtocol$

**where**

$mkAutoSim \equiv \lambda a.$

$\langle \mid pInit = simInit\ a,$

$pTrans = \lambda obs\ ec. (SOME\ ec'.\ ec' \in set\ (simTrans\ a\ ec)$   
 $\quad \wedge\ simObs\ a\ ec' = obs),$

$pAct = simAction\ a\ \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle$

The automaton faithfully constructs the simulated equivalence class of the given trace:

**lemma** (in *AlgSimIncrEnvironment*)  $mkAutoSim\ ec:$

**assumes**  $tC: t \in jkbpC$

**shows**  $simAbs\ (runJP\ mkAutoSim\ t\ a) = sim-equiv-class\ a\ t \langle proof \rangle$

This follows from a simple induction on  $t$ .

The following is a version of the Theorem 2 of van der Meyden (1996).

**theorem** (in *AlgSimIncrEnvironment*)  $mkAutoSim$ -implements:

$implements\ mkAutoSim \langle proof \rangle$

The reader may care to contrast these structures with the *progression structures* of van der Meyden (1997), where states contain entire Kripke structures, and expanding the automaton is alternated with bisimulation reduction to ensure termination when a finite-state implementation exists (see §??) We also use simulations in Appendix ?? to show the complexity of some related model checking problems.

We now review a simple *depth-first search* (DFS) theory, and an abstraction of finite maps, before presenting the algorithm for KBP synthesis.

```

locale DFS =
  fixes succs :: 'a ⇒ 'a list
  and isNode :: 'a ⇒ bool
  and invariant :: 'b ⇒ bool
  and ins :: 'a ⇒ 'b ⇒ 'b
  and memb :: 'a ⇒ 'b ⇒ bool
  and empt :: 'b
  and nodeAbs :: 'a ⇒ 'c
  assumes ins-eg:  $\bigwedge x y S. \llbracket isNode\ x; isNode\ y; invariant\ S; \neg\ memb\ y\ S \rrbracket$ 
     $\implies\ memb\ x\ (ins\ y\ S)$ 
     $\longleftrightarrow\ ((nodeAbs\ x = nodeAbs\ y) \vee memb\ x\ S)$ 
  and succs:  $\bigwedge x y. \llbracket isNode\ x; isNode\ y; nodeAbs\ x = nodeAbs\ y \rrbracket$ 
     $\implies\ nodeAbs\ `set\ (succs\ x) = nodeAbs\ `set\ (succs\ y)$ 
  and empt:  $\bigwedge x. isNode\ x \implies \neg\ memb\ x\ empt$ 
  and succs-isNode:  $\bigwedge x. isNode\ x \implies list-all\ isNode\ (succs\ x)$ 
  and empt-invariant: invariant\ empt
  and ins-invariant:  $\bigwedge x S. \llbracket isNode\ x; invariant\ S; \neg\ memb\ x\ S \rrbracket$ 
     $\implies\ invariant\ (ins\ x\ S)$ 
  and graph-finite: finite\ (nodeAbs\ ` { x . isNode\ x})

```

Figure 2: The *DFS* locale.

## 6.7 Generic DFS

We use a generic DFS to construct the transitions and action function of the implementation of the JKBP. This theory is largely due to Stefan Berghofer and Alex Krauss (Berghofer and Reiter 2009). All proofs are elided as the fine details of how we explore the state space are inessential to the synthesis algorithm.

The DFS itself is defined in the standard tail-recursive way:

```

partial-function (tailrec) gen-dfs where
  gen-dfs succs ins memb S wl = (case wl of
     $\llbracket \rrbracket \Rightarrow S$ 
  |  $(x \# xs) \Rightarrow$ 
    if memb x S then gen-dfs succs ins memb S xs
    else gen-dfs succs ins memb (ins x S) (succs x @ xs))<proof>

```

The proofs are carried out in the locale of Figure 2, which details our requirements on the parameters for the DFS to behave as one would expect. Intuitively we are traversing a graph defined by *succs* from some initial work list *wl*, constructing an object of type 'b as we go. The function *ins* integrates the current node into this construction. The predicate *isNode* is invariant over the set of states reachable from the initial work list, and is respected by *empt* and *ins*. We can also supply an invariant for the constructed object (*invariant*). Inside the locale, *dfs* abbreviates *gen-dfs* partially applied to the fixed parameters.

To support our data refinement (§6.6) we also require that the representation of nodes be adequate via the abstraction function *nodeAbs*, which the transition

relation *succs* and visited predicate *memb* must respect. To ensure termination it must be the case that there are only a finite number of states, though there might be an infinity of representations.

We characterise the DFS traversal using the reflexive transitive closure operator:

**definition** (in *DFS*) *reachable* :: 'a set  $\Rightarrow$  'a set **where**

*reachable xs*  $\equiv$   $\{(x, y). y \in \text{set } (\text{succs } x)^*\}$  “*xs* $\langle\text{proof}\rangle\langle\text{proof}\rangle\langle\text{proof}\rangle\langle\text{proof}\rangle\langle\text{proof}\rangle\langle\text{proof}\rangle\langle\text{proof}\rangle\langle\text{proof}\rangle\langle\text{proof}\rangle\langle\text{proof}\rangle$

We make use of two results about the traversal. Firstly, that some representation of each reachable node has been incorporated into the final construction:

**theorem** (in *DFS*) *reachable-imp-dfs*:

**assumes** *y*: *isNode y*

**and** *xs*: *list-all isNode xs*

**and** *m*:  $y \in \text{reachable } (\text{set } xs)$

**shows**  $\exists y'. \text{nodeAbs } y' = \text{nodeAbs } y \wedge \text{memb } y' (\text{dfs } \text{empty } xs)\langle\text{proof}\rangle\langle\text{proof}\rangle$

Secondly, that if an invariant holds on the initial object then it holds on the final one:

**theorem** (in *DFS*) *dfs-invariant*:

**assumes** *invariant S*

**assumes** *list-all isNode xs*

**shows** *invariant (dfs S xs)* $\langle\text{proof}\rangle$

## 6.8 Finite map operations

The algorithm represents an automaton as a pair of maps, which we capture abstractly with a record and a predicate:

**record** ('m, 'k, 'e) *MapOps* =

*empty* :: 'm

*lookup* :: 'm  $\Rightarrow$  'k  $\rightarrow$  'e

*update* :: 'k  $\Rightarrow$  'e  $\Rightarrow$  'm  $\Rightarrow$  'm

**definition**

*MapOps* :: ('k  $\Rightarrow$  'kabs)  $\Rightarrow$  'kabs set  $\Rightarrow$  ('m, 'k, 'e) *MapOps*  $\Rightarrow$  bool

**where**

*MapOps*  $\alpha$  *d ops*  $\equiv$

$(\forall k. \alpha k \in d \rightarrow \text{lookup } ops (\text{empty } ops) k = \text{None})$

$\wedge (\forall e k k' M. \alpha k \in d \wedge \alpha k' \in d$

$\rightarrow \text{lookup } ops (\text{update } ops k e M) k'$

$= (\text{if } \alpha k' = \alpha k \text{ then } \text{Some } e \text{ else } \text{lookup } ops M k'))\langle\text{proof}\rangle\langle\text{proof}\rangle\langle\text{proof}\rangle$

The function  $\alpha$  abstracts concrete keys of type 'k, and the parameter *d* specifies the valid abstract keys.

This approach has the advantage over a locale that we can pass records to functions, while for a locale we would need to pass the three functions separately (as in the DFS theory of §6.7).

We use the following function to test for membership in the domain of the map:

**definition** *isSome* :: 'a option ⇒ bool **where**  
*isSome opt* ≡ case opt of None ⇒ False | Some - ⇒ True⟨proof⟩⟨proof⟩⟨proof⟩

## 6.9 An algorithm for automata synthesis

We now show how to construct the automaton defined by *mkAutoSim* (§6.6) using the DFS of §6.7.

From here on we assume that the environment consists of only a finite set of states:

```
locale FiniteEnvironment =
  Environment jkbp envInit envAction envTrans envVal envObs
  for jkbp :: ('a, 'p, 'aAct) JKBP
  and envInit :: ('s :: finite) list
  and envAction :: 's ⇒ 'eAct list
  and envTrans :: 'eAct ⇒ ('a ⇒ 'aAct) ⇒ 's ⇒ 's
  and envVal :: 's ⇒ 'p ⇒ bool
  and envObs :: 'a ⇒ 's ⇒ 'obs
```

The *Algorithm* locale, shown in Figure 3, also extends the *AlgSimIncrEnvironment* locale with a pair of finite map operations: *aOps* is used to map automata states to lists of actions, and *tOps* handles simulated transitions. In both cases the maps are only required to work on the abstract domain of simulated canonical traces. Note also that the space of simulated equivalence classes of type 'ss must be finite, but there is no restriction on the representation type 'rep.

We develop the algorithm for a single, fixed agent, which requires us to define a new locale *AlgorithmForAgent* that extends *Algorithm* with an extra parameter designating the agent:

```
locale AlgorithmForAgent =
  Algorithm jkbp envInit envAction envTrans envVal jview envObs
  jviewInit jviewIncr
  simf simRels simVal simAbs simObs simInit simTrans simAction
  aOps tOps
  — ...
+ fixes a :: 'a
```

### 6.9.1 DFS operations

We represent the automaton under construction using a record:

```
record ('ma, 'mt) AlgState =
  aActs :: 'ma
  aTrans :: 'mt
```

```
context AlgorithmForAgent
begin
```

We instantiate the DFS theory with the following functions.

A node is an equivalence class of represented simulated traces.

```

locale Algorithm =
  FiniteEnvironment jkbp envInit envAction envTrans envVal envObs
+ AlgSimIncrEnvironment jkbp envInit envAction envTrans envVal jview envObs
  jviewInit jviewIncr
  simf simRels simVal simAbs simObs simInit simTrans simAction
and jkbp :: ('a, 'p, 'aAct) JKBP
and envInit :: ('s :: finite) list
and envAction :: 's ⇒ 'eAct list
and envTrans :: 'eAct ⇒ ('a ⇒ 'aAct) ⇒ 's ⇒ 's
and envVal :: 's ⇒ 'p ⇒ bool
and jview :: ('a, 's, 'tobs) JointView

and envObs :: 'a ⇒ 's ⇒ 'obs
and jviewInit :: ('a, 'obs, 'tobs) InitialIncrJointView
and jviewIncr :: ('a, 'obs, 'tobs) IncrJointView

and simf :: 's Trace ⇒ 'ss :: finite
and simRels :: 'a ⇒ 'ss Relation
and simVal :: 'ss ⇒ 'p ⇒ bool

and simAbs :: 'rep ⇒ 'ss set

and simObs :: 'a ⇒ 'rep ⇒ 'obs
and simInit :: 'a ⇒ 'obs ⇒ 'rep
and simTrans :: 'a ⇒ 'rep ⇒ 'rep list
and simAction :: 'a ⇒ 'rep ⇒ 'aAct list

+ fixes aOps :: ('ma, 'rep, 'aAct list) MapOps
  and tOps :: ('mt, 'rep × 'obs, 'rep) MapOps

assumes aOps: MapOps simAbs jkbpSEC aOps
  and tOps: MapOps (λk. (simAbs (fst k), snd k)) (jkbpSEC × UNIV) tOps

```

Figure 3: The *Algorithm* locale.



**definition**  $k\text{-isNode} :: 'rep \Rightarrow \text{bool}$  **where**  
 $k\text{-isNode } ec \equiv \text{simAbs } ec \in \text{sim-equiv-class } a \text{ ' jkbpC}$

The successors of a node are those produced by the simulated transition function.

**abbreviation**  $k\text{-succs} :: 'rep \Rightarrow 'rep \text{ list}$  **where**  
 $k\text{-succs} \equiv \text{simTrans } a$

The initial automaton has no transitions and no actions.

**definition**  $k\text{-empt} :: ('ma, 'mt) \text{ AlgState}$  **where**  
 $k\text{-empt} \equiv (\mid aActs = \text{empty } aOps, aTrans = \text{empty } tOps \mid)$

We use the domain of the action map to track the set of nodes the DFS has visited.

**definition**  $k\text{-memb} :: 'rep \Rightarrow ('ma, 'mt) \text{ AlgState} \Rightarrow \text{bool}$  **where**  
 $k\text{-memb } s \ A \equiv \text{isSome } (\text{lookup } aOps \ (aActs \ A) \ s)$

We integrate a new equivalence class into the automaton by updating the action and transition maps:

**definition**  $\text{actsUpdate} :: 'rep \Rightarrow ('ma, 'mt) \text{ AlgState} \Rightarrow 'ma$  **where**  
 $\text{actsUpdate } ec \ A \equiv \text{update } aOps \ ec \ (\text{simAction } a \ ec) \ (aActs \ A)$

**definition**  $\text{transUpdate} :: 'rep \Rightarrow 'rep \Rightarrow 'mt \Rightarrow 'mt$  **where**  
 $\text{transUpdate } ec \ ec' \ \text{at} \equiv \text{update } tOps \ (ec, \text{simObs } a \ ec') \ ec' \ \text{at}$

**definition**  $k\text{-ins} :: 'rep \Rightarrow ('ma, 'mt) \text{ AlgState} \Rightarrow ('ma, 'mt) \text{ AlgState}$  **where**  
 $k\text{-ins } ec \ A \equiv (\mid aActs = \text{actsUpdate } ec \ A,$   
 $aTrans = \text{foldr } (\text{transUpdate } ec) \ (k\text{-succs } ec) \ (aTrans \ A) \mid)$

The required properties are straightforward to show.

$\langle \text{proof} \rangle \langle \text{proof} \rangle \langle \text{proof} \rangle \langle \text{proof} \rangle \langle \text{proof} \rangle \langle \text{proof} \rangle$

### 6.9.2 Algorithm invariant

The invariant for the automata construction is straightforward, viz that at each step of the process the state represents an automaton that concords with  $mkAutoSim$  on the visited equivalence classes. We also need to know that the state has preserved the  $MapOps$  invariants.

**definition**  $k\text{-invariant} :: ('ma, 'mt) \text{ AlgState} \Rightarrow \text{bool}$  **where**  
 $k\text{-invariant } A \equiv$   
 $(\forall ec \ ec'. k\text{-isNode } ec \wedge k\text{-isNode } ec' \wedge \text{simAbs } ec' = \text{simAbs } ec$   
 $\longrightarrow \text{lookup } aOps \ (aActs \ A) \ ec = \text{lookup } aOps \ (aActs \ A) \ ec')$   
 $\wedge (\forall ec \ ec' \ \text{obs}. k\text{-isNode } ec \wedge k\text{-isNode } ec' \wedge \text{simAbs } ec' = \text{simAbs } ec$   
 $\longrightarrow \text{lookup } tOps \ (aTrans \ A) \ (ec, \text{obs}) = \text{lookup } tOps \ (aTrans \ A) \ (ec', \text{obs}))$   
 $\wedge (\forall ec. k\text{-isNode } ec \wedge k\text{-memb } ec \ A$   
 $\longrightarrow (\exists \text{acts}. \text{lookup } aOps \ (aActs \ A) \ ec = \text{Some } \text{acts}$   
 $\wedge \text{set } \text{acts} = \text{set } (\text{simAction } a \ ec)))$

$$\begin{aligned}
& \wedge (\forall ec \text{ obs. } k\text{-isNode } ec \wedge k\text{-memb } ec \ A \\
& \quad \wedge \text{ obs} \in \text{simObs } a \text{ ' set (simTrans } a \ ec) \\
& \quad \longrightarrow (\exists ec'. \text{lookup } tOps \ (aTrans \ A) \ (ec, \text{ obs}) = \text{Some } ec' \\
& \quad \quad \wedge \text{simAbs } ec' \in \text{simAbs ' set (simTrans } a \ ec) \\
& \quad \quad \wedge \text{simObs } a \ ec' = \text{obs}) \langle \text{proof} \rangle \langle \text{proof} \rangle \langle \text{proof} \rangle \langle \text{proof} \rangle \langle \text{proof} \rangle \langle \text{proof} \rangle \langle \text{proof} \rangle \langle \text{proof} \rangle \langle \text{proof} \rangle \langle \text{proof} \rangle \langle \text{proof} \rangle \langle \text{proof} \rangle \langle \text{proof} \rangle
\end{aligned}$$

Showing that the invariant holds of  $k\text{-empt}$  and is respected by  $k\text{-ins}$  is routine. The initial frontier is the partition of the set of initial states under the initial observation function.

**definition** (in *Algorithm*)  $k\text{-frontier} :: 'a \Rightarrow 'rep \text{ list}$  **where**  
 $k\text{-frontier } a \equiv \text{map (simInit } a \circ \text{envObs } a) \ \text{envInit} \langle \text{proof} \rangle$   
**end**

We now instantiate the *DFS* locale with respect to the *AlgorithmForAgent* locale. The instantiated lemmas are given the mandatory prefix *KBPAIlg* in the *AlgorithmForAgent* locale.

**sublocale** *AlgorithmForAgent*  
 $< \text{KBPAIlg: DFS } k\text{-succs } k\text{-isNode } k\text{-invariant } k\text{-ins } k\text{-memb } k\text{-empt } \text{simAbs}$   
 $\langle \text{proof} \rangle \langle \text{proof} \rangle$  **context** *AlgorithmForAgent*  
**begin**

The final algorithm, with the constants inlined, is shown in Figure 4. The rest of this section shows its correctness.

Firstly it follows immediately from *dfs-invariant* that the invariant holds of the result of the DFS:

$\langle \text{proof} \rangle$  **lemma**  $k\text{-dfs-invariant: } k\text{-invariant } k\text{-dfs} \langle \text{proof} \rangle$

Secondly we can see that the set of reachable equivalence classes coincides with the partition of  $jkbpC$  under the simulation and representation functions:

**lemma**  $k\text{-reachable:}$   
 $\text{simAbs ' KBPAIlg.reachable (set (k-frontier } a)) = \text{sim-equiv-class } a \text{ ' } jkbpC \langle \text{proof} \rangle$

Left to right follows from an induction on the reflexive, transitive closure, and right to left by induction over canonical traces.

This result immediately yields the same result at the level of representations:

**lemma**  $k\text{-memb-rep:}$   
**assumes**  $N: k\text{-isNode } \text{rec}$   
**shows**  $k\text{-memb } \text{rec } k\text{-dfs} \langle \text{proof} \rangle$   
**end**

This concludes our agent-specific reasoning; we now show that the algorithm works for all agents. The following command generalises all our lemmas in the *AlgorithmForAgent* to the *Algorithm* locale, giving them the mandatory prefix *KBP*:

**sublocale** *Algorithm*  
 $< \text{KBP: AlgorithmForAgent}$

**definition**

```

alg-dfs :: ('ma, 'rep, 'aAct list) MapOps
        => ('mt, 'rep × 'obs, 'rep) MapOps
        => ('rep => 'obs)
        => ('rep => 'rep list)
        => ('rep => 'aAct list)
        => 'rep list
        => ('ma, 'mt) AlgState

```

**where**

```

alg-dfs aOps tOps simObs simTrans simAction ≡
  let k-empt = (| aActs = empty aOps, aTrans = empty tOps |);
      k-memb = (λs A. isSome (lookup aOps (aActs A) s));
      k-succs = simTrans;
      actsUpdate = λec A. update aOps ec (simAction ec) (aActs A);
      transUpdate = λec ec' at. update tOps (ec, simObs ec') ec' at;
      k-ins = λec A. (| aActs = actsUpdate ec A,
                      aTrans = foldr (transUpdate ec) (k-succs ec) (aTrans A) |)
  in gen-dfs k-succs k-ins k-memb k-empt

```

**definition**

```

mkAlgAuto :: ('ma, 'rep, 'aAct list) MapOps
           => ('mt, 'rep × 'obs, 'rep) MapOps
           => ('a => 'rep => 'obs)
           => ('a => 'obs => 'rep)
           => ('a => 'rep => 'rep list)
           => ('a => 'rep => 'aAct list)
           => ('a => 'rep list)
           => ('a, 'obs, 'aAct, 'rep) JointProtocol

```

**where**

```

mkAlgAuto aOps tOps simObs simInit simTrans simAction frontier ≡ λa.
  let auto = alg-dfs aOps tOps (simObs a) (simTrans a) (simAction a)
              (frontier a)
  in (| pInit = simInit a,
        pTrans = λobs ec. the (lookup tOps (aTrans auto) (ec, obs)),
        pAct = λec. the (lookup aOps (aActs auto) ec) |)

```

Figure 4: The algorithm. The function *the* projects a value from the *'a option* type, diverging on *None*.

```

    jkbp envInit envAction envTrans envVal jview envObs
    jviewInit jviewIncr simf simRels simVal simAbs simObs
    simInit simTrans simAction aOps tOps a for a⟨proof⟩

```

**context** *Algorithm*  
**begin**

**abbreviation**

```

    k-mkAlgAuto ≡
    mkAlgAuto aOps tOps simObs simInit simTrans simAction k-frontier⟨proof⟩

```

Running the automata produced by the DFS on a canonical trace  $t$  yields some representation of the expected equivalence class:

**lemma** *k-mkAlgAuto-ec:*

```

    assumes tC: t ∈ jkbpC
    shows simAbs (runJP k-mkAlgAuto t a) = sim-equiv-class a t⟨proof⟩

```

This involves an induction over the canonical trace  $t$ .

That the DFS and *mkAutoSim* yield the same actions on canonical traces follows immediately from this result and the invariant:

**lemma** *k-mkAlgAuto-mkAutoSim-act-eq:*

```

    assumes tC: t ∈ jkbpC
    shows set ∘ actJP k-mkAlgAuto t = set ∘ actJP mkAutoSim t⟨proof⟩

```

Therefore these two constructions are behaviourally equivalent, and so the DFS generates an implementation of *jkbp* in the given environment:

**theorem** *k-mkAlgAuto-implements: implements k-mkAlgAuto⟨proof⟩*

**end**

Clearly the automata generated by this algorithm are large. We discuss this issue in §??.

## 7 Concrete views

Following [van der Meyden \(1996\)](#), we provide two concrete synchronous views that illustrate how the theory works. For each view we give a simulation and a representation that satisfy the requirements of the *Algorithm* locale in [Figure 3](#).

### 7.1 The Clock View

The *clock view* records the current time and the observation for the most recent state:

**definition** (in *Environment*)

```

    clock-jview :: ('a, 's, nat × 'obs) JointView

```

**where**

```

    clock-jview ≡ λa t. (tLength t, envObs a (tLast t))⟨proof⟩⟨proof⟩⟨proof⟩⟨proof⟩⟨proof⟩

```

This is the least-information synchronous view, given the requirements of §4. We show that finite-state implementations exist for all environments with respect to this view as per van der Meyden (1996).

The corresponding incremental view simply increments the counter records the new observation.

**definition** (in *Environment*)

$clock\text{-}jviewInit :: 'a \Rightarrow 'obs \Rightarrow nat \times 'obs$

**where**

$clock\text{-}jviewInit \equiv \lambda a\ obs. (0, obs)$

**definition** (in *Environment*)

$clock\text{-}jviewIncr :: 'a \Rightarrow 'obs \Rightarrow nat \times 'obs \Rightarrow nat \times 'obs$

**where**

$clock\text{-}jviewIncr \equiv \lambda a\ obs' (l, obs). (l + 1, obs')$

It is straightforward to demonstrate the assumptions of the incremental environment locale (§6.1) with respect to an arbitrary environment.

**sublocale** *Environment*

$< Clock: IncrEnvironment\ jkbp\ envInit\ envAction\ envTrans\ envVal$   
 $clock\text{-}jview\ envObs\ clock\text{-}jviewInit\ clock\text{-}jviewIncr\langle proof \rangle$

As we later show, satisfaction of a formula at a trace  $t \in Clock.jkbpC_n$  is determined by the set of final states of traces in  $Clock.jkbpC_n$ :

**context** *Environment*

**begin**

**abbreviation**  $clock\text{-}commonAbs :: 's\ Trace \Rightarrow 's\ set$  **where**

$clock\text{-}commonAbs\ t \equiv tLast\ ' Clock.jkbpC_n\ (tLength\ t)$

Intuitively this set contains the states that the agents commonly consider possible at time  $n$ , which is sufficient for determining knowledge as the clock view ignores paths. Therefore we can simulate trace  $t$  by pairing this abstraction of  $t$  with its final state:

**type-synonym** (in  $-$ )  $'s\ clock\text{-}simWorlds = 's\ set \times 's$

**definition**  $clock\text{-}sim :: 's\ Trace \Rightarrow 's\ clock\text{-}simWorlds$  **where**

$clock\text{-}sim \equiv \lambda t. (clock\text{-}commonAbs\ t, tLast\ t)$

In the Kripke structure for our simulation, we relate worlds for  $a$  if the sets of commonly-held states coincide, and the observation of the final states of the traces is the same. Propositions are evaluated at the final state.

**definition**  $clock\text{-}simRels :: 'a \Rightarrow 's\ clock\text{-}simWorlds\ Relation$  **where**

$clock\text{-}simRels \equiv \lambda a. \{ ((X, s), (X', s')) \mid X\ X'\ s\ s'.$   
 $X = X' \wedge \{s, s'\} \subseteq X \wedge envObs\ a\ s = envObs\ a\ s' \}$

**definition**  $clock\text{-}simVal :: 's\ clock\text{-}simWorlds \Rightarrow 'p \Rightarrow bool$  **where**

$clock\text{-}simVal \equiv envVal \circ snd$

**abbreviation** *clock-simMC* :: ('a, 'p, 's clock-simWorlds) KripkeStructure **where**  
*clock-simMC* ≡ mkKripke (clock-sim ' Clock.jkbpC) clock-simRels clock-simVal⟨proof⟩⟨proof⟩⟨proof⟩⟨proof⟩

That this is in fact a simulation (§2.3) is entirely straightforward.

**lemma** *clock-sim*:  
*sim Clock.MC clock-simMC clock-sim⟨proof⟩*  
**end**

The *SimIncrEnvironment* of §6.5 only requires that we provide it an *Environment* and a simulation.

**sublocale** *Environment*  
 < Clock: *SimIncrEnvironment* jkbp envInit envAction envTrans envVal  
 clock-jview envObs clock-jviewInit clock-jviewIncr  
 clock-sim clock-simRels clock-simVal⟨proof⟩

We next consider algorithmic issues.

### 7.1.1 Representations

We now turn to the issue of how to represent equivalence classes of states. As these are used as map keys, it is easiest to represent them canonically. A simple approach is to use *ordered distinct lists* of type '*a* odlist for the sets and *tries* for the maps. Therefore we ask that environment states '*s* belong to the class *linorder* of linearly-ordered types, and moreover that the set *agents* be effectively presented. We introduce a new locale capturing these requirements:

**locale** *FiniteLinorderEnvironment* =  
*Environment* jkbp envInit envAction envTrans envVal envObs  
**for** jkbp :: ('a::{finite, linorder}, 'p, 'aAct) JKBP  
**and** envInit :: ('s::{finite, linorder}) list  
**and** envAction :: 's ⇒ 'eAct list  
**and** envTrans :: 'eAct ⇒ ('a ⇒ 'aAct) ⇒ 's ⇒ 's  
**and** envVal :: 's ⇒ 'p ⇒ bool  
**and** envObs :: 'a ⇒ 's ⇒ 'obs  
+ **fixes** agents :: 'a odlist  
**assumes** agents: ODList.toSet agents = UNIV

**context** *FiniteLinorderEnvironment*  
**begin**

For a fixed agent *a*, we can reduce the number of worlds in *clock-simMC* by taking its quotient with respect to the equivalence relation for *a*. In other words, we represent a simulated equivalence class by a pair of the set of all states reachable at a particular time, and the subset of these that *a* considers possible. The worlds in our representational Kripke structure are therefore a pair of ordered, distinct lists:

**type-synonym** (in -) '*s* clock-simWorldsRep = '*s* odlist × '*s* odlist

We can readily abstract a representation to a set of simulated equivalence classes:

**definition** (in  $-$ )

$clock\text{-}simAbs :: 's::linorder\ clock\text{-}simWorldsRep \Rightarrow 's\ clock\text{-}simWorlds\ set$

**where**

$clock\text{-}simAbs\ X \equiv \{ (ODList.toSet\ (fst\ X),\ s) \mid s.\ s \in ODList.toSet\ (snd\ X) \}$

Assuming  $X$  represents a simulated equivalence class for  $t \in jkbpC$ ,  $clock\text{-}simAbs$   $X$  decomposes into these two functions:

**definition**

$agent\text{-}abs :: 'a \Rightarrow 's\ Trace \Rightarrow 's\ set$

**where**

$agent\text{-}abs\ a\ t \equiv$   
 $\{ tLast\ t' \mid t'.\ t' \in Clock.jkbpC \wedge clock\text{-}jview\ a\ t' = clock\text{-}jview\ a\ t \}$

**definition**

$common\text{-}abs :: 's\ Trace \Rightarrow 's\ set$

**where**

$common\text{-}abs\ t \equiv tLast\ ' Clock.jkbpCn\ (tLength\ t)\langle proof \rangle\langle proof \rangle\langle proof \rangle\langle proof \rangle\langle proof \rangle$

This representation is canonical on the domain of interest (though not in general):

**lemma**  $clock\text{-}simAbs\text{-}inj\text{-}on$ :

$inj\text{-}on\ clock\text{-}simAbs\ \{ x.\ clock\text{-}simAbs\ x \in Clock.jkbpSEC \}\langle proof \rangle$

We could further compress this representation by labelling each element of the set of states reachable at time  $n$  with a bit to indicate whether the agent considers that state possible. Note, however, that the representation would be non-canonical: if  $(s, True)$  is in the representation, indicating that the agent considers  $s$  possible, then  $(s, False)$  may or may not be. The associated abstraction function is not injective and hence would obfuscate the following. Repairing this would entail introducing a new type, which would again complicate this development.

The following lemmas make use of this Kripke structure, constructed from the set of final states of a temporal slice  $X$ :

**definition**

$clock\text{-}repRels :: 'a \Rightarrow ('s \times 's)\ set$

**where**

$clock\text{-}repRels \equiv \lambda a.\ \{ (s, s').\ envObs\ a\ s = envObs\ a\ s' \}$

**abbreviation**

$clock\text{-}repMC :: 's\ set \Rightarrow ('a, 'p, 's)\ KripkeStructure$

**where**

$clock\text{-}repMC \equiv \lambda X.\ mkKripke\ X\ clock\text{-}repRels\ envVal\langle proof \rangle\langle proof \rangle$

We can show that this Kripke structure retains sufficient information from  $clock\text{-}simMC$  by showing simulation. This is eased by introducing an intermediary structure that focusses on a particular trace:

**abbreviation**

$$\text{clock-jkbpCSt} :: 'b \text{ Trace} \Rightarrow 's \text{ clock-simWorlds set}$$
**where**

$$\text{clock-jkbpCSt } t \equiv \text{clock-sim } ' \text{ Clock.jkbpCn } (t\text{Length } t)$$
**abbreviation**

$$\text{clock-simMCt} :: 'b \text{ Trace} \Rightarrow ('a, 'p, 's \text{ clock-simWorlds}) \text{ KripkeStructure}$$
**where**

$$\text{clock-simMCt } t \equiv \text{mkKripke } (\text{clock-jkbpCSt } t) \text{ clock-simRels } \text{clock-simVal}$$
**definition**  $\text{clock-repSim} :: 's \text{ clock-simWorlds} \Rightarrow 's$  **where**

$$\text{clock-repSim} \equiv \text{snd} \langle \text{proof} \rangle \langle \text{proof} \rangle \langle \text{proof} \rangle \langle \text{proof} \rangle$$
**lemma**  $\text{clock-repSim}$ :
$$\text{assumes } tC: t \in \text{Clock.jkbpC}$$

$$\text{shows } \text{sim } (\text{clock-simMCt } t)$$

$$((\text{clock-repMC} \circ \text{fst}) (\text{clock-sim } t))$$

$$\text{clock-repSim} \langle \text{proof} \rangle$$

The following sections show how we satisfy the remaining requirements of the *Algorithm* locale of Figure 3. Where the proof is routine, we simply present the lemma without proof or comment.

Due to a limitation in the code generator in the present version of Isabelle (2011), we need to define the equations we wish to execute outside of a locale; the syntax (*in*  $-$ ) achieves this by making definitions at the theory top-level. We then define (but elide) locale-local abbreviations that supply the locale-bound variables to these definitions.

**7.1.2 Initial states**

The initial states of the automaton for an agent is simply  $\text{envInit}$  paired with the partition of  $\text{envInit}$  under the agent's observation.

**definition** (*in*  $-$ )
$$\text{clock-simInit} :: ('s::\text{linorder}) \text{ list} \Rightarrow ('a \Rightarrow 's \Rightarrow 'obs)$$

$$\Rightarrow 'a \Rightarrow 'obs \Rightarrow 's \text{ clock-simWorldsRep}$$
**where**

$$\text{clock-simInit } \text{envInit } \text{envObs} \equiv \lambda a \text{ iobs.}$$

$$\text{let } \text{cec} = \text{ODList.fromList } \text{envInit}$$

$$\text{in } (\text{cec}, \text{ODList.filter } (\lambda s. \text{envObs } a \text{ } s = \text{iobs}) \text{ cec})$$
**lemma**  $\text{clock-simInit}$ :
$$\text{assumes } \text{iobs} \in \text{envObs } a \text{ ' set } \text{envInit}$$

$$\text{shows } \text{clock-simAbs } (\text{clock-simInit } a \text{ iobs})$$

$$= \text{clock-sim } ' \{ t' \in \text{Clock.jkbpC.}$$

$$\text{clock-jview } a \text{ } t' = \text{clock-jviewInit } a \text{ iobs } \} \langle \text{proof} \rangle$$



### 7.1.3 Simulated observations

Agent  $a$  will make the same observation at any of the worlds that it considers possible, so we choose the first one in the list:

**definition** (in  $-$ )

$$\begin{aligned} \text{clock-simObs} &:: ('a \Rightarrow ('s :: \text{linorder}) \Rightarrow 'obs) \\ &\Rightarrow 'a \Rightarrow 's \text{ clock-simWorldsRep} \Rightarrow 'obs \end{aligned}$$

**where**

$$\text{clock-simObs envObs} \equiv \lambda a. \text{envObs } a \circ \text{ODList.hd} \circ \text{snd}$$

**lemma** *clock-simObs*:

**assumes**  $tC: t \in \text{Clock.jkbpC}$

**and**  $ec: \text{clock-simAbs } ec = \text{Clock.sim-equiv-class } a \ t$

**shows**  $\text{clock-simObs } a \ ec = \text{envObs } a \ (\text{tLast } t)\langle \text{proof} \rangle$

### 7.1.4 Evaluation

We define our *eval* function in terms of *evalS*, which implements boolean logic over  $'s \text{ odlist}$  in the usual way – see §7.3.4 for the relevant clauses. It requires three functions specific to the representation: one each for propositions, knowledge and common knowledge.

Propositions define subsets of the worlds considered possible:

**abbreviation** (in  $-$ )

$$\begin{aligned} \text{clock-evalProp} &:: (('s :: \text{linorder}) \Rightarrow 'p \Rightarrow \text{bool}) \\ &\Rightarrow 's \text{ odlist} \Rightarrow 'p \Rightarrow 's \text{ odlist} \end{aligned}$$

**where**

$$\text{clock-evalProp envVal} \equiv \lambda X \ p. \text{ODList.filter } (\lambda s. \text{envVal } s \ p) \ X$$

The knowledge relation computes the subset of the commonly-held-possible worlds  $cec$  that agent  $a$  considers possible at world  $s$ :

**definition** (in  $-$ )

$$\begin{aligned} \text{clock-knowledge} &:: ('a \Rightarrow ('s :: \text{linorder}) \Rightarrow 'obs) \Rightarrow 's \text{ odlist} \\ &\Rightarrow 'a \Rightarrow 's \Rightarrow 's \text{ odlist} \end{aligned}$$

**where**

$$\text{clock-knowledge envObs } cec \equiv \lambda a \ s.$$

$$\text{ODList.filter } (\lambda s'. \text{envObs } a \ s = \text{envObs } a \ s') \ cec$$

Similarly the common knowledge operation computes the transitive closure of the union of the knowledge relations for the agents  $as$ :

**definition** (in  $-$ )

$$\begin{aligned} \text{clock-commonKnowledge} &:: ('a \Rightarrow ('s :: \text{linorder}) \Rightarrow 'obs) \Rightarrow 's \text{ odlist} \\ &\Rightarrow 'a \ \text{list} \Rightarrow 's \Rightarrow 's \text{ odlist} \end{aligned}$$

**where**

$$\text{clock-commonKnowledge envObs } cec \equiv \lambda as \ s.$$

$$\text{let } r = \lambda a. \text{ODList.fromList } [(s', s'') . s' \leftarrow \text{toList } cec, s'' \leftarrow \text{toList } cec, \\ \text{envObs } a \ s' = \text{envObs } a \ s''];$$

$$R = \text{toList } (\text{ODList.big-union } r \ as)$$

in  $ODList.fromList$  (*memo-list-trancl*  $R$   $s$ )

The function *memo-list-trancl* comes from the executable transitive closure theory of (Sternagel and Thiemann 2011).

The evaluation function evaluates a subjective knowledge formula on the representation of an equivalence class:

**definition** (in  $-$ )

$$\begin{aligned} eval &:: (('s :: linorder) \Rightarrow 'p \Rightarrow bool) \\ &\Rightarrow ('a \Rightarrow 's \Rightarrow 'obs) \\ &\Rightarrow 's \text{ clock-simWorldsRep} \Rightarrow ('a, 'p) \text{ Kform} \Rightarrow bool \end{aligned}$$

**where**

$$\begin{aligned} eval \ envVal \ envObs &\equiv \lambda(cec, aec). \ evalS \ (clock\ evalProp \ envVal) \\ &\quad (clock\ knowledge \ envObs \ cec) \\ &\quad (clock\ commonKnowledge \ envObs \ cec) \\ &\quad aec \end{aligned}$$

This function corresponds with the standard semantics:

$\langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle$  **lemma** *eval-models*:

$$\begin{aligned} \text{assumes } tC &: t \in \text{Clock.jkbpC} \\ \text{and } aec &: ODList.toSet \ aec = \text{agent-abs } a \ t \\ \text{and } cec &: ODList.toSet \ cec = \text{common-abs } t \\ \text{and } subj\ phi &: \text{subjective } a \ \varphi \\ \text{and } s &: s \in ODList.toSet \ aec \\ \text{shows } eval \ envVal \ envObs \ (cec, aec) \ \varphi \\ &\longleftrightarrow \text{clock-repMC} \ (ODList.toSet \ cec), s \models \varphi \langle proof \rangle \end{aligned}$$

### 7.1.5 Simulated actions

From a common equivalence class and a subjective equivalence class for agent  $a$ , we can compute the actions enabled for  $a$ :

**definition** (in  $-$ )

$$\begin{aligned} clock\ simAction &:: ('a, 'p, 'aAct) \ JKBP \Rightarrow (('s :: linorder) \Rightarrow 'p \Rightarrow bool) \\ &\Rightarrow ('a \Rightarrow 's \Rightarrow 'obs) \\ &\Rightarrow 'a \Rightarrow 's \text{ clock-simWorldsRep} \Rightarrow 'aAct \ \text{list} \end{aligned}$$

**where**

$$\begin{aligned} clock\ simAction \ jkbp \ envVal \ envObs &\equiv \lambda a \ (Y, X). \\ &[\ \text{action } gc. \ gc \leftarrow \ jkbp \ a, \ eval \ envVal \ envObs \ (Y, X) \ (\text{guard } gc) \ ] \end{aligned}$$

Using the above result about evaluation, we can relate *clock-simAction* to *jAction*. Firstly, *clock-simAction* behaves the same as *jAction* using the *clock-repMC* structure:

**lemma** *clock-simAction-jAction*:

$$\begin{aligned} \text{assumes } tC &: t \in \text{Clock.jkbpC} \\ \text{and } aec &: ODList.toSet \ aec = \text{agent-abs } a \ t \\ \text{and } cec &: ODList.toSet \ cec = \text{common-abs } t \\ \text{shows } set \ (clock\ simAction \ a \ (cec, aec)) \\ &= \text{set} \ (jAction \ (clock\ repMC \ (ODList.toSet \ cec)) \ (tLast \ t) \ a) \langle proof \rangle \langle proof \rangle \end{aligned}$$

We can connect the agent's choice of actions on the *clock-repMC* structure to those on the *Clock.MC* structure using our earlier results about actions being preserved by generated models and simulations.

**lemma** *clock-simAction'*:  
**assumes** *tC*:  $t \in \text{Clock.jkbpC}$   
**assumes** *aec*:  $\text{ODList.toSet } aec = \text{agent-abs } a \ t$   
**assumes** *cec*:  $\text{ODList.toSet } cec = \text{common-abs } t$   
**shows**  $\text{set } (\text{clock-simAction } a \ (cec, aec)) = \text{set } (\text{jAction } \text{Clock.MC } t \ a)$ *<proof>*

The *Algorithm* locale requires a specialisation of this lemma:

**lemma** *clock-simAction*:  
**assumes** *tC*:  $t \in \text{Clock.jkbpC}$   
**assumes** *ec*:  $\text{clock-simAbs } ec = \text{Clock.sim-equiv-class } a \ t$   
**shows**  $\text{set } (\text{clock-simAction } a \ ec) = \text{set } (\text{jAction } \text{Clock.MC } t \ a)$ *<proof>*

### 7.1.6 Simulated transitions

We need to determine the image of the set of commonly-held-possible states under the transition function, and also for the agent's subjective equivalence class. We do this with the *clock-trans* function:

**definition** (in  $-$ )  
 $\text{clock-trans} :: ('a :: \text{linorder}) \text{odlist} \Rightarrow ('a, 'p, 'aAct) \text{JKBP}$   
 $\Rightarrow (('s :: \text{linorder}) \Rightarrow 'eAct \text{list})$   
 $\Rightarrow ('eAct \Rightarrow ('a \Rightarrow 'aAct) \Rightarrow 's \Rightarrow 's)$   
 $\Rightarrow ('s \Rightarrow 'p \Rightarrow \text{bool}) \Rightarrow ('a \Rightarrow 's \Rightarrow 'obs)$   
 $\Rightarrow 's \text{odlist} \Rightarrow 's \text{odlist} \Rightarrow 's \text{odlist}$

**where**

$\text{clock-trans agents } jkbp \ \text{envAction } \ \text{envTrans } \ \text{envVal } \ \text{envObs} \equiv \lambda cec \ X.$   
 $\text{ODList.fromList } (\text{concat}$   
 $\quad [ [ \text{envTrans } \ \text{eact } \ \text{aact } \ s .$   
 $\quad \quad \text{eact} \leftarrow \text{envAction } \ s,$   
 $\quad \quad \text{aact} \leftarrow \text{listToFuns } (\lambda a. \ \text{clock-simAction } \ jkbp \ \text{envVal } \ \text{envObs } \ a$   
 $\quad \quad \quad (\text{cec}, \ \text{clock-knowledge } \ \text{envObs } \ cec \ a \ s))$   
 $\quad \quad \quad (\text{toList } \ \text{agents}) \ ] .$   
 $\quad \quad \text{s} \leftarrow \text{toList } \ X \ ])$ *<proof>*

The function *listToFuns* exhibits the isomorphism between  $('a \times 'b \ \text{list}) \ \text{list}$  and  $('a \Rightarrow 'b) \ \text{list}$  for finite types *a*.

We can show that the transition function works for both the commonly-held set of states and the agent subjective one. The proofs are straightforward.

**lemma** *clock-trans-common*:  
**assumes** *tC*:  $t \in \text{Clock.jkbpC}$   
**assumes** *ec*:  $\text{clock-simAbs } ec = \text{Clock.sim-equiv-class } a \ t$   
**shows**  $\text{ODList.toSet } (\text{clock-trans } (\text{fst } ec) \ (\text{fst } ec))$   
 $= \{ s \mid t' \ s. \ t' \rightsquigarrow s \in \text{Clock.jkbpC} \wedge \text{tLength } t' = \text{tLength } t \}$ *<proof>*

**lemma** *clock-trans-agent*:

**assumes**  $tC$ :  $t \in \text{Clock.jkbp}C$   
**assumes**  $ec$ :  $\text{clock-simAbs } ec = \text{Clock.sim-equiv-class } a \ t$   
**shows**  $\text{ODList.toSet } (\text{clock-trans } (\text{fst } ec) (\text{snd } ec))$   
 $= \{ s \mid t' \ s. \ t' \rightsquigarrow s \in \text{Clock.jkbp}C \wedge \text{clock-jview } a \ t' = \text{clock-jview } a \ t \} \langle \text{proof} \rangle$

Note that the clock semantics disregards paths, so we simply compute the successors of the temporal slice and partition that. Similarly the successors of the agent's subjective equivalence class tell us what the set of possible observations are:

**definition (in -)**  
 $\text{clock-mkSuccs} :: ('s :: \text{linorder} \Rightarrow 'obs) \Rightarrow 'obs \Rightarrow 's \ \text{odlist}$   
 $\Rightarrow 's \ \text{clock-simWorldsRep}$

**where**

$\text{clock-mkSuccs } \text{envObs } obs \ Y' \equiv (Y', \text{ODList.filter } (\lambda s. \ \text{envObs } s = obs) \ Y')$

Finally we can define our transition function on simulated states:

**definition (in -)**  
 $\text{clock-simTrans} :: ('a :: \text{linorder}) \ \text{odlist} \Rightarrow ('a, 'p, 'aAct) \ \text{JKBP}$   
 $\Rightarrow (('s :: \text{linorder}) \Rightarrow 'eAct \ \text{list})$   
 $\Rightarrow ('eAct \Rightarrow ('a \Rightarrow 'aAct) \Rightarrow 's \Rightarrow 's)$   
 $\Rightarrow ('s \Rightarrow 'p \Rightarrow \text{bool}) \Rightarrow ('a \Rightarrow 's \Rightarrow 'obs)$   
 $\Rightarrow 'a \Rightarrow 's \ \text{clock-simWorldsRep} \Rightarrow 's \ \text{clock-simWorldsRep} \ \text{list}$

**where**

$\text{clock-simTrans } \text{agents } \text{jkbp } \text{envAction } \text{envTrans } \text{envVal } \text{envObs} \equiv \lambda a \ (Y, X).$   
 $\text{let } X' = \text{clock-trans } \text{agents } \text{jkbp } \text{envAction } \text{envTrans } \text{envVal } \text{envObs } \ Y \ X;$   
 $Y' = \text{clock-trans } \text{agents } \text{jkbp } \text{envAction } \text{envTrans } \text{envVal } \text{envObs } \ Y \ Y$   
 $\text{in } [ \text{clock-mkSuccs } (\text{envObs } a) \ \text{obs } \ Y' .$   
 $\text{obs} \leftarrow \text{map } (\text{envObs } a) \ (\text{toList } X') ]$

Showing that this respects the property asked of it by the *Algorithm* locale is straightforward:

**lemma**  $\text{clock-simTrans}$ :

**assumes**  $tC$ :  $t \in \text{Clock.jkbp}C$   
**and**  $ec$ :  $\text{clock-simAbs } ec = \text{Clock.sim-equiv-class } a \ t$   
**shows**  $\text{clock-simAbs } ' \ \text{set } (\text{clock-simTrans } a \ ec)$   
 $= \{ \text{Clock.sim-equiv-class } a \ (t' \rightsquigarrow s)$   
 $\mid t' \ s. \ t' \rightsquigarrow s \in \text{Clock.jkbp}C \wedge \text{clock-jview } a \ t' = \text{clock-jview } a \ t \} \langle \text{proof} \rangle$

**end**

### 7.1.7 Maps

As mentioned above, the canonicity of our ordered, distinct list representation of automaton states allows us to use them as keys in a digital trie; a value of type  $('key, 'val) \ \text{trie}$  maps keys of type  $'key \ \text{list}$  to values of type  $'val$ .

In this specific case we track automaton transitions using a two-level structure mapping sets of states to an association list mapping observations to sets of states, and for actions automaton states map directly to agent actions.

**type-synonym**  $('s, 'obs) \ \text{clock-trans-trie}$

= ('s, ('s, ('obs, 's clock-simWorldsRep) mapping) trie) trie  
**type-synonym** ('s, 'aAct) clock-acts-trie = ('s, ('s, 'aAct) trie) trie⟨proof⟩⟨proof⟩

We define two records *acts-MapOps* and *trans-MapOps* satisfying the *MapOps* predicate (§6.8). Discharging the obligations in the *Algorithm* locale is routine, leaning on the work of Lammich and Lochbihler (2010).

### 7.1.8 Locale instantiation

Finally we assemble the algorithm and discharge the proof obligations.

**sublocale** *FiniteLinorderEnvironment*  
 < *Clock: Algorithm*  
 jkbp envInit envAction envTrans envVal  
 clock-jview envObs clock-jviewInit clock-jviewIncr  
 clock-sim clock-simRels clock-simVal  
 clock-simAbs clock-simObs clock-simInit clock-simTrans clock-simAction  
 acts-MapOps trans-MapOps⟨proof⟩

Explicitly, the algorithm for this case is:

**definition**

*mkClockAuto* ≡ λagents jkbp envInit envAction envTrans envVal envObs.  
*mkAlgAuto* acts-MapOps  
 trans-MapOps  
 (clock-simObs envObs)  
 (clock-simInit envInit envObs)  
 (clock-simTrans agents jkbp envAction envTrans envVal envObs)  
 (clock-simAction jkbp envVal envObs)  
 (λa. map (clock-simInit envInit envObs a ∘ envObs a) envInit)

**lemma** (in *FiniteLinorderEnvironment*) *mkClockAuto*-implements:

*Clock.implements*  
 (mkClockAuto agents jkbp envInit envAction envTrans envVal envObs)⟨proof⟩⟨proof⟩

We discuss the clock semantics further in §??.

## 7.2 The Synchronous Perfect-Recall View

The synchronous perfect-recall (SPR) view records all observations the agent has made on a given trace. This is the canonical full-information synchronous view; all others are functions of this one.

Intuitively it maintains a list of all observations made on the trace, with the length of the list recording the time:

**definition** (in *Environment*) *spr-jview* :: ('a, 's, 'obs Trace) *JointView* **where**  
*spr-jview* a = tMap (envObs a)⟨proof⟩⟨proof⟩⟨proof⟩⟨proof⟩

The corresponding incremental view appends a new observation to the existing ones:

**definition** (in *Environment*)  $spr\text{-}jviewInit :: 'a \Rightarrow 'obs \Rightarrow 'obs\ Trace$  **where**  
 $spr\text{-}jviewInit \equiv \lambda a\ obs.\ tInit\ obs$

**definition** (in *Environment*)  
 $spr\text{-}jviewIncr :: 'a \Rightarrow 'obs \Rightarrow 'obs\ Trace \Rightarrow 'obs\ Trace$   
**where**  
 $spr\text{-}jviewIncr \equiv \lambda a\ obs'\ tobs.\ tobs \rightsquigarrow obs'$

**sublocale** *Environment*  
 $< SPR: IncrEnvironment\ jkbp\ envInit\ envAction\ envTrans\ envVal$   
 $spr\text{-}jview\ envObs\ spr\text{-}jviewInit\ spr\text{-}jviewIncr\langle proof \rangle\langle proof \rangle\langle proof \rangle$

van der Meyden (1996, Theorem 5) showed that it is not the case that finite-state implementations always exist with respect to the SPR view, and so we consider three special cases:

- §7.3 where there is a single agent;
- §7.4 when the protocols of the agents are deterministic and communication is by broadcast; and
- §7.5 when the agents use non-deterministic protocols and again use broadcast to communicate.

Note that these cases do overlap but none is wholly contained in another.

### 7.3 Perfect Recall for a Single Agent

We capture our expectations of a single-agent scenario in the following locale:

**locale** *FiniteSingleAgentEnvironment* =  
 $FiniteEnvironment\ jkbp\ envInit\ envAction\ envTrans\ envVal\ envObs$   
**for**  $jkbp :: ('a, 'p, 'aAct)\ JKBP$   
**and**  $envInit :: ('s :: \{finite, linorder\})\ list$   
**and**  $envAction :: 's \Rightarrow 'eAct\ list$   
**and**  $envTrans :: 'eAct \Rightarrow ('a \Rightarrow 'aAct) \Rightarrow 's \Rightarrow 's$   
**and**  $envVal :: 's \Rightarrow 'p \Rightarrow bool$   
**and**  $envObs :: 'a \Rightarrow 's \Rightarrow 'obs$   
**+ fixes**  $agent :: 'a$   
**assumes**  $envSingleAgent: a = agent$

As per the clock semantics of §7.1, we assume that the set of states is finite and linearly ordered. We give the sole agent the name *agent*.

Our simulation is quite similar to the one for the clock semantics of §7.1: it records the set of worlds that the agent considers possible relative to a trace and the SPR view. The key difference is that it is path-sensitive:

**context** *FiniteSingleAgentEnvironment*  
**begin**

**definition**  $spr-abs :: 's Trace \Rightarrow 's set$  **where**  
 $spr-abs t \equiv$   
 $tLast \text{ ' } \{ t' \in SPR.jkbpC . spr-jview agent t' = spr-jview agent t \}$

**type-synonym** (in  $-$ )  $'s spr-simWorlds = 's set \times 's$

**definition**  $spr-sim :: 's Trace \Rightarrow 's spr-simWorlds$  **where**  
 $spr-sim \equiv \lambda t. (spr-abs t, tLast t)\langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle$

The Kripke structure for this simulation relates worlds for *agent* if the sets of states it considers possible coincide, and the observation of the final states of the trace is the same. Propositions are evaluated at the final state.

**definition**  $spr-simRels :: 'a \Rightarrow 's spr-simWorlds Relation$  **where**  
 $spr-simRels \equiv \lambda a. \{ ((U, u), (V, v)) \mid U u V v.$   
 $U = V \wedge \{u, v\} \subseteq U \wedge envObs a u = envObs a v \}$

**definition**  $spr-simVal :: 's spr-simWorlds \Rightarrow 'p \Rightarrow bool$  **where**  
 $spr-simVal \equiv envVal \circ snd$

**abbreviation**  $spr-simMC :: ('a, 'p, 's spr-simWorlds) KripkeStructure$  **where**  
 $spr-simMC \equiv mkKripke (spr-sim \text{ ' } SPR.jkbpC) spr-simRels spr-simVal \langle proof \rangle \langle proof \rangle$

Demonstrating that this is a simulation (§2.3) is straightforward.

**lemma**  $spr-sim: sim SPR.MC spr-simMC spr-sim \langle proof \rangle$

**end**

**sublocale**  $FiniteSingleAgentEnvironment$   
 $< SPRsingle: SimIncrEnvironment jkbp envInit envAction envTrans envVal$   
 $spr-jview envObs spr-jviewInit spr-jviewIncr$   
 $spr-sim spr-simRels spr-simVal \langle proof \rangle$

### 7.3.1 Representations

As in §7.1.1, we quotient  $'s spr-simWorlds$  by  $spr-simRels$ . Because there is only a single agent, an element of this quotient corresponding to a cononical trace  $t$  is isomorphic to the set of states that are possible given the sequence of observations made by *agent* on  $t$ . Therefore we have a very simple representation:

**context**  $FiniteSingleAgentEnvironment$   
**begin**

**type-synonym** (in  $-$ )  $'s spr-simWorldsRep = 's odlist$

It is very easy to map these representations back to simulated equivalence classes:

**definition**  
 $spr-simAbs :: 's spr-simWorldsRep \Rightarrow 's spr-simWorlds set$   
**where**

$spr\text{-}simAbs \equiv \lambda ss. \{ (toSet\ ss, s) \mid s. s \in toSet\ ss \}$

This time our representation is unconditionally canonical:

**lemma**  $spr\text{-}simAbs\text{-}inj$ :  $inj\ spr\text{-}simAbs\langle proof \rangle\langle proof \rangle\langle proof \rangle\langle proof \rangle$

We again make use of the following Kripke structure, where the worlds are the final states of the subset of the temporal slice that *agent* believes possible:

**definition**  $spr\text{-}repRels :: 'a \Rightarrow ('s \times 's)\ set$  **where**  
 $spr\text{-}repRels \equiv \lambda a. \{ (s, s'). envObs\ a\ s' = envObs\ a\ s \}$

**abbreviation**  $spr\text{-}repMC :: 's\ set \Rightarrow ('a, 'p, 's)\ KripkeStructure$  **where**  
 $spr\text{-}repMC \equiv \lambda X. mkKripke\ X\ spr\text{-}repRels\ envVal$

Similarly we show that this Kripke structure is adequate by introducing an intermediate structure and connecting them all with a tower of simulations:

**abbreviation**  $spr\text{-}jkbpCSt :: 's\ Trace \Rightarrow 's\ spr\text{-}simWorlds\ set$  **where**  
 $spr\text{-}jkbpCSt\ t \equiv SPRsingle.sim\equiv class\ agent\ t$

**abbreviation**  
 $spr\text{-}simMCt :: 's\ Trace \Rightarrow ('a, 'p, 's\ spr\text{-}simWorlds)\ KripkeStructure$   
**where**  
 $spr\text{-}simMCt\ t \equiv mkKripke\ (spr\text{-}jkbpCSt\ t)\ spr\text{-}simRels\ spr\text{-}simVal$

**definition**  $spr\text{-}repSim :: 's\ spr\text{-}simWorlds \Rightarrow 's$  **where**  
 $spr\text{-}repSim \equiv snd\langle proof \rangle\langle proof \rangle\langle proof \rangle\langle proof \rangle$

**lemma**  $spr\text{-}repSim$ :  
**assumes**  $tC$ :  $t \in SPR.jkbpC$   
**shows**  $sim\ (spr\text{-}simMCt\ t)$   
 $((spr\text{-}repMC \circ fst)\ (spr\text{-}sim\ t))$   
 $spr\text{-}repSim\langle proof \rangle$

As before, the following sections discharge the requirements of the *Algorithm* locale of Figure 3.

### 7.3.2 Initial states

The initial states of the automaton for *agent* is simply the partition of  $envInit$  under *agent*'s observation.

**definition** (**in**  $-$ )  
 $spr\text{-}simInit :: ('s :: linorder)\ list \Rightarrow ('a \Rightarrow 's \Rightarrow 'obs)$   
 $\Rightarrow 'a \Rightarrow 'obs \Rightarrow 's\ spr\text{-}simWorldsRep$

**where**  
 $spr\text{-}simInit\ envInit\ envObs \equiv \lambda a\ iobs.$   
 $ODList.fromList\ [ s. s \leftarrow envInit, envObs\ a\ s = iobs ]$

**lemma**  $spr\text{-}simInit$ :



**assumes**  $iobs \in envObs\ a\ 'set\ envInit$   
**shows**  $spr\simAbs\ (spr\simInit\ a\ iobs)$   
 $= spr\sim\ ' \{ t' \in SPR.jkbpC.\ spr\simview\ a\ t' = spr\simviewInit\ a\ iobs \} \langle proof \rangle$

### 7.3.3 Simulated observations

As the agent makes the same observation on the entire equivalence class, we arbitrarily choose the first element of the representation:

**definition** (in  $-$ )  
 $spr\simObs :: ('a \Rightarrow 's \Rightarrow 'obs)$   
 $\Rightarrow 'a \Rightarrow ('s :: linorder)\ spr\simWorldsRep \Rightarrow 'obs$

**where**

$spr\simObs\ envObs \equiv \lambda a.\ envObs\ a \circ ODLList.hd$

**lemma**  $spr\simObs$ :

**assumes**  $tC: t \in SPR.jkbpC$   
**assumes**  $ec: spr\simAbs\ ec = SPRsingle.sim\equiv\ class\ a\ t$   
**shows**  $spr\simObs\ a\ ec = envObs\ a\ (tLast\ t) \langle proof \rangle$

### 7.3.4 Evaluation

As the single-agent case is much simpler than the multi-agent ones, we define an evaluation function specialised to its representation.

Intuitively  $eval$  yields the subset of  $X$  where the formula holds, where  $X$  is taken to be a representation of a canonical equivalence class for  $agent$ .

**fun** (in  $-$ )  
 $eval :: (('s :: linorder) \Rightarrow 'p \Rightarrow bool)$   
 $\Rightarrow 's\ odlist \Rightarrow ('a,\ 'p)\ Kform \Rightarrow 's\ odlist$

**where**

$eval\ val\ X\ (Kprop\ p) = ODLList.filter\ (\lambda s.\ val\ s\ p)\ X$   
 $| eval\ val\ X\ (Knot\ \varphi) = ODLList.difference\ X\ (eval\ val\ X\ \varphi)$   
 $| eval\ val\ X\ (Kand\ \varphi\ \psi) = ODLList.intersect\ (eval\ val\ X\ \varphi)\ (eval\ val\ X\ \psi)$   
 $| eval\ val\ X\ (Kknows\ a\ \varphi) = (if\ eval\ val\ X\ \varphi = X\ then\ X\ else\ ODLList.empty)$   
 $| eval\ val\ X\ (Kknows\ as\ \varphi) =$   
 $(if\ as = [] \vee eval\ val\ X\ \varphi = X\ then\ X\ else\ ODLList.empty)$

In general this is less efficient than the tableau approach of Fagin et al. (1995, Proposition 3.2.1), which labels all states with all formulas. However it is often the case that the set of relevant worlds is much smaller than the set of all system states.

Showing that this corresponds with the standard models relation is routine.

$\langle proof \rangle \langle proof \rangle \langle proof \rangle$

**lemma**  $eval\ models$ :

**assumes**  $ec: spr\simAbs\ ec = SPRsingle.sim\equiv\ class\ agent\ t$   
**assumes**  $subj: subjective\ agent\ \varphi$   
**assumes**  $s: s \in toSet\ ec$   
**shows**  $toSet\ (eval\ envVal\ ec\ \varphi) \neq \{\} \iff spr\sim repMC\ (toSet\ ec),\ s \models \varphi \langle proof \rangle$

### 7.3.5 Simulated actions

The actions enabled on a canonical equivalence class are those for which *eval* yields a non-empty set of states:

**definition (in -)**

$$\begin{aligned} \text{spr-simAction} &:: ('a, 'p, 'aAct) \text{ KBP} \Rightarrow (('s :: \text{linorder}) \Rightarrow 'p \Rightarrow \text{bool}) \\ &\Rightarrow 'a \Rightarrow 's \text{ spr-simWorldsRep} \Rightarrow 'aAct \text{ list} \end{aligned}$$

**where**

$$\begin{aligned} \text{spr-simAction kbp envVal} &\equiv \lambda a X. \\ &[ \text{action gc. gc} \leftarrow \text{kbp}, \text{eval envVal } X \text{ (guard gc)} \neq \text{ODList.empty} ] \end{aligned}$$

The key lemma relates the agent's behaviour on an equivalence class to that on its representation:

**lemma spr-simAction-jAction:**

$$\begin{aligned} \text{assumes } tC: t &\in \text{SPR.jkbpC} \\ \text{assumes } ec: \text{spr-simAbs } ec &= \text{SPRsingle.sim-equiv-class agent } t \\ \text{shows set (spr-simAction agent } ec) & \\ &= \text{set (jAction (spr-repMC (toSet } ec)) (tLast } t) \text{ agent)} \langle \text{proof} \rangle \langle \text{proof} \rangle \end{aligned}$$

The *Algorithm* locale requires the following lemma, which is a straightforward chaining of the above simulations.

**lemma spr-simAction:**

$$\begin{aligned} \text{assumes } tC: t &\in \text{SPR.jkbpC} \\ \text{and } ec: \text{spr-simAbs } ec &= \text{SPRsingle.sim-equiv-class } a \text{ } t \\ \text{shows set (spr-simAction } a \text{ } ec) &= \text{set (jAction SPR.MC } t \text{ } a) \langle \text{proof} \rangle \end{aligned}$$

### 7.3.6 Simulated transitions

It is straightforward to determine the possible successor states of a given canonical equivalence class *X*:

**definition (in -)**

$$\begin{aligned} \text{spr-trans} &:: ('a, 'p, 'aAct) \text{ KBP} \\ &\Rightarrow ('s \Rightarrow 'eAct \text{ list}) \\ &\Rightarrow ('eAct \Rightarrow ('a \Rightarrow 'aAct) \Rightarrow 's \Rightarrow 's) \\ &\Rightarrow ('s \Rightarrow 'p \Rightarrow \text{bool}) \\ &\Rightarrow 'a \Rightarrow ('s :: \text{linorder}) \text{ spr-simWorldsRep} \Rightarrow 's \text{ list} \end{aligned}$$

**where**

$$\begin{aligned} \text{spr-trans kbp envAction envTrans val} &\equiv \lambda a X. \\ &[ \text{envTrans eact } (\lambda a'. \text{aact}) \text{ } s. \\ &\quad s \leftarrow \text{toList } X, \text{eact} \leftarrow \text{envAction } s, \text{aact} \leftarrow \text{spr-simAction kbp val } a \text{ } X ] \end{aligned}$$

Using this function we can determine the set of possible successor equivalence classes from *X*:

**abbreviation (in -)** *envObs-rel* :: ('s ⇒ 'obs) ⇒ 's × 's ⇒ bool **where**

$$\text{envObs-rel envObs} \equiv \lambda (s, s'). \text{envObs } s' = \text{envObs } s$$

**definition (in -)**

```

spr-simTrans :: ('a, 'p, 'aAct) KBP
  => (('s::linorder) => 'eAct list)
  => ('eAct => ('a => 'aAct) => 's => 's)
  => ('s => 'p => bool)
  => ('a => 's => 'obs)
  => 'a => 's spr-simWorldsRep => 's spr-simWorldsRep list

```

**where**

```

spr-simTrans kbp envAction envTrans val envObs ≡ λa X.
  map ODLList.fromList (partition (envObs-rel (envObs a))
    (spr-trans kbp envAction envTrans val a X))⟨proof⟩⟨proof⟩

```

The *partition* function splits a list into equivalence classes under the given equivalence relation.

The property asked for by the *Algorithm* locale follows from the properties of *partition* and *spr-trans*:

**lemma** *spr-simTrans*:

```

assumes tC: t ∈ SPR.jkbpC
assumes ec: spr-simAbs ec = SPRsingle.sim-equiv-class a t
shows spr-simAbs ' set (spr-simTrans a ec)
  = { SPRsingle.sim-equiv-class a (t' ∼ s)
    | t' s. t' ∼ s ∈ SPR.jkbpC ∧ spr-jview a t' = spr-jview a t }⟨proof⟩

```

**end**

### 7.3.7 Maps

As in §7.1.7, we use a pair of tries and an association list to handle the automata representation. Recall that the keys of these tries are lists of system states.

**type-synonym** ('s, 'obs) *spr-trans-trie* = ('s, ('obs, 's odlist) mapping) trie

**type-synonym** ('s, 'aAct) *spr-acts-trie* = ('s, ('s, 'aAct) trie) trie⟨proof⟩

### 7.3.8 Locale instantiation

The above is sufficient to instantiate the *Algorithm* locale.

**sublocale** *FiniteSingleAgentEnvironment*

```

< SPRsingle: Algorithm
  jkbp envInit envAction envTrans envVal
  spr-jview envObs spr-jviewInit spr-jviewIncr
  spr-sim spr-simRels spr-simVal
  spr-simAbs spr-simObs spr-simInit spr-simTrans spr-simAction
  trie-odlist-MapOps trans-MapOps⟨proof⟩⟨proof⟩⟨proof⟩

```

We use this theory to synthesise a solution to the robot of §1 in §8.1.

```

record (overloaded) ('a, 'es, 'ps) BEState =
  es :: 'es
  ps :: ('a × 'ps) odlist

locale FiniteDetBroadcastEnvironment =
  Environment jkbp envInit envAction envTrans envVal envObs
  for jkbp :: 'a ⇒ ('a :: {finite,linorder}, 'p, 'aAct) KBP
  and envInit
    :: ('a, 'es :: {finite,linorder}, 'as :: {finite,linorder}) BEState list
  and envAction :: ('a, 'es, 'as) BEState ⇒ 'eAct list
  and envTrans :: 'eAct ⇒ ('a ⇒ 'aAct)
    ⇒ ('a, 'es, 'as) BEState ⇒ ('a, 'es, 'as) BEState
  and envVal :: ('a, 'es, 'as) BEState ⇒ 'p ⇒ bool
  and envObs :: 'a ⇒ ('a, 'es, 'as) BEState ⇒ ('cobs × 'as option)

+ fixes agents :: 'a odlist
fixes envObsC :: 'es ⇒ 'cobs
defines envObs a s ≡ (envObsC (es s), ODLList.lookup (ps s) a)
assumes agents: ODLList.toSet agents = UNIV
assumes envTrans: ∀ s s' a eact eact' aact aact'.
  ODLList.lookup (ps s) a = ODLList.lookup (ps s') a ∧ aact a = aact' a
  → ODLList.lookup (ps (envTrans eact aact s)) a
  = ODLList.lookup (ps (envTrans eact' aact' s')) a
assumes jkbpDet: ∀ a. ∀ t ∈ SPR.jkbpC. length (jAction SPR.MC t a) ≤ 1

```

Figure 5: Finite broadcast environments with a deterministic JKBP.

## 7.4 Perfect Recall in Deterministic Broadcast Environments

It is well known that simultaneous broadcast has the effect of making information *common knowledge*; roughly put, the agents all learn the same things at the same time as the system evolves, so the relation amongst the agents' states of knowledge never becomes more complex than it is in the initial state (Fagin et al. 1995, Chapter 6). For this reason we might hope to find finite-state implementations of JKBP's in broadcast environments.

The broadcast assumption by itself is insufficient in general, however (van der Meyden 1996, §7), and so we need to further constrain the scenario. Here we require that for each canonical trace the JKBP prescribes at most one action. In practice this constraint is easier to verify than the circularity would suggest; we return to this point at the end of this section.

*<proof></proof>*

We encode our expectations of the scenario in the *FiniteBroadcastEnvironment* locale of Figure 5. The broadcast is modelled by having all agents make the same common observation of the shared state of type 'es. We also allow each agent to maintain a private state of type 'ps; that other agents cannot influence it or directly observe it is enforced by the constraint *envTrans* and the definition

of  $envObs$ .

We do however allow the environment's protocol to be non-deterministic and a function of the entire system state, including private states.

**context** *FiniteDetBroadcastEnvironment*  
**begin**(*proof*)

We seek a suitable simulation space by considering what determines an agent's knowledge. Intuitively any set of traces that is relevant to the agents' states of knowledge with respect to  $t \in jkbpC$  need include only those with the same common observation as  $t$ :

**definition**  $tObsC :: ('a, 'es, 'as) BEState Trace \Rightarrow 'cobs Trace$  **where**  
 $tObsC \equiv tMap (envObsC \circ es)$

Clearly this is an abstraction of the SPR jview of the given trace.

**lemma** *spr-jview-tObsC*:

**assumes**  $spr-jview\ a\ t = spr-jview\ a\ t'$

**shows**  $tObsC\ t = tObsC\ t' \langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle$

Unlike the single-agent case of §7.3, it is not sufficient for a simulation to record only the final states; we need to relate the initial private states of the agents with the final states they consider possible, as the initial states may contain information that is not common knowledge. This motivates the following abstraction:

**definition**

$tObsC-abs :: ('a, 'es, 'as) BEState Trace \Rightarrow ('a, 'es, 'as) BEState Relation$

**where**

$tObsC-abs\ t \equiv \{ (tFirst\ t', tLast\ t')$

$| t'. t' \in SPR.jkbpC \wedge tObsC\ t' = tObsC\ t \} \langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle$

**end**

We use the following record to represent the worlds of the simulated Kripke structure:

**record (overloaded)**  $('a, 'es, 'as) spr-simWorld =$

$sprFst :: ('a, 'es, 'as) BEState$

$sprLst :: ('a, 'es, 'as) BEState$

$sprCRel :: ('a, 'es, 'as) BEState Relation$

$\langle proof \rangle$

**context** *FiniteDetBroadcastEnvironment*

**begin**

The simulation of a trace  $t \in jkbpC$  records its initial and final states, and the relation between initial and final states of all commonly-plausible traces:

**definition**

$spr-sim :: ('a, 'es, 'as) BEState Trace \Rightarrow ('a, 'es, 'as) spr-simWorld$

**where**

$spr\text{-}sim \equiv \lambda t. (\mid sprFst = tFirst\ t, sprLst = tLast\ t, sprCRel = tObsC\text{-}abs\ t \mid)$

The associated Kripke structure relates two worlds for an agent if the agent's observation on the the first and last states corresponds, and the worlds have the same common observation relation. As always, we evaluate propositions on the final state of the trace.

**definition**

$spr\text{-}simRels :: 'a \Rightarrow ('a, 'es, 'as)\ spr\text{-}simWorld\ Relation$

**where**

$spr\text{-}simRels \equiv \lambda a. \{ (s, s') \mid s\ s'. \\ \quad envObs\ a\ (sprFst\ s) = envObs\ a\ (sprFst\ s') \\ \wedge\ envObs\ a\ (sprLst\ s) = envObs\ a\ (sprLst\ s') \\ \wedge\ sprCRel\ s = sprCRel\ s' \}$

**definition**  $spr\text{-}simVal :: ('a, 'es, 'as)\ spr\text{-}simWorld \Rightarrow 'p \Rightarrow bool$  **where**

$spr\text{-}simVal \equiv envVal \circ sprLst$

**abbreviation**

$spr\text{-}simMC \equiv mkKripke\ (spr\text{-}sim\ 'SPR.jkbpC)\ spr\text{-}simRels\ spr\text{-}simVal\langle proof \rangle\langle proof \rangle\langle proof \rangle\langle proof \rangle\langle proof \rangle\langle proof \rangle\langle proof \rangle\langle proof \rangle$

All the properties of a simulation are easy to show for  $spr\text{-}sim$  except for reverse simulation.

The critical lemma states that if we have two traces that yield the same common observations, and an agent makes the same observation on their initial states, then that agent's private states at each point on the two traces are identical.

**lemma**  $spr\text{-}jview\text{-}det\text{-}ps$ :

**assumes**  $tt'C: \{t, t'\} \subseteq SPR.jkbpC$   
**assumes**  $obsCtt': tObsC\ t = tObsC\ t'$   
**assumes**  $first: envObs\ a\ (tFirst\ t) = envObs\ a\ (tFirst\ t')$   
**shows**  $tMap\ (\lambda s. ODLList.lookup\ (ps\ s)\ a)\ t \\ = tMap\ (\lambda s. ODLList.lookup\ (ps\ s)\ a)\ t'\langle proof \rangle\langle proof \rangle$

The proof proceeds by lock-step induction over  $t$  and  $t'$ , appealing to the  $jkbpDet$  assumption, the definition of  $envObs$  and the constraint  $envTrans$ .

It is then a short step to showing reverse simulation, and hence simulation:

**lemma**  $spr\text{-}sim: sim\ SPR.MC\ spr\text{-}simMC\ spr\text{-}sim\langle proof \rangle$

**end**

**sublocale**  $FiniteDetBroadcastEnvironment$

$< SPRdet: SimIncrEnvironment\ jkbp\ envInit\ envAction\ envTrans\ envVal \\ spr\text{-}jview\ envObs\ spr\text{-}jviewInit\ spr\text{-}jviewIncr \\ spr\text{-}sim\ spr\text{-}simRels\ spr\text{-}simVal\langle proof \rangle$

### 7.4.1 Representations

As before we canonically represent the quotient of the simulated worlds  $('a, 'es, 'as)$   $spr\text{-}simWorld$  under  $spr\text{-}simRels$  using ordered, distinct lists. In particu-

lar, we use the type  $('a \times 'a)$  *odlist* (abbreviated *'a odrelation*) to canonically represent relations.

**context** *FiniteDetBroadcastEnvironment*  
**begin**

**type-synonym** (in  $-$ )  $('a, 'es, 'as)$  *spr-simWorldsECRep*  
 $= ('a, 'es, 'as)$  *BESState odrelation*  
**type-synonym** (in  $-$ )  $('a, 'es, 'as)$  *spr-simWorldsRep*  
 $= ('a, 'es, 'as)$  *spr-simWorldsECRep*  $\times$   $('a, 'es, 'as)$  *spr-simWorldsECRep*

We can abstract such a representation into a set of simulated equivalence classes:

**definition**

$spr-simAbs :: ('a, 'es, 'as)$  *spr-simWorldsRep*  
 $\Rightarrow ('a, 'es, 'as)$  *spr-simWorld set*

**where**

$spr-simAbs \equiv \lambda(cec, aec). \{ \langle \mid sprFst = s, sprLst = s', sprCRel = toSet\ cec \mid \rangle$   
 $\mid s\ s'. (s, s') \in toSet\ aec \}$

Assuming  $X$  represents a simulated equivalence class for  $t \in jkbpC$ , we can decompose  $spr-simAbs\ X$  in terms of  $tObsC-abs\ t$  and  $agent-abs\ t$ :

**definition**

$agent-abs :: 'a \Rightarrow ('a, 'es, 'as)$  *BESState Trace*  
 $\Rightarrow ('a, 'es, 'as)$  *BESState Relation*

**where**

$agent-abs\ a\ t \equiv \{ (tFirst\ t', tLast\ t')$   
 $\mid t'. t' \in SPR.jkbpC \wedge spr-jview\ a\ t' = spr-jview\ a\ t \}$ *<proof><proof><proof><proof>*

This representation is canonical on the domain of interest (though not in general):

**lemma** *spr-simAbs-inj-on*:

$inj-on\ spr-simAbs\ \{ x . spr-simAbs\ x \in SPRdet.jkbpSEC \}$ *<proof>*

The following sections make use of a Kripke structure constructed over  $tObsC-abs\ t$  for some canonical trace  $t$ . Note that we use the relation in the generated code.

**type-synonym** (in  $-$ )  $('a, 'es, 'as)$  *spr-simWorlds*  
 $= ('a, 'es, 'as)$  *BESState*  $\times$   $('a, 'es, 'as)$  *BESState*

**definition** (in  $-$ )

$spr-repRels :: ('a \Rightarrow ('a, 'es, 'as)$  *BESState*  $\Rightarrow 'cobs \times 'as\ option)$   
 $\Rightarrow 'a \Rightarrow ('a, 'es, 'as)$  *spr-simWorlds Relation*

**where**

$spr-repRels\ envObs \equiv \lambda a. \{ ((u, v), (u', v')).$   
 $envObs\ a\ u = envObs\ a\ u' \wedge envObs\ a\ v = envObs\ a\ v' \}$

**definition**

$spr-repVal :: ('a, 'es, 'as)$  *spr-simWorlds*  $\Rightarrow 'p \Rightarrow bool$

**where**

$$\text{spr-repVal} \equiv \text{envVal} \circ \text{snd}$$

**abbreviation**

$$\begin{aligned} \text{spr-repMC} &:: ('a, 'es, 'as) \text{ BState Relation} \\ &\Rightarrow ('a, 'p, ('a, 'es, 'as) \text{ spr-simWorlds}) \text{ KripkeStructure} \end{aligned}$$

**where**

$$\text{spr-repMC} \equiv \lambda \text{tcobsR}. \text{mkKripke tcobsR (spr-repRels envObs) spr-repVal} \langle \text{proof} \rangle \langle \text{proof} \rangle$$

As before we can show that this Kripke structure is adequate for a particular canonical trace  $t$  by showing that it simulates  $\text{spr-repMC}$ . We introduce an intermediate structure:

**abbreviation**

$$\text{spr-jkbpCSt} :: ('a, 'es, 'as) \text{ BState Trace} \Rightarrow ('a, 'es, 'as) \text{ spr-simWorld set}$$

**where**

$$\text{spr-jkbpCSt } t \equiv \text{spr-sim } \{ t' . t' \in \text{SPR.jkbpC} \wedge t \text{ObsC } t = t \text{ObsC } t' \}$$

**abbreviation**

$$\begin{aligned} \text{spr-simMCt} &:: ('a, 'es, 'as) \text{ BState Trace} \\ &\Rightarrow ('a, 'p, ('a, 'es, 'as) \text{ spr-simWorld}) \text{ KripkeStructure} \end{aligned}$$

**where**

$$\text{spr-simMCt } t \equiv \text{mkKripke (spr-jkbpCSt } t) \text{ spr-simRels spr-simVal}$$

**definition**

$$\text{spr-repSim} :: ('a, 'es, 'as) \text{ spr-simWorld} \Rightarrow ('a, 'es, 'as) \text{ spr-simWorlds}$$

**where**

$$\text{spr-repSim} \equiv \lambda s. (\text{sprFst } s, \text{sprLst } s) \langle \text{proof} \rangle \langle \text{proof} \rangle$$

**lemma**  $\text{spr-repSim}$ :

$$\begin{aligned} \text{assumes } tC: &t \in \text{SPR.jkbpC} \\ \text{shows } \text{sim } &(\text{spr-simMCt } t) \\ &((\text{spr-repMC} \circ \text{sprCRel}) (\text{spr-sim } t)) \\ &\text{spr-repSim} \langle \text{proof} \rangle \end{aligned}$$

As before we define a set of constants that satisfy the *Algorithm* locale given the assumptions of the *FiniteDetBroadcastEnvironment* locale.

## 7.4.2 Initial states

The initial states for agent  $a$  given an initial observation  $iobs$  consist of the set of states that yield a common observation consonant with  $iobs$  paired with the set of states where  $a$  observes  $iobs$ :

**definition** (in  $-$ )

$$\begin{aligned} \text{spr-simInit} &:: \\ &('a, 'es, 'as) \text{ BState list} \Rightarrow ('es \Rightarrow 'cobs) \\ &\Rightarrow ('a \Rightarrow ('a, 'es, 'as) \text{ BState} \Rightarrow 'cobs \times 'obs) \\ &\Rightarrow 'a \Rightarrow ('cobs \times 'obs) \\ &\Rightarrow ('a :: \text{linorder}, 'es :: \text{linorder}, 'as :: \text{linorder}) \text{ spr-simWorldsRep} \end{aligned}$$

**where**



$spr\text{-}simInit\ envObsC\ envObs \equiv \lambda a\ iobs.$   
 $(ODList.fromList [(s, s). s \leftarrow envInit, envObsC (es\ s) = fst\ iobs ],$   
 $ODList.fromList [(s, s). s \leftarrow envInit, envObs\ a\ s = iobs ])$

**lemma** *spr-simInit*:

**assumes**  $iobs \in envObs\ a\ 'set\ envInit$

**shows**  $spr\text{-}simAbs\ (spr\text{-}simInit\ a\ iobs)$

$= spr\text{-}sim\ ' \{ t' \in SPR.jkbpC. spr\text{-}jview\ a\ t' = spr\text{-}jviewInit\ a\ iobs \}$ *(proof)*

### 7.4.3 Simulated observations

An observation can be made at any element of the representation of a simulated equivalence class of a canonical trace:

**definition** (in  $-$ )

$spr\text{-}simObs ::$

$('es \Rightarrow 'cobs)$

$\Rightarrow 'a \Rightarrow ('a :: linorder, 'es :: linorder, 'as :: linorder)\ spr\text{-}simWorldsRep$

$\Rightarrow 'cobs \times 'as\ option$

**where**

$spr\text{-}simObs\ envObsC \equiv \lambda a. (\lambda s. (envObsC (es\ s), ODList.lookup (ps\ s)\ a))$   
 $\circ\ snd \circ ODList.hd \circ snd$

**lemma** *spr-simObs*:

**assumes**  $tC: t \in SPR.jkbpC$

**assumes**  $ec: spr\text{-}simAbs\ ec = SPRdet.sim\equiv\ class\ a\ t$

**shows**  $spr\text{-}simObs\ a\ ec = envObs\ a\ (tLast\ t)$ *(proof)*

### 7.4.4 Evaluation

As for the clock semantics (§7.1.4), we use the general evaluation function *evalS*.

Once again we propositions are used to filter the set of possible worlds  $X$ :

**abbreviation** (in  $-$ )

$spr\text{-}evalProp ::$

$(('a::linorder, 'es::linorder, 'as::linorder)\ BEState \Rightarrow 'p \Rightarrow bool)$

$\Rightarrow ('a, 'es, 'as)\ BEState\ odrelation$

$\Rightarrow 'p \Rightarrow ('a, 'es, 'as)\ BEState\ odrelation$

**where**

$spr\text{-}evalProp\ envVal \equiv \lambda X\ p. ODList.filter (\lambda s. envVal (snd\ s)\ p)\ X$

The knowledge operation computes the subset of possible worlds *cec* that yield the same observation as  $s$  for agent  $a$ :

**definition** (in  $-$ )

$spr\text{-}knowledge ::$

$('a \Rightarrow ('a::linorder, 'es::linorder, 'as::linorder)\ BEState$

$\Rightarrow 'cobs \times 'as\ option)$

$\Rightarrow ('a, 'es, 'as)\ BEState\ odrelation$

$\Rightarrow 'a \Rightarrow ('a, 'es, 'as)\ spr\text{-}simWorlds$

$\Rightarrow ('a, 'es, 'as)\ spr\text{-}simWorldsECCRep$

**where**

$spr\text{-}knowledge\ envObs\ cec \equiv \lambda a\ s.$   
 $ODList.fromList\ [s' . s' \leftarrow toList\ cec, (s, s') \in spr\text{-}repRels\ envObs\ a]\langle proof \rangle$

Similarly the common knowledge operation computes the transitive closure (Sternagel and Thiemann 2011) of the union of the knowledge relations for the agents  $as$ :

**definition (in  $-$ )**

$spr\text{-}commonKnowledge ::$   
 $( 'a \Rightarrow ( 'a::linorder, 'es::linorder, 'as::linorder )\ BEState$   
 $\Rightarrow 'cobs \times 'as\ option)$   
 $\Rightarrow ( 'a, 'es, 'as )\ BEState\ odrelation$   
 $\Rightarrow 'a\ list$   
 $\Rightarrow ( 'a, 'es, 'as )\ spr\text{-}sim\ Worlds$   
 $\Rightarrow ( 'a, 'es, 'as )\ spr\text{-}sim\ Worlds\ ECRrep$

**where**

$spr\text{-}commonKnowledge\ envObs\ cec \equiv \lambda as\ s.$   
 $let\ r = \lambda a. ODLList.fromList$   
 $\quad [ (s', s'') . s' \leftarrow toList\ cec, s'' \leftarrow toList\ cec,$   
 $\quad \quad (s', s'') \in spr\text{-}repRels\ envObs\ a ];$   
 $R = toList\ (ODList.big\ union\ r\ as)$   
 $in\ ODLList.fromList\ (memo\ list\ trancl\ R\ s)\langle proof \rangle$

The evaluation function evaluates a subjective knowledge formula on the representation of an equivalence class:

**definition (in  $-$ )**

$eval\ envVal\ envObs \equiv \lambda (cec, X).$   
 $evalS\ (spr\text{-}evalProp\ envVal)$   
 $\quad (spr\text{-}knowledge\ envObs\ cec)$   
 $\quad (spr\text{-}commonKnowledge\ envObs\ cec)$   
 $X\langle proof \rangle\langle proof \rangle\langle proof \rangle\langle proof \rangle\langle proof \rangle\langle proof \rangle\langle proof \rangle\langle proof \rangle\langle proof \rangle\langle proof \rangle\langle proof \rangle$

This function corresponds with the standard semantics:

**lemma  $eval\text{-}models$ :**

**assumes**  $tC$ :  $t \in SPR.jkbpC$   
**assumes**  $ec$ :  $spr\text{-}simAbs\ ec = SPRdet.sim\ equiv\ class\ a\ t$   
**assumes**  $subj\text{-}phi$ : subjective  $a\ \varphi$   
**assumes**  $s$ :  $s \in toSet\ (snd\ ec)$   
**shows**  $eval\ envVal\ envObs\ ec\ \varphi \iff spr\text{-}repMC\ (toSet\ (fst\ ec)),\ s \models \varphi\langle proof \rangle$

### 7.4.5 Simulated actions

From a common equivalence class and a subjective equivalence class for agent  $a$ , we can compute the actions enabled for  $a$ :

**definition (in  $-$ )**

$spr\text{-}simAction ::$   
 $( 'a, 'p, 'aAct )\ JKBP \Rightarrow ( ( 'a, 'es, 'as )\ BEState \Rightarrow 'p \Rightarrow bool)$   
 $\Rightarrow ( 'a \Rightarrow ( 'a, 'es, 'as )\ BEState \Rightarrow 'cobs \times 'as\ option)$

$$\begin{aligned}
&\Rightarrow 'a \\
&\Rightarrow ('a::\text{linorder}, 'es::\text{linorder}, 'as::\text{linorder}) \text{ spr-simWorldsRep} \\
&\Rightarrow 'a\text{Act list}
\end{aligned}$$

**where**

$$\begin{aligned}
&\text{spr-simAction jkbp envVal envObs} \equiv \lambda a \text{ ec.} \\
&[ \text{action gc. gc} \leftarrow \text{jkbp a, eval envVal envObs ec (guard gc)} ]
\end{aligned}$$

Using the above result about evaluation, we can relate *spr-simAction* to *jAction*. Firstly, *spr-simAction* behaves the same as *jAction* using the *spr-repMC* structure:

**lemma spr-action-jaction:**

$$\begin{aligned}
&\text{assumes } tC: t \in \text{SPR.jkbpC} \\
&\text{assumes } ec: \text{spr-simAbs ec} = \text{SPRdet.sim-equiv-class a t} \\
&\text{shows } \text{set (spr-simAction a ec)} \\
&\quad = \text{set (jAction (spr-repMC (toSet (fst ec))) (tFirst t, tLast t) a) (proof) (proof)}
\end{aligned}$$

We can connect the agent's choice of actions on the *spr-repMC* structure to those on the *SPR.MC* structure using our earlier results about actions being preserved by generated models and simulations.

**lemma spr-simAction:**

$$\begin{aligned}
&\text{assumes } tC: t \in \text{SPR.jkbpC} \\
&\text{assumes } ec: \text{spr-simAbs ec} = \text{SPRdet.sim-equiv-class a t} \\
&\text{shows } \text{set (spr-simAction a ec)} = \text{set (jAction SPR.MC t a) (proof)}
\end{aligned}$$

#### 7.4.6 Simulated transitions

The story of simulated transitions takes some doing. We begin by computing the successor relation of a given equivalence class  $X$  with respect to the common equivalence class *cec*:

**abbreviation (in -)**

$$\begin{aligned}
&\text{spr-jAction jkbp envVal envObs cec s} \equiv \lambda a. \\
&\quad \text{spr-simAction jkbp envVal envObs a (cec, spr-knowledge envObs cec a s)}
\end{aligned}$$

**definition (in -)**

$$\begin{aligned}
&\text{spr-trans} :: 'a \text{ odlist} \\
&\quad \Rightarrow ('a, 'p, 'aAct) \text{ JKBP} \\
&\quad \Rightarrow (('a::\text{linorder}, 'es::\text{linorder}, 'as::\text{linorder}) \text{ BEState} \Rightarrow 'eAct \text{ list}) \\
&\quad \Rightarrow ('eAct \Rightarrow ('a \Rightarrow 'aAct)) \\
&\quad \quad \Rightarrow ('a, 'es, 'as) \text{ BEState} \Rightarrow ('a, 'es, 'as) \text{ BEState} \\
&\quad \Rightarrow (('a, 'es, 'as) \text{ BEState} \Rightarrow 'p \Rightarrow \text{bool}) \\
&\quad \Rightarrow ('a \Rightarrow ('a, 'es, 'as) \text{ BEState} \Rightarrow 'cobs \times 'as \text{ option}) \\
&\quad \quad \Rightarrow ('a, 'es, 'as) \text{ spr-simWorldsECCRep} \\
&\quad \quad \Rightarrow ('a, 'es, 'as) \text{ spr-simWorldsECCRep} \\
&\quad \quad \Rightarrow (('a, 'es, 'as) \text{ BEState} \times ('a, 'es, 'as) \text{ BEState}) \text{ list}
\end{aligned}$$

**where**

$$\begin{aligned}
&\text{spr-trans agents jkbp envAction envTrans envVal envObs} \equiv \lambda \text{cec X.} \\
&[ (\text{initialS, succS}) . \\
&\quad (\text{initialS, finalS}) \leftarrow \text{toList X,}
\end{aligned}$$

$$\begin{aligned}
eact &\leftarrow envAction\ finalS, \\
succS &\leftarrow [ envTrans\ eact\ aact\ finalS . \\
&\quad aact \leftarrow listToFuns\ (spr-jAction\ jkbp\ envVal\ envObs\ cec \\
&\quad\quad\quad (initialS, finalS)) \\
&\quad\quad\quad (toList\ agents) ] ]
\end{aligned}$$

We will split the result of this function according to the common observation and also agent  $a$ 's observation, where  $a$  is the agent we are constructing the automaton for.

**definition (in -)**

$$\begin{aligned}
spr-simObsC &:: ('es \Rightarrow 'cobs) \\
&\Rightarrow (('a::linorder, 'es::linorder, 'as::linorder)\ BEState \\
&\quad \times ('a, 'es, 'as)\ BEState)\ odlist \\
&\Rightarrow 'cobs
\end{aligned}$$

**where**

$$spr-simObsC\ envObsC \equiv envObsC \circ es \circ snd \circ ODLList.hd$$

**abbreviation (in -)**

$$\begin{aligned}
envObs-rel &:: (('a, 'es, 'as)\ BEState \Rightarrow 'cobs \times 'as\ option) \\
&\Rightarrow ('a, 'es, 'as)\ spr-simWorlds \times ('a, 'es, 'as)\ spr-simWorlds \Rightarrow bool
\end{aligned}$$

**where**

$$envObs-rel\ envObs \equiv \lambda(s, s').\ envObs\ (snd\ s') = envObs\ (snd\ s)$$

The above combine to yield the successor equivalence classes like so:

**definition (in -)**

$$\begin{aligned}
spr-simTrans &:: 'a\ odlist \\
&\Rightarrow ('a, 'p, 'aAct)\ JKBP \\
&\Rightarrow (('a::linorder, 'es::linorder, 'as::linorder)\ BEState \Rightarrow 'eAct\ list) \\
&\Rightarrow ('eAct \Rightarrow ('a \Rightarrow 'aAct)) \\
&\quad \Rightarrow ('a, 'es, 'as)\ BEState \Rightarrow ('a, 'es, 'as)\ BEState) \\
&\Rightarrow (('a, 'es, 'as)\ BEState \Rightarrow 'p \Rightarrow bool) \\
&\Rightarrow ('es \Rightarrow 'cobs) \\
&\Rightarrow ('a \Rightarrow ('a, 'es, 'as)\ BEState \Rightarrow 'cobs \times 'as\ option) \\
&\Rightarrow 'a \\
&\Rightarrow ('a, 'es, 'as)\ spr-simWorldsRep \\
&\Rightarrow ('a, 'es, 'as)\ spr-simWorldsRep\ list
\end{aligned}$$

**where**

$$spr-simTrans\ agents\ jkbp\ envAction\ envTrans\ envVal\ envObsC\ envObs \equiv \lambda a\ ec.$$

$$\begin{aligned}
let\ aSuccs &= spr-trans\ agents\ jkbp\ envAction\ envTrans\ envVal\ envObs \\
&\quad\quad\quad (fst\ ec)\ (snd\ ec);
\end{aligned}$$

$$cec' = ODLList.fromList$$

$$\begin{aligned}
&\quad (spr-trans\ agents\ jkbp\ envAction\ envTrans\ envVal\ envObs \\
&\quad\quad\quad (fst\ ec)\ (fst\ ec))
\end{aligned}$$

$$\begin{aligned}
in\ [ &(ODList.filter\ (\lambda s.\ envObsC\ (es\ (snd\ s)) = spr-simObsC\ envObsC\ aec')\ cec', \\
&\quad aec') .
\end{aligned}$$

$$aec' \leftarrow map\ ODLList.fromList\ (partition\ (envObs-rel\ (envObs\ a))\ aSuccs)$$

[<proof><proof><proof><proof>

Showing that  $spr-simTrans$  works requires a series of auxiliary lemmas that show we do in fact compute the correct successor equivalence classes. We elide

the unedifying details, skipping straight to the lemma that the *Algorithm* locale expects:

**lemma** *spr-simTrans*:  
**assumes** *tC*:  $t \in \text{SPR.jkbpC}$   
**assumes** *ec*:  $\text{spr-simAbs } ec = \text{SPRdet.sim-equiv-class } a \ t$   
**shows**  $\text{spr-simAbs } ' \text{ set } (\text{spr-simTrans } a \ ec)$   
 $= \{ \text{SPRdet.sim-equiv-class } a \ (t' \rightsquigarrow s)$   
 $\mid t' \ s. \ t' \rightsquigarrow s \in \text{SPR.jkbpC} \wedge \text{spr-jview } a \ t' = \text{spr-jview } a \ t \}$ *<proof>*

The explicit-state approach sketched above is quite inefficient, and also some distance from the symbolic techniques we use in §???. However it does suffice to demonstrate the theory on the muddy children example in §8.2.

**end**

#### 7.4.7 Maps

As always we use a pair of tries. The domain of these maps is the pair of relations.

**type-synonym**  $( 'a, 'es, 'obs, 'as) \text{ trans-trie}$   
 $= (( 'a, 'es, 'as) \text{ BState},$   
 $(( 'a, 'es, 'as) \text{ BState},$   
 $(( 'a, 'es, 'as) \text{ BState},$   
 $(( 'a, 'es, 'as) \text{ BState},$   
 $( 'obs, ( 'a, 'es, 'as) \text{ spr-simWorldsRep}) \text{ mapping}) \text{ trie}) \text{ trie}) \text{ trie}) \text{ trie}$

**type-synonym**  
 $( 'a, 'es, 'aAct, 'as) \text{ acts-trie}$   
 $= (( 'a, 'es, 'as) \text{ BState},$   
 $(( 'a, 'es, 'as) \text{ BState},$   
 $(( 'a, 'es, 'as) \text{ BState},$   
 $(( 'a, 'es, 'as) \text{ BState}, 'aAct) \text{ trie}) \text{ trie}) \text{ trie}) \text{ trie} \langle \text{proof} \rangle \langle \text{proof} \rangle$

This suffices to placate the *Algorithm* locale.

**sublocale** *FiniteDetBroadcastEnvironment*  
 $< \text{SPRdet: Algorithm}$   
 $\text{jkbp envInit envAction envTrans envVal}$   
 $\text{spr-jview envObs spr-jviewInit spr-jviewIncr}$   
 $\text{spr-sim spr-simRels spr-simVal}$   
 $\text{spr-simAbs spr-simObs spr-simInit spr-simTrans spr-simAction}$   
 $\text{acts-MapOps trans-MapOps} \langle \text{proof} \rangle \langle \text{proof} \rangle \langle \text{proof} \rangle$

As we remarked earlier in this section, in general it may be difficult to establish the determinacy of a KBP as it is a function of the environment. However in many cases determinism is syntactically manifest as the guards are logically disjoint, independently of the knowledge subformulas. The following lemma generates the required proof obligations for this case:

**lemma** *(in PreEnvironmentJView) jkbpDetI*:

```

assumes  $tC$ :  $t \in jkbpC$ 
assumes  $jkbpSynDet$ :  $\forall a. distinct (map\ guard (jkbp\ a))$ 
assumes  $jkbpSemDet$ :  $\forall a\ gc\ gc'.$ 
     $gc \in set (jkbp\ a) \wedge gc' \in set (jkbp\ a) \wedge t \in jkbpC$ 
     $\longrightarrow guard\ gc = guard\ gc' \vee \neg(MC, t \models guard\ gc \wedge MC, t \models guard\ gc')$ 
shows  $length (jAction\ MC\ t\ a) \leq 1$ (proof)

```

The scenario presented here is a variant of the broadcast environments treated by van der Meyden (1996), which we cover in the next section.

## 7.5 Perfect Recall in Non-deterministic Broadcast Environments

```

record ( $'a, 'ePubAct, 'es, 'pPubAct, 'ps$ )  $BESState =$ 
   $es :: 'es$ 
   $ps :: 'a \Rightarrow 'ps$ 
   $pubActs :: 'ePubAct \times ('a \Rightarrow 'pPubAct)$ 

locale  $FiniteBroadcastEnvironment =$ 
   $Environment\ jkbp\ envInit\ envAction\ envTrans\ envVal\ envObs$ 
  for  $jkbp :: ('a :: finite, 'p, ('pPubAct :: finite \times 'ps :: finite))\ JKBP$ 
  and  $envInit$ 
     $:: ('a, 'ePubAct :: finite, 'es :: finite, 'pPubAct, 'ps)\ BESState\ list$ 
  and  $envAction :: ('a, 'ePubAct, 'es, 'pPubAct, 'ps)\ BESState$ 
     $\Rightarrow ('ePubAct \times 'ePrivAct)\ list$ 
  and  $envTrans :: ('ePubAct \times 'ePrivAct)$ 
     $\Rightarrow ('a \Rightarrow ('pPubAct \times 'ps))$ 
     $\Rightarrow ('a, 'ePubAct, 'es, 'pPubAct, 'ps)\ BESState$ 
     $\Rightarrow ('a, 'ePubAct, 'es, 'pPubAct, 'ps)\ BESState$ 
  and  $envVal :: ('a, 'ePubAct, 'es, 'pPubAct, 'ps)\ BESState \Rightarrow 'p \Rightarrow bool$ 
  and  $envObs :: 'a \Rightarrow ('a, 'ePubAct, 'es, 'pPubAct, 'ps)\ BESState$ 
     $\Rightarrow ('cobs \times 'ps \times ('ePubAct \times ('a \Rightarrow 'pPubAct)))$ 

+ fixes  $envObsC :: 'es \Rightarrow 'cobs$ 
  and  $envActionES :: 'es \Rightarrow ('ePubAct \times ('a \Rightarrow 'pPubAct))$ 
     $\Rightarrow ('ePubAct \times 'ePrivAct)\ list$ 
  and  $envTransES :: ('ePubAct \times 'ePrivAct) \Rightarrow ('a \Rightarrow 'pPubAct)$ 
     $\Rightarrow 'es \Rightarrow 'es$ 
defines  $envObs-def: envObs\ a \equiv (\lambda s. (envObsC\ (es\ s), ps\ s\ a, pubActs\ s))$ 
  and  $envAction-def: envAction\ s \equiv envActionES\ (es\ s)\ (pubActs\ s)$ 
  and  $envTrans-def:$ 
     $envTrans\ eact\ aact\ s \equiv []\ es = envTransES\ eact\ (fst \circ aact)\ (es\ s)$ 
     $,\ ps = snd \circ aact$ 
     $,\ pubActs = (fst\ eact, fst \circ aact)\ []$ 

```

Figure 6: Finite broadcast environments with non-deterministic KBPs.

(*proof*)

For completeness we reproduce the results of [van der Meyden \(1996\)](#) regarding non-deterministic KBPs in broadcast environments.

The determinism requirement is replaced by the constraint that actions be split into public and private components, where the private part influences the agents' private states, and the public part is broadcast and recorded in the system state. Moreover the protocol of the environment is only a function of the environment state, and not the agents' private states. Once again an agent's view consists of the common observation and their private state. The situation is described by the locale in [Figure 6](#). Note that as we do not intend to generate code for this case, we adopt more transparent but less effective representations.

Our goal in the following is to instantiate the *SimIncrEnvironment* locale with respect to the assumptions made in the *FiniteBroadcastEnvironment* locale. We begin by defining similar simulation machinery to the previous section.

**context** *FiniteBroadcastEnvironment*  
**begin**

As for the deterministic variant, we abstract traces using the common observation. Note that this now includes the public part of the agents' actions.

**definition**

$$\begin{aligned} tObsC &:: ('a, 'ePubAct, 'es, 'pPubAct, 'ps) BState Trace \\ &\Rightarrow ('cobs \times 'ePubAct \times ('a \Rightarrow 'pPubAct)) Trace \end{aligned}$$

**where**

$$tObsC \equiv tMap (\lambda s. (envObsC (es\ s), pubActs\ s)) \langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle$$

Similarly we introduce common and agent-specific abstraction functions:

**definition**

$$\begin{aligned} tObsC-abs &:: ('a, 'ePubAct, 'es, 'pPubAct, 'ps) BState Trace \\ &\Rightarrow ('a, 'ePubAct, 'es, 'pPubAct, 'ps) BState Relation \end{aligned}$$

**where**

$$\begin{aligned} tObsC-abs\ t &\equiv \{ (tFirst\ t', tLast\ t') \\ &\quad | t'. t' \in SPR.jkbpC \wedge tObsC\ t' = tObsC\ t \} \end{aligned}$$

**definition**

$$\begin{aligned} agent-abs &:: 'a \Rightarrow ('a, 'ePubAct, 'es, 'pPubAct, 'ps) BState Trace \\ &\Rightarrow ('a, 'ePubAct, 'es, 'pPubAct, 'ps) BState Relation \end{aligned}$$

**where**

$$\begin{aligned} agent-abs\ a\ t &\equiv \{ (tFirst\ t', tLast\ t') \\ &\quad | t'. t' \in SPR.jkbpC \wedge spr-jview\ a\ t' = spr-jview\ a\ t \} \langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle \end{aligned}$$

**end**

The simulation is identical to that in the previous section:

$$\begin{aligned} \mathbf{record} \quad &('a, 'ePubAct, 'es, 'pPubAct, 'ps) SPRstate = \\ &sprFst :: ('a, 'ePubAct, 'es, 'pPubAct, 'ps) BState \\ &sprLst :: ('a, 'ePubAct, 'es, 'pPubAct, 'ps) BState \\ &sprCRel :: ('a, 'ePubAct, 'es, 'pPubAct, 'ps) BState Relation \end{aligned}$$

**context** *FiniteBroadcastEnvironment*

**begin**

**definition**

$$\begin{aligned} \text{spr-sim} &:: ('a, 'ePubAct, 'es, 'pPubAct, 'ps) \text{ BEState Trace} \\ &\Rightarrow ('a, 'ePubAct, 'es, 'pPubAct, 'ps) \text{ SPRstate} \end{aligned}$$

**where**

$$\text{spr-sim} \equiv \lambda t. \langle \text{sprFst} = \text{tFirst } t, \text{sprLst} = \text{tLast } t, \text{sprCRel} = \text{tObsC-abs } t \rangle \langle \text{proof} \rangle$$

The Kripke structure over simulated traces is also the same:

**definition**

$$\text{spr-simRels} :: 'a \Rightarrow ('a, 'ePubAct, 'es, 'pPubAct, 'ps) \text{ SPRstate Relation}$$

**where**

$$\begin{aligned} \text{spr-simRels} &\equiv \lambda a. \{ (s, s') \mid s \text{ s}' \\ &\quad \text{envObs } a \text{ (sprFst } s) = \text{envObs } a \text{ (sprFst } s') \\ &\quad \wedge \text{envObs } a \text{ (sprLst } s) = \text{envObs } a \text{ (sprLst } s') \\ &\quad \wedge \text{sprCRel } s = \text{sprCRel } s' \} \end{aligned}$$

**definition**

$$\text{spr-simVal} :: ('a, 'ePubAct, 'es, 'pPubAct, 'ps) \text{ SPRstate} \Rightarrow 'p \Rightarrow \text{bool}$$

**where**

$$\text{spr-simVal} \equiv \text{envVal} \circ \text{sprLst}$$

**abbreviation**

$$\text{spr-simMC} \equiv \text{mkKripke} (\text{spr-sim} \text{ 'SPR.jkbpC}) \text{ spr-simRels spr-simVal} \langle \text{proof} \rangle$$

As usual, showing that *spr-sim* is in fact a simulation is routine for all properties except for reverse simulation. For that we use proof techniques similar to those of [Lomuscio et al. \(2000\)](#): the goal is to show that, given  $t \in \text{jkbpC}$ , we can construct a trace  $t' \in \text{jkbpC}$  indistinguishable from  $t$  by agent  $a$ , based on the public actions, the common observation and  $a$ 's private and initial states.

To do this we define a splicing operation:

**definition**

$$\begin{aligned} \text{sSplice} &:: 'a \\ &\Rightarrow ('a, 'ePubAct, 'es, 'pPubAct, 'ps) \text{ BEState} \\ &\Rightarrow ('a, 'ePubAct, 'es, 'pPubAct, 'ps) \text{ BEState} \\ &\Rightarrow ('a, 'ePubAct, 'es, 'pPubAct, 'ps) \text{ BEState} \end{aligned}$$

**where**

$$\text{sSplice } a \text{ s } s' \equiv s \langle \text{ps} := (\text{ps } s)(a := \text{ps } s' \text{ a}) \rangle \langle \text{proof} \rangle \langle \text{proof} \rangle \langle \text{proof} \rangle \langle \text{proof} \rangle \langle \text{proof} \rangle$$

The effect of *sSplice*  $a \text{ s } s'$  is to update  $s$  with  $a$ 's private state in  $s'$ . The key properties are that provided the common observation on  $s$  and  $s'$  are the same, then agent  $a$ 's observation on *sSplice*  $a \text{ s } s'$  is the same as at  $s'$ , while everyone else's is the same as at  $s$ .

We hoist this operation pointwise to traces:

**abbreviation**

$$\begin{aligned} \text{tSplice} &:: ('a, 'ePubAct, 'es, 'pPubAct, 'ps) \text{ BEState Trace} \\ &\Rightarrow 'a \\ &\Rightarrow ('a, 'ePubAct, 'es, 'pPubAct, 'ps) \text{ BEState Trace} \end{aligned}$$



$\Rightarrow ('a, 'ePubAct, 'es, 'pPubAct, 'ps) BEState Trace$   
 $(- \bowtie_a - [55, 1000, 56] 55)$

**where**

$t \bowtie_a t' \equiv tZip (sSplice a) t t' \langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle$

The key properties are that after splicing, if  $t$  and  $t'$  have the same common observation, then so does  $t \bowtie_a t'$ , and for all agents  $a' \neq a$ , the view  $a'$  has of  $t \bowtie_a t'$  is the same as it has of  $t$ , while for  $a$  it is the same as  $t'$ .

We can conclude that provided the two traces are initially indistinguishable to  $a$ , and not commonly distinguishable, then  $t \bowtie_a t'$  is a canonical trace:

**lemma** *tSplice-jkbpC*:

**assumes**  $tt: \{t, t'\} \subseteq SPR.jkbpC$

**assumes**  $init: envObs a (tFirst t) = envObs a (tFirst t')$

**assumes**  $tObsC: tObsC t = tObsC t'$

**shows**  $t \bowtie_a t' \in SPR.jkbpC \langle proof \rangle \langle proof \rangle$

The proof is by induction over  $t$  and  $t'$ , and depends crucially on the public actions being recorded in the state and commonly observed. Showing the reverse simulation property is then straightforward.

**lemma** *spr-sim*:  $sim SPR.MC spr-simMC spr-sim \langle proof \rangle \mathbf{end}$

**sublocale** *FiniteBroadcastEnvironment*

$< SPR: SimIncrEnvironment jkbp envInit envAction envTrans envVal$   
 $spr-jview envObs spr-jviewInit spr-jviewIncr$   
 $spr-sim spr-simRels spr-simVal \langle proof \rangle$

The algorithmic representations and machinery of the deterministic JKBP case suffice for this one too, and so we omit the details.

### 7.5.1 Perfect Recall in Independently-Initialised Non-deterministic Broadcast Environments

If the private and environment parts of the initial states are independent we can simplify the construction of the previous section and consider only sets of states rather than relations. This greatly reduces the state space that the algorithm traverses.

We capture this independence by adding some assumptions to the *FiniteBroadcastEnvironment* locale of Figure 6; the result is the *FiniteBroadcastEnvironmentIndependentInit* locale shown in Figure 7. We ask that the initial states be the Cartesian product of possible private and environment states; in other words there is nothing for the agents to learn about correlations amongst the initial states. As there are initially no public actions from the previous round, we use the *default* class to indicate that there is a fixed but arbitrary choice to be made here.

Again we introduce common and agent-specific abstraction functions:

**context** *FiniteBroadcastEnvironmentIndependentInit*

```

locale FiniteBroadcastEnvironmentIndependentInit =
  FiniteBroadcastEnvironment jkbp envInit envAction envTrans envVal envObs
    envObsC envActionES envTransES
  for jkbp :: ('a::finite, 'p, ('pPubAct::{default,finite} × 'ps::finite)) JKBP
  and envInit :: ('a, 'ePubAct :: {default, finite}, 'es :: finite,
    'pPubAct, 'ps) BESState list
  and envAction :: ('a, 'ePubAct, 'es, 'pPubAct, 'ps) BESState
    ⇒ ('ePubAct × 'ePrivAct) list
  and envTrans :: ('ePubAct × 'ePrivAct)
    ⇒ ('a ⇒ ('pPubAct × 'ps))
    ⇒ ('a, 'ePubAct, 'es, 'pPubAct, 'ps) BESState
    ⇒ ('a, 'ePubAct, 'es, 'pPubAct, 'ps) BESState
  and envVal :: ('a, 'ePubAct, 'es, 'pPubAct, 'ps) BESState ⇒ 'p ⇒ bool
  and envObs :: 'a ⇒ ('a, 'ePubAct, 'es, 'pPubAct, 'ps) BESState
    ⇒ ('cobs × 'ps × ('ePubAct × ('a ⇒ 'pPubAct)))
  and envObsC :: 'es ⇒ 'cobs
  and envActionES :: 'es ⇒ ('ePubAct × ('a ⇒ 'pPubAct))
    ⇒ ('ePubAct × 'ePrivAct) list
  and envTransES :: ('ePubAct × 'ePrivAct) ⇒ ('a ⇒ 'pPubAct)
    ⇒ 'es ⇒ 'es

+ fixes agents :: 'a list
fixes envInitBits :: 'es list × ('a ⇒ 'ps list)
defines envInit-def:
  envInit ≡ [ (| es = esf, ps = psf, pubActs = (default, λ-. default) |)
    . psf ← listToFuns (snd envInitBits) agents
    , esf ← fst envInitBits ]
assumes agents: set agents = UNIV distinct agents

```

Figure 7: Finite broadcast environments with non-deterministic KBPs, where the initial private and environment states are independent.

**begin**

**definition**

*tObsC-ii-abs* :: ('a, 'ePubAct, 'es, 'pPubAct, 'ps) *BESState Trace*  
 ⇒ ('a, 'ePubAct, 'es, 'pPubAct, 'ps) *BESState set*

**where** *tObsC-ii-abs t* ≡

{ *tLast t' | t'. t' ∈ SPR.jkbpC ∧ tObsC t' = tObsC t* }

**definition**

*agent-ii-abs* :: 'a ⇒ ('a, 'ePubAct, 'es, 'pPubAct, 'ps) *BESState Trace*  
 ⇒ ('a, 'ePubAct, 'es, 'pPubAct, 'ps) *BESState set*

**where** *agent-ii-abs a t* ≡

{ *tLast t' | t'. t' ∈ SPR.jkbpC ∧ spr-jview a t' = spr-jview a t* }⟨*proof*⟩⟨*proof*⟩⟨*proof*⟩⟨*proof*⟩⟨*proof*⟩⟨*proof*⟩

The simulation is similar to the single-agent case (§7.3); for a given canonical trace  $t$  it pairs the set of worlds that any agent considers possible with the final state of  $t$ :

**type-synonym** (in  $-$ ) ( $'a, 'ePubAct, 'es, 'pPubAct, 'ps$ ) *SPRstate* =  
 ( $'a, 'ePubAct, 'es, 'pPubAct, 'ps$ ) *BESState set*  
 $\times$  ( $'a, 'ePubAct, 'es, 'pPubAct, 'ps$ ) *BESState*

**definition**

$spr\text{-}ii\text{-}sim :: ('a, 'ePubAct, 'es, 'pPubAct, 'ps) BESState Trace$   
 $\Rightarrow ('a, 'ePubAct, 'es, 'pPubAct, 'ps) SPRstate$

**where**  $spr\text{-}ii\text{-}sim \equiv \lambda t. (tObsC\text{-}ii\text{-}abs\ t, tLast\ t)\langle proof \rangle$

The Kripke structure over simulated traces is also quite similar:

**definition**

$spr\text{-}ii\text{-}simRels :: 'a \Rightarrow ('a, 'ePubAct, 'es, 'pPubAct, 'ps) SPRstate Relation$

**where**  $spr\text{-}ii\text{-}simRels \equiv \lambda a.$

$\{ (s, s') \mid s\ s'.\ envObs\ a\ (snd\ s) = envObs\ a\ (snd\ s') \wedge fst\ s = fst\ s' \}$

**definition**

$spr\text{-}ii\text{-}simVal :: ('a, 'ePubAct, 'es, 'pPubAct, 'ps) SPRstate \Rightarrow 'p \Rightarrow bool$

**where**  $spr\text{-}ii\text{-}simVal \equiv envVal \circ snd$

**abbreviation**

$spr\text{-}ii\text{-}simMC \equiv mkKripke\ (spr\text{-}ii\text{-}sim\ 'SPR.jkbpC)\ spr\text{-}ii\text{-}simRels\ spr\text{-}ii\text{-}simVal\ \langle proof \rangle\ \langle proof \rangle\ \langle proof \rangle$

The proofs that this simulation is adequate are similar to those in the previous section. We elide the details.

**lemma**  $spr\text{-}ii\text{-}sim: sim\ SPR.MC\ spr\text{-}ii\text{-}simMC\ spr\text{-}ii\text{-}sim\ \langle proof \rangle$

**end**

**sublocale** *FiniteBroadcastEnvironmentIndependentInit*

$< SPRii: SimIncrEnvironment\ jkbp\ envInit\ envAction\ envTrans\ envVal$   
 $spr\text{-}jview\ envObs\ spr\text{-}jviewInit\ spr\text{-}jviewIncr$   
 $spr\text{-}ii\text{-}sim\ spr\text{-}ii\text{-}simRels\ spr\text{-}ii\text{-}simVal\ \langle proof \rangle$

## 8 Examples

We demonstrate the theory by using Isabelle's code generator to run it on two standard examples: the Robot from §1, and the classic muddy children puzzle.

### 8.1 The Robot

Recall the autonomous robot of §1: we are looking for an implementation of the KBP:

```
do
  []  $\mathbf{K}_{robot}$  goal     $\rightarrow$  Halt
  []  $\neg\mathbf{K}_{robot}$  goal   $\rightarrow$  Nothing
od
```

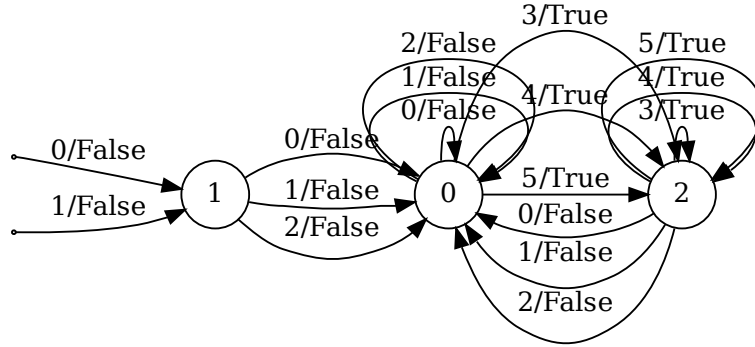


Figure 8: The implementation of the robot using the clock semantics.

in an environment where positions are identified with the natural numbers, the robot’s sensor is within one of the position, and the proposition `goal` is true when the position is in  $\{2, 3, 4\}$ . The robot is initially at position zero, and the effect of its `Halt` action is to cause the robot to instantaneously stop at its current position. A later `Nothing` action may allow the environment to move the robot further to the right.

To obtain a finite environment, we truncate the number line at 5, which is intuitively sound for determining the robot’s behaviour due to the synchronous view, and the fact that if it reaches this rightmost position then it can never satisfy its objective. Running the Haskell code generated by Isabelle yields the automata shown in Figure 8 and Figure 9 for the clock and synchronous perfect recall semantics respectively. These have been minimised using Hopcroft’s algorithm (Gries 1973).

The (inessential) labels on the states are an upper bound on the set of positions that the robot considers possible when it is in that state. Transitions are annotated with the observations yielded by the sensor. Double-circled states are those in which the robot performs the `Halt` action, the others `Nothing`. We observe that the synchronous perfect-recall view yields a “ratchet” protocol, i.e. if the robot learns that it is in the goal region then it halts for all time, and that it never overshoots the goal region. Conversely the clock semantics allows the robot to infinitely alternate its actions depending on the sensor reading. This is effectively the behaviour of the intuitive implementation that halts iff the sensor reads three or more.

We can also see that minimisation does not yield the smallest automata we could

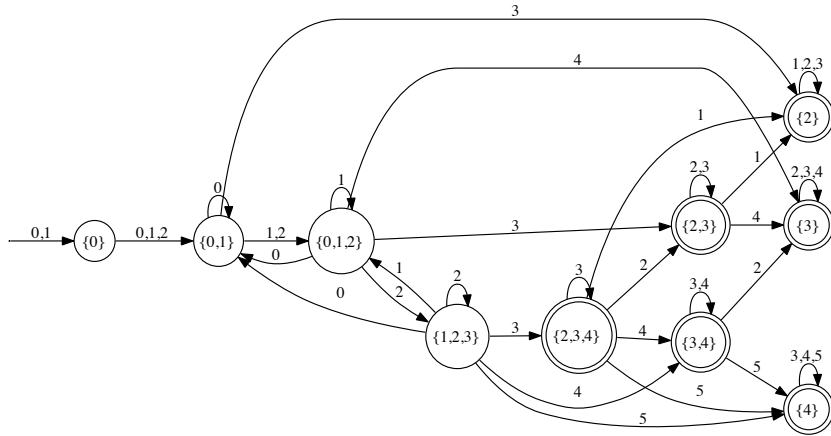


Figure 9: The implementation of the robot using the SPR semantics.

hope for; in particular there are a lot of redundant states where the prescribed behaviour is the same but the robot’s state of knowledge different. This is because our implementations do not specify what happens on invalid observations, which we have modelled as errors instead of don’t-cares, and these extraneous distinctions are preserved by bisimulation reduction. We discuss this further in §??.

*<proof><proof><proof><proof><proof><proof><proof><proof>*

## 8.2 The Muddy Children

Our first example of a multi-agent broadcast scenario is the classic muddy children puzzle, one of a class of such puzzles that exemplify non-obvious reasoning about mutual states of knowledge. It goes as follows (Fagin et al. 1995, §1.1, Example 7.2.5):

*N* children are playing together, *k* of whom get mud on their foreheads. Each can see the mud on the others’ foreheads but not their own.

A parental figure appears and says “At least one of you has mud on your forehead.”, expressing something already known to each of them if *k* > 1.

The parental figure then asks “Does any of you know whether you have mud on your own forehead?” over and over.

Assuming the children are perceptive, intelligent, truthful and they answer simultaneously, what will happen?

This puzzle relies essentially on *synchronous public broadcasts* making particular facts *common knowledge*, and that agents are capable of the requisite logical inference.

As the mother has complete knowledge of the situation, we integrate her behaviour into the environment. Each agent  $\text{child}_i$  reasons with the following KBP:

```

do
  []  $\hat{\mathbf{K}}_{\text{child}_i} \text{muddy}_i$    → Say “I know if my forehead is muddy”
  []  $\neg \hat{\mathbf{K}}_{\text{child}_i} \text{muddy}_i$  → Say nothing
od

```

where  $\hat{\mathbf{K}}_a \varphi$  abbreviates  $\mathbf{K}_a \varphi \vee \mathbf{K}_a \neg \varphi$ .

As this protocol is deterministic, we use the SPR algorithm of §7.4.

The model records a child’s initial observation of the mother’s pronouncement and the muddiness of the other children in her initial private state, and these states are not changed by *envTrans*. The recurring common observation is all of the children’s public responses to the mother’s questions. Being able to distinguish these observations is crucial to making this a broadcast scenario.

Running the algorithm for three children and minimising using Hopcroft’s algorithm yields the automaton in Figure 10 for  $\text{child}_0$ . The initial transitions are labelled with the initial observation, i.e., the cleanliness “C” or muddiness “M” of the other two children. The dashed initial transition covers the case where everyone is clean; in the others the mother has announced that someone is dirty. Later transitions simply record the actions performed by each of the agents, where “K” is the first action in the above KBP, and “N” the second. Double-circled states are those in which  $\text{child}_0$  knows whether she is muddy, and single-circled where she does not.

In essence the child counts the number of muddy foreheads she sees and waits that many rounds before announcing that she knows.

Note that a solution to this puzzle is beyond the reach of the clock semantics as it requires (in general) remembering the sequence of previous broadcasts of length proportional to the number of children. We discuss this further in §??.

*<proof><proof><proof><proof><proof><proof><proof><proof><proof><proof><proof><proof><proof>*

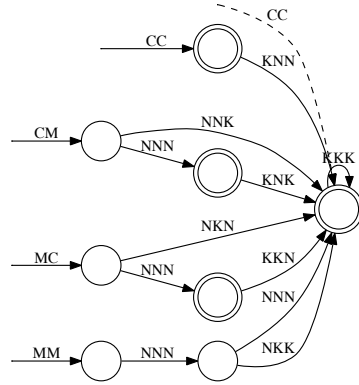


Figure 10: The protocol of  $\text{child}_0$ .

## 9 Perspective and related work

The most challenging and time-consuming aspect of mechanising this theory was making definitions suitable for the code generator. For example, we could have used a locale to model the interface to the maps in §6.9, but as the code generator presently does not cope with functions arising from locale interpretation, we are forced to say things at least twice if we try to use both features, as we implicitly did in §6.9. Whether it is more convenient or even necessary to use a record and predicate or a locale presently requires experimentation and guidance from experienced users.

As reflected by the traffic on the Isabelle mailing list, a common stumbling block when using the code generator is the treatment of sets. The existing libraries are insufficiently general: Florian Haftmann’s *Cset* theory<sup>1</sup> does not readily support a choice operator, such as the one we used in §???. Even the heroics of the Isabelle Collections Framework Lammich and Lochbihler (2010) are insufficient as their equality on keys is structural (i.e., HOL equality), forcing us to either use a canonical representation (such as ordered distinct lists) or redo the relevant proofs with reified operations (equality, orderings, etc.). Neither of these is satisfying from the perspective of reuse.

Working with suitably general theories, e.g., using data refinement, is difficult as the simplifier is significantly less helpful for reasoning under abstract quotients, such as those in §6.9; what could typically be shown by equational rewriting now involves reasoning about existentials. For this reason we have only allowed some types to be refined; the representations of observations and system states are constant throughout our development, which may preclude some optimisations. The recent work of Kaliszyk and Urban Kaliszyk and Urban (2011) addresses these issues for concrete quotients, but not for the abstract ones that arise in this kind of top-down development.

As for the use of knowledge in formally reasoning about systems, this and similar semantics are under increasing scrutiny due to their relation to security properties. Despite the explosion in number of epistemic model checkers van Eijck and Orzan (2005); Gammie and van der Meyden (2004); Kacprzak et al. (2008); Lomuscio et al. (2009), finding implementations of knowledge-based programs remains a substantially manual affair Al-Bataineh and van der Meyden (2010). A refinement framework has also been developed Bickford et al. (2004); Engelhardt et al. (2000).

The theory presented here supports a more efficient implementation using symbolic techniques, ala MCK; recasting the operations of the `SimEnvironment` locale into boolean decision diagrams is straightforward. It is readily generalised to other synchronous views, as alluded to in §7.3, and adding a common knowledge modality, useful for talking about consensus (Fagin et al. 1995, Chapter 6), is routine. We hope that such an implementation will lead to more exploration of the KBP formalism.

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<sup>1</sup>The theory *Cset* accompanies the Isabelle/HOL distribution.

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