Decomposition of totally ordered hoops

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Abstract

We formalize a well known result in theory of hoops: every totally ordered hoop can be written as an ordinal sum of irreducible (equivalently Wajsberg) hoops. This formalization is based on the proof for BL-chains (i.e., bounded totally ordered hoops) by Busaniche [5].

Contents

1 Some order tools: posets with explicit universe

theory *Posets*
imports *Main HOL−Library.LaTeXsugar*

begin

locale *poset-on* =
  fixes P :: $'b$ *set*
  fixes *P-lesseq* :: $'b \Rightarrow 'b \Rightarrow$ *bool* (**infix** ‹$\leq^P$› *60*)
  fixes *P-less* :: $'b \Rightarrow 'b \Rightarrow$ *bool* (**infix** ‹$<^P$› *60*)
  assumes *not-empty* [*simp*]: $P \neq \emptyset$
  and *reflex*: *reflp-on* P ($\leq^P$)
  and *antisymm*: *antisymp-on* P ($\leq^P$)
  and *trans*: *transp-on* P ($\leq^P$)
  and *strict-iff-order*: $x \in P \Longrightarrow y \in P \Longrightarrow x <^P y = (x \leq^P y \wedge x \neq y)$
begin

lemma *strict-trans*:
  assumes $a \in P$ $b \in P$ $c \in P$ $a <^P b$ $b <^P c$
  shows $a <^P c$
  ⟨*proof*⟩

end

locale *bot-poset-on* = *poset-on* +
  fixes *bot* :: $'b$ (‹0^P›)
  assumes *bot-closed*: $0^P \in P$
  and *bot-first*: $x \in P \Longrightarrow 0^P \leq^P x$

locale *top-poset-on* = *poset-on* +
  fixes *top* :: $'b$ (‹1^P›)
  assumes *top-closed*: $1^P \in P$
  and *top-last*: $x \in P \Longrightarrow x \leq^P 1^P$

locale *bounded-poset-on* = *bot-poset-on* + *top-poset-on*

locale *total-poset-on* = *poset-on* +
  assumes *total*: *totalp-on* P ($\leq^P$)
begin

lemma *trichotomy*:
  assumes $a \in P$ $b \in P$
  shows $(a <^P b \wedge \neg(a = b \vee b <^P a)) \vee$
        $(a = b \wedge \neg(a <^P b \vee b <^P a)) \vee$
        $(b <^P a \wedge \neg(a = b \vee a <^P b))$
  ⟨*proof*⟩

lemma *strict-order-equiv-not-converse*:

assumes $a \in P$ $b \in P$
shows $a <^P b \longleftrightarrow \neg(b \leq^P a)$
$\langle proof \rangle$

end

end

2 Hoops

A *hoop* is a naturally ordered *pocrim* (i.e., a partially ordered commutative residuated integral monoid). This structures have been introduced by Büchi and Owens in [4] and constitute the algebraic counterpart of fragments without negation and falsum of some nonclassical logics.

theory *Hoops*
imports *Posets*
begin

2.1 Definitions

locale *hoop* =
fixes *universe* :: $'a$ *set* (‹A›)
and *multiplication* :: $'a \Rightarrow 'a \Rightarrow 'a$ (**infix** ‹$*^A$› *60*)
and *implication* :: $'a \Rightarrow 'a \Rightarrow 'a$ (**infix** ‹$\to^A$› *60*)
and *one* :: $'a$ (‹1^A›)
assumes *mult-closed*: $x \in A \Longrightarrow y \in A \Longrightarrow x *^A y \in A$
and *imp-closed*: $x \in A \Longrightarrow y \in A \Longrightarrow x \to^A y \in A$
and *one-closed* [*simp*]: $1^A \in A$
and *mult-comm*: $x \in A \Longrightarrow y \in A \Longrightarrow x *^A y = y *^A x$
and *mult-assoc*: $x \in A \Longrightarrow y \in A \Longrightarrow z \in A \Longrightarrow x *^A (y *^A z) = (x *^A y) *^A z$
and *mult-neutr* [*simp*]: $x \in A \Longrightarrow x *^A 1^A = x$
and *imp-reflex* [*simp*]: $x \in A \Longrightarrow x \to^A x = 1^A$
and *divisibility*: $x \in A \Longrightarrow y \in A \Longrightarrow x *^A (x \to^A y) = y *^A (y \to^A x)$
and *residuation*: $x \in A \Longrightarrow y \in A \Longrightarrow z \in A \Longrightarrow$
$x \to^A (y \to^A z) = (x *^A y) \to^A z$
begin

definition *hoop-order* :: $'a \Rightarrow 'a \Rightarrow bool$ (**infix** ‹$\leq^A$› *60*)
where $x \leq^A y \equiv (x \to^A y = 1^A)$

definition *hoop-order-strict* :: $'a \Rightarrow 'a \Rightarrow bool$ (**infix** ‹$<^A$› *60*)
where $x <^A y \equiv (x \leq^A y \wedge x \neq y)$

definition *hoop-inf* :: $'a \Rightarrow 'a \Rightarrow 'a$ (**infix** ‹$\wedge^A$› *60*)
where $x \wedge^A y = x *^A (x \to^A y)$

definition *hoop-pseudo-sup* :: $'a \Rightarrow 'a \Rightarrow 'a$ (**infix** ‹$\vee^{*A}$› *60*)
where $x \vee^{*A} y = ((x \to^A y) \to^A y) \wedge^A ((y \to^A x) \to^A x)$

end

locale *wajsberg-hoop* = *hoop* +
  assumes *T*: $x \in A \Longrightarrow y \in A \Longrightarrow (x \to^A y) \to^A y = (y \to^A x) \to^A x$
begin

definition *wajsberg-hoop-sup* :: $'a \Rightarrow 'a \Rightarrow 'a$ (**infix** ‹$\vee^A$› *60*)
  where $x \vee^A y = (x \to^A y) \to^A y$

end

2.2 Basic properties

context *hoop*
begin

lemma *mult-neutr-2* [*simp*]:
  assumes $a \in A$
  shows $1^A *^A a = a$
  ⟨*proof*⟩

lemma *imp-one-A*:
  assumes $a \in A$
  shows $(1^A \to^A a) \to^A 1^A = 1^A$
⟨*proof*⟩

lemma *imp-one-B*:
  assumes $a \in A$
  shows $(1^A \to^A a) \to^A a = 1^A$
⟨*proof*⟩

lemma *imp-one-C*:
  assumes $a \in A$
  shows $1^A \to^A a = a$
⟨*proof*⟩

lemma *imp-one-top*:
  assumes $a \in A$
  shows $a \to^A 1^A = 1^A$
⟨*proof*⟩

The proofs of *imp-one-A*, *imp-one-B*, *imp-one-C* and *imp-one-top* are based on proofs found in [3] (see Section 1: (4), (6), (7) and (12)).

lemma *swap*:
  assumes $a \in A$ $b \in A$ $c \in A$
  shows $a \to^A (b \to^A c) = b \to^A (a \to^A c)$
⟨*proof*⟩

lemma *imp-A*:
assumes $a \in A$ $b \in A$
shows $a \to^A (b \to^A a) = 1^A$
⟨*proof*⟩

2.3 Multiplication monotonicity

lemma *mult-mono*:
assumes $a \in A$ $b \in A$ $c \in A$
shows $(a \to^A b) \to^A ((a *^A c) \to^A (b *^A c)) = 1^A$
⟨*proof*⟩

2.4 Implication monotonicity and anti-monotonicity

lemma *imp-mono*:
assumes $a \in A$ $b \in A$ $c \in A$
shows $(a \to^A b) \to^A ((c \to^A a) \to^A (c \to^A b)) = 1^A$
⟨*proof*⟩

lemma *imp-anti-mono*:
assumes $a \in A$ $b \in A$ $c \in A$
shows $(a \to^A b) \to^A ((b \to^A c) \to^A (a \to^A c)) = 1^A$
⟨*proof*⟩

2.5 $(\leq^A)$ defines a partial order over A

lemma *ord-reflex*:
assumes $a \in A$
shows $a \leq^A a$
⟨*proof*⟩

lemma *ord-trans*:
assumes $a \in A$ $b \in A$ $c \in A$ $a \leq^A b$ $b \leq^A c$
shows $a \leq^A c$
⟨*proof*⟩

lemma *ord-antisymm*:
assumes $a \in A$ $b \in A$ $a \leq^A b$ $b \leq^A a$
shows $a = b$
⟨*proof*⟩

lemma *ord-antisymm-equiv*:
assumes $a \in A$ $b \in A$ $a \to^A b = 1^A$ $b \to^A a = 1^A$
shows $a = b$
⟨*proof*⟩

lemma *ord-top*:
assumes $a \in A$
shows $a \leq^A 1^A$
⟨*proof*⟩

sublocale *top-poset-on* A $(\le^A)$ $(<^A)$ 1^A
⟨*proof*⟩

2.6 Order properties

lemma *ord-mult-mono-A*:
assumes $a \in A$ $b \in A$ $c \in A$
shows $(a \to^A b) \le^A ((a *^A c) \to^A (b *^A c))$
⟨*proof*⟩

lemma *ord-mult-mono-B*:
assumes $a \in A$ $b \in A$ $c \in A$ $a \le^A b$
shows $(a *^A c) \le^A (b *^A c)$
⟨*proof*⟩

lemma *ord-residuation*:
assumes $a \in A$ $b \in A$ $c \in A$
shows $(a *^A b) \le^A c \longleftrightarrow a \le^A (b \to^A c)$
⟨*proof*⟩

lemma *ord-imp-mono-A*:
assumes $a \in A$ $b \in A$ $c \in A$
shows $(a \to^A b) \le^A ((c \to^A a) \to^A (c \to^A b))$
⟨*proof*⟩

lemma *ord-imp-mono-B*:
assumes $a \in A$ $b \in A$ $c \in A$ $a \le^A b$
shows $(c \to^A a) \le^A (c \to^A b)$
⟨*proof*⟩

lemma *ord-imp-anti-mono-A*:
assumes $a \in A$ $b \in A$ $c \in A$
shows $(a \to^A b) \le^A ((b \to^A c) \to^A (a \to^A c))$
⟨*proof*⟩

lemma *ord-imp-anti-mono-B*:
assumes $a \in A$ $b \in A$ $c \in A$ $a \le^A b$
shows $(b \to^A c) \le^A (a \to^A c)$
⟨*proof*⟩

lemma *ord-A*:
assumes $a \in A$ $b \in A$
shows $b \le^A (a \to^A b)$
⟨*proof*⟩

lemma *ord-B*:
assumes $a \in A$ $b \in A$
shows $b \le^A ((a \to^A b) \to^A b)$

⟨*proof*⟩

lemma *ord-C*:
assumes $a \in A$ $b \in A$
shows $a \leq^A ((a \to^A b) \to^A b)$
⟨*proof*⟩

lemma *ord-D*:
assumes $a \in A$ $b \in A$ $a <^A b$
shows $b \to^A a \neq 1^A$
⟨*proof*⟩

2.7 Additional multiplication properties

lemma *mult-lesseq-inf*:
assumes $a \in A$ $b \in A$
shows $(a *^A b) \leq^A (a \wedge^A b)$
⟨*proof*⟩

lemma *mult-A*:
assumes $a \in A$ $b \in A$
shows $(a *^A b) \leq^A a$
⟨*proof*⟩

lemma *mult-B*:
assumes $a \in A$ $b \in A$
shows $(a *^A b) \leq^A b$
⟨*proof*⟩

lemma *mult-C*:
assumes $a \in A-\{1^A\}$ $b \in A-\{1^A\}$
shows $a *^A b \in A-\{1^A\}$
⟨*proof*⟩

2.8 Additional implication properties

lemma *imp-B*:
assumes $a \in A$ $b \in A$
shows $a \to^A b = ((a \to^A b) \to^A b) \to^A b$
⟨*proof*⟩

The following two results can be found in [2] (see Proposition 1.7 and 2.2).

lemma *imp-C*:
assumes $a \in A$ $b \in A$
shows $(a \to^A b) \to^A (b \to^A a) = b \to^A a$
⟨*proof*⟩

lemma *imp-D*:
assumes $a \in A$ $b \in A$

shows $(((b \rightarrow^A a) \rightarrow^A a) \rightarrow^A b) \rightarrow^A (b \rightarrow^A a) = b \rightarrow^A a$
⟨*proof*⟩

2.9 $(\wedge^A)$ defines a semilattice over A

lemma *inf-closed*:
assumes $a \in A$ $b \in A$
shows $a \wedge^A b \in A$
⟨*proof*⟩

lemma *inf-comm*:
assumes $a \in A$ $b \in A$
shows $a \wedge^A b = b \wedge^A a$
⟨*proof*⟩

lemma *inf-A*:
assumes $a \in A$ $b \in A$
shows $(a \wedge^A b) \leq^A a$
⟨*proof*⟩

lemma *inf-B*:
assumes $a \in A$ $b \in A$
shows $(a \wedge^A b) \leq^A b$
⟨*proof*⟩

lemma *inf-C*:
assumes $a \in A$ $b \in A$ $c \in A$ $a \leq^A b$ $a \leq^A c$
shows $a \leq^A (b \wedge^A c)$
⟨*proof*⟩

lemma *inf-order*:
assumes $a \in A$ $b \in A$
shows $a \leq^A b \longleftrightarrow (a \wedge^A b = a)$
⟨*proof*⟩

2.10 Properties of $(\vee^{*A})$

lemma *pseudo-sup-closed*:
assumes $a \in A$ $b \in A$
shows $a \vee^{*A} b \in A$
⟨*proof*⟩

lemma *pseudo-sup-comm*:
assumes $a \in A$ $b \in A$
shows $a \vee^{*A} b = b \vee^{*A} a$
⟨*proof*⟩

lemma *pseudo-sup-A*:
assumes $a \in A$ $b \in A$
shows $a \leq^A (a \vee^{*A} b)$

⟨*proof*⟩

lemma *pseudo-sup-B*:
assumes $a \in A$ $b \in A$
shows $b \leq^A (a \vee^{*A} b)$
⟨*proof*⟩

lemma *pseudo-sup-order*:
assumes $a \in A$ $b \in A$
shows $a \leq^A b \longleftrightarrow a \vee^{*A} b = b$
⟨*proof*⟩

end

end

3 Ordinal sums

We define *tower of hoops*, a family of almost disjoint hoops indexed by a total order. This is based on the definition of *bounded tower of irreducible hoops* in [5] (see paragraph after Lemma 3.3). Parting from a tower of hoops we can define a hoop known as *ordinal sum*. Ordinal sums are a fundamental tool in the study of totally ordered hoops.

theory *Ordinal-Sums*
imports *Hoops*
begin

3.1 Tower of hoops

locale *tower-of-hoops* =
fixes *index-set* :: $'b$ *set* (‹*I*›)
fixes *index-lesseq* :: $'b \Rightarrow 'b \Rightarrow$ *bool* (**infix** ‹$\leq^I$› *60*)
fixes *index-less* :: $'b \Rightarrow 'b \Rightarrow$ *bool* (**infix** ‹$<^I$› *60*)
fixes *universes* :: $'b \Rightarrow ('a$ *set*) (‹*UNI*›)
fixes *multiplications* :: $'b \Rightarrow ('a \Rightarrow 'a \Rightarrow 'a)$ (‹*MUL*›)
fixes *implications* :: $'b \Rightarrow ('a \Rightarrow 'a \Rightarrow 'a)$ (‹*IMP*›)
fixes *sum-one* :: $'a$ (‹1^S›)
assumes *index-set-total-order*: *total-poset-on* I $(\leq^I)$ $(<^I)$
and *almost-disjoint*: $i \in I \Longrightarrow j \in I \Longrightarrow i \neq j \Longrightarrow UNI\ i \cap UNI\ j = \{1^S\}$
and *family-of-hoops*: $i \in I \Longrightarrow$ *hoop* $(UNI\ i)$ $(MUL\ i)$ $(IMP\ i)$ 1^S
begin

sublocale *total-poset-on* I $(\leq^I)$ $(<^I)$
⟨*proof*⟩

abbreviation (*uni-i*)
uni-i :: $['b] \Rightarrow ('a$ *set*) (‹($\mathbb{A}$(_))› [*61*] *60*)
where $\mathbb{A}_i \equiv UNI\ i$

abbreviation (*mult-i*)
mult-i :: $['b] \Rightarrow ('a \Rightarrow 'a \Rightarrow 'a)$ (‹(∗(ˉ))› [*61*] *60*)
where $*^i \equiv MUL\ i$

abbreviation (*imp-i*)
imp-i :: $['b] \Rightarrow ('a \Rightarrow 'a \Rightarrow 'a)$ (‹(→(ˉ))› [*61*] *60*)
where $\rightarrow^i \equiv IMP\ i$

abbreviation (*mult-i-xy*)
mult-i-xy :: $['a, 'b, 'a] \Rightarrow 'a$ (‹((-)/ ∗(ˉ) / (-))› [*61*, *50*, *61*] *60*)
where $x *^i y \equiv MUL\ i\ x\ y$

abbreviation (*imp-i-xy*)
imp-i-xy :: $['a, 'b, 'a] \Rightarrow 'a$ (‹((-)/ →(ˉ) / (-))› [*61*, *50*, *61*] *60*)
where $x \rightarrow^i y \equiv IMP\ i\ x\ y$

3.2 Ordinal sum universe

definition *sum-univ* :: *'a set* (‹*S*›)
where $S = \{x.\ \exists\ i \in I.\ x \in \mathbb{A}_i\}$

lemma *sum-one-closed* [*simp*]: $1^S \in S$
⟨*proof*⟩

lemma *sum-subsets*:
assumes $i \in I$
shows $\mathbb{A}_i \subseteq S$
⟨*proof*⟩

3.3 Floor function: definition and properties

lemma *floor-unique*:
assumes $a \in S-\{1^S\}$
shows $\exists!\ i.\ i \in I \wedge a \in \mathbb{A}_i$
⟨*proof*⟩

function *floor* :: $'a \Rightarrow 'b$ **where**
floor $x = (THE\ i.\ i \in I \wedge x \in \mathbb{A}_i)$ **if** $x \in S-\{1^S\}$
| *floor* $x = undefined$ **if** $x = 1^S \vee x \notin S$
⟨*proof*⟩
termination ⟨*proof*⟩

abbreviation (*uni-floor*)
uni-floor :: $['a] \Rightarrow ('a\ set)$ (‹($\mathbb{A}_{floor}$ (-))› [*61*] *60*)
where $\mathbb{A}_{floor\ x} \equiv UNI\ (floor\ x)$

abbreviation (*mult-floor*)
mult-floor :: $['a] \Rightarrow ('a \Rightarrow 'a \Rightarrow 'a)$ (‹($*^{floor}$ (ˉ))› [*61*] *60*)
where $*^{floor\ a} \equiv MUL\ (floor\ a)$

abbreviation (*imp-floor*)
  imp-floor :: $['a] \Rightarrow ('a \Rightarrow 'a \Rightarrow 'a)$ (‹($\rightarrow^{floor}$ (-))› [61] 60)
  where $\rightarrow^{floor\ a} \equiv IMP\ (floor\ a)$

abbreviation (*mult-floor-xy*)
  mult-floor-xy :: $['a, 'a, 'a] \Rightarrow 'a$ (‹((-)/ $*^{floor}$ (-) / (-))› [61, 50, 61] 60)
  where $x *^{floor\ y} z \equiv MUL\ (floor\ y)\ x\ z$

abbreviation (*imp-floor-xy*)
  imp-floor-xy :: $['a, 'a, 'a] \Rightarrow 'a$ (‹((-)/ $\rightarrow^{floor}$ (-) / (-))› [61, 50, 61] 60)
  where $x \rightarrow^{floor\ y} z \equiv IMP\ (floor\ y)\ x\ z$

lemma *floor-prop*:
  assumes $a \in S-\{1^S\}$
  shows $floor\ a \in I \wedge a \in \mathbb{A}_{floor\ a}$
⟨*proof*⟩

lemma *floor-one-closed*:
  assumes $i \in I$
  shows $1^S \in \mathbb{A}_i$
  ⟨*proof*⟩

lemma *floor-mult-closed*:
  assumes $i \in I$ $a \in \mathbb{A}_i$ $b \in \mathbb{A}_i$
  shows $a *^i b \in \mathbb{A}_i$
  ⟨*proof*⟩

lemma *floor-imp-closed*:
  assumes $i \in I$ $a \in \mathbb{A}_i$ $b \in \mathbb{A}_i$
  shows $a \rightarrow^i b \in \mathbb{A}_i$
  ⟨*proof*⟩

3.4 Ordinal sum multiplication and implication

function *sum-mult* :: $'a \Rightarrow 'a \Rightarrow 'a$ (**infix** ‹$*^S$› 60) **where**
  $x *^S y = x *^{floor\ x} y$ **if** $x \in S-\{1^S\}$ $y \in S-\{1^S\}$ $floor\ x = floor\ y$
| $x *^S y = x$ **if** $x \in S-\{1^S\}$ $y \in S-\{1^S\}$ $floor\ x <^I floor\ y$
| $x *^S y = y$ **if** $x \in S-\{1^S\}$ $y \in S-\{1^S\}$ $floor\ y <^I floor\ x$
| $x *^S y = y$ **if** $x = 1^S$ $y \in S-\{1^S\}$
| $x *^S y = x$ **if** $x \in S-\{1^S\}$ $y = 1^S$
| $x *^S y = 1^S$ **if** $x = 1^S$ $y = 1^S$
| $x *^S y = undefined$ **if** $x \notin S \vee y \notin S$
  ⟨*proof*⟩
termination ⟨*proof*⟩

function *sum-imp* :: $'a \Rightarrow 'a \Rightarrow 'a$ (**infix** ‹$\rightarrow^S$› 60) **where**
  $x \rightarrow^S y = x \rightarrow^{floor\ x} y$ **if** $x \in S-\{1^S\}$ $y \in S-\{1^S\}$ $floor\ x = floor\ y$
| $x \rightarrow^S y = 1^S$ **if** $x \in S-\{1^S\}$ $y \in S-\{1^S\}$ $floor\ x <^I floor\ y$

| $x \to^S y = y$ **if** $x \in S-\{1^S\}$ $y \in S-\{1^S\}$ *floor* $y <^I$ *floor* x
| $x \to^S y = y$ **if** $x = 1^S$ $y \in S-\{1^S\}$
| $x \to^S y = 1^S$ **if** $x \in S-\{1^S\}$ $y = 1^S$
| $x \to^S y = 1^S$ **if** $x = 1^S$ $y = 1^S$
| $x \to^S y =$ *undefined* **if** $x \notin S \vee y \notin S$
⟨*proof*⟩
termination ⟨*proof*⟩

3.4.1 Some multiplication properties

lemma *sum-mult-not-one-aux*:
assumes $a \in S-\{1^S\}$ $b \in \mathbb{A}_{floor\ a}$
shows $a *^S b \in (\mathbb{A}_{floor\ a})-\{1^S\}$
⟨*proof*⟩

corollary *sum-mult-not-one*:
assumes $a \in S-\{1^S\}$ $b \in \mathbb{A}_{floor\ a}$
shows $a *^S b \in S-\{1^S\} \wedge$ *floor* $(a *^S b) =$ *floor* a
⟨*proof*⟩

lemma *sum-mult-A*:
assumes $a \in S-\{1^S\}$ $b \in \mathbb{A}_{floor\ a}$
shows $a *^S b = a *^{floor\ a} b \wedge b *^S a = b *^{floor\ a} a$
⟨*proof*⟩

3.4.2 Some implication properties

lemma *sum-imp-floor*:
assumes $a \in S-\{1^S\}$ $b \in S-\{1^S\}$ *floor* $a =$ *floor* b $a \to^S b \in S-\{1^S\}$
shows *floor* $(a \to^S b) =$ *floor* a
⟨*proof*⟩

lemma *sum-imp-A*:
assumes $a \in S-\{1^S\}$ $b \in \mathbb{A}_{floor\ a}$
shows $a \to^S b = a \to^{floor\ a} b$
⟨*proof*⟩

lemma *sum-imp-B*:
assumes $a \in S-\{1^S\}$ $b \in \mathbb{A}_{floor\ a}$
shows $b \to^S a = b \to^{floor\ a} a$
⟨*proof*⟩

lemma *sum-imp-floor-antisymm*:
assumes $a \in S-\{1^S\}$ $b \in S-\{1^S\}$ *floor* $a =$ *floor* b
$a \to^S b = 1^S$ $b \to^S a = 1^S$
shows $a = b$
⟨*proof*⟩

corollary *sum-imp-C*:
assumes $a \in S-\{1^S\}$ $b \in S-\{1^S\}$ $a \neq b$ *floor* $a =$ *floor* b $a \to^S b = 1^S$

shows $b \to^S a \neq 1^S$
⟨*proof*⟩

lemma *sum-imp-D*:
assumes $a \in S$
shows $1^S \to^S a = a$
⟨*proof*⟩

lemma *sum-imp-E*:
assumes $a \in S$
shows $a \to^S 1^S = 1^S$
⟨*proof*⟩

3.5 The ordinal sum of a tower of hoops is a hoop

3.5.1 S is not empty

lemma *sum-not-empty*: $S \neq \emptyset$
⟨*proof*⟩

3.5.2 $(*^S)$ and $(\to^S)$ are well defined

lemma *sum-mult-closed-one*:
assumes $a \in S$ $b \in S$ $a = 1^S \vee b = 1^S$
shows $a *^S b \in S$
⟨*proof*⟩

lemma *sum-mult-closed-not-one*:
assumes $a \in S-\{1^S\}$ $b \in S-\{1^S\}$
shows $a *^S b \in S-\{1^S\}$
⟨*proof*⟩

lemma *sum-mult-closed*:
assumes $a \in S$ $b \in S$
shows $a *^S b \in S$
⟨*proof*⟩

lemma *sum-imp-closed-one*:
assumes $a \in S$ $b \in S$ $a = 1^S \vee b = 1^S$
shows $a \to^S b \in S$
⟨*proof*⟩

lemma *sum-imp-closed-not-one*:
assumes $a \in S-\{1^S\}$ $b \in S-\{1^S\}$
shows $a \to^S b \in S$
⟨*proof*⟩

lemma *sum-imp-closed*:
assumes $a \in S$ $b \in S$
shows $a \to^S b \in S$

$\langle proof \rangle$

3.5.3 Neutrality of 1^S

lemma *sum-mult-neutr*:
assumes $a \in S$
shows $a *^S 1^S = a \wedge 1^S *^S a = a$
$\langle proof \rangle$

3.5.4 Commutativity of $(*^S)$

Now we prove $x *^S y = y *^S x$ by showing that it holds when one of the variables is equal to 1^S. Then we consider when none of them is 1^S.

lemma *sum-mult-comm-one*:
assumes $a \in S$ $b \in S$ $a = 1^S \vee b = 1^S$
shows $a *^S b = b *^S a$
$\langle proof \rangle$

lemma *sum-mult-comm-not-one*:
assumes $a \in S-\{1^S\}$ $b \in S-\{1^S\}$
shows $a *^S b = b *^S a$
$\langle proof \rangle$

lemma *sum-mult-comm*:
assumes $a \in S$ $b \in S$
shows $a *^S b = b *^S a$
$\langle proof \rangle$

3.5.5 Associativity of $(*^S)$

Next we prove $x *^S (y *^S z) = (x *^S y) *^S z$.

lemma *sum-mult-assoc-one*:
assumes $a \in S$ $b \in S$ $c \in S$ $a = 1^S \vee b = 1^S \vee c = 1^S$
shows $a *^S (b *^S c) = (a *^S b) *^S c$
$\langle proof \rangle$

lemma *sum-mult-assoc-not-one*:
assumes $a \in S-\{1^S\}$ $b \in S-\{1^S\}$ $c \in S-\{1^S\}$
shows $a *^S (b *^S c) = (a *^S b) *^S c$
$\langle proof \rangle$

lemma *sum-mult-assoc*:
assumes $a \in S$ $b \in S$ $c \in S$
shows $a *^S (b *^S c) = (a *^S b) *^S c$
$\langle proof \rangle$

3.5.6 Reflexivity of $(\rightarrow^S)$

lemma *sum-imp-reflex*:

assumes $a \in S$
shows $a \rightarrow^S a = 1^S$
⟨*proof*⟩

3.5.7 Divisibility

We prove $x *^S (x \rightarrow^S y) = y *^S (y \rightarrow^S x)$ using the same methods as before.

lemma *sum-divisibility-one*:
assumes $a \in S$ $b \in S$ $a = 1^S \vee b = 1^S$
shows $a *^S (a \rightarrow^S b) = b *^S (b \rightarrow^S a)$
⟨*proof*⟩

lemma *sum-divisibility-aux*:
assumes $a \in S-\{1^S\}$ $b \in \mathbb{A}_{floor\ a}$
shows $a *^S (a \rightarrow^S b) = a *^{floor\ a} (a \rightarrow^{floor\ a} b)$
⟨*proof*⟩

lemma *sum-divisibility-not-one*:
assumes $a \in S-\{1^S\}$ $b \in S-\{1^S\}$
shows $a *^S (a \rightarrow^S b) = b *^S (b \rightarrow^S a)$
⟨*proof*⟩

lemma *sum-divisibility*:
assumes $a \in S$ $b \in S$
shows $a *^S (a \rightarrow^S b) = b *^S (b \rightarrow^S a)$
⟨*proof*⟩

3.5.8 Residuation

Finally we prove $(x *^S y) \rightarrow^S z = x \rightarrow^S (y \rightarrow^S z)$.

lemma *sum-residuation-one*:
assumes $a \in S$ $b \in S$ $c \in S$ $a = 1^S \vee b = 1^S \vee c = 1^S$
shows $(a *^S b) \rightarrow^S c = a \rightarrow^S (b \rightarrow^S c)$
⟨*proof*⟩

lemma *sum-residuation-not-one*:
assumes $a \in S-\{1^S\}$ $b \in S-\{1^S\}$ $c \in S-\{1^S\}$
shows $(a *^S b) \rightarrow^S c = a \rightarrow^S (b \rightarrow^S c)$
⟨*proof*⟩

lemma *sum-residuation*:
assumes $a \in S$ $b \in S$ $c \in S$
shows $(a *^S b) \rightarrow^S c = a \rightarrow^S (b \rightarrow^S c)$
⟨*proof*⟩

3.5.9 Main result

sublocale *hoop* S $(*^S)$ $(\rightarrow^S)$ 1^S

⟨*proof*⟩

end

end

4 Totally ordered hoops

theory *Totally-Ordered-Hoops*
  imports *Ordinal-Sums*
begin

4.1 Definitions

locale *totally-ordered-hoop* = *hoop* +
  assumes *total-order*: $x \in A \Longrightarrow y \in A \Longrightarrow x \leq^A y \vee y \leq^A x$
begin

function *fixed-points* :: $'a \Rightarrow 'a$ *set* (‹F›) **where**
  $F\ a = \{b \in A{-}\{1^A\}.\ a \to^A b = b\}$ **if** $a \in A{-}\{1^A\}$
| $F\ a = \{1^A\}$ **if** $a = 1^A$
| $F\ a =$ *undefined* **if** $a \notin A$
  ⟨*proof*⟩
termination ⟨*proof*⟩

definition *rel-F* :: $'a \Rightarrow 'a \Rightarrow$ *bool* (**infix** ‹$\sim F$› *60*)
  where $x \sim F\ y \equiv \forall\ z \in A.\ (x \to^A z = z) \longleftrightarrow (y \to^A z = z)$

definition *rel-F-canonical-map* :: $'a \Rightarrow 'a$ *set* (‹π›)
  where $\pi\ x = \{b \in A.\ x \sim F\ b\}$

end

4.2 Properties of F

context *totally-ordered-hoop*
begin

lemma *F-equiv*:
  assumes $a \in A{-}\{1^A\}$ $b \in A$
  shows $b \in F\ a \longleftrightarrow (b \in A \wedge b \neq 1^A \wedge a \to^A b = b)$
  ⟨*proof*⟩

lemma *F-subset*:
  assumes $a \in A$
  shows $F\ a \subseteq A$
⟨*proof*⟩

lemma *F-of-one*:

assumes $a \in A$
shows $F\ a = \{1^A\} \longleftrightarrow a = 1^A$
⟨*proof*⟩

lemma *F-of-mult*:
assumes $a \in A-\{1^A\}$ $b \in A-\{1^A\}$
shows $F\ (a *^A b) = \{c \in A-\{1^A\}.\ (a *^A b) \to^A c = c\}$
⟨*proof*⟩

lemma *F-of-imp*:
assumes $a \in A$ $b \in A$ $a \to^A b \neq 1^A$
shows $F\ (a \to^A b) = \{c \in A-\{1^A\}.\ (a \to^A b) \to^A c = c\}$
⟨*proof*⟩

lemma *F-bound*:
assumes $a \in A$ $b \in A$ $a \in F\ b$
shows $a \leq^A b$
⟨*proof*⟩

The following results can be found in Lemma 3.3 in [5].

lemma *LEMMA-3-3-1*:
assumes $a \in A-\{1^A\}$ $b \in A$ $c \in A$ $b \in F\ a$ $c \leq^A b$
shows $c \in F\ a$
⟨*proof*⟩

lemma *LEMMA-3-3-2*:
assumes $a \in A-\{1^A\}$ $b \in A-\{1^A\}$ $F\ a = F\ b$
shows $F\ a = F\ (a *^A b)$
⟨*proof*⟩

lemma *LEMMA-3-3-3*:
assumes $a \in A-\{1^A\}$ $b \in A-\{1^A\}$ $a \leq^A b$
shows $F\ a \subseteq F\ b$
⟨*proof*⟩

lemma *LEMMA-3-3-4*:
assumes $a \in A-\{1^A\}$ $b \in A-\{1^A\}$ $a <^A b$ $F\ a \neq F\ b$
shows $a \in F\ b$
⟨*proof*⟩

lemma *LEMMA-3-3-5*:
assumes $a \in A-\{1^A\}$ $b \in A-\{1^A\}$ $F\ a \neq F\ b$
shows $a *^A b = a \wedge^A b$
⟨*proof*⟩

lemma *LEMMA-3-3-6*:
assumes $a \in A-\{1^A\}$ $b \in A-\{1^A\}$ $a <^A b$ $F\ a = F\ b$
shows $F\ (b \to^A a) = F\ b$
⟨*proof*⟩

4.3 Properties of $(\sim F)$

4.3.1 $(\sim F)$ is an equivalence relation

lemma *rel-F-reflex*:
assumes $a \in A$
shows $a \sim F\ a$
⟨*proof*⟩

lemma *rel-F-symm*:
assumes $a \in A\ \ b \in A\ \ a \sim F\ b$
shows $b \sim F\ a$
⟨*proof*⟩

lemma *rel-F-trans*:
assumes $a \in A\ \ b \in A\ \ c \in A\ \ a \sim F\ b\ \ b \sim F\ c$
shows $a \sim F\ c$
⟨*proof*⟩

4.3.2 Equivalent definition

lemma *rel-F-equiv*:
assumes $a \in A\ \ b \in A$
shows $(a \sim F\ b) = (F\ a = F\ b)$
⟨*proof*⟩

4.3.3 Properties of equivalence classes given by $(\sim F)$

lemma *class-one*: $\pi\ 1^A = \{1^A\}$
⟨*proof*⟩

lemma *classes-subsets*:
assumes $a \in A$
shows $\pi\ a \subseteq A$
⟨*proof*⟩

lemma *classes-not-empty*:
assumes $a \in A$
shows $a \in \pi\ a$
⟨*proof*⟩

corollary *class-not-one*:
assumes $a \in A - \{1^A\}$
shows $\pi\ a \neq \{1^A\}$
⟨*proof*⟩

lemma *classes-disjoint*:
assumes $a \in A\ \ b \in A\ \ \pi\ a \cap \pi\ b \neq \emptyset$
shows $\pi\ a = \pi\ b$
⟨*proof*⟩

lemma *classes-cover*: $A = \{x.\ \exists\ y \in A.\ x \in \pi\ y\}$
⟨*proof*⟩

lemma *classes-convex*:
assumes $a \in A$ $b \in A$ $c \in A$ $d \in A$ $b \in \pi\ a$ $c \in \pi\ a$ $b \le^A d$ $d \le^A c$
shows $d \in \pi\ a$
⟨*proof*⟩

lemma *related-iff-same-class*:
assumes $a \in A$ $b \in A$
shows $a \sim F\ b \longleftrightarrow \pi\ a = \pi\ b$
⟨*proof*⟩

corollary *same-F-iff-same-class*:
assumes $a \in A$ $b \in A$
shows $F\ a = F\ b \longleftrightarrow \pi\ a = \pi\ b$
⟨*proof*⟩

end

4.4 Irreducible hoops: definition and equivalences

A totally ordered hoop is *irreducible* if it cannot be written as the ordinal sum of two nontrivial totally ordered hoops.

locale *totally-ordered-irreducible-hoop* = *totally-ordered-hoop* +
assumes *irreducible*: $\nexists\ B\ C.$
$(A = B \cup C) \wedge$
$(\{1^A\} = B \cap C) \wedge$
$(\exists\ y \in B.\ y \neq 1^A) \wedge$
$(\exists\ y \in C.\ y \neq 1^A) \wedge$
$(hoop\ B\ (*^A)\ (\rightarrow^A)\ 1^A) \wedge$
$(hoop\ C\ (*^A)\ (\rightarrow^A)\ 1^A) \wedge$
$(\forall\ x \in B-\{1^A\}.\ \forall\ y \in C.\ x *^A y = x) \wedge$
$(\forall\ x \in B-\{1^A\}.\ \forall\ y \in C.\ x \rightarrow^A y = 1^A) \wedge$
$(\forall\ x \in C.\ \forall\ y \in B.\ x \rightarrow^A y = y)$

lemma *irr-test*:
assumes *totally-ordered-hoop A PA RA a*
$\neg$*totally-ordered-irreducible-hoop A PA RA a*
shows $\exists\ B\ C.$
$(A = B \cup C) \wedge$
$(\{a\} = B \cap C) \wedge$
$(\exists\ y \in B.\ y \neq a) \wedge$
$(\exists\ y \in C.\ y \neq a) \wedge$
$(hoop\ B\ PA\ RA\ a) \wedge$
$(hoop\ C\ PA\ RA\ a) \wedge$
$(\forall\ x \in B-\{a\}.\ \forall\ y \in C.\ PA\ x\ y = x) \wedge$
$(\forall\ x \in B-\{a\}.\ \forall\ y \in C.\ RA\ x\ y = a) \wedge$

$(\forall\ x \in C.\ \forall\ y \in B.\ RA\ x\ y = y)$
⟨*proof*⟩

locale *totally-ordered-one-fixed-hoop* = *totally-ordered-hoop* +
assumes *one-fixed*: $x \in A \Longrightarrow y \in A \Longrightarrow y \to^A x = x \Longrightarrow x = 1^A \vee y = 1^A$

locale *totally-ordered-wajsberg-hoop* = *totally-ordered-hoop* + *wajsberg-hoop*

context *totally-ordered-hoop*
begin

The following result can be found in [1] (see Lemma 3.5).

lemma *not-one-fixed-implies-not-irreducible*:
assumes ¬*totally-ordered-one-fixed-hoop* $A\ (*^A)\ (\to^A)\ 1^A$
shows ¬*totally-ordered-irreducible-hoop* $A\ (*^A)\ (\to^A)\ 1^A$
⟨*proof*⟩

Next result can be found in [2] (see Proposition 2.2).

lemma *one-fixed-implies-wajsberg*:
assumes *totally-ordered-one-fixed-hoop* $A\ (*^A)\ (\to^A)\ 1^A$
shows *totally-ordered-wajsberg-hoop* $A\ (*^A)\ (\to^A)\ 1^A$
⟨*proof*⟩

The proof of the following result can be found in [1] (see Theorem 3.6).

lemma *not-irreducible-implies-not-wajsberg*:
assumes ¬*totally-ordered-irreducible-hoop* $A\ (*^A)\ (\to^A)\ 1^A$
shows ¬*totally-ordered-wajsberg-hoop* $A\ (*^A)\ (\to^A)\ 1^A$
⟨*proof*⟩

Summary of all results in this subsection:

theorem *one-fixed-equivalent-to-wajsberg*:
shows *totally-ordered-one-fixed-hoop* $A\ (*^A)\ (\to^A)\ 1^A \equiv$
totally-ordered-wajsberg-hoop $A\ (*^A)\ (\to^A)\ 1^A$
⟨*proof*⟩

theorem *wajsberg-equivalent-to-irreducible*:
shows *totally-ordered-wajsberg-hoop* $A\ (*^A)\ (\to^A)\ 1^A \equiv$
totally-ordered-irreducible-hoop $A\ (*^A)\ (\to^A)\ 1^A$
⟨*proof*⟩

theorem *irreducible-equivalent-to-one-fixed*:
shows *totally-ordered-irreducible-hoop* $A\ (*^A)\ (\to^A)\ 1^A \equiv$
totally-ordered-one-fixed-hoop $A\ (*^A)\ (\to^A)\ 1^A$
⟨*proof*⟩

end

4.5 Decomposition

locale *tower-of-irr-hoops* = *tower-of-hoops* +
assumes *family-of-irr-hoops*: $i \in I \Longrightarrow$
totally-ordered-irreducible-hoop $(\mathbb{A}_i)$ $(*^i)$ $(\rightarrow^i)$ 1^S

locale *tower-of-nontrivial-irr-hoops* = *tower-of-irr-hoops* +
assumes *nontrivial*: $i \in I \Longrightarrow \exists\ x \in \mathbb{A}_i.\ x \neq 1^S$

context *totally-ordered-hoop*
begin

4.5.1 Definition of index set I

definition *index-set* :: ('*a set*) *set* (‹I›)
where $I = \{y.\ (\exists\ x \in A.\ \pi\ x = y)\}$

lemma *indexes-subsets*:
assumes $i \in I$
shows $i \subseteq A$
⟨*proof*⟩

lemma *indexes-not-empty*:
assumes $i \in I$
shows $i \neq \emptyset$
⟨*proof*⟩

lemma *indexes-disjoint*:
assumes $i \in I$ $j \in I$ $i \neq j$
shows $i \cap j = \emptyset$
⟨*proof*⟩

lemma *indexes-cover*: $A = \{x.\ \exists\ i \in I.\ x \in i\}$
⟨*proof*⟩

lemma *indexes-class-of-elements*:
assumes $i \in I$ $a \in A$ $a \in i$
shows $\pi\ a = i$
⟨*proof*⟩

lemma *indexes-convex*:
assumes $i \in I$ $a \in i$ $b \in i$ $d \in A$ $a \leq^A d$ $d \leq^A b$
shows $d \in i$
⟨*proof*⟩

4.5.2 Definition of total partial order over I

Since each equivalence class is convex, $(\leq^A)$ induces a total order on I.

function *index-order* :: ('*a set*) ⇒ ('*a set*) ⇒ *bool* (**infix** ‹$\leq^I$› *60*) **where**

$x \leq^I y = ((x = y) \vee (\forall\ v \in x.\ \forall\ w \in y.\ v \leq^A w))$ **if** $x \in I\ y \in I$
| $x \leq^I y = \mathit{undefined}$ **if** $x \notin I \vee y \notin I$
⟨*proof*⟩
termination ⟨*proof*⟩

definition *index-order-strict* (**infix** ‹$<^I$› *60*)
where $x <^I y = (x \leq^I y \wedge x \neq y)$

lemma *index-ord-reflex*:
assumes $i \in I$
shows $i \leq^I i$
⟨*proof*⟩

lemma *index-ord-antisymm*:
assumes $i \in I\ j \in I\ i \leq^I j\ j \leq^I i$
shows $i = j$
⟨*proof*⟩

lemma *index-ord-trans*:
assumes $i \in I\ j \in I\ k \in I\ i \leq^I j\ j \leq^I k$
shows $i \leq^I k$
⟨*proof*⟩

lemma *index-order-total* :
assumes $i \in I\ j \in I\ \neg(j \leq^I i)$
shows $i \leq^I j$
⟨*proof*⟩

sublocale *total-poset-on* I $(\leq^I)$ $(<^I)$
⟨*proof*⟩

4.5.3 Definition of universes

definition *universes* :: *'a set* ⇒ *'a set* (‹UNI_A›)
where $UNI_A\ x = x \cup \{1^A\}$

abbreviation (*uniA-i*)
uniA-i :: [*'a set*] ⇒ (*'a set*) (‹($\mathbb{A}$(_))› [*61*] *60*)
where $\mathbb{A}_i \equiv UNI_A\ i$

abbreviation (*uniA-pi*)
uniA-pi :: [*'a*] ⇒ (*'a set*) (‹($\mathbb{A}_\pi$ (_))› [*61*] *60*)
where $\mathbb{A}_{\pi x} \equiv UNI_A\ (\pi\ x)$

abbreviation (*uniA-pi-one*)
uniA-pi-one :: *'a set* (‹($\mathbb{A}_{\pi 1 A}$)› *60*)
where $\mathbb{A}_{\pi 1 A} \equiv UNI_A\ (\pi\ 1^A)$

lemma *universes-subsets*:

assumes $i \in I$ $a \in \mathbb{A}_i$
shows $a \in A$
⟨*proof*⟩

lemma *universes-not-empty*:
assumes $i \in I$
shows $\mathbb{A}_i \neq \emptyset$
⟨*proof*⟩

lemma *universes-almost-disjoint*:
assumes $i \in I$ $j \in I$ $i \neq j$
shows $(\mathbb{A}_i) \cap (\mathbb{A}_j) = \{1^A\}$
⟨*proof*⟩

lemma *universes-cover*: $A = \{x.\ \exists\ i \in I.\ x \in \mathbb{A}_i\}$
⟨*proof*⟩

lemma *universes-aux*:
assumes $i \in I$ $a \in i$
shows $\mathbb{A}_i = \pi\ a \cup \{1^A\}$
⟨*proof*⟩

4.5.4 Universes are subhoops of A

lemma *universes-one-closed*:
assumes $i \in I$
shows $1^A \in \mathbb{A}_i$
⟨*proof*⟩

lemma *universes-mult-closed*:
assumes $i \in I$ $a \in \mathbb{A}_i$ $b \in \mathbb{A}_i$
shows $a *^A b \in \mathbb{A}_i$
⟨*proof*⟩

lemma *universes-imp-closed*:
assumes $i \in I$ $a \in \mathbb{A}_i$ $b \in \mathbb{A}_i$
shows $a \rightarrow^A b \in \mathbb{A}_i$
⟨*proof*⟩

4.5.5 Universes are irreducible hoops

lemma *universes-one-fixed*:
assumes $i \in I$ $a \in \mathbb{A}_i$ $b \in \mathbb{A}_i$ $a \rightarrow^A b = b$
shows $a = 1^A \vee b = 1^A$
⟨*proof*⟩

corollary *universes-one-fixed-hoops*:
assumes $i \in I$
shows *totally-ordered-one-fixed-hoop* $(\mathbb{A}_i)$ $(*^A)$ $(\rightarrow^A)$ 1^A
⟨*proof*⟩

corollary *universes-irreducible-hoops*:
  assumes $i \in I$
  shows *totally-ordered-irreducible-hoop* $(\mathbb{A}_i)$ $(*^A)$ $(\to^A)$ 1^A
  ⟨*proof*⟩

4.5.6 Some useful results

lemma *index-aux*:
  assumes $i \in I$ $j \in I$ $i <^I j$ $a \in (\mathbb{A}_i) - \{1^A\}$ $b \in (\mathbb{A}_j) - \{1^A\}$
  shows $a <^A b \wedge \neg(a \sim F\ b)$
⟨*proof*⟩

lemma *different-indexes-mult*:
  assumes $i \in I$ $j \in I$ $i <^I j$ $a \in (\mathbb{A}_i) - \{1^A\}$ $b \in (\mathbb{A}_j) - \{1^A\}$
  shows $a *^A b = a$
⟨*proof*⟩

lemma *different-indexes-imp-1*:
  assumes $i \in I$ $j \in I$ $i <^I j$ $a \in (\mathbb{A}_i) - \{1^A\}$ $b \in (\mathbb{A}_j) - \{1^A\}$
  shows $a \to^A b = 1^A$
⟨*proof*⟩

lemma *different-indexes-imp-2* :
  assumes $i \in I$ $j \in I$ $i <^I j$ $a \in (\mathbb{A}_j) - \{1^A\}$ $b \in (\mathbb{A}_i) - \{1^A\}$
  shows $a \to^A b = b$
⟨*proof*⟩

4.5.7 Definition of multiplications, implications and one

definition *mult-map* :: *'a set* ⇒ (*'a* ⇒ *'a* ⇒ *'a*) (‹MUL_A›)
  where $MUL_A\ x = (*^A)$

definition *imp-map* :: *'a set* ⇒ (*'a* ⇒ *'a* ⇒ *'a*) (‹IMP_A›)
  where $IMP_A\ x = (\to^A)$

definition *sum-one* :: *'a* (‹1^S›)
  where $1^S = 1^A$

abbreviation (*multA-i*)
  multA-i :: [*'a set*] ⇒ (*'a* ⇒ *'a* ⇒ *'a*) (‹(∗(-))› [*50*] *60*)
  where $*^i \equiv MUL_A\ i$

abbreviation (*impA-i*)
  impA-i:: [*'a set*] ⇒ (*'a* ⇒ *'a* ⇒ *'a*) (‹(→(-))› [*50*] *60*)
  where $\to^i \equiv IMP_A\ i$

abbreviation (*multA-i-xy*)
  multA-i-xy :: [*'a*, *'a set*, *'a*] ⇒ *'a* (‹((-)/ ∗(-) / (-))› [*61*, *50*, *61*] *60*)
  where $x *^i y \equiv MUL_A\ i\ x\ y$

abbreviation (*impA-i-xy*)
impA-i-xy :: ['a, 'a set, 'a] ⇒ 'a (‹((-)/ →(^) / (-))› [61, 50, 61] 60)
where $x \rightarrow^i y \equiv IMP_A\ i\ x\ y$

abbreviation (*ord-i-xy*)
ord-i-xy :: ['a, 'a set, 'a] ⇒ bool (‹((-)/ ≤(^) / (-))› [61, 50, 61] 60)
where $x \leq^i y \equiv$ *hoop.hoop-order* $(IMP_A\ i)\ 1^S\ x\ y$

4.5.8 Main result

We prove the main result: a totally ordered hoop is equal to an ordinal sum of a tower of irreducible hoops.

sublocale *A-SUM*: *tower-of-irr-hoops* $I\ (\leq^I)\ (<^I)\ UNI_A\ MUL_A\ IMP_A\ 1^S$
⟨*proof*⟩

lemma *same-uni* [*simp*]: *A-SUM.sum-univ* = *A*
⟨*proof*⟩

lemma *floor-is-class*:
assumes $a \in A-\{1^A\}$
shows *A-SUM.floor* $a = \pi\ a$
⟨*proof*⟩

lemma *same-mult*:
assumes $a \in A\ b \in A$
shows $a *^A b =$ *A-SUM.sum-mult a b*
⟨*proof*⟩

lemma *same-imp*:
assumes $a \in A\ b \in A$
shows $a \rightarrow^A b =$ *A-SUM.sum-imp a b*
⟨*proof*⟩

lemma *ordinal-sum-is-totally-ordered-hoop*:
totally-ordered-hoop A-SUM.sum-univ A-SUM.sum-mult A-SUM.sum-imp 1^S
⟨*proof*⟩

theorem *totally-ordered-hoop-is-equal-to-ordinal-sum-of-tower-of-irr-hoops*:
shows *eq-universe*: *A* = *A-SUM.sum-univ*
and *eq-mult*: $x \in A \Longrightarrow y \in A \Longrightarrow x *^A y =$ *A-SUM.sum-mult x y*
and *eq-imp*: $x \in A \Longrightarrow y \in A \Longrightarrow x \rightarrow^A y =$ *A-SUM.sum-imp x y*
and *eq-one*: $1^A = 1^S$
⟨*proof*⟩

4.5.9 Remarks on the nontrivial case

In the nontrivial case we have that every totally ordered hoop can be written as the ordinal sum of a tower of nontrivial irreducible hoops. The proof of this fact is almost immediate. By definition, $\mathbb{A}_{\pi 1^A} = \{\mathit{1}^A\}$ is the only trivial hoop in our tower. Moreover, $\mathbb{A}_{\pi a}$ is non-trivial for every $a \in A - \{\mathit{1}^A\}$. Given that $\mathit{1}^A \in \mathbb{A}_i$ for every $i \in I$ we can simply remove $\pi\ \mathit{1}^A$ from I and obtain the desired result.

lemma *nontrivial-tower*:
  assumes $\exists\ x \in A.\ x \neq \mathit{1}^A$
  shows
    $\mathit{tower\text{-}of\text{-}nontrivial\text{-}irr\text{-}hoops}\ (I - \{\pi\ \mathit{1}^A\})\ (\leq^I)\ (<^I)\ \mathit{UNI}_A\ \mathit{MUL}_A\ \mathit{IMP}_A\ \mathit{1}^S$
⟨*proof*⟩

lemma *ordinal-sum-of-nontrivial*:
  assumes $\exists\ x \in A.\ x \neq \mathit{1}^A$
  shows $\mathit{A\text{-}SUM.sum\text{-}univ} = \{x.\ \exists\ i \in I - \{\pi\ \mathit{1}^A\}.\ x \in \mathbb{A}_i\}$
⟨*proof*⟩

end

4.5.10 Converse of main result

We show that the converse of the main result also holds, that is, the ordinal sum of a tower of irreducible hoops is a totally ordered hoop.

context *tower-of-irr-hoops*
begin

proposition *ordinal-sum-of-tower-of-irr-hoops-is-totally-ordered-hoop*:
  shows $\mathit{totally\text{-}ordered\text{-}hoop}\ S\ (*^S)\ (\rightarrow^S)\ \mathit{1}^S$
⟨*proof*⟩

end

end

5 BL-chains

BL-chains generate the variety of BL-algebras, the algebraic counterpart of the Basic Fuzzy Logic (see [6]). As mentioned in the abstract, this formalization is based on the proof for BL-chains found in [5]. We define $\mathit{BL-chain}$ and *bounded tower of irreducible hoops* and formalize the main result on that paper (Theorem 3.4).

theory *BL-Chains*
  imports *Totally-Ordered-Hoops*

begin

5.1 Definitions

locale *bl-chain* = *totally-ordered-hoop* +
fixes *zeroA* :: *'a* (‹0^A›)
assumes *zero-closed*: $0^A \in A$
assumes *zero-first*: $x \in A \Longrightarrow 0^A \leq^A x$

locale *bounded-tower-of-irr-hoops* = *tower-of-irr-hoops* +
fixes *zeroI* (‹0^I›)
fixes *zeroS* (‹0^S›)
assumes *I-zero-closed* : $0^I \in I$
and *zero-first*: $i \in I \Longrightarrow 0^I \leq^I i$
and *first-zero-closed*: $0^S \in UNI\ 0^I$
and *first-bounded*: $x \in UNI\ 0^I \Longrightarrow IMP\ 0^I\ 0^S\ x = 1^S$
begin

abbreviation (*uni-zero*)
uni-zero :: *'b set* (‹$\mathbb{A}_{0^I}$›)
where $\mathbb{A}_{0^I} \equiv UNI\ 0^I$

abbreviation (*imp-zero*)
imp-zero :: [*'b*, *'b*] ⇒ *'b* (‹((-)/ $\rightarrow^{0^I}$ / (-))› [*61*,*61*] *60*)
where $x \rightarrow^{0^I} y \equiv IMP\ 0^I\ x\ y$

end

context *bl-chain*
begin

5.2 First element of *I*

definition *zeroI* :: *'a set* (‹0^I›)
where $0^I = \pi\ 0^A$

lemma *I-zero-closed*: $0^I \in I$
⟨*proof*⟩

lemma *I-has-first-element*:
assumes $i \in I$ $i \neq 0^I$
shows $0^I <^I i$
⟨*proof*⟩

5.3 Main result for BL-chains

definition *zeroS* :: *'a* (‹0^S›)
where $0^S = 0^A$

abbreviation (*uniA-zero*)

uniA-zero :: *'a set* (‹($\mathbb{A}_{0I}$)›)
where $\mathbb{A}_{0I} \equiv$ *UNI*$_A$ 0^I

abbreviation (*impA-zero-xy*)
impA-zero-xy :: [*'a*, *'a*] ⇒ *'a* (‹((-)/ $\rightarrow^{0I}$ / (-))› [*61*, *61*] *60*)
where $x \rightarrow^{0I} y \equiv IMP_A\ 0^I\ x\ y$

lemma *tower-is-bounded*:
shows *bounded-tower-of-irr-hoops I* ($\leq^I$) ($<^I$) *UNI*$_A$ *MUL*$_A$ *IMP*$_A$ 1^S 0^I 0^S
⟨*proof*⟩

lemma *ordinal-sum-is-bl-totally-ordered*:
shows *bl-chain A-SUM.sum-univ A-SUM.sum-mult A-SUM.sum-imp* 1^S 0^S
⟨*proof*⟩

theorem *bl-chain-is-equal-to-ordinal-sum-of-bounded-tower-of-irr-hoops*:
shows *eq-universe*: A = *A-SUM.sum-univ*
and *eq-mult*: $x \in A \Longrightarrow y \in A \Longrightarrow x *^A y$ = *A-SUM.sum-mult x y*
and *eq-imp*: $x \in A \Longrightarrow y \in A \Longrightarrow x \rightarrow^A y$ = *A-SUM.sum-imp x y*
and *eq-zero*: $0^A = 0^S$
and *eq-one*: $1^A = 1^S$
⟨*proof*⟩

end

5.4 Converse of main result for BL-chains

context *bounded-tower-of-irr-hoops*
begin

We show that the converse of the main result holds if $0^S \neq 1^S$. If $0^S = 1^S$ then the converse may not be true. For example, take a trivial hoop A and an arbitrary not bounded Wajsberg hoop B such that $A \cap B = \{1\}$. The ordinal sum of both hoops is equal to B and therefore not bounded.

proposition *ordinal-sum-of-bounded-tower-of-irr-hoops-is-bl-chain*:
assumes $0^S \neq 1^S$
shows *bl-chain S* ($*^S$) ($\rightarrow^S$) 1^S 0^S
⟨*proof*⟩

end

end

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