

The Jordan-Hölder Theorem

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Abstract

This submission contains theories that lead to a formalization of the proof of the Jordan-Hölder theorem about composition series of finite groups. The theories formalize the notions of isomorphism classes of groups, simple groups, normal series, composition series, maximal normal subgroups. Furthermore, they provide proofs of the second isomorphism theorem for groups, the characterization theorem for maximal normal subgroups as well as many useful lemmas about normal subgroups and factor groups. The formalization is based on the work in my first AFP submission [vR14] while the proof of the Jordan-Hölder theorem itself is inspired by course notes of Stuart Rankin [Ran05].

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```

theory SndIsomorphismGrp
imports
  ~/src/HOL/Algebra/Coset
  ../Secondary-Sylow/SubgroupConjugation
begin

```

1 The Second Isomorphism Theorem for Groups

1.1 Preliminaries

```

lemma (in group) triv-subgroup:
  shows subgroup {1} G
unfolding subgroup-def by auto

```

```

lemma (in group) triv-normal-subgroup:
  shows {1} < G
unfolding normal-def normal-axioms-def l-coset-def r-coset-def
using is-group triv-subgroup by auto

```

```

lemma (in group) normal-restrict-supergroup:
  assumes SsubG:subgroup S G
  assumes Nnormal:N < G
  assumes N ⊆ S
  shows N < (G⟦carrier := S⟧)
proof –
  interpret Sgrp: group G⟦carrier := S⟧ using SsubG by (rule subgroup-imp-group)
  show ?thesis
  proof(rule Sgrp.normalI)
    show subgroup N (G⟦carrier := S⟧) using assms is-group by (metis subgroup.subgroup-of-subset normal-inv-iff)
  next
    from SsubG have S ⊆ carrier G by (rule subgroup-imp-subset)
    thus ∀x∈carrier (G⟦carrier := S⟧). N #> G⟦carrier := S⟧ x = x <# G⟦carrier := S⟧
  N
    using Nnormal unfolding normal-def normal-axioms-def l-coset-def r-coset-def
by fastforce
  qed
qed

```

As this is maybe the best place this fits in: Factorizing by the trivial subgroup is an isomorphism.

```

lemma (in group) trivial-factor-iso:
  shows the-elem ∈ (G Mod {1}) ≅ G
proof –
  have group-hom G G (λx. x) unfolding group-hom-def group-hom-axioms-def
  hom-def using is-group by simp

```

moreover have $(\lambda x. x) \text{ ' carrier } G = \text{carrier } G$ **by simp**
moreover have $\text{kernel } G \ G \ (\lambda x. x) = \{1\}$ **unfolding kernel-def by auto**
ultimately show *?thesis* **using** *group-hom.FactGroup-iso* **by force**
qed

And the dual theorem to the previous one: Factorizing by the group itself gives the trivial group

lemma (*in group*) *self-factor-iso*:

shows $(\lambda X. \text{the-elem } ((\lambda x. 1) \text{ ' } X)) \in (G \text{ Mod } (\text{carrier } G)) \cong G(\text{carrier} := \{1\})$

proof –

have *group* $(G(\text{carrier} := \{1\}))$ **by** (*metis subgroup-imp-group triv-subgroup*)
hence *group-hom* $G \ (G(\text{carrier} := \{1\})) \ (\lambda x. 1)$ **unfolding** *group-hom-def group-hom-axioms-def hom-def* **using** *is-group* **by auto**
moreover have $(\lambda x. 1) \text{ ' carrier } G = \text{carrier } (G(\text{carrier} := \{1\}))$ **by auto**
moreover have $\text{kernel } G \ (G(\text{carrier} := \{1\})) \ (\lambda x. 1) = \text{carrier } G$ **unfolding kernel-def by auto**
ultimately show *?thesis* **using** *group-hom.FactGroup-iso* **by force**
qed

This theory provides a proof of the second isomorphism theorems for groups. The theorems consist of several facts about normal subgroups.

The first lemma states that whenever we have a subgroup S and a normal subgroup H of a group G , their intersection is normal in G

locale *second-isomorphism-grp = normal +*
fixes $S::'a \text{ set}$
assumes *subgrpS:subgroup S G*

context *second-isomorphism-grp*
begin

interpretation *groupS: group G(carrier := S)*
using *subgrpS* **by** (*metis subgroup-imp-group*)

lemma *normal-subgrp-intersection-normal*:

shows $S \cap H \triangleleft (G(\text{carrier} := S))$
proof(*auto simp: groupS.normal-inv-iff*)
from *subgrpS is-subgroup* **have** $\bigwedge x. x \in \{S, H\} \implies \text{subgroup } x \ G$ **by auto**
hence *subgroup* $(\bigcap \{S, H\}) \ G$ **using** *subgroups-Inter* **by blast**
hence *subgroup* $(S \cap H) \ G$ **by auto**
moreover have $S \cap H \subseteq S$ **by simp**
ultimately show *subgroup* $(S \cap H) \ (G(\text{carrier} := S))$ **using** *is-group subgroup.subgroup-of-subset subgrpS* **by metis**
next
fix $g \ h$
assume $g: g \in S$ **and** $hH: h \in H$ **and** $hS: h \in S$ {
from $g \ hH$ *subgrpS* **show** $g \otimes h \otimes \text{inv } G(\text{carrier} := S) \ g \in H$ **by** (*metis inv-op-closed2 subgroup.mem-carrier subgroup-inv-equality*)

```

} {
  from  $g \in hS$   $subgrpS$  show  $g \otimes h \in inv_G(\text{carrier} := S)$   $g \in S$  by (metis subgroup.m-closed subgroup.m-inv-closed subgroup-inv-equality)
}
qed

```

lemma *normal-set-mult-subgroup*:

```

shows subgroup ( $H \langle \# \rangle S$ )  $G$ 
proof(rule subgroupI)
  show  $H \langle \# \rangle S \subseteq \text{carrier } G$  by (metis setmult-subset-G subgroup-imp-subset subgrpS subset)
next
  have  $1 \in H$   $1 \in S$  using is-subgroup subgrpS subgroup.one-closed by auto
  hence  $1 \otimes 1 \in H \langle \# \rangle S$  unfolding set-mult-def by blast
  thus  $H \langle \# \rangle S \neq \{\}$  by auto
next
  fix  $g$ 
  assume  $g : g \in H \langle \# \rangle S$ 
  then obtain  $h \ s$  where  $h : h \in H$  and  $s : s \in S$  and  $ghs : g = h \otimes s$  unfolding set-mult-def by auto
  hence  $s \in \text{carrier } G$  by (metis subgroup.mem-carrier subgrpS)
  with  $h \ ghs$  obtain  $h'$  where  $h' : h' \in H$  and  $g = s \otimes h'$  using coset-eq unfolding r-coset-def l-coset-def by auto
  with  $s$  have  $inv \ g = (inv \ h') \otimes (inv \ s)$  by (metis inv-mult-group mem-carrier subgroup.mem-carrier subgrpS)
  moreover from  $h' \ s$  subgrpS have  $inv \ h' \in H$   $inv \ s \in S$  using subgroup.m-inv-closed m-inv-closed by auto
  ultimately show  $inv \ g \in H \langle \# \rangle S$  unfolding set-mult-def by auto
next
  fix  $g \ g'$ 
  assume  $g : g \in H \langle \# \rangle S$  and  $h : g' \in H \langle \# \rangle S$ 
  then obtain  $h \ h' \ s \ s'$  where  $hh'ss' : h \in H \ h' \in H \ s \in S \ s' \in S$  and  $g = h \otimes s$  and  $g' = h' \otimes s'$  unfolding set-mult-def by auto
  hence  $g \otimes g' = (h \otimes s) \otimes (h' \otimes s')$  by metis
  also from  $hh'ss'$  have  $inG : h \in \text{carrier } G \ h' \in \text{carrier } G \ s \in \text{carrier } G \ s' \in \text{carrier } G$  using subgrpS mem-carrier subgroup.mem-carrier by force+
  hence  $(h \otimes s) \otimes (h' \otimes s') = h \otimes (s \otimes h') \otimes s'$  using m-assoc by auto
  also from  $hh'ss'$  inG obtain  $h''$  where  $h'' : h'' \in H$  and  $s \otimes h' = h'' \otimes s$  using coset-eq unfolding r-coset-def l-coset-def by fastforce
  hence  $h \otimes (s \otimes h') \otimes s' = h \otimes (h'' \otimes s) \otimes s'$  by simp
  also from  $h''$  inG have  $\dots = (h \otimes h'') \otimes (s \otimes s')$  using m-assoc mem-carrier by auto
  finally have  $g \otimes g' = h \otimes h'' \otimes (s \otimes s')$ .
  moreover with  $h'' \ hh'ss'$  have  $\dots \in H \langle \# \rangle S$  unfolding set-mult-def using subgrpS subgroup.m-closed by fastforce
  ultimately show  $g \otimes g' \in H \langle \# \rangle S$  by simp
qed

```

lemma *oneH*: $1 \in H$ by (metis is-subgroup subgroup.one-closed)

lemma *H-contained-in-set-mult:*

shows $H \subseteq H \langle \# \rangle S$

proof *auto*

have $1 \in S$ **by** (*metis subgroup.one-closed subgrpS*)

fix x

assume $x \in H$

with $\langle 1 \in S \rangle$ **have** $x \otimes 1 \in H \langle \# \rangle S$ **unfolding** *set-mult-def* **by** *force*

with x **show** $x \in H \langle \# \rangle S$ **by** (*metis mem-carrier r-one*)

qed

lemma *S-contained-in-set-mult:*

shows $S \subseteq H \langle \# \rangle S$

proof *auto*

fix s

assume $s \in S$

with *oneH* **have** $1 \otimes s \in H \langle \# \rangle S$ **unfolding** *set-mult-def* **by** *force*

with s **show** $s \in H \langle \# \rangle S$ **using** *subgrpS subgroup.mem-carrier l-one* **by** *force*

qed

lemma *normal-intersection-hom:*

shows *group-hom* ($G \langle \text{carrier} := S \rangle$) ($(G \langle \text{carrier} := H \langle \# \rangle S \rangle) \text{Mod } H$) ($\lambda g. H \# \rangle g$)

proof (*auto del: equalityI simp: group-hom-def group-hom-axioms-def hom-def groupS.is-group*)

have *gr:group* ($G \langle \text{carrier} := H \langle \# \rangle S \rangle$) **by** (*metis normal-set-mult-subgroup subgroup-imp-group*)

moreover **have** $H \subseteq H \langle \# \rangle S$ **by** (*rule H-contained-in-set-mult*)

moreover **have** *subgroup* ($H \langle \# \rangle S$) G **by** (*metis normal-set-mult-subgroup*)

ultimately **have** $H \triangleleft (G \langle \text{carrier} := H \langle \# \rangle S \rangle)$ **using** *normal-restrict-supergroup* **by** (*metis inv-op-closed2 is-subgroup normal-inv-iff*)

with *gr* **show** *group* ($(G \langle \text{carrier} := H \langle \# \rangle S \rangle) \text{Mod } H$) **by** (*metis normal.factorgroup-is-group*)

next

fix g

assume $g \in S$

with *subgrpS* **have** $1 \otimes g \in H \langle \# \rangle S$ **unfolding** *set-mult-def* **by** *fastforce*

with g **have** $g \in H \langle \# \rangle S$ **by** (*metis l-one subgroup.mem-carrier subgrpS*)

thus $H \# \rangle g \in \text{carrier} ((G \langle \text{carrier} := H \langle \# \rangle S \rangle) \text{Mod } H)$ **unfolding** *FactGroup-def RCOSETS-def r-coset-def* **by** *auto*

next

fix $g \ g' \ x$

assume $g \in S$ **and** $g' \in S$

hence $(H \# \rangle g) \langle \# \rangle (H \# \rangle g') = H \# \rangle (g \otimes g')$ **by** (*metis rcos-sum subgroup.mem-carrier subgrpS*)

thus $H \# \rangle (g \otimes g') = (H \# \rangle g) \langle \# \rangle (G \langle \text{carrier} := H \langle \# \rangle S \rangle) (H \# \rangle g')$

unfolding *set-mult-def* **by** *auto*

qed

lemma *normal-intersection-hom-kernel:*

shows $\text{kernel } (G(\text{carrier} := S)) ((G(\text{carrier} := H \langle \# \rangle S)) \text{ Mod } H) (\lambda g. H \# \rangle g) = H \cap S$
proof –
have $\text{kernel } (G(\text{carrier} := S)) ((G(\text{carrier} := H \langle \# \rangle S)) \text{ Mod } H) (\lambda g. H \# \rangle g)$
 $= \{g \in S. H \# \rangle g = \mathbf{1}_{(G(\text{carrier} := H \langle \# \rangle S)) \text{ Mod } H}\}$ **unfolding**
 kernel-def **by** *auto*
also have $\dots = \{g \in S. H \# \rangle g = H\}$ **unfolding** FactGroup-def **by** *auto*
also have $\dots = \{g \in S. g \in H\}$ **by** (*metis coset-eq is-subgroup lcoset-join2 rcos-self subgroup.mem-carrier subgrpS*)
also have $\dots = H \cap S$ **by** *auto*
finally show *?thesis*.
qed

lemma *normal-intersection-hom-surj*:

shows $(\lambda g. H \# \rangle g) \text{ 'carrier } (G(\text{carrier} := S)) = \text{carrier } ((G(\text{carrier} := H \langle \# \rangle S)) \text{ Mod } H)$
proof *auto*
fix g
assume $g \in S$
hence $g \in H \langle \# \rangle S$ **using** *S-contained-in-set-mult* **by** *auto*
thus $H \# \rangle g \in \text{carrier } ((G(\text{carrier} := H \langle \# \rangle S)) \text{ Mod } H)$ **unfolding**
 $\text{FactGroup-def RCOSETS-def r-coset-def}$ **by** *auto*
next
fix x
assume $x \in \text{carrier } (G(\text{carrier} := H \langle \# \rangle S)) \text{ Mod } H$
then obtain $h\ s$ **where** $h:h \in H$ **and** $s:s \in S$ **and** $x = H \# \rangle (h \otimes s)$
unfolding $\text{FactGroup-def RCOSETS-def r-coset-def set-mult-def}$ **by** *auto*
hence $x = (H \# \rangle h) \# \rangle s$ **by** (*metis h s coset-mult-assoc mem-carrier subgroup.mem-carrier subgrpS subset*)
also have $\dots = H \# \rangle s$ **by** (*metis h is-group rcos-const*)
finally have $x = H \# \rangle s$.
with s **show** $x \in \text{op } \# \rangle H \text{ ' } S$ **by** *simp*
qed

Finally we can prove the actual isomorphism theorem:

theorem *normal-intersection-quotient-iso*:

shows $(\lambda X. \text{the-elem } ((\lambda g. H \# \rangle g) \text{ ' } X)) \in ((G(\text{carrier} := S)) \text{ Mod } (H \cap S))$
 $\cong (((G(\text{carrier} := H \langle \# \rangle S)) \text{ Mod } H)$
using *normal-intersection-hom-kernel[symmetric] normal-intersection-hom normal-intersection-hom-surj*
by (*metis group-hom.FactGroup-iso*)

end

end

theory *SubgroupsAndNormalSubgroups*
imports

```

../Secondary-Sylow/SndSylow
SndIsomorphismGrp
Coset
begin

```

2 Preliminary lemmas

A group of order 1 is always the trivial group.

```

lemma (in group) order-one-triv-iff:
  shows (order G = 1) = (carrier G = {1})
proof
  assume order:order G = 1
  then obtain x where x:carrier G = {x} unfolding order-def by (auto simp
add: card-Suc-eq)
  hence 1 = x using one-closed by auto
  with x show carrier G = {1} by simp
next
  assume carrier G = {1}
  thus order G = 1 unfolding order-def by auto
qed

```

```

lemma (in group) finite-pos-order:
  assumes finite:finite (carrier G)
  shows 0 < order G
proof -
  from one-closed finite show ?thesis unfolding order-def by (metis card-gt-0-iff
subgroup-nonempty subgroup-self)
qed

```

```

lemma iso-order-closed:
  assumes  $\varphi \in G \cong H$ 
  shows order G = order H
using assms
unfolding order-def iso-def by (metis (no-types) bij-betw-same-card mem-Collect-eq)

```

3 More Facts about Subgroups

```

lemma (in subgroup) subgroup-of-restricted-group:
  assumes subgroup U (G| carrier := H)
  shows  $U \subseteq H$ 
using assms subgroup-imp-subset by force

```

```

lemma (in subgroup) subgroup-of-subgroup:
  assumes group G
  assumes subgroup U (G| carrier := H)
  shows subgroup U G
proof

```

```

from assms(2) have  $U \subseteq H$  by (rule subgroup-of-restricted-group)
thus  $U \subseteq \text{carrier } G$  by (auto simp:subset)
next
  fix  $x y$ 
  have  $a:x \otimes y = x \otimes_{G \langle \text{carrier} := H \rangle} y$  by simp
  assume  $x \in U y \in U$ 
  with assms  $a$  show  $x \otimes y \in U$  by (metis subgroup.m-closed)
next
  have  $1_{G \langle \text{carrier} := H \rangle} = 1$  by simp
  with assms show  $1 \in U$  by (metis subgroup.one-closed)
next
  have subgroup  $H G..$ 
  fix  $x$ 
  assume  $x \in U$ 
  with assms(2) have  $\text{inv}_{G \langle \text{carrier} := H \rangle} x \in U$  by (rule subgroup.m-inv-closed)
  moreover from assms  $\langle x \in U \rangle$  have  $x \in H$  by (metis in-mono subgroup-of-restricted-group)
  with assms(1)  $\langle \text{subgroup } H G \rangle$  have  $\text{inv}_{G \langle \text{carrier} := H \rangle} x = \text{inv } x$  by (rule
group.subgroup-inv-equality)
  ultimately show  $\text{inv } x \in U$  by simp
qed

```

Being a subgroup is preserved by surjective homomorphisms

lemma (*in subgroup*) *surj-hom-subgroup*:

```

assumes  $\varphi:\text{group-hom } G F$ 
assumes  $\varphi\text{surj}:\varphi^{-1}(\text{carrier } G) = \text{carrier } F$ 
shows subgroup  $(\varphi^{-1} H) F$ 
proof
  from  $\varphi\text{surj}$  show  $\text{img-subset}:\varphi^{-1} H \subseteq \text{carrier } F$  unfolding iso-def bij-betw-def
by auto
next
  fix  $f f'$ 
  assume  $h:f \in \varphi^{-1} H$  and  $h':f' \in \varphi^{-1} H$ 
  with  $\varphi\text{surj}$  obtain  $g g'$  where  $g:g \in H f = \varphi g$  and  $g':g' \in H f' = \varphi g'$  by
auto
  hence  $g \otimes_G g' \in H$  by (metis m-closed)
  hence  $\varphi(g \otimes_G g') \in \varphi^{-1} H$  by simp
  with  $g g' \varphi$  show  $f \otimes_F f' \in \varphi^{-1} H$  using group-hom.hom-mult by fastforce
next
  have  $\varphi 1 \in \varphi^{-1} H$  by auto
  with  $\varphi$  show  $1_F \in \varphi^{-1} H$  by (metis group-hom.hom-one)
next
  fix  $f$ 
  assume  $f:f \in \varphi^{-1} H$ 
  then obtain  $g$  where  $g:g \in H f = \varphi g$  by auto
  hence  $\text{inv } g \in H$  by auto
  hence  $\varphi(\text{inv } g) \in \varphi^{-1} H$  by auto
  with  $\varphi g$  subset show  $\text{inv}_F f \in \varphi^{-1} H$  using group-hom.hom-inv by fastforce
qed

```


... and thus of course by isomorphisms of groups.

lemma *iso-subgroup*:

assumes *groups:group* G *group* F

assumes HG :*subgroup* H G

assumes φ : $\varphi \in G \cong F$

shows *subgroup* $(\varphi \text{ ` } H)$ F

proof –

from *groups* φ **have** *group-hom* G F φ **unfolding** *group-hom-def* *group-hom-axioms-def* *iso-def* **by** *auto*

moreover from φ **have** $\varphi \text{ ` } (\text{carrier } G) = \text{carrier } F$ **unfolding** *iso-def* *bij-betw-def* **by** *simp*

moreover note HG

ultimately show *?thesis* **by** (*metis subgroup.surj-hom-subgroup*)

qed

An isomorphism restricts to an isomorphism of subgroups.

lemma *iso-restrict*:

assumes *groups:group* G *group* F

assumes HG :*subgroup* H G

assumes φ : $\varphi \in G \cong F$

shows (*restrict* φ H) $\in (G(\text{carrier } := H)) \cong (F(\text{carrier } := \varphi \text{ ` } H))$

unfolding *iso-def* *hom-def* *bij-betw-def* *inj-on-def*

proof *auto*

fix g h

assume $g \in H$ $h \in H$

hence $g \in \text{carrier } G$ $h \in \text{carrier } G$ **by** (*metis* HG *subgroup.mem-carrier*)+

thus $\varphi (g \otimes_G h) = \varphi g \otimes_F \varphi h$ **using** φ **unfolding** *iso-def* *hom-def* **by** *auto*

next

fix g h

assume $g \in H$ $h \in H$ $g \otimes_G h \notin H$

hence *False* **using** HG **unfolding** *subgroup-def* **by** *auto*

thus *undefined* = $\varphi g \otimes_F \varphi h$ **by** *auto*

next

fix g h

assume g : $g \in H$ **and** h : $h \in H$ **and** eq : $\varphi g = \varphi h$

hence $g \in \text{carrier } G$ $h \in \text{carrier } G$ **by** (*metis* HG *subgroup.mem-carrier*)+

with eq **show** $g = h$ **using** φ **unfolding** *iso-def* *bij-betw-def* *inj-on-def* **by** *auto*

qed

The intersection of two subgroups is, again, a subgroup

lemma (*in group*) *subgroup-intersect*:

assumes *subgroup* H G

assumes *subgroup* H' G

shows *subgroup* $(H \cap H')$ G

using *assms* **unfolding** *subgroup-def* **by** *auto*

4 Facts about Normal Subgroups

lemma (in *normal*) *is-normal*:
 shows $H \triangleleft G$
 by (*metis coset-eq is-subgroup normalI*)

Being a normal subgroup is preserved by surjective homomorphisms.

lemma (in *normal*) *surj-hom-normal-subgroup*:
 assumes $\varphi: \text{group-hom } G F$ φ
 assumes $\varphi \text{ surj}: \varphi \text{ ' } (\text{carrier } G) = \text{carrier } F$
 shows $(\varphi \text{ ' } H) \triangleleft F$
proof (*rule group.normalI*)
 from φ show *group F unfolding group-hom-def group-hom-axioms-def by simp*
next
 from φ $\varphi \text{ surj}$ show *subgroup* $(\varphi \text{ ' } H) F$ **by** (*rule surj-hom-subgroup*)
next
 show $\forall x \in \text{carrier } F. \varphi \text{ ' } H \#>_F x = x <\#_F \varphi \text{ ' } H$
proof
 fix f
 assume $f: f \in \text{carrier } F$
 with $\varphi \text{ surj}$ **obtain** g **where** $g: g \in \text{carrier } G$ $f = \varphi g$ **by** *auto*
 hence $\varphi \text{ ' } H \#>_F f = \varphi \text{ ' } H \#>_F \varphi g$ **by** *simp*
 also have $\dots = (\lambda x. (\varphi x) \otimes_F (\varphi g)) \text{ ' } H$ **unfolding** *r-coset-def image-def by auto*
 also have $\dots = (\lambda x. \varphi (x \otimes g)) \text{ ' } H$ **using** *subset g φ group-hom.hom-mult unfolding image-def by fastforce*
 also have $\dots = \varphi \text{ ' } (H \#> g)$ **using** φ **unfolding** *r-coset-def by auto*
 also have $\dots = \varphi \text{ ' } (g <\# H)$ **by** (*metis coset-eq g(1)*)
 also have $\dots = (\lambda x. \varphi (g \otimes x)) \text{ ' } H$ **using** φ **unfolding** *l-coset-def by auto*
 also have $\dots = (\lambda x. (\varphi g) \otimes_F (\varphi x)) \text{ ' } H$ **using** *subset g φ group-hom.hom-mult by fastforce*
 also have $\dots = \varphi g <\#_F \varphi \text{ ' } H$ **unfolding** *l-coset-def image-def by auto*
 also have $\dots = f <\#_F \varphi \text{ ' } H$ **using** g **by** *simp*
 finally show $\varphi \text{ ' } H \#>_F f = f <\#_F \varphi \text{ ' } H$.
qed
qed

Being a normal subgroup is preserved by group isomorphisms.

lemma *iso-normal-subgroup*:
 assumes *groups:group* G *group* F
 assumes $HG: H \triangleleft G$
 assumes $\varphi: \varphi \in G \cong F$
 shows $(\varphi \text{ ' } H) \triangleleft F$
proof –
 from *groups* φ **have** *group-hom* $G F$ φ **unfolding** *group-hom-def group-hom-axioms-def iso-def by auto*
 moreover from φ **have** $\varphi \text{ ' } (\text{carrier } G) = \text{carrier } F$ **unfolding** *iso-def bij-betw-def by simp*
 moreover **note** HG

ultimately show *?thesis* **using** *normal.surj-hom-normal-subgroup* **by** *metis*
qed

The trivial subgroup is a subgroup:

lemma (**in** *group*) *triv-subgroup*:
shows *subgroup* $\{1\}$ *G*
unfolding *subgroup-def* **by** *auto*

The cardinality of the right cosets of the trivial subgroup is the cardinality of the group itself:

lemma (**in** *group*) *card-rcosets-triv*:
assumes *finite* (*carrier G*)
shows *card* (*rcosets* $\{1\}$) = *order G*
proof –
have *subgroup* $\{1\}$ *G* **by** (*rule triv-subgroup*)
with *assms* **have** *card* (*rcosets* $\{1\}$) * *card* $\{1\}$ = *order G* **by** (*rule lagrange*)
thus *?thesis* **by** (*auto simp:card-Suc-eq*)
qed

The intersection of two normal subgroups is, again, a normal subgroup.

lemma (**in** *group*) *normal-subgroup-intersect*:
assumes $M \triangleleft G$ **and** $N \triangleleft G$
shows $M \cap N \triangleleft G$
using *assms* *subgroup-intersect is-group normal-inv-iff* **by** *simp*

The set product of two normal subgroups is a normal subgroup.

lemma (**in** *group*) *setmult-lcos-assoc*:
 $\llbracket H \subseteq \text{carrier } G; K \subseteq \text{carrier } G; x \in \text{carrier } G \rrbracket$
 $\implies (x \langle \# \rangle H) \langle \# \rangle K = x \langle \# \rangle (H \langle \# \rangle K)$
by (*force simp add: l-coset-def set-mult-def m-assoc*)

lemma (**in** *group*) *normal-subgroup-set-mult-closed*:
assumes $M \triangleleft G$ **and** $N \triangleleft G$
shows $M \langle \# \rangle N \triangleleft G$

proof (*rule normalI*)
from *assms* **show** *subgroup* ($M \langle \# \rangle N$) *G*
using *second-isomorphism-grp.normal-set-mult-subgroup normal-imp-subgroup*
unfolding *second-isomorphism-grp-def second-isomorphism-grp-axioms-def* **by**
force
next
show $\forall x \in \text{carrier } G. M \langle \# \rangle N \# \rangle x = x \langle \# \rangle (M \langle \# \rangle N)$
proof
fix *x*
assume $x : x \in \text{carrier } G$
have $M \langle \# \rangle N \# \rangle x = M \langle \# \rangle (N \# \rangle x)$ **by** (*metis assms(1,2) normal-inv-iff*
setmult-rcos-assoc subgroup-imp-subset x)
also have $\dots = M \langle \# \rangle (x \langle \# \rangle N)$ **by** (*metis assms(2) normal.coset-eq x*)
also have $\dots = (M \# \rangle x) \langle \# \rangle N$ **by** (*metis assms(1,2) normal-imp-subgroup*
rcos-assoc-lcos subgroup-imp-subset x)

also have $\dots = (x \langle \# \rangle M) \langle \# \rangle N$ **by** (*metis assms(1) normal.coset-eq x*)
also have $\dots = x \langle \# \rangle (M \langle \# \rangle N)$ **by** (*metis assms(1,2) normal-imp-subgroup setmult-lcos-assoc subgroup-imp-subset x*)
finally show $M \langle \# \rangle N \# \rangle x = x \langle \# \rangle (M \langle \# \rangle N)$.
qed
qed

The following is a very basic lemma about subgroups: If restricting the carrier of a group yields a group it's a subgroup of the group we've started with.

lemma (*in group*) *restrict-group-imp-subgroup*:
assumes $H \subseteq \text{carrier } G$ *group* ($G \langle \! \langle \text{carrier} := H \rangle \! \rangle$)
shows *subgroup* H G
proof
from *assms(1)* **show** $H \subseteq \text{carrier } G$.
next
fix $x y$
assume $x \in H$ $y \in H$
hence $x \in \text{carrier } (G \langle \! \langle \text{carrier} := H \rangle \! \rangle)$ $y \in \text{carrier } (G \langle \! \langle \text{carrier} := H \rangle \! \rangle)$ **by** *auto*
with *assms(2)* **show** $x \otimes y \in H$ **using** *assms(2) group.is-monoid monoid.m-closed*
by *fastforce*
next
show $1 \in H$ **using** *assms(2) group.is-monoid monoid.one-closed* **by** *fastforce*
next
fix x
assume $x \in H$
hence $x : x \in \text{carrier } (G \langle \! \langle \text{carrier} := H \rangle \! \rangle)$ **by** *auto*
hence $\text{inv } G \langle \! \langle \text{carrier} := H \rangle \! \rangle$ $x \in \text{carrier } (G \langle \! \langle \text{carrier} := H \rangle \! \rangle)$ **using** *assms(2) group.inv-closed* **by** *fastforce*
hence $\text{inv } G \langle \! \langle \text{carrier} := H \rangle \! \rangle$ $x \in \text{carrier } G$ **using** x *assms(1)* **by** *auto*
moreover have $\text{inv } G \langle \! \langle \text{carrier} := H \rangle \! \rangle$ $x \otimes x = 1$ **using** *assms(2) group.l-inv x*
by *fastforce*
moreover have $x \in \text{carrier } G$ **using** x *assms(1)* **by** *auto*
ultimately have $\text{inv } G \langle \! \langle \text{carrier} := H \rangle \! \rangle$ $x = \text{inv } x$ **using** *inv-equality[symmetric]*
by *auto*
thus $\text{inv } x \in H$ **using** *assms(2) group.inv-closed x* **by** *fastforce*
qed

A subgroup relation survives factoring by a normal subgroup.

lemma (*in group*) *normal-subgroup-factorize*:
assumes $N \triangleleft G$ **and** $N \subseteq H$ **and** *subgroup* H G
shows *subgroup* ($\text{rcosets } G \langle \! \langle \text{carrier} := H \rangle \! \rangle$ N) ($G \text{ Mod } N$)
proof –
interpret $G \text{ Mod } N$: *group* $G \text{ Mod } N$ **using** *assms(1)* **by** (*rule normal.factorgroup-is-group*)
have $N \triangleleft G \langle \! \langle \text{carrier} := H \rangle \! \rangle$ **using** *assms* **by** (*metis normal-restrict-supergroup*)
hence grpHN : *group* ($G \langle \! \langle \text{carrier} := H \rangle \! \rangle \text{ Mod } N$) **by** (*rule normal.factorgroup-is-group*)
have $\text{op } \langle \# \rangle G \langle \! \langle \text{carrier} := H \rangle \! \rangle = (\lambda U K. (\bigcup h \in U. \bigcup k \in K. \{h \otimes_{G \langle \! \langle \text{carrier} := H \rangle \! \rangle} k\}))$ **using** *set-mult-def* **by** *metis*

moreover have $\dots = (\lambda U K. (\bigcup h \in U. \bigcup k \in K. \{h \otimes_G k\}))$ **by auto**
moreover have $op \langle \# \rangle = (\lambda U K. (\bigcup h \in U. \bigcup k \in K. \{h \otimes k\}))$ **using set-mult-def**
by metis
ultimately have $op \langle \# \rangle_{G(\text{carrier} := H)} = op \langle \# \rangle_G$ **by simp**
with grpHN have group $((G \text{ Mod } N)(\text{carrier} := (\text{rcosets}_G(\text{carrier} := H) N)))$
unfolding FactGroup-def by auto
moreover have $\text{rcosets}_G(\text{carrier} := H) N \subseteq \text{carrier } (G \text{ Mod } N)$ **unfolding**
FactGroup-def RCOSETS-def r-coset-def
using *assms(3) subgroup-imp-subset* **by fastforce**
ultimately show *?thesis* **using** *GModN.is-group group.restrict-group-imp-subgroup*
by auto
qed

A normality relation survives factoring by a normal subgroup.

lemma (in group) *normality-factorization*:

assumes $NG:N \triangleleft G$ **and** $NH:N \subseteq H$ **and** $HG:H \triangleleft G$

shows $(\text{rcosets}_G(\text{carrier} := H) N) \triangleleft (G \text{ Mod } N)$

proof –

from *assms(1)* **interpret** *GModN: group G Mod N* **by** *(metis normal.factorgroup-is-group)*

show *?thesis*

proof *(auto simp: GModN.normal-inv-iff)*

from *assms* **show** $\text{subgroup } (\text{rcosets}_G(\text{carrier} := H) N) (G \text{ Mod } N)$ **using**
normal-imp-subgroup normal-subgroup-factorize **by force**

next

fix $U V$

assume $U:U \in \text{carrier } (G \text{ Mod } N)$ **and** $V:V \in \text{rcosets}_G(\text{carrier} := H) N$

then obtain g **where** $g:g \in \text{carrier } G$ $U = N \#> g$ **unfolding** *FactGroup-def*
RCOSETS-def **by auto**

from V **obtain** h **where** $h:h \in H$ $V = N \#> h$ **unfolding** *FactGroup-def*
RCOSETS-def r-coset-def **by auto**

hence $hG:h \in \text{carrier } G$ **using** *HG normal-imp-subgroup subgroup.mem-carrier*
by force

hence $ghG:g \otimes h \in \text{carrier } G$ **using** *g m-closed* **by auto**

from $g h$ **have** $g \otimes h \otimes \text{inv } g \in H$ **using** *HG normal-inv-iff* **by auto**

moreover have $U \langle \# \rangle V \langle \# \rangle \text{inv}_G \text{ Mod } N$ $U = N \#> (g \otimes h \otimes \text{inv } g)$

proof –

from $g U$ **have** $\text{inv}_G \text{ Mod } N$ $U = N \#> \text{inv } g$ **using** *NG normal.inv-FactGroup*
normal.rcos-inv **by fastforce**

hence $U \langle \# \rangle V \langle \# \rangle \text{inv}_G \text{ Mod } N$ $U = (N \#> g) \langle \# \rangle (N \#> h) \langle \# \rangle$
 $(N \#> \text{inv } g)$ **using** *g h* **by simp**

also have $\dots = N \#> (g \otimes h) \langle \# \rangle (N \#> \text{inv } g)$ **using** *g hG NG*
normal.rcos-sum **by force**

also have $\dots = N \#> (g \otimes h \otimes \text{inv } g)$ **using** *g inv-closed ghG NG*
normal.rcos-sum **by force**

finally show *?thesis* .

qed

ultimately show $U \langle \# \rangle V \langle \# \rangle \text{inv}_G \text{ Mod } N$ $U \in \text{rcosets}_G(\text{carrier} := H)$
 N **unfolding** *RCOSETS-def r-coset-def* **by auto**

qed
qed

Factoring by a normal subgroups yields the trivial group iff the subgroup is the whole group.

lemma (in *normal*) *fact-group-trivial-iff*:

assumes *finite* (*carrier G*)

shows (*carrier* ($G \text{ Mod } H$) = $\{1_{G \text{ Mod } H}\}$) = ($H = \text{carrier } G$)

proof

assume *carrier* ($G \text{ Mod } H$) = $\{1_{G \text{ Mod } H}\}$

moreover with *assms lagrange* **have** *order* ($G \text{ Mod } H$) * *card* $H = \text{order } G$

unfolding *FactGroup-def order-def* **using** *is-subgroup* **by force**

ultimately have *card* $H = \text{order } G$ **unfolding** *order-def* **by auto**

thus $H = \text{carrier } G$ **using** *subgroup-imp-subset is-subgroup assms card-subset-eq*

unfolding *order-def*

by *metis*

next

from *assms* **have** *ordergt0:order* $G > 0$ **unfolding** *order-def* **by** (*metis subgroup.finite-imp-card-positive subgroup-self*)

assume $H = \text{carrier } G$

hence *card* $H = \text{order } G$ **unfolding** *order-def* **by simp**

with *assms is-subgroup lagrange* **have** *card* (*rcosets* H) * *order* $G = \text{order } G$ **by** *metis*

with *ordergt0* **have** *card* (*rcosets* H) = 1 **by** (*metis mult-eq-self-implies-10 mult.commute neq0-conv*)

hence *order* ($G \text{ Mod } H$) = 1 **unfolding** *order-def FactGroup-def* **by auto**

thus *carrier* ($G \text{ Mod } H$) = $\{1_{G \text{ Mod } H}\}$ **using** *factorgroup-is-group* **by** (*metis group.order-one-triv-iff*)

qed

Finite groups have finite quotients.

lemma (in *normal*) *factgroup-finite*:

assumes *finite* (*carrier G*)

shows *finite* (*rcosets H*)

using *assms* **unfolding** *RCOSETS-def* **by auto**

The union of all the cosets contained in a subgroup of a quotient group acts as a representation for that subgroup.

lemma (in *normal*) *factgroup-subgroup-union-char*:

assumes *subgroup* A ($G \text{ Mod } H$)

shows $(\bigcup A) = \{x \in \text{carrier } G. H \#> x \in A\}$

proof

show $\bigcup A \subseteq \{x \in \text{carrier } G. H \#> x \in A\}$

proof

fix x

assume $x : x \in \bigcup A$

then obtain a **where** $a : a \in A$ $x \in a$ **by auto**

with *assms* **have** $xx : x \in \text{carrier } G$ **using** *subgroup-imp-subset* **unfolding** *FactGroup-def RCOSETS-def r-coset-def* **by force**

from *assms a* **obtain** *y* **where** $y:y \in \text{carrier } G \ a = H \#> y$ **using** *subgroup-imp-subset*
unfolding *FactGroup-def RCOSETS-def* **by** *force*
with *a* **have** $x \in H \#> y$ **by** *simp*
hence $H \#> y = H \#> x$ **using** *y is-subgroup repr-independence* **by** *auto*
with $y(2) \ a(1)$ **have** $H \#> x \in A$ **by** *auto*
with xx **show** $x \in \{x \in \text{carrier } G. H \#> x \in A\}$ **by** *simp*
qed
next
show $\{x \in \text{carrier } G. H \#> x \in A\} \subseteq \bigcup A$
proof
fix *x*
assume $x:x \in \{x \in \text{carrier } G. H \#> x \in A\}$
hence $xx:x \in \text{carrier } G \ H \#> x \in A$ **by** *auto*
moreover **have** $x \in H \#> x$ **by** (*metis is-subgroup rcos-self xx(1)*)
ultimately **show** $x \in \bigcup A$ **by** *auto*
qed
qed

lemma (*in normal*) *factgroup-subgroup-union-subgroup*:
assumes *subgroup A (G Mod H)*
shows *subgroup ($\bigcup A$) G*
proof –
have *subgroup $\{x \in \text{carrier } G. H \#> x \in A\} G$*
proof
show $\{x \in \text{carrier } G. H \#> x \in A\} \subseteq \text{carrier } G$ **by** *auto*
next
fix *x y*
assume $x \in \{x \in \text{carrier } G. H \#> x \in A\}$ **and** $y \in \{x \in \text{carrier } G. H \#>$
 $x \in A\}$
hence $x:x \in \text{carrier } G \ H \#> x \in A$ **and** $y:y \in \text{carrier } G \ H \#> y \in A$ **by**
auto
hence $xyG:x \otimes y \in \text{carrier } G$ **by** (*metis m-closed*)
from *assms x y* **have** $(H \#> x) <\#> (H \#> y) \in A$ **using** *subgroup.m-closed*
unfolding *FactGroup-def* **by** *fastforce*
hence $H \#> (x \otimes y) \in A$ **by** (*metis rcos-sum x(1) y(1)*)
with xyG **show** $x \otimes y \in \{x \in \text{carrier } G. H \#> x \in A\}$ **by** *simp*
next
have $H \#> 1 \in A$ **using** *assms subgroup.one-closed* **unfolding** *FactGroup-def*
by (*metis coset-mult-one monoid.select-convs(2) subset*)
with *assms one-closed* **show** $1 \in \{x \in \text{carrier } G. H \#> x \in A\}$ **by** *simp*
next
fix *x*
assume $x \in \{x \in \text{carrier } G. H \#> x \in A\}$
hence $x:x \in \text{carrier } G \ H \#> x \in A$ **by** *auto*
hence $invx:inv \ x \in \text{carrier } G$ **using** *inv-closed* **by** *simp*
from *assms x* **have** $\text{set-inv } (H \#> x) \in A$ **using** *subgroup.m-inv-closed* **by**
(*metis inv-FactGroup subgroup.mem-carrier*)
hence $H \#> (inv \ x) \in A$ **by** (*metis rcos-inv x(1)*)
with $invx$ **show** $inv \ x \in \{x \in \text{carrier } G. H \#> x \in A\}$ **by** *simp*

```

qed
with assms factgroup-subgroup-union-char show ?thesis by auto
qed

lemma (in normal) factgroup-subgroup-union-normal:
  assumes  $A \triangleleft (G \text{ Mod } H)$ 
  shows  $\bigcup A \triangleleft G$ 
proof -
  have  $\{x \in \text{carrier } G. H \#> x \in A\} \triangleleft G$ 
  unfolding normal-def normal-axioms-def
  proof auto
    from assms show subgroup  $\{x \in \text{carrier } G. H \#> x \in A\} G$ 
    by (metis (full-types) factgroup-subgroup-union-char factgroup-subgroup-union-subgroup
normal-imp-subgroup)
  next
    show group  $G$  by (rule is-group)
  next
    interpret Anormal: normal  $A (G \text{ Mod } H)$  using assms by simp
    fix  $x y$ 
    assume  $x: x \in \text{carrier } G \ y \in \{x \in \text{carrier } G. H \#> x \in A\} \#> x$ 
    then obtain  $x'$  where  $x' \in \{x \in \text{carrier } G. H \#> x \in A\} \ y = x' \otimes x$ 
  unfolding r-coset-def by auto
    hence  $x': x' \in \text{carrier } G \ H \#> x' \in A$  by auto
    from  $x(1)$  have  $Hx: H \#> x \in \text{carrier } (G \text{ Mod } H)$  unfolding FactGroup-def
RCOSETS-def by force
    with  $x'$  have  $(\text{inv}_{G \text{ Mod } H} (H \#> x)) \otimes_{G \text{ Mod } H} (H \#> x') \otimes_{G \text{ Mod } H} (H \#> x) \in A$ 
using Anormal.inv-op-closed1 by auto
    hence  $(\text{set-inv } (H \#> x)) <\#> (H \#> x') <\#> (H \#> x) \in A$  using
inv-FactGroup Hx unfolding FactGroup-def by auto
    hence  $(H \#> (\text{inv } x)) <\#> (H \#> x') <\#> (H \#> x) \in A$  using  $x(1)$  by
(metis rcos-inv)
    hence  $(H \#> (\text{inv } x \otimes x')) <\#> (H \#> x) \in A$  by (metis inv-closed rcos-sum
 $x'(1) \ x(1)$ )
    hence  $H \#> (\text{inv } x \otimes x' \otimes x) \in A$  by (metis inv-closed m-closed rcos-sum
 $x'(1) \ x(1)$ )
    moreover have  $\text{inv } x \otimes x' \otimes x \in \text{carrier } G$  using  $x \ x'$  by (metis inv-closed
m-closed)
    ultimately have  $\text{inv } x \otimes x' \otimes x \in \{x \in \text{carrier } G. H \#> x \in A\}$  by auto
    hence  $x \text{ coset}: x \otimes (\text{inv } x \otimes x' \otimes x) \in x <\# \{x \in \text{carrier } G. H \#> x \in A\}$ 
unfolding l-coset-def using  $x(1)$  by auto
    have  $x \otimes (\text{inv } x \otimes x' \otimes x) = (x \otimes \text{inv } x) \otimes x' \otimes x$  by (metis Units-eq
Units-inv-Units m-assoc m-closed  $x'(1) \ x(1)$ )
    also have  $\dots = x' \otimes x$  by (metis l-one r-inv  $x'(1) \ x(1)$ )
    also have  $\dots = y$  by (metis  $\langle y = x' \otimes x \rangle$ )
    finally have  $x \otimes (\text{inv } x \otimes x' \otimes x) = y$ .
    with xcoset show  $y \in x <\# \{x \in \text{carrier } G. H \#> x \in A\}$  by auto
  next
    interpret Anormal: normal  $A (G \text{ Mod } H)$  using assms by simp
    fix  $x y$ 

```


assume $x: x \in \text{carrier } G \ y \in x <\# \{x \in \text{carrier } G. H \#> x \in A\}$
then obtain x' **where** $x' \in \{x \in \text{carrier } G. H \#> x \in A\}$ $y = x \otimes x'$
unfolding *l-coset-def* **by auto**
hence $x': x' \in \text{carrier } G \ H \#> x' \in A$ **by auto**
from $x(1)$ **have** $\text{inv } x: \text{inv } x \in \text{carrier } G$ **by** (*rule inv-closed*)
hence $H \text{inv } x: H \#> (\text{inv } x) \in \text{carrier } (G \text{ Mod } H)$ **unfolding** *FactGroup-def*
RCOSETS-def **by force**
with x' **have** $(\text{inv }_{G \text{ Mod } H} (H \#> \text{inv } x)) \otimes_{G \text{ Mod } H} (H \#> x') \otimes_{G \text{ Mod } H}$
 $(H \#> \text{inv } x) \in A$ **using** $\text{inv } x$ *Anormal.inv-op-closed1* **by auto**
hence $(\text{set-inv } (H \#> \text{inv } x)) <\#> (H \#> x') <\#> (H \#> \text{inv } x) \in A$
using $\text{inv-FactGroup } H \text{inv } x$ **unfolding** *FactGroup-def* **by auto**
hence $(H \#> \text{inv } (\text{inv } x)) <\#> (H \#> x') <\#> (H \#> \text{inv } x) \in A$ **using**
 $\text{inv } x$ **by** (*metis rcos-inv*)
hence $(H \#> x) <\#> (H \#> x') <\#> (H \#> \text{inv } x) \in A$ **by** (*metis inv-inv*
 $x(1)$)
hence $(H \#> (x \otimes x')) <\#> (H \#> \text{inv } x) \in A$ **by** (*metis rcos-sum* $x'(1)$
 $x(1)$)
hence $H \#> (x \otimes x' \otimes \text{inv } x) \in A$ **by** (*metis inv-closed m-closed rcos-sum*
 $x'(1) \ x(1)$)
moreover have $x \otimes x' \otimes \text{inv } x \in \text{carrier } G$ **using** $x \ x'$ **by** (*metis inv-closed*
m-closed)
ultimately have $x \otimes x' \otimes \text{inv } x \in \{x \in \text{carrier } G. H \#> x \in A\}$ **by auto**
hence $x\text{coset}:(x \otimes x' \otimes \text{inv } x) \otimes x \in \{x \in \text{carrier } G. H \#> x \in A\} \#> x$
unfolding *r-coset-def* **using** $\text{inv } x$ **by auto**
have $(x \otimes x' \otimes \text{inv } x) \otimes x = (x \otimes x') \otimes (\text{inv } x \otimes x)$ **by** (*metis Units-eq*
Units-inv-Units m-assoc m-closed $x'(1) \ x(1)$)
also have $\dots = x \otimes x'$ **using** $x(1)$ *l-inv* $x'(1)$ *m-closed r-one* **by auto**
also have $\dots = y$ **by** (*metis* $\langle y = x \otimes x' \rangle$)
finally have $x \otimes x' \otimes \text{inv } x \otimes x = y$.
with $x\text{coset}$ **show** $y \in \{x \in \text{carrier } G. H \#> x \in A\} \#> x$ **by auto**
qed
with assms **show** *?thesis* **by** (*metis (full-types) factgroup-subgroup-union-char*
normal-imp-subgroup)
qed

lemma (*in normal*) *factgroup-subgroup-union-factor*:

assumes *subgroup* $A \ (G \text{ Mod } H)$

shows $A = \text{rcosets } G(\text{carrier} := \bigcup A) \ H$

proof –

have $A = \text{rcosets } G(\text{carrier} := \{x \in \text{carrier } G. H \#> x \in A\}) \ H$

proof auto

fix U

assume $U: U \in A$

then obtain x' **where** $x': x' \in \text{carrier } G \ U = H \#> x'$ **using** assms *subgroup-imp-subset*

unfolding *FactGroup-def* *RCOSETS-def* **by force**

with U **have** $H \#> x' \in A$ **by** *simp*

with x' **show** $U \in \text{rcosets } G(\text{carrier} := \{x \in \text{carrier } G. H \#> x \in A\}) \ H$ **un-**
folding *RCOSETS-def* *r-coset-def* **by auto**

next

```

fix U
assume U:U ∈ rcosets G(⟦carrier := {x ∈ carrier G. H #> x ∈ A}⟧) H
then obtain x' where x':x' ∈ {x ∈ carrier G. H #> x ∈ A} U = H #> x'
unfolding RCOSETS-def r-coset-def by auto
hence x' ∈ carrier G H #> x' ∈ A by auto
with x' show U ∈ A by simp
qed
with assms show ?thesis using factgroup-subgroup-union-char by auto
qed

```

5 Flattening the type of group carriers

Flattening here means to convert the type of group elements from 'a set to 'a. This is possible whenever the empty set is not an element of the group.

definition *flatten* **where**

```

flatten (G::('a set, 'b) monoid-scheme) rep = (⟦carrier=(rep ' (carrier G)),
  monoid.mult=(λ x y. rep ((the-inv-into (carrier G) rep x) ⊗G (the-inv-into
(carrier G) rep y))),
  one=rep 1G ⟧)

```

lemma *flatten-set-group-hom*:

```

assumes group:group G
assumes inj:inj-on rep (carrier G)
shows rep ∈ hom G (flatten G rep)
unfolding hom-def
proof auto
fix g
assume g:g ∈ carrier G
thus rep g ∈ carrier (flatten G rep) unfolding flatten-def by auto
next
fix g h
assume g:g ∈ carrier G and h:h ∈ carrier G
hence rep g ∈ carrier (flatten G rep) rep h ∈ carrier (flatten G rep) unfolding
flatten-def by auto
hence rep g ⊗flatten G rep rep h
  = rep (the-inv-into (carrier G) rep (rep g) ⊗G the-inv-into (carrier G) rep (rep
h)) unfolding flatten-def by auto
also have ... = rep (g ⊗G h) using inj g h by (metis the-inv-into-f-f)
finally show rep (g ⊗G h) = rep g ⊗flatten G rep rep h..
qed

```

lemma *flatten-set-group*:

```

assumes group:group G
assumes inj:inj-on rep (carrier G)
shows group (flatten G rep)
proof (rule groupI)
fix x y
assume x:x ∈ carrier (flatten G rep) and y:y ∈ carrier (flatten G rep)

```

def $g \equiv \text{the-inv-into } (\text{carrier } G) \text{ rep } x$ **and** $h \equiv \text{the-inv-into } (\text{carrier } G) \text{ rep } y$
hence $x \otimes_{\text{flatten } G \text{ rep}} y = \text{rep } (g \otimes_G h)$ **unfolding** *flatten-def* **by** *auto*
moreover from *g-def h-def* **have** $g \in \text{carrier } G$ $h \in \text{carrier } G$
using *inj x y the-inv-into-into* **unfolding** *flatten-def* **by** (*metis partial-object.select-convs(1)*
subset-refl)
hence $g \otimes_G h \in \text{carrier } G$ **by** (*metis group group.is-monoid monoid.m-closed*)
hence $\text{rep } (g \otimes_G h) \in \text{carrier } (\text{flatten } G \text{ rep})$ **unfolding** *flatten-def* **by** *simp*
ultimately show $x \otimes_{\text{flatten } G \text{ rep}} y \in \text{carrier } (\text{flatten } G \text{ rep})$ **by** *simp*
next
show $\mathbf{1}_{\text{flatten } G \text{ rep}} \in \text{carrier } (\text{flatten } G \text{ rep})$ **unfolding** *flatten-def* **by** (*simp*
add: group group.is-monoid)
next
fix $x y z$
assume $x: x \in \text{carrier } (\text{flatten } G \text{ rep})$ **and** $y: y \in \text{carrier } (\text{flatten } G \text{ rep})$ **and** $z: z \in \text{carrier } (\text{flatten } G \text{ rep})$
def $g \equiv \text{the-inv-into } (\text{carrier } G) \text{ rep } x$ **and** $h \equiv \text{the-inv-into } (\text{carrier } G) \text{ rep } y$
and $k \equiv \text{the-inv-into } (\text{carrier } G) \text{ rep } z$
hence $x \otimes_{\text{flatten } G \text{ rep}} y \otimes_{\text{flatten } G \text{ rep}} z = (\text{rep } (g \otimes_G h)) \otimes_{\text{flatten } G \text{ rep}} z$
unfolding *flatten-def* **by** *auto*
also have $\dots = \text{rep } (\text{the-inv-into } (\text{carrier } G) \text{ rep } (\text{rep } (g \otimes_G h)) \otimes_G k)$ **using**
k-def **unfolding** *flatten-def* **by** *auto*
also from *g-def h-def k-def* **have** $ghkG: g \in \text{carrier } G$ $h \in \text{carrier } G$ $k \in \text{carrier } G$
using *inj x y z the-inv-into-into* **unfolding** *flatten-def* **by** *fastforce*
hence $gh: g \otimes_G h \in \text{carrier } G$ **and** $hk: h \otimes_G k \in \text{carrier } G$ **by** (*metis group*
group.is-monoid monoid.m-closed)
hence $\text{rep } (\text{the-inv-into } (\text{carrier } G) \text{ rep } (\text{rep } (g \otimes_G h)) \otimes_G k) = \text{rep } ((g \otimes_G h)$
 $\otimes_G k)$
unfolding *flatten-def* **using** *inj the-inv-into-f-f* **by** *fastforce*
also have $\dots = \text{rep } (g \otimes_G (h \otimes_G k))$ **using** *group group.is-monoid ghkG*
monoid.m-assoc **by** *fastforce*
also have $\dots = x \otimes_{\text{flatten } G \text{ rep}} (\text{rep } (h \otimes_G k))$ **unfolding** *g-def* *flatten-def*
using *hk inj the-inv-into-f-f* **by** *fastforce*
also have $\dots = x \otimes_{\text{flatten } G \text{ rep}} (y \otimes_{\text{flatten } G \text{ rep}} z)$ **unfolding** *h-def* *k-def*
flatten-def **using** *x y* **by** *force*
finally show $x \otimes_{\text{flatten } G \text{ rep}} y \otimes_{\text{flatten } G \text{ rep}} z = x \otimes_{\text{flatten } G \text{ rep}} (y \otimes_{\text{flatten } G \text{ rep}} z)$.
next
fix x
assume $x: x \in \text{carrier } (\text{flatten } G \text{ rep})$
def $g \equiv \text{the-inv-into } (\text{carrier } G) \text{ rep } x$
hence $gG: g \in \text{carrier } G$ **using** *inj x* **unfolding** *flatten-def* **using** *the-inv-into-into*
by *force*
have $\mathbf{1}_G \in (\text{carrier } G)$ **by** (*simp add: group group.is-monoid*)
hence $\text{the-inv-into } (\text{carrier } G) \text{ rep } (\mathbf{1}_{\text{flatten } G \text{ rep}}) = \mathbf{1}_G$ **unfolding** *flatten-def*
using *the-inv-into-f-f inj* **by** *force*
hence $\mathbf{1}_{\text{flatten } G \text{ rep}} \otimes_{\text{flatten } G \text{ rep}} x = \text{rep } (\mathbf{1}_G \otimes_G g)$ **unfolding** *flatten-def*
g-def **by** *simp*
also have $\dots = \text{rep } g$ **using** *gG group* **by** (*metis group.is-monoid monoid.l-one*)

also have $\dots = x$ **unfolding** $g\text{-def}$ **using** $\text{inj } x$ $f\text{-the-inv-into-}f$ **unfolding**
 flatten-def **by force**
finally show $\mathbf{1}_{\text{flatten } G \text{ rep}} \otimes_{\text{flatten } G \text{ rep}} x = x$.
next
from group inj **have** $\text{hom}:\text{rep} \in \text{hom } G (\text{flatten } G \text{ rep})$ **using** $\text{flatten-set-group-hom}$
by auto
fix x
assume $x:x \in \text{carrier } (\text{flatten } G \text{ rep})$
def $g \equiv \text{the-inv-into } (\text{carrier } G) \text{ rep } x$
hence $gG:g \in \text{carrier } G$ **using** $\text{inj } x$ **unfolding** flatten-def **using** the-inv-into-into
by force
hence $\text{inv}_G:\text{inv}_G g \in \text{carrier } G$ **by** $(\text{metis group group.inv-closed})$
hence $\text{rep } (\text{inv}_G g) \in \text{carrier } (\text{flatten } G \text{ rep})$ **unfolding** flatten-def **by auto**
moreover have $\text{rep } (\text{inv}_G g) \otimes_{\text{flatten } G \text{ rep}} x = \text{rep } (\text{inv}_G g) \otimes_{\text{flatten } G \text{ rep}}$
 $(\text{rep } g)$
unfolding $g\text{-def}$ **using** $f\text{-the-inv-into-}f$ $\text{inj } x$ **unfolding** flatten-def **by fastforce**
hence $\text{rep } (\text{inv}_G g) \otimes_{\text{flatten } G \text{ rep}} x = \text{rep } (\text{inv}_G g \otimes_G g)$
using hom **unfolding** hom-def **using** $gG \text{ inv}_G \text{ hom-def}$ **by auto**
hence $\text{rep } (\text{inv}_G g) \otimes_{\text{flatten } G \text{ rep}} x = \text{rep } \mathbf{1}_G$ **using** $\text{inv}_G gG$ **by** $(\text{metis group}$
 $\text{group.l-inv})$
hence $\text{rep } (\text{inv}_G g) \otimes_{\text{flatten } G \text{ rep}} x = \mathbf{1}_{\text{flatten } G \text{ rep}}$ **unfolding** flatten-def **by**
 auto
ultimately show $\exists y \in \text{carrier } (\text{flatten } G \text{ rep}). y \otimes_{\text{flatten } G \text{ rep}} x = \mathbf{1}_{\text{flatten } G \text{ rep}}$
by auto
qed

lemma **(in normal)** $\text{flatten-set-group-mod-inj}$:
shows $\text{inj-on } (\lambda U. \text{SOME } g. g \in U) (\text{carrier } (G \text{ Mod } H))$
proof (rule inj-onI)
fix $U V$
assume $U:U \in \text{carrier } (G \text{ Mod } H)$ **and** $V:V \in \text{carrier } (G \text{ Mod } H)$
then obtain $g h$ **where** $g:U = H \#> g$ $g \in \text{carrier } G$ **and** $h:V = H \#> h$
 $h \in \text{carrier } G$
unfolding $\text{FactGroup-def RCOSETS-def}$ **by auto**
hence $\text{notempty}:U \neq \{\}$ $V \neq \{\}$ **by** $(\text{metis empty-iff is-subgroup rcos-self})+$
assume $(\text{SOME } g. g \in U) = (\text{SOME } g. g \in V)$
with notempty **have** $(\text{SOME } g. g \in U) \in U \cap V$ **by** $(\text{metis IntI ex-in-conv}$
 $\text{someI})$
thus $U = V$ **by** $(\text{metis Int-iff } g h \text{ is-subgroup repr-independence})$
qed

lemma **(in normal)** $\text{flatten-set-group-mod}$:
shows $\text{group } (\text{flatten } (G \text{ Mod } H) (\lambda U. \text{SOME } g. g \in U))$
using $\text{factorgroup-is-group}$ $\text{flatten-set-group-mod-inj}$ **by** $(\text{rule flatten-set-group})$

lemma **(in normal)** $\text{flatten-set-group-mod-iso}$:
shows $(\lambda U. \text{SOME } g. g \in U) \in (G \text{ Mod } H) \cong (\text{flatten } (G \text{ Mod } H) (\lambda U. \text{SOME}$
 $g. g \in U))$
unfolding $\text{iso-def bij-betw-def}$

```

apply (auto)
apply (metis flatten-set-group-mod-inj factorgroup-is-group flatten-set-group-hom)
apply (rule flatten-set-group-mod-inj)
unfolding flatten-def apply (auto)
done

end

```

```

theory SimpleGroups
imports
  SubgroupsAndNormalSubgroups
  ../Secondary-Sylow/SndSylow
  SndIsomorphismGrp
begin

```

6 Simple Groups

```

locale simple-group = group +
  assumes order-gt-one:order  $G > 1$ 
  assumes no-real-normal-subgroup:  $\bigwedge H. H \triangleleft G \implies (H = \text{carrier } G \vee H = \{1\})$ 

```

```

lemma (in simple-group) is-simple-group: simple-group  $G$  by (rule simple-group-axioms)

```

Simple groups are non-trivial.

```

lemma (in simple-group) simple-not-triv: carrier  $G \neq \{1\}$  using order-gt-one
unfolding order-def by auto

```

Every group of prime order is simple

```

lemma (in group) prime-order-simple:
  assumes prime:prime (order  $G$ )
  shows simple-group  $G$ 
proof
  from prime show  $1 < \text{order } G$  unfolding prime-nat-iff by auto
next
  fix  $H$ 
  assume  $H \triangleleft G$ 
  hence  $HG: \text{subgroup } H \ G$  unfolding normal-def by simp
  hence  $\text{card } H \ \text{dvd} \ \text{order } G$  by (rule card-subgrp-dvd)
  with prime have  $\text{card } H = 1 \vee \text{card } H = \text{order } G$  unfolding prime-nat-iff by
simp
  thus  $H = \text{carrier } G \vee H = \{1\}$ 
  proof
    assume  $\text{card } H = 1$ 
    moreover from  $HG$  have  $1 \in H$  by (metis subgroup.one-closed)
    ultimately show ?thesis by (auto simp: card-Suc-eq)
  next
    assume  $\text{card } H = \text{order } G$ 

```

moreover from HG **have** $H \subseteq \text{carrier } G$ **unfolding** *subgroup-def* **by** *simp*
moreover from *prime* **have** $\text{card } (\text{carrier } G) > 1$ **unfolding** *order-def*
prime-nat-iff ..
hence *finite* $(\text{carrier } G)$ **by** $(\text{auto } \text{simp}:\text{card-ge-0-finite})$
ultimately show *?thesis* **unfolding** *order-def* **by** $(\text{metis } \text{card-subset-eq})$
qed
qed

Being simple is a property that is preserved by isomorphisms.

lemma **(in** *simple-group*) *iso-simple*:
assumes $H:\text{group } H$
assumes $\text{iso}:\varphi \in G \cong H$
shows *simple-group* H
unfolding *simple-group-def* *simple-group-axioms-def* **using** *assms(1)*
proof $(\text{auto } \text{del}:\text{equalityI})$
from *iso* **have** $\text{order } G = \text{order } H$ **unfolding** *iso-def* *order-def* **using** *bij-betw-same-card*
by *auto*
with *order-gt-one* **show** $\text{Suc } 0 < \text{order } H$ **by** *simp*
next
have $\text{inv-iso}:(\text{inv-into } (\text{carrier } G) \varphi) \in H \cong G$ **using** *iso* **by** $(\text{rule } \text{iso-sym})$
fix N
assume $NH:N \triangleleft H$ **and** $N\text{neq1}:N \neq \{1_H\}$
then interpret $N\text{normal}:\text{normal } N H$ **by** *simp*
def $M \equiv (\text{inv-into } (\text{carrier } G) \varphi) \text{ ' } N$
hence $MG:M \triangleleft G$ **using** *inv-iso* $NH H$ **by** $(\text{metis } \text{is-group } \text{iso-normal-subgroup})$
have $\text{surj}:\varphi \text{ ' carrier } G = \text{carrier } H$ **using** *iso* **unfolding** *iso-def* *bij-betw-def*
by *simp*
hence $MN:\varphi \text{ ' } M = N$ **unfolding** *M-def* **using** *Nnormal.subset image-inv-into-cancel*
by *metis*
moreover have $M \neq \{1\}$
proof $(\text{rule } \text{notI})$
assume $M = \{1\}$
hence $\varphi \text{ ' } M = \{\varphi 1\}$ **by** $(\text{metis } (\text{full-types}) \text{ image-empty } \text{ image-insert})$
hence $\varphi \text{ ' } M = \{1_H\}$ **by** $(\text{metis } (\text{lifting}) \text{ Nnormal.is-subgroup } MN \text{ calculation } \text{ singleton-iff } \text{ subgroup.one-closed})$
thus *False* **using** *Nneq1* MN **by** *simp*
qed
hence $M = \text{carrier } G$ **using** *no-real-normal-subgroup* MG **by** *auto*
ultimately show $N = \text{carrier } H$ **using** *surj* **by** *simp*
qed

As a corollary of this: Factorizing a group by itself does not result in a simple group!

lemma **(in** *group*) *self-factor-not-simple*: $\neg \text{simple-group } (G \text{ Mod } (\text{carrier } G))$
proof
assume $\text{assm}:\text{simple-group } (G \text{ Mod } (\text{carrier } G))$
have $\text{group } (G \setminus (\text{carrier } := \{1\}))$ **by** $(\text{metis } \text{subgroup-imp-group } \text{triv-subgroup})$
with assm *self-factor-iso* *simple-group.iso-simple* **have** $\text{simple-group } (G \setminus (\text{carrier } := \{1\}))$ **by** *auto*

```

    thus False using simple-group.simple-not-triv by force
qed

end

```

```

theory MaximalNormalSubgroups
imports
  SubgroupsAndNormalSubgroups
  SimpleGroups
begin

```

7 Facts about maximal normal subgroups

A maximal normal subgroup of G is a normal subgroup which is not contained in other any proper normal subgroup of G .

```

locale max-normal-subgroup = normal +
  assumes proper:  $H \neq \text{carrier } G$ 
  assumes max-normal:  $\bigwedge J. J \triangleleft G \implies J \neq H \implies J \neq \text{carrier } G \implies \neg (H \subseteq J)$ 

```

Another characterization of maximal normal subgroups: The factor group is simple.

```

theorem (in normal) max-normal-simple-quotient:
  assumes finite: finite (carrier  $G$ )
  shows max-normal-subgroup  $H \ G = \text{simple-group } (G \text{ Mod } H)$ 
proof
  assume max-normal-subgroup  $H \ G$ 
  then interpret maxH: max-normal-subgroup  $H \ G$ .
  show simple-group  $(G \text{ Mod } H)$  unfolding simple-group-def simple-group-axioms-def
  apply (rule conjI)
  apply (rule factorgroup-is-group)
  proof (auto del: equalityI)
    from finite factgroup-finite factorgroup-is-group group.finite-pos-order have
    gt0:  $0 < \text{card } (\text{rcosets } H)$ 
    unfolding FactGroup-def order-def by force
    from maxH.proper finite have carrier  $(G \text{ Mod } H) \neq \{\mathbf{1}_{G \text{ Mod } H}\}$  using
    fact-group-trivial-iff by auto
    hence  $1 \neq \text{order } (G \text{ Mod } H)$  using factorgroup-is-group group.order-one-triv-iff
  bymetis
  with gt0 show  $\text{Suc } 0 < \text{order } (G \text{ Mod } H)$  unfolding order-def FactGroup-def
by auto
  next
  fix  $A'$ 
  assume A'normal:  $A' \triangleleft G \text{ Mod } H$  and A'nottriv:  $A' \neq \{\mathbf{1}_{G \text{ Mod } H}\}$ 
  def  $A \equiv \bigcup A'$ 
  have A2:  $A \triangleleft G$  using A'normal unfolding A-def by (rule factgroup-subgroup-union-normal)

```

have $H \in A'$ **using** A' normal normal-imp-subgroup subgroup.one-closed **un-**
folding FactGroup-def **by force**
hence $H \subseteq A$ **unfolding** A-def **by auto**
hence $A1:H \triangleleft (G \setminus \text{carrier} := A)$ **using** A2 is-normal **by** (metis is-subgroup
maxH.max-normal normal-restrict-supergroup subgroup-self)
have $A3:A' = \text{rcosets}_{G \setminus \text{carrier} := A} H$
unfolding A-def **using** factgroup-subgroup-union-factor A'normal normal-imp-subgroup
by auto
from A1 **interpret** normalHA: normal H (G \setminus \text{carrier} := A) **by metis**
have $H \subseteq A$ **using** normalHA.is-subgroup subgroup-imp-subset **by force**
with A2 **have** $A = H \vee A = \text{carrier } G$ **using** maxH.max-normal **by auto**
thus $A' = \text{carrier } (G \text{ Mod } H)$
proof (rule disjE)
assume $A = H$
hence $\text{carrier } (G \setminus \text{carrier} := A) \text{ Mod } H = \{1_{(G \setminus \text{carrier} := A) \text{ Mod } H}\}$
by (metis finite is-group normalHA.fact-group-trivial-iff normalHA.subgroup-self
normalHA.subset subgroup-finite subgroup-of-restricted-group subgroup-of-subgroup
subset-antisym)
also have $\dots = \{1_{G \text{ Mod } H}\}$ **unfolding** FactGroup-def **by auto**
finally have $A' = \{1_{G \text{ Mod } H}\}$ **using** A3 **unfolding** FactGroup-def **by simp**
with A'nottriv **show** ?thesis..
next
assume $A = \text{carrier } G$
hence $(G \setminus \text{carrier} := A) \text{ Mod } H = G \text{ Mod } H$ **by auto**
thus $A' = \text{carrier } (G \text{ Mod } H)$ **using** A3 **unfolding** FactGroup-def **by simp**
qed
qed
next
assume simple:simple-group (G Mod H)
show max-normal-subgroup H G
proof
from simple **have** $\text{carrier } (G \text{ Mod } H) \neq \{1_{G \text{ Mod } H}\}$ **unfolding** simple-group-def
simple-group-axioms-def order-def **by auto**
with finite fact-group-trivial-iff **show** $H \neq \text{carrier } G$ **by auto**
next
fix A
assume $A:A \triangleleft G \ A \neq H \ A \neq \text{carrier } G$
show $\neg H \subseteq A$
proof
assume HA:H $\subseteq A$
hence $H \triangleleft (G \setminus \text{carrier} := A)$ **by** (metis A(1) inv-op-closed2 is-subgroup
normal-inv-iff normal-restrict-supergroup)
then interpret normalHA: normal H (G \setminus \text{carrier} := A) **by simp**
from finite **have** finiteA:finite A **using** A(1) normal-imp-subgroup **by** (metis
subgroup-finite)
have $\text{rcosets}_{(G \setminus \text{carrier} := A)} H \triangleleft G \text{ Mod } H$ **using** normality-factorization
is-normal HA A(1) **by auto**
with simple **have** $\text{rcosets}_{(G \setminus \text{carrier} := A)} H = \{1_{G \text{ Mod } H}\} \vee \text{rcosets}_{(G \setminus \text{carrier} := A)}$
 $H = \text{carrier } (G \text{ Mod } H)$


```

    unfolding simple-group-def simple-group-axioms-def by auto
thus False
proof
  assume  $\text{rcosets}_{G(\text{carrier} := A)} H = \{1_{G \text{ Mod } H}\}$ 
  hence  $\text{rcosets}_{G(\text{carrier} := A)} H = \{1_{(G(\text{carrier} := A)) \text{ Mod } H}\}$  unfolding
FactGroup-def by auto
  with finiteA have  $H = A$  using normalHA.fact-group-trivial-iff unfolding
FactGroup-def by auto
  with  $A(2)$  show ?thesis by simp
next
  assume  $AHGH:\text{rcosets}_{G(\text{carrier} := A)} H = \text{carrier } (G \text{ Mod } H)$ 
  have  $A = \text{carrier } G$  unfolding FactGroup-def RCOSETS-def
  proof
  show  $A \subseteq \text{carrier } G$  using  $A(1)$  normal-imp-subgroup subgroup-imp-subset
by metis
  next
  show  $\text{carrier } G \subseteq A$ 
  proof
  fix  $x$ 
  assume  $x:x \in \text{carrier } G$ 
  hence  $H \#> x \in \text{rcosets } H$  unfolding RCOSETS-def by auto
  with  $AHGH$  have  $H \#> x \in \text{rcosets}_{G(\text{carrier} := A)} H$  unfolding
FactGroup-def by simp
  then obtain  $x'$  where  $x':x' \in A \ H \#>x = H \#>_{G(\text{carrier} := A)} x'$ 
unfolding RCOSETS-def by auto
  hence  $H \#> x = H \#> x'$  unfolding r-coset-def by auto
  hence  $x \in H \#> x'$  by (metis is-subgroup rcos-self x)
  hence  $x \in A \#> x'$  using  $HA$  unfolding r-coset-def by auto
  thus  $x \in A$  using  $x'(1)$  unfolding r-coset-def using subgroup.m-closed
A(1) normal-imp-subgroup by force
  qed
  qed
  with  $A(3)$  show ?thesis by simp
  qed
  qed
  qed
  qed
end

```

```

theory CompositionSeries
imports
  SimpleGroups
  MaximalNormalSubgroups
begin

```

8 Normal series and Composition series

8.1 Preliminaries

A subgroup which is unique in cardinality is normal:

lemma (in group) *unique-sizes-subgrp-normal*:
assumes *fin:finite* (carrier G)
assumes $\exists!Q. Q \in \text{subgroups-of-size } q$
shows (*THE* $Q. Q \in \text{subgroups-of-size } q$) $\triangleleft G$
proof –
from *assms* **obtain** Q **where** $Q \in \text{subgroups-of-size } q$ **by** *auto*
def $Q \equiv \text{THE } Q. Q \in \text{subgroups-of-size } q$
with *assms* **have** $Q\text{size}: Q \in \text{subgroups-of-size } q$ **using** *theI* **by** *metis*
hence $QG:\text{subgroup } Q G$ **and** $\text{card}Q:\text{card } Q = q$ **unfolding** *subgroups-of-size-def*
by *auto*
from QG **have** $Q \triangleleft G$ **apply**(*rule normalI*)
proof
fix g
assume $g:g \in \text{carrier } G$
hence $\text{inv}g:\text{inv } g \in \text{carrier } G$ **by** (*metis inv-closed*)
with *fin* $Q\text{size}$ **have** *conjugation-action* q ($\text{inv } g$) $Q \in \text{subgroups-of-size } q$ **by**
(*metis conjugation-is-size-invariant*)
with g $Q\text{size}$ **have** $(\text{inv } g) <\# (Q \#> \text{inv } (\text{inv } g)) \in \text{subgroups-of-size } q$
unfolding *conjugation-action-def* **by** *auto*
with $\text{inv}g$ g **have** $\text{inv } g <\# (Q \#> g) = Q$ **by** (*metis Qsize assms(2) inv-inv*)
with QG QG g **show** $Q \#> g = g <\# Q$ **by** (*rule conj-wo-inv*)
qed
with $Q\text{-def}$ **show** *?thesis* **by** *simp*
qed

A group whose order is the product of two distinct primes p and q where $p < q$ has a unique subgroup of size q :

lemma (in group) *pq-order-unique-subgrp*:
assumes *finite:finite* (carrier G)
assumes *orderG:order* $G = q * p$
assumes *primep:prime* p **and** *primeq:prime* q **and** $pq:p < q$
shows $\exists!Q. Q \in (\text{subgroups-of-size } q)$
proof –
from *primep* *primeq* pq **have** $nqdvdp:\neg (q \text{ dvd } p)$ **by** (*metis less-not-refl3 prime-nat-iff*)
def $\text{cal}M \equiv \{s. s \subseteq \text{carrier } G \wedge \text{card } s = q \wedge 1\}$
def $\text{Rel}M \equiv \{(N1, N2). N1 \in \text{cal}M \wedge N2 \in \text{cal}M \wedge (\exists g \in \text{carrier } G. N1 = N2 \#> g)\}$
interpret *syl: snd-sylow* G q 1 p $\text{cal}M$ $\text{Rel}M$
unfolding *snd-sylow-def* *sylow-def* *snd-sylow-axioms-def* *sylow-axioms-def*
using *is-group* *primeq* *orderG* *finite* *nqdvdp* *calM-def* *RelM-def* **by** *auto*
obtain Q **where** $Q:Q \in \text{subgroups-of-size } q$ **by** (*metis (lifting, mono-tags)*
mem-Collect-eq *power-one-right* *subgroups-of-size-def* *syl.sylow-thm*)
thus *?thesis*
proof (*rule ex1I*)

```

fix P
assume P:P ∈ subgroups-of-size q
have card (subgroups-of-size q) mod q = 1 by (metis power-one-right syl.p-sylow-mod-p)

    moreover have card (subgroups-of-size q) dvd p by (metis power-one-right
    syl.num-sylow-dvd-remainder)
    ultimately have card (subgroups-of-size q) = 1 using pq primep by (metis
    Divides.mod-less prime-nat-iff)
    with Q P show P = Q by (auto simp:card-Suc-eq)
qed
qed

```

... And this unique subgroup is normal.

```

corollary (in group) pq-order-subgrp-normal:
  assumes finite:finite (carrier G)
  assumes orderG:order G = q * p
  assumes primep:prime p and primeq:prime q and pq:p < q
  shows (THE Q. Q ∈ subgroups-of-size q) ◁ G
using asms by (metis pq-order-unique-subgrp unique-sizes-subgrp-normal)

```

The trivial subgroup is normal in every group.

```

lemma (in group) trivial-subgroup-is-normal:
  shows {1} ◁ G
unfolding normal-def normal-axioms-def r-coset-def l-coset-def by (auto intro:
  normalI subgroupI simp: is-group)

```

8.2 Normal Series

We define a normal series as a locale which fixes one group G and a list \mathfrak{G} of subsets of G 's carrier. This list must begin with the trivial subgroup, end with the carrier of the group itself and each of the list items must be a normal subgroup of its successor.

```

locale normal-series = group +
  fixes  $\mathfrak{G}$ 
  assumes notempty: $\mathfrak{G} \neq []$ 
  assumes hd:hd  $\mathfrak{G} = \{1\}$ 
  assumes last:last  $\mathfrak{G} = \text{carrier } G$ 
  assumes normal: $\bigwedge i. i + 1 < \text{length } \mathfrak{G} \implies (\mathfrak{G} ! i) \triangleleft G(\text{carrier} := \mathfrak{G} ! (i + 1))$ 

```

lemma (in normal-series) is-normal-series: normal-series $G \ \mathfrak{G}$ **by** (rule normal-series-axioms)

For every group there is a "trivial" normal series consisting only of the group itself and its trivial subgroup.

```

lemma (in group) trivial-normal-series:
  shows normal-series G [{1}, carrier G]
unfolding normal-series-def normal-series-axioms-def
using is-group trivial-subgroup-is-normal by auto

```

We can also show that the normal series presented above is the only such with a length of two:

```

lemma (in normal-series) length-two-unique:
  assumes length  $\mathfrak{G} = 2$ 
  shows  $\mathfrak{G} = [\{1\}, \text{carrier } G]$ 
proof(rule nth-equalityI)
  from assms show length  $\mathfrak{G} = \text{length } [\{1\}, \text{carrier } G]$  by auto
next
  show  $\forall i < \text{length } \mathfrak{G}. \mathfrak{G} ! i = [\{1\}, \text{carrier } G] ! i$ 
  proof(rule allI, rule impI)
    fix  $i$ 
    assume  $i : i < \text{length } \mathfrak{G}$ 
    with assms have  $i = 0 \vee i = 1$  by auto
    thus  $\mathfrak{G} ! i = [\{1\}, \text{carrier } G] ! i$ 
    proof(rule disjE)
      assume  $i : i = 0$ 
      hence  $\mathfrak{G} ! i = \text{hd } \mathfrak{G}$  by (metis hd-conv-nth notempty)
      thus  $\mathfrak{G} ! i = [\{1\}, \text{carrier } G] ! i$  using hd  $i$  by simp
    next
      assume  $i : i = 1$ 
      with assms have  $\mathfrak{G} ! i = \text{last } \mathfrak{G}$  by (metis diff-add-inverse last-conv-nth
nat-1-add-1 notempty)
      thus  $\mathfrak{G} ! i = [\{1\}, \text{carrier } G] ! i$  using last  $i$  by simp
    qed
  qed
qed

```

We can construct new normal series by expanding existing ones: If we append the carrier of a group G to a normal series for a normal subgroup $H \triangleleft G$ we receive a normal series for G .

```

lemma (in group) normal-series-extend:
  assumes normal:normal-series ( $G(\text{carrier} := H)$ )  $\mathfrak{H}$ 
  assumes  $HG : H \triangleleft G$ 
  shows normal-series  $G$  ( $\mathfrak{H} @ [\text{carrier } G]$ )
proof –
  from normal interpret normalH: normal-series ( $G(\text{carrier} := H)$ )  $\mathfrak{H}$ .
  from normalH.hd have hd  $\mathfrak{H} = \{1\}$  by simp
  with normalH.notempty have hdTriv:hd ( $\mathfrak{H} @ [\text{carrier } G]$ ) =  $\{1\}$  by (metis
hd-append2)
  show ?thesis unfolding normal-series-def normal-series-axioms-def using is-group
  proof auto
    fix  $x$ 
    assume  $x \in \text{hd } (\mathfrak{H} @ [\text{carrier } G])$ 
    with hdTriv show  $x = 1$  by simp
  next
    from hdTriv show  $1 \in \text{hd } (\mathfrak{H} @ [\text{carrier } G])$  by simp
  next
    fix  $i$ 
    assume  $i : i < \text{length } \mathfrak{H}$ 

```

```

show ( $\mathfrak{H} @ [carrier\ G] ! i < G(\text{carrier} := \mathfrak{H} @ [carrier\ G] ! Suc\ i)$ )
proof (cases  $i + 1 < length\ \mathfrak{H}$ )
  case True
    with normalH.normal have  $\mathfrak{H} ! i < G(\text{carrier} := \mathfrak{H} ! (i + 1))$  by auto
    with  $i$  have ( $\mathfrak{H} @ [carrier\ G] ! i < G(\text{carrier} := \mathfrak{H} ! (i + 1))$ ) using
nth-append by metis
    with True show ( $\mathfrak{H} @ [carrier\ G] ! i < G(\text{carrier} := \mathfrak{H} @ [carrier\ G] !$ 
(Suc  $i))$ ) using nth-append Suc-eq-plus1 by metis
  next
    case False
    with  $i$  have  $i2:i + 1 = length\ \mathfrak{H}$  by simp
    from  $i$  have ( $\mathfrak{H} @ [carrier\ G] ! i = \mathfrak{H} ! i$ ) by (metis nth-append)
    also from  $i2$  normalH.notempty have  $\dots = last\ \mathfrak{H}$  by (metis add-diff-cancel-right'
last-conv-nth)
    also from normalH.last have  $\dots = H$  by simp
    finally have ( $\mathfrak{H} @ [carrier\ G] ! i = H$ ).
    moreover from  $i2$  have ( $\mathfrak{H} @ [carrier\ G] ! (i + 1) = carrier\ G$ ) by (metis
nth-append-length)
    ultimately show ?thesis using HG by auto
  qed
qed
qed

```

All entries of a normal series for G are subgroups of G .

lemma (*in normal-series*) *normal-series-subgroups*:

shows $i < length\ \mathfrak{G} \implies subgroup\ (\mathfrak{G} ! i)\ G$

proof –

have $i + 1 < length\ \mathfrak{G} \implies subgroup\ (\mathfrak{G} ! i)\ G$

proof (*induction* $length\ \mathfrak{G} - (i + 2)$ *arbitrary: i*)

case 0

hence $i:i + 2 = length\ \mathfrak{G}$ **by** *simp*

hence $ii:i + 1 = length\ \mathfrak{G} - 1$ **by** *force*

from i *normal* **have** $\mathfrak{G} ! i < G(\text{carrier} := \mathfrak{G} ! (i + 1))$ **by** *auto*

with ii *last* *notempty* **show** $subgroup\ (\mathfrak{G} ! i)\ G$ **using** *last-conv-nth* *normal-imp-subgroup*
by *fastforce*

next

case (*Suc* k)

from *Suc*(3) *normal* **have** $i:subgroup\ (\mathfrak{G} ! i)\ (G(\text{carrier} := \mathfrak{G} ! (i + 1)))$

using *normal-imp-subgroup* **by** *auto*

from *Suc*(2) **have** $k:k = length\ \mathfrak{G} - ((i + 1) + 2)$ **by** *arith*

with *Suc* **have** $subgroup\ (\mathfrak{G} ! (i + 1))\ G$ **by** *simp*

with i **show** $subgroup\ (\mathfrak{G} ! i)\ G$ **by** (*metis* *is-group* *subgroup.subgroup-of-subgroup*)

qed

moreover **have** $i + 1 = length\ \mathfrak{G} \implies subgroup\ (\mathfrak{G} ! i)\ G$

using *last* *notempty* *last-conv-nth* **by** (*metis* *add-diff-cancel-right'* *subgroup-self*)

ultimately **show** $i < length\ \mathfrak{G} \implies subgroup\ (\mathfrak{G} ! i)\ G$ **by** *force*

qed

The second to last entry of a normal series is a normal subgroup of G .

```

lemma (in normal-series) normal-series-snd-to-last:
  shows  $\mathfrak{G} ! (\text{length } \mathfrak{G} - 2) \triangleleft G$ 
proof (cases  $2 \leq \text{length } \mathfrak{G}$ )
  case False
    with notempty have length:length  $\mathfrak{G} = 1$  by (metis Suc-eq-plus1 leI length-0-conv
less-2-cases plus-nat.add-0)
    with hd have  $\mathfrak{G} ! (\text{length } \mathfrak{G} - 2) = \{1\}$  using hd-conv-nth notempty by auto
    with length show ?thesis by (metis trivial-subgroup-is-normal)
  next
  case True
    hence  $(\text{length } \mathfrak{G} - 2) + 1 < \text{length } \mathfrak{G}$  by arith
    with normal last have  $\mathfrak{G} ! (\text{length } \mathfrak{G} - 2) \triangleleft G(\text{carrier} := \mathfrak{G} ! ((\text{length } \mathfrak{G} - 2)
+ 1))$  by auto
    have  $1 + (1 + (\text{length } \mathfrak{G} - (1 + 1))) = \text{length } \mathfrak{G}$ 
      using True le-add-diff-inverse by presburger
    then have  $\mathfrak{G} ! (\text{length } \mathfrak{G} - 2) \triangleleft G(\text{carrier} := \mathfrak{G} ! (\text{length } \mathfrak{G} - 1))$ 
      by (metis  $\mathfrak{G} ! (\text{length } \mathfrak{G} - 2) \triangleleft G(\text{carrier} := \mathfrak{G} ! (\text{length } \mathfrak{G} - 2 + 1))$ )
      add commute add-diff-cancel-left' one-add-one)
    with notempty last show ?thesis using last-conv-nth by force
qed

```

Just like the expansion of normal series, every prefix of a normal series is again a normal series.

```

lemma (in normal-series) normal-series-prefix-closed:
  assumes  $i \leq \text{length } \mathfrak{G}$  and  $0 < i$ 
  shows normal-series  $(G(\text{carrier} := \mathfrak{G} ! (i - 1)))$  (take  $i$   $\mathfrak{G}$ )
unfolding normal-series-def normal-series-axioms-def
using assms
apply (auto del:equalityI)
apply (metis diff-Suc-Suc diff-is-0-eq' gr-implies-not0 minus-nat.diff-0 normal-series-subgroups
not0-implies-Suc not-less-eq subgroup-imp-group zero-less-diff)
proof -
  from assms have hd (take  $i$   $\mathfrak{G}$ ) = (take  $i$   $\mathfrak{G}$ ) ! 0 by (metis gr-implies-not0
hd-conv-nth notempty take-eq-Nil)
  also from assms have ... =  $\mathfrak{G} ! 0$  by (metis nth-take)
  also from hd have ... =  $\{1\}$  by (metis hd-conv-nth notempty)
  finally show hd (take  $i$   $\mathfrak{G}$ ) =  $\{1\}$ .
next
  from assms have last (take  $i$   $\mathfrak{G}$ ) = (take  $i$   $\mathfrak{G}$ ) ! (length (take  $i$   $\mathfrak{G}$ ) - 1) by
(metis last-conv-nth neq0-conv notempty take-eq-Nil)
  also from assms have ... = (take  $i$   $\mathfrak{G}$ ) ! (i - 1) by (metis length-take min.absorb2)
  also from assms have ... =  $\mathfrak{G} ! (i - 1)$  by (metis diff-less nth-take zero-less-one)
  finally show last (take  $i$   $\mathfrak{G}$ ) =  $\mathfrak{G} ! (i - \text{Suc } 0)$  by simp
next
  fix  $j$ 
  assume  $j:\text{Suc } j < i$ 
  hence  $j + 1 < \text{length } \mathfrak{G}$  using assms by simp
  with normal show  $\mathfrak{G} ! j \triangleleft G(\text{carrier} := \mathfrak{G} ! (\text{Suc } j))$  by auto
qed

```

If a group's order is the product of two distinct primes p and q , where $p < q$, we can construct a normal series using the only subgroup of size q .

lemma (in *group*) *pq-order-normal-series*:

assumes *finite:finite* (*carrier* G)

assumes *orderG:order* $G = q * p$

assumes *primep:prime* p **and** *primeq:prime* q **and** *pq:p < q*

shows *normal-series* G $[\{1\}]$, (*THE* H . $H \in$ *subgroups-of-size* q), *carrier* G]

proof –

def $H \equiv$ (*THE* H . $H \in$ *subgroups-of-size* q)

with *assms* **have** *HG:H* \triangleleft G **by** (*metis pq-order-subgrp-normal*)

then interpret *groupH: group* G (*carrier* := H) **unfolding** *normal-def* **by**
(*metis subgroup-imp-group*)

have *normal-series* (G (*carrier* := H)) $[\{1\}]$, H **using** *groupH.trivial-normal-series*

by *auto*

with *HG* **show** *?thesis* **unfolding** *H-def* **by** (*metis append-Cons append-Nil normal-series-extend*)

qed

The following defines the list of all quotient groups of the normal series:

definition (in *normal-series*) *quotients*

where *quotients* = *map* (λi . G (*carrier* := $\mathfrak{G} ! (i + 1)$) *Mod* $\mathfrak{G} ! i$) $[0..<((\text{length } \mathfrak{G}) - 1)]$

The list of quotient groups has one less entry than the series itself:

lemma (in *normal-series*) *quotients-length*:

shows *length* *quotients* + 1 = *length* \mathfrak{G}

proof –

have *length* *quotients* + 1 = *length* $[0..<((\text{length } \mathfrak{G}) - 1)]$ + 1 **unfolding**
quotients-def **by** *simp*

also have ... = $(\text{length } \mathfrak{G} - 1) + 1$ **by** (*metis diff-zero length-upt*)

also with *notempty* **have** ... = *length* \mathfrak{G}

by (*simp add: ac-simps*)

finally show *?thesis* .

qed

lemma (in *normal-series*) *last-quotient*:

assumes *length* $\mathfrak{G} > 1$

shows *last* *quotients* = G *Mod* $\mathfrak{G} ! (\text{length } \mathfrak{G} - 1 - 1)$

proof –

from *assms* **have** *lsimp:length* $\mathfrak{G} - 1 - 1 + 1 = \text{length } \mathfrak{G} - 1$ **by** *auto*

from *assms* **have** *quotients* $\neq []$ **unfolding** *quotients-def* **by** *auto*

hence *last* *quotients* = *quotients* ! $(\text{length } \text{quotients} - 1)$ **by** (*metis last-conv-nth*)

also have ... = *quotients* ! $(\text{length } \mathfrak{G} - 1 - 1)$ **by** (*metis add-diff-cancel-left'*
quotients-length add.commute)

also have ... = G (*carrier* := $\mathfrak{G} ! ((\text{length } \mathfrak{G} - 1 - 1) + 1)$) *Mod* $\mathfrak{G} ! (\text{length } \mathfrak{G} - 1 - 1)$

unfolding *quotients-def* **using** *assms* **by** *auto*

also have ... = G (*carrier* := $\mathfrak{G} ! (\text{length } \mathfrak{G} - 1)$) *Mod* $\mathfrak{G} ! (\text{length } \mathfrak{G} - 1 - 1)$ **using** *lsimp* **by** *simp*

also have $\dots = G \text{ Mod } \mathfrak{G}! (\text{length } \mathfrak{G} - 1 - 1)$ **using** *last last-conv-nth notempty*
by *force*
finally show *?thesis* .
qed

The next lemma transports the constituting properties of a normal series along an isomorphism of groups.

lemma (*in normal-series*) *normal-series-iso*:
assumes $H:\text{group } H$
assumes $\text{iso}:\Psi \in G \cong H$
shows *normal-series* H (*map* (*image* Ψ) \mathfrak{G})
apply (*simp add: normal-series-def normal-series-axioms-def*)
using H *notempty* **apply** *simp*
proof (*rule conjI*)
from H *is-group iso* **have** *group-hom:group-hom* $G H \Psi$ **unfolding** *group-hom-def*
group-hom-axioms-def iso-def **by** *auto*
have $\text{hd} (\text{map} (\text{image } \Psi) \mathfrak{G}) = \Psi \text{ ' } \{1\}$ **by** (*metis hd-map hd notempty*)
also have $\dots = \{\Psi 1\}$ **by** (*metis image-empty image-insert*)
also have $\dots = \{1_H\}$ **using** *group-hom group-hom.hom-one* **by** *auto*
finally show $\text{hd} (\text{map} (\text{op ' } \Psi) \mathfrak{G}) = \{1_H\}$.
next
show $\text{last} (\text{map} (\text{op ' } \Psi) \mathfrak{G}) = \text{carrier } H \wedge (\forall i. \text{Suc } i < \text{length } \mathfrak{G} \longrightarrow \Psi \text{ ' } \mathfrak{G} !$
 $i \triangleleft H(\text{carrier} := \Psi \text{ ' } \mathfrak{G} ! \text{Suc } i))$
proof (*auto del: equalityI*)
have $\text{last} (\text{map} (\text{op ' } \Psi) \mathfrak{G}) = \Psi \text{ ' } (\text{carrier } G)$ **using** *last last-map notempty*
by *metis*
also have $\dots = \text{carrier } H$ **using** *iso unfolding iso-def bij-betw-def* **by** *simp*
finally show $\text{last} (\text{map} (\text{op ' } \Psi) \mathfrak{G}) = \text{carrier } H$.
next
fix i
assume $i:\text{Suc } i < \text{length } \mathfrak{G}$
hence $\text{norm}:\mathfrak{G} ! i \triangleleft G(\text{carrier} := \mathfrak{G} ! \text{Suc } i)$ **using** *normal* **by** *simp*
moreover have $\text{restrict } \Psi (\mathfrak{G} ! \text{Suc } i) \in (G(\text{carrier} := \mathfrak{G} ! \text{Suc } i)) \cong H(\text{carrier}$
 $:= \Psi \text{ ' } \mathfrak{G} ! \text{Suc } i)$
by (*metis H i is-group iso iso-restrict normal-series-subgroups*)
moreover have *group* $(G(\text{carrier} := \mathfrak{G} ! \text{Suc } i))$ **by** (*metis i normal-series-subgroups*
subgroup-imp-group)
moreover hence *subgroup* $(\mathfrak{G} ! \text{Suc } i) G$ **by** (*metis i normal-series-subgroups*)
hence *subgroup* $(\Psi \text{ ' } \mathfrak{G} ! \text{Suc } i) H$ **by** (*metis H is-group iso iso-subgroup*)
hence *group* $(H(\text{carrier} := \Psi \text{ ' } \mathfrak{G} ! \text{Suc } i))$ **by** (*metis H subgroup.subgroup-is-group*)
ultimately have $\text{restrict } \Psi (\mathfrak{G} ! \text{Suc } i) \text{ ' } \mathfrak{G} ! i \triangleleft H(\text{carrier} := \Psi \text{ ' } \mathfrak{G} ! \text{Suc } i)$
using *is-group H iso-normal-subgroup* **by** *auto*
moreover from norm **have** $\mathfrak{G} ! i \subseteq \mathfrak{G} ! \text{Suc } i$ **unfolding** *normal-def subgroup-def*
by *auto*
hence $\{y. \exists x \in \mathfrak{G} ! i. y = (\text{if } x \in \mathfrak{G} ! \text{Suc } i \text{ then } \Psi x \text{ else undefined})\} = \{y.$
 $\exists x \in \mathfrak{G} ! i. y = \Psi x\}$ **by** *auto*
ultimately show $\Psi \text{ ' } \mathfrak{G} ! i \triangleleft H(\text{carrier} := \Psi \text{ ' } \mathfrak{G} ! \text{Suc } i)$ **unfolding**
restrict-def image-def **by** *auto*
qed

qed

8.3 Composition Series

A composition series is a normal series where all consecutive factor groups are simple:

locale *composition-series* = *normal-series* +
assumes *simplefact*: $\bigwedge i. i + 1 < \text{length } \mathfrak{G} \implies \text{simple-group } (G \langle \text{carrier} := \mathfrak{G} ! (i + 1) \rangle \text{ Mod } \mathfrak{G} ! i)$

lemma (*in composition-series*) *is-composition-series*:
shows *composition-series* $G \ \mathfrak{G}$
by (*rule composition-series-axioms*)

A composition series for a group G has length one if and only if G is the trivial group.

lemma (*in composition-series*) *composition-series-length-one*:
shows $(\text{length } \mathfrak{G} = 1) = (\mathfrak{G} = [\{1\}])$
proof
assume $\text{length } \mathfrak{G} = 1$
with *hd* **have** $\text{length } \mathfrak{G} = \text{length } [\{1\}] \wedge (\forall i < \text{length } \mathfrak{G}. \mathfrak{G} ! i = [\{1\}] ! i)$
using *hd-conv-nth notempty* **by** *force*
thus $\mathfrak{G} = [\{1\}]$ **using** *list-eq-iff-nth-eq* **by** *blast*
next
assume $\mathfrak{G} = [\{1\}]$
thus $\text{length } \mathfrak{G} = 1$ **by** *simp*
qed

lemma (*in composition-series*) *composition-series-triv-group*:
shows $(\text{carrier } G = \{1\}) = (\mathfrak{G} = [\{1\}])$
proof
assume $G:\text{carrier } G = \{1\}$
have $\text{length } \mathfrak{G} = 1$
proof (*rule ccontr*)
assume $\text{length } \mathfrak{G} \neq 1$
with *notempty* **have** $\text{length}:\text{length } \mathfrak{G} \geq 2$ **by** (*metis Suc-eq-plus1 length-0-conv less-2-cases not-less plus-nat.add-0*)
with *simplefact hd hd-conv-nth notempty* **have** *simple-group* $(G \langle \text{carrier} := \mathfrak{G} ! 1 \rangle \text{ Mod } \{1\})$ **by** *force*
moreover **have** *SG:subgroup* $(\mathfrak{G} ! 1) \ G$ **using** *length normal-series-subgroups*
by *auto*
hence *group* $(G \langle \text{carrier} := \mathfrak{G} ! 1 \rangle)$ **by** (*metis subgroup-imp-group*)
ultimately **have** *simple-group* $(G \langle \text{carrier} := \mathfrak{G} ! 1 \rangle)$ **using** *group.trivial-factor-iso simple-group.iso-simple* **by** *fastforce*
moreover **from** *SG* G **have** $\text{carrier } (G \langle \text{carrier} := \mathfrak{G} ! 1 \rangle) = \{1\}$ **unfolding**
subgroup-def **by** *auto*
ultimately **show** *False* **using** *simple-group.simple-not-triv* **by** *force*
qed

```

thus  $\mathfrak{G} = [\{1\}]$  by (metis composition-series-length-one)
next
  assume  $\mathfrak{G} = [\{1\}]$ 
  with last show carrier  $G = \{1\}$  by auto
qed

```

The inner elements of a composition series may not consist of the trivial subgroup or the group itself.

lemma (*in composition-series*) *inner-elements-not-triv*:

```

assumes  $i + 1 < \text{length } \mathfrak{G}$ 
assumes  $i > 0$ 
shows  $\mathfrak{G} ! i \neq \{1\}$ 
proof
  from assms have  $(i - 1) + 1 < \text{length } \mathfrak{G}$  by simp
  hence simple:simple-group  $(G \langle \text{carrier} := \mathfrak{G} ! ((i - 1) + 1) \rangle \text{Mod } \mathfrak{G} ! (i - 1))$ 
using simplefact by auto
  assume  $i : \mathfrak{G} ! i = \{1\}$ 
  moreover from assms have  $(i - 1) + 1 = i$  by auto
  ultimately have  $G \langle \text{carrier} := \mathfrak{G} ! ((i - 1) + 1) \rangle \text{Mod } \mathfrak{G} ! (i - 1) = G \langle \text{carrier} := \{1\} \rangle \text{Mod } \mathfrak{G} ! (i - 1)$  using i by auto
  hence  $\text{order } (G \langle \text{carrier} := \mathfrak{G} ! ((i - 1) + 1) \rangle \text{Mod } \mathfrak{G} ! (i - 1)) = 1$  unfolding FactGroup-def order-def RCOSETS-def by force
  thus False using i simple unfolding simple-group-def simple-group-axioms-def by auto
qed

```

A composition series of a simple group always is its trivial one.

lemma (*in composition-series*) *composition-series-simple-group*:

```

shows  $(\text{simple-group } G) = (\mathfrak{G} = [\{1\}], \text{carrier } G)$ 
proof
  assume  $\mathfrak{G} = [\{1\}, \text{carrier } G]$ 
  with simplefact have simple-group  $(G \text{Mod } \{1\})$  by auto
  moreover have  $\text{the-elem } \in (G \text{Mod } \{1\}) \cong G$  by (rule trivial-factor-iso)
  ultimately show simple-group  $G$  by (metis is-group simple-group.iso-simple)
next
  assume simple:simple-group  $G$ 
  have  $\text{length } \mathfrak{G} > 1$ 
  proof (rule ccontr)
    assume  $\neg 1 < \text{length } \mathfrak{G}$ 
    hence  $\text{length } \mathfrak{G} = 1$  by (simp add: Suc-leI antisym notempty)
    hence carrier  $G = \{1\}$  using hd last by (metis composition-series-length-one composition-series-triv-group)
    hence  $\text{order } G = 1$  unfolding order-def by auto
    with simple show False unfolding simple-group-def simple-group-axioms-def
  by auto
  qed
  moreover have  $\text{length } \mathfrak{G} \leq 2$ 
  proof (rule ccontr)
    def  $k \equiv \text{length } \mathfrak{G} - 2$ 

```

```

    assume  $\neg$  (length  $\mathfrak{G} \leq 2$ )
    hence gt2:length  $\mathfrak{G} > 2$  by simp
    hence ksmall:k + 1 < length  $\mathfrak{G}$  unfolding k-def by auto
    from gt2 have carrier: $\mathfrak{G} ! (k + 1) = \text{carrier } G$  using notempty last last-conv-nth
    k-def
      by (metis Nat.add-diff-assoc Nat.diff-cancel  $\langle \neg \text{length } \mathfrak{G} \leq 2 \rangle$  add commute
    nat-le-linear one-add-one)
      from normal ksmall have  $\mathfrak{G} ! k \triangleleft G \langle \text{carrier} := \mathfrak{G} ! (k + 1) \rangle$  by simp
      from simplefact ksmall have simplek:simple-group ( $G \langle \text{carrier} := \mathfrak{G} ! (k + 1) \rangle$ )
    Mod  $\mathfrak{G} ! k$ ) by simp
      from simplefact ksmall have simplek':simple-group ( $G \langle \text{carrier} := \mathfrak{G} ! ((k - 1) + 1) \rangle$ )
    Mod  $\mathfrak{G} ! (k - 1)$ ) by auto
      have  $\mathfrak{G} ! k \triangleleft G$  using carrier k-def gt2 normal ksmall by force
      with simple have  $(\mathfrak{G} ! k) = \text{carrier } G \vee (\mathfrak{G} ! k) = \{1\}$  unfolding simple-group-def
    simple-group-axioms-def by simp
      thus False
      proof (rule disjE)
        assume  $\mathfrak{G} ! k = \text{carrier } G$ 
        hence  $G \langle \text{carrier} := \mathfrak{G} ! (k + 1) \rangle \text{ Mod } \mathfrak{G} ! k = G \text{ Mod } (\text{carrier } G)$  using
    carrier by auto
        with simplek self-factor-not-simple show False by auto
      next
        assume  $\mathfrak{G} ! k = \{1\}$ 
        with ksmall k-def gt2 show False using inner-elements-not-triv by auto
      qed
    qed
    ultimately have length  $\mathfrak{G} = 2$  by simp
    thus  $\mathfrak{G} = [\{1\}, \text{carrier } G]$  by (rule length-two-unique)
  qed

```

Two consecutive elements in a composition series are distinct.

lemma (in *composition-series*) *entries-distinct*:

```

  assumes finite:finite (carrier  $G$ )
  assumes i:i + 1 < length  $\mathfrak{G}$ 
  shows  $\mathfrak{G} ! i \neq \mathfrak{G} ! (i + 1)$ 
  proof
    from finite have finite ( $\mathfrak{G} ! (i + 1)$ )
      using i normal-series-subgroups subgroup-imp-subset rev-finite-subset by metis
    hence fin:finite (carrier ( $G \langle \text{carrier} := \mathfrak{G} ! (i + 1) \rangle$ )) by auto
    from i have norm: $\mathfrak{G} ! i \triangleleft (G \langle \text{carrier} := \mathfrak{G} ! (i + 1) \rangle)$  by (rule normal)
    assume  $\mathfrak{G} ! i = \mathfrak{G} ! (i + 1)$ 
    hence  $\mathfrak{G} ! i = \text{carrier } (G \langle \text{carrier} := \mathfrak{G} ! (i + 1) \rangle)$  by auto
    hence carrier ( $(G \langle \text{carrier} := (\mathfrak{G} ! (i + 1)) \rangle) \text{ Mod } (\mathfrak{G} ! i) = \{1\}_{(G \langle \text{carrier} := \mathfrak{G} ! (i + 1) \rangle) \text{ Mod } \mathfrak{G} ! i}$ )
      using norm fin normal.fact-group-trivial-iff by metis
    hence  $\neg$  simple-group ( $(G \langle \text{carrier} := (\mathfrak{G} ! (i + 1)) \rangle) \text{ Mod } (\mathfrak{G} ! i)$ ) by (metis
    simple-group.simple-not-triv)
    thus False by (metis i simplefact)
  qed

```

The normal series for groups of order $p * q$ is even a composition series:

lemma (in group) *pq-order-composition-series*:

assumes *finite:finite* (carrier G)

assumes *orderG:order* $G = q * p$

assumes *primep:prime* p **and** *primeq:prime* q **and** *pq:p < q*

shows *composition-series* G $\{\{1\}, (THE\ H.\ H \in \text{subgroups-of-size } q), \text{carrier } G\}$

unfolding *composition-series-def* *composition-series-axioms-def*

apply(*auto*)

using *assms* **apply**(*rule pq-order-normal-series*)

proof –

def $H \equiv THE\ H.\ H \in \text{subgroups-of-size } q$

from *assms* **have** *exi:∃!Q. Q ∈ (subgroups-of-size q)* **by** (*auto simp: pq-order-unique-subgrp*)

hence *Hsize:H ∈ subgroups-of-size q* **unfolding** *H-def* **using** *theI'* **by** *metis*

hence *HsubG:subgroup H G* **unfolding** *subgroups-of-size-def* **by** *auto*

then interpret *Hgroup: group G (carrier := H)* **by** (*metis subgroup-imp-group*)

fix i

assume $i < Suc\ (Suc\ 0)$

hence $i = 0 \vee i = 1$ **by** *auto*

thus *simple-group (G (carrier := [H, carrier G] ! i) Mod [{1}, H, carrier G] !*

i)

proof

assume $i:i = 0$

from *Hsize* **have** *orderH:order (G (carrier := H)) = q* **unfolding** *subgroups-of-size-def* *order-def* **by** *simp*

hence *order (G (carrier := H) Mod {1}) = q* **unfolding** *FactGroup-def* **using** *card-rcosets-triv* *order-def*

by (*metis Hgroup.card-rcosets-triv HsubG finite monoid.cases-scheme monoid.select-convs(2) partial-object.select-convs(1) partial-object.update-convs(1) subgroup-finite*)

have *normal {1} (G (carrier := H))* **by** (*metis Hgroup.is-group Hgroup.normal-inv-iff HsubG group.trivial-subgroup-is-normal is-group singleton-iff subgroup.one-closed subgroup.subgroup-of-subgroup*)

hence *group (G (carrier := H) Mod {1})* **by** (*metis normal.factorgroup-is-group*)

with *orderH primeq* **have** *simple-group (G (carrier := H) Mod {1})* **by** (*metis (order (G (carrier := H) Mod {1}) = q) group.prime-order-simple*)

with i **show** *?thesis* **by** *simp*

next

assume $i:i = 1$

from *assms exi* **have** $H \triangleleft G$ **unfolding** *H-def* **by** (*metis pq-order-subgrp-normal*)

hence *groupGH:group (G Mod H)* **by** (*metis normal.factorgroup-is-group*)

from *primeq* **have** $q \neq 0$ **by** (*metis not-prime-0*)

from *HsubG finite orderG* **have** $\text{card } (\text{rcosets } H) * \text{card } H = q * p$ **unfolding** *subgroups-of-size-def* **using** *lagrange* **by** *simp*

with *Hsize* **have** $\text{card } (\text{rcosets } H) * q = q * p$ **unfolding** *subgroups-of-size-def*

by *simp*

with $(q \neq 0)$ **have** $\text{card } (\text{rcosets } H) = p$ **by** *auto*

hence $\text{order } (G \text{ Mod } H) = p$ **unfolding** *order-def* *FactGroup-def* **by** *auto*

with *groupGH primep* **have** *simple-group (G Mod H)* **by** (*metis group.prime-order-simple*)

with i **show** *?thesis* **by** *auto*

qed

qed

Prefixes of composition series are also composition series.

lemma (in *composition-series*) *composition-series-prefix-closed*:

assumes $i \leq \text{length } \mathfrak{G}$ **and** $0 < i$

shows *composition-series* ($G(\text{carrier} := \mathfrak{G} ! (i - 1))$) (*take* i \mathfrak{G})

unfolding *composition-series-def* *composition-series-axioms-def*

proof *auto*

from *assms* **show** *normal-series* ($G(\text{carrier} := \mathfrak{G} ! (i - \text{Suc } 0))$) (*take* i \mathfrak{G}) **by**
(*metis* *One-nat-def* *normal-series-prefix-closed*)

next

fix j

assume $j : \text{Suc } j < \text{length } \mathfrak{G}$ $\text{Suc } j < i$

with *simplefact* **show** *simple-group* ($G(\text{carrier} := \mathfrak{G} ! \text{Suc } j) \text{ Mod } \mathfrak{G} ! j$) **by**
(*metis* *Suc-eq-plus1*)

qed

The second element in a composition series is simple group.

lemma (in *composition-series*) *composition-series-snd-simple*:

assumes $2 \leq \text{length } \mathfrak{G}$

shows *simple-group* ($G(\text{carrier} := \mathfrak{G} ! 1)$)

proof –

from *assms* **interpret** *compTake: composition-series* $G(\text{carrier} := \mathfrak{G} ! 1)$ *take*
 2 \mathfrak{G} **by** (*metis* *add-diff-cancel-right'* *composition-series-prefix-closed* *one-add-one*
zero-less-numeral)

from *assms* **have** $\text{length } (\text{take } 2 \ \mathfrak{G}) = 2$ **by** (*metis* *add-diff-cancel-right'* *append-take-drop-id*
diff-diff-cancel *length-append* *length-drop*)

hence ($\text{take } 2 \ \mathfrak{G}$) = $[\{\mathbf{1}(G(\text{carrier} := \mathfrak{G} ! 1))\}, \text{carrier } (G(\text{carrier} := \mathfrak{G} ! 1))]$
by (*rule* *compTake.length-two-unique*)

thus *?thesis* **by** (*metis* *compTake.composition-series-simple-group*)

qed

As a stronger way to state the previous lemma: An entry of a composition series is simple if and only if it is the second one.

lemma (in *composition-series*) *composition-snd-simple-iff*:

assumes $i < \text{length } \mathfrak{G}$

shows (*simple-group* ($G(\text{carrier} := \mathfrak{G} ! i)$)) = ($i = 1$)

proof

assume *simpI: simple-group* ($G(\text{carrier} := \mathfrak{G} ! i)$)

hence $\mathfrak{G} ! i \neq \{\mathbf{1}\}$ **using** *simple-group.simple-not-triv* **by** *force*

hence $i \neq 0$ **using** *hd hd-conv-nth notempty* **by** *auto*

then **interpret** *compTake: composition-series* $G(\text{carrier} := \mathfrak{G} ! i)$ *take* ($\text{Suc } i$)

\mathfrak{G}

using *assms* *composition-series-prefix-closed* **by** (*metis* *diff-Suc-1 less-eq-Suc-le*
zero-less-Suc)

from *simpI* **have** ($\text{take } (\text{Suc } i) \ \mathfrak{G}$) = $[\{\mathbf{1}G(\text{carrier} := \mathfrak{G} ! i)\}, \text{carrier } (G(\text{carrier}$
 $:= \mathfrak{G} ! i))]$

by (*metis* *compTake.composition-series-simple-group*)

```

hence  $\text{length } (\text{take } (\text{Suc } i) \mathfrak{G}) = 2$  by auto
hence  $\text{min } (\text{length } \mathfrak{G}) (\text{Suc } i) = 2$  by (metis length-take)
with assms have  $\text{Suc } i = 2$  by force
thus  $i = 1$  by simp
next
assume  $i:i = 1$ 
with assms have  $2 \leq \text{length } \mathfrak{G}$  by simp
with  $i$  show simple-group ( $G(\text{carrier} := \mathfrak{G} ! i)$ ) by (metis composition-series-snd-simple)
qed

```

The second to last entry of a normal series is not only a normal subgroup but actually even a *maximal* normal subgroup.

lemma (*in composition-series*) *snd-to-last-max-normal*:

```

assumes finite:finite (carrier  $G$ )
assumes length:length  $\mathfrak{G} > 1$ 
shows max-normal-subgroup ( $\mathfrak{G} ! (\text{length } \mathfrak{G} - 2)$ )  $G$ 
unfolding max-normal-subgroup-def max-normal-subgroup-axioms-def
proof (auto del: equalityI)
show  $\mathfrak{G} ! (\text{length } \mathfrak{G} - 2) \triangleleft G$  by (rule normal-series-snd-to-last)
next
def  $G' \equiv \mathfrak{G} ! (\text{length } \mathfrak{G} - 2)$ 
from length have  $\text{length}21:\text{length } \mathfrak{G} - 2 + 1 = \text{length } \mathfrak{G} - 1$  by arith
from length have  $\text{length } \mathfrak{G} - 2 + 1 < \text{length } \mathfrak{G}$  by arith
with simplefact have simple-group ( $G(\text{carrier} := \mathfrak{G} ! ((\text{length } \mathfrak{G} - 2) + 1))$ )
Mod  $G'$ ) unfolding  $G'$ -def by auto
with  $\text{length}21$  have simple-last:simple-group ( $G \text{ Mod } G'$ ) using last notempty
last-conv-nth by fastforce
{
assume snd-to-last-eq: $G' = \text{carrier } G$ 
hence  $\text{carrier } (G \text{ Mod } G') = \{\mathbf{1}_{G \text{ Mod } G'}\}$ 
using normal-series-snd-to-last finite normal.fact-group-trivial-iff unfolding
 $G'$ -def by metis
with snd-to-last-eq have  $\neg \text{simple-group } (G \text{ Mod } G')$  by (metis self-factor-not-simple)
with simple-last show False unfolding  $G'$ -def by auto
}
{
have  $G'G:G' \triangleleft G$  unfolding  $G'$ -def by (rule normal-series-snd-to-last)
fix  $J$ 
assume  $J:J \triangleleft G \ J \neq G' \ J \neq \text{carrier } G \ G' \subseteq J$ 
hence  $JG'GG':\text{rcosets}_{G(\text{carrier} := J)} G' \triangleleft G \text{ Mod } G'$  using normality-factorization
normal-series-snd-to-last unfolding  $G'$ -def by auto
from  $G'G \ J(1,4)$  have  $G'J:G' \triangleleft (G(\text{carrier} := J))$  by (metis normal-imp-subgroup
normal-restrict-supergroup)
from finite  $J(1)$  have  $\text{fin}J:\text{finite } J$  by (auto simp: normal-imp-subgroup
subgroup-finite)
from  $JG'GG'$  simple-last have  $\text{rcosets}_{G(\text{carrier} := J)} G' = \{\mathbf{1}_{G \text{ Mod } G'}\} \vee$ 
 $\text{rcosets}_{G(\text{carrier} := J)} G' = \text{carrier } (G \text{ Mod } G')$ 
unfolding simple-group-def simple-group-axioms-def by auto
thus False

```

```

proof
  assume  $rcosets_{G(\text{carrier} := J)} G' = \{\mathbf{1}_{G \text{ Mod } G'}\}$ 
  hence  $rcosets_{G(\text{carrier} := J)} G' = \{\mathbf{1}_{(G(\text{carrier} := J)) \text{ Mod } G'}\}$  unfolding
FactGroup-def by simp
  hence  $G' = J$  using  $G'J \text{ fin}J \text{ normal.fact-group-trivial-iff}$  unfolding
FactGroup-def by fastforce
  with  $J(2)$  show False by simp
next
  assume  $facts\text{-}eq:rcosets_{G(\text{carrier} := J)} G' = \text{carrier } (G \text{ Mod } G')$ 
  have  $J = \text{carrier } G$ 
  proof
    show  $J \subseteq \text{carrier } G$  using  $J(1) \text{ normal-imp-subgroup subgroup-imp-subset}$ 
by force
  next
    show  $\text{carrier } G \subseteq J$ 
    proof
      fix  $x$ 
      assume  $x : x \in \text{carrier } G$ 
      hence  $G' \#> x \in \text{carrier } (G \text{ Mod } G')$  unfolding FactGroup-def
RCOSETS-def by auto
      hence  $G' \#> x \in rcosets_{G(\text{carrier} := J)} G'$  using facts-eq by auto
      then obtain  $j$  where  $j : j \in J \ G' \#> x = G' \#> j$  unfolding RCOSETS-def
r-coset-def by force
      hence  $x \in G' \#> j$  using  $G'G \text{ normal-imp-subgroup } x \text{ repr-independence}D$ 
by fastforce
      then obtain  $g'$  where  $g' : g' \in G' \ x = g' \otimes j$  unfolding r-coset-def by
auto
      hence  $g' \in J$  using  $G'J \text{ normal-imp-subgroup subgroup-imp-subset}$  by
force
      with  $g'(2) \ j(1)$  show  $x \in J$  using  $J(1) \text{ normal-imp-subgroup subgroup.m-closed}$ 
by fastforce
    qed
  qed
  with  $J(3)$  show False by simp
qed
}
qed

```

For the next lemma we need a few facts about removing adjacent duplicates.

lemma *remdups-adj-obtain-adjacency*:

assumes $i + 1 < \text{length } (\text{remdups-adj } xs) \ \text{length } xs > 0$

obtains j **where** $j + 1 < \text{length } xs$

$(\text{remdups-adj } xs) ! i = xs ! j \ (\text{remdups-adj } xs) ! (i + 1) = xs ! (j + 1)$

using *assms* **proof** (*induction xs arbitrary: i thesis*)

case *Nil*

hence *False* **by** (*metis length-greater-0-conv*)

thus *thesis..*

next

case (*Cons x xs*)

hence $xs \neq []$ **using** *Divides.div-less Suc-eq-plus1 Zero-not-Suc div-eq-dividend-iff list.size(3,4) plus-nat.add-0 remdups-adj.simps(2)* **by** *metis*
then obtain $y \ xs'$ **where** $xs:xs = y \# \ xs'$ **by** (*metis list.exhaust*)
from $\langle xs \neq [] \rangle$ **have** $lenxs:length \ xs > 0$ **by** *simp*
from xs **have** $rem:remdups-adj \ (x \# \ xs) = (if \ x = y \ then \ remdups-adj \ (y \# \ xs') \ else \ x \# \ remdups-adj \ (y \# \ xs'))$ **using** *remdups-adj.simps(3)* **by** *auto*
show *thesis*
proof (*cases \ x = y*)
 case *True*
 with $rem \ xs$ **have** $rem2:remdups-adj \ (x \# \ xs) = remdups-adj \ xs$ **by** *auto*
 with *Cons(3)* **have** $i + 1 < length \ (remdups-adj \ xs)$ **by** *simp*
 with *Cons.IH lenxs* **obtain** k **where** $j:k + 1 < length \ xs \ remdups-adj \ xs ! i = xs ! k$
 $remdups-adj \ xs ! (i + 1) = xs ! (k + 1)$ **by** *auto*
 thus *thesis* **using** *Cons(2) rem2* **by** *auto*
 next
 case *False*
 with $rem \ xs$ **have** $rem2:remdups-adj \ (x \# \ xs) = x \# \ remdups-adj \ xs$ **by** *auto*
 show *thesis*
 proof (*cases \ i*)
 case *0*
 have $0 + 1 < length \ (x \# \ xs)$ **using** *lenxs* **by** *auto*
 moreover **have** $remdups-adj \ (x \# \ xs) ! i = (x \# \ xs) ! 0$
 proof –
 have $remdups-adj \ (x \# \ xs) ! i = (x \# \ remdups-adj \ (y \# \ xs')) ! 0$ **using**
 $xs \ rem2 \ 0$ **by** *simp*
 also **have** $\dots = x$ **by** *simp*
 also **have** $\dots = (x \# \ xs) ! 0$ **by** *simp*
 finally **show** *?thesis*.
 qed
 moreover **have** $remdups-adj \ (x \# \ xs) ! (i + 1) = (x \# \ xs) ! (0 + 1)$
 proof –
 have $remdups-adj \ (x \# \ xs) ! (i + 1) = (x \# \ remdups-adj \ (y \# \ xs')) ! 1$
using $xs \ rem2 \ 0$ **by** *simp*
 also **have** $\dots = remdups-adj \ (y \# \ xs') ! 0$ **by** *simp*
 also **have** $\dots = (y \# \ (remdups \ (y \# \ xs')) ! 0$ **by** (*metis nth-Cons' remdups-adj-Cons-alt*)
 also **have** $\dots = y$ **by** *simp*
 also **have** $\dots = (x \# \ xs) ! (0 + 1)$ **unfolding** xs **by** *simp*
 finally **show** *?thesis*.
 qed
 ultimately **show** *thesis* **by** (*rule Cons.premis(1)*)
 next
 case (*Suc \ k*)
 with *Cons(3)* **have** $k + 1 < length \ (remdups-adj \ (x \# \ xs)) - 1$ **by** *auto*
 also **have** $\dots \leq length \ (remdups-adj \ xs) + 1 - 1$ **by** (*metis One-nat-def le-refl list.size(4) rem2*)
 also **have** $\dots = length \ (remdups-adj \ xs)$ **by** *simp*
 finally **have** $k + 1 < length \ (remdups-adj \ xs)$.

with *Cons.IH lenxs* **obtain** j **where** $j:j + 1 < \text{length } xs \text{ remdups-adj } xs ! k$
 $= xs ! j$
 $\text{remdups-adj } xs ! (k + 1) = xs ! (j + 1)$ **by** *auto*
from $j(1)$ **have** $Suc j + 1 < \text{length } (x \# xs)$ **by** *simp*
moreover **have** $\text{remdups-adj } (x \# xs) ! i = (x \# xs) ! (Suc j)$
proof –
have $\text{remdups-adj } (x \# xs) ! i = (x \# \text{remdups-adj } xs) ! i$ **using** *rem2* **by**
simp
also **have** $\dots = (\text{remdups-adj } xs) ! k$ **using** *Suc* **by** *simp*
also **have** $\dots = xs ! j$ **using** $j(2)$.
also **have** $\dots = (x \# xs) ! (Suc j)$ **by** *simp*
finally **show** *?thesis* .
qed
moreover **have** $\text{remdups-adj } (x \# xs) ! (i + 1) = (x \# xs) ! (Suc j + 1)$
proof –
have $\text{remdups-adj } (x \# xs) ! (i + 1) = (x \# \text{remdups-adj } xs) ! (i + 1)$
using *rem2* **by** *simp*
also **have** $\dots = (\text{remdups-adj } xs) ! (k + 1)$ **using** *Suc* **by** *simp*
also **have** $\dots = xs ! (j + 1)$ **using** $j(3)$.
also **have** $\dots = (x \# xs) ! (Suc j + 1)$ **by** *simp*
finally **show** *?thesis* .
qed
ultimately **show** *thesis* **by** (*rule Cons.prem1*)
qed
qed
qed

lemma *hd-remdups-adj[simp]*: $hd (\text{remdups-adj } xs) = hd xs$
by (*induction xs rule: remdups-adj.induct*) *simp-all*

lemma *remdups-adj-adjacent*:

$Suc i < \text{length } (\text{remdups-adj } xs) \implies \text{remdups-adj } xs ! i \neq \text{remdups-adj } xs ! Suc i$
proof (*induction xs arbitrary: i rule: remdups-adj.induct*)
case $(\exists x y xs i)$
thus *?case* **by** (*cases i, cases x = y*) (*simp, auto simp: hd-conv-nth[symmetric]*)
qed *simp-all*

Intersecting each entry of a composition series with a normal subgroup of G and removing all adjacent duplicates yields another composition series.

lemma (*in composition-series*) *intersect-normal*:

assumes *finite:finite* (*carrier G*)
assumes *KG:K* $K \triangleleft G$
shows *composition-series* ($G \langle \text{carrier} := K \rangle$) ($\text{remdups-adj } (\text{map } (\lambda H. K \cap H) \mathfrak{G})$)
unfolding *composition-series-def composition-series-axioms-def normal-series-def normal-series-axioms-def*
apply (*auto simp only: conjI del: equalityI*)
proof –
show *group* ($G \langle \text{carrier} := K \rangle$) **using** *KG normal-imp-subgroup subgroup-imp-group*

by auto
next
 — Show, that removing adjacent duplicates doesn't result in an empty list.
assume $\text{remdups-adj } (\text{map } (op \cap K) \mathfrak{G}) = []$
hence $\text{map } (op \cap K) \mathfrak{G} = []$ **by** (*metis remdups-adj-Nil-iff*)
hence $\mathfrak{G} = []$ **by** (*metis Nil-is-map-conv*)
with notempty show False..
next
 — Show, that the head of the reduced list is still the trivial group
have $\mathfrak{G} = \{1\} \# \text{tl } \mathfrak{G}$ **using** *notempty hd* **by** (*metis list.sel(1,3) neq-Nil-conv*)
hence $\text{map } (op \cap K) \mathfrak{G} = \text{map } (op \cap K) (\{1\} \# \text{tl } \mathfrak{G})$ **by** *simp*
hence $\text{remdups-adj } (\text{map } (op \cap K) \mathfrak{G}) = \text{remdups-adj } ((K \cap \{1\}) \# (\text{map } (op \cap K) (\text{tl } \mathfrak{G})))$ **by** *simp*
also have $\dots = (K \cap \{1\}) \# \text{tl } (\text{remdups-adj } ((K \cap \{1\}) \# (\text{map } (op \cap K) (\text{tl } \mathfrak{G}))))$ **by** *simp*
finally have $\text{hd } (\text{remdups-adj } (\text{map } (op \cap K) \mathfrak{G})) = K \cap \{1\}$ **using** *list.sel(1)*
by metis
thus $\text{hd } (\text{remdups-adj } (\text{map } (op \cap K) \mathfrak{G})) = \{1_{G(\text{carrier} := K)}\}$
using *KG normal-imp-subgroup subgroup.one-closed* **by force**
next
 — Show that the last entry is really $K \cap G$. Since we don't have a lemma ready to talk about the last entry of a reduced list, we reverse the list twice.
have $\text{rev } \mathfrak{G} = (\text{carrier } G) \# \text{tl } (\text{rev } \mathfrak{G})$ **by** (*metis list.sel(1,3) last last-rev neq-Nil-conv notempty rev-is-Nil-conv rev-rev-ident*)
hence $\text{rev } (\text{map } (op \cap K) \mathfrak{G}) = \text{map } (op \cap K) ((\text{carrier } G) \# \text{tl } (\text{rev } \mathfrak{G}))$ **by** (*metis rev-map*)
hence $\text{rev:rev } (\text{map } (op \cap K) \mathfrak{G}) = (K \cap (\text{carrier } G)) \# (\text{map } (op \cap K) (\text{tl } (\text{rev } \mathfrak{G})))$ **by** *simp*
have $\text{last } (\text{remdups-adj } (\text{map } (op \cap K) \mathfrak{G})) = \text{hd } (\text{rev } (\text{remdups-adj } (\text{map } (op \cap K) \mathfrak{G})))$
by (*metis hd-rev map-is-Nil-conv notempty remdups-adj-Nil-iff*)
also have $\dots = \text{hd } (\text{remdups-adj } (\text{rev } (\text{map } (op \cap K) \mathfrak{G})))$ **by** (*metis remdups-adj-rev*)
also have $\dots = \text{hd } (\text{remdups-adj } ((K \cap (\text{carrier } G)) \# (\text{map } (op \cap K) (\text{tl } (\text{rev } \mathfrak{G}))))$ **by** (*metis rev*)
also have $\dots = \text{hd } ((K \cap (\text{carrier } G)) \# (\text{remdups-adj } ((K \cap (\text{carrier } G)) \# (\text{map } (op \cap K) (\text{tl } (\text{rev } \mathfrak{G}))))))$ **by** (*metis list.sel(1) remdups-adj-Cons-alt*)
also have $\dots = K$ **using** *KG normal-imp-subgroup subgroup-imp-subset* **by force**
finally show $\text{last } (\text{remdups-adj } (\text{map } (op \cap K) \mathfrak{G})) = \text{carrier } (G(\text{carrier} := K))$ **by auto**
next
 — The induction step, using the second isomorphism theorem for groups.
fix j
assume $j: j + 1 < \text{length } (\text{remdups-adj } (\text{map } (op \cap K) \mathfrak{G}))$
have $KG \text{notempty} : (\text{map } (op \cap K) \mathfrak{G}) \neq []$ **using** *notempty* **by** (*metis Nil-is-map-conv*)
with j obtain i where $i: i + 1 < \text{length } (\text{map } (op \cap K) \mathfrak{G})$
 $(\text{remdups-adj } (\text{map } (op \cap K) \mathfrak{G})) ! j = (\text{map } (op \cap K) \mathfrak{G}) ! i$
 $(\text{remdups-adj } (\text{map } (op \cap K) \mathfrak{G})) ! (j + 1) = (\text{map } (op \cap K) \mathfrak{G}) ! (i + 1)$
using *remdups-adj-obtain-adjacency* **by force**
from $i(1)$ **have** $i: i + 1 < \text{length } \mathfrak{G}$ **by** (*metis length-map*)

hence $G_i S_i : \mathfrak{G} ! i \triangleleft G(\text{carrier} := \mathfrak{G} ! (i + 1))$ **by** (*metis normal*)
hence $G_i S_i' : \mathfrak{G} ! i \subseteq \mathfrak{G} ! (i + 1)$ **using** *normal-imp-subgroup subgroup-imp-subset*
by force
from i' **have** $\text{fin} G_i S_i : \text{finite} (\mathfrak{G} ! (i + 1))$ **using** *normal-series-subgroups finite*
by (*metis subgroup-finite*)
from $G_i S_i$ $K G_i i'$ *normal-series-subgroups* **have** $G_i S_i K \text{norm} G_i S_i : \mathfrak{G} ! (i + 1) \cap K$
 $\triangleleft G(\text{carrier} := \mathfrak{G} ! (i + 1))$
using *second-isomorphism-grp.normal-subgrp-intersection-normal*
unfolding *second-isomorphism-grp-def second-isomorphism-grp-axioms-def* **by**
auto
with $G_i S_i$ **have** $\mathfrak{G} ! i \cap (\mathfrak{G} ! (i + 1) \cap K) \triangleleft G(\text{carrier} := \mathfrak{G} ! (i + 1))$
by (*metis group.normal-subgroup-intersect group.subgroup-imp-group i' is-group*
is-normal-series normal-series.normal-series-subgroups)
hence $K \cap (\mathfrak{G} ! i \cap \mathfrak{G} ! (i + 1)) \triangleleft G(\text{carrier} := \mathfrak{G} ! (i + 1))$ **by** (*metis*
inf-commute inf-left-commute)
hence $K G_i \text{norm} G_i S_i : K \cap \mathfrak{G} ! i \triangleleft G(\text{carrier} := \mathfrak{G} ! (i + 1))$ **using** $G_i S_i'$ **by**
(*metis le-iff-inf*)
moreover **have** $K \cap \mathfrak{G} ! i \subseteq K \cap \mathfrak{G} ! (i + 1)$ **using** $G_i S_i'$ **by** *auto*
moreover **have** $\text{group} G_i S_i : \text{group} (G(\text{carrier} := \mathfrak{G} ! (i + 1)))$ **using** i *normal-series-subgroups*
subgroup-imp-group **by** *auto*
moreover **have** $\text{sub} K G_i S_i G_i S_i : \text{subgroup} (K \cap \mathfrak{G} ! (i + 1)) (G(\text{carrier} := \mathfrak{G} ! (i + 1)))$
by (*metis G_i S_i K norm G_i S_i inf-sup-aci(1) normal-imp-subgroup*)
ultimately **have** $\text{fstgoal} : K \cap \mathfrak{G} ! i \triangleleft G(\text{carrier} := \mathfrak{G} ! (i + 1), \text{carrier} := K \cap \mathfrak{G} ! (i + 1))$
using *group.normal-restrict-supergroup* **by force**
thus $\text{remdups-adj} (\text{map} (op \cap K) \mathfrak{G}) ! j \triangleleft G(\text{carrier} := K, \text{carrier} := \text{remdups-adj} (\text{map} (op \cap K) \mathfrak{G}) ! (j + 1))$
using i **by** *auto*
from *simplefact* **have** $G_i \text{simple} : \text{simple-group} (G(\text{carrier} := \mathfrak{G} ! (i + 1))) \text{Mod} \mathfrak{G} ! i$ **using** i' **by** *simp*
hence $G_i \text{max} : \text{max-normal-subgroup} (\mathfrak{G} ! i) (G(\text{carrier} := \mathfrak{G} ! (i + 1)))$
using *normal.max-normal-simple-quotient G_i S_i fin G_i S_i* **by force**
from $G_i S_i K \text{norm} G_i S_i G_i S_i$ **have** $\mathfrak{G} ! i <\#\> G(\text{carrier} := \mathfrak{G} ! (i + 1)) \mathfrak{G} ! (i + 1)$
 $\cap K \triangleleft (G(\text{carrier} := \mathfrak{G} ! (i + 1)))$
using $\text{group} G_i S_i$ *group.normal-subgroup-set-mult-closed* **by** *simp*
hence $\mathfrak{G} ! i <\#\> \mathfrak{G} ! (i + 1) \cap K \triangleleft G(\text{carrier} := \mathfrak{G} ! (i + 1))$ **unfolding**
set-mult-def **by** *auto*
hence $\mathfrak{G} ! i <\#\> K \cap \mathfrak{G} ! (i + 1) \triangleleft G(\text{carrier} := \mathfrak{G} ! (i + 1))$ **using**
inf-commute **by** *metis*
moreover **have** $\mathfrak{G} ! i \subseteq \mathfrak{G} ! i <\#\> G(\text{carrier} := \mathfrak{G} ! (i + 1)) K \cap \mathfrak{G} ! (i + 1)$
using *second-isomorphism-grp.H-contained-in-set-mult*
unfolding *second-isomorphism-grp-def second-isomorphism-grp-axioms-def*
using *sub K G_i S_i G_i S_i G_i S_i normal-imp-subgroup* **by** *fastforce*
hence $\mathfrak{G} ! i \subseteq \mathfrak{G} ! i <\#\> K \cap \mathfrak{G} ! (i + 1)$ **unfolding** *set-mult-def* **by** *auto*
ultimately **have** $K G_i \text{disj} : \mathfrak{G} ! i <\#\> K \cap \mathfrak{G} ! (i + 1) = \mathfrak{G} ! i \vee \mathfrak{G} ! i <\#\> K \cap \mathfrak{G} ! (i + 1) = \mathfrak{G} ! (i + 1)$
using $G_i \text{max}$ **unfolding** *max-normal-subgroup-def max-normal-subgroup-axioms-def*
by *auto*
obtain φ **where** $\varphi \in (G(\text{carrier} := K \cap \mathfrak{G} ! (i + 1))) \text{Mod} (\mathfrak{G} ! i \cap (K \cap \mathfrak{G} ! (i + 1)))$

$(i + 1)))$
 $\cong (G(\text{carrier} := \mathfrak{G} ! i <\#\rangle G(\text{carrier} := \mathfrak{G} ! (i + 1)) K \cap \mathfrak{G} ! (i + 1)) \text{Mod}$
 $\mathfrak{G} ! i)$
using *second-isomorphism-grp.normal-intersection-quotient-isom*
unfolding *second-isomorphism-grp-def second-isomorphism-grp-axioms-def*
using *GiSi subKGSiGSi normal-imp-subgroup* **by** *fastforce*
hence $\varphi \in (G(\text{carrier} := K \cap \mathfrak{G} ! (i + 1)) \text{Mod} (K \cap \mathfrak{G} ! (i + 1) \cap \mathfrak{G} ! i))$
 $\cong (G(\text{carrier} := \mathfrak{G} ! i <\#\rangle G(\text{carrier} := \mathfrak{G} ! (i + 1)) K \cap \mathfrak{G} ! (i + 1)) \text{Mod}$
 $\mathfrak{G} ! i)$ **by** (*metis inf-commute*)
hence $\varphi \in (G(\text{carrier} := K \cap \mathfrak{G} ! (i + 1)) \text{Mod} (K \cap (\mathfrak{G} ! (i + 1) \cap \mathfrak{G} ! i)))$
 $\cong (G(\text{carrier} := \mathfrak{G} ! i <\#\rangle G(\text{carrier} := \mathfrak{G} ! (i + 1)) K \cap \mathfrak{G} ! (i + 1)) \text{Mod}$
 $\mathfrak{G} ! i)$ **by** (*metis Int-assoc*)
hence $\varphi \in (G(\text{carrier} := K \cap \mathfrak{G} ! (i + 1)) \text{Mod} (K \cap \mathfrak{G} ! i))$
 $\cong (G(\text{carrier} := \mathfrak{G} ! i <\#\rangle G(\text{carrier} := \mathfrak{G} ! (i + 1)) K \cap \mathfrak{G} ! (i + 1)) \text{Mod}$
 $\mathfrak{G} ! i)$ **by** (*metis GiSi' Int-absorb2 Int-commute*)
hence $\varphi : \varphi \in (G(\text{carrier} := K \cap \mathfrak{G} ! (i + 1)) \text{Mod} (K \cap \mathfrak{G} ! i)) \cong (G(\text{carrier}$
 $:= \mathfrak{G} ! i <\#\rangle K \cap \mathfrak{G} ! (i + 1)) \text{Mod} \mathfrak{G} ! i)$
unfolding *set-mult-def* **by** *auto*
from *fstgoal* **have** *KGsiKGigroup:group* $(G(\text{carrier} := K \cap \mathfrak{G} ! (i + 1)) \text{Mod}$
 $(K \cap \mathfrak{G} ! i))$ **using** *normal.factorgroup-is-group* **by** *auto*
from *KGdisj* **show** *simple-group* $(G(\text{carrier} := K, \text{carrier} := \text{remdups-adj} (\text{map}$
 $(\text{op} \cap K) \mathfrak{G} ! (j + 1)) \text{Mod} \text{remdups-adj} (\text{map} (\text{op} \cap K) \mathfrak{G} ! j))$
proof *auto*
have *groupGi:group* $(G(\text{carrier} := \mathfrak{G} ! i))$ **using** *i' normal-series-subgroups*
subgroup-imp-group **by** *auto*
assume $\mathfrak{G} ! i <\#\rangle K \cap \mathfrak{G} ! \text{Suc } i = \mathfrak{G} ! i$
with φ **have** $\varphi \in (G(\text{carrier} := K \cap \mathfrak{G} ! (i + 1)) \text{Mod} (K \cap \mathfrak{G} ! i)) \cong$
 $(G(\text{carrier} := \mathfrak{G} ! i) \text{Mod} \mathfrak{G} ! i)$ **by** *auto*
moreover obtain ψ **where** $\psi \in (G(\text{carrier} := \mathfrak{G} ! i) \text{Mod} (\text{carrier} (G(\text{carrier}$
 $:= \mathfrak{G} ! i)))) \cong (G(\text{carrier} := \{\mathbf{1}_{G(\text{carrier} := \mathfrak{G} ! i)}\}))$
using *group.self-factor-iso* *groupGi* **by** *force*
ultimately obtain π **where** $\pi \in (G(\text{carrier} := K \cap \mathfrak{G} ! (i + 1)) \text{Mod} (K \cap$
 $\mathfrak{G} ! i)) \cong (G(\text{carrier} := \{\mathbf{1}\}))$
using *KGsiKGigroup* *group.iso-trans* **by** *fastforce*
hence $\text{order} (G(\text{carrier} := K \cap \mathfrak{G} ! (i + 1)) \text{Mod} (K \cap \mathfrak{G} ! i)) = \text{order}$
 $(G(\text{carrier} := \{\mathbf{1}\}))$ **by** (*metis iso-order-closed*)
hence $\text{order} (G(\text{carrier} := K \cap \mathfrak{G} ! (i + 1)) \text{Mod} (K \cap \mathfrak{G} ! i)) = 1$ **unfolding**
order-def **by** *auto*
hence $\text{carrier} (G(\text{carrier} := K \cap \mathfrak{G} ! (i + 1)) \text{Mod} (K \cap \mathfrak{G} ! i)) =$
 $\{\mathbf{1}_{G(\text{carrier} := K \cap \mathfrak{G} ! (i + 1)) \text{Mod} (K \cap \mathfrak{G} ! i)}\}$
using *group.order-one-triv-iff* *KGsiKGigroup* **by** *auto*
moreover from *fstgoal* **have** $K \cap \mathfrak{G} ! i \triangleleft G(\text{carrier} := K \cap \mathfrak{G} ! (i + 1))$ **by**
auto
moreover from *finGSi* **have** *finite* $(\text{carrier} (G(\text{carrier} := K \cap \mathfrak{G} ! (i + 1))))$
by *auto*
ultimately have $K \cap \mathfrak{G} ! i = \text{carrier} (G(\text{carrier} := K \cap \mathfrak{G} ! (i + 1)))$ **by**
(*metis normal.fact-group-trivial-iff*)
hence $(\text{remdups-adj} (\text{map} (\text{op} \cap K) \mathfrak{G} ! j) = (\text{remdups-adj} (\text{map} (\text{op} \cap K)$

$\mathfrak{G}) ! (j + 1)$ **using** i **by** *auto*
with j **have** *False* **using** *remdups-adj-adjacent KGnotempty Suc-eq-plus1* **by**
metis
thus *simple-group* $(G(\text{carrier} := \text{remdups-adj} (\text{map} (op \cap K) \mathfrak{G}) ! \text{Suc } j)) \text{ Mod}$
remdups-adj $(\text{map} (op \cap K) \mathfrak{G}) ! j)$.
next
assume $\mathfrak{G} ! i < \# > K \cap \mathfrak{G} ! \text{Suc } i = \mathfrak{G} ! \text{Suc } i$
moreover with φ **have** $\varphi \in (G(\text{carrier} := K \cap \mathfrak{G} ! (i + 1)) \text{ Mod} (K \cap \mathfrak{G} !$
 $i)) \cong (G(\text{carrier} := \mathfrak{G} ! (i + 1)) \text{ Mod} \mathfrak{G} ! i)$ **by** *auto*
then obtain φ' **where** $\varphi' \in (G(\text{carrier} := \mathfrak{G} ! (i + 1)) \text{ Mod} \mathfrak{G} ! i) \cong$
 $(G(\text{carrier} := K \cap \mathfrak{G} ! (i + 1)) \text{ Mod} (K \cap \mathfrak{G} ! i))$
using *KGsiKGigroup group.iso-sym* **by** *auto*
with *Gisimple KGsiKGigroup* **have** *simple-group* $(G(\text{carrier} := K \cap \mathfrak{G} ! (i +$
 $1)) \text{ Mod} (K \cap \mathfrak{G} ! i))$ **by** $(\text{metis } \text{simple-group.iso-simple})$
with i **show** *simple-group* $(G(\text{carrier} := \text{remdups-adj} (\text{map} (op \cap K) \mathfrak{G}) !$
 $\text{Suc } j)) \text{ Mod } \text{remdups-adj} (\text{map} (op \cap K) \mathfrak{G}) ! j)$ **by** *auto*
qed
qed

lemma *(in group) composition-series-extend:*

assumes *composition-series* $(G(\text{carrier} := H)) \mathfrak{H}$
assumes *simple-group* $(G \text{ Mod } H) H \triangleleft G$
shows *composition-series* $G (\mathfrak{H} @ [\text{carrier } G])$
unfolding *composition-series-def composition-series-axioms-def*
proof *auto*
from *assms(1)* **interpret** *comp \mathfrak{H}* : *composition-series* $G(\text{carrier} := H) \mathfrak{H}$.
show *normal-series* $G (\mathfrak{H} @ [\text{carrier } G])$ **using** *assms(3)* *comp \mathfrak{H} .is-normal-series*
by $(\text{metis } \text{normal-series-extend})$
fix i
assume $i < \text{length } \mathfrak{H}$
show *simple-group* $(G(\text{carrier} := (\mathfrak{H} @ [\text{carrier } G]) ! \text{Suc } i)) \text{ Mod} (\mathfrak{H} @ [\text{carrier}$
 $G]) ! i)$
proof $(\text{cases } i = \text{length } \mathfrak{H} - 1)$
case *True*
hence $(\mathfrak{H} @ [\text{carrier } G]) ! \text{Suc } i = \text{carrier } G$ **by** $(\text{metis } i \text{ diff-Suc-1 lessE}$
 $\text{nth-append-length})$
moreover have $(\mathfrak{H} @ [\text{carrier } G]) ! i = \mathfrak{H} ! i$ **by** $(\text{metis } \text{butlast-snoc } i \text{ nth-butlast})$
hence $(\mathfrak{H} @ [\text{carrier } G]) ! i = H$ **using** *True last-conv-nth comp \mathfrak{H} .notempty*
comp \mathfrak{H} .last **by** *auto*
ultimately show *?thesis* **using** *assms(2)* **by** *auto*
next
case *False*
hence $\text{Suc } i < \text{length } \mathfrak{H}$ **using** i **by** *auto*
hence $(\mathfrak{H} @ [\text{carrier } G]) ! \text{Suc } i = \mathfrak{H} ! \text{Suc } i$ **using** *nth-append* **by** *metis*
moreover from i **have** $(\mathfrak{H} @ [\text{carrier } G]) ! i = \mathfrak{H} ! i$ **using** *nth-append* **by**
metis
ultimately show *?thesis* **using** $(\text{Suc } i < \text{length } \mathfrak{H}) \text{ comp \mathfrak{H} .simplefact}$ **by** *auto*
qed
qed

```

lemma (in composition-series) entries-mono:
  assumes  $i \leq j$   $j < \text{length } \mathfrak{G}$ 
  shows  $\mathfrak{G} ! i \subseteq \mathfrak{G} ! j$ 
using assms proof (induction  $j - i$  arbitrary:  $i j$ )
  case 0
  hence  $i = j$  by auto
  thus  $\mathfrak{G} ! i \subseteq \mathfrak{G} ! j$  by auto
next
  case (Suc  $k i j$ )
  hence  $i' : i + (\text{Suc } k) = j$   $i + 1 < \text{length } \mathfrak{G}$  by auto
  hence  $ij : i + 1 \leq j$  by auto
  have  $\mathfrak{G} ! i \subseteq \mathfrak{G} ! (i + 1)$  using  $i'$  normal normal-imp-subgroup subgroup-imp-subset
  by force
  moreover have  $j - (i + 1) = k$   $j < \text{length } \mathfrak{G}$  using Suc assms by auto
  hence  $\mathfrak{G} ! (i + 1) \subseteq \mathfrak{G} ! j$  using Suc(1)  $ij$  by auto
  ultimately show  $\mathfrak{G} ! i \subseteq \mathfrak{G} ! j$  by simp
qed

end

```

```

theory GroupIsoClasses
imports
  Groups
  List
  Coset
begin

```

9 Isomorphism Classes of Groups

We construct a quotient type for isomorphism classes of groups.

```

typedef 'a group = {G :: 'a monoid. group G}
proof
  show  $\bigwedge a. (\text{carrier} = \{a\}, \text{mult} = (\lambda x y. x), \text{one} = a) \in \{G. \text{group } G\}$ 
  unfolding group-def group-axioms-def monoid-def Units-def by auto
qed

```

```

definition group-iso-rel :: 'a group  $\Rightarrow$  'a group  $\Rightarrow$  bool
  where group-iso-rel  $G H = (\exists \varphi. \varphi \in \text{Rep-group } G \cong \text{Rep-group } H)$ 

```

```

quotient-type 'a group-iso-class = 'a group / group-iso-rel
morphisms Rep-group-iso Abs-group-iso
proof (rule equivpI)
  show reflp group-iso-rel
  proof (rule reflpI)
    fix  $G :: 'b$  group
    show group-iso-rel  $G G$ 
  end
end

```

```

    unfolding group-iso-rel-def using Rep-group iso-refl by auto
  qed
next
  show symp group-iso-rel
  proof (rule sympI)
    fix G H :: 'b group
    assume group-iso-rel G H
    then obtain  $\varphi$  where  $\varphi \in \text{Rep-group } G \cong \text{Rep-group } H$  unfolding group-iso-rel-def
  by auto
    then obtain  $\varphi'$  where  $\varphi' \in \text{Rep-group } H \cong \text{Rep-group } G$  using group.iso-sym
  Rep-group by fastforce
    thus group-iso-rel H G unfolding group-iso-rel-def by auto
  qed
next
  show transp group-iso-rel
  proof (rule transpI)
    fix G H I :: 'b group
    assume group-iso-rel G H group-iso-rel H I
    then obtain  $\varphi \psi$  where  $\varphi \in \text{Rep-group } G \cong \text{Rep-group } H$   $\psi \in \text{Rep-group } H$ 
 $\cong \text{Rep-group } I$  unfolding group-iso-rel-def by auto
    then obtain  $\pi$  where  $\pi \in \text{Rep-group } G \cong \text{Rep-group } I$  using group.iso-trans
  Rep-group by fastforce
    thus group-iso-rel G I unfolding group-iso-rel-def by auto
  qed
qed

```

This assigns to a given group the group isomorphism class

```

definition (in group) iso-class :: 'a group-iso-class
  where iso-class = Abs-group-iso (Abs-group (monoid.truncate G))

```

Two isomorphic groups do indeed have the same isomorphism class:

lemma iso-classes-iff:

```

  assumes group G
  assumes group H
  shows  $(\exists \varphi. \varphi \in G \cong H) = (\text{group.iso-class } G = \text{group.iso-class } H)$ 
proof –
  from assms(1,2) have groups:group (monoid.truncate G) group (monoid.truncate
H)
    unfolding monoid.truncate-def group-def group-axioms-def Units-def monoid-def
  by auto
  have  $(\exists \varphi. \varphi \in G \cong H) = (\exists \varphi. \varphi \in (\text{monoid.truncate } G) \cong (\text{monoid.truncate }
H))$ 
    unfolding iso-def hom-def monoid.truncate-def by auto
  also have  $\dots = \text{group-iso-rel } (\text{Abs-group } (\text{monoid.truncate } G)) (\text{Abs-group } (\text{monoid.truncate }
H))$ 
    unfolding group-iso-rel-def using groups group.Abs-group-inverse by (metis
mem-Collect-eq)
  also have  $\dots = (\text{group.iso-class } G = \text{group.iso-class } H)$  using group.iso-class-def
  assms group-iso-class.abs-eq-iff by metis

```

```

finally show ?thesis.
qed

end

```

```

theory JordanHolder
imports
  CompositionSeries
  MaximalNormalSubgroups
  Multiset
  GroupIsoClasses
begin

```

10 The Jordan-Hölder Theorem

```

locale jordan-hoelder = group
  + comp  $\mathfrak{H}$ ? : composition-series  $G$   $\mathfrak{H}$ 
  + comp  $\mathfrak{G}$ ? : composition-series  $G$   $\mathfrak{G}$  for  $\mathfrak{H}$  and  $\mathfrak{G}$ 
  + assumes finite:finite (carrier  $G$ )

```

Before we finally start the actual proof of the theorem, one last lemma: Cancelling the last entry of a normal series results in a normal series with quotients being all but the last of the original ones.

```

lemma (in normal-series) quotients-butlast:
  assumes length  $\mathfrak{G} > 1$ 
  shows butlast quotients = normal-series.quotients ( $G$  (carrier :=  $\mathfrak{G} ! (\text{length } \mathfrak{G} - 1 - 1)$ )) (take (length  $\mathfrak{G} - 1$ )  $\mathfrak{G}$ )
proof (rule nth-equalityI )
  def  $n \equiv \text{length } \mathfrak{G} - 1$ 
  hence  $n = \text{length} (\text{take } n \ \mathfrak{G})$   $n > 0$   $n < \text{length } \mathfrak{G}$  using assms notempty by
auto
  interpret normal  $\mathfrak{G}$  butlast: normal-series ( $G$  (carrier :=  $\mathfrak{G} ! (n - 1)$ )) (take  $n$   $\mathfrak{G}$ )

  using normal-series-prefix-closed ( $n > 0$ ) ( $n < \text{length } \mathfrak{G}$ ) by auto
  have length (butlast quotients) = length quotients - 1 by (metis length-butlast)
  also have  $\dots = \text{length } \mathfrak{G} - 1 - 1$  by (metis add-diff-cancel-right' quotients-length)
  also have  $\dots = \text{length} (\text{take } n \ \mathfrak{G}) - 1$  by (metis ( $n = \text{length} (\text{take } n \ \mathfrak{G})$ ) n-def)
  also have  $\dots = \text{length } \text{normal}\mathfrak{G}\text{butlast.quotients}$  by (metis normal  $\mathfrak{G}$  butlast.quotients-length
diff-add-inverse2)
  finally show length (butlast quotients) = length normal  $\mathfrak{G}$  butlast.quotients .
  have  $\forall i < \text{length} (\text{butlast quotients}). \text{butlast quotients} ! i = \text{normal}\mathfrak{G}\text{butlast.quotients} ! i$ 
proof auto
  fix  $i$ 
  assume  $i : i < \text{length } \text{quotients} - \text{Suc } 0$ 
  hence  $i' : i < \text{length } \mathfrak{G} - 1$   $i < n$   $i + 1 < n$  unfolding n-def using
quotients-length by auto

```



```

from  $i$  have butlast quotients ! i = quotients ! i by (metis One-nat-def
length-butlast nth-butlast)
also have  $\dots = G(\text{carrier} := \mathfrak{G} ! (i + 1)) \text{ Mod } \mathfrak{G} ! i$  unfolding quotients-def
using  $i'(1)$  by auto
also have  $\dots = G(\text{carrier} := (\text{take } n \ \mathfrak{G}) ! (i + 1)) \text{ Mod } (\text{take } n \ \mathfrak{G}) ! i$  using
 $i'(2,3)$  nth-take by metis
also have  $\dots = \text{normal}\mathfrak{G}\text{butlast.quotients} ! i$  unfolding normal\mathfrak{G}\text{butlast.quotients-def}
using  $i'$  by fastforce
finally show butlast (normal-series.quotients G \mathfrak{G}) ! i = normal-series.quotients
(G(\text{carrier} := \mathfrak{G} ! (n - Suc 0))) (take n \mathfrak{G}) ! i by auto
qed
thus  $\forall i < \text{length} (\text{butlast quotients}). \text{butlast quotients} ! i$ 
 $= \text{normal-series.quotients} (G(\text{carrier} := \mathfrak{G} ! (\text{length } \mathfrak{G} - 1 - 1))) (\text{take}$ 
 $(\text{length } \mathfrak{G} - 1) \ \mathfrak{G}) ! i$ 
unfolding n-def by auto
qed

```

The main part of the Jordan Hlder theorem is its statement about the uniqueness of a composition series. Here, uniqueness up to reordering and isomorphism is modelled by stating that the multisets of isomorphism classes of all quotients are equal.

theorem *jordan-hoelder-multisets:*

```

assumes group G
assumes finite (carrier G)
assumes composition-series G \mathfrak{G}
assumes composition-series G \mathfrak{H}
shows  $\text{mset} (\text{map } \text{group.iso-class} (\text{normal-series.quotients } G \ \mathfrak{G}))$ 
 $= \text{mset} (\text{map } \text{group.iso-class} (\text{normal-series.quotients } G \ \mathfrak{H}))$ 
using assms
proof (induction length \mathfrak{G} arbitrary: \mathfrak{G} \mathfrak{H} G rule: full-nat-induct)
case ( $1 \ \mathfrak{G} \ \mathfrak{H} \ G$ )
then interpret comp\mathfrak{G}: composition-series G \mathfrak{G} by simp
from  $1$  interpret comp\mathfrak{H}: composition-series G \mathfrak{H} by simp
from  $1$  interpret grpG: group G by simp
show ?case
proof (cases length \mathfrak{G} \le 2)
next
case True
hence  $\text{length } \mathfrak{G} = 0 \vee \text{length } \mathfrak{G} = 1 \vee \text{length } \mathfrak{G} = 2$  by arith
with comp\mathfrak{G}.notempty have  $\text{length } \mathfrak{G} = 1 \vee \text{length } \mathfrak{G} = 2$  by simp
thus ?thesis
proof (auto simp del: mset-map)
— First trivial case:  $\mathfrak{G}$  is the trivial group.
assume  $\text{length } \mathfrak{G} = \text{Suc } 0$ 
hence  $\text{length}:\text{length } \mathfrak{G} = 1$  by simp
hence  $\text{length } [] + 1 = \text{length } \mathfrak{G}$  by auto
moreover from  $\text{length}$  have  $\text{char}\mathfrak{G}:\mathfrak{G} = [\{1_G\}]$  by (metis comp\mathfrak{G}.composition-series-length-one)
hence  $\text{carrier } G = \{1_G\}$  by (metis comp\mathfrak{G}.composition-series-triv-group)
with  $\text{length } \text{char}\mathfrak{G}$  have  $\mathfrak{G} = \mathfrak{H}$  using comp\mathfrak{H}.composition-series-triv-group

```

```

by simp
  thus ?thesis by simp
next
  — Second trivial case:  $\mathfrak{G}$  is simple.
  assume length  $\mathfrak{G} = 2$ 
  hence  $\mathfrak{G}\text{char}:\mathfrak{G} = [\{1_G\}, \text{carrier } G]$  by (metis comp $\mathfrak{G}$ .length-two-unique)
  hence simple:simple-group  $G$  by (metis comp $\mathfrak{G}$ .composition-series-simple-group)
  hence  $\mathfrak{H} = [\{1_G\}, \text{carrier } G]$  using comp $\mathfrak{H}$ .composition-series-simple-group
by auto
  with  $\mathfrak{G}\text{char}$  have  $\mathfrak{G} = \mathfrak{H}$  by simp
  thus ?thesis by simp
qed
next
  case False
  — Non-trivial case:  $\mathfrak{G}$  has length at least 3.
  hence length:length  $\mathfrak{G} \geq 3$  by simp
  — First we show that  $\mathfrak{H}$  must have a length of at least 3.
  hence  $\neg$  simple-group  $G$  using comp $\mathfrak{G}$ .composition-series-simple-group by auto
  hence  $\mathfrak{H} \neq [\{1_G\}, \text{carrier } G]$  using comp $\mathfrak{H}$ .composition-series-simple-group by
auto
  hence length  $\mathfrak{H} \neq 2$  using comp $\mathfrak{H}$ .length-two-unique by auto
  moreover from length have carrier  $G \neq \{1_G\}$  using comp $\mathfrak{G}$ .composition-series-length-one
comp $\mathfrak{G}$ .composition-series-triv-group by auto
  hence length  $\mathfrak{H} \neq 1$  using comp $\mathfrak{H}$ .composition-series-length-one comp $\mathfrak{H}$ .composition-series-triv-group
by auto
  moreover from comp $\mathfrak{H}$ .notempty have length  $\mathfrak{H} \neq 0$  by simp
  ultimately have length $\mathfrak{H}$ big:length  $\mathfrak{H} \geq 3$  using comp $\mathfrak{H}$ .notempty by arith
  def  $m \equiv$  length  $\mathfrak{H} - 1$ 
  def  $n \equiv$  length  $\mathfrak{G} - 1$ 
  from length $\mathfrak{H}$ big have  $m':m > 0$   $m < \text{length } \mathfrak{H}$   $(m - 1) + 1 < \text{length } \mathfrak{H}$   $m - 1 = \text{length } \mathfrak{H} - 2$   $m - 1 + 1 = \text{length } \mathfrak{H} - 1$   $m - 1 < \text{length } \mathfrak{H}$ 
  unfolding  $m$ -def by auto
  from length have  $n':n > 0$   $n < \text{length } \mathfrak{G}$   $(n - 1) + 1 < \text{length } \mathfrak{G}$   $n - 1 < \text{length } \mathfrak{G}$   $\text{Suc } n \leq \text{length } \mathfrak{G}$ 
   $n - 1 = \text{length } \mathfrak{G} - 2$   $n - 1 + 1 = \text{length } \mathfrak{G} - 1$  unfolding  $n$ -def by auto
  def  $\mathfrak{G}Pn \equiv G(\downarrow \text{carrier} := \mathfrak{G}!(n - 1))$ 
  def  $\mathfrak{H}Pm \equiv G(\downarrow \text{carrier} := \mathfrak{H}!(m - 1))$ 
  then interpret grp $\mathfrak{G}Pn$ : group  $\mathfrak{G}Pn$  unfolding  $\mathfrak{G}Pn$ -def using  $n'$  by (metis
comp $\mathfrak{G}$ .normal-series-subgroups comp $\mathfrak{G}$ .subgroup-imp-group)
  interpret grp $\mathfrak{H}Pm$ : group  $\mathfrak{H}Pm$  unfolding  $\mathfrak{H}Pm$ -def using  $m'$  comp $\mathfrak{H}$ .normal-series-subgroups
1(2) group.subgroup-imp-group by force
  have finGbl:finite (carrier  $\mathfrak{G}Pn$ ) using  $\langle n - 1 < \text{length } \mathfrak{G} \rangle$  1(3) unfolding
 $\mathfrak{G}Pn$ -def using comp $\mathfrak{G}$ .normal-series-subgroups comp $\mathfrak{G}$ .subgroup-finite by auto
  have finHbl:finite (carrier  $\mathfrak{H}Pm$ ) using  $\langle m - 1 < \text{length } \mathfrak{H} \rangle$  1(3) unfolding
 $\mathfrak{H}Pm$ -def using comp $\mathfrak{H}$ .normal-series-subgroups comp $\mathfrak{G}$ .subgroup-finite by auto
  have quot $\mathfrak{G}$ notempty:comp $\mathfrak{G}$ .quotients  $\neq \square$  using comp $\mathfrak{G}$ .quotients-length
length by auto
  have quot $\mathfrak{H}$ notempty:comp $\mathfrak{H}$ .quotients  $\neq \square$  using comp $\mathfrak{H}$ .quotients-length
length $\mathfrak{H}$ big by auto

```

— Instantiate truncated composition series since they are used for both cases

interpret \mathfrak{H} *butlast*: *composition-series* $\mathfrak{H}Pm$ *take* m \mathfrak{H} **using** *comp* \mathfrak{H} .*composition-series-prefix-closed* $m'(1,2)$ $\mathfrak{H}Pm$ -*def* **by** *auto*

interpret \mathfrak{G} *butlast*: *composition-series* $\mathfrak{G}Pn$ *take* n \mathfrak{G} **using** *comp* \mathfrak{G} .*composition-series-prefix-closed* $n'(1,2)$ $\mathfrak{G}Pn$ -*def* **by** *auto*

have *ltaken*: $n = \text{length } (\text{take } n \ \mathfrak{G})$ **using** *length-take* $n'(2)$ **by** *auto*

have *ltakem*: $m = \text{length } (\text{take } m \ \mathfrak{H})$ **using** *length-take* $m'(2)$ **by** *auto*

show *?thesis*

proof (*cases* $\mathfrak{H} ! (m - 1) = \mathfrak{G} ! (n - 1)$)

— If $\mathfrak{H} ! (l - 1) = \mathfrak{G} ! 1$, everything is simple...

case *True*

— The last quotients of \mathfrak{G} and \mathfrak{H} are equal.

have *lasteq*:*last comp* \mathfrak{G} .*quotients* = *last comp* \mathfrak{H} .*quotients*

proof —

from *length* **have** *lg*:*length* $\mathfrak{G} - 1 - 1 + 1 = \text{length } \mathfrak{G} - 1$ **by** (*metis* *Suc-diff-1 Suc-eq-plus1 n'(1) n-def*)

from *length* \mathfrak{H} *big* **have** *lh*:*length* $\mathfrak{H} - 1 - 1 + 1 = \text{length } \mathfrak{H} - 1$ **by** (*metis* *Suc-diff-1 Suc-eq-plus1 <0 < m> m-def*)

have *last comp* \mathfrak{G} .*quotients* = *G Mod* $\mathfrak{G} ! (n - 1)$ **using** *length comp* \mathfrak{G} .*last-quotient*

unfolding *n-def* **by** *auto*

also **have** $\dots = \text{G Mod } \mathfrak{H} ! (m - 1)$ **using** *True* **by** *simp*

also **have** $\dots = \text{last comp } \mathfrak{H}$.*quotients* **using** *length* \mathfrak{H} *big comp* \mathfrak{H} .*last-quotient*

unfolding *m-def* **by** *auto*

finally **show** *?thesis* .

qed

from *ltaken* **have** *ind*:*mset* (*map group.iso-class* \mathfrak{G} *butlast.quotients*) = *mset* (*map group.iso-class* \mathfrak{H} *butlast.quotients*)

using *1(1) True n'(5) grp* $\mathfrak{G}Pn$.*is-group finGbl* \mathfrak{G} *butlast.is-composition-series* \mathfrak{H} *butlast.is-composition-series* **unfolding** $\mathfrak{G}Pn$ -*def* $\mathfrak{H}Pm$ -*def* **by** *metis*

have *mset* (*map group.iso-class* *comp* \mathfrak{G} .*quotients*)

= *mset* (*map group.iso-class* (*butlast comp* \mathfrak{G} .*quotients* @ [*last comp* \mathfrak{G} .*quotients*])) **by** (*simp add: quots* \mathfrak{G} *notempty*)

also **have** $\dots = \text{mset } (\text{map group.iso-class } (\mathfrak{G}$ *butlast.quotients* @ [*last comp* \mathfrak{G} .*quotients*])) **using** *comp* \mathfrak{G} .*quotients-butlast length* **unfolding** *n-def* $\mathfrak{G}Pn$ -*def* **by** *auto*

also **have** $\dots = \text{mset } ((\text{map group.iso-class } \mathfrak{G}$ *butlast.quotients*) @ [*group.iso-class* (*last comp* \mathfrak{G} .*quotients*)])) **by** *auto*

also **have** $\dots = \text{mset } (\text{map group.iso-class } \mathfrak{G}$ *butlast.quotients*) + {*# group.iso-class* (*last comp* \mathfrak{G} .*quotients*) #} **by** *auto*

also **have** $\dots = \text{mset } (\text{map group.iso-class } \mathfrak{H}$ *butlast.quotients*) + {*# group.iso-class* (*last comp* \mathfrak{G} .*quotients*) #} **using** *ind* **by** *simp*

also **have** $\dots = \text{mset } (\text{map group.iso-class } \mathfrak{H}$ *butlast.quotients*) + {*# group.iso-class* (*last comp* \mathfrak{H} .*quotients*) #} **using** *lasteq* **by** *simp*

also **have** $\dots = \text{mset } ((\text{map group.iso-class } \mathfrak{H}$ *butlast.quotients*) @ [*group.iso-class* (*last comp* \mathfrak{H} .*quotients*)])) **by** *auto*

also **have** $\dots = \text{mset } (\text{map group.iso-class } (\mathfrak{H}$ *butlast.quotients* @ [*last comp* \mathfrak{H} .*quotients*])) **by** *auto*

also **have** $\dots = \text{mset } (\text{map group.iso-class } (\text{butlast comp } \mathfrak{H}$.*quotients* @ [*last*

```

comp $\mathfrak{H}$ .quotients])) using length $\mathfrak{H}$ big comp $\mathfrak{H}$ .quotients-butlast unfolding m-def
 $\mathfrak{H}Pm$ -def by auto
  also have ... = mset (map group.iso-class comp $\mathfrak{H}$ .quotients) using append-butlast-last-id
  quotes $\mathfrak{H}$ notempty by simp
  finally show ?thesis .
next
case False
def  $\mathfrak{H}PmInt\mathfrak{G}Pn \equiv G(\text{carrier} := \mathfrak{H}!(m-1) \cap \mathfrak{G}!(n-1))$ 
interpret  $\mathfrak{G}Pnmax$ : max-normal-subgroup  $\mathfrak{G}!(n-1) G$  unfolding n-def
  by (metis add-lessD1 diff-diff-add  $n'(3)$  add commute one-add-one 1(3)
  comp $\mathfrak{G}$ .snd-to-last-max-normal)
interpret  $\mathfrak{H}Pmmax$ : max-normal-subgroup  $\mathfrak{H}!(m-1) G$  unfolding m-def
  by (metis add-lessD1 diff-diff-add  $m'(3)$  add commute one-add-one 1(3)
  comp $\mathfrak{H}$ .snd-to-last-max-normal)
  have  $\mathfrak{H}PmnormG$ :  $\mathfrak{H}!(m-1) \triangleleft G$  using comp $\mathfrak{H}$ .normal-series-snd-to-last
   $m'(4)$  unfolding m-def by auto
  have  $\mathfrak{G}PnnormG$ :  $\mathfrak{G}!(n-1) \triangleleft G$  using comp $\mathfrak{G}$ .normal-series-snd-to-last
   $n'(6)$  unfolding n-def by auto
  have  $\mathfrak{H}Pmint\mathfrak{G}PnnormG$ :  $\mathfrak{H}!(m-1) \cap \mathfrak{G}!(n-1) \triangleleft G$  using  $\mathfrak{H}PmnormG$ 
   $\mathfrak{G}PnnormG$  by (rule comp $\mathfrak{G}$ .normal-subgroup-intersect)
  have  $Intnorm\mathfrak{G}Pn$ :  $\mathfrak{H}!(m-1) \cap \mathfrak{G}!(n-1) \triangleleft \mathfrak{G}Pn$  using  $\mathfrak{G}PnnormG$ 
   $\mathfrak{H}PmnormG$  Int-lower2 unfolding  $\mathfrak{G}Pn$ -def
  by (metis comp $\mathfrak{G}$ .normal-restrict-supergroup comp $\mathfrak{G}$ .normal-series-subgroups
  comp $\mathfrak{G}$ .normal-subgroup-intersect  $n'(4)$ )
  then interpret  $grp\mathfrak{G}PnMod\mathfrak{H}Pmint\mathfrak{G}Pn$ : group  $\mathfrak{G}Pn Mod \mathfrak{H}!(m-1) \cap$ 
   $\mathfrak{G}!(n-1)$  by (rule normal.factorgroup-is-group)
  have  $Intnorm\mathfrak{H}Pm$ :  $\mathfrak{H}!(m-1) \cap \mathfrak{G}!(n-1) \triangleleft \mathfrak{H}Pm$  using  $\mathfrak{H}PmnormG$ 
   $\mathfrak{G}PnnormG$  Int-lower2 Int-commute unfolding  $\mathfrak{H}Pm$ -def
  by (metis comp $\mathfrak{G}$ .normal-restrict-supergroup comp $\mathfrak{G}$ .normal-subgroup-intersect
  comp $\mathfrak{H}$ .normal-series-subgroups  $m'(6)$ )
  then interpret  $grp\mathfrak{H}PmMod\mathfrak{H}Pmint\mathfrak{G}Pn$ : group  $\mathfrak{H}Pm Mod \mathfrak{H}!(m-1) \cap$ 
   $\mathfrak{G}!(n-1)$  by (rule normal.factorgroup-is-group)

```

— Show that the second to last entries are not contained in each other.

```

have not $\mathfrak{H}PmSub\mathfrak{G}Pn$ :  $\neg(\mathfrak{H}!(m-1) \subseteq \mathfrak{G}!(n-1))$  using  $\mathfrak{H}Pmmax$ .max-normal
 $\mathfrak{G}PnnormG$  False[symmetric]  $\mathfrak{G}Pnmax$ .proper by simp
have not $\mathfrak{G}PnSub\mathfrak{H}Pm$ :  $\neg(\mathfrak{G}!(n-1) \subseteq \mathfrak{H}!(m-1))$  using  $\mathfrak{G}Pnmax$ .max-normal
 $\mathfrak{H}PmnormG$  False  $\mathfrak{H}Pmmax$ .proper by simp

```

— Show that $G Mod \mathfrak{H}!(m-1) \cap \mathfrak{G}!(n-1)$ is a simple group.

```

have  $\mathfrak{H}PmSubSetmult$ :  $\mathfrak{H}!(m-1) \subseteq \mathfrak{H}!(m-1) \langle \# \rangle_G \mathfrak{G}!(n-1)$ 
using second-isomorphism-grp.H-contained-in-set-mult  $\mathfrak{G}Pnmax$ .is-normal
 $\mathfrak{H}PmnormG$  normal-imp-subgroup
unfolding second-isomorphism-grp-def second-isomorphism-grp-axioms-def
max-normal-subgroup-def by metis
have  $\mathfrak{G}PnSubSetmult$ :  $\mathfrak{G}!(n-1) \subseteq \mathfrak{H}!(m-1) \langle \# \rangle_G \mathfrak{G}!(n-1)$ 
using second-isomorphism-grp.S-contained-in-set-mult  $\mathfrak{G}Pnmax$ .is-normal
 $\mathfrak{H}PmnormG$  normal-imp-subgroup
unfolding second-isomorphism-grp-def second-isomorphism-grp-axioms-def

```

max-normal-subgroup-def **by** *metis*

have $\mathfrak{G}!(n-1) \neq (\mathfrak{H}!(m-1)) \langle \# \rangle_G (\mathfrak{G}!(n-1))$ **using** $\mathfrak{H}PmSubSetmult$
not $\mathfrak{H}PmSub\mathfrak{G}Pn$ **by** *auto*

hence $set-multG:(\mathfrak{H}!(m-1)) \langle \# \rangle_G (\mathfrak{G}!(n-1)) = carrier\ G$

using $\mathfrak{G}Pnmax.max-normal\ \mathfrak{G}Pnmax.is-normal\ \mathfrak{H}PmnormG\ comp\mathfrak{G}.normal-subgroup-set-mult-closed$
 $\mathfrak{G}PnSubSetmult$ **by** *metis*

then obtain φ **where** $\varphi \in (\mathfrak{G}Pn\ Mod\ (\mathfrak{H}!(m-1) \cap \mathfrak{G}!(n-1))) \cong$
 $(G(\backslash carrier := carrier\ G) \backslash Mod\ \mathfrak{H}!(m-1))$

using $second-isomorphism-grp.normal-intersection-quotient-isom\ \mathfrak{H}PmnormG$
 $\mathfrak{G}Pnmax.is-normal\ normal-imp-subgroup$

unfolding $second-isomorphism-grp-def\ second-isomorphism-grp-axioms-def$
max-normal-subgroup-def\ \mathfrak{G}Pn-def **by** *metis*

hence $\varphi:\varphi \in (\mathfrak{G}Pn\ Mod\ (\mathfrak{H}!(m-1) \cap \mathfrak{G}!(n-1))) \cong (G\ Mod\ \mathfrak{H}!(m-1))$
by *auto*

then obtain $\varphi2$ **where** $\varphi2:\varphi2 \in (G\ Mod\ \mathfrak{H}!(m-1)) \cong (\mathfrak{G}Pn\ Mod\ (\mathfrak{H}!$
 $(m-1) \cap \mathfrak{G}!(n-1)))$

using $group.iso-sym\ grp\mathfrak{G}PnMod\mathfrak{H}Pmint\mathfrak{G}Pn.is-group$ **by** *auto*

moreover have $simple-group\ (G(\backslash carrier := \mathfrak{H}!(m-1+1)) \backslash Mod\ \mathfrak{H}!(m-1))$ **using** $comp\mathfrak{H}.simplefact\ m'(3)$ **by** *simp*

hence $simple-group\ (G\ Mod\ \mathfrak{H}!(m-1))$ **using** $comp\mathfrak{H}.last\ last-conv-nth$
 $comp\mathfrak{H}.notempty\ m'(5)$ **by** *fastforce*

ultimately have $simple\mathfrak{G}PnModInt:simple-group\ (\mathfrak{G}Pn\ Mod\ (\mathfrak{H}!(m-1)$
 $\cap \mathfrak{G}!(n-1)))$

using $simple-group.iso-simple\ grp\mathfrak{G}PnMod\mathfrak{H}Pmint\mathfrak{G}Pn.is-group$ **by** *auto*

interpret $grpGMod\mathfrak{H}Pm: group\ (G\ Mod\ \mathfrak{H}!(m-1))$ **by** $(metis\ \mathfrak{H}PmnormG$
 $normal.factorgroup-is-group)$

— Show analogues of the previous statements for $\mathfrak{H}!(m-1)$ instead of $\mathfrak{G}!(n-1)$.

have $\mathfrak{H}PmSubSetmult':\mathfrak{H}!(m-1) \subseteq \mathfrak{G}!(n-1) \langle \# \rangle_G \mathfrak{H}!(m-1)$

using $second-isomorphism-grp.S-contained-in-set-mult\ \mathfrak{G}Pnmax.is-normal$
 $\mathfrak{H}PmnormG\ normal-imp-subgroup$

unfolding $second-isomorphism-grp-def\ second-isomorphism-grp-axioms-def$
max-normal-subgroup-def **by** *metis*

have $\mathfrak{G}PnSubSetmult':\mathfrak{G}!(n-1) \subseteq \mathfrak{G}!(n-1) \langle \# \rangle_G \mathfrak{H}!(m-1)$

using $second-isomorphism-grp.H-contained-in-set-mult\ \mathfrak{G}Pnmax.is-normal$
 $\mathfrak{H}PmnormG\ normal-imp-subgroup$

unfolding $second-isomorphism-grp-def\ second-isomorphism-grp-axioms-def$
max-normal-subgroup-def **by** *metis*

have $\mathfrak{H}!(m-1) \neq (\mathfrak{G}!(n-1)) \langle \# \rangle_G (\mathfrak{H}!(m-1))$ **using** $\mathfrak{G}PnSubSetmult'$
not $\mathfrak{G}PnSub\mathfrak{H}Pm$ **by** *auto*

hence $set-multG:(\mathfrak{G}!(n-1)) \langle \# \rangle_G (\mathfrak{H}!(m-1)) = carrier\ G$

using $\mathfrak{H}Pmmax.max-normal\ \mathfrak{H}Pmmax.is-normal\ \mathfrak{G}PnnormG\ comp\mathfrak{G}.normal-subgroup-set-mult-closed$
 $\mathfrak{H}PmSubSetmult'$ **by** *metis*

from $set-multG$ **obtain** ψ **where** $\psi \in (\mathfrak{H}Pm\ Mod\ (\mathfrak{G}!(n-1) \cap \mathfrak{H}!(m-1))) \cong$
 $(G(\backslash carrier := carrier\ G) \backslash Mod\ \mathfrak{G}!(n-1))$

using $second-isomorphism-grp.normal-intersection-quotient-isom\ \mathfrak{G}PnnormG$
 $\mathfrak{H}Pmmax.is-normal\ normal-imp-subgroup$

unfolding $second-isomorphism-grp-def\ second-isomorphism-grp-axioms-def$

max-normal-subgroup-def $\mathfrak{H}Pm$ -def by metis
hence $\psi:\psi \in (\mathfrak{H}Pm \text{ Mod } (\mathfrak{H}! (m-1) \cap (\mathfrak{G}! (n-1)))) \cong (G(\text{carrier} := \text{carrier } G) \text{ Mod } \mathfrak{G}! (n-1))$ **using** *Int-commute* **by metis**
then obtain $\psi2$ **where** $\psi2:\psi2 \in (G \text{ Mod } \mathfrak{G}! (n-1)) \cong (\mathfrak{H}Pm \text{ Mod } (\mathfrak{H}! (m-1) \cap \mathfrak{G}! (n-1)))$
using *group.iso-sym grp* $\mathfrak{H}Pm$ *Mod* $\mathfrak{H}Pm$ *int* $\mathfrak{G}Pn$ *.is-group* **by auto**
moreover have *simple-group* $(G(\text{carrier} := \mathfrak{G}! (n-1+1)) \text{ Mod } \mathfrak{G}! (n-1))$ **using** *comp* \mathfrak{G} *.simplefact* $n'(3)$ **by simp**
hence *simple-group* $(G \text{ Mod } \mathfrak{G}! (n-1))$ **using** *comp* \mathfrak{G} *.last last-conv-nth comp* \mathfrak{G} *.notempty* $n'(7)$ **by fastforce**
ultimately have *simple* $\mathfrak{H}Pm$ *Mod**Int:simple-group* $(\mathfrak{H}Pm \text{ Mod } (\mathfrak{H}! (m-1) \cap \mathfrak{G}! (n-1)))$
using *simple-group.iso-simple grp* $\mathfrak{H}Pm$ *Mod* $\mathfrak{H}Pm$ *int* $\mathfrak{G}Pn$ *.is-group* **by auto**
interpret *grp* G *Mod* $\mathfrak{G}Pn$: *group* $(G \text{ Mod } \mathfrak{G}! (n-1))$ **by** $(\text{metis } \mathfrak{G}Pn$ *norm* G *normal.factorgroup-is-group)*

— Instantiate several composition series used to build up the equality of quotient multisets.

def $\mathfrak{K} \equiv \text{remdups-adj } (\text{map } (op \cap (\mathfrak{H}! (m-1))) \mathfrak{G})$
def $\mathfrak{L} \equiv \text{remdups-adj } (\text{map } (op \cap (\mathfrak{G}! (n-1))) \mathfrak{H})$
interpret \mathfrak{K} : *composition-series* $\mathfrak{H}Pm$ \mathfrak{K} **using** *comp* \mathfrak{G} *.intersect-normal* $1(3)$
 $\mathfrak{H}Pm$ *norm* G **unfolding** \mathfrak{K} -def $\mathfrak{H}Pm$ -def **by auto**
interpret \mathfrak{L} : *composition-series* $\mathfrak{G}Pn$ \mathfrak{L} **using** *comp* \mathfrak{H} *.intersect-normal* $1(3)$
 $\mathfrak{G}Pn$ *norm* G **unfolding** \mathfrak{L} -def $\mathfrak{G}Pn$ -def **by auto**

— Apply the induction hypothesis on \mathfrak{G} butlast and \mathfrak{L}

from $n'(2)$ **have** *Suc* $(\text{length } (\text{take } n \mathfrak{G})) \leq \text{length } \mathfrak{G}$ **by auto**
hence *msets* \mathfrak{G} *butlast* \mathfrak{L} : *mset* $(\text{map } \text{group.iso-class } \mathfrak{G}$ *butlast.quotients*) = *mset* $(\text{map } \text{group.iso-class } \mathfrak{L}$ *quotients*)
using *1.hyps grp* $\mathfrak{G}Pn$ *.is-group fin* Gbl \mathfrak{G} *butlast.is-composition-series* \mathfrak{L} *.is-composition-series* **by metis**
hence $\text{length } \mathfrak{L}:n = \text{length } \mathfrak{L}$ **using** \mathfrak{G} *butlast.quotients-length* \mathfrak{L} *.quotients-length length-map size-mset* *ltaken* **by metis**
hence $\text{length } \mathfrak{L}':\text{length } \mathfrak{L} > 1$ $\text{length } \mathfrak{L} - 1 > 0$ $\text{length } \mathfrak{L} - 1 \leq \text{length } \mathfrak{L}$
using $n'(6)$ *length* **by auto**
have *Inteq* \mathfrak{L} *sndlast*: $\mathfrak{H}! (m-1) \cap \mathfrak{G}! (n-1) = \mathfrak{L}! (\text{length } \mathfrak{L} - 1 - 1)$
proof —
have $\text{length } \mathfrak{L} - 1 - 1 + 1 < \text{length } \mathfrak{L}$ **using** *length* \mathfrak{L}' **by auto**
moreover have *KGnotempty*: $(\text{map } (op \cap (\mathfrak{G}! (n-1))) \mathfrak{H}) \neq []$ **using** *comp* \mathfrak{H} *.notempty* **by** $(\text{metis } Nil$ *-is-map-conv)
ultimately obtain i **where** $i:i + 1 < \text{length } (\text{map } (op \cap (\mathfrak{G}! (n-1))) \mathfrak{H})$
 $\mathfrak{L}! (\text{length } \mathfrak{L} - 1 - 1) = (\text{map } (op \cap (\mathfrak{G}! (n-1))) \mathfrak{H})! i$ $\mathfrak{L}! (\text{length } \mathfrak{L} - 1 - 1 + 1) = (\text{map } (op \cap (\mathfrak{G}! (n-1))) \mathfrak{H})! (i + 1)$
using *remdups-adj-obtain-adjacency* **unfolding** \mathfrak{L} -def **by force**
hence $\mathfrak{L}! (\text{length } \mathfrak{L} - 1 - 1) = \mathfrak{H}! i \cap \mathfrak{G}! (n-1)$ $\mathfrak{L}! (\text{length } \mathfrak{L} - 1 - 1 + 1) = \mathfrak{H}! (i + 1) \cap \mathfrak{G}! (n-1)$ **by auto**
hence $\mathfrak{L}! (\text{length } \mathfrak{L} - 1) = \mathfrak{H}! (i + 1) \cap \mathfrak{G}! (n-1)$ **using** *length* $\mathfrak{L}'(2)$*

by (metis Suc-diff-1 Suc-eq-plus1)
 hence $\mathfrak{G}Pn\text{sub}\mathfrak{H}Pm:\mathfrak{G}!(n-1) \subseteq \mathfrak{H}!(i+1)$ using $\mathfrak{L}.last$ $\mathfrak{L}.notempty$
last-conv-nth unfolding $\mathfrak{G}Pn\text{-def}$ by auto
 from $i(1)$ have $i+1 < m+1$ unfolding $m\text{-def}$ by auto
 moreover have $\neg(i+1 \leq m-1)$ using $comp\mathfrak{H}.entries\text{-mono } m'(6)$
not $\mathfrak{G}Pn\text{Sub}\mathfrak{H}Pm$ $\mathfrak{G}Pn\text{sub}\mathfrak{H}Pm$ by fastforce
 ultimately have $m-1 = i$ by auto
 with i show ?thesis by auto
 qed
 hence $\mathfrak{L}sndlast:\mathfrak{H}PmInt\mathfrak{G}Pn = (\mathfrak{G}Pn(\text{carrier} := \mathfrak{L}!(length \mathfrak{L} - 1 - 1)))$
unfolding $\mathfrak{H}PmInt\mathfrak{G}Pn\text{-def}$ $\mathfrak{G}Pn\text{-def}$ by auto
 then interpret $\mathfrak{L}butlast: composition\text{-series } \mathfrak{H}PmInt\mathfrak{G}Pn$ take (length $\mathfrak{L} - 1$) \mathfrak{L} using $length\mathfrak{L}'$ $\mathfrak{L}.composition\text{-series-prefix-closed}$ by metis
 from $\langle length \mathfrak{L} > 1 \rangle$ have $quots\mathfrak{L}notempty:\mathfrak{L}.quotients \neq []$ unfolding
 $\mathfrak{L}.quotients\text{-def}$ by auto

— Apply the induction hypothesis on $\mathfrak{L}butlast$ and $\mathfrak{R}butlast$

have $length \mathfrak{R} > 1$
 proof (rule ccontr)
 assume $\neg length \mathfrak{R} > 1$
 with $\mathfrak{R}.notempty$ have $length \mathfrak{R} = 1$ by (metis One-nat-def Suc-less1
length-greater-0-conv)
 hence $carrier \mathfrak{H}Pm = \{1_{\mathfrak{H}Pm}\}$ using $\mathfrak{R}.composition\text{-series-length-one}$
 $\mathfrak{R}.composition\text{-series-triv-group}$ by auto
 hence $carrier \mathfrak{H}Pm = \{1_G\}$ unfolding $\mathfrak{H}Pm\text{-def}$ by auto
 hence $carrier \mathfrak{H}Pm \subseteq \mathfrak{G}!(n-1)$ using $\mathfrak{G}Pnmax.is\text{-subgroup sub-}$
group.one-closed by auto
 with $not\mathfrak{H}Pm\text{Sub}\mathfrak{G}Pn$ show False unfolding $\mathfrak{H}Pm\text{-def}$ by auto
 qed
 hence $length\mathfrak{R}':length \mathfrak{R} - 1 > 0$ $length \mathfrak{R} - 1 \leq length \mathfrak{R}$ by auto
 have $Inteq\mathfrak{R}sndlast:\mathfrak{H}!(m-1) \cap \mathfrak{G}!(n-1) = \mathfrak{R}!(length \mathfrak{R} - 1 - 1)$
 proof –
 have $length \mathfrak{R} - 1 - 1 + 1 < length \mathfrak{R}$ using $length\mathfrak{R}'$ by auto
 moreover have $KGnotempty:(map (op \cap (\mathfrak{H}!(m-1))) \mathfrak{G}) \neq []$ using
 $comp\mathfrak{G}.notempty$ by (metis Nil-is-map-conv)
 ultimately obtain i where $i:i+1 < length (map (op \cap (\mathfrak{H}!(m-1)))$
 $\mathfrak{G})$
 $\mathfrak{R}!(length \mathfrak{R} - 1 - 1) = (map (op \cap (\mathfrak{H}!(m-1))) \mathfrak{G}) ! i \mathfrak{R}!(length$
 $\mathfrak{R} - 1 - 1 + 1) = (map (op \cap (\mathfrak{H}!(m-1))) \mathfrak{G}) ! (i+1)$
 using $remdups\text{-adj-obtain-adjacency}$ unfolding $\mathfrak{R}\text{-def}$ by force
 hence $\mathfrak{R}!(length \mathfrak{R} - 1 - 1) = \mathfrak{G}! i \cap \mathfrak{H}!(m-1) \mathfrak{R}!(length \mathfrak{R} - 1 -$
 $1 + 1) = \mathfrak{G}!(i+1) \cap \mathfrak{H}!(m-1)$ by auto
 hence $\mathfrak{R}!(length \mathfrak{R} - 1) = \mathfrak{G}!(i+1) \cap \mathfrak{H}!(m-1)$ using $length\mathfrak{R}'(1)$
 by (metis Suc-diff-1 Suc-eq-plus1)
 hence $\mathfrak{H}Pm\text{sub}\mathfrak{G}Pn:\mathfrak{H}!(m-1) \subseteq \mathfrak{G}!(i+1)$ using $\mathfrak{R}.last$ $\mathfrak{R}.notempty$
last-conv-nth unfolding $\mathfrak{H}Pm\text{-def}$ by auto
 from $i(1)$ have $i+1 < n+1$ unfolding $n\text{-def}$ by auto
 moreover have $\neg(i+1 \leq n-1)$ using $comp\mathfrak{G}.entries\text{-mono } n'(2)$
not $\mathfrak{H}Pm\text{Sub}\mathfrak{G}Pn$ $\mathfrak{H}Pm\text{sub}\mathfrak{G}Pn$ by fastforce

ultimately have $n - 1 = i$ **by** *auto*
with i **show** *?thesis* **by** *auto*
qed
have *composition-series* ($G(\text{carrier} := \mathfrak{K} ! (\text{length } \mathfrak{K} - 1 - 1))$) (*take* ($\text{length } \mathfrak{K} - 1$) \mathfrak{K})
using $\text{length}.\mathfrak{K}' \mathfrak{K}.\text{composition-series-prefix-closed}$ **unfolding** $\mathfrak{H}PmInt\mathfrak{G}Pn\text{-def}$ $\mathfrak{H}Pm\text{-def}$ **by** *fastforce*
then interpret $\mathfrak{K}butlast: \text{composition-series } \mathfrak{H}PmInt\mathfrak{G}Pn$ (*take* ($\text{length } \mathfrak{K} - 1$) \mathfrak{K}) **using** $Inteq\mathfrak{K}sndlast$ **unfolding** $\mathfrak{H}PmInt\mathfrak{G}Pn\text{-def}$ **by** *auto*
from $finGbl$ **have** $finInt:finite$ (*carrier* $\mathfrak{H}PmInt\mathfrak{G}Pn$) **unfolding** $\mathfrak{H}PmInt\mathfrak{G}Pn\text{-def}$ $\mathfrak{G}Pn\text{-def}$ **by** *simp*
moreover have Suc (length (*take* ($\text{length } \mathfrak{L} - 1$) \mathfrak{L})) \leq $\text{length } \mathfrak{G}$ **using** $\text{length}.\mathfrak{L}$ **unfolding** $n\text{-def}$ **using** $n'(2)$ **by** *auto*
ultimately have $\text{multisets}.\mathfrak{K}\mathfrak{L}butlast:mset$ (*map* $group.iso\text{-class } \mathfrak{L}butlast.quotients$) $=$ $mset$ (*map* $group.iso\text{-class } \mathfrak{K}butlast.quotients$)
using $1.hyps \mathfrak{L}butlast.is\text{-group } \mathfrak{K}butlast.is\text{-composition-series } \mathfrak{L}butlast.is\text{-composition-series}$
by *auto*
hence length (*take* ($\text{length } \mathfrak{K} - 1$) \mathfrak{K}) $=$ length (*take* ($\text{length } \mathfrak{L} - 1$) \mathfrak{L})
using $\mathfrak{K}butlast.quotients\text{-length } \mathfrak{L}butlast.quotients\text{-length } \text{length}\text{-map } size\text{-mset}$
by *metis*
hence length (*take* ($\text{length } \mathfrak{K} - 1$) \mathfrak{K}) $=$ $n - 1$ **using** $\text{length}.\mathfrak{L}$ $n'(1)$ **by** *auto*
hence $\text{length}.\mathfrak{K}:\text{length } \mathfrak{K} = n$ **by** (*metis* $Suc\text{-diff-1 } \mathfrak{K}.\text{notempty } butlast\text{-conv}\text{-take } \text{length}\text{-butlast } \text{length}\text{-greater-0}\text{-conv } n'(1)$)

— Apply the induction hypothesis on \mathfrak{K} and $\mathfrak{H}butlast$
from $Inteq\mathfrak{K}sndlast$ **have** $\mathfrak{K}sndlast:\mathfrak{H}PmInt\mathfrak{G}Pn = (\mathfrak{H}Pm(\text{carrier} := \mathfrak{K} ! (\text{length } \mathfrak{K} - 1 - 1)))$ **unfolding** $\mathfrak{H}PmInt\mathfrak{G}Pn\text{-def } \mathfrak{H}Pm\text{-def } \mathfrak{K}\text{-def}$ **by** *auto*
from $\text{length}.\mathfrak{K}$ **have** Suc ($\text{length } \mathfrak{K}$) \leq $\text{length } \mathfrak{G}$ **using** $n'(2)$ **by** *auto*
hence $\text{multisets}.\mathfrak{H}butlast.\mathfrak{K}:mset$ (*map* $group.iso\text{-class } \mathfrak{H}butlast.quotients$) $=$ $mset$ (*map* $group.iso\text{-class } \mathfrak{K}.quotients$)
using $1.hyps \text{grp}\mathfrak{H}Pm.is\text{-group } finHbl \mathfrak{H}butlast.is\text{-composition-series } \mathfrak{K}.is\text{-composition-series}$
by *metis*
hence $\text{length}.\mathfrak{K}:m = \text{length } \mathfrak{K}$ **using** $\mathfrak{H}butlast.quotients\text{-length } \mathfrak{K}.quotients\text{-length } \text{length}\text{-map } size\text{-mset } ltakem$ **by** *metis*
hence $\text{length } \mathfrak{K} > 1$ $\text{length } \mathfrak{K} - 1 > 0$ $\text{length } \mathfrak{K} - 1 \leq \text{length } \mathfrak{K}$ **using** $m'(4)$ $\text{length}.\mathfrak{H}big$ **by** *auto*
hence $quots\mathfrak{K}notempty:\mathfrak{K}.quotients \neq []$ **unfolding** $\mathfrak{K}.quotients\text{-def}$ **by** *auto*

interpret $\mathfrak{K}butlastadd\mathfrak{G}Pn: \text{composition-series } \mathfrak{G}Pn$ (*take* ($\text{length } \mathfrak{K} - 1$) \mathfrak{K})
@ [$\mathfrak{G} ! (n - 1)$]
using $\text{grp}\mathfrak{G}Pn.\text{composition-series-extend } \mathfrak{K}butlast.is\text{-composition-series } simple\mathfrak{G}PnModInt \text{Intnorm}\mathfrak{G}Pn$
unfolding $\mathfrak{G}Pn\text{-def } \mathfrak{H}PmInt\mathfrak{G}Pn\text{-def}$ **by** *auto*
interpret $\mathfrak{L}butlastadd\mathfrak{H}Pm: \text{composition-series } \mathfrak{H}Pm$ (*take* ($\text{length } \mathfrak{L} - 1$) \mathfrak{L}) @ [$\mathfrak{H} ! (m - 1)$]
using $\text{grp}\mathfrak{H}Pm.\text{composition-series-extend } \mathfrak{L}butlast.is\text{-composition-series } simple\mathfrak{H}PmModInt \text{Intnorm}\mathfrak{H}Pm$
unfolding $\mathfrak{H}Pm\text{-def } \mathfrak{H}PmInt\mathfrak{G}Pn\text{-def}$ **by** *auto*

— Prove equality of those composition series.

```

have mset (map group.iso-class comp $\mathfrak{G}$ .quotients)
      = mset (map group.iso-class ((butlast comp $\mathfrak{G}$ .quotients) @ [last
comp $\mathfrak{G}$ .quotients])) using quotes $\mathfrak{G}$ notempty by simp
also have ... = mset (map group.iso-class ( $\mathfrak{G}$ butlast.quotients @ [G Mod  $\mathfrak{G}$ 
!(n - 1)]))
using comp $\mathfrak{G}$ .quotients-butlast comp $\mathfrak{G}$ .last-quotient length unfolding n-def
 $\mathfrak{G}$ Pn-def by auto
also have ... = mset (map group.iso-class ((butlast  $\mathfrak{L}$ .quotients) @ [last
 $\mathfrak{L}$ .quotients])) + {# group.iso-class (G Mod  $\mathfrak{G}$  !(n - 1)) #}
using multisets $\mathfrak{G}$ butlast $\mathfrak{L}$  quotes $\mathfrak{L}$ notempty by simp
also have ... = mset (map group.iso-class ( $\mathfrak{L}$ butlast.quotients @ [ $\mathfrak{G}$ Pn Mod
 $\mathfrak{H}$  !(m - 1)  $\cap$   $\mathfrak{G}$  !(n - 1)])) + {# group.iso-class (G Mod  $\mathfrak{G}$  !(n - 1)) #}
using  $\mathfrak{L}$ .quotients-butlast  $\mathfrak{L}$ .last-quotient (length  $\mathfrak{L}$  > 1)  $\mathfrak{L}$ sndlast In-
teq $\mathfrak{L}$ sndlast unfolding n-def by auto
also have ... = mset (map group.iso-class  $\mathfrak{R}$ butlast.quotients) + {# group.iso-class
( $\mathfrak{G}$ Pn Mod  $\mathfrak{H}$  !(m - 1)  $\cap$   $\mathfrak{G}$  !(n - 1)) #} + {# group.iso-class (G Mod  $\mathfrak{G}$  !(n
- 1)) #}
using multisets $\mathfrak{R}$  $\mathfrak{L}$ butlast by simp
also have ... = mset (map group.iso-class  $\mathfrak{R}$ butlast.quotients) + {# group.iso-class
(G Mod  $\mathfrak{H}$  !(m - 1)) #} + {# group.iso-class ( $\mathfrak{H}$ Pm Mod  $\mathfrak{H}$  !(m - 1)  $\cap$   $\mathfrak{G}$  !(n
- 1)) #}
using  $\varphi$   $\psi$ 2 iso-classes-iff grp $\mathfrak{G}$ PnMod $\mathfrak{H}$ Pmint $\mathfrak{G}$ Pn.is-group grpGMod $\mathfrak{H}$ Pm.is-group
grpGMod $\mathfrak{G}$ Pn.is-group grp $\mathfrak{H}$ PmMod $\mathfrak{H}$ Pmint $\mathfrak{G}$ Pn.is-group
by metis
also have ... = mset (map group.iso-class  $\mathfrak{R}$ butlast.quotients) + {# group.iso-class
( $\mathfrak{H}$ Pm Mod  $\mathfrak{H}$  !(m - 1)  $\cap$   $\mathfrak{G}$  !(n - 1)) #} + {# group.iso-class (G Mod  $\mathfrak{H}$  !(m
- 1)) #}
by simp
also have ... = mset (map group.iso-class ((butlast  $\mathfrak{R}$ .quotients) @ [last
 $\mathfrak{R}$ .quotients])) + {# group.iso-class (G Mod  $\mathfrak{H}$  !(m - 1)) #}
using  $\mathfrak{R}$ .quotients-butlast  $\mathfrak{R}$ .last-quotient (length  $\mathfrak{R}$  > 1)  $\mathfrak{R}$ sndlast In-
teq $\mathfrak{R}$ sndlast unfolding m-def by auto
also have ... = mset (map group.iso-class  $\mathfrak{H}$ butlast.quotients) + {# group.iso-class
(G Mod  $\mathfrak{H}$  !(m - 1)) #}
using multisets $\mathfrak{H}$ butlast $\mathfrak{R}$  quotes $\mathfrak{R}$ notempty by simp
also have ... = mset (map group.iso-class ((butlast comp $\mathfrak{H}$ .quotients) @ [last
comp $\mathfrak{H}$ .quotients]))
using comp $\mathfrak{H}$ .quotients-butlast comp $\mathfrak{H}$ .last-quotient length $\mathfrak{H}$ big unfolding
m-def  $\mathfrak{H}$ Pm-def by auto
also have ... = mset (map group.iso-class comp $\mathfrak{H}$ .quotients) using quotes $\mathfrak{H}$ notempty
by simp
finally show ?thesis .
qed
qed
qed

```

As a corollary, we see that the composition series of a fixed group all have the same length.

corollary (in *jordan-hoelder*) *jordan-hoelder-size*:
 shows $\text{length } \mathfrak{G} = \text{length } \mathfrak{H}$

proof –
 have $\text{length } \mathfrak{G} = \text{length } \text{comp}\mathfrak{G}.\text{quotients} + 1$ **by** (*metis compG.quotients-length*)
 also have $\dots = \text{length } (\text{map } \text{group.iso-class } \text{comp}\mathfrak{G}.\text{quotients}) + 1$ **by** (*metis length-map*)
 also have $\dots = \text{size } (\text{mset } (\text{map } \text{group.iso-class } \text{comp}\mathfrak{G}.\text{quotients})) + 1$ **by** (*metis size-mset*)
 also have $\dots = \text{size } (\text{mset } (\text{map } \text{group.iso-class } \text{comp}\mathfrak{H}.\text{quotients})) + 1$
 using *jordan-hoelder-multisets is-group finite is-composition-series compH.is-composition-series*
by *metis*
 also have $\dots = \text{length } (\text{map } \text{group.iso-class } \text{comp}\mathfrak{H}.\text{quotients}) + 1$ **by** (*metis size-mset*)
 also have $\dots = \text{length } \text{comp}\mathfrak{H}.\text{quotients} + 1$ **by** (*metis length-map*)
 also have $\dots = \text{length } \mathfrak{H}$ **by** (*metis compH.quotients-length*)
 finally show *?thesis*.
qed

end

References

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