IsaGeoCoq: Partial porting of GeoCoq 2.4.0. Case studies: Tarski's postulate of parallels implies the 5th postulate of Euclid, the postulate of Playfair and the original postulate of Euclid.

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Abstract

The GeoCoq library contains a formalization of geometry using the Coq proof assistant. It contains both proofs about the foundations of geometry [20, 15, 6, 16] and high-level proofs in the same style as in high-school. [1](Code Repository https://github.com/GeoCoq/GeoCoq).

Some theorems also inspired by [20] are also formalized with others ITP(Metamath, Mizar) or ATP [24, 25, 3, 23, 4, 2, 17, 5, 11, 19, 8, 9, 10].

We port a part of the GeoCoq 2.4.0 library within the Isabelle/Hol proof assistant: more precisely, the files Chap02.v to Chap13_3.v, suma.v as well as the associated definitions and some useful files for the demonstration of certain parallel postulates.

While the demonstrations in Coq are written in procedural language [26], the transcript is done in declarative language Isar[18].

The synthetic approach of the demonstrations are directly inspired by those contained in GeoCoq. Some demonstrations are credited to G.E Martin(«lemma bet_le_lt:» in Ch11_angles.thy, proved by Martin as Theorem 18.17 in [14]) or Gupta H.N (Krippen Lemma, proved by Gupta in its PhD in 1965 as Theorem 3.45). (See [12]).

In this work, the proofs are not contructive. The sledeghammer tool being used to find some demonstrations.

The names of the lemmas and theorems used are kept as far as possible as well as the definitions. A different translation has been proposed when the name was already used in Isabel/Hol ("Len" is translated as "TarskiLen") or that characters were not allowed in Isabel/Hol ("anga'" in Ch13_angles.v is translated as "angaP"). For some definitions the highlighting of a variable has changed the order or the position of the variables (Midpoint, Out, Inter,...).

All the lemmas are valid in absolute/neutral space defined with Tarski's axioms.

It should be noted that T.J.M. Makarios [13] has begun some demonstrations of certain proposals mainly those corresponding to SST chapters 2 and 3. It uses a definition that does not quite coincide with the definition used in Geocoq and here. As an example, Makarios introduces the axiom A11 (Axiom of continuity) in the definition of the locale "Tarski_absolute_space".

Furthermore, the definition of the locale "TarskiAbsolute" [22, 21] is not not identical to the one defined in the "Tarski_neutral_dimensionless" class of GeoCoq. Indeed this one does not contain the axiom "upper_dimension". In some cases particular, it is nevertheless to use the axiom "upper_dimension". The addition of the word "_2D" in the file indicates its presence.

In the last part, it is formalized that, in the neutral/absolute space, the axiom of the parallels of the system of Tarski implies the Playfair axiom, the 5th postulate of euclide and the postulate original from Euclid. These proofs, which are not constructive, are directly inspired by [12, 7].

Contents

theory *Tarski-Neutral*

imports
Main

begin

1 Tarski's axiom system for neutral geometry

1.1 Tarski's axiom system for neutral geometry: dimensionless

locale *Tarski-neutral-dimensionless* =
fixes $Bet :: 'p \Rightarrow 'p \Rightarrow 'p \Rightarrow bool$
fixes $Cong :: 'p \Rightarrow 'p \Rightarrow 'p \Rightarrow 'p \Rightarrow bool$
assumes *cong-pseudo-reflexivity*: $\forall\ a\ b.$
$Cong\ a\ b\ b\ a$
and *cong-inner-transitivity*: $\forall\ a\ b\ p\ q\ r\ s.$
$Cong\ a\ b\ p\ q \wedge$
$Cong\ a\ b\ r\ s$
$\longrightarrow$
$Cong\ p\ q\ r\ s$
and *cong-identity*: $\forall\ a\ b\ c.$
$Cong\ a\ b\ c\ c$
$\longrightarrow$
$a = b$
and *segment-construction*: $\forall\ a\ b\ c\ q.$
$\exists x. (Bet\ q\ a\ x \wedge Cong\ a\ x\ b\ c)$
and *five-segment*: $\forall\ a\ b\ c\ a'\ b'\ c'.$
$a \neq b \wedge$
$Bet\ a\ b\ c \wedge$
$Bet\ a'\ b'\ c' \wedge$
$Cong\ a\ b\ a'\ b' \wedge$
$Cong\ b\ c\ b'\ c' \wedge$
$Cong\ a\ d\ a'\ d' \wedge$
$Cong\ b\ d\ b'\ d'$
$\longrightarrow$
$Cong\ c\ d\ c'\ d'$
and *between-identity*: $\forall\ a\ b.$
$Bet\ a\ b\ a$
$\longrightarrow$
$a = b$
and *inner-pasch*: $\forall\ a\ b\ c\ p\ q.$
$Bet\ a\ p\ c \wedge$
$Bet\ b\ q\ c$
$\longrightarrow$
$(\exists\ x.\ Bet\ p\ x\ b \wedge Bet\ q\ x\ a)$
and *lower-dim*: $\exists\ a\ b\ c.\ (\neg\ Bet\ a\ b\ c \wedge \neg\ Bet\ b\ c\ a \wedge \neg\ Bet\ c\ a\ b)$

1.2 Tarski's axiom system for neutral geometry: 2D

locale *Tarski-2D* = *Tarski-neutral-dimensionless* +
assumes *upper-dim*: $\forall\ a\ b\ c\ p\ q.$
$p \neq q \wedge$
$Cong\ a\ p\ a\ q \wedge$
$Cong\ b\ p\ b\ q \wedge$
$Cong\ c\ p\ c\ q$
$\longrightarrow$
$(Bet\ a\ b\ c \vee Bet\ b\ c\ a \vee Bet\ c\ a\ b)$

2 Definitions

2.1 Tarski's axiom system for neutral geometry: dimensionless

context *Tarski-neutral-dimensionless*
begin

2.1.1 Congruence

definition *OFSC* ::
$['p,'p,'p,'p,'p,'p,'p,'p] \Rightarrow bool$
(‹- - - - *OFSC* - - - -› [*99*,*99*,*99*,*99*,*99*,*99*,*99*,*99*] *50*)
where
$A\ B\ C\ D\ OFSC\ A'\ B'\ C'\ D' \equiv$

$Bet\ A\ B\ C \wedge$
$Bet\ A'\ B'\ C' \wedge$
$Cong\ A\ B\ A'\ B' \wedge$
$Cong\ B\ C\ B'\ C' \wedge$
$Cong\ A\ D\ A'\ D' \wedge$
$Cong\ B\ D\ B'\ D'$

definition *Cong3* ::
$['p,'p,'p,'p,'p,'p] \Rightarrow bool$
(‹- - - *Cong3* - - -› [*99*,*99*,*99*,*99*,*99*,*99*] *50*)
where
$A\ B\ C\ Cong3\ A'\ B'\ C' \equiv$

$Cong\ A\ B\ A'\ B' \wedge$
$Cong\ A\ C\ A'\ C' \wedge$
$Cong\ B\ C\ B'\ C'$

2.1.2 Betweenness

definition *Col* ::
$['p,'p,'p] \Rightarrow bool$
(‹*Col* - - -› [*99*,*99*,*99*] *50*)
where
$Col\ A\ B\ C \equiv$

$Bet\ A\ B\ C \vee Bet\ B\ C\ A \vee Bet\ C\ A\ B$

definition *Bet4* ::
$['p,'p,'p,'p] \Rightarrow bool$
(‹*Bet4* - - - -› [*99*,*99*,*99*,*99*] *50*)
where
$Bet4\ A1\ A2\ A3\ A4 \equiv$

$Bet\ A1\ A2\ A3 \wedge$
$Bet\ A2\ A3\ A4 \wedge$
$Bet\ A1\ A3\ A4 \wedge$
$Bet\ A1\ A2\ A4$

definition *BetS* ::
$['p,'p,'p] \Rightarrow bool$ (‹*BetS* - - -› [*99*,*99*,*99*] *50*)
where
$BetS\ A\ B\ C \equiv$

$Bet\ A\ B\ C \wedge$
$A \neq B \wedge$
$B \neq C$

2.1.3 Collinearity

definition *FSC* ::
$['p,'p,'p,'p,'p,'p,'p,'p] \Rightarrow bool$
(‹- - - - *FSC* - - - -› [*99*,*99*,*99*,*99*,*99*,*99*,*99*,*99*] *50*)

where
$A\ B\ C\ D\ FSC\ A'\ B'\ C'\ D' \equiv$

$Col\ A\ B\ C\ \wedge$
$A\ B\ C\ Cong3\ A'\ B'\ C'\ \wedge$
$Cong\ A\ D\ A'\ D'\ \wedge$
$Cong\ B\ D\ B'\ D'$

2.1.4 Congruence and Betweenness

definition *IFSC* ::
$['p,'p,'p,'p,'p,'p,'p,'p] \Rightarrow bool$
(‹- - - - *IFSC* - - - -› [*99*,*99*,*99*,*99*,*99*,*99*,*99*,*99*] *50*)
where
$A\ B\ C\ D\ IFSC\ A'\ B'\ C'\ D' \equiv$

$Bet\ A\ B\ C\ \wedge$
$Bet\ A'\ B'\ C'\ \wedge$
$Cong\ A\ C\ A'\ C'\ \wedge$
$Cong\ B\ C\ B'\ C'\ \wedge$
$Cong\ A\ D\ A'\ D'\ \wedge$
$Cong\ C\ D\ C'\ D'$

2.1.5 Between transivitity LE

definition *Le* ::
$['p,'p,'p,'p] \Rightarrow bool$ (‹- - *Le* - -› [*99*,*99*,*99*,*99*] *50*)
where $A\ B\ Le\ C\ D \equiv$

$\exists\ E.\ (Bet\ C\ E\ D \wedge Cong\ A\ B\ C\ E)$

definition *Lt* ::
$['p,'p,'p,'p] \Rightarrow bool$ (‹- - *Lt* - -› [*99*,*99*,*99*,*99*] *50*)
where $A\ B\ Lt\ C\ D \equiv$

$A\ B\ Le\ C\ D \wedge \neg\ Cong\ A\ B\ C\ D$

definition *Ge* ::
$['p,'p,'p,'p] \Rightarrow bool$ (‹- -*Ge* - -› [*99*,*99*,*99*,*99*] *50*)
where $A\ B\ Ge\ C\ D \equiv$

$C\ D\ Le\ A\ B$

definition *Gt* ::
$['p,'p,'p,'p] \Rightarrow bool$ (‹- - *Gt* - -› [*99*,*99*,*99*,*99*] *50*)
where $A\ B\ Gt\ C\ D \equiv$

$C\ D\ Lt\ A\ B$

2.1.6 Out lines

definition *Out* ::
$['p,'p,'p] \Rightarrow bool$ (‹- *Out* - -› [*99*,*99*,*99*] *50*)
where $P\ Out\ A\ B \equiv$

$A \neq P\ \wedge$
$B \neq P\ \wedge$
$(Bet\ P\ A\ B \vee Bet\ P\ B\ A)$

2.1.7 Midpoint

definition *Midpoint* ::
$['p,'p,'p] \Rightarrow bool$ (‹- *Midpoint* - -› [*99*,*99*,*99*] *50*)
where $M\ Midpoint\ A\ B \equiv$

$Bet\ A\ M\ B\ \wedge$

Cong A M M B

2.1.8 Orthogonality

definition *Per* ::
$[{}'p,{}'p,{}'p] \Rightarrow bool$ (‹*Per* - - -› [*99*,*99*,*99*] *50*)
where *Per A B C* ≡

$\exists\ C'{::}'p.\ (B\ Midpoint\ C\ C' \wedge Cong\ A\ C\ A\ C')$

definition *PerpAt* ::
$[{}'p,{}'p,{}'p,{}'p,{}'p] \Rightarrow bool$ (‹- *PerpAt* - - - - › [*99*,*99*,*99*,*99*,*99*] *50*)
where *X PerpAt A B C D* ≡

$A \neq B \wedge$
$C \neq D \wedge$
$Col\ X\ A\ B \wedge$
$Col\ X\ C\ D \wedge$
$(\forall\ U\ V.\ ((Col\ U\ A\ B \wedge Col\ V\ C\ D) \longrightarrow Per\ U\ X\ V))$

definition *Perp* ::
$[{}'p,{}'p,{}'p,{}'p] \Rightarrow bool$ (‹- - *Perp* - -› [*99*,*99*,*99*,*99*] *50*)
where *A B Perp C D* ≡

$\exists\ X{::}'p.\ X\ PerpAt\ A\ B\ C\ D$

2.1.9 Coplanar

definition *Coplanar* ::
$[{}'p,{}'p,{}'p,{}'p] \Rightarrow bool$ (‹*Coplanar* - - - -› [*99*,*99*,*99*,*99*] *50*)
where *Coplanar A B C D* ≡
$\exists\ X.\ (Col\ A\ B\ X \wedge Col\ C\ D\ X) \vee$
$(Col\ A\ C\ X \wedge Col\ B\ D\ X) \vee$
$(Col\ A\ D\ X \wedge Col\ B\ C\ X)$

definition *TS* ::
$[{}'p,{}'p,{}'p,{}'p] \Rightarrow bool$ (‹- - *TS* - -› [*99*,*99*,*99*,*99*] *50*)
where *A B TS P Q* ≡
$\neg\ Col\ P\ A\ B \wedge \neg\ Col\ Q\ A\ B \wedge (\exists\ T{::}'p.\ Col\ T\ A\ B \wedge Bet\ P\ T\ Q)$

definition *ReflectL* ::
$[{}'p,{}'p,{}'p,{}'p] \Rightarrow bool$ (‹- - *ReflectL* - -› [*99*,*99*,*99*,*99*] *50*)
where *P′ P ReflectL A B* ≡
$(\exists\ X.\ X\ Midpoint\ P\ P' \wedge Col\ A\ B\ X) \wedge (A\ B\ Perp\ P\ P' \vee P = P')$

definition *Reflect* ::
$[{}'p,{}'p,{}'p,{}'p] \Rightarrow bool$ (‹- - *Reflect* - -› [*99*,*99*,*99*,*99*] *50*)
where *P′ P Reflect A B* ≡
$(A \neq B \wedge P'\ P\ ReflectL\ A\ B) \vee (A = B \wedge A\ Midpoint\ P\ P')$

definition *InAngle* ::
$[{}'p,{}'p,{}'p,{}'p] \Rightarrow bool$ (‹- *InAngle* - - -› [*99*,*99*,*99*,*99*] *50*)
where *P InAngle A B C* ≡
$A \neq B \wedge C \neq B \wedge P \neq B \wedge$
$(\exists\ X.\ Bet\ A\ X\ C \wedge (X = B \vee B\ Out\ X\ P))$

definition *ParStrict*::
$[{}'p,{}'p,{}'p,{}'p] \Rightarrow bool$ (‹- - *ParStrict* - -› [*99*,*99*,*99*,*99*] *50*)
where *A B ParStrict C D* ≡ $Coplanar\ A\ B\ C\ D \wedge \neg\ (\exists\ X.\ Col\ X\ A\ B \wedge Col\ X\ C\ D)$

definition *Par*::
$[{}'p,{}'p,{}'p,{}'p] \Rightarrow bool$ (‹- - *Par* - -› [*99*,*99*,*99*,*99*] *50*)
where *A B Par C D* ≡
$A\ B\ ParStrict\ C\ D \vee (A \neq B \wedge C \neq D \wedge Col\ A\ C\ D \wedge Col\ B\ C\ D)$

definition *Plg*::
$[{}'p,{}'p,{}'p,{}'p] \Rightarrow bool$ (‹*Plg* - - - -› [*99*,*99*,*99*,*99*] *50*)

where *Plg A B C D* ≡
$(A \neq C \lor B \neq D) \land (\exists\ M.\ M\ Midpoint\ A\ C \land M\ Midpoint\ B\ D)$

definition *ParallelogramStrict*::
$['p,'p,'p,'p] \Rightarrow bool$ (‹*ParallelogramStrict* - - - -› [99,99,99,99] 50)
where *ParallelogramStrict A B A′ B′* ≡
$A\ A'\ TS\ B\ B' \land A\ B\ Par\ A'\ B' \land Cong\ A\ B\ A'\ B'$

definition *ParallelogramFlat*::
$['p,'p,'p,'p] \Rightarrow bool$ (‹*ParallelogramFlat* - - - -› [99,99,99,99] 50)
where *ParallelogramFlat A B A′ B′* ≡
$Col\ A\ B\ A' \land Col\ A\ B\ B' \land$
$Cong\ A\ B\ A'\ B' \land Cong\ A\ B'\ A'\ B \land$
$(A \neq A' \lor B \neq B')$

definition *Parallelogram*::
$['p,'p,'p,'p] \Rightarrow bool$ (‹*Parallelogram* - - - -› [99,99,99,99] 50)
where *Parallelogram A B A′ B′* ≡
$ParallelogramStrict\ A\ B\ A'\ B' \lor ParallelogramFlat\ A\ B\ A'\ B'$

definition *Rhombus*::
$['p,'p,'p,'p] \Rightarrow bool$ (‹*Rhombus* - - - -› [99,99,99,99] 50)
where *Rhombus A B C D* ≡ $Plg\ A\ B\ C\ D \land Cong\ A\ B\ B\ C$

definition *Rectangle*::
$['p,'p,'p,'p] \Rightarrow bool$ (‹*Rectangle* - - - -› [99,99,99,99] 50)
where *Rectangle A B C D* ≡ $Plg\ A\ B\ C\ D \land Cong\ A\ C\ B\ D$

definition *Square*::
$['p,'p,'p,'p] \Rightarrow bool$ (‹*Square* - - - -› [99,99,99,99] 50)
where *Square A B C D* ≡ $Rectangle\ A\ B\ C\ D \land Cong\ A\ B\ B\ C$

definition *Lambert*::
$['p,'p,'p,'p] \Rightarrow bool$ (‹*Lambert* - - - -› [99,99,99,99] 50)
where *Lambert A B C D* ≡
$A \neq B \land B \neq C \land C \neq D \land$
$A \neq D \land Per\ B\ A\ D \land Per\ A\ D\ C \land Per\ A\ B\ C \land Coplanar\ A\ B\ C\ D$

2.1.10 Plane

definition *OS* ::
$['p,'p,'p,'p] \Rightarrow bool$ (‹- - *OS* - -› [99,99,99,99] 50)
where *A B OS P Q* ≡
$\exists\ R{::}'p.\ A\ B\ TS\ P\ R \land A\ B\ TS\ Q\ R$

definition *TSP* ::
$['p,'p,'p,'p,'p] \Rightarrow bool$ (‹- - -*TSP* - -› [99,99,99,99,99] 50)
where *A B C TSP P Q* ≡
$(\neg\ Coplanar\ A\ B\ C\ P) \land (\neg\ Coplanar\ A\ B\ C\ Q) \land$
$(\exists\ T.\ Coplanar\ A\ B\ C\ T \land Bet\ P\ T\ Q)$

definition *OSP* ::
$['p,'p,'p,'p,'p] \Rightarrow bool$ (‹- - - *OSP* - -› [99,99,99,99,99] 50)
where *A B C OSP P Q* ≡
$\exists\ R.\ ((A\ B\ C\ TSP\ P\ R) \land (A\ B\ C\ TSP\ Q\ R))$

definition *Saccheri*::
$['p,'p,'p,'p] \Rightarrow bool$ (‹*Saccheri* - - - -› [99,99,99,99] 50)
where *Saccheri A B C D* ≡
$Per\ B\ A\ D \land Per\ A\ D\ C \land Cong\ A\ B\ C\ D \land A\ D\ OS\ B\ C$

2.1.11 Line reflexivity 2D

definition *ReflectLAt* ::
$['p,'p,'p,'p,'p] \Rightarrow bool$ (‹- *ReflectLAt* - - - -› [99,99,99,99,99] 50)
where *M ReflectLAt P′ P A B* ≡
$(M\ Midpoint\ P\ P' \land Col\ A\ B\ M) \land (A\ B\ Perp\ P\ P' \lor P = P')$

definition *ReflectAt* ::
  $['p,'p,'p,'p,'p] \Rightarrow bool$ (‹- *ReflectAt* - - - -› [*99,99,99,99,99*] *50*)
  where $M\ ReflectAt\ P'\ P\ A\ B \equiv$
$(A \neq B \wedge M\ ReflectLAt\ P'\ P\ A\ B) \vee (A = B \wedge A = M \wedge M\ Midpoint\ P\ P')$

2.1.12 Line reflexivity

definition *upper-dim-axiom* ::
  bool (‹*UpperDimAxiom*› [] *50*)
  where
    upper-dim-axiom ≡

$\forall\ A\ B\ C\ P\ Q.$
$P \neq Q \wedge$
$Cong\ A\ P\ A\ Q \wedge$
$Cong\ B\ P\ B\ Q \wedge$
$Cong\ C\ P\ C\ Q$
$\longrightarrow$
$(Bet\ A\ B\ C \vee Bet\ B\ C\ A \vee Bet\ C\ A\ B)$

definition *all-coplanar-axiom* ::
  bool (‹*AllCoplanarAxiom*› [] *50*)
  where
    AllCoplanarAxiom ≡

$\forall\ A\ B\ C\ P\ Q.$
$P \neq Q \wedge$
$Cong\ A\ P\ A\ Q \wedge$
$Cong\ B\ P\ B\ Q \wedge$
$Cong\ C\ P\ C\ Q$
$\longrightarrow$
$(Bet\ A\ B\ C \vee Bet\ B\ C\ A \vee Bet\ C\ A\ B)$

2.1.13 Angles

definition *CongA* ::
  $['p,'p,'p,'p,'p,'p] \Rightarrow bool$ (‹- - - *CongA* - - -› [*99,99,99,99,99,99*] *50*)
  where $A\ B\ C\ CongA\ D\ E\ F \equiv$
  $A \neq B \wedge C \neq B \wedge D \neq E \wedge F \neq E \wedge$
$(\exists\ A'\ C'\ D'\ F'.\ Bet\ B\ A\ A' \wedge Cong\ A\ A'\ E\ D \wedge$
  $Bet\ B\ C\ C' \wedge Cong\ C\ C'\ E\ F \wedge$
  $Bet\ E\ D\ D' \wedge Cong\ D\ D'\ B\ A \wedge$
  $Bet\ E\ F\ F' \wedge Cong\ F\ F'\ B\ C \wedge$
  $Cong\ A'\ C'\ D'\ F')$

definition *LeA* ::
  $['p,'p,'p,'p,'p,'p] \Rightarrow bool$ (‹- - - *LeA* - - -› [*99,99,99,99,99,99*] *50*)
  where $A\ B\ C\ LeA\ D\ E\ F \equiv$
$\exists\ P.\ (P\ InAngle\ D\ E\ F \wedge A\ B\ C\ CongA\ D\ E\ P)$

definition *LtA* ::
  $['p,'p,'p,'p,'p,'p] \Rightarrow bool$ (‹- - - *LtA* - - -› [*99,99,99,99,99,99*] *50*)
  where $A\ B\ C\ LtA\ D\ E\ F \equiv A\ B\ C\ LeA\ D\ E\ F \wedge \neg\ A\ B\ C\ CongA\ D\ E\ F$

definition *GtA* ::
  $['p,'p,'p,'p,'p,'p] \Rightarrow bool$ (‹- - - *GtA* - - -› [*99,99,99,99,99,99*] *50*)
  where $A\ B\ C\ GtA\ D\ E\ F \equiv D\ E\ F\ LtA\ A\ B\ C$

definition *Acute* ::
  $['p,'p,'p] \Rightarrow bool$ (‹*Acute* - - -› [*99,99,99*] *50*)
  where $Acute\ A\ B\ C \equiv$
$\exists\ A'\ B'\ C'.\ (Per\ A'\ B'\ C' \wedge A\ B\ C\ LtA\ A'\ B'\ C')$

definition *Obtuse* ::
  $['p,'p,'p] \Rightarrow bool$ (‹*Obtuse* - - -› [*99,99,99*] *50*)
  where $Obtuse\ A\ B\ C \equiv$

$\exists\ A'\ B'\ C'.\ (Per\ A'\ B'\ C' \wedge A'\ B'\ C'\ LtA\ A\ B\ C)$

definition *OrthAt* ::
$[{'}p,{'}p,{'}p,{'}p,{'}p,{'}p] \Rightarrow bool$ (‹- *OrthAt* - - - - -› [*99*,*99*,*99*,*99*,*99*,*99*] *50*)
where $X\ OrthAt\ A\ B\ C\ U\ V \equiv$
$\neg\ Col\ A\ B\ C \wedge U \neq V \wedge Coplanar\ A\ B\ C\ X \wedge Col\ U\ V\ X \wedge$
$(\forall\ P\ Q.\ (Coplanar\ A\ B\ C\ P \wedge Col\ U\ V\ Q) \longrightarrow Per\ P\ X\ Q)$

definition *Orth* ::
$[{'}p,{'}p,{'}p,{'}p,{'}p] \Rightarrow bool$ (‹- - - *Orth* - -› [*99*,*99*,*99*,*99*,*99*] *50*)
where $A\ B\ C\ Orth\ U\ V \equiv \exists\ X.\ X\ OrthAt\ A\ B\ C\ U\ V$

definition *SuppA* ::
$[{'}p,{'}p,{'}p,{'}p,{'}p,{'}p] \Rightarrow bool$
(‹- - - *SuppA* - - - › [*99*,*99*,*99*,*99*,*99*,*99*] *50*)
where
$A\ B\ C\ SuppA\ D\ E\ F \equiv$
$A \neq B \wedge (\exists\ A'.\ Bet\ A\ B\ A' \wedge\ D\ E\ F\ CongA\ C\ B\ A')$

2.1.14 Sum of angles

definition *SumA* ::
$[{'}p,{'}p,{'}p,{'}p,{'}p,{'}p,{'}p,{'}p,{'}p] \Rightarrow bool$ (‹- - - - - - *SumA* - - -› [*99*,*99*,*99*,*99*,*99*,*99*,*99*,*99*,*99*] *50*)
where
$A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I \equiv$

$\exists\ J.\ (C\ B\ J\ CongA\ D\ E\ F \wedge \neg\ B\ C\ OS\ A\ J \wedge Coplanar\ A\ B\ C\ J \wedge A\ B\ J\ CongA\ G\ H\ I)$

definition *TriSumA* ::
$[{'}p,{'}p,{'}p,{'}p,{'}p,{'}p] \Rightarrow bool$ (‹- - - *TriSumA* - - -› [*99*,*99*,*99*,*99*,*99*,*99*] *50*)
where
$A\ B\ C\ TriSumA\ D\ E\ F \equiv$

$\exists\ G\ H\ I.\ (A\ B\ C\ B\ C\ A\ SumA\ G\ H\ I \wedge G\ H\ I\ C\ A\ B\ SumA\ D\ E\ F)$

definition *SAMS* ::
$[{'}p,{'}p,{'}p,{'}p,{'}p,{'}p] \Rightarrow bool$ (‹*SAMS* - - - - - -› [*99*,*99*,*99*,*99*,*99*,*99*] *50*)
where
$SAMS\ A\ B\ C\ D\ E\ F \equiv$

$(A \neq B \wedge$
$(E\ Out\ D\ F \vee \neg\ Bet\ A\ B\ C)) \wedge$
$(\exists\ J.\ (C\ B\ J\ CongA\ D\ E\ F \wedge \neg\ (B\ C\ OS\ A\ J) \wedge \neg\ (A\ B\ TS\ C\ J) \wedge Coplanar\ A\ B\ C\ J))$

2.1.15 Parallelism

definition *Inter* ::
$[{'}p,{'}p,{'}p,{'}p,{'}p] \Rightarrow bool$ (‹- *Inter* - - - -› [*99*,*99*,*99*,*99*,*99*] *50*)
where $X\ Inter\ A1\ A2\ B1\ B2 \equiv$

$B1 \neq B2 \wedge$
$(\exists\ P{::}{'}p.\ (Col\ P\ B1\ B2 \wedge \neg\ Col\ P\ A1\ A2)) \wedge$
$Col\ A1\ A2\ X \wedge Col\ B1\ B2\ X$

2.1.16 Perpendicularity

definition *Perp2* ::
$[{'}p,{'}p,{'}p,{'}p,{'}p] \Rightarrow bool$ (‹- *Perp2* - - - -› [*99*,*99*,*99*,*99*,*99*] *50*)
where
$P\ Perp2\ A\ B\ C\ D \equiv$

$\exists\ X\ Y.\ (Col\ P\ X\ Y \wedge X\ Y\ Perp\ A\ B \wedge X\ Y\ Perp\ C\ D)$

2.1.17 Lentgh

definition *QCong*::
$([{'}p,{'}p] \Rightarrow bool) \Rightarrow bool$ (‹*QCong* -› [*99*] *50*)

where
QCong l ≡

∃ *A B*. (∀ *X Y*. (*Cong A B X Y* ⟷ *l X Y*))

definition *TarskiLen*::
['p,'p,(['p,'p] ⇒ *bool*)] ⇒ *bool* (‹*TarskiLen* - - -› [*99*,*99*,*99*] *50*)
where
TarskiLen A B l ≡

QCong l ∧ *l A B*

definition *QCongNull* ::
(['p,'p] ⇒ *bool*) ⇒ *bool* (‹*QCongNull* -› [*99*] *50*)
where
QCongNull l ≡

QCong l ∧ (∃ *A*. *l A A*)

2.1.18 Equivalence Class of Angles

definition *QCongA* ::
(['p, 'p, 'p] ⇒ *bool*) ⇒ *bool* (‹*QCongA* -› [*99*] *50*)
where
QCongA a ≡

∃ *A B C*. (*A* ≠ *B* ∧ *C* ≠ *B* ∧ (∀ *X Y Z*. *A B C CongA X Y Z* ⟷ *a X Y Z*))

definition *Ang* ::
['p,'p,'p, (['p, 'p, 'p] ⇒ *bool*)] ⇒ *bool* (‹- - - *Ang* -› [*99*,*99*,*99*,*99*] *50*)
where
A B C Ang a ≡

QCongA a ∧
a A B C

definition *QCongAAcute* ::
(['p, 'p, 'p] ⇒ *bool*) ⇒ *bool* (‹*QCongAACute* -› [*99*] *50*)
where
QCongAAcute a ≡

∃ *A B C*. (*Acute A B C* ∧ (∀ *X Y Z*. (*A B C CongA X Y Z* ⟷ *a X Y Z*)))

definition *AngAcute* ::
['p,'p,'p, (['p,'p,'p] ⇒ *bool*)] ⇒ *bool* (‹- - - *AngAcute* -› [*99*,*99*,*99*,*99*] *50*)
where
A B C AngAcute a ≡

((*QCongAAcute a*) ∧ (*a A B C*))

definition *QCongANullAcute* ::
(['p,'p,'p] ⇒ *bool*) ⇒ *bool* (‹*QCongANullAcute* -› [*99*] *50*)
where
QCongANullAcute a ≡

QCongAAcute a ∧
(∀ *A B C*. (*a A B C* ⟶ *B Out A C*))

definition *QCongAnNull* ::
(['p,'p,'p] ⇒ *bool*) ⇒ *bool* (‹*QCongAnNull* -› [*99*] *50*)
where
QCongAnNull a ≡

QCongA a ∧
(∀ *A B C*. (*a A B C* ⟶ ¬ *B Out A C*))

definition *QCongAnFlat* ::
([′p,′p,′p] ⇒ *bool*) ⇒ *bool* (‹*QCongAnFlat* -› [*99*] *50*)
where
QCongAnFlat a ≡

QCongA a ∧
(∀ *A B C*. (*a A B C* ⟶ ¬ *Bet A B C*))

definition *IsNullAngaP* ::
([′p,′p,′p] ⇒ *bool*) ⇒ *bool* (‹*IsNullAngaP* -› [*99*] *50*)
where
IsNullAngaP a≡

QCongAAcute a ∧
(∃ *A B C*. (*a A B C* ∧ *B Out A C*))

definition *QCongANull* ::
([′p,′p,′p] ⇒ *bool*) ⇒ *bool* (‹*QCongANull* -› [*99*] *50*)
where
QCongANull a ≡

QCongA a ∧
(∀ *A B C*. (*a A B C* ⟶ *B Out A C*))

definition *AngFlat* ::
([′p, ′p, ′p] ⇒ *bool*) ⇒ *bool* (‹*AngFlat* -› [*99*] *50*)
where
AngFlat a ≡

QCongA a ∧
(∀ *A B C*. (*a A B C* ⟶ *Bet A B C*))

2.2 Parallel's definition Postulate

definition *tarski-s-parallel-postulate* ::
bool
(‹*TarskiSParallelPostulate*›)
where
tarski-s-parallel-postulate ≡
∀ *A B C D T*. (*Bet A D T* ∧ *Bet B D C* ∧ *A* ≠ *D*) ⟶
(∃ *X Y*. *Bet A B X* ∧ *Bet A C Y* ∧ *Bet X T Y*)

definition *euclid-5* ::
bool (‹*Euclid5*›)
where
euclid-5 ≡

∀ *P Q R S T U*.
(*BetS P T Q* ∧
BetS R T S ∧
BetS Q U R ∧
¬ *Col P Q S* ∧
Cong P T Q T ∧
Cong R T S T)
⟶
(∃ *I*. *BetS S Q I* ∧ *BetS P U I*)

definition *euclid-s-parallel-postulate* ::
bool (‹*EuclidSParallelPostulate*›)
where
euclid-s-parallel-postulate ≡

∀ *A B C D P Q R*.
(*B C OS A D* ∧
SAMS A B C B C D ∧
A B C B C D SumA P Q R ∧

¬ *Bet P Q R*)
⟶
(∃ *Y*. *B Out A Y* ∧ *C Out D Y*)

definition *playfair-s-postulate* ::
bool
(‹*PlayfairSPostulate*›)
where
playfair-s-postulate ≡

∀ *A1 A2 B1 B2 C1 C2 P*.
(*A1 A2 Par B1 B2* ∧
Col P B1 B2 ∧
A1 A2 Par C1 C2 ∧
Col P C1 C2)
⟶
(*Col C1 B1 B2* ∧ *Col C2 B1 B2*)

3 Propositions

3.1 Congruence properties

lemma *cong-reflexivity*:
shows *Cong A B A B*
⟨*proof*⟩

lemma *cong-symmetry*:
assumes *Cong A B C D*
shows *Cong C D A B*
⟨*proof*⟩

lemma *cong-transitivity*:
assumes *Cong A B C D* **and** *Cong C D E F*
shows *Cong A B E F*
⟨*proof*⟩

lemma *cong-left-commutativity*:
assumes *Cong A B C D*
shows *Cong B A C D*
⟨*proof*⟩

lemma *cong-right-commutativity*:
assumes *Cong A B C D*
shows *Cong A B D C*
⟨*proof*⟩

lemma *cong-3421*:
assumes *Cong A B C D*
shows *Cong C D B A*
⟨*proof*⟩

lemma *cong-4312*:
assumes *Cong A B C D*
shows *Cong D C A B*
⟨*proof*⟩

lemma *cong-4321*:
assumes *Cong A B C D*
shows *Cong D C B A*
⟨*proof*⟩

lemma *cong-trivial-identity*:
shows *Cong A A B B*
⟨*proof*⟩

lemma *cong-reverse-identity*:

assumes *Cong A A C D*
shows $C = D$
⟨*proof*⟩

lemma *cong-commutativity*:
assumes *Cong A B C D*
shows *Cong B A D C*
⟨*proof*⟩

lemma *not-cong-2134*:
assumes ¬ *Cong A B C D*
shows ¬ *Cong B A C D*
⟨*proof*⟩

lemma *not-cong-1243*:
assumes ¬ *Cong A B C D*
shows ¬ *Cong A B D C*
⟨*proof*⟩

lemma *not-cong-2143*:
assumes ¬ *Cong A B C D*
shows ¬ *Cong B A D C*
⟨*proof*⟩

lemma *not-cong-3412*:
assumes ¬ *Cong A B C D*
shows ¬ *Cong C D A B*
⟨*proof*⟩

lemma *not-cong-4312*:
assumes ¬ *Cong A B C D*
shows ¬ *Cong D C A B*
⟨*proof*⟩

lemma *not-cong-3421*:
assumes ¬ *Cong A B C D*
shows ¬ *Cong C D B A*
⟨*proof*⟩

lemma *not-cong-4321*:
assumes ¬ *Cong A B C D*
shows ¬ *Cong D C B A*
⟨*proof*⟩

lemma *five-segment-with-def*:
assumes $A\ B\ C\ D$ *OFSC* $A'\ B'\ C'\ D'$ **and** $A \neq B$
shows *Cong* $C\ D\ C'\ D'$
⟨*proof*⟩

lemma *cong-diff*:
assumes $A \neq B$ **and** *Cong A B C D*
shows $C \neq D$
⟨*proof*⟩

lemma *cong-diff-2*:
assumes $B \neq A$ **and** *Cong A B C D*
shows $C \neq D$
⟨*proof*⟩

lemma *cong-diff-3*:
assumes $C \neq D$ **and** *Cong A B C D*
shows $A \neq B$
⟨*proof*⟩

lemma *cong-diff-4*:
assumes $D \neq C$ **and** *Cong A B C D*

shows $A \neq B$
⟨*proof*⟩

lemma *cong-3-sym*:
assumes $A\ B\ C\ Cong3\ A'\ B'\ C'$
shows $A'\ B'\ C'\ Cong3\ A\ B\ C$
⟨*proof*⟩

lemma *cong-3-swap*:
assumes $A\ B\ C\ Cong3\ A'\ B'\ C'$
shows $B\ A\ C\ Cong3\ B'\ A'\ C'$
⟨*proof*⟩

lemma *cong-3-swap-2*:
assumes $A\ B\ C\ Cong3\ A'\ B'\ C'$
shows $A\ C\ B\ Cong3\ A'\ C'\ B'$
⟨*proof*⟩

lemma *cong3-transitivity*:
assumes $A0\ B0\ C0\ Cong3\ A1\ B1\ C1$ **and**
$A1\ B1\ C1\ Cong3\ A2\ B2\ C2$
shows $A0\ B0\ C0\ Cong3\ A2\ B2\ C2$
⟨*proof*⟩

lemma *eq-dec-points*:
shows $A = B \vee \neg\ A = B$
⟨*proof*⟩

lemma *distinct*:
assumes $P \neq Q$
shows $R \neq P \vee R \neq Q$
⟨*proof*⟩

lemma *l2-11*:
assumes $Bet\ A\ B\ C$ **and**
$Bet\ A'\ B'\ C'$ **and**
$Cong\ A\ B\ A'\ B'$ **and**
$Cong\ B\ C\ B'\ C'$
shows $Cong\ A\ C\ A'\ C'$
⟨*proof*⟩

lemma *bet-cong3*:
assumes $Bet\ A\ B\ C$ **and**
$Cong\ A\ B\ A'\ B'$
shows $\exists\ C'.\ A\ B\ C\ Cong3\ A'\ B'\ C'$
⟨*proof*⟩

lemma *construction-uniqueness*:
assumes $Q \neq A$ **and**
$Bet\ Q\ A\ X$ **and**
$Cong\ A\ X\ B\ C$ **and**
$Bet\ Q\ A\ Y$ **and**
$Cong\ A\ Y\ B\ C$
shows $X = Y$
⟨*proof*⟩

lemma *Cong-cases*:
assumes $Cong\ A\ B\ C\ D \vee Cong\ A\ B\ D\ C \vee Cong\ B\ A\ C\ D \vee Cong\ B\ A\ D\ C \vee Cong\ C\ D\ A\ B \vee Cong\ C\ D\ B\ A \vee Cong\ D\ C\ A\ B \vee Cong\ D\ C\ B\ A$
shows $Cong\ A\ B\ C\ D$
⟨*proof*⟩

lemma *Cong-perm* :
assumes $Cong\ A\ B\ C\ D$
shows $Cong\ A\ B\ C\ D \wedge Cong\ A\ B\ D\ C \wedge Cong\ B\ A\ C\ D \wedge Cong\ B\ A\ D\ C \wedge Cong\ C\ D\ A\ B \wedge Cong\ C\ D\ B\ A \wedge Cong\ D\ C\ A\ B \wedge Cong\ D\ C\ B\ A$

⟨*proof*⟩

3.2 Betweeness properties

lemma *bet-col*:
assumes *Bet A B C*
shows *Col A B C*
⟨*proof*⟩

lemma *between-trivial*:
shows *Bet A B B*
⟨*proof*⟩

lemma *between-symmetry*:
assumes *Bet A B C*
shows *Bet C B A*
⟨*proof*⟩

lemma *Bet-cases*:
assumes *Bet A B C* ∨ *Bet C B A*
shows *Bet A B C*
⟨*proof*⟩

lemma *Bet-perm*:
assumes *Bet A B C*
shows *Bet A B C* ∧ *Bet C B A*
⟨*proof*⟩

lemma *between-trivial2*:
shows *Bet A A B*
⟨*proof*⟩

lemma *between-equality*:
assumes *Bet A B C* **and** *Bet B A C*
shows $A = B$
⟨*proof*⟩

lemma *between-equality-2*:
assumes *Bet A B C* **and**
Bet A C B
shows $B = C$
⟨*proof*⟩

lemma *between-exchange3*:
assumes *Bet A B C* **and**
Bet A C D
shows *Bet B C D*
⟨*proof*⟩

lemma *bet-neq12--neq*:
assumes *Bet A B C* **and**
$A \neq B$
shows $A \neq C$
⟨*proof*⟩

lemma *bet-neq21--neq*:
assumes *Bet A B C* **and**
$B \neq A$
shows $A \neq C$
⟨*proof*⟩

lemma *bet-neq23--neq*:
assumes *Bet A B C* **and**
$B \neq C$
shows $A \neq C$
⟨*proof*⟩

lemma *bet-neq32--neq*:
  assumes *Bet A B C* **and**
    $C \neq B$
  shows $A \neq C$
  ⟨*proof*⟩

lemma *not-bet-distincts*:
  assumes ¬ *Bet A B C*
  shows $A \neq B \wedge B \neq C$
  ⟨*proof*⟩

lemma *between-inner-transitivity*:
  assumes *Bet A B D* **and**
    Bet B C D
  shows *Bet A B C*
  ⟨*proof*⟩

lemma *outer-transitivity-between2*:
  assumes *Bet A B C* **and**
    Bet B C D **and**
    $B \neq C$
  shows *Bet A C D*
⟨*proof*⟩

lemma *between-exchange2*:
  assumes *Bet A B D* **and**
    Bet B C D
  shows *Bet A C D*
  ⟨*proof*⟩

lemma *outer-transitivity-between*:
  assumes *Bet A B C* **and**
    Bet B C D **and**
    $B \neq C$
  shows *Bet A B D*
  ⟨*proof*⟩

lemma *between-exchange4*:
  assumes *Bet A B C* **and**
    Bet A C D
  shows *Bet A B D*
  ⟨*proof*⟩

lemma *l3-9-4*:
  assumes *Bet4 A1 A2 A3 A4*
  shows *Bet4 A4 A3 A2 A1*
  ⟨*proof*⟩

lemma *l3-17*:
  assumes *Bet A B C* **and**
    Bet A' B' *C* **and**
    Bet A P A'
  shows $(\exists\ Q.\ Bet\ P\ Q\ C \wedge Bet\ B\ Q\ B')$
⟨*proof*⟩

lemma *lower-dim-ex*:
  $\exists\ A\ B\ C.\ \neg\ (Bet\ A\ B\ C \vee Bet\ B\ C\ A \vee Bet\ C\ A\ B)$
  ⟨*proof*⟩

lemma *two-distinct-points*:
  $\exists\ X{::}'p.\ \exists\ Y{::}'p.\ X \neq Y$
  ⟨*proof*⟩

lemma *point-construction-different*:
  $\exists\ C.\ Bet\ A\ B\ C \wedge B \neq C$

⟨*proof*⟩

lemma *another-point*:
$\exists$ B::'p. $A \neq B$
⟨*proof*⟩

lemma *Cong-stability*:
assumes $\neg$ $\neg$ *Cong* A B C D
shows *Cong* A B C D
⟨*proof*⟩

lemma *l2-11-b*:
assumes *Bet* A B C **and**
Bet A' B' C' **and**
Cong A B A' B' **and**
Cong B C B' C'
shows *Cong* A C A' C'
⟨*proof*⟩

lemma *cong-dec-eq-dec-b*:
assumes $\neg$ $A \neq B$
shows $A = B$
⟨*proof*⟩

lemma *BetSEq*:
assumes *BetS* A B C
shows *Bet* A B C $\wedge$ $A \neq B$ $\wedge$ $A \neq C$ $\wedge$ $B \neq C$
⟨*proof*⟩

3.3 Collinearity

3.3.1 Collinearity and betweenness

lemma *l4-2*:
assumes A B C D *IFSC* A' B' C' D'
shows *Cong* B D B' D'
⟨*proof*⟩

lemma *l4-3*:
assumes *Bet* A B C **and**
Bet A' B' C' **and**
Cong A C A' C'
and *Cong* B C B' C'
shows *Cong* A B A' B'
⟨*proof*⟩

lemma *l4-3-1*:
assumes *Bet* A B C **and**
Bet A' B' C' **and**
Cong A B A' B' **and**
Cong A C A' C'
shows *Cong* B C B' C'
⟨*proof*⟩

lemma *l4-5*:
assumes *Bet* A B C **and**
Cong A C A' C'
shows $\exists$ B'. (*Bet* A' B' C' $\wedge$ A B C *Cong3* A' B' C')
⟨*proof*⟩

lemma *l4-6*:
assumes *Bet* A B C **and**
A B C *Cong3* A' B' C'
shows *Bet* A' B' C'
⟨*proof*⟩

lemma *cong3-bet-eq*:
assumes *Bet A B C* **and**
A B C Cong3 A X C
shows *X* = *B*
⟨*proof*⟩

3.3.2 Collinearity

lemma *col-permutation-1*:
assumes *Col A B C*
shows *Col B C A*
⟨*proof*⟩

lemma *col-permutation-2*:
assumes *Col A B C*
shows *Col C A B*
⟨*proof*⟩

lemma *col-permutation-3*:
assumes *Col A B C*
shows *Col C B A*
⟨*proof*⟩

lemma *col-permutation-4*:
assumes *Col A B C*
shows *Col B A C*
⟨*proof*⟩

lemma *col-permutation-5*:
assumes *Col A B C*
shows *Col A C B*
⟨*proof*⟩

lemma *not-col-permutation-1*:
assumes ¬ *Col A B C*
shows ¬ *Col B C A*
⟨*proof*⟩

lemma *not-col-permutation-2*:
assumes ~ *Col A B C*
shows ~ *Col C A B*
⟨*proof*⟩

lemma *not-col-permutation-3*:
assumes ¬ *Col A B C*
shows ¬ *Col C B A*
⟨*proof*⟩

lemma *not-col-permutation-4*:
assumes ¬ *Col A B C*
shows ¬ *Col B A C*
⟨*proof*⟩

lemma *not-col-permutation-5*:
assumes ¬ *Col A B C*
shows ¬ *Col A C B*
⟨*proof*⟩

lemma *Col-cases*:
assumes *Col A B C* ∨ *Col A C B* ∨ *Col B A C* ∨ *Col B C A* ∨ *Col C A B* ∨ *Col C B A*
shows *Col A B C*
⟨*proof*⟩

lemma *Col-perm*:
assumes *Col A B C*
shows *Col A B C* ∧ *Col A C B* ∧ *Col B A C* ∧ *Col B C A* ∧ *Col C A B* ∧ *Col C B A*

⟨*proof*⟩

lemma *col-trivial-1*:
Col A A B
⟨*proof*⟩

lemma *col-trivial-2*:
Col A B B
⟨*proof*⟩

lemma *col-trivial-3*:
Col A B A
⟨*proof*⟩

lemma *l4-13*:
assumes *Col A B C* **and**
$A\ B\ C\ Cong3\ A'\ B'\ C'$
shows $Col\ A'\ B'\ C'$
⟨*proof*⟩

lemma *l4-14R1*:
assumes *Bet A B C* **and**
$Cong\ A\ B\ A'\ B'$
shows $\exists\ C'.\ A\ B\ C\ Cong3\ A'\ B'\ C'$
⟨*proof*⟩

lemma *l4-14R2*:
assumes *Bet B C A* **and**
$Cong\ A\ B\ A'\ B'$
shows $\exists\ C'.\ A\ B\ C\ Cong3\ A'\ B'\ C'$
⟨*proof*⟩

lemma *l4-14R3*:
assumes *Bet C A B* **and**
$Cong\ A\ B\ A'\ B'$
shows $\exists\ C'.\ A\ B\ C\ Cong3\ A'\ B'\ C'$
⟨*proof*⟩

lemma *l4-14*:
assumes *Col A B C* **and**
$Cong\ A\ B\ A'\ B'$
shows $\exists\ C'.\ A\ B\ C\ Cong3\ A'\ B'\ C'$
⟨*proof*⟩

lemma *l4-16R1*:
assumes $A\ B\ C\ D\ FSC\ A'\ B'\ C'\ D'$ **and**
$A \neq B$ **and**
Bet A B C
shows $Cong\ C\ D\ C'\ D'$
⟨*proof*⟩

lemma *l4-16R2*:
assumes $A\ B\ C\ D\ FSC\ A'\ B'\ C'\ D'$
and *Bet B C A*
shows $Cong\ C\ D\ C'\ D'$
⟨*proof*⟩

lemma *l4-16R3*:
assumes $A\ B\ C\ D\ FSC\ A'\ B'\ C'\ D'$ **and** $A \neq B$
and *Bet C A B*
shows $Cong\ C\ D\ C'\ D'$
⟨*proof*⟩

lemma *l4-16*:
assumes $A\ B\ C\ D\ FSC\ A'\ B'\ C'\ D'$ **and**
$A \neq B$

shows *Cong C D C′ D′*
⟨*proof*⟩

lemma *l4-17*:
assumes $A \neq B$ **and**
Col A B C **and**
Cong A P A Q **and**
Cong B P B Q
shows *Cong C P C Q*
⟨*proof*⟩

lemma *l4-18*:
assumes $A \neq B$ **and**
Col A B C **and**
Cong A C A C′ **and**
Cong B C B C′
shows $C = C'$
⟨*proof*⟩

lemma *l4-19*:
assumes *Bet A C B* **and**
Cong A C A C′ **and**
Cong B C B C′
shows $C = C'$
⟨*proof*⟩

lemma *not-col-distincts*:
assumes ¬ *Col A B C*
shows ¬ *Col A B C* ∧ $A \neq B$ ∧ $B \neq C$ ∧ $A \neq C$
⟨*proof*⟩

lemma *NCol-cases*:
assumes ¬ *Col A B C* ∨ ¬ *Col A C B* ∨ ¬ *Col B A C* ∨ ¬ *Col B C A* ∨ ¬ *Col C A B* ∨ ¬ *Col C B A*
shows ¬ *Col A B C*
⟨*proof*⟩

lemma *NCol-perm*:
assumes ¬ *Col A B C*
shows ¬ *Col A B C* ∧ ~ *Col A C B* ∧ ~ *Col B A C* ∧ ~ *Col B C A* ∧ ~ *Col C A B* ∧ ~ *Col C B A*
⟨*proof*⟩

lemma *col-cong-3-cong-3-eq*:
assumes $A \neq B$
and *Col A B C*
and *A B C Cong3 A′ B′ C1*
and *A B C Cong3 A′ B′ C2*
shows $C1 = C2$
⟨*proof*⟩

3.4 Between transitivity le

lemma *l5-1*:
assumes $A \neq B$ **and**
Bet A B C **and**
Bet A B D
shows *Bet A C D* ∨ *Bet A D C*
⟨*proof*⟩

lemma *l5-2*:
assumes $A \neq B$ **and**
Bet A B C **and**
Bet A B D
shows *Bet B C D* ∨ *Bet B D C*
⟨*proof*⟩

lemma *segment-construction-2*:
  assumes $A \neq Q$
  shows $\exists X. ((Bet\ Q\ A\ X \vee Bet\ Q\ X\ A) \wedge Cong\ Q\ X\ B\ C)$
⟨*proof*⟩

lemma *l5-3*:
  assumes $Bet\ A\ B\ D$ **and**
    $Bet\ A\ C\ D$
  shows $Bet\ A\ B\ C \vee Bet\ A\ C\ B$
  ⟨*proof*⟩

lemma *bet3--bet*:
  assumes $Bet\ A\ B\ E$ **and**
    $Bet\ A\ D\ E$ **and**
    $Bet\ B\ C\ D$
  shows $Bet\ A\ C\ E$
  ⟨*proof*⟩

lemma *le-bet*:
  assumes $C\ D\ Le\ A\ B$
  shows $\exists X. (Bet\ A\ X\ B \wedge Cong\ A\ X\ C\ D)$
  ⟨*proof*⟩

lemma *l5-5-1*:
  assumes $A\ B\ Le\ C\ D$
  shows $\exists X. (Bet\ A\ B\ X \wedge Cong\ A\ X\ C\ D)$
⟨*proof*⟩

lemma *l5-5-2*:
  assumes $\exists X. (Bet\ A\ B\ X \wedge Cong\ A\ X\ C\ D)$
  shows $A\ B\ Le\ C\ D$
⟨*proof*⟩

lemma *l5-6*:
  assumes $A\ B\ Le\ C\ D$ **and**
    $Cong\ A\ B\ A'\ B'$ **and**
    $Cong\ C\ D\ C'\ D'$
  shows $A'\ B'\ Le\ C'\ D'$
  ⟨*proof*⟩

lemma *le-reflexivity*:
  shows $A\ B\ Le\ A\ B$
  ⟨*proof*⟩

lemma *le-transitivity*:
  assumes $A\ B\ Le\ C\ D$ **and**
    $C\ D\ Le\ E\ F$
  shows $A\ B\ Le\ E\ F$
  ⟨*proof*⟩

lemma *between-cong*:
  assumes $Bet\ A\ C\ B$ **and**
    $Cong\ A\ C\ A\ B$
  shows $C = B$
  ⟨*proof*⟩

lemma *cong3-symmetry*:
  assumes $A\ B\ C\ Cong3\ A'\ B'\ C'$
  shows $A'\ B'\ C'\ Cong3\ A\ B\ C$
  ⟨*proof*⟩

lemma *between-cong-2*:
  assumes $Bet\ A\ D\ B$ **and**
    $Bet\ A\ E\ B$
    and $Cong\ A\ D\ A\ E$
  shows $D = E$ ⟨*proof*⟩

lemma *between-cong-3*:
  assumes $A \neq B$
    and *Bet* A B D
    and *Bet* A B E
    and *Cong* B D B E
  shows $D = E$
  ⟨*proof*⟩

lemma *le-anti-symmetry*:
  assumes A B *Le* C D **and**
    C D *Le* A B
  shows *Cong* A B C D
  ⟨*proof*⟩

lemma *cong-dec*:
  shows *Cong* A B C D $\vee \neg$ *Cong* A B C D
  ⟨*proof*⟩

lemma *bet-dec*:
  shows *Bet* A B C $\vee \neg$ *Bet* A B C
  ⟨*proof*⟩

lemma *col-dec*:
  shows *Col* A B C $\vee \neg$ *Col* A B C
  ⟨*proof*⟩

lemma *le-trivial*:
  shows A A *Le* C D
  ⟨*proof*⟩

lemma *le-cases*:
  shows A B *Le* C D $\vee$ C D *Le* A B
  ⟨*proof*⟩

lemma *le-zero*:
  assumes A B *Le* C C
  shows $A = B$
  ⟨*proof*⟩

lemma *le-diff*:
  assumes $A \neq B$ **and** A B *Le* C D
  shows $C \neq D$
  ⟨*proof*⟩

lemma *lt-diff*:
  assumes A B *Lt* C D
  shows $C \neq D$
  ⟨*proof*⟩

lemma *bet-cong-eq*:
  assumes *Bet* A B C **and**
    Bet A C D **and**
    Cong B C A D
  shows $C = D \wedge A = B$
⟨*proof*⟩

lemma *cong--le*:
  assumes *Cong* A B C D
  shows A B *Le* C D
  ⟨*proof*⟩

lemma *cong--le3412*:
  assumes *Cong* A B C D
  shows C D *Le* A B
  ⟨*proof*⟩

lemma *le1221*:
shows *A B Le B A*
⟨*proof*⟩

lemma *le-left-comm*:
assumes *A B Le C D*
shows *B A Le C D*
⟨*proof*⟩

lemma *le-right-comm*:
assumes *A B Le C D*
shows *A B Le D C*
⟨*proof*⟩

lemma *le-comm*:
assumes *A B Le C D*
shows *B A Le D C*
⟨*proof*⟩

lemma *ge-left-comm*:
assumes *A B Ge C D*
shows *B A Ge C D*
⟨*proof*⟩

lemma *ge-right-comm*:
assumes *A B Ge C D*
shows *A B Ge D C*
⟨*proof*⟩

lemma *ge-comm0*:
assumes *A B Ge C D*
shows *B A Ge D C*
⟨*proof*⟩

lemma *lt-right-comm*:
assumes *A B Lt C D*
shows *A B Lt D C*
⟨*proof*⟩

lemma *lt-left-comm*:
assumes *A B Lt C D*
shows *B A Lt C D*
⟨*proof*⟩

lemma *lt-comm*:
assumes *A B Lt C D*
shows *B A Lt D C*
⟨*proof*⟩

lemma *gt-left-comm0*:
assumes *A B Gt C D*
shows *B A Gt C D*
⟨*proof*⟩

lemma *gt-right-comm*:
assumes *A B Gt C D*
shows *A B Gt D C*
⟨*proof*⟩

lemma *gt-comm*:
assumes *A B Gt C D*
shows *B A Gt D C*
⟨*proof*⟩

lemma *cong2-lt--lt*:

assumes $A\ B\ Lt\ C\ D$ **and**
$Cong\ A\ B\ A'\ B'$ **and**
$Cong\ C\ D\ C'\ D'$
shows $A'\ B'\ Lt\ C'\ D'$
⟨*proof*⟩

lemma *fourth-point*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
$Col\ A\ B\ P$ **and**
$Bet\ A\ B\ C$
shows $Bet\ P\ A\ B \lor Bet\ A\ P\ B \lor Bet\ B\ P\ C \lor Bet\ B\ C\ P$
⟨*proof*⟩

lemma *third-point*:
assumes $Col\ A\ B\ P$
shows $Bet\ P\ A\ B \lor Bet\ A\ P\ B \lor Bet\ A\ B\ P$
⟨*proof*⟩

lemma *l5-12-a*:
assumes $Bet\ A\ B\ C$
shows $A\ B\ Le\ A\ C \land B\ C\ Le\ A\ C$
⟨*proof*⟩

lemma *bet--le1213*:
assumes $Bet\ A\ B\ C$
shows $A\ B\ Le\ A\ C$
⟨*proof*⟩

lemma *bet--le2313*:
assumes $Bet\ A\ B\ C$
shows $B\ C\ Le\ A\ C$
⟨*proof*⟩

lemma *bet--lt1213*:
assumes $B \neq C$ **and**
$Bet\ A\ B\ C$
shows $A\ B\ Lt\ A\ C$
⟨*proof*⟩

lemma *bet--lt2313*:
assumes $A \neq B$ **and**
$Bet\ A\ B\ C$
shows $B\ C\ Lt\ A\ C$
⟨*proof*⟩

lemma *l5-12-b*:
assumes $Col\ A\ B\ C$ **and**
$A\ B\ Le\ A\ C$ **and**
$B\ C\ Le\ A\ C$
shows $Bet\ A\ B\ C$
⟨*proof*⟩

lemma *bet-le-eq*:
assumes $Bet\ A\ B\ C$
and $A\ C\ Le\ B\ C$
shows $A = B$
⟨*proof*⟩

lemma *or-lt-cong-gt*:
$A\ B\ Lt\ C\ D \lor A\ B\ Gt\ C\ D \lor Cong\ A\ B\ C\ D$
⟨*proof*⟩

lemma *lt--le*:
assumes $A\ B\ Lt\ C\ D$
shows $A\ B\ Le\ C\ D$

⟨*proof*⟩

lemma *le1234-lt--lt*:
assumes $A\ B\ Le\ C\ D$ **and**
$C\ D\ Lt\ E\ F$
shows $A\ B\ Lt\ E\ F$
⟨*proof*⟩

lemma *le3456-lt--lt*:
assumes $A\ B\ Lt\ C\ D$ **and**
$C\ D\ Le\ E\ F$
shows $A\ B\ Lt\ E\ F$
⟨*proof*⟩

lemma *lt-transitivity*:
assumes $A\ B\ Lt\ C\ D$ **and**
$C\ D\ Lt\ E\ F$
shows $A\ B\ Lt\ E\ F$
⟨*proof*⟩

lemma *not-and-lt*:
$\neg (A\ B\ Lt\ C\ D \wedge C\ D\ Lt\ A\ B)$
⟨*proof*⟩

lemma *nlt*:
$\neg\ A\ B\ Lt\ A\ B$
⟨*proof*⟩

lemma *le--nlt*:
assumes $A\ B\ Le\ C\ D$
shows $\neg\ C\ D\ Lt\ A\ B$
⟨*proof*⟩

lemma *cong--nlt*:
assumes $Cong\ A\ B\ C\ D$
shows $\neg\ A\ B\ Lt\ C\ D$
⟨*proof*⟩

lemma *nlt--le*:
assumes $\neg\ A\ B\ Lt\ C\ D$
shows $C\ D\ Le\ A\ B$
⟨*proof*⟩

lemma *lt--nle*:
assumes $A\ B\ Lt\ C\ D$
shows $\neg\ C\ D\ Le\ A\ B$
⟨*proof*⟩

lemma *nle--lt*:
assumes $\neg\ A\ B\ Le\ C\ D$
shows $C\ D\ Lt\ A\ B$
⟨*proof*⟩

lemma *lt1123*:
assumes $B \neq C$
shows $A\ A\ Lt\ B\ C$
⟨*proof*⟩

lemma *bet2-le2--le-R1*:
assumes $Bet\ a\ P\ b$ **and**
$Bet\ A\ Q\ B$ **and**
$P\ a\ Le\ Q\ A$ **and**
$P\ b\ Le\ Q\ B$ **and**
$B = Q$
shows $a\ b\ Le\ A\ B$
⟨*proof*⟩

lemma *bet2-le2--le-R2*:
  assumes *Bet a Po b* **and**
    Bet A PO B **and**
    Po a Le PO A **and**
    Po b Le PO B **and**
    $A \neq PO$ **and**
    $B \neq PO$
  shows *a b Le A B*
⟨*proof*⟩

lemma *bet2-le2--le*:
  assumes *Bet a P b* **and**
    Bet A Q B **and**
    P a Le Q A **and**
    P b Le Q B
  shows *a b Le A B*
⟨*proof*⟩

lemma *Le-cases*:
  assumes $A\ B\ Le\ C\ D \vee B\ A\ Le\ C\ D \vee A\ B\ Le\ D\ C \vee B\ A\ Le\ D\ C$
  shows *A B Le C D*
  ⟨*proof*⟩

lemma *Lt-cases*:
  assumes $A\ B\ Lt\ C\ D \vee B\ A\ Lt\ C\ D \vee A\ B\ Lt\ D\ C \vee B\ A\ Lt\ D\ C$
  shows *A B Lt C D*
  ⟨*proof*⟩

3.5 Out lines

lemma *bet-out*:
  assumes $B \neq A$ **and**
    Bet A B C
  shows *A Out B C*
  ⟨*proof*⟩

lemma *bet-out-1*:
  assumes $B \neq A$ **and**
    Bet C B A
  shows *A Out B C*
  ⟨*proof*⟩

lemma *out-dec*:
  shows $P\ Out\ A\ B \vee \neg\ P\ Out\ A\ B$
  ⟨*proof*⟩

lemma *out-diff1*:
  assumes *A Out B C*
  shows $B \neq A$
  ⟨*proof*⟩

lemma *out-diff2*:
  assumes *A Out B C*
  shows $C \neq A$
  ⟨*proof*⟩

lemma *out-distinct*:
  assumes *A Out B C*
  shows $B \neq A \wedge C \neq A$
  ⟨*proof*⟩

lemma *out-col*:
  assumes *A Out B C*
  shows *Col A B C*
  ⟨*proof*⟩

lemma *l6-2*:
assumes $A \neq P$ **and**
$B \neq P$ **and**
$C \neq P$ **and**
Bet A P C
shows *Bet B P C* $\longleftrightarrow$ *P Out A B*
⟨*proof*⟩

lemma *bet-out--bet*:
assumes *Bet A P C* **and**
P Out A B
shows *Bet B P C*
⟨*proof*⟩

lemma *l6-3-1*:
assumes *P Out A B*
shows $A \neq P \wedge B \neq P \wedge (\exists\ C.\ (C \neq P \wedge Bet\ A\ P\ C \wedge Bet\ B\ P\ C))$
⟨*proof*⟩

lemma *l6-3-2*:
assumes $A \neq P$ **and**
$B \neq P$ **and**
$\exists\ C.\ (C \neq P \wedge Bet\ A\ P\ C \wedge Bet\ B\ P\ C)$
shows *P Out A B*
⟨*proof*⟩

lemma *l6-4-1*:
assumes *P Out A B* **and**
Col A P B
shows ¬ *Bet A P B*
⟨*proof*⟩

lemma *l6-4-2*:
assumes *Col A P B*
and ¬ *Bet A P B*
shows *P Out A B*
⟨*proof*⟩

lemma *out-trivial*:
assumes $A \neq P$
shows *P Out A A*
⟨*proof*⟩

lemma *l6-6*:
assumes *P Out A B*
shows *P Out B A*
⟨*proof*⟩

lemma *l6-7*:
assumes *P Out A B* **and**
P Out B C
shows *P Out A C*
⟨*proof*⟩

lemma *bet-out-out-bet*:
assumes *Bet A B C* **and**
B Out A A' **and**
B Out C C'
shows *Bet* A' *B* C'
⟨*proof*⟩

lemma *out2-bet-out*:
assumes *B Out A C* **and**
B Out X P **and**
Bet A X C

shows B *Out* $A\ P \wedge B$ *Out* $C\ P$
⟨*proof*⟩

lemma *l6-11-uniqueness*:
assumes A *Out* $X\ R$ **and**
Cong $A\ X\ B\ C$ **and**
A *Out* $Y\ R$ **and**
Cong $A\ Y\ B\ C$
shows $X = Y$
⟨*proof*⟩

lemma *l6-11-existence*:
assumes $R \neq A$ **and**
$B \neq C$
shows $\exists\ X.\ (A$ *Out* $X\ R \wedge$ *Cong* $A\ X\ B\ C)$
⟨*proof*⟩

lemma *segment-construction-3*:
assumes $A \neq B$ **and**
$X \neq Y$
shows $\exists\ C.\ (A$ *Out* $B\ C \wedge$ *Cong* $A\ C\ X\ Y)$
⟨*proof*⟩

lemma *l6-13-1*:
assumes P *Out* $A\ B$ **and**
$P\ A$ *Le* $P\ B$
shows *Bet* $P\ A\ B$
⟨*proof*⟩

lemma *l6-13-2*:
assumes P *Out* $A\ B$ **and**
Bet $P\ A\ B$
shows $P\ A$ *Le* $P\ B$
⟨*proof*⟩

lemma *l6-16-1*:
assumes $P \neq Q$ **and**
Col $S\ P\ Q$ **and**
Col $X\ P\ Q$
shows *Col* $X\ P\ S$
⟨*proof*⟩

lemma *col-transitivity-1*:
assumes $P \neq Q$ **and**
Col $P\ Q\ A$ **and**
Col $P\ Q\ B$
shows *Col* $P\ A\ B$
⟨*proof*⟩

lemma *col-transitivity-2*:
assumes $P \neq Q$ **and**
Col $P\ Q\ A$ **and**
Col $P\ Q\ B$
shows *Col* $Q\ A\ B$
⟨*proof*⟩

lemma *l6-21*:
assumes $\neg$ *Col* $A\ B\ C$ **and**
$C \neq D$ **and**
Col $A\ B\ P$ **and**
Col $A\ B\ Q$ **and**
Col $C\ D\ P$ **and**
Col $C\ D\ Q$
shows $P = Q$
⟨*proof*⟩

lemma *col2--eq*:
assumes *Col A X Y* **and**
Col B X Y **and**
¬ *Col A X B*
shows $X = Y$
⟨*proof*⟩

lemma *not-col-exists*:
assumes $A \neq B$
shows ∃ *C*. ¬ *Col A B C*
⟨*proof*⟩

lemma *col3*:
assumes $X \neq Y$ **and**
Col X Y A **and**
Col X Y B **and**
Col X Y C
shows *Col A B C*
⟨*proof*⟩

lemma *colx*:
assumes $A \neq B$ **and**
Col X Y A **and**
Col X Y B **and**
Col A B C
shows *Col X Y C*
⟨*proof*⟩

lemma *out2--bet*:
assumes *A Out B C* **and**
C Out A B
shows *Bet A B C*
⟨*proof*⟩

lemma *bet2-le2--le1346*:
assumes *Bet A B C* **and**
Bet A' B' C' **and**
A B Le A' B' **and**
B C Le B' C'
shows *A C Le* A' C'
⟨*proof*⟩

lemma *bet2-le2--le2356-R1*:
assumes *Bet A A C* **and**
Bet A' B' C' **and**
A A Le A' B' **and**
A' C' *Le A C*
shows B' C' *Le A C*
⟨*proof*⟩

lemma *bet2-le2--le2356-R2*:
assumes $A \neq B$ **and**
Bet A B C **and**
Bet A' B' C' **and**
A B Le A' B' **and**
A' C' *Le A C*
shows B' C' *Le B C*
⟨*proof*⟩

lemma *bet2-le2--le2356*:
assumes *Bet A B C* **and**
Bet A' B' C' **and**
A B Le A' B' **and**
A' C' *Le A C*
shows B' C' *Le B C*

⟨*proof*⟩

lemma *bet2-le2--le1245*:
  assumes $Bet\ A\ B\ C$ **and**
    $Bet\ A'\ B'\ C'$ **and**
    $B\ C\ Le\ B'\ C'$ **and**
    $A'\ C'\ Le\ A\ C$
  shows $A'\ B'\ Le\ A\ B$
  ⟨*proof*⟩

lemma *cong-preserves-bet*:
  assumes $Bet\ B\ A'\ A0$ **and**
    $Cong\ B\ A'\ E\ D'$ **and**
    $Cong\ B\ A0\ E\ D0$ **and**
    $E\ Out\ D'\ D0$
  shows $Bet\ E\ D'\ D0$
  ⟨*proof*⟩

lemma *out-cong-cong*:
  assumes $B\ Out\ A\ A0$ **and**
    $E\ Out\ D\ D0$ **and**
    $Cong\ B\ A\ E\ D$ **and**
    $Cong\ B\ A0\ E\ D0$
  shows $Cong\ A\ A0\ D\ D0$
  ⟨*proof*⟩

lemma *not-out-bet*:
  assumes $Col\ A\ B\ C$ **and**
    $\neg\ B\ Out\ A\ C$
  shows $Bet\ A\ B\ C$
  ⟨*proof*⟩

lemma *or-bet-out*:
  shows $Bet\ A\ B\ C \lor B\ Out\ A\ C \lor \neg\ Col\ A\ B\ C$
  ⟨*proof*⟩

lemma *not-bet-out*:
  assumes $Col\ A\ B\ C$ **and**
    $\neg\ Bet\ A\ B\ C$
  shows $B\ Out\ A\ C$
  ⟨*proof*⟩

lemma *not-bet-and-out*:
  shows $\neg\ (Bet\ A\ B\ C \land B\ Out\ A\ C)$
  ⟨*proof*⟩

lemma *out-to-bet*:
  assumes $Col\ A'\ B'\ C'$ **and**
    $B\ Out\ A\ C \longleftrightarrow B'\ Out\ A'\ C'$ **and**
    $Bet\ A\ B\ C$
  shows $Bet\ A'\ B'\ C'$
  ⟨*proof*⟩

lemma *col-out2-col*:
  assumes $Col\ A\ B\ C$ **and**
    $B\ Out\ A\ AA$ **and**
    $B\ Out\ C\ CC$
  shows $Col\ AA\ B\ CC$ ⟨*proof*⟩

lemma *bet2-out-out*:
  assumes $B \neq A$ **and**
    $B' \neq A$ **and**
    $A\ Out\ C\ C'$ **and**
    $Bet\ A\ B\ C$ **and**
    $Bet\ A\ B'\ C'$
  shows $A\ Out\ B\ B'$

⟨*proof*⟩

lemma *bet2--out*:
assumes $A \neq B$ **and**
$A \neq B'$ **and**
Bet A B C
and *Bet A B' C*
shows *A Out B B'*
⟨*proof*⟩

lemma *out-bet-out-1*:
assumes *P Out A C* **and**
Bet A B C
shows *P Out A B*
⟨*proof*⟩

lemma *out-bet-out-2*:
assumes *P Out A C* **and**
Bet A B C
shows *P Out B C*
⟨*proof*⟩

lemma *out-bet--out*:
assumes *Bet P Q A* **and**
Q Out A B
shows *P Out A B*
⟨*proof*⟩

lemma *segment-reverse*:
assumes *Bet A B C*
shows $\exists\ B'.\ Bet\ A\ B'\ C \wedge Cong\ C\ B'\ A\ B$
⟨*proof*⟩

lemma *diff-col-ex*:
shows $\exists\ C.\ A \neq C \wedge B \neq C \wedge Col\ A\ B\ C$
⟨*proof*⟩

lemma *diff-bet-ex3*:
assumes *Bet A B C*
shows $\exists\ D.\ A \neq D \wedge B \neq D \wedge C \neq D \wedge Col\ A\ B\ D$
⟨*proof*⟩

lemma *diff-col-ex3*:
assumes *Col A B C*
shows $\exists\ D.\ A \neq D \wedge B \neq D \wedge C \neq D \wedge Col\ A\ B\ D$
⟨*proof*⟩

lemma *Out-cases*:
assumes $A\ Out\ B\ C \vee A\ Out\ C\ B$
shows *A Out B C*
⟨*proof*⟩

3.6 Midpoint

lemma *midpoint-dec*:
$I\ Midpoint\ A\ B \vee \neg\ I\ Midpoint\ A\ B$
⟨*proof*⟩

lemma *is-midpoint-id*:
assumes *A Midpoint A B*
shows $A = B$
⟨*proof*⟩

lemma *is-midpoint-id-2*:
assumes *A Midpoint B A*
shows $A = B$

⟨*proof*⟩

lemma *l7-2*:
assumes *M Midpoint A B*
shows *M Midpoint B A*
⟨*proof*⟩

lemma *l7-3*:
assumes *M Midpoint A A*
shows $M = A$
⟨*proof*⟩

lemma *l7-3-2*:
A Midpoint A A
⟨*proof*⟩

lemma *symmetric-point-construction*:
$\exists$ P'. *A Midpoint P* P'
⟨*proof*⟩

lemma *symmetric-point-uniqueness*:
assumes *P Midpoint A P1* **and**
P Midpoint A P2
shows $P1 = P2$
⟨*proof*⟩

lemma *l7-9*:
assumes *A Midpoint P X* **and**
A Midpoint Q X
shows $P = Q$
⟨*proof*⟩

lemma *l7-9-bis*:
assumes *A Midpoint P X* **and**
A Midpoint X Q
shows $P = Q$
⟨*proof*⟩

lemma *l7-13-R1*:
assumes $A \neq P$ **and**
A Midpoint P' *P* **and**
A Midpoint Q' *Q*
shows *Cong P Q* P' Q'
⟨*proof*⟩

lemma *l7-13*:
assumes *A Midpoint* P' *P* **and**
A Midpoint Q' *Q*
shows *Cong P Q* P' Q'
⟨*proof*⟩

lemma *l7-15*:
assumes *A Midpoint P* P' **and**
A Midpoint Q Q' **and**
A Midpoint R R' **and**
Bet P Q R
shows *Bet* P' Q' R'
⟨*proof*⟩

lemma *l7-16*:
assumes *A Midpoint P* P' **and**
A Midpoint Q Q' **and**
A Midpoint R R' **and**
A Midpoint S S' **and**
Cong P Q R S
shows *Cong* P' Q' R' S'

⟨*proof*⟩

lemma *symmetry-preserves-midpoint*:
assumes *Z Midpoint A D* **and**
Z Midpoint B E **and**
Z Midpoint C F **and**
B Midpoint A C
shows *E Midpoint D F*
⟨*proof*⟩

lemma *Mid-cases*:
assumes *A Midpoint B C* ∨ *A Midpoint C B*
shows *A Midpoint B C*
⟨*proof*⟩

lemma *Mid-perm*:
assumes *A Midpoint B C*
shows *A Midpoint B C* ∧ *A Midpoint C B*
⟨*proof*⟩

lemma *l7-17*:
assumes *A Midpoint P P′* **and**
B Midpoint P P′
shows *A* = *B*
⟨*proof*⟩

lemma *l7-17-bis*:
assumes *A Midpoint P P′* **and**
B Midpoint P′ P
shows *A* = *B*
⟨*proof*⟩

lemma *l7-20*:
assumes *Col A M B* **and**
Cong M A M B
shows *A* = *B* ∨ *M Midpoint A B*
⟨*proof*⟩

lemma *l7-20-bis*:
assumes *A* ≠ *B* **and**
Col A M B **and**
Cong M A M B
shows *M Midpoint A B*
⟨*proof*⟩

lemma *cong-col-mid*:
assumes *A* ≠ *C* **and**
Col A B C **and**
Cong A B B C
shows *B Midpoint A C*
⟨*proof*⟩

lemma *l7-21-R1*:
assumes ¬ *Col A B C* **and**
B ≠ *D* **and**
Cong A B C D **and**
Cong B C D A **and**
Col A P C **and**
Col B P D
shows *P Midpoint A C*
⟨*proof*⟩

lemma *l7-21*:
assumes ¬ *Col A B C* **and**
B ≠ *D* **and**
Cong A B C D **and**

Cong B C D A **and**
Col A P C **and**
Col B P D
shows *P Midpoint A C* ∧ *P Midpoint B D*
⟨*proof*⟩

lemma *l7-22-aux-R1*:
assumes *Bet A1 C C* **and**
Bet B1 C B2 **and**
Cong C A1 C B1 **and**
Cong C C C B2 **and**
M1 Midpoint A1 B1 **and**
*M2 Midpoint A2 B2***and**
C A1 Le C C
shows *Bet M1 C M2*
⟨*proof*⟩

lemma *l7-22-aux-R2*:
assumes *A2* ≠ *C* **and**
Bet A1 C A2 **and**
Bet B1 C B2 **and**
Cong C A1 C B1 **and**
Cong C A2 C B2 **and**
M1 Midpoint A1 B1 **and**
M2 Midpoint A2 B2 **and**
C A1 Le C A2
shows *Bet M1 C M2*
⟨*proof*⟩

lemma *l7-22-aux*:
assumes *Bet A1 C A2* **and**
Bet B1 C B2 **and**
Cong C A1 C B1 **and**
Cong C A2 C B2 **and**
M1 Midpoint A1 B1 **and**
M2 Midpoint A2 B2 **and**
C A1 Le C A2
shows *Bet M1 C M2*
⟨*proof*⟩

lemma *l7-22*:
assumes *Bet A1 C A2* **and**
Bet B1 C B2 **and**
Cong C A1 C B1 **and**
Cong C A2 C B2 **and**
M1 Midpoint A1 B1 **and**
M2 Midpoint A2 B2
shows *Bet M1 C M2*
⟨*proof*⟩

lemma *bet-col1*:
assumes *Bet A B D* **and**
Bet A C D
shows *Col A B C*
⟨*proof*⟩

lemma *l7-25-R1*:
assumes *Cong C A C B* **and**
Col A B C
shows ∃ *X*. *X Midpoint A B*
⟨*proof*⟩

lemma *l7-25-R2*:
assumes *Cong C A C B* **and**
¬ *Col A B C*
shows ∃ *X*. *X Midpoint A B*

⟨*proof*⟩

lemma *l7-25*:
assumes *Cong C A C B*
shows ∃ *X*. *X Midpoint A B*
⟨*proof*⟩

lemma *midpoint-distinct-1*:
assumes $A \neq B$ **and**
I Midpoint A B
shows $I \neq A \wedge I \neq B$
⟨*proof*⟩

lemma *midpoint-distinct-2*:
assumes $I \neq A$ **and**
I Midpoint A B
shows $A \neq B \wedge I \neq B$
⟨*proof*⟩

lemma *midpoint-distinct-3*:
assumes $I \neq B$ **and**
I Midpoint A B
shows $A \neq B \wedge I \neq A$
⟨*proof*⟩

lemma *midpoint-def*:
assumes *Bet A B C* **and**
Cong A B B C
shows *B Midpoint A C*
⟨*proof*⟩

lemma *midpoint-bet*:
assumes *B Midpoint A C*
shows *Bet A B C*
⟨*proof*⟩

lemma *midpoint-col*:
assumes *M Midpoint A B*
shows *Col M A B*
⟨*proof*⟩

lemma *midpoint-cong*:
assumes *B Midpoint A C*
shows *Cong A B B C*
⟨*proof*⟩

lemma *midpoint-out*:
assumes $A \neq C$ **and**
B Midpoint A C
shows *A Out B C*
⟨*proof*⟩

lemma *midpoint-out-1*:
assumes $A \neq C$ **and**
B Midpoint A C
shows *C Out A B*
⟨*proof*⟩

lemma *midpoint-not-midpoint*:
assumes $A \neq B$ **and**
I Midpoint A B
shows ¬ *B Midpoint A I*
⟨*proof*⟩

lemma *swap-diff*:
assumes $A \neq B$

shows $B \neq A$
⟨*proof*⟩

lemma *cong-cong-half-1*:
assumes *M Midpoint A B* **and**
M' Midpoint A' B' **and**
Cong A B A' B'
shows *Cong A M A' M'*
⟨*proof*⟩

lemma *cong-cong-half-2*:
assumes *M Midpoint A B* **and**
M' Midpoint A' B' **and**
Cong A B A' B'
shows *Cong B M B' M'*
⟨*proof*⟩

lemma *cong-mid2--cong*:
assumes *M Midpoint A B* **and**
M' Midpoint A' B' **and**
Cong A M A' M'
shows *Cong A B A' B'*
⟨*proof*⟩

lemma *mid--lt*:
assumes $A \neq B$ **and**
M Midpoint A B
shows *A M Lt A B*
⟨*proof*⟩

lemma *le-mid2--le13*:
assumes *M Midpoint A B* **and**
M' Midpoint A' B' **and**
A M Le A' M'
shows *A B Le A' B'*
⟨*proof*⟩

lemma *le-mid2--le12*:
assumes *M Midpoint A B* **and**
M' Midpoint A' B'
and *A B Le A' B'*
shows *A M Le A' M'*
⟨*proof*⟩

lemma *lt-mid2--lt13*:
assumes *M Midpoint A B* **and**
M' Midpoint A' B' **and**
A M Lt A' M'
shows *A B Lt A' B'*
⟨*proof*⟩

lemma *lt-mid2--lt12*:
assumes *M Midpoint A B* **and**
M' Midpoint A' B' **and**
A B Lt A' B'
shows *A M Lt A' M'*
⟨*proof*⟩

lemma *midpoint-preserves-out*:
assumes *A Out B C* **and**
M Midpoint A A' **and**
M Midpoint B B' **and**
M Midpoint C C'
shows *A' Out B' C'*
⟨*proof*⟩

lemma *col-cong-bet*:
assumes *Col A B D* **and**
Cong A B C D **and**
Bet A C B
shows *Bet C A D* ∨ *Bet C B D*
⟨*proof*⟩

lemma *col-cong2-bet1*:
assumes *Col A B D* **and**
Bet A C B **and**
Cong A B C D **and**
Cong A C B D
shows *Bet C B D*
⟨*proof*⟩

lemma *col-cong2-bet2*:
assumes *Col A B D* **and**
Bet A C B **and**
Cong A B C D **and**
Cong A D B C
shows *Bet C A D*
⟨*proof*⟩

lemma *col-cong2-bet3*:
assumes *Col A B D* **and**
Bet A B C **and**
Cong A B C D **and**
Cong A C B D
shows *Bet B C D*
⟨*proof*⟩

lemma *col-cong2-bet4*:
assumes *Col A B C* **and**
Bet A B D **and**
Cong A B C D **and**
Cong A D B C
shows *Bet B D C*
⟨*proof*⟩

lemma *col-bet2-cong1*:
assumes *Col A B D* **and**
Bet A C B **and**
Cong A B C D **and**
Bet C B D
shows *Cong A C D B*
⟨*proof*⟩

lemma *col-bet2-cong2*:
assumes *Col A B D* **and**
Bet A C B **and**
Cong A B C D **and**
Bet C A D
shows *Cong D A B C*
⟨*proof*⟩

lemma *bet2-lt2--lt*:
assumes *Bet a Po b* **and**
Bet A PO B **and**
Po a Lt PO A **and**
Po b Lt PO B
shows *a b Lt A B*
⟨*proof*⟩

lemma *bet2-lt-le--lt*:
assumes *Bet a Po b* **and**
Bet A PO B **and**

Cong Po a PO A **and**
Po b Lt PO B
shows *a b Lt A B*
⟨*proof*⟩

3.7 Orthogonality

lemma *per-dec*:
Per A B C ∨ ¬ *Per A B C*
⟨*proof*⟩

lemma *l8-2*:
assumes *Per A B C*
shows *Per C B A*
⟨*proof*⟩

lemma *Per-cases*:
assumes *Per A B C* ∨ *Per C B A*
shows *Per A B C*
⟨*proof*⟩

lemma *Per-perm* :
assumes *Per A B C*
shows *Per A B C* ∧ *Per C B A*
⟨*proof*⟩

lemma *l8-3* :
assumes *Per A B C* **and**
$A \neq B$ **and**
Col B A A'
shows *Per A' B C*
⟨*proof*⟩

lemma *l8-4*:
assumes *Per A B C* **and**
B Midpoint C C'
shows *Per A B C'*
⟨*proof*⟩

lemma *l8-5*:
Per A B B
⟨*proof*⟩

lemma *l8-6*:
assumes *Per A B C* **and**
Per A' B C **and**
Bet A C A'
shows $B = C$
⟨*proof*⟩

lemma *l8-7*:
assumes *Per A B C* **and**
Per A C B
shows $B = C$
⟨*proof*⟩

lemma *l8-8*:
assumes *Per A B A*
shows $A = B$
⟨*proof*⟩

lemma *per-distinct*:
assumes *Per A B C* **and**
$A \neq B$
shows $A \neq C$
⟨*proof*⟩

lemma *per-distinct-1*:
assumes $Per\ A\ B\ C$ **and**
$B \neq C$
shows $A \neq C$
⟨*proof*⟩

lemma *l8-9*:
assumes $Per\ A\ B\ C$ **and**
$Col\ A\ B\ C$
shows $A = B \vee C = B$
⟨*proof*⟩

lemma *l8-10*:
assumes $Per\ A\ B\ C$ **and**
$A\ B\ C\ Cong3\ A'\ B'\ C'$
shows $Per\ A'\ B'\ C'$
⟨*proof*⟩

lemma *col-col-per-per*:
assumes $A \neq X$ **and**
$C \neq X$ **and**
$Col\ U\ A\ X$ **and**
$Col\ V\ C\ X$ **and**
$Per\ A\ X\ C$
shows $Per\ U\ X\ V$
⟨*proof*⟩

lemma *perp-in-dec*:
$X\ PerpAt\ A\ B\ C\ D \vee \neg\ X\ PerpAt\ A\ B\ C\ D$
⟨*proof*⟩

lemma *perp-distinct*:
assumes $A\ B\ Perp\ C\ D$
shows $A \neq B \wedge C \neq D$
⟨*proof*⟩

lemma *l8-12*:
assumes $X\ PerpAt\ A\ B\ C\ D$
shows $X\ PerpAt\ C\ D\ A\ B$
⟨*proof*⟩

lemma *per-col*:
assumes $B \neq C$ **and**
$Per\ A\ B\ C$ **and**
$Col\ B\ C\ D$
shows $Per\ A\ B\ D$
⟨*proof*⟩

lemma *l8-13-2*:
assumes $A \neq B$ **and**
$C \neq D$ **and**
$Col\ X\ A\ B$ **and**
$Col\ X\ C\ D$ **and**
$\exists\ U.\ \exists\ V.\ Col\ U\ A\ B \wedge Col\ V\ C\ D \wedge U \neq X \wedge V \neq X \wedge Per\ U\ X\ V$
shows $X\ PerpAt\ A\ B\ C\ D$
⟨*proof*⟩

lemma *l8-14-1*:
$\neg\ A\ B\ Perp\ A\ B$
⟨*proof*⟩

lemma *l8-14-2-1a*:
assumes $X\ PerpAt\ A\ B\ C\ D$
shows $A\ B\ Perp\ C\ D$
⟨*proof*⟩

lemma *perp-in-distinct*:
  assumes X *PerpAt* $A\ B\ C\ D$
  shows $A \neq B \wedge C \neq D$
  ⟨*proof*⟩

lemma *l8-14-2-1b*:
  assumes X *PerpAt* $A\ B\ C\ D$ **and**
    Col $Y\ A\ B$ **and**
    Col $Y\ C\ D$
  shows $X = Y$
  ⟨*proof*⟩

lemma *l8-14-2-1b-bis*:
  assumes $A\ B$ *Perp* $C\ D$ **and**
    Col $X\ A\ B$ **and**
    Col $X\ C\ D$
  shows X *PerpAt* $A\ B\ C\ D$
  ⟨*proof*⟩

lemma *l8-14-2-2*:
  assumes $A\ B$ *Perp* $C\ D$ **and**
    $\forall\ Y.\ (Col\ Y\ A\ B \wedge Col\ Y\ C\ D) \longrightarrow X = Y$
  shows X *PerpAt* $A\ B\ C\ D$
  ⟨*proof*⟩

lemma *l8-14-3*:
  assumes X *PerpAt* $A\ B\ C\ D$ **and**
    Y *PerpAt* $A\ B\ C\ D$
  shows $X = Y$
  ⟨*proof*⟩

lemma *l8-15-1*:
  assumes *Col* $A\ B\ X$ **and**
    $A\ B$ *Perp* $C\ X$
  shows X *PerpAt* $A\ B\ C\ X$
  ⟨*proof*⟩

lemma *l8-15-2*:
  assumes *Col* $A\ B\ X$ **and**
    X *PerpAt* $A\ B\ C\ X$
  shows $A\ B$ *Perp* $C\ X$
  ⟨*proof*⟩

lemma *perp-in-per*:
  assumes B *PerpAt* $A\ B\ B\ C$
  shows *Per* $A\ B\ C$
  ⟨*proof*⟩

lemma *perp-sym*:
  assumes $A\ B$ *Perp* $A\ B$
  shows $C\ D$ *Perp* $C\ D$
  ⟨*proof*⟩

lemma *perp-col0*:
  assumes $A\ B$ *Perp* $C\ D$ **and**
    $X \neq Y$ **and**
    Col $A\ B\ X$ **and**
    Col $A\ B\ Y$
  shows $C\ D$ *Perp* $X\ Y$
⟨*proof*⟩

lemma *per-perp-in*:
  assumes $A \neq B$ **and**
    $B \neq C$ **and**
    Per $A\ B\ C$

shows *B PerpAt A B B C*
⟨*proof*⟩

lemma *per-perp*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
Per A B C
shows *A B Perp B C*
⟨*proof*⟩

lemma *perp-left-comm*:
assumes *A B Perp C D*
shows *B A Perp C D*
⟨*proof*⟩

lemma *perp-right-comm*:
assumes *A B Perp C D*
shows *A B Perp D C*
⟨*proof*⟩

lemma *perp-comm*:
assumes *A B Perp C D*
shows *B A Perp D C*
⟨*proof*⟩

lemma *perp-in-sym*:
assumes *X PerpAt A B C D*
shows *X PerpAt C D A B*
⟨*proof*⟩

lemma *perp-in-left-comm*:
assumes *X PerpAt A B C D*
shows *X PerpAt B A C D*
⟨*proof*⟩

lemma *perp-in-right-comm*:
assumes *X PerpAt A B C D*
shows *X PerpAt A B D C*
⟨*proof*⟩

lemma *perp-in-comm*:
assumes *X PerpAt A B C D*
shows *X PerpAt B A D C*
⟨*proof*⟩

lemma *Perp-cases*:
assumes *A B Perp C D* ∨ *B A Perp C D* ∨ *A B Perp D C* ∨ *B A Perp D C* ∨ *C D Perp A B* ∨ *C D Perp B A* ∨ *D C Perp A B* ∨ *D C Perp B A*
shows *A B Perp C D*
⟨*proof*⟩

lemma *Perp-perm* :
assumes *A B Perp C D*
shows *A B Perp C D* ∧ *B A Perp C D* ∧ *A B Perp D C* ∧ *B A Perp D C* ∧ *C D Perp A B* ∧ *C D Perp B A* ∧ *D C Perp A B* ∧ *D C Perp B A*
⟨*proof*⟩

lemma *Perp-in-cases*:
assumes *X PerpAt A B C D* ∨ *X PerpAt B A C D* ∨ *X PerpAt A B D C* ∨ *X PerpAt B A D C* ∨ *X PerpAt C D A B* ∨ *X PerpAt C D B A* ∨ *X PerpAt D C A B* ∨ *X PerpAt D C B A*
shows *X PerpAt A B C D*
⟨*proof*⟩

lemma *Perp-in-perm*:
assumes *X PerpAt A B C D*
shows *X PerpAt A B C D* ∧ *X PerpAt B A C D* ∧ *X PerpAt A B D C* ∧ *X PerpAt B A D C* ∧ *X PerpAt C D A B*

$\wedge$ *X PerpAt C D B A* $\wedge$ *X PerpAt D C A B* $\wedge$ *X PerpAt D C B A*
⟨*proof*⟩

lemma *perp-in-col*:
assumes *X PerpAt A B C D*
shows *Col A B X* $\wedge$ *Col C D X*
⟨*proof*⟩

lemma *perp-perp-in*:
assumes *A B Perp C A*
shows *A PerpAt A B C A*
⟨*proof*⟩

lemma *perp-per-1*:
assumes *A B Perp C A*
shows *Per B A C*
⟨*proof*⟩

lemma *perp-per-2*:
assumes *A B Perp A C*
shows *Per B A C*
⟨*proof*⟩

lemma *perp-col*:
assumes $A \neq E$ **and**
A B Perp C D **and**
Col A B E
shows *A E Perp C D*
⟨*proof*⟩

lemma *perp-col2*:
assumes *A B Perp X Y* **and**
$C \neq D$ **and**
Col A B C **and**
Col A B D
shows *C D Perp X Y*
⟨*proof*⟩

lemma *perp-col4*:
assumes $P \neq Q$ **and**
$R \neq S$ **and**
Col A B P **and**
Col A B Q **and**
Col C D R **and**
Col C D S **and**
A B Perp C D
shows *P Q Perp R S*
⟨*proof*⟩

lemma *perp-not-eq-1*:
assumes *A B Perp C D*
shows $A \neq B$
⟨*proof*⟩

lemma *perp-not-eq-2*:
assumes *A B Perp C D*
shows $C \neq D$
⟨*proof*⟩

lemma *diff-per-diff*:
assumes $A \neq B$ **and**
Cong A P B R **and**
Per B A P
and *Per A B R*
shows $P \neq R$
⟨*proof*⟩

lemma *per-not-colp*:
assumes $A \neq B$ **and**
$A \neq P$ **and**
$B \neq R$ **and**
Per B A P
and *Per A B R*
shows ¬ *Col P A R*
⟨*proof*⟩

lemma *per-not-col*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
Per A B C
shows ¬ *Col A B C*
⟨*proof*⟩

lemma *perp-not-col2*:
assumes *A B Perp C D*
shows ¬ *Col A B C* ∨ ¬ *Col A B D*
⟨*proof*⟩

lemma *perp-not-col*:
assumes *A B Perp P A*
shows ¬ *Col A B P*
⟨*proof*⟩

lemma *perp-in-col-perp-in*:
assumes $C \neq E$ **and**
Col C D E **and**
P PerpAt A B C D
shows *P PerpAt A B C E*
⟨*proof*⟩

lemma *perp-col2-bis*:
assumes *A B Perp C D* **and**
Col C D P **and**
Col C D Q **and**
$P \neq Q$
shows *A B Perp P Q*
⟨*proof*⟩

lemma *perp-in-perp-bis-R1*:
assumes $X \neq A$ **and**
X PerpAt A B C D
shows *X B Perp C D* ∨ *A X Perp C D*
⟨*proof*⟩

lemma *perp-in-perp-bis*:
assumes *X PerpAt A B C D*
shows *X B Perp C D* ∨ *A X Perp C D*
⟨*proof*⟩

lemma *col-per-perp*:
assumes $A \neq B$ **and**
$B \neq C$ **and**

$D \neq C$ **and**
Col B C D **and**
Per A B C
shows *C D Perp A B*
⟨*proof*⟩

lemma *per-cong-mid-R1*:
assumes $B = H$ **and**

Bet A B C **and**
Cong A H C H **and**
Per H B C
shows *B Midpoint A C*
⟨*proof*⟩

lemma *per-cong-mid-R2*:
assumes
$B \neq C$ **and**
Bet A B C **and**
Cong A H C H **and**
Per H B C
shows *B Midpoint A C*
⟨*proof*⟩

lemma *per-cong-mid*:
assumes $B \neq C$ **and**
Bet A B C **and**
Cong A H C H **and**
Per H B C
shows *B Midpoint A C*
⟨*proof*⟩

lemma *per-double-cong*:
assumes *Per A B C* **and**
B Midpoint C C'
shows *Cong A C A C'*
⟨*proof*⟩

lemma *cong-perp-or-mid-R1*:
assumes *Col A B X* **and**
$A \neq B$ **and**
M Midpoint A B **and**
Cong A X B X
shows $X = M \lor \neg$ *Col A B X* $\land$ *M PerpAt X M A B*
⟨*proof*⟩

lemma *cong-perp-or-mid-R2*:
assumes $\neg$ *Col A B X* **and**
$A \neq B$ **and**
M Midpoint A B **and**
Cong A X B X
shows $X = M \lor \neg$ *Col A B X* $\land$ *M PerpAt X M A B*
⟨*proof*⟩

lemma *cong-perp-or-mid*:
assumes $A \neq B$ **and**
M Midpoint A B **and**
Cong A X B X
shows $X = M \lor \neg$ *Col A B X* $\land$ *M PerpAt X M A B*
⟨*proof*⟩

lemma *col-per2-cases*:
assumes $B \neq C$ **and**
$B' \neq C$ **and**
$C \neq D$ **and**
Col B C D **and**
Per A B C **and**
Per A B' C
shows $B = B' \lor \neg$ *Col B' C D*
⟨*proof*⟩

lemma *l8-16-1*:
assumes *Col A B X* **and**
Col A B U **and**
A B Perp C X

shows ¬ *Col A B C* ∧ *Per C X U*
⟨*proof*⟩

lemma *l8-16-2*:
assumes *Col A B X* **and**
Col A B U
and $U \neq X$ **and**
¬ *Col A B C* **and**
Per C X U
shows *A B Perp C X*
⟨*proof*⟩

lemma *l8-18-uniqueness*:
assumes
Col A B X **and**
A B Perp C X **and**
Col A B Y **and**
A B Perp C Y
shows $X = Y$
⟨*proof*⟩

lemma *midpoint-distinct*:
assumes ¬ *Col A B C* **and**
Col A B X **and**
X Midpoint C C'
shows $C \neq C'$
⟨*proof*⟩

lemma *l8-20-1-R1*:
assumes $A = B$
shows *Per B A P*
⟨*proof*⟩

lemma *l8-20-1-R2*:
assumes $A \neq B$ **and**
Per A B C **and**
P Midpoint C' D **and**
A Midpoint C' C **and**
B Midpoint D C
shows *Per B A P*
⟨*proof*⟩

lemma *l8-20-1*:
assumes *Per A B C* **and**
P Midpoint C' D **and**
A Midpoint C' C **and**
B Midpoint D C
shows *Per B A P*
⟨*proof*⟩

lemma *l8-20-2*:
assumes *P Midpoint C' D* **and**
A Midpoint C' C **and**
B Midpoint D C **and**
$B \neq C$
shows $A \neq P$
⟨*proof*⟩

lemma *perp-col1*:
assumes $C \neq X$ **and**
A B Perp C D **and**
Col C D X
shows *A B Perp C X*
⟨*proof*⟩

lemma *l8-18-existence*:

assumes ¬ *Col A B C*
shows ∃ *X*. *Col A B X* ∧ *A B Perp C X*
⟨*proof*⟩

lemma *l8-21-aux*:
assumes ¬ *Col A B C*
shows ∃ *P*. ∃ *T*. (*A B Perp P A* ∧ *Col A B T* ∧ *Bet C T P*)
⟨*proof*⟩

lemma *l8-21*:
assumes *A* ≠ *B*
shows ∃ *P T*. *A B Perp P A* ∧ *Col A B T* ∧ *Bet C T P*
⟨*proof*⟩

lemma *per-cong*:
assumes *A* ≠ *B* **and**
A ≠ *P* **and**
Per B A P **and**
Per A B R **and**
Cong A P B R **and**
Col A B X **and**
Bet P X R
shows *Cong A R P B*
⟨*proof*⟩

lemma *perp-cong*:
assumes *A* ≠ *B* **and**
A ≠ *P* **and**
A B Perp P A **and**
A B Perp R B **and**
Cong A P B R **and**
Col A B X **and**
Bet P X R
shows *Cong A R P B*
⟨*proof*⟩

lemma *perp-exists*:
assumes *A* ≠ *B*
shows ∃ *X*. *PO X Perp A B*
⟨*proof*⟩

lemma *perp-vector*:
assumes *A* ≠ *B*
shows ∃ *X Y*. *A B Perp X Y*
⟨*proof*⟩

lemma *midpoint-existence-aux*:
assumes *A* ≠ *B* **and**
A B Perp Q B **and**
A B Perp P A **and**
Col A B T **and**
Bet Q T P **and**
A P Le B Q
shows ∃ *X*. *X Midpoint A B*
⟨*proof*⟩

lemma *midpoint-existence*:
∃ *X*. *X Midpoint A B*
⟨*proof*⟩

lemma *perp-in-id*:
assumes *X PerpAt A B C A*
shows *X* = *A*
⟨*proof*⟩

lemma *l8-22*:

assumes $A \neq B$ **and**
$A \neq P$ **and**
Per B A P **and**
Per A B R **and**
Cong A P B R **and**
Col A B X **and**
Bet P X R **and**
Cong A R P B
shows *X Midpoint A B* $\wedge$ *X Midpoint P R*
⟨*proof*⟩

lemma *l8-22-bis*:
assumes $A \neq B$ **and**
$A \neq P$ **and**
A B Perp P A **and**
A B Perp R B **and**
Cong A P B R **and**
Col A B X **and**
Bet P X R
shows *Cong A R P B* $\wedge$ *X Midpoint A B* $\wedge$ *X Midpoint P R*
⟨*proof*⟩

lemma *perp-in-perp*:
assumes *X PerpAt A B C D*
shows *A B Perp C D*
⟨*proof*⟩

lemma *perp-proj*:
assumes *A B Perp C D* **and**
$\neg$ *Col A C D*
shows $\exists$ *X*. *Col A B X* $\wedge$ *A X Perp C D*
⟨*proof*⟩

lemma *l8-24* :
assumes *P A Perp A B* **and**
Q B Perp A B **and**
Col A B T **and**
Bet P T Q **and**
Bet B R Q **and**
Cong A P B R
shows $\exists$ *X*. *X Midpoint A B* $\wedge$ *X Midpoint P R*
⟨*proof*⟩

lemma *col-per2--per*:
assumes $A \neq B$ **and**
Col A B C **and**
Per A X P **and**
Per B X P
shows *Per C X P*
⟨*proof*⟩

lemma *perp-in-per-1*:
assumes *X PerpAt A B C D*
shows *Per A X C*
⟨*proof*⟩

lemma *perp-in-per-2*:
assumes *X PerpAt A B C D*
shows *Per A X D*
⟨*proof*⟩

lemma *perp-in-per-3*:
assumes *X PerpAt A B C D*
shows *Per B X C*
⟨*proof*⟩

lemma *perp-in-per-4*:
assumes *X PerpAt A B C D*
shows *Per B X D*
⟨*proof*⟩

3.8 Planes

3.8.1 Coplanar

lemma *coplanar-perm-1*:
assumes *Coplanar A B C D*
shows *Coplanar A B D C*
⟨*proof*⟩

lemma *coplanar-perm-2*:
assumes *Coplanar A B C D*
shows *Coplanar A C B D*
⟨*proof*⟩

lemma *coplanar-perm-3*:
assumes *Coplanar A B C D*
shows *Coplanar A C D B*
⟨*proof*⟩

lemma *coplanar-perm-4*:
assumes *Coplanar A B C D*
shows *Coplanar A D B C*
⟨*proof*⟩

lemma *coplanar-perm-5*:
assumes *Coplanar A B C D*
shows *Coplanar A D C B*
⟨*proof*⟩

lemma *coplanar-perm-6*:
assumes *Coplanar A B C D*
shows *Coplanar B A C D*
⟨*proof*⟩

lemma *coplanar-perm-7*:
assumes *Coplanar A B C D*
shows *Coplanar B A D C*
⟨*proof*⟩

lemma *coplanar-perm-8*:
assumes *Coplanar A B C D*
shows *Coplanar B C A D*
⟨*proof*⟩

lemma *coplanar-perm-9*:
assumes *Coplanar A B C D*
shows *Coplanar B C D A*
⟨*proof*⟩

lemma *coplanar-perm-10*:
assumes *Coplanar A B C D*
shows *Coplanar B D A C*
⟨*proof*⟩

lemma *coplanar-perm-11*:
assumes *Coplanar A B C D*
shows *Coplanar B D C A*
⟨*proof*⟩

lemma *coplanar-perm-12*:
assumes *Coplanar A B C D*
shows *Coplanar C A B D*

⟨*proof*⟩

lemma *coplanar-perm-13*:
assumes *Coplanar A B C D*
shows *Coplanar C A D B*
⟨*proof*⟩

lemma *coplanar-perm-14*:
assumes *Coplanar A B C D*
shows *Coplanar C B A D*
⟨*proof*⟩

lemma *coplanar-perm-15*:
assumes *Coplanar A B C D*
shows *Coplanar C B D A*
⟨*proof*⟩

lemma *coplanar-perm-16*:
assumes *Coplanar A B C D*
shows *Coplanar C D A B*
⟨*proof*⟩

lemma *coplanar-perm-17*:
assumes *Coplanar A B C D*
shows *Coplanar C D B A*
⟨*proof*⟩

lemma *coplanar-perm-18*:
assumes *Coplanar A B C D*
shows *Coplanar D A B C*
⟨*proof*⟩

lemma *coplanar-perm-19*:
assumes *Coplanar A B C D*
shows *Coplanar D A C B*
⟨*proof*⟩

lemma *coplanar-perm-20*:
assumes *Coplanar A B C D*
shows *Coplanar D B A C*
⟨*proof*⟩

lemma *coplanar-perm-21*:
assumes *Coplanar A B C D*
shows *Coplanar D B C A*
⟨*proof*⟩

lemma *coplanar-perm-22*:
assumes *Coplanar A B C D*
shows *Coplanar D C A B*
⟨*proof*⟩

lemma *coplanar-perm-23*:
assumes *Coplanar A B C D*
shows *Coplanar D C B A*
⟨*proof*⟩

lemma *ncoplanar-perm-1*:
assumes ¬ *Coplanar A B C D*
shows ¬ *Coplanar A B D C*
⟨*proof*⟩

lemma *ncoplanar-perm-2*:
assumes ¬ *Coplanar A B C D*
shows ¬ *Coplanar A C B D*
⟨*proof*⟩

lemma *ncoplanar-perm-3*:
assumes ¬ *Coplanar A B C D*
shows ¬ *Coplanar A C D B*
⟨*proof*⟩

lemma *ncoplanar-perm-4*:
assumes ¬ *Coplanar A B C D*
shows ¬ *Coplanar A D B C*
⟨*proof*⟩

lemma *ncoplanar-perm-5*:
assumes ¬ *Coplanar A B C D*
shows ¬ *Coplanar A D C B*
⟨*proof*⟩

lemma *ncoplanar-perm-6*:
assumes ¬ *Coplanar A B C D*
shows ¬ *Coplanar B A C D*
⟨*proof*⟩

lemma *ncoplanar-perm-7*:
assumes ¬ *Coplanar A B C D*
shows ¬ *Coplanar B A D C*
⟨*proof*⟩

lemma *ncoplanar-perm-8*:
assumes ¬ *Coplanar A B C D*
shows ¬ *Coplanar B C A D*
⟨*proof*⟩

lemma *ncoplanar-perm-9*:
assumes ¬ *Coplanar A B C D*
shows ¬ *Coplanar B C D A*
⟨*proof*⟩

lemma *ncoplanar-perm-10*:
assumes ¬ *Coplanar A B C D*
shows ¬ *Coplanar B D A C*
⟨*proof*⟩

lemma *ncoplanar-perm-11*:
assumes ¬ *Coplanar A B C D*
shows ¬ *Coplanar B D C A*
⟨*proof*⟩

lemma *ncoplanar-perm-12*:
assumes ¬ *Coplanar A B C D*
shows ¬ *Coplanar C A B D*
⟨*proof*⟩

lemma *ncoplanar-perm-13*:
assumes ¬ *Coplanar A B C D*
shows ¬ *Coplanar C A D B*
⟨*proof*⟩

lemma *ncoplanar-perm-14*:
assumes ¬ *Coplanar A B C D*
shows ¬ *Coplanar C B A D*
⟨*proof*⟩

lemma *ncoplanar-perm-15*:
assumes ¬ *Coplanar A B C D*
shows ¬ *Coplanar C B D A*
⟨*proof*⟩

lemma *ncoplanar-perm-16*:
assumes ¬ *Coplanar A B C D*
shows ¬ *Coplanar C D A B*
⟨*proof*⟩

lemma *ncoplanar-perm-17*:
assumes ¬ *Coplanar A B C D*
shows ¬ *Coplanar C D B A*
⟨*proof*⟩

lemma *ncoplanar-perm-18*:
assumes ¬ *Coplanar A B C D*
shows ¬ *Coplanar D A B C*
⟨*proof*⟩

lemma *ncoplanar-perm-19*:
assumes ¬ *Coplanar A B C D*
shows ¬ *Coplanar D A C B*
⟨*proof*⟩

lemma *ncoplanar-perm-20*:
assumes ¬ *Coplanar A B C D*
shows ¬ *Coplanar D B A C*
⟨*proof*⟩

lemma *ncoplanar-perm-21*:
assumes ¬ *Coplanar A B C D*
shows ¬ *Coplanar D B C A*
⟨*proof*⟩

lemma *ncoplanar-perm-22*:
assumes ¬ *Coplanar A B C D*
shows ¬ *Coplanar D C A B*
⟨*proof*⟩

lemma *ncoplanar-perm-23*:
assumes ¬ *Coplanar A B C D*
shows ¬ *Coplanar D C B A*
⟨*proof*⟩

lemma *coplanar-trivial*:
shows *Coplanar A A B C*
⟨*proof*⟩

lemma *col--coplanar*:
assumes *Col A B C*
shows *Coplanar A B C D*
⟨*proof*⟩

lemma *ncop--ncol*:
assumes ¬ *Coplanar A B C D*
shows ¬ *Col A B C*
⟨*proof*⟩

lemma *ncop--ncols*:
assumes ¬ *Coplanar A B C D*
shows ¬ *Col A B C* ∧ ¬ *Col A B D* ∧ ¬ *Col A C D* ∧ ¬ *Col B C D*
⟨*proof*⟩

lemma *bet--coplanar*:
assumes *Bet A B C*
shows *Coplanar A B C D*
⟨*proof*⟩

lemma *out--coplanar*:
assumes *A Out B C*

shows *Coplanar A B C D*
⟨*proof*⟩

lemma *midpoint--coplanar*:
assumes *A Midpoint B C*
shows *Coplanar A B C D*
⟨*proof*⟩

lemma *perp--coplanar*:
assumes *A B Perp C D*
shows *Coplanar A B C D*
⟨*proof*⟩

lemma *ts--coplanar*:
assumes *A B TS C D*
shows *Coplanar A B C D*
⟨*proof*⟩

lemma *reflectl--coplanar*:
assumes *A B ReflectL C D*
shows *Coplanar A B C D*
⟨*proof*⟩

lemma *reflect--coplanar*:
assumes *A B Reflect C D*
shows *Coplanar A B C D*
⟨*proof*⟩

lemma *inangle--coplanar*:
assumes *A InAngle B C D*
shows *Coplanar A B C D*
⟨*proof*⟩

lemma *pars--coplanar*:
assumes *A B ParStrict C D*
shows *Coplanar A B C D*
⟨*proof*⟩

lemma *par--coplanar*:
assumes *A B Par C D*
shows *Coplanar A B C D*
⟨*proof*⟩

lemma *plg--coplanar*:
assumes *Plg A B C D*
shows *Coplanar A B C D*
⟨*proof*⟩

lemma *plgs--coplanar*:
assumes *ParallelogramStrict A B C D*
shows *Coplanar A B C D*
⟨*proof*⟩

lemma *plgf--coplanar*:
assumes *ParallelogramFlat A B C D*
shows *Coplanar A B C D*
⟨*proof*⟩

lemma *parallelogram--coplanar*:
assumes *Parallelogram A B C D*
shows *Coplanar A B C D*
⟨*proof*⟩

lemma *rhombus--coplanar*:
assumes *Rhombus A B C D*
shows *Coplanar A B C D*

⟨*proof*⟩

lemma *rectangle--coplanar*:
assumes *Rectangle A B C D*
shows *Coplanar A B C D*
⟨*proof*⟩

lemma *square--coplanar*:
assumes *Square A B C D*
shows *Coplanar A B C D*
⟨*proof*⟩

lemma *lambert--coplanar*:
assumes *Lambert A B C D*
shows *Coplanar A B C D*
⟨*proof*⟩

3.8.2 Planes

lemma *ts-distincts*:
assumes *A B TS P Q*
shows $A \neq B \wedge A \neq P \wedge A \neq Q \wedge B \neq P \wedge B \neq Q \wedge P \neq Q$
⟨*proof*⟩

lemma *l9-2*:
assumes *A B TS P Q*
shows *A B TS Q P*
⟨*proof*⟩

lemma *invert-two-sides*:
assumes *A B TS P Q*
shows *B A TS P Q*
⟨*proof*⟩

lemma *l9-3*:
assumes *P Q TS A C* **and**
Col M P Q **and**
M Midpoint A C **and**
Col R P Q **and**
R Out A B
shows *P Q TS B C*
⟨*proof*⟩

lemma *mid-preserves-col*:
assumes *Col A B C* **and**
M Midpoint A A' **and**
M Midpoint B B' **and**
M Midpoint C C'
shows *Col A' B' C'*
⟨*proof*⟩

lemma *per-mid-per*:
assumes
Per X A B **and**
M Midpoint A B **and**
M Midpoint X Y
shows *Cong A X B Y* $\wedge$ *Per Y B A*
⟨*proof*⟩

lemma *sym-preserve-diff*:
assumes $A \neq B$ **and**
M Midpoint A A' **and**
M Midpoint B B'
shows $A' \neq B'$
⟨*proof*⟩

lemma *l9-4-1-aux-R1*:
assumes $R = S$ **and**
$S\ C\ Le\ R\ A$ **and**
$P\ Q\ TS\ A\ C$ **and**
$Col\ R\ P\ Q$ **and**
$P\ Q\ Perp\ A\ R$ **and**
$Col\ S\ P\ Q$ **and**
$P\ Q\ Perp\ C\ S$ **and**
$M\ Midpoint\ R\ S$
shows $\forall\ U\ C'.\ M\ Midpoint\ U\ C' \longrightarrow (R\ Out\ U\ A \longleftrightarrow S\ Out\ C\ C')$
⟨*proof*⟩

lemma *l9-4-1-aux-R21*:
assumes $R \neq S$ **and**
$S\ C\ Le\ R\ A$ **and**
$P\ Q\ TS\ A\ C$ **and**
$Col\ R\ P\ Q$ **and**
$P\ Q\ Perp\ A\ R$ **and**
$Col\ S\ P\ Q$ **and**
$P\ Q\ Perp\ C\ S$ **and**
$M\ Midpoint\ R\ S$
shows $\forall\ U\ C'.\ M\ Midpoint\ U\ C' \longrightarrow (R\ Out\ U\ A \longleftrightarrow S\ Out\ C\ C')$
⟨*proof*⟩

lemma *l9-4-1-aux*:
assumes $S\ C\ Le\ R\ A$ **and**
$P\ Q\ TS\ A\ C$ **and**
$Col\ R\ P\ Q$ **and**
$P\ Q\ Perp\ A\ R$ **and**
$Col\ S\ P\ Q$ **and**
$P\ Q\ Perp\ C\ S$ **and**
$M\ Midpoint\ R\ S$
shows $\forall\ U\ C'.\ (M\ Midpoint\ U\ C' \longrightarrow (R\ Out\ U\ A \longleftrightarrow S\ Out\ C\ C'))$
⟨*proof*⟩

lemma *per-col-eq*:
assumes $Per\ A\ B\ C$ **and**
$Col\ A\ B\ C$ **and**
$B \neq C$
shows $A = B$
⟨*proof*⟩

lemma *l9-4-1*:
assumes $P\ Q\ TS\ A\ C$ **and**
$Col\ R\ P\ Q$ **and**
$P\ Q\ Perp\ A\ R$ **and**
$Col\ S\ P\ Q$ **and**
$P\ Q\ Perp\ C\ S$ **and**
$M\ Midpoint\ R\ S$
shows $\forall\ U\ C'.\ M\ Midpoint\ U\ C' \longrightarrow (R\ Out\ U\ A \longleftrightarrow S\ Out\ C\ C')$
⟨*proof*⟩

lemma *mid-two-sides*:
assumes $M\ Midpoint\ A\ B$ **and**
$\neg\ Col\ A\ B\ X$ **and**
$M\ Midpoint\ X\ Y$
shows $A\ B\ TS\ X\ Y$
⟨*proof*⟩

lemma *col-preserves-two-sides*:
assumes $C \neq D$ **and**
$Col\ A\ B\ C$ **and**
$Col\ A\ B\ D$ **and**
$A\ B\ TS\ X\ Y$
shows $C\ D\ TS\ X\ Y$

⟨*proof*⟩

lemma *out-out-two-sides*:
  assumes $A \neq B$ **and**
    A B TS X Y **and**
    Col I A B **and**
    Col I X Y **and**
    I Out X U **and**
    I Out Y V
  shows *A B TS U V*
⟨*proof*⟩

lemma *l9-4-2-aux-R1*:
  assumes $R = S$ **and**
    S C Le R A **and**
    P Q TS A C **and**
    Col R P Q **and**
    P Q Perp A R **and**
    Col S P Q **and**
    P Q Perp C S **and**
    R Out U A **and**
    S Out V C
  shows *P Q TS U V*
⟨*proof*⟩

lemma *l9-4-2-aux-R2*:
  assumes $R \neq S$ **and**
    S C Le R A **and**
    P Q TS A C **and**
    Col R P Q **and**
    P Q Perp A R **and**
    Col S P Q **and**
    P Q Perp C S **and**
    R Out U A **and**
    S Out V C
  shows *P Q TS U V*
⟨*proof*⟩

lemma *l9-4-2-aux*:
  assumes *S C Le R A* **and**
    P Q TS A C **and**
    Col R P Q **and**
    P Q Perp A R **and**
    Col S P Q **and**
    P Q Perp C S **and**
    R Out U A **and**
    S Out V C
  shows *P Q TS U V*
  ⟨*proof*⟩

lemma *l9-4-2*:
  assumes *P Q TS A C* **and**
    Col R P Q **and**
    P Q Perp A R **and**
    Col S P Q **and**
    P Q Perp C S **and**
    R Out U A **and**
    S Out V C
  shows *P Q TS U V*
⟨*proof*⟩

lemma *l9-5*:
  assumes *P Q TS A C* **and**
    Col R P Q **and**
    R Out A B
  shows *P Q TS B C*

⟨*proof*⟩

lemma *outer-pasch-R1*:
assumes *Col P Q C* **and**
Bet A C P **and**
Bet B Q C
shows ∃ *X*. *Bet A X B* ∧ *Bet P Q X*
⟨*proof*⟩

lemma *outer-pasch-R2*:
assumes ¬ *Col P Q C* **and**
Bet A C P **and**
Bet B Q C
shows ∃ *X*. *Bet A X B* ∧ *Bet P Q X*
⟨*proof*⟩

lemma *outer-pasch*:
assumes *Bet A C P* **and**
Bet B Q C
shows ∃ *X*. *Bet A X B* ∧ *Bet P Q X*
⟨*proof*⟩

lemma *os-distincts*:
assumes *A B OS X Y*
shows $A \neq B \land A \neq X \land A \neq Y \land B \neq X \land B \neq Y$
⟨*proof*⟩

lemma *invert-one-side*:
assumes *A B OS P Q*
shows *B A OS P Q*
⟨*proof*⟩

lemma *l9-8-1*:
assumes *P Q TS A C* **and**
P Q TS B C
shows *P Q OS A B*
⟨*proof*⟩

lemma *not-two-sides-id*:
shows ¬ *P Q TS A A*
⟨*proof*⟩

lemma *l9-8-2*:
assumes *P Q TS A C* **and**
P Q OS A B
shows *P Q TS B C*
⟨*proof*⟩

lemma *l9-9*:
assumes *P Q TS A B*
shows ¬ *P Q OS A B*
⟨*proof*⟩

lemma *l9-9-bis*:
assumes *P Q OS A B*
shows ¬ *P Q TS A B*
⟨*proof*⟩

lemma *one-side-chara*:
assumes *P Q OS A B*
shows ∀ *X*. *Col X P Q* ⟶ ¬ *Bet A X B*
⟨*proof*⟩

lemma *l9-10*:
assumes ¬ *Col A P Q*
shows ∃ *C*. *P Q TS A C*

⟨*proof*⟩

lemma *one-side-reflexivity*:
assumes ¬ *Col A P Q*
shows *P Q OS A A*
⟨*proof*⟩

lemma *one-side-symmetry*:
assumes *P Q OS A B*
shows *P Q OS B A*
⟨*proof*⟩

lemma *one-side-transitivity*:
assumes *P Q OS A B* **and**
P Q OS B C
shows *P Q OS A C*
⟨*proof*⟩

lemma *l9-17*:
assumes *P Q OS A C* **and**
Bet A B C
shows *P Q OS A B*
⟨*proof*⟩

lemma *l9-18-R1*:
assumes *Col X Y P* **and**
Col A B P
and *X Y TS A B*
shows *Bet A P B* ∧ ¬ *Col X Y A* ∧ ¬ *Col X Y B*
⟨*proof*⟩

lemma *l9-18-R2*:
assumes *Col X Y P* **and**
Col A B P **and**
Bet A P B **and**
¬ *Col X Y A* **and**
¬ *Col X Y B*
shows *X Y TS A B*
⟨*proof*⟩

lemma *l9-18*:
assumes *Col X Y P* **and**
Col A B P
shows *X Y TS A B* ⟷ (*Bet A P B* ∧ ¬ *Col X Y A* ∧ ¬ *Col X Y B*)
⟨*proof*⟩

lemma *l9-19-R1*:
assumes *Col X Y P* **and**
Col A B P **and**
X Y OS A B
shows *P Out A B* ∧ ¬ *Col X Y A*
⟨*proof*⟩

lemma *l9-19-R2*:
assumes *Col X Y P* **and**

P Out A B **and**
¬ *Col X Y A*
shows *X Y OS A B*
⟨*proof*⟩

lemma *l9-19*:
assumes *Col X Y P* **and**
Col A B P
shows *X Y OS A B* ⟷ (*P Out A B* ∧ ¬ *Col X Y A*)
⟨*proof*⟩

lemma *one-side-not-col123*:
assumes *A B OS X Y*
shows ¬ *Col A B X*
⟨*proof*⟩

lemma *one-side-not-col124*:
assumes *A B OS X Y*
shows ¬ *Col A B Y*
⟨*proof*⟩

lemma *col-two-sides*:
assumes *Col A B C* **and**
$A \neq C$ **and**
A B TS P Q
shows *A C TS P Q*
⟨*proof*⟩

lemma *col-one-side*:
assumes *Col A B C* **and**
$A \neq C$ **and**
A B OS P Q
shows *A C OS P Q*
⟨*proof*⟩

lemma *out-out-one-side*:
assumes *A B OS X Y* **and**
A Out Y Z
shows *A B OS X Z*
⟨*proof*⟩

lemma *out-one-side*:
assumes ¬ *Col A B X* ∨ ¬ *Col A B Y* **and**
A Out X Y
shows *A B OS X Y*
⟨*proof*⟩

lemma *bet--ts*:
assumes $A \neq Y$ **and**
¬ *Col A B X* **and**
Bet X A Y
shows *A B TS X Y*
⟨*proof*⟩

lemma *bet-ts--ts*:
assumes *A B TS X Y* **and**
Bet X Y Z
shows *A B TS X Z*
⟨*proof*⟩

lemma *bet-ts--os*:
assumes *A B TS X Y* **and**
Bet X Y Z
shows *A B OS Y Z*
⟨*proof*⟩

lemma *l9-31* :
assumes *A X OS Y Z* **and**
A Z OS Y X
shows *A Y TS X Z*
⟨*proof*⟩

lemma *col123--nos*:
assumes *Col P Q A*
shows ¬ *P Q OS A B*

⟨*proof*⟩

lemma *col124--nos*:
assumes *Col P Q B*
shows ¬ *P Q OS A B*
⟨*proof*⟩

lemma *col2-os--os*:
assumes $C \neq D$ **and**
Col A B C **and**
Col A B D **and**
A B OS X Y
shows *C D OS X Y*
⟨*proof*⟩

lemma *os-out-os*:
assumes *Col A B P* **and**
A B OS C D **and**
P Out C C'
shows *A B OS C' D*
⟨*proof*⟩

lemma *ts-ts-os*:
assumes *A B TS C D* **and**
C D TS A B
shows *A C OS B D*
⟨*proof*⟩

lemma *col-one-side-out*:
assumes *Col A X Y* **and**
A B OS X Y
shows *A Out X Y*
⟨*proof*⟩

lemma *col-two-sides-bet*:
assumes *Col A X Y* **and**
A B TS X Y
shows *Bet X A Y*
⟨*proof*⟩

lemma *os-ts1324--os*:
assumes *A X OS Y Z* **and**
A Y TS X Z
shows *A Z OS X Y*
⟨*proof*⟩

lemma *ts2--ex-bet2*:
assumes *A C TS B D* **and**
B D TS A C
shows ∃ *X*. *Bet A X C* ∧ *Bet B X D*
⟨*proof*⟩

lemma *out-one-side-1*:
assumes ¬ *Col A B C* **and**
Col A B X **and**
X Out C D
shows *A B OS C D*
⟨*proof*⟩

lemma *out-two-sides-two-sides*:
assumes
Col A B PX **and**
PX Out X P **and**
A B TS P Y
shows *A B TS X Y*
⟨*proof*⟩

lemma *l8-21-bis*:
  assumes $X \neq Y$ **and**
    $\neg$ *Col C A B*
  shows $\exists$ *P*. *Cong A P X Y* $\wedge$ *A B Perp P A* $\wedge$ *A B TS C P*
⟨*proof*⟩

lemma *ts--ncol*:
  assumes *A B TS X Y*
  shows $\neg$ *Col A X Y* $\vee$ $\neg$ *Col B X Y*
  ⟨*proof*⟩

lemma *one-or-two-sides-aux*:
  assumes $\neg$ *Col C A B* **and**
    $\neg$ *Col D A B* **and**
    Col A C X
    and *Col B D X*
  shows *A B TS C D* $\vee$ *A B OS C D*
⟨*proof*⟩

lemma *cop--one-or-two-sides*:
  assumes *Coplanar A B C D* **and**
    $\neg$ *Col C A B* **and**
    $\neg$ *Col D A B*
  shows *A B TS C D* $\vee$ *A B OS C D*
⟨*proof*⟩

lemma *os--coplanar*:
  assumes *A B OS C D*
  shows *Coplanar A B C D*
⟨*proof*⟩

lemma *coplanar-trans-1*:
  assumes $\neg$ *Col P Q R* **and**
    Coplanar P Q R A **and**
    Coplanar P Q R B
  shows *Coplanar Q R A B*
⟨*proof*⟩

lemma *col-cop--cop*:
  assumes *Coplanar A B C D* **and**
    $C \neq D$ **and**
    Col C D E
  shows *Coplanar A B C E*
⟨*proof*⟩

lemma *bet-cop--cop*:
  assumes *Coplanar A B C E* **and**
    Bet C D E
  shows *Coplanar A B C D*
  ⟨*proof*⟩

lemma *col2-cop--cop*:
  assumes *Coplanar A B C D* **and**
    $C \neq D$ **and**
    Col C D E **and**
    Col C D F
  shows *Coplanar A B E F*
⟨*proof*⟩

lemma *col-cop2--cop*:
  assumes $U \neq V$ **and**
    Coplanar A B C U **and**
    Coplanar A B C V **and**
    Col U V P
  shows *Coplanar A B C P*

⟨proof⟩

lemma *bet-cop2--cop*:
assumes *Coplanar A B C U* **and**
Coplanar A B C W **and**
Bet U V W
shows *Coplanar A B C V*
⟨proof⟩

lemma *coplanar-pseudo-trans*:
assumes ¬ *Col P Q R* **and**
Coplanar P Q R A **and**
Coplanar P Q R B **and**
Coplanar P Q R C **and**
Coplanar P Q R D
shows *Coplanar A B C D*
⟨proof⟩

lemma *l9-30*:
assumes ¬ *Coplanar A B C P* **and**
¬ *Col D E F* **and**
Coplanar D E F P **and**
Coplanar A B C X **and**
Coplanar A B C Y **and**
Coplanar A B C Z **and**
Coplanar D E F X **and**
Coplanar D E F Y **and**
Coplanar D E F Z
shows *Col X Y Z*
⟨proof⟩

lemma *cop-per2--col*:
assumes *Coplanar A X Y Z* **and**
$A \neq Z$ **and**
Per X Z A **and**
Per Y Z A
shows *Col X Y Z*
⟨proof⟩

lemma *cop-perp2--col*:
assumes *Coplanar A B Y Z* **and**
X Y Perp A B **and**
X Z Perp A B
shows *Col X Y Z*
⟨proof⟩

lemma *two-sides-dec*:
shows *A B TS C D* ∨ ¬ *A B TS C D*
⟨proof⟩

lemma *cop-nts--os*:
assumes *Coplanar A B C D* **and**
¬ *Col C A B* **and**
¬ *Col D A B* **and**
¬ *A B TS C D*
shows *A B OS C D*
⟨proof⟩

lemma *cop-nos--ts*:
assumes *Coplanar A B C D* **and**
¬ *Col C A B* **and**
¬ *Col D A B* **and**
¬ *A B OS C D*
shows *A B TS C D*
⟨proof⟩

lemma *one-side-dec*:
A B OS C D ∨ ¬ *A B OS C D*
⟨*proof*⟩

lemma *cop-dec*:
Coplanar A B C D ∨ ¬ *Coplanar A B C D*
⟨*proof*⟩

lemma *ex-diff-cop*:
∃ *E*. *Coplanar A B C E* ∧ *D* ≠ *E*
⟨*proof*⟩

lemma *ex-ncol-cop*:
assumes *D* ≠ *E*
shows ∃ *F*. *Coplanar A B C F* ∧ ¬ *Col D E F*
⟨*proof*⟩

lemma *ex-ncol-cop2*:
∃ *E F*. (*Coplanar A B C E* ∧ *Coplanar A B C F* ∧ ¬ *Col D E F*)
⟨*proof*⟩

lemma *col2-cop2--eq*:
assumes ¬ *Coplanar A B C U* **and**
U ≠ *V* **and**
Coplanar A B C P **and**
Coplanar A B C Q **and**
Col U V P **and**
Col U V Q
shows *P* = *Q*
⟨*proof*⟩

lemma *cong3-cop2--col*:
assumes *Coplanar A B C P* **and**
Coplanar A B C Q **and**
P ≠ *Q* **and**
Cong A P A Q **and**
Cong B P B Q **and**
Cong C P C Q
shows *Col A B C*
⟨*proof*⟩

lemma *l9-38*:
assumes *A B C TSP P Q*
shows *A B C TSP Q P*
⟨*proof*⟩

lemma *l9-39*:
assumes *A B C TSP P R* **and**
Coplanar A B C D **and**
D Out P Q
shows *A B C TSP Q R*
⟨*proof*⟩

lemma *l9-41-1*:
assumes *A B C TSP P R* **and**
A B C TSP Q R
shows *A B C OSP P Q*
⟨*proof*⟩

lemma *l9-41-2*:
assumes *A B C TSP P R* **and**
A B C OSP P Q
shows *A B C TSP Q R*
⟨*proof*⟩

lemma *tsp-exists*:

assumes ¬ *Coplanar A B C P*
shows ∃ *Q*. *A B C TSP P Q*
⟨*proof*⟩

lemma *osp-reflexivity*:
assumes ¬ *Coplanar A B C P*
shows *A B C OSP P P*
⟨*proof*⟩

lemma *osp-symmetry*:
assumes *A B C OSP P Q*
shows *A B C OSP Q P*
⟨*proof*⟩

lemma *osp-transitivity*:
assumes *A B C OSP P Q* **and**
A B C OSP Q R
shows *A B C OSP P R*
⟨*proof*⟩

lemma *cop3-tsp--tsp*:
assumes ¬ *Col D E F* **and**
Coplanar A B C D **and**
Coplanar A B C E **and**
Coplanar A B C F **and**
A B C TSP P Q
shows *D E F TSP P Q*
⟨*proof*⟩

lemma *cop3-osp--osp*:
assumes ¬ *Col D E F* **and**
Coplanar A B C D **and**
Coplanar A B C E **and**
Coplanar A B C F **and**
A B C OSP P Q
shows *D E F OSP P Q*
⟨*proof*⟩

lemma *ncop-distincts*:
assumes ¬ *Coplanar A B C D*
shows $A \neq B \wedge A \neq C \wedge A \neq D \wedge B \neq C \wedge B \neq D \wedge C \neq D$
⟨*proof*⟩

lemma *tsp-distincts*:
assumes *A B C TSP P Q*
shows $A \neq B \wedge A \neq C \wedge B \neq C \wedge A \neq P \wedge B \neq P \wedge C \neq P \wedge A \neq Q \wedge B \neq Q \wedge C \neq Q \wedge P \neq Q$
⟨*proof*⟩

lemma *osp-distincts*:
assumes *A B C OSP P Q*
shows $A \neq B \wedge A \neq C \wedge B \neq C \wedge A \neq P \wedge B \neq P \wedge C \neq P \wedge A \neq Q \wedge B \neq Q \wedge C \neq Q$
⟨*proof*⟩

lemma *tsp--ncop1*:
assumes *A B C TSP P Q*
shows ¬ *Coplanar A B C P*
⟨*proof*⟩

lemma *tsp--ncop2*:
assumes *A B C TSP P Q*
shows ¬ *Coplanar A B C Q*
⟨*proof*⟩

lemma *osp--ncop1*:
assumes *A B C OSP P Q*

shows ¬ *Coplanar A B C P*
⟨*proof*⟩

lemma *osp--ncop2*:
assumes *A B C OSP P Q*
shows ¬ *Coplanar A B C Q*
⟨*proof*⟩

lemma *tsp--nosp*:
assumes *A B C TSP P Q*
shows ¬ *A B C OSP P Q*
⟨*proof*⟩

lemma *osp--ntsp*:
assumes *A B C OSP P Q*
shows ¬ *A B C TSP P Q*
⟨*proof*⟩

lemma *osp-bet--osp*:
assumes *A B C OSP P R* **and**
Bet P Q R
shows *A B C OSP P Q*
⟨*proof*⟩

lemma *l9-18-3*:
assumes *Coplanar A B C P* **and**
Col X Y P
shows *A B C TSP X Y* ⟷ (*Bet X P Y* ∧ ¬ *Coplanar A B C X* ∧ ¬ *Coplanar A B C Y*)
⟨*proof*⟩

lemma *bet-cop--tsp*:
assumes ¬ *Coplanar A B C X* **and**
$P \neq Y$ **and**
Coplanar A B C P **and**
Bet X P Y
shows *A B C TSP X Y*
⟨*proof*⟩

lemma *cop-out--osp*:
assumes ¬ *Coplanar A B C X* **and**
Coplanar A B C P **and**
P Out X Y
shows *A B C OSP X Y*
⟨*proof*⟩

lemma *l9-19-3*:
assumes *Coplanar A B C P* **and**
Col X Y P
shows *A B C OSP X Y* ⟷ (*P Out X Y* ∧ ¬ *Coplanar A B C X*)
⟨*proof*⟩

lemma *cop2-ts--tsp*:
assumes ¬ *Coplanar A B C X* **and** *Coplanar A B C D* **and**
Coplanar A B C E **and** *D E TS X Y*
shows *A B C TSP X Y*
⟨*proof*⟩

lemma *cop2-os--osp*:
assumes ¬ *Coplanar A B C X* **and**
Coplanar A B C D **and**
Coplanar A B C E **and**
D E OS X Y
shows *A B C OSP X Y*
⟨*proof*⟩

lemma *cop3-tsp--ts*:

assumes $D \neq E$ **and**
Coplanar A B C D **and**
Coplanar A B C E **and**
Coplanar D E X Y **and**
A B C TSP X Y
shows *D E TS X Y*
⟨*proof*⟩

lemma *cop3-osp--os*:
assumes $D \neq E$ **and**
Coplanar A B C D **and**
Coplanar A B C E **and**
Coplanar D E X Y **and**
A B C OSP X Y
shows *D E OS X Y*
⟨*proof*⟩

lemma *cop-tsp--ex-cop2*:
assumes
A B C TSP D E
shows $\exists\ Q.$ (*Coplanar A B C Q* $\wedge$ *Coplanar D E P Q* $\wedge$ $P \neq Q$)
⟨*proof*⟩

lemma *cop-osp--ex-cop2*:
assumes *Coplanar A B C P* **and**
A B C OSP D E
shows $\exists\ Q.$ *Coplanar A B C Q* $\wedge$ *Coplanar D E P Q* $\wedge$ $P \neq Q$
⟨*proof*⟩

lemma *sac--coplanar*:
assumes *Saccheri A B C D*
shows *Coplanar A B C D*
⟨*proof*⟩

3.9 Line reflexivity

3.9.1 Dimensionless

lemma *Ch10-Goal1*:
assumes $\neg$ *Coplanar D C B A*
shows $\neg$ *Coplanar A B C D*
⟨*proof*⟩

lemma *ex-sym*:
$\exists\ Y.$ (*A B Perp X Y* $\vee$ $X = Y$) $\wedge$ ($\exists\ M.$ *Col A B M* $\wedge$ *M Midpoint X Y*)
⟨*proof*⟩

lemma *is-image-is-image-spec*:
assumes $A \neq B$
shows *P′ P Reflect A B* $\longleftrightarrow$ *P′ P ReflectL A B*
⟨*proof*⟩

lemma *ex-sym1*:
assumes $A \neq B$
shows $\exists\ Y.$ (*A B Perp X Y* $\vee$ $X = Y$) $\wedge$ ($\exists\ M.$ *Col A B M* $\wedge$ *M Midpoint X Y* $\wedge$ *X Y Reflect A B*)
⟨*proof*⟩

lemma *l10-2-uniqueness*:
assumes *P1 P Reflect A B* **and**
P2 P Reflect A B
shows $P1 = P2$
⟨*proof*⟩

lemma *l10-2-uniqueness-spec*:
assumes *P1 P ReflectL A B* **and**
P2 P ReflectL A B
shows $P1 = P2$

⟨*proof*⟩

lemma *l10-2-existence-spec*:
$\exists$ P'. P' P *ReflectL* A B
⟨*proof*⟩

lemma *l10-2-existence*:
$\exists$ P'. P' P *Reflect* A B
⟨*proof*⟩

lemma *l10-4-spec*:
assumes P P' *ReflectL* A B
shows P' P *ReflectL* A B
⟨*proof*⟩

lemma *l10-4*:
assumes P P' *Reflect* A B
shows P' P *Reflect* A B
⟨*proof*⟩

lemma *l10-5*:
assumes P' P *Reflect* A B **and**
P'' P' *Reflect* A B
shows $P = P''$
⟨*proof*⟩

lemma *l10-6-uniqueness*:
assumes P *P1* *Reflect* A B **and**
P *P2* *Reflect* A B
shows *P1* = *P2*
⟨*proof*⟩

lemma *l10-6-uniqueness-spec*:
assumes P *P1* *ReflectL* A B **and**
P *P2* *ReflectL* A B
shows *P1* = *P2*
⟨*proof*⟩

lemma *l10-6-existence-spec*:
assumes $A \neq B$
shows $\exists$ P. P' P *ReflectL* A B
⟨*proof*⟩

lemma *l10-6-existence*:
$\exists$ P. P' P *Reflect* A B
⟨*proof*⟩

lemma *l10-7*:
assumes P' P *Reflect* A B **and**
Q' Q *Reflect* A B **and**
$P' = Q'$
shows $P = Q$
⟨*proof*⟩

lemma *l10-8*:
assumes P P *Reflect* A B
shows *Col* P A B
⟨*proof*⟩

lemma *col--refl*:
assumes *Col* P A B
shows P P *ReflectL* A B
⟨*proof*⟩

lemma *is-image-col-cong*:
assumes $A \neq B$ **and**

P P′ Reflect A B **and**
Col A B X
shows *Cong P X P′ X*
⟨*proof*⟩

lemma *is-image-spec-col-cong*:
assumes *P P′ ReflectL A B* **and**
Col A B X
shows *Cong P X P′ X*
⟨*proof*⟩

lemma *image-id*:
assumes $A \neq B$ **and**
Col A B T **and**
T T′ Reflect A B
shows $T = T'$
⟨*proof*⟩

lemma *osym-not-col*:
assumes *P P′ Reflect A B* **and**
¬ *Col A B P*
shows ¬ *Col A B P′*
⟨*proof*⟩

lemma *midpoint-preserves-image*:
assumes $A \neq B$ **and**
Col A B M **and**
P P′ Reflect A B **and**
M Midpoint P Q **and**
M Midpoint P′ Q′
shows *Q Q′ Reflect A B*
⟨*proof*⟩

lemma *image-in-is-image-spec*:
assumes *M ReflectLAt P P′ A B*
shows *P P′ ReflectL A B*
⟨*proof*⟩

lemma *image-in-gen-is-image*:
assumes *M ReflectAt P P′ A B*
shows *P P′ Reflect A B*
⟨*proof*⟩

lemma *image-image-in*:
assumes $P \neq P'$ **and**
P P′ ReflectL A B **and**
Col A B M **and**
Col P M P′
shows *M ReflectLAt P P′ A B*
⟨*proof*⟩

lemma *image-in-col*:
assumes *Y ReflectLAt P P′ A B*
shows *Col P P′ Y*
⟨*proof*⟩

lemma *is-image-spec-rev*:
assumes *P P′ ReflectL A B*
shows *P P′ ReflectL B A*
⟨*proof*⟩

lemma *is-image-rev*:
assumes *P P′ Reflect A B*
shows *P P′ Reflect B A*
⟨*proof*⟩

lemma *midpoint-preserves-per*:
assumes *Per A B C* **and**
M Midpoint A A1 **and**
M Midpoint B B1 **and**
M Midpoint C C1
shows *Per A1 B1 C1*
⟨*proof*⟩

lemma *col--image-spec*:
assumes *Col A B X*
shows *X X ReflectL A B*
⟨*proof*⟩

lemma *image-triv*:
A A Reflect A B
⟨*proof*⟩

lemma *cong-midpoint--image*:
assumes *Cong A X A Y* **and**
B Midpoint X Y
shows *Y X Reflect A B*
⟨*proof*⟩

lemma *col-image-spec--eq*:
assumes *Col A B P* **and**
P P′ ReflectL A B
shows $P = P'$
⟨*proof*⟩

lemma *image-spec-triv*:
A A ReflectL B B
⟨*proof*⟩

lemma *image-spec--eq*:
assumes *P P′ ReflectL A A*
shows $P = P'$
⟨*proof*⟩

lemma *image--midpoint*:
assumes *P P′ Reflect A A*
shows *A Midpoint P′ P*
⟨*proof*⟩

lemma *is-image-spec-dec*:
A B ReflectL C D ∨ ¬ *A B ReflectL C D*
⟨*proof*⟩

lemma *l10-14*:
assumes $P \neq P'$ **and**
$A \neq B$ **and**
P P′ Reflect A B
shows *A B TS P P′*
⟨*proof*⟩

lemma *l10-15*:
assumes *Col A B C* **and**
¬ *Col A B P*
shows ∃ *Q. A B Perp Q C* ∧ *A B OS P Q*
⟨*proof*⟩

lemma *ex-per-cong*:
assumes $A \neq B$ **and**
$X \neq Y$ **and**
Col A B C **and**
¬ *Col A B D*

shows ∃ *P*. *Per P C A* ∧ *Cong P C X Y* ∧ *A B OS P D*
⟨*proof*⟩

lemma *exists-cong-per*:
∃ *C*. *Per A B C* ∧ *Cong B C X Y*
⟨*proof*⟩

3.9.2 Upper dim 2

lemma *upper-dim-implies-per2--col*:
assumes *upper-dim-axiom*
shows ∀ *A B C X*. (*Per A X C* ∧ *X* ≠ *C* ∧ *Per B X C*) ⟶ *Col A B X*
⟨*proof*⟩

lemma *upper-dim-implies-col-perp2--col*:
assumes *upper-dim-axiom*
shows ∀ *A B X Y P*. (*Col A B P* ∧ *A B Perp X P* ∧ *P A Perp Y P*) ⟶ *Col Y X P*
⟨*proof*⟩

lemma *upper-dim-implies-perp2--col*:
assumes *upper-dim-axiom*
shows ∀ *X Y Z A B*. (*X Y Perp A B* ∧ *X Z Perp A B*) ⟶ *Col X Y Z*
⟨*proof*⟩

lemma *upper-dim-implies-not-two-sides-one-side-aux*:
assumes *upper-dim-axiom*
shows ∀ *A B X Y PX*. (*A* ≠ *B* ∧ *PX* ≠ *A* ∧ *A B Perp X PX* ∧ *Col A B PX* ∧ ¬ *Col X A B* ∧ ¬ *Col Y A B* ∧ ¬ *A B TS X Y*) ⟶ *A B OS X Y*
⟨*proof*⟩

lemma *upper-dim-implies-not-two-sides-one-side*:
assumes *upper-dim-axiom*
shows ∀ *A B X Y*. (¬ *Col X A B* ∧ ¬ *Col Y A B* ∧ ¬ *A B TS X Y*) ⟶ *A B OS X Y*
⟨*proof*⟩

lemma *upper-dim-implies-not-one-side-two-sides*:
assumes *upper-dim-axiom*
shows ∀ *A B X Y*. (¬ *Col X A B* ∧ ¬ *Col Y A B* ∧ ¬ *A B OS X Y*) ⟶ *A B TS X Y*
⟨*proof*⟩

lemma *upper-dim-implies-one-or-two-sides*:
assumes *upper-dim-axiom*
shows ∀ *A B X Y*. (¬ *Col X A B* ∧ ¬ *Col Y A B*) ⟶ (*A B TS X Y* ∨ *A B OS X Y*)
⟨*proof*⟩

lemma *upper-dim-implies-all-coplanar*:
assumes *upper-dim-axiom*
shows *all-coplanar-axiom*
⟨*proof*⟩

lemma *all-coplanar-implies-upper-dim*:
assumes *all-coplanar-axiom*
shows *upper-dim-axiom*
⟨*proof*⟩

lemma *all-coplanar-upper-dim*:
shows *all-coplanar-axiom* ⟷ *upper-dim-axiom*
⟨*proof*⟩

lemma *upper-dim-stab*:
shows ¬ ¬ *upper-dim-axiom* ⟶ *upper-dim-axiom* ⟨*proof*⟩

lemma *cop--cong-on-bissect*:
assumes *Coplanar A B X P* **and**
M Midpoint A B **and**
M PerpAt A B P M **and**

Cong X A X B
shows *Col M P X*
⟨*proof*⟩

lemma *cong-cop-mid-perp--col*:
assumes *Coplanar A B X P* **and**
Cong A X B X **and**
M Midpoint A B **and**
A B Perp P M
shows *Col M P X*
⟨*proof*⟩

lemma *cop-image-in2--col*:
assumes *Coplanar A B P Q* **and**
M ReflectLAt P P′ A B **and**
M ReflectLAt Q Q′ A B
shows *Col M P Q*
⟨*proof*⟩

lemma *l10-10-spec*:
assumes *P′ P ReflectL A B* **and**
Q′ Q ReflectL A B
shows *Cong P Q P′ Q′*
⟨*proof*⟩

lemma *l10-10*:
assumes *P′ P Reflect A B* **and**
Q′ Q Reflect A B
shows *Cong P Q P′ Q′*
⟨*proof*⟩

lemma *image-preserves-bet*:
assumes *A A′ ReflectL X Y* **and**
B B′ ReflectL X Y **and**
C C′ ReflectL X Y **and**
Bet A B C
shows *Bet A′ B′ C′*
⟨*proof*⟩

lemma *image-gen-preserves-bet*:
assumes *A A′ Reflect X Y* **and**
B B′ Reflect X Y **and**
C C′ Reflect X Y **and**
Bet A B C
shows *Bet A′ B′ C′*
⟨*proof*⟩

lemma *image-preserves-col*:
assumes *A A′ ReflectL X Y* **and**
B B′ ReflectL X Y **and**
C C′ ReflectL X Y **and**
Col A B C
shows *Col A′ B′ C′* ⟨*proof*⟩

lemma *image-gen-preserves-col*:
assumes *A A′ Reflect X Y* **and**
B B′ Reflect X Y **and**
C C′ Reflect X Y **and**
Col A B C
shows *Col A′ B′ C′*
⟨*proof*⟩

lemma *image-gen-preserves-ncol*:
assumes *A A′ Reflect X Y* **and**
B B′ Reflect X Y **and**
C C′ Reflect X Y **and**

¬ *Col A B C*
shows ¬ *Col A' B' C'*
⟨*proof*⟩

lemma *image-gen-preserves-inter*:
assumes *A A' Reflect X Y* **and**
B B' Reflect X Y **and**
C C' Reflect X Y **and**
D D' Reflect X Y **and**
¬ *Col A B C* **and**
$C \neq D$ **and**
Col A B I **and**
Col C D I **and**
Col A' B' I' **and**
Col C' D' I'
shows *I I' Reflect X Y*
⟨*proof*⟩

lemma *intersection-with-image-gen*:
assumes *A A' Reflect X Y* **and**
B B' Reflect X Y **and**
¬ *Col A B A'* **and**
Col A B C **and**
Col A' B' C
shows *Col C X Y*
⟨*proof*⟩

lemma *image-preserves-midpoint* :
assumes *A A' ReflectL X Y* **and**
B B' ReflectL X Y **and**
C C' ReflectL X Y **and**
A Midpoint B C
shows *A' Midpoint B' C'*
⟨*proof*⟩

lemma *image-spec-preserves-per*:
assumes *A A' ReflectL X Y* **and**
B B' ReflectL X Y **and**
C C' ReflectL X Y **and**
Per A B C
shows *Per A' B' C'*
⟨*proof*⟩

lemma *image-preserves-per*:
assumes *A A' Reflect X Y* **and**
*B B' Reflect X Y***and**
C C' Reflect X Y **and**
Per A B C
shows *Per A' B' C'*
⟨*proof*⟩

lemma *l10-12*:
assumes *Per A B C* **and**
Per A' B' C' **and**
Cong A B A' B' **and**
Cong B C B' C'
shows *Cong A C A' C'*
⟨*proof*⟩

lemma *l10-16*:
assumes ¬ *Col A B C* **and**
¬ *Col A' B' P* **and**
Cong A B A' B'
shows ∃ *C'. A B C Cong3 A' B' C'* ∧ *A' B' OS P C'*
⟨*proof*⟩

lemma *cong-cop-image--col*:
assumes $P \neq P'$ **and**
$P\ P'$ *Reflect* $A\ B$ **and**
Cong $P\ X\ P'\ X$ **and**
Coplanar $A\ B\ P\ X$
shows *Col* $A\ B\ X$
⟨*proof*⟩

lemma *cong-cop-per2-1*:
assumes $A \neq B$ **and**
Per $A\ B\ X$ **and**
Per $A\ B\ Y$ **and**
Cong $B\ X\ B\ Y$ **and**
Coplanar $A\ B\ X\ Y$
shows $X = Y \lor B$ *Midpoint* $X\ Y$
⟨*proof*⟩

lemma *cong-cop-per2*:
assumes $A \neq B$ **and**
Per $A\ B\ X$ **and**
Per $A\ B\ Y$ **and**
Cong $B\ X\ B\ Y$ **and**
Coplanar $A\ B\ X\ Y$
shows $X = Y \lor X\ Y$ *ReflectL* $A\ B$
⟨*proof*⟩

lemma *cong-cop-per2-gen*:
assumes $A \neq B$ **and**
Per $A\ B\ X$ **and**
Per $A\ B\ Y$ **and**
Cong $B\ X\ B\ Y$ **and**
Coplanar $A\ B\ X\ Y$
shows $X = Y \lor X\ Y$ *Reflect* $A\ B$
⟨*proof*⟩

lemma *ex-perp-cop*:
assumes $A \neq B$
shows $\exists\ Q.\ A\ B$ *Perp* $Q\ C \land$ *Coplanar* $A\ B\ P\ Q$
⟨*proof*⟩

lemma *hilbert-s-version-of-pasch-aux*:
assumes *Coplanar* $A\ B\ C\ P$ **and**
$\neg$ *Col* $A\ I\ P$ **and**
$\neg$ *Col* $B\ C\ P$ **and**
Bet $B\ I\ C$ **and**
$B \neq I$ **and**
$I \neq C$ **and**
$B \neq C$
shows $\exists\ X.$ *Col* $I\ P\ X \land (($*Bet* $A\ X\ B \land A \neq X \land X \neq B \land A \neq B) \lor ($*Bet* $A\ X\ C \land A \neq X \land X \neq C \land A \neq C))$
⟨*proof*⟩

lemma *hilbert-s-version-of-pasch*:
assumes *Coplanar* $A\ B\ C\ P$ **and**
$\neg$ *Col* $C\ Q\ P$ **and**
$\neg$ *Col* $A\ B\ P$ **and**
BetS $A\ Q\ B$
shows $\exists\ X.$ *Col* $P\ Q\ X \land ($*BetS* $A\ X\ C \lor$ *BetS* $B\ X\ C)$
⟨*proof*⟩

lemma *two-sides-cases*:
assumes $\neg$ *Col* *PO* $A\ B$ **and**
PO P *OS* $A\ B$
shows *PO* A *TS* $P\ B \lor$ *PO* B *TS* $P\ A$
⟨*proof*⟩

lemma *not-par-two-sides*:

assumes $C \neq D$ **and**
Col A B I **and**
Col C D I **and**
$\neg$ *Col A B C*
shows $\exists$ *X Y*. *Col C D X* $\wedge$ *Col C D Y* $\wedge$ *A B TS X Y*
⟨*proof*⟩

lemma *cop-not-par-other-side*:
assumes $C \neq D$ **and**
Col A B I **and**
Col C D I **and**
$\neg$ *Col A B C* **and**
$\neg$ *Col A B P* **and**
Coplanar A B C P
shows $\exists$ *Q*. *Col C D Q* $\wedge$ *A B TS P Q*
⟨*proof*⟩

lemma *cop-not-par-same-side*:
assumes $C \neq D$ **and**
Col A B I **and**
Col C D I **and**
$\neg$ *Col A B C* **and**
$\neg$ *Col A B P* **and**
Coplanar A B C P
shows $\exists$ *Q*. *Col C D Q* $\wedge$ *A B OS P Q*
⟨*proof*⟩

end

3.9.3 Line reflexivity: 2D

context *Tarski-2D*

begin

lemma *all-coplanar*:
Coplanar A B C D
⟨*proof*⟩

lemma *per2--col*:
assumes *Per A X C* **and**
$X \neq C$ **and**
Per B X C
shows *Col A B X*
⟨*proof*⟩

lemma *perp2--col*:
assumes *X Y Perp A B* **and**
X Z Perp A B
shows *Col X Y Z*
⟨*proof*⟩
end

3.10 Angles

3.10.1 Some generalites

context *Tarski-neutral-dimensionless*

begin

lemma *l11-3*:
assumes *A B C CongA D E F*
shows $\exists$ A' C' D' F'. *B Out* A' *A* $\wedge$ *B Out C* C' $\wedge$ *E Out* D' *D* $\wedge$ *E Out F* F' $\wedge$ A' *B* C' *Cong3* D' *E* F'
⟨*proof*⟩

lemma *l11-aux*:
assumes B *Out* A A' **and**
E *Out* D D' **and**
Cong B A' E D' **and**
Bet B A *A0* **and**
Bet E D *D0* **and**
Cong A *A0* E D **and**
Cong D *D0* B A
shows *Cong* B *A0* E *D0* $\wedge$ *Cong* A' *A0* D' *D0*
⟨*proof*⟩

lemma *l11-3-bis*:
assumes $\exists\ A'\ C'\ D'\ F'.$ (B *Out* A' A $\wedge$ B *Out* C' C $\wedge$ E *Out* D' D $\wedge$ E *Out* F' F $\wedge$ A' B C' *Cong3* D' E F')
shows A B C *CongA* D E F
⟨*proof*⟩

lemma *l11-4-1*:
assumes A B C *CongA* D E F **and**

B *Out* A' A **and**
B *Out* C' C **and**
E *Out* D' D **and**
E *Out* F' F **and**
Cong B A' E D' **and** *Cong* B C' E F'
shows *Cong* A' C' D' F'
⟨*proof*⟩

lemma *l11-4-2*:
assumes $A \neq B$ **and**
$C \neq B$ **and**
$D \neq E$ **and**
$F \neq E$ **and**
$\forall\ A'\ C'\ D'\ F'.$ (B *Out* A' A $\wedge$ B *Out* C' C $\wedge$ E *Out* D' D $\wedge$ E *Out* F' F $\wedge$ *Cong* B A' E D' $\wedge$ *Cong* B C' E F' $\longrightarrow$ *Cong* A' C' D' F')
shows A B C *CongA* D E F
⟨*proof*⟩

lemma *conga-refl*:
assumes $A \neq B$ **and**
$C \neq B$
shows A B C *CongA* A B C
⟨*proof*⟩

lemma *conga-sym*:
assumes A B C *CongA* A' B' C'
shows A' B' C' *CongA* A B C
⟨*proof*⟩

lemma *l11-10*:
assumes A B C *CongA* D E F **and**
B *Out* A' A **and**
B *Out* C' C **and**
E *Out* D' D **and**
E *Out* F' F
shows A' B C' *CongA* D' E F'
⟨*proof*⟩

lemma *out2--conga*:
assumes B *Out* A' A **and**
B *Out* C' C
shows A B C *CongA* A' B C'
⟨*proof*⟩

lemma *cong3-diff*:
assumes $A \neq B$ **and**
A B C *Cong3* A' B' C'

shows $A' \neq B'$
⟨*proof*⟩

lemma *cong3-diff2*:
assumes $B \neq C$ **and**
$A\ B\ C\ Cong3\ A'\ B'\ C'$
shows $B' \neq C'$
⟨*proof*⟩

lemma *cong3-conga*:
assumes $A \neq B$ **and**
$C \neq B$ **and**
$A\ B\ C\ Cong3\ A'\ B'\ C'$
shows $A\ B\ C\ CongA\ A'\ B'\ C'$
⟨*proof*⟩

lemma *cong3-conga2*:
assumes $A\ B\ C\ Cong3\ A'\ B'\ C'$ **and**
$A\ B\ C\ CongA\ A''\ B''\ C''$
shows $A'\ B'\ C'\ CongA\ A''\ B''\ C''$
⟨*proof*⟩

lemma *conga-diff1*:
assumes $A\ B\ C\ CongA\ A'\ B'\ C'$
shows $A \neq B$
⟨*proof*⟩

lemma *conga-diff2*:
assumes $A\ B\ C\ CongA\ A'\ B'\ C'$
shows $C \neq B$
⟨*proof*⟩

lemma *conga-diff45*:
assumes $A\ B\ C\ CongA\ A'\ B'\ C'$
shows $A' \neq B'$
⟨*proof*⟩

lemma *conga-diff56*:
assumes $A\ B\ C\ CongA\ A'\ B'\ C'$
shows $C' \neq B'$
⟨*proof*⟩

lemma *conga-trans*:
assumes $A\ B\ C\ CongA\ A'\ B'\ C'$ **and**
$A'\ B'\ C'\ CongA\ A''\ B''\ C''$
shows $A\ B\ C\ CongA\ A''\ B''\ C''$
⟨*proof*⟩

lemma *conga-pseudo-refl*:
assumes $A \neq B$ **and**
$C \neq B$
shows $A\ B\ C\ CongA\ C\ B\ A$
⟨*proof*⟩

lemma *conga-trivial-1*:
assumes $A \neq B$ **and**
$C \neq D$
shows $A\ B\ A\ CongA\ C\ D\ C$
⟨*proof*⟩

lemma *l11-13*:
assumes $A\ B\ C\ CongA\ D\ E\ F$ **and**
$Bet\ A\ B\ A'$ **and**
$A' \neq B$ **and**
$Bet\ D\ E\ D'$ **and**
$D' \neq E$

shows A' B C *CongA* D' E F
⟨*proof*⟩

lemma *conga-right-comm*:
assumes A B C *CongA* D E F
shows A B C *CongA* F E D
⟨*proof*⟩

lemma *conga-left-comm*:
assumes A B C *CongA* D E F
shows C B A *CongA* D E F
⟨*proof*⟩

lemma *conga-comm*:
assumes A B C *CongA* D E F
shows C B A *CongA* F E D
⟨*proof*⟩

lemma *conga-line*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
$A' \neq B'$ **and**
$B' \neq C'$
and *Bet* A B C **and**
Bet A' B' C'
shows A B C *CongA* A' B' C'
⟨*proof*⟩

lemma *l11-14*:
assumes *Bet* A B A' **and**
$A \neq B$ **and**
$A' \neq B$ **and**
Bet C B C' **and**
$B \neq C$ **and**
$B \neq C'$
shows A B C *CongA* A' B C'
⟨*proof*⟩

lemma *l11-16*:
assumes *Per* A B C **and**
$A \neq B$ **and**
$C \neq B$ **and**
Per A' B' C' **and**
$A' \neq B'$ **and**
$C' \neq B'$
shows A B C *CongA* A' B' C'
⟨*proof*⟩

lemma *l11-17*:
assumes *Per* A B C **and**
A B C *CongA* A' B' C'
shows *Per* A' B' C'
⟨*proof*⟩

lemma *l11-18-1*:
assumes *Bet* C B D **and**
$B \neq C$ **and**
$B \neq D$ **and**
$A \neq B$ **and**
Per A B C
shows A B C *CongA* A B D
⟨*proof*⟩

lemma *l11-18-2*:
assumes *Bet* C B D **and**
A B C *CongA* A B D

shows *Per A B C*
⟨*proof*⟩

lemma *cong3-preserves-out*:
assumes *A Out B C* **and**
A B C Cong3 A′ B′ C′
shows *A′ Out B′ C′*
⟨*proof*⟩

lemma *l11-21-a*:
assumes *B Out A C* **and**
A B C CongA A′ B′ C′
shows *B′ Out A′ C′*
⟨*proof*⟩

lemma *l11-21-b*:
assumes *B Out A C* **and**
B′ Out A′ C′
shows *A B C CongA A′ B′ C′*
⟨*proof*⟩

lemma *conga-cop--or-out-ts*:
assumes *Coplanar A B C C′* **and**
A B C CongA A B C′
shows *B Out C C′* ∨ *A B TS C C′*
⟨*proof*⟩

lemma *conga-os--out*:
assumes *A B C CongA A B C′* **and**
A B OS C C′
shows *B Out C C′*
⟨*proof*⟩

lemma *cong2-conga-cong*:
assumes *A B C CongA A′ B′ C′* **and**
Cong A B A′ B′ **and**
Cong B C B′ C′
shows *Cong A C A′ C′*
⟨*proof*⟩

lemma *angle-construction-1*:
assumes ¬ *Col A B C* **and**
¬ *Col A′ B′ P*
shows ∃ *C′*. (*A B C CongA A′ B′ C′* ∧ *A′ B′ OS C′ P*)
⟨*proof*⟩

lemma *angle-construction-2*:
assumes *A* ≠ *B* **and**
B ≠ *C* **and**
¬ *Col A′ B′ P*
shows ∃ *C′*. (*A B C CongA A′ B′ C′* ∧ (*A′ B′ OS C′ P* ∨ *Col A′ B′ C′*))
⟨*proof*⟩

lemma *ex-conga-ts*:
assumes ¬ *Col A B C* **and**
¬ *Col A′ B′ P*
shows ∃ *C′*. *A B C CongA A′ B′ C′* ∧ *A′ B′ TS C′ P*
⟨*proof*⟩

lemma *l11-15*:
assumes ¬ *Col A B C* **and**
¬ *Col D E P*
shows
∃ *F*. (*A B C CongA D E F* ∧ *E D OS F P*) ∧
(∀ *F1 F2*. ((*A B C CongA D E F1* ∧ *E D OS F1 P*) ∧
(*A B C CongA D E F2* ∧ *E D OS F2 P*))

$\longrightarrow$ *E Out F1 F2*)
⟨*proof*⟩

lemma *l11-19*:
assumes *Per A B P1* **and**
Per A B P2 **and**
A B OS P1 P2
shows *B Out P1 P2*
⟨*proof*⟩

lemma *l11-22-bet*:
assumes *Bet A B C* **and**
P' B' *TS* A' C' **and**
A B P CongA A' B' P' **and**
P B C CongA P' B' C'
shows *Bet* A' B' C'
⟨*proof*⟩

lemma *l11-22a*:
assumes *B P TS A C* **and**
B' P' *TS* A' C' **and**
A B P CongA A' B' P' **and**
P B C CongA P' B' C'
shows *A B C CongA* A' B' C'
⟨*proof*⟩

lemma *l11-22b*:
assumes *B P OS A C* **and**
B' P' *OS* A' C' **and**
A B P CongA A' B' P' **and**
P B C CongA P' B' C'
shows *A B C CongA* A' B' C'
⟨*proof*⟩

lemma *l11-22*:
assumes ((*B P TS A C* $\wedge$ B' P' *TS* A' C')$\vee$(*B P OS A C* $\wedge$ B' P' *OS* A' C')) **and**
A B P CongA A' B' P' **and**
P B C CongA P' B' C'
shows *A B C CongA* A' B' C'
⟨*proof*⟩

lemma *l11-24*:
assumes *P InAngle A B C*
shows *P InAngle C B A*
⟨*proof*⟩

lemma *col-in-angle*:
assumes $A \neq B$ **and**
$C \neq B$ **and**
$P \neq B$ **and**
B Out A P $\vee$ *B Out C P*
shows *P InAngle A B C*
⟨*proof*⟩

lemma *out321--inangle*:
assumes $C \neq B$ **and**
B Out A P
shows *P InAngle A B C*
⟨*proof*⟩

lemma *inangle1123*:
assumes $A \neq B$ **and**
$C \neq B$
shows *A InAngle A B C*
⟨*proof*⟩

lemma *out341--inangle*:
assumes $A \neq B$ **and**
B Out C P
shows *P InAngle A B C*
⟨*proof*⟩

lemma *inangle3123*:
assumes $A \neq B$ **and**
$C \neq B$
shows *C InAngle A B C*
⟨*proof*⟩

lemma *in-angle-two-sides*:
assumes ¬ *Col B A P* **and**
¬ *Col B C P* **and**
P InAngle A B C
shows *P B TS A C*
⟨*proof*⟩

lemma *in-angle-out*:
assumes *B Out A C* **and**
P InAngle A B C
shows *B Out A P*
⟨*proof*⟩

lemma *col-in-angle-out*:
assumes *Col B A P* **and**
¬ *Bet A B C* **and**
P InAngle A B C
shows *B Out A P*
⟨*proof*⟩

lemma *l11-25-aux*:
assumes *P InAngle A B C* **and**
¬ *Bet A B C* **and**
B Out A' *A*
shows *P InAngle* A' *B C*
⟨*proof*⟩

lemma *l11-25*:
assumes *P InAngle A B C* **and**
B Out A' *A* **and**
B Out C' *C* **and**
B Out P' *P*
shows P' *InAngle* A' *B* C'
⟨*proof*⟩

lemma *inangle-distincts*:
assumes *P InAngle A B C*
shows $A \neq B \wedge C \neq B \wedge P \neq B$
⟨*proof*⟩

lemma *segment-construction-0*:
shows $\exists$ B'. *Cong* A' B' *A B*
⟨*proof*⟩

lemma *angle-construction-3*:
assumes $A \neq B$ **and**
$C \neq B$ **and**
$A' \neq B'$
shows $\exists$ C'. *A B C CongA* A' B' C'
⟨*proof*⟩

lemma *l11-28*:
assumes *A B C Cong3* A' B' C' **and**
Col A C D

shows ∃ D'. (*Cong A D A' D'* ∧ *Cong B D B' D'* ∧ *Cong C D C' D'*)
⟨*proof*⟩

lemma *bet-conga--bet*:
assumes *Bet A B C* **and**
A B C CongA A' B' C'
shows *Bet A' B' C'*
⟨*proof*⟩

lemma *in-angle-one-side*:
assumes ¬ *Col A B C* **and**
¬ *Col B A P* **and**
P InAngle A B C
shows *A B OS P C*
⟨*proof*⟩

lemma *inangle-one-side*:
assumes ¬ *Col A B C* **and**
¬ *Col A B P* **and**
¬ *Col A B Q* **and**
P InAngle A B C **and**
Q InAngle A B C
shows *A B OS P Q*
⟨*proof*⟩

lemma *inangle-one-side2*:
assumes ¬ *Col A B C* **and**
¬ *Col A B P* **and**
¬ *Col A B Q* **and**
¬ *Col C B P* **and**
¬ *Col C B Q* **and**
P InAngle A B C **and**
Q InAngle A B C
shows *A B OS P Q* ∧ *C B OS P Q*
⟨*proof*⟩

lemma *col-conga-col*:
assumes *Col A B C* **and**
A B C CongA D E F
shows *Col D E F*
⟨*proof*⟩

lemma *ncol-conga-ncol*:
assumes ¬ *Col A B C* **and**
A B C CongA D E F
shows ¬ *Col D E F*
⟨*proof*⟩

lemma *angle-construction-4*:
assumes $A \neq B$ **and**
$C \neq B$ **and**
$A' \neq B'$
shows ∃ C'. (*A B C CongA A' B' C'* ∧ *Coplanar A' B' C' P*)
⟨*proof*⟩

lemma *lea-distincts*:
assumes *A B C LeA D E F*
shows $A \neq B \wedge C \neq B \wedge D \neq E \wedge F \neq E$
⟨*proof*⟩

lemma *l11-29-a*:
assumes *A B C LeA D E F*
shows ∃ Q. (*C InAngle A B Q* ∧ *A B Q CongA D E F*)
⟨*proof*⟩

lemma *in-angle-line*:

assumes $P \neq B$ **and**
$A \neq B$ **and**
$C \neq B$ **and**
Bet A B C
shows *P InAngle A B C*
⟨*proof*⟩

lemma *l11-29-b*:
assumes $\exists$ *Q*. (*C InAngle A B Q* $\wedge$ *A B Q CongA D E F*)
shows *A B C LeA D E F*
⟨*proof*⟩

lemma *bet-in-angle-bet*:
assumes *Bet A B P* **and**
P InAngle A B C
shows *Bet A B C*
⟨*proof*⟩

lemma *lea-line*:
assumes *Bet A B P* **and**
A B P LeA A B C
shows *Bet A B C*
⟨*proof*⟩

lemma *eq-conga-out*:
assumes *A B A CongA D E F*
shows *E Out D F*
⟨*proof*⟩

lemma *out-conga-out*:
assumes *B Out A C* **and**
A B C CongA D E F
shows *E Out D F*
⟨*proof*⟩

lemma *conga-ex-cong3*:
assumes *A B C CongA A′ B′ C′*
shows $\exists$ *AA CC*. ((*B Out A AA* $\wedge$ *B Out C CC*) $\longrightarrow$ *AA B CC Cong3 A′ B′ C′*)
⟨*proof*⟩

lemma *conga-preserves-in-angle*:
assumes *A B C CongA A′ B′ C′* **and**
A B I CongA A′ B′ I′ **and**
I InAngle A B C **and** *A′ B′ OS I′ C′*
shows *I′ InAngle A′ B′ C′*
⟨*proof*⟩

lemma *l11-30*:
assumes *A B C LeA D E F* **and**
A B C CongA A′ B′ C′ **and**
D E F CongA D′ E′ F′
shows *A′ B′ C′ LeA D′ E′ F′*
⟨*proof*⟩

lemma *l11-31-1*:
assumes *B Out A C* **and**
$D \neq E$ **and**
$F \neq E$
shows *A B C LeA D E F*
⟨*proof*⟩

lemma *l11-31-2*:
assumes $A \neq B$ **and**
$C \neq B$ **and**
$D \neq E$ **and**
$F \neq E$ **and**

Bet D E F
shows *A B C LeA D E F*
⟨*proof*⟩

lemma *lea-refl*:
assumes $A \neq B$ **and**
$C \neq B$
shows *A B C LeA A B C*
⟨*proof*⟩

lemma *conga--lea*:
assumes *A B C CongA D E F*
shows *A B C LeA D E F*
⟨*proof*⟩

lemma *conga--lea456123*:
assumes *A B C CongA D E F*
shows *D E F LeA A B C*
⟨*proof*⟩

lemma *lea-left-comm*:
assumes *A B C LeA D E F*
shows *C B A LeA D E F*
⟨*proof*⟩

lemma *lea-right-comm*:
assumes *A B C LeA D E F*
shows *A B C LeA F E D*
⟨*proof*⟩

lemma *lea-comm*:
assumes*A B C LeA D E F*
shows *C B A LeA F E D*
⟨*proof*⟩

lemma *lta-left-comm*:
assumes *A B C LtA D E F*
shows *C B A LtA D E F*
⟨*proof*⟩

lemma *lta-right-comm*:
assumes *A B C LtA D E F*
shows *A B C LtA F E D*
⟨*proof*⟩

lemma *lta-comm*:
assumes *A B C LtA D E F*
shows *C B A LtA F E D*
⟨*proof*⟩

lemma *lea-out4--lea*:
assumes *A B C LeA D E F* **and**
B *Out* $A\ A'$ **and**
B *Out* $C\ C'$ **and**
E *Out* $D\ D'$ **and**
E *Out* $F\ F'$
shows $A'\ B\ C'$ *LeA* $D'\ E\ F'$
⟨*proof*⟩

lemma *lea121345*:
assumes $A \neq B$ **and**
$C \neq D$ **and**
$D \neq E$
shows *A B A LeA C D E*
⟨*proof*⟩

lemma *inangle--lea*:
  assumes *P InAngle A B C*
  shows *A B P LeA A B C*
  ⟨*proof*⟩

lemma *inangle--lea-1*:
  assumes *P InAngle A B C*
  shows *P B C LeA A B C*
  ⟨*proof*⟩

lemma *inangle--lta*:
  assumes ¬ *Col P B C* **and**
    P InAngle A B C
  shows *A B P LtA A B C*
  ⟨*proof*⟩

lemma *in-angle-trans*:
  assumes *C InAngle A B D* **and**
    D InAngle A B E
  shows *C InAngle A B E*
⟨*proof*⟩

lemma *lea-trans*:
  assumes *A B C LeA A1 B1 C1* **and**
    A1 B1 C1 LeA A2 B2 C2
  shows *A B C LeA A2 B2 C2*
⟨*proof*⟩

lemma *in-angle-asym*:
  assumes *D InAngle A B C* **and**
    C InAngle A B D
  shows *A B C CongA A B D*
⟨*proof*⟩

lemma *lea-asym*:
  assumes *A B C LeA D E F* **and**
    D E F LeA A B C
  shows *A B C CongA D E F*
⟨*proof*⟩

lemma *col-lta--bet*:
  assumes *Col X Y Z* **and**
    A B C LtA X Y Z
  shows *Bet X Y Z*
⟨*proof*⟩

lemma *col-lta--out*:
  assumes *Col A B C* **and**
    A B C LtA X Y Z
  shows *B Out A C*
⟨*proof*⟩

lemma *lta-distincts*:
  assumes *A B C LtA D E F*
  shows $A \neq B \wedge C \neq B \wedge D \neq E \wedge F \neq E \wedge D \neq F$
  ⟨*proof*⟩

lemma *gta-distincts*:
  assumes *A B C GtA D E F*
  shows $A \neq B \wedge C \neq B \wedge D \neq E \wedge F \neq E \wedge A \neq C$
  ⟨*proof*⟩

lemma *acute-distincts*:
  assumes *Acute A B C*
  shows $A \neq B \wedge C \neq B$

⟨*proof*⟩

lemma *obtuse-distincts*:
assumes *Obtuse A B C*
shows $A \neq B \wedge C \neq B \wedge A \neq C$
⟨*proof*⟩

lemma *two-sides-in-angle*:
assumes $B \neq P'$ **and**
B P TS A C **and**
Bet P B P'
shows *P InAngle A B C* ∨ *P' InAngle A B C*
⟨*proof*⟩

lemma *in-angle-reverse*:
assumes $A' \neq B$ **and**
Bet A B A' **and**
C InAngle A B D
shows *D InAngle A' B C*
⟨*proof*⟩

lemma *in-angle-trans2*:
assumes *C InAngle A B D* **and**
D InAngle A B E
shows *D InAngle C B E*
⟨*proof*⟩

lemma *l11-36-aux1*:
assumes $A \neq B$ **and**
$A' \neq B$ **and**
$D \neq E$ **and**
$D' \neq E$ **and**
Bet A B A' **and**
Bet D E D' **and**
A B C LeA D E F
shows *D' E F LeA A' B C*
⟨*proof*⟩

lemma *l11-36-aux2*:
assumes $A \neq B$ **and**
$A' \neq B$ **and**
$D \neq E$ **and**
$D' \neq E$ **and**
Bet A B A' **and**
Bet D E D' **and**
D' E F LeA A' B C
shows *A B C LeA D E F*
⟨*proof*⟩

lemma *l11-36*:
assumes $A \neq B$ **and**
$A' \neq B$ **and**
$D \neq E$ **and**
$D' \neq E$ **and**
Bet A B A' **and**
Bet D E D'
shows *A B C LeA D E F* ⟷ *D' E F LeA A' B C*
⟨*proof*⟩

lemma *l11-41-aux*:
assumes ¬ *Col A B C* **and**
Bet B A D **and**
$A \neq D$
shows *A C B LtA C A D*
⟨*proof*⟩

lemma *l11-41*:
assumes $\neg$ *Col A B C* **and**
Bet B A D **and**
$A \neq D$
shows *A C B LtA C A D* $\wedge$ *A B C LtA C A D*
⟨*proof*⟩

lemma *not-conga*:
assumes *A B C CongA A′ B′ C′* **and**
$\neg$ *A B C CongA D E F*
shows $\neg$ *A′ B′ C′ CongA D E F*
⟨*proof*⟩

lemma *not-conga-sym*:
assumes $\neg$ *A B C CongA D E F*
shows $\neg$ *D E F CongA A B C*
⟨*proof*⟩

lemma *not-and-lta*:
shows $\neg$ (*A B C LtA D E F* $\wedge$ *D E F LtA A B C*)
⟨*proof*⟩

lemma *conga-preserves-lta*:
assumes *A B C CongA A′ B′ C′* **and**
D E F CongA D′ E′ F′ **and**
A B C LtA D E F
shows *A′ B′ C′ LtA D′ E′ F′*
⟨*proof*⟩

lemma *lta-trans*:
assumes *A B C LtA A1 B1 C1* **and**
A1 B1 C1 LtA A2 B2 C2
shows *A B C LtA A2 B2 C2*
⟨*proof*⟩

lemma *obtuse-sym*:
assumes *Obtuse A B C*
shows *Obtuse C B A*
⟨*proof*⟩

lemma *acute-sym*:
assumes *Acute A B C*
shows *Acute C B A*
⟨*proof*⟩

lemma *acute-col--out*:
assumes *Col A B C* **and**
Acute A B C
shows *B Out A C*
⟨*proof*⟩

lemma *col-obtuse--bet*:
assumes *Col A B C* **and**
Obtuse A B C
shows *Bet A B C*
⟨*proof*⟩

lemma *out--acute*:
assumes *B Out A C*
shows *Acute A B C*
⟨*proof*⟩

lemma *bet--obtuse*:
assumes *Bet A B C* **and**
$A \neq B$ **and** $B \neq C$
shows *Obtuse A B C*

⟨*proof*⟩

lemma *l11-43-aux*:
assumes $A \neq B$ **and**
$A \neq C$ **and**
Per B A C ∨ *Obtuse B A C*
shows *Acute A B C*
⟨*proof*⟩

lemma *l11-43*:
assumes $A \neq B$ **and**
$A \neq C$ **and**
Per B A C ∨ *Obtuse B A C*
shows *Acute A B C* ∧ *Acute A C B*
⟨*proof*⟩

lemma *acute-lea-acute*:
assumes *Acute D E F* **and**
A B C LeA D E F
shows *Acute A B C*
⟨*proof*⟩

lemma *lea-obtuse-obtuse*:
assumes *Obtuse D E F* **and**
D E F LeA A B C
shows *Obtuse A B C*
⟨*proof*⟩

lemma *l11-44-1-a*:
assumes $A \neq B$ **and**
$A \neq C$ **and**
Cong B A B C
shows *B A C CongA B C A*
⟨*proof*⟩

lemma *l11-44-2-a*:
assumes ¬ *Col A B C* **and**
B A Lt B C
shows *B C A LtA B A C*
⟨*proof*⟩

lemma *not-lta-and-conga*:
¬ (*A B C LtA D E F* ∧ *A B C CongA D E F*)
⟨*proof*⟩

lemma *conga-sym-equiv*:
A B C CongA A' B' C' ⟷ *A' B' C' CongA A B C*
⟨*proof*⟩

lemma *conga-dec*:
A B C CongA D E F ∨ ¬ *A B C CongA D E F*
⟨*proof*⟩

lemma *lta-not-conga*:
assumes *A B C LtA D E F*
shows ¬ *A B C CongA D E F*
⟨*proof*⟩

lemma *lta--lea*:
assumes *A B C LtA D E F*
shows *A B C LeA D E F*
⟨*proof*⟩

lemma *nlta*:
¬ *A B C LtA A B C*
⟨*proof*⟩

lemma *lea--nlta*:
assumes $A\ B\ C\ LeA\ D\ E\ F$
shows $\neg\ D\ E\ F\ LtA\ A\ B\ C$
⟨*proof*⟩

lemma *lta--nlea*:
assumes $A\ B\ C\ LtA\ D\ E\ F$
shows $\neg\ D\ E\ F\ LeA\ A\ B\ C$
⟨*proof*⟩

lemma *l11-44-1-b*:
assumes $\neg\ Col\ A\ B\ C$ **and**
$B\ A\ C\ CongA\ B\ C\ A$
shows $Cong\ B\ A\ B\ C$
⟨*proof*⟩

lemma *l11-44-2-b*:
assumes $B\ A\ C\ LtA\ B\ C\ A$
shows $B\ C\ Lt\ B\ A$
⟨*proof*⟩

lemma *l11-44-1*:
assumes $\neg\ Col\ A\ B\ C$
shows $B\ A\ C\ CongA\ B\ C\ A \longleftrightarrow Cong\ B\ A\ B\ C$
⟨*proof*⟩

lemma *l11-44-2*:
assumes $\neg\ Col\ A\ B\ C$
shows $B\ A\ C\ LtA\ B\ C\ A \longleftrightarrow B\ C\ Lt\ B\ A$
⟨*proof*⟩

lemma *l11-44-2bis*:
assumes $\neg\ Col\ A\ B\ C$
shows $B\ A\ C\ LeA\ B\ C\ A \longleftrightarrow B\ C\ Le\ B\ A$
⟨*proof*⟩

lemma *l11-46*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
$Per\ A\ B\ C \lor Obtuse\ A\ B\ C$
shows $B\ A\ Lt\ A\ C \land B\ C\ Lt\ A\ C$
⟨*proof*⟩

lemma *l11-47*:
assumes $Per\ A\ C\ B$ **and**
$H\ PerpAt\ C\ H\ A\ B$
shows $Bet\ A\ H\ B \land A \neq H \land B \neq H$
⟨*proof*⟩

lemma *l11-49*:
assumes $A\ B\ C\ CongA\ A'\ B'\ C'$ **and**
$Cong\ B\ A\ B'\ A'$ **and**
$Cong\ B\ C\ B'\ C'$
shows $Cong\ A\ C\ A'\ C' \land (A \neq C \longrightarrow (B\ A\ C\ CongA\ B'\ A'\ C' \land B\ C\ A\ CongA\ B'\ C'\ A'))$
⟨*proof*⟩

lemma *l11-50-1*:
assumes $\neg\ Col\ A\ B\ C$ **and**
$B\ A\ C\ CongA\ B'\ A'\ C'$ **and**
$A\ B\ C\ CongA\ A'\ B'\ C'$ **and**
$Cong\ A\ B\ A'\ B'$
shows $Cong\ A\ C\ A'\ C' \land Cong\ B\ C\ B'\ C' \land A\ C\ B\ CongA\ A'\ C'\ B'$
⟨*proof*⟩

lemma *l11-50-2*:

assumes ¬ *Col A B C* **and**
B C A CongA B′ C′ A′ **and**
A B C CongA A′ B′ C′ **and**
Cong A B A′ B′
shows *Cong A C A′ C′* ∧ *Cong B C B′ C′* ∧ *C A B CongA C′ A′ B′*
⟨*proof*⟩

lemma *l11-51*:
assumes $A \neq B$ **and**
$A \neq C$ **and**
$B \neq C$ **and**
Cong A B A′ B′ **and**
Cong A C A′ C′ **and**
Cong B C B′ C′
shows
B A C CongA B′ A′ C′ ∧ *A B C CongA A′ B′ C′* ∧ *B C A CongA B′ C′ A′*
⟨*proof*⟩

lemma *conga-distinct*:
assumes *A B C CongA D E F*
shows $A \neq B \wedge C \neq B \wedge D \neq E \wedge F \neq E$
⟨*proof*⟩

lemma *l11-52*:
assumes *A B C CongA A′ B′ C′* **and**
Cong A C A′ C′ **and**
Cong B C B′ C′ **and**
B C Le A C
shows *Cong B A B′ A′* ∧ *B A C CongA B′ A′ C′* ∧ *B C A CongA B′ C′ A′*
⟨*proof*⟩

lemma *l11-53*:
assumes *Per D C B* **and**
$C \neq D$ **and**
$A \neq B$ **and**
$B \neq C$ **and**
Bet A B C
shows *C A D LtA C B D* ∧ *B D Lt A D*
⟨*proof*⟩

lemma *cong2-conga-obtuse--cong-conga2*:
assumes *Obtuse A B C* **and**
A B C CongA A′ B′ C′ **and**
Cong A C A′ C′ **and**
Cong B C B′ C′
shows *Cong B A B′ A′* ∧ *B A C CongA B′ A′ C′* ∧
B C A CongA B′ C′ A′
⟨*proof*⟩

lemma *cong2-per2--cong-conga2*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
Per A B C **and**
Per A′ B′ C′ **and**
Cong A C A′ C′ **and**
Cong B C B′ C′
shows *Cong B A B′ A′* ∧ *B A C CongA B′ A′ C′* ∧
B C A CongA B′ C′ A′
⟨*proof*⟩

lemma *cong2-per2--cong*:
assumes *Per A B C* **and**
Per A′ B′ C′ **and**
Cong A C A′ C′ **and**
Cong B C B′ C′
shows *Cong B A B′ A′*

⟨*proof*⟩

lemma *cong2-per2--cong-3*:
assumes *Per A B C*
Per A′ B′ C′ **and**
Cong A C A′ C′ **and**
Cong B C B′ C′
shows *A B C Cong3 A′ B′ C′*
⟨*proof*⟩

lemma *cong-lt-per2--lt*:
assumes *Per A B C* **and**
Per A′ B′ C′ **and**
Cong A B A′ B′ **and**
B C Lt B′ C′
shows *A C Lt A′ C′*
⟨*proof*⟩

lemma *cong-le-per2--le*:
assumes *Per A B C* **and**
Per A′ B′ C′ **and**
Cong A B A′ B′ **and**
B C Le B′ C′
shows *A C Le A′ C′*
⟨*proof*⟩

lemma *lt2-per2--lt*:
assumes *Per A B C* **and**
Per A′ B′ C′ **and**
A B Lt A′ B′ **and**
B C Lt B′ C′
shows *A C Lt A′ C′*
⟨*proof*⟩

lemma *le-lt-per2--lt*:
assumes *Per A B C* **and**
Per A′ B′ C′ **and**
A B Le A′ B′ **and**
B C Lt B′ C′
shows *A C Lt A′ C′*
⟨*proof*⟩

lemma *le2-per2--le*:
assumes *Per A B C* **and**
Per A′ B′ C′ **and**
A B Le A′ B′ **and**
B C Le B′ C′
shows *A C Le A′ C′*
⟨*proof*⟩

lemma *cong-lt-per2--lt-1*:
assumes *Per A B C* **and**
Per A′ B′ C′ **and**
A B Lt A′ B′ **and**
Cong A C A′ C′
shows *B′ C′ Lt B C*
⟨*proof*⟩

lemma *symmetry-preserves-conga*:
assumes $A \neq B$ **and** $C \neq B$ **and**
M Midpoint A A′ **and**
M Midpoint B B′ **and**
M Midpoint C C′
shows *A B C CongA A′ B′ C′*
⟨*proof*⟩

lemma *l11-57*:
assumes $A\ A'\ OS\ B\ B'$ **and**
$Per\ B\ A\ A'$ **and**
$Per\ B'\ A'\ A$ **and**
$A\ A'\ OS\ C\ C'$ **and**
$Per\ C\ A\ A'$ **and**
$Per\ C'\ A'\ A$
shows $B\ A\ C\ CongA\ B'\ A'\ C'$
⟨*proof*⟩

lemma *cop3-orth-at--orth-at*:
assumes $\neg\ Col\ D\ E\ F$ **and**
$Coplanar\ A\ B\ C\ D$ **and**
$Coplanar\ A\ B\ C\ E$ **and**
$Coplanar\ A\ B\ C\ F$ **and**
$X\ OrthAt\ A\ B\ C\ U\ V$
shows $X\ OrthAt\ D\ E\ F\ U\ V$
⟨*proof*⟩

lemma *col2-orth-at--orth-at*:
assumes $U \neq V$ **and**
$Col\ P\ Q\ U$ **and**
$Col\ P\ Q\ V$ **and**
$X\ OrthAt\ A\ B\ C\ P\ Q$
shows $X\ OrthAt\ A\ B\ C\ U\ V$
⟨*proof*⟩

lemma *col-orth-at--orth-at*:
assumes $U \neq W$ **and**
$Col\ U\ V\ W$ **and**
$X\ OrthAt\ A\ B\ C\ U\ V$
shows $X\ OrthAt\ A\ B\ C\ U\ W$
⟨*proof*⟩

lemma *orth-at-symmetry*:
assumes $X\ OrthAt\ A\ B\ C\ U\ V$
shows $X\ OrthAt\ A\ B\ C\ V\ U$
⟨*proof*⟩

lemma *orth-at-distincts*:
assumes $X\ OrthAt\ A\ B\ C\ U\ V$
shows $A \neq B \wedge B \neq C \wedge A \neq C \wedge U \neq V$
⟨*proof*⟩

lemma *orth-at-chara*:
$X\ OrthAt\ A\ B\ C\ X\ P \longleftrightarrow$
$(\neg\ Col\ A\ B\ C \wedge X \neq P \wedge Coplanar\ A\ B\ C\ X \wedge (\forall\ D.(Coplanar\ A\ B\ C\ D \longrightarrow Per\ D\ X\ P)))$
⟨*proof*⟩

lemma *cop3-orth--orth*:
assumes $\neg\ Col\ D\ E\ F$ **and**
$Coplanar\ A\ B\ C\ D$ **and**
$Coplanar\ A\ B\ C\ E$ **and**
$Coplanar\ A\ B\ C\ F$ **and**
$A\ B\ C\ Orth\ U\ V$
shows $D\ E\ F\ Orth\ U\ V$
⟨*proof*⟩

lemma *col2-orth--orth*:
assumes $U \neq V$ **and**
$Col\ P\ Q\ U$ **and**
$Col\ P\ Q\ V$ **and**
$A\ B\ C\ Orth\ P\ Q$
shows $A\ B\ C\ Orth\ U\ V$
⟨*proof*⟩

lemma *col-orth--orth*:
  assumes $U \neq W$ **and**
    Col U V W **and**
    A B C Orth U V
  shows *A B C Orth U W*
  ⟨*proof*⟩

lemma *orth-symmetry*:
  assumes *A B C Orth U V*
  shows *A B C Orth V U*
  ⟨*proof*⟩

lemma *orth-distincts*:
  assumes *A B C Orth U V*
  shows $A \neq B \wedge B \neq C \wedge A \neq C \wedge U \neq V$
  ⟨*proof*⟩

lemma *col-cop-orth--orth-at*:
  assumes *A B C Orth U V* **and**
    Coplanar A B C X **and**
    Col U V X
  shows *X OrthAt A B C U V*
⟨*proof*⟩

lemma *l11-60-aux*:
  assumes ¬ *Col A B C* **and**
    Cong A P A Q **and**
    Cong B P B Q **and**
    Cong C P C Q **and**
    Coplanar A B C D
  shows *Cong D P D Q*
⟨*proof*⟩

lemma *l11-60*:
  assumes ¬ *Col A B C* **and**
    Per A D P **and**
    Per B D P **and**
    Per C D P **and**
    Coplanar A B C E
  shows *Per E D P*
  ⟨*proof*⟩

lemma *l11-60-bis*:
  assumes ¬ *Col A B C* **and**
    $D \neq P$ **and**
    Coplanar A B C D **and**
    Per A D P **and**
    Per B D P **and**
    Per C D P
  shows *D OrthAt A B C D P*
  ⟨*proof*⟩

lemma *l11-61*:
  assumes $A \neq A'$ **and**
    $A \neq B$ **and**
    $A \neq C$ **and**
    Coplanar $A\ A'\ B\ B'$ **and**
    Per $B\ A\ A'$ **and**
    Per $B'\ A'\ A$ **and**
    Coplanar $A\ A'\ C\ C'$ **and**
    Per $C\ A\ A'$ **and**
    Per B A C
  shows *Per* $B'\ A'\ C'$
⟨*proof*⟩

lemma *l11-61-bis*:

assumes *D OrthAt A B C D P* **and**
D E Perp E Q **and**
Coplanar A B C E **and**
Coplanar D E P Q
shows *E OrthAt A B C E Q*
⟨*proof*⟩

lemma *l11-62-unicity*:
assumes *Coplanar A B C D* **and**
Coplanar A B C D′ **and**
$\forall$ *E. Coplanar A B C E* $\longrightarrow$ *Per E D P* **and**
$\forall$ *E. Coplanar A B C E* $\longrightarrow$ *Per E D′ P*
shows $D = D'$
⟨*proof*⟩

lemma *l11-62-unicity-bis*:
assumes *X OrthAt A B C X U* **and**
Y OrthAt A B C Y U
shows $X = Y$
⟨*proof*⟩

lemma *orth-at2--eq*:
assumes *X OrthAt A B C U V* **and**
Y OrthAt A B C U V
shows $X = Y$
⟨*proof*⟩

lemma *col-cop-orth-at--eq*:
assumes *X OrthAt A B C U V* **and**
Coplanar A B C Y **and**
Col U V Y
shows $X = Y$
⟨*proof*⟩

lemma *orth-at--ncop1*:
assumes $U \neq X$ **and**
X OrthAt A B C U V
shows ¬ *Coplanar A B C U*
⟨*proof*⟩

lemma *orth-at--ncop2*:
assumes $V \neq X$ **and**
X OrthAt A B C U V
shows ¬ *Coplanar A B C V*
⟨*proof*⟩

lemma *orth-at--ncop*:
assumes *X OrthAt A B C X P*
shows ¬ *Coplanar A B C P*
⟨*proof*⟩

lemma *l11-62-existence*:
$\exists$ *D.* (*Coplanar A B C D* $\wedge$ ($\forall$ *E.* (*Coplanar A B C E* $\longrightarrow$ *Per E D P*)))
⟨*proof*⟩

lemma *l11-62-existence-bis*:
assumes ¬ *Coplanar A B C P*
shows $\exists$ *X. X OrthAt A B C X P*
⟨*proof*⟩

lemma *l11-63-aux*:
assumes *Coplanar A B C D* **and**
$D \neq E$ **and**
E OrthAt A B C E P
shows $\exists$ *Q.* (*D E OS P Q* $\wedge$ *A B C Orth D Q*)
⟨*proof*⟩

lemma *l11-63-existence*:
assumes *Coplanar A B C D* **and**
¬ *Coplanar A B C P*
shows ∃ *Q*. *A B C Orth D Q*
⟨*proof*⟩

lemma *l8-21-3*:
assumes *Coplanar A B C D* **and**
¬ *Coplanar A B C X*
shows
∃ *P T*. (*A B C Orth D P* ∧ *Coplanar A B C T* ∧ *Bet X T P*)
⟨*proof*⟩

lemma *mid2-orth-at2--cong*:
assumes *X OrthAt A B C X P* **and**
Y OrthAt A B C Y Q **and**
X Midpoint P P′ **and**
Y Midpoint Q Q′
shows *Cong P Q P′ Q′*
⟨*proof*⟩

lemma *orth-at2-tsp--ts*:
assumes $P \neq Q$ **and**
P OrthAt A B C P X **and**
Q OrthAt A B C Q Y **and**
A B C TSP X Y
shows *P Q TS X Y*
⟨*proof*⟩

lemma *orth-dec*:
shows *A B C Orth U V* ∨ ¬ *A B C Orth U V* ⟨*proof*⟩

lemma *orth-at-dec*:
shows *X OrthAt A B C U V* ∨ ¬ *X OrthAt A B C U V* ⟨*proof*⟩

lemma *tsp-dec*:
shows *A B C TSP X Y* ∨ ¬ *A B C TSP X Y* ⟨*proof*⟩

lemma *osp-dec*:
shows *A B C OSP X Y* ∨ ¬ *A B C OSP X Y* ⟨*proof*⟩

lemma *ts2--inangle*:
assumes *A C TS B P* **and**
B P TS A C
shows *P InAngle A B C*
⟨*proof*⟩

lemma *os-ts--inangle*:
assumes *B P TS A C* **and**
B A OS C P
shows *P InAngle A B C*
⟨*proof*⟩

lemma *os2--inangle*:
assumes *B A OS C P* **and**
B C OS A P
shows *P InAngle A B C*
⟨*proof*⟩

lemma *acute-conga--acute*:
assumes *Acute A B C* **and**
A B C CongA D E F
shows *Acute D E F*
⟨*proof*⟩

lemma *acute-out2--acute*:
assumes B *Out* A' A **and**
B *Out* C' C **and**
Acute A B C
shows *Acute* A' B C'
⟨*proof*⟩

lemma *conga-obtuse--obtuse*:
assumes *Obtuse* A B C **and**
A B C *CongA* D E F
shows *Obtuse* D E F
⟨*proof*⟩

lemma *obtuse-out2--obtuse*:
assumes B *Out* A' A **and**
B *Out* C' C **and**
Obtuse A B C
shows *Obtuse* A' B C'
⟨*proof*⟩

lemma *bet-lea--bet*:
assumes *Bet* A B C **and**
A B C *LeA* D E F
shows *Bet* D E F
⟨*proof*⟩

lemma *out-lea--out*:
assumes E *Out* D F **and**
A B C *LeA* D E F
shows B *Out* A C
⟨*proof*⟩

lemma *bet2-lta--lta*:
assumes A B C *LtA* D E F **and**
Bet A B A' **and**
$A' \neq B$ **and**
Bet D E D' **and**
$D' \neq E$
shows D' E F *LtA* A' B C
⟨*proof*⟩

lemma *lea123456-lta--lta*:
assumes A B C *LeA* D E F **and**
D E F *LtA* G H I
shows A B C *LtA* G H I
⟨*proof*⟩

lemma *lea456789-lta--lta*:
assumes A B C *LtA* D E F **and**
D E F *LeA* G H I
shows A B C *LtA* G H I
⟨*proof*⟩

lemma *acute-per--lta*:
assumes *Acute* A B C **and**
$D \neq E$ **and**
$E \neq F$ **and**
Per D E F
shows A B C *LtA* D E F
⟨*proof*⟩

lemma *obtuse-per--lta*:
assumes *Obtuse* A B C **and**
$D \neq E$ **and**
$E \neq F$ **and**
Per D E F

shows *D E F LtA A B C*
⟨*proof*⟩

lemma *acute-obtuse--lta*:
assumes *Acute A B C* **and**
Obtuse D E F
shows *A B C LtA D E F*
⟨*proof*⟩

lemma *lea-in-angle*:
assumes *A B P LeA A B C* **and**
A B OS C P
shows *P InAngle A B C*
⟨*proof*⟩

lemma *acute-bet--obtuse*:
assumes *Bet A B* A' **and**
$A' \neq B$ **and**
Acute A B C
shows *Obtuse* A' *B C*
⟨*proof*⟩

lemma *bet-obtuse--acute*:
assumes *Bet A B* A' **and**
$A' \neq B$ **and**
Obtuse A B C
shows *Acute* A' *B C*
⟨*proof*⟩

lemma *inangle-dec*:
P InAngle A B C ∨ ¬ *P InAngle A B C* ⟨*proof*⟩

lemma *lea-dec*:
A B C LeA D E F ∨ ¬ *A B C LeA D E F* ⟨*proof*⟩

lemma *lta-dec*:
A B C LtA D E F ∨ ¬ *A B C LtA D E F* ⟨*proof*⟩

lemma *lea-total*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
$D \neq E$ **and**
$E \neq F$
shows *A B C LeA D E F* ∨ *D E F LeA A B C*
⟨*proof*⟩

lemma *or-lta2-conga*:
assumes $A \neq B$ **and**
$C \neq B$ **and**
$D \neq E$ **and**
$F \neq E$
shows *A B C LtA D E F* ∨ *D E F LtA A B C* ∨ *A B C CongA D E F*
⟨*proof*⟩

lemma *angle-partition*:
assumes $A \neq B$ **and**
$B \neq C$
shows *Acute A B C* ∨ *Per A B C* ∨ *Obtuse A B C*
⟨*proof*⟩

lemma *acute-chara-1*:
assumes *Bet A B* A' **and**
$B \neq A'$ **and**
Acute A B C
shows *A B C LtA* A' *B C*
⟨*proof*⟩

lemma *acute-chara-2*:
assumes *Bet A B A′* **and**
A B C LtA A′ B C
shows *Acute A B C*
⟨*proof*⟩

lemma *acute-chara*:
assumes *Bet A B A′* **and**
$B \neq A'$
shows *Acute A B C* ⟷ *A B C LtA A′ B C*
⟨*proof*⟩

lemma *obtuse-chara*:
assumes *Bet A B A′* **and**
$B \neq A'$
shows *Obtuse A B C* ⟷ *A′ B C LtA A B C*
⟨*proof*⟩

lemma *conga--acute*:
assumes *A B C CongA A C B*
shows *Acute A B C*
⟨*proof*⟩

lemma *cong--acute*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
Cong A B A C
shows *Acute A B C*
⟨*proof*⟩

lemma *nlta--lea*:
assumes ¬ *A B C LtA D E F* **and**
$A \neq B$ **and**
$B \neq C$ **and**
$D \neq E$ **and**
$E \neq F$
shows *D E F LeA A B C*
⟨*proof*⟩

lemma *nlea--lta*:
assumes ¬ *A B C LeA D E F* **and**
$A \neq B$ **and**
$B \neq C$ **and**
$D \neq E$ **and**
$E \neq F$
shows *D E F LtA A B C*
⟨*proof*⟩

lemma *triangle-strict-inequality*:
assumes *Bet A B D* **and**
Cong B C B D **and**
¬ *Bet A B C*
shows *A C Lt A D*
⟨*proof*⟩

lemma *triangle-inequality*:
assumes *Bet A B D* **and**
Cong B C B D
shows *A C Le A D*
⟨*proof*⟩

lemma *triangle-strict-inequality-2*:
assumes *Bet A′ B′ C′* **and**
Cong A B A′ B′ **and**
Cong B C B′ C′ **and**

¬ *Bet A B C*
shows *A C Lt A′ C′*
⟨*proof*⟩

lemma *triangle-inequality-2*:
assumes *Bet A′ B′ C′* **and**
Cong A B A′ B′ **and**
Cong B C B′ C′
shows *A C Le A′ C′*
⟨*proof*⟩

lemma *triangle-strict-reverse-inequality*:
assumes *A Out B D* **and**
Cong A C A D **and**
¬ *A Out B C*
shows *B D Lt B C*
⟨*proof*⟩

lemma *triangle-reverse-inequality*:
assumes *A Out B D* **and**
Cong A C A D
shows *B D Le B C*
⟨*proof*⟩

lemma *os3--lta*:
assumes *A B OS C D* **and**
B C OS A D **and**
A C OS B D
shows *B A C LtA B D C*
⟨*proof*⟩

lemma *bet-le--lt*:
assumes *Bet A D B* **and**
$A \neq D$ **and**
$D \neq B$ **and**
A C Le B C
shows *D C Lt B C*
⟨*proof*⟩

lemma *cong2--ncol*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
$A \neq C$ **and**
Cong A P B P **and**
Cong A P C P
shows ¬ *Col A B C*
⟨*proof*⟩

lemma *cong4-cop2--eq*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
$A \neq C$ **and**
Cong A P B P **and**
Cong A P C P **and**
Coplanar A B C P **and**
Cong A Q B Q **and**
Cong A Q C Q **and**
Coplanar A B C Q
shows $P = Q$
⟨*proof*⟩

lemma *t18-18-aux*:
assumes *Cong A B D E* **and**
Cong A C D F **and**
F D E LtA C A B **and**
¬ *Col A B C* **and**

$\neg$ *Col D E F* **and**
D F Le D E
shows *E F Lt B C*
⟨*proof*⟩

lemma *t18-18*:
assumes *Cong A B D E* **and**
Cong A C D F **and**
F D E LtA C A B
shows *E F Lt B C*
⟨*proof*⟩

lemma *t18-19*:
assumes $A \neq B$ **and**
$A \neq C$ **and**
Cong A B D E **and**
Cong A C D F **and**
E F Lt B C
shows *F D E LtA C A B*
⟨*proof*⟩

lemma *acute-trivial*:
assumes $A \neq B$
shows *Acute A B A*
⟨*proof*⟩

lemma *acute-not-per*:
assumes *Acute A B C*
shows $\neg$ *Per A B C*
⟨*proof*⟩

lemma *angle-bisector*:
assumes $A \neq B$ **and**
$C \neq B$
shows $\exists$ *P*. (*P InAngle A B C* $\wedge$ *P B A CongA P B C*)
⟨*proof*⟩

lemma *reflectl--conga*:
assumes $A \neq B$ **and**
$B \neq P$ **and**
P P′ ReflectL A B
shows *A B P CongA A B P′*
⟨*proof*⟩

lemma *conga-cop-out-reflectl--out*:
assumes $\neg$ *B Out A C* **and**
Coplanar A B C P **and**
P B A CongA P B C **and**
B Out A T **and**
T T′ ReflectL B P
shows *B Out C T′*
⟨*proof*⟩

lemma *col-conga-cop-reflectl--col*:
assumes $\neg$ *B Out A C* **and**
Coplanar A B C P **and**
P B A CongA P B C **and**
Col B A T **and**
T T′ ReflectL B P
shows *Col B C T′*
⟨*proof*⟩

lemma *conga2-cop2--col*:
assumes $\neg$ *B Out A C* **and**
P B A CongA P B C **and**
P′ B A CongA P′ B C **and**

Coplanar A B P P' **and**
Coplanar B C P P'
shows *Col B P P'*
⟨*proof*⟩

lemma *conga2-cop2--col-1*:
assumes ¬ *Col A B C* **and**
P B A CongA P B C **and**
P' B A CongA P' B C **and**
Coplanar A B C P **and**
Coplanar A B C P'
shows *Col B P P'*
⟨*proof*⟩

lemma *col-conga--conga*:
assumes *P B A CongA P B C* **and**
Col B P P' **and**
$B \neq P'$
shows *P' B A CongA P' B C*
⟨*proof*⟩

lemma *cop-inangle--ex-col-inangle*:
assumes ¬ *B Out A C* **and**
P InAngle A B C **and**
Coplanar A B C Q
shows ∃ *R*. (*R InAngle A B C* ∧ $P \neq R$ ∧ *Col P Q R*)
⟨*proof*⟩

lemma *col-inangle2--out*:
assumes ¬ *Bet A B C* **and**
P InAngle A B C **and**
Q InAngle A B C **and**
Col B P Q
shows *B Out P Q*
⟨*proof*⟩

lemma *inangle2--lea*:
assumes *P InAngle A B C* **and**
Q InAngle A B C
shows *P B Q LeA A B C*
⟨*proof*⟩

lemma *conga-inangle-per--acute*:
assumes *Per A B C* **and**
P InAngle A B C **and**
P B A CongA P B C
shows *Acute A B P*
⟨*proof*⟩

lemma *conga-inangle2-per--acute*:
assumes *Per A B C* **and**
P InAngle A B C **and**
P B A CongA P B C **and**
Q InAngle A B C
shows *Acute P B Q*
⟨*proof*⟩

lemma *lta-os--ts*:
assumes
A O1 P LtA A O1 B **and**
O1 A OS B P
shows *O1 P TS A B*
⟨*proof*⟩

lemma *bet--suppa*:
assumes $A \neq B$ **and**

$B \neq C$ **and**
$B \neq A'$ **and**
Bet $A\ B\ A'$
shows $A\ B\ C$ *SuppA* $C\ B\ A'$
⟨*proof*⟩

lemma *ex-suppa*:
assumes $A \neq B$ **and**
$B \neq C$
shows $\exists\ D\ E\ F.\ A\ B\ C$ *SuppA* $D\ E\ F$
⟨*proof*⟩

lemma *suppa-distincts*:
assumes $A\ B\ C$ *SuppA* $D\ E\ F$
shows $A \neq B \wedge B \neq C \wedge D \neq E \wedge E \neq F$
⟨*proof*⟩

lemma *suppa-right-comm*:
assumes $A\ B\ C$ *SuppA* $D\ E\ F$
shows $A\ B\ C$ *SuppA* $F\ E\ D$
⟨*proof*⟩

lemma *suppa-left-comm*:
assumes $A\ B\ C$ *SuppA* $D\ E\ F$
shows $C\ B\ A$ *SuppA* $D\ E\ F$
⟨*proof*⟩

lemma *suppa-comm*:
assumes $A\ B\ C$ *SuppA* $D\ E\ F$
shows $C\ B\ A$ *SuppA* $F\ E\ D$
⟨*proof*⟩

lemma *suppa-sym*:
assumes $A\ B\ C$ *SuppA* $D\ E\ F$
shows $D\ E\ F$ *SuppA* $A\ B\ C$
⟨*proof*⟩

lemma *conga2-suppa--suppa*:
assumes $A\ B\ C$ *CongA* $A'\ B'\ C'$ **and**
$D\ E\ F$ *CongA* $D'\ E'\ F'$ **and**
$A\ B\ C$ *SuppA* $D\ E\ F$
shows $A'\ B'\ C'$ *SuppA* $D'\ E'\ F'$
⟨*proof*⟩

lemma *suppa2--conga456*:
assumes $A\ B\ C$ *SuppA* $D\ E\ F$ **and**
$A\ B\ C$ *SuppA* $D'\ E'\ F'$
shows $D\ E\ F$ *CongA* $D'\ E'\ F'$
⟨*proof*⟩

lemma *suppa2--conga123*:
assumes $A\ B\ C$ *SuppA* $D\ E\ F$ **and**
$A'\ B'\ C'$ *SuppA* $D\ E\ F$
shows $A\ B\ C$ *CongA* $A'\ B'\ C'$
⟨*proof*⟩

lemma *bet-out--suppa*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
Bet $A\ B\ C$ **and**
E *Out* $D\ F$
shows $A\ B\ C$ *SuppA* $D\ E\ F$
⟨*proof*⟩

lemma *bet-suppa--out*:
assumes *Bet* $A\ B\ C$ **and**

A B C SuppA D E F
shows *E Out D F*
⟨*proof*⟩

lemma *out-suppa--bet*:
assumes *B Out A C* **and**
A B C SuppA D E F
shows *Bet D E F*
⟨*proof*⟩

lemma *per-suppa--per*:
assumes *Per A B C* **and**
A B C SuppA D E F
shows *Per D E F*
⟨*proof*⟩

lemma *per2--suppa*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
$D \neq E$ **and**
$E \neq F$ **and**
Per A B C **and**
Per D E F
shows *A B C SuppA D E F*
⟨*proof*⟩

lemma *suppa--per*:
assumes *A B C SuppA A B C*
shows *Per A B C*
⟨*proof*⟩

lemma *acute-suppa--obtuse*:
assumes *Acute A B C* **and**
A B C SuppA D E F
shows *Obtuse D E F*
⟨*proof*⟩

lemma *obtuse-suppa--acute*:
assumes *Obtuse A B C* **and**
A B C SuppA D E F
shows *Acute D E F*
⟨*proof*⟩

lemma *lea-suppa2--lea*:
assumes *A B C SuppA* $A' B' C'$ **and**
D E F SuppA $D' E' F'$
A B C LeA D E F
shows $D' E' F'$ *LeA* $A' B' C'$
⟨*proof*⟩

lemma *lta-suppa2--lta*:
assumes *A B C SuppA* $A' B' C'$
and *D E F SuppA* $D' E' F'$
and *A B C LtA D E F*
shows $D' E' F'$ *LtA* $A' B' C'$
⟨*proof*⟩

lemma *suppa-dec*:
A B C SuppA D E F $\vee \neg$ *A B C SuppA D E F*
⟨*proof*⟩

lemma *acute-one-side-aux*:
assumes *C A OS P B* **and**
Acute A C P **and**
C A Perp B C
shows *C B OS A P*

⟨*proof*⟩

lemma *acute-one-side-aux0*:
assumes *Col A C P* **and**
Acute A C P **and**
C A Perp B C
shows *C B OS A P*
⟨*proof*⟩

lemma *acute-cop-perp--one-side*:
assumes *Acute A C P* **and**
C A Perp B C **and**
Coplanar A B C P
shows *C B OS A P*
⟨*proof*⟩

lemma *acute--not-obtuse*:
assumes *Acute A B C*
shows ¬ *Obtuse A B C*
⟨*proof*⟩

3.10.2 Sum of angles

lemma *suma-distincts*:
assumes *A B C D E F SumA G H I*
shows $A \neq B \land B \neq C \land D \neq E \land E \neq F \land G \neq H \land H \neq I$
⟨*proof*⟩

lemma *trisuma-distincts*:
assumes *A B C TriSumA D E F*
shows $A \neq B \land B \neq C \land A \neq C \land D \neq E \land E \neq F$
⟨*proof*⟩

lemma *ex-suma*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
$D \neq E$ **and**
$E \neq F$
shows ∃ *G H I. A B C D E F SumA G H I*
⟨*proof*⟩

lemma *suma2--conga*:
assumes *A B C D E F SumA G H I* **and**
A B C D E F SumA G′ H′ I′
shows *G H I CongA G′ H′ I′*
⟨*proof*⟩

lemma *suma-sym*:
assumes *A B C D E F SumA G H I*
shows *D E F A B C SumA G H I*
⟨*proof*⟩

lemma *conga3-suma--suma*:
assumes *A B C D E F SumA G H I* **and**
A B C CongA A′ B′ C′ **and**
D E F CongA D′ E′ F′ **and**
G H I CongA G′ H′ I′
shows *A′ B′ C′ D′ E′ F′ SumA G′ H′ I′*
⟨*proof*⟩

lemma *out6-suma--suma*:
assumes *A B C D E F SumA G H I* **and**
B Out A A′ **and**
B Out C C′ **and**
E Out D D′ **and**
E Out F F′ **and**

H *Out* G G' **and**
H *Out* I I'
shows A' B C' D' E F' *SumA* G' H I'
⟨*proof*⟩

lemma *out546-suma--conga*:
assumes A B C D E F *SumA* G H I **and**
E *Out* D F
shows A B C *CongA* G H I
⟨*proof*⟩

lemma *out546--suma*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
E *Out* D F
shows A B C D E F *SumA* A B C
⟨*proof*⟩

lemma *out213-suma--conga*:
assumes A B C D E F *SumA* G H I **and**
B *Out* A C
shows D E F *CongA* G H I
⟨*proof*⟩

lemma *out213--suma*:
assumes $D \neq E$ **and**
$E \neq F$ **and**
B *Out* A C
shows A B C D E F *SumA* D E F
⟨*proof*⟩

lemma *suma-left-comm*:
assumes A B C D E F *SumA* G H I
shows C B A D E F *SumA* G H I
⟨*proof*⟩

lemma *suma-middle-comm*:
assumes A B C D E F *SumA* G H I
shows A B C F E D *SumA* G H I
⟨*proof*⟩

lemma *suma-right-comm*:
assumes A B C D E F *SumA* G H I
shows A B C D E F *SumA* I H G
⟨*proof*⟩

lemma *suma-comm*:
assumes A B C D E F *SumA* G H I
shows C B A F E D *SumA* I H G
⟨*proof*⟩

lemma *ts--suma*:
assumes A B *TS* C D
shows C B A A B D *SumA* C B D
⟨*proof*⟩

lemma *ts--suma-1*:
assumes A B *TS* C D
shows C A B B A D *SumA* C A D
⟨*proof*⟩

lemma *inangle--suma*:
assumes P *InAngle* A B C
shows A B P P B C *SumA* A B C
⟨*proof*⟩

lemma *bet--suma*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
$P \neq B$ **and** *Bet A B C*
shows *A B P P B C SumA A B C*
⟨*proof*⟩

lemma *sams-chara*:
assumes $A \neq B$ **and**
$A' \neq B$ **and**
Bet A B A'
shows $SAMS\ A\ B\ C\ D\ E\ F \longleftrightarrow D\ E\ F\ LeA\ C\ B\ A'$
⟨*proof*⟩

lemma *sams-distincts*:
assumes *SAMS A B C D E F*
shows $A \neq B \wedge B \neq C \wedge D \neq E \wedge E \neq F$
⟨*proof*⟩

lemma *sams-sym*:
assumes *SAMS A B C D E F*
shows *SAMS D E F A B C*
⟨*proof*⟩

lemma *sams-right-comm*:
assumes *SAMS A B C D E F*
shows *SAMS A B C F E D*
⟨*proof*⟩

lemma *sams-left-comm*:
assumes *SAMS A B C D E F*
shows *SAMS C B A D E F*
⟨*proof*⟩

lemma *sams-comm*:
assumes *SAMS A B C D E F*
shows *SAMS C B A F E D*
⟨*proof*⟩

lemma *conga2-sams--sams*:
assumes $A\ B\ C\ CongA\ A'\ B'\ C'$ **and**
$D\ E\ F\ CongA\ D'\ E'\ F'$ **and**
SAMS A B C D E F
shows $SAMS\ A'\ B'\ C'\ D'\ E'\ F'$
⟨*proof*⟩

lemma *out546--sams*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
E Out D F
shows *SAMS A B C D E F*
⟨*proof*⟩

lemma *out213--sams*:
assumes $D \neq E$ **and**
$E \neq F$ **and**
B Out A C
shows *SAMS A B C D E F*
⟨*proof*⟩

lemma *bet-suma--sams*:
assumes *A B C D E F SumA G H I* **and**
Bet G H I
shows *SAMS A B C D E F*
⟨*proof*⟩

lemma *bet--sams*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
$P \neq B$ **and**
Bet A B C
shows *SAMS A B P P B C*
⟨*proof*⟩

lemma *suppa--sams*:
assumes *A B C SuppA D E F*
shows *SAMS A B C D E F*
⟨*proof*⟩

lemma *os-ts--sams*:
assumes *B P TS A C* **and**
A B OS P C
shows *SAMS A B P P B C*
⟨*proof*⟩

lemma *os2--sams*:
assumes *A B OS P C* **and**
C B OS P A
shows *SAMS A B P P B C*
⟨*proof*⟩

lemma *inangle--sams*:
assumes *P InAngle A B C*
shows *SAMS A B P P B C*
⟨*proof*⟩

lemma *sams-suma--lea123789*:
assumes *A B C D E F SumA G H I* **and**
SAMS A B C D E F
shows *A B C LeA G H I*
⟨*proof*⟩

lemma *sams-suma--lea456789*:
assumes *A B C D E F SumA G H I* **and**
SAMS A B C D E F
shows *D E F LeA G H I*
⟨*proof*⟩

lemma *sams-lea2--sams*:
assumes *SAMS* A' B' C' D' E' F' **and**
A B C LeA A' B' C' **and**
D E F LeA D' E' F'
shows *SAMS A B C D E F*
⟨*proof*⟩

lemma *sams-lea456-suma2--lea*:
assumes *D E F LeA* D' E' F' **and**
SAMS A B C D' E' F' **and**
A B C D E F SumA G H I **and**
A B C D' E' F' *SumA* G' H' I'
shows *G H I LeA* G' H' I'
⟨*proof*⟩

lemma *sams-lea123-suma2--lea*:
assumes *A B C LeA* A' B' C' **and**
SAMS A' B' C' *D E F* **and**
A B C D E F SumA G H I **and**
A' B' C' *D E F SumA* G' H' I'
shows *G H I LeA* G' H' I'
⟨*proof*⟩

lemma *sams-lea2-suma2--lea*:

assumes $A\ B\ C\ LeA\ A'\ B'\ C'$ **and**
$D\ E\ F\ LeA\ D'\ E'\ F'$ **and**
$SAMS\ A'\ B'\ C'\ D'\ E'\ F'$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
$A'\ B'\ C'\ D'\ E'\ F'\ SumA\ G'\ H'\ I'$
shows $G\ H\ I\ LeA\ G'\ H'\ I'$
⟨*proof*⟩

lemma *sams2-suma2--conga456*:
assumes $SAMS\ A\ B\ C\ D\ E\ F$ **and**
$SAMS\ A\ B\ C\ D'\ E'\ F'$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
$A\ B\ C\ D'\ E'\ F'\ SumA\ G\ H\ I$
shows $D\ E\ F\ CongA\ D'\ E'\ F'$
⟨*proof*⟩

lemma *sams2-suma2--conga123*:
assumes $SAMS\ A\ B\ C\ D\ E\ F$ **and**
$SAMS\ A'\ B'\ C'\ D\ E\ F$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
$A'\ B'\ C'\ D\ E\ F\ SumA\ G\ H\ I$
shows $A\ B\ C\ CongA\ A'\ B'\ C'$
⟨*proof*⟩

lemma *suma-assoc-1*:
assumes $SAMS\ A\ B\ C\ D\ E\ F$ **and**
$SAMS\ D\ E\ F\ G\ H\ I$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ A'\ B'\ C'$ **and**
$D\ E\ F\ G\ H\ I\ SumA\ D'\ E'\ F'$ **and**
$A'\ B'\ C'\ G\ H\ I\ SumA\ K\ L\ M$
shows $A\ B\ C\ D'\ E'\ F'\ SumA\ K\ L\ M$
⟨*proof*⟩

lemma *suma-assoc-2*:
assumes $SAMS\ A\ B\ C\ D\ E\ F$ **and**
$SAMS\ D\ E\ F\ G\ H\ I$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ A'\ B'\ C'$ **and**
$D\ E\ F\ G\ H\ I\ SumA\ D'\ E'\ F'$ **and**
$A\ B\ C\ D'\ E'\ F'\ SumA\ K\ L\ M$
shows $A'\ B'\ C'\ G\ H\ I\ SumA\ K\ L\ M$
⟨*proof*⟩

lemma *suma-assoc*:
assumes $SAMS\ A\ B\ C\ D\ E\ F$ **and**
$SAMS\ D\ E\ F\ G\ H\ I$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ A'\ B'\ C'$ **and**
$D\ E\ F\ G\ H\ I\ SumA\ D'\ E'\ F'$
shows
$A'\ B'\ C'\ G\ H\ I\ SumA\ K\ L\ M \longleftrightarrow A\ B\ C\ D'\ E'\ F'\ SumA\ K\ L\ M$
⟨*proof*⟩

lemma *sams-assoc-1*:
assumes $SAMS\ A\ B\ C\ D\ E\ F$ **and**
$SAMS\ D\ E\ F\ G\ H\ I$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ A'\ B'\ C'$ **and**
$D\ E\ F\ G\ H\ I\ SumA\ D'\ E'\ F'$ **and**
$SAMS\ A'\ B'\ C'\ G\ H\ I$
shows $SAMS\ A\ B\ C\ D'\ E'\ F'$
⟨*proof*⟩

lemma *sams-assoc-2*:
assumes $SAMS\ A\ B\ C\ D\ E\ F$ **and**
$SAMS\ D\ E\ F\ G\ H\ I$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ A'\ B'\ C'$ **and**
$D\ E\ F\ G\ H\ I\ SumA\ D'\ E'\ F'$ **and**
$SAMS\ A\ B\ C\ D'\ E'\ F'$

shows $SAMS\ A'\ B'\ C'\ G\ H\ I$
⟨*proof*⟩

lemma *sams-assoc*:
assumes $SAMS\ A\ B\ C\ D\ E\ F$ **and**
$SAMS\ D\ E\ F\ G\ H\ I$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ A'\ B'\ C'$ **and**
$D\ E\ F\ G\ H\ I\ SumA\ D'\ E'\ F'$
shows $(SAMS\ A'\ B'\ C'\ G\ H\ I) \longleftrightarrow (SAMS\ A\ B\ C\ D'\ E'\ F')$
⟨*proof*⟩

lemma *sams-nos--nts*:
assumes $SAMS\ A\ B\ C\ C\ B\ J$ **and**
$\neg\ B\ C\ OS\ A\ J$
shows $\neg\ \ A\ B\ TS\ C\ J$
⟨*proof*⟩

lemma *conga-sams-nos--nts*:
assumes $SAMS\ A\ B\ C\ D\ E\ F$ **and**
$C\ B\ J\ CongA\ D\ E\ F$ **and**
$\neg\ B\ C\ OS\ A\ J$
shows $\neg\ A\ B\ TS\ C\ J$
⟨*proof*⟩

lemma *sams-lea2-suma2--conga123*:
assumes $A\ B\ C\ LeA\ A'\ B'\ C'$ **and**
$D\ E\ F\ LeA\ D'\ E'\ F'$ **and**
$SAMS\ A'\ B'\ C'\ D'\ E'\ F'$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
$A'\ B'\ C'\ D'\ E'\ F'\ SumA\ G\ H\ I$
shows $A\ B\ C\ CongA\ A'\ B'\ C'$
⟨*proof*⟩

lemma *sams-lea2-suma2--conga456*:
assumes $A\ B\ C\ LeA\ A'\ B'\ C'$ **and**
$D\ E\ F\ LeA\ D'\ E'\ F'$ **and**
$SAMS\ A'\ B'\ C'\ D'\ E'\ F'$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
$A'\ B'\ C'\ D'\ E'\ F'\ SumA\ G\ H\ I$
shows $D\ E\ F\ CongA\ D'\ E'\ F'$
⟨*proof*⟩

lemma *sams-suma--out213*:
assumes $A\ B\ C\ D\ E\ F\ SumA\ D\ E\ F$ **and**
$SAMS\ A\ B\ C\ D\ E\ F$
shows $B\ Out\ A\ C$
⟨*proof*⟩

lemma *sams-suma--out546*:
assumes $A\ B\ C\ D\ E\ F\ SumA\ A\ B\ C$ **and**
$SAMS\ A\ B\ C\ D\ E\ F$
shows $E\ Out\ D\ F$
⟨*proof*⟩

lemma *sams-lea-lta123-suma2--lta*:
assumes $A\ B\ C\ LtA\ A'\ B'\ C'$ **and**
$D\ E\ F\ LeA\ D'\ E'\ F'$ **and**
$SAMS\ A'\ B'\ C'\ D'\ E'\ F'$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
$A'\ B'\ C'\ D'\ E'\ F'\ SumA\ G'\ H'\ I'$
shows $G\ H\ I\ LtA\ G'\ H'\ I'$
⟨*proof*⟩

lemma *sams-lea-lta456-suma2--lta*:
assumes $A\ B\ C\ LeA\ A'\ B'\ C'$ **and**
$D\ E\ F\ LtA\ D'\ E'\ F'$ **and**

$SAMS\ A'\ B'\ C'\ D'\ E'\ F'$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
$A'\ B'\ C'\ D'\ E'\ F'\ SumA\ G'\ H'\ I'$
shows $G\ H\ I\ LtA\ G'\ H'\ I'$
⟨*proof*⟩

lemma *sams-lta2-suma2--lta*:
assumes $A\ B\ C\ LtA\ A'\ B'\ C'$ **and**
$D\ E\ F\ LtA\ D'\ E'\ F'$ **and**
$SAMS\ A'\ B'\ C'\ D'\ E'\ F'$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
$A'\ B'\ C'\ D'\ E'\ F'\ SumA\ G'\ H'\ I'$
shows $G\ H\ I\ LtA\ G'\ H'\ I'$
⟨*proof*⟩

lemma *sams-lea2-suma2--lea123*:
assumes $D'\ E'\ F'\ LeA\ D\ E\ F$ **and**
$G\ H\ I\ LeA\ G'\ H'\ I'$ **and**
$SAMS\ A\ B\ C\ D\ E\ F$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
$A'\ B'\ C'\ D'\ E'\ F'\ SumA\ G'\ H'\ I'$
shows $A\ B\ C\ LeA\ A'\ B'\ C'$
⟨*proof*⟩

lemma *sams-lea2-suma2--lea456*:
assumes $A'\ B'\ C'\ LeA\ A\ B\ C$ **and**
$G\ H\ I\ LeA\ G'\ H'\ I'$ **and**
$SAMS\ A\ B\ C\ D\ E\ F$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
$A'\ B'\ C'\ D'\ E'\ F'\ SumA\ G'\ H'\ I'$
shows $D\ E\ F\ LeA\ D'\ E'\ F'$
⟨*proof*⟩

lemma *sams-lea-lta456-suma2--lta123*:
assumes $D'\ E'\ F'\ LtA\ D\ E\ F$ **and**
$G\ H\ I\ LeA\ G'\ H'\ I'$ **and**
$SAMS\ A\ B\ C\ D\ E\ F$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
$A'\ B'\ C'\ D'\ E'\ F'\ SumA\ G'\ H'\ I'$
shows $A\ B\ C\ LtA\ A'\ B'\ C'$
⟨*proof*⟩

lemma *sams-lea-lta123-suma2--lta456*:
assumes $A'\ B'\ C'\ LtA\ A\ B\ C$ **and**
$G\ H\ I\ LeA\ G'\ H'\ I'$ **and**
$SAMS\ A\ B\ C\ D\ E\ F$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
$A'\ B'\ C'\ D'\ E'\ F'\ SumA\ G'\ H'\ I'$
shows $D\ E\ F\ LtA\ D'\ E'\ F'$
⟨*proof*⟩

lemma *sams-lea-lta789-suma2--lta123*:
assumes $D'\ E'\ F'\ LeA\ D\ E\ F$ **and**
$G\ H\ I\ LtA\ G'\ H'\ I'$ **and**
$SAMS\ A\ B\ C\ D\ E\ F$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
$A'\ B'\ C'\ D'\ E'\ F'\ SumA\ G'\ H'\ I'$
shows $A\ B\ C\ LtA\ A'\ B'\ C'$
⟨*proof*⟩

lemma *sams-lea-lta789-suma2--lta456*:
assumes $A'\ B'\ C'\ LeA\ A\ B\ C$ **and**
$G\ H\ I\ LtA\ G'\ H'\ I'$ **and**
$SAMS\ A\ B\ C\ D\ E\ F$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
$A'\ B'\ C'\ D'\ E'\ F'\ SumA\ G'\ H'\ I'$

shows $D\ E\ F\ LtA\ D'\ E'\ F'$
⟨*proof*⟩

lemma *sams-lta2-suma2--lta123*:
assumes $D'\ E'\ F'\ LtA\ D\ E\ F$ **and**
$G\ H\ I\ LtA\ G'\ H'\ I'$ **and**
$SAMS\ A\ B\ C\ D\ E\ F$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
$A'\ B'\ C'\ D'\ E'\ F'\ SumA\ G'\ H'\ I'$
shows $A\ B\ C\ LtA\ A'\ B'\ C'$
⟨*proof*⟩

lemma *sams-lta2-suma2--lta456*:
assumes $A'\ B'\ C'\ LtA\ A\ B\ C$ **and**
$G\ H\ I\ LtA\ G'\ H'\ I'$ **and**
$SAMS\ A\ B\ C\ D\ E\ F$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
$A'\ B'\ C'\ D'\ E'\ F'\ SumA\ G'\ H'\ I'$
shows $D\ E\ F\ LtA\ D'\ E'\ F'$
⟨*proof*⟩

lemma *sams123231*:
assumes $A \neq B$ **and**
$A \neq C$ **and**
$B \neq C$
shows $SAMS\ A\ B\ C\ B\ C\ A$
⟨*proof*⟩

lemma *col-suma--col*:
assumes $Col\ D\ E\ F$ **and**
$A\ B\ C\ B\ C\ A\ SumA\ D\ E\ F$
shows $Col\ A\ B\ C$
⟨*proof*⟩

lemma *ncol-suma--ncol*:
assumes $\neg\ Col\ A\ B\ C$ **and**
$A\ B\ C\ B\ C\ A\ SumA\ D\ E\ F$
shows $\neg\ Col\ D\ E\ F$
⟨*proof*⟩

lemma *per2-suma--bet*:
assumes $Per\ A\ B\ C$ **and**
$Per\ D\ E\ F$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$
shows $Bet\ G\ H\ I$
⟨*proof*⟩

lemma *bet-per2--suma*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
$D \neq E$ **and**
$E \neq F$ **and**
$G \neq H$ **and**
$H \neq I$ **and**
$Per\ A\ B\ C$ **and**
$Per\ D\ E\ F$ **and**
$Bet\ G\ H\ I$
shows $A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$
⟨*proof*⟩

lemma *per2--sams*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
$D \neq E$ **and**
$E \neq F$ **and**
$Per\ A\ B\ C$ **and**

Per D E F
shows *SAMS A B C D E F*
⟨*proof*⟩

lemma *bet-per-suma--per456*:
assumes *Per A B C* **and**
Bet G H I **and**
A B C D E F SumA G H I
shows *Per D E F*
⟨*proof*⟩

lemma *bet-per-suma--per123*:
assumes *Per D E F* **and**
Bet G H I **and**
A B C D E F SumA G H I
shows *Per A B C*
⟨*proof*⟩

lemma *bet-suma--per*:
assumes *Bet D E F* **and**
A B C A B C SumA D E F
shows *Per A B C*
⟨*proof*⟩

lemma *acute--sams*:
assumes *Acute A B C*
shows *SAMS A B C A B C*
⟨*proof*⟩

lemma *acute-suma--nbet*:
assumes *Acute A B C* **and**
A B C A B C SumA D E F
shows ¬ *Bet D E F*
⟨*proof*⟩

lemma *acute2--sams*:
assumes *Acute A B C* **and**
Acute D E F
shows *SAMS A B C D E F*
⟨*proof*⟩

lemma *acute2-suma--nbet-a*:
assumes *Acute A B C* **and**
D E F LeA A B C **and**
A B C D E F SumA G H I
shows ¬ *Bet G H I*
⟨*proof*⟩

lemma *acute2-suma--nbet*:
assumes *Acute A B C* **and**
Acute D E F **and**
A B C D E F SumA G H I
shows ¬ *Bet G H I*
⟨*proof*⟩

lemma *acute-per--sams*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
Per A B C **and**
Acute D E F
shows *SAMS A B C D E F*
⟨*proof*⟩

lemma *acute-per-suma--nbet*:
assumes $A \neq B$ **and**
$B \neq C$ **and**

Per A B C **and**
Acute D E F **and**
A B C D E F SumA G H I
shows ¬ *Bet G H I*
⟨*proof*⟩

lemma *obtuse--nsams*:
assumes *Obtuse A B C*
shows ¬ *SAMS A B C A B C*
⟨*proof*⟩

lemma *nbet-sams-suma--acute*:
assumes ¬ *Bet D E F* **and**
SAMS A B C A B C **and**
A B C A B C SumA D E F
shows *Acute A B C*
⟨*proof*⟩

lemma *nsams--obtuse*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
¬ *SAMS A B C A B C*
shows *Obtuse A B C*
⟨*proof*⟩

lemma *sams2-suma2--conga*:
assumes *SAMS A B C A B C* **and**
A B C A B C SumA D E F **and**
SAMS A' B' C' A' B' C' **and**
A' B' C' A' B' C' *SumA D E F*
shows *A B C CongA* A' B' C'
⟨*proof*⟩

lemma *acute2-suma2--conga*:
assumes *Acute A B C* **and**
A B C A B C SumA D E F **and**
Acute A' B' C' **and**
A' B' C' A' B' C' *SumA D E F*
shows *A B C CongA* A' B' C'
⟨*proof*⟩

lemma *bet2-suma--out*:
assumes *Bet A B C* **and**
Bet D E F **and**
A B C D E F SumA G H I
shows *H Out G I*
⟨*proof*⟩

lemma *col2-suma--col*:
assumes *Col A B C* **and**
Col D E F **and**
A B C D E F SumA G H I
shows *Col G H I*
⟨*proof*⟩

lemma *suma-suppa--bet*:
assumes *A B C SuppA D E F* **and**
A B C D E F SumA G H I
shows *Bet G H I*
⟨*proof*⟩

lemma *bet-suppa--suma*:
assumes $G \neq H$ **and**
$H \neq I$ **and**
A B C SuppA D E F **and**
Bet G H I

shows $A\ B\ C\ D\ E\ F$ *SumA* $G\ H\ I$
⟨*proof*⟩

lemma *bet-suma--suppa*:
assumes $A\ B\ C\ D\ E\ F$ *SumA* $G\ H\ I$ **and**
Bet $G\ H\ I$
shows $A\ B\ C$ *SuppA* $D\ E\ F$
⟨*proof*⟩

lemma *bet2-suma--suma*:
assumes $A' \neq B$ **and**
$D' \neq E$ **and**
Bet $A\ B\ A'$ **and**
Bet $D\ E\ D'$ **and**
$A\ B\ C\ D\ E\ F$ *SumA* $G\ H\ I$
shows $A'\ B\ C\ D'\ E\ F$ *SumA* $G\ H\ I$
⟨*proof*⟩

lemma *suma-suppa2--suma*:
assumes $A\ B\ C$ *SuppA* $A'\ B'\ C'$ **and**
$D\ E\ F$ *SuppA* $D'\ E'\ F'$ **and**
$A\ B\ C\ D\ E\ F$ *SumA* $G\ H\ I$
shows $A'\ B'\ C'\ D'\ E'\ F'$ *SumA* $G\ H\ I$
⟨*proof*⟩

lemma *suma2-obtuse2--conga*:
assumes *Obtuse* $A\ B\ C$ **and**
$A\ B\ C\ A\ B\ C$ *SumA* $D\ E\ F$ **and**
Obtuse $A'\ B'\ C'$ **and**
$A'\ B'\ C'\ A'\ B'\ C'$ *SumA* $D\ E\ F$
shows $A\ B\ C$ *CongA* $A'\ B'\ C'$
⟨*proof*⟩

lemma *bet-suma2--or-conga*:
assumes $A0 \neq B$ **and**
Bet $A\ B\ A0$ **and**
$A\ B\ C\ A\ B\ C$ *SumA* $D\ E\ F$ **and**
$A'\ B'\ C'\ A'\ B'\ C'$ *SumA* $D\ E\ F$
shows $A\ B\ C$ *CongA* $A'\ B'\ C' \vee A0\ B\ C$ *CongA* $A'\ B'\ C'$
⟨*proof*⟩

lemma *suma2--or-conga-suppa*:
assumes $A\ B\ C\ A\ B\ C$ *SumA* $D\ E\ F$ **and**
$A'\ B'\ C'\ A'\ B'\ C'$ *SumA* $D\ E\ F$
shows $A\ B\ C$ *CongA* $A'\ B'\ C' \vee A\ B\ C$ *SuppA* $A'\ B'\ C'$
⟨*proof*⟩

lemma *ex-trisuma*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
$A \neq C$
shows $\exists\ D\ E\ F.\ A\ B\ C$ *TriSumA* $D\ E\ F$
⟨*proof*⟩

lemma *trisuma-perm-231*:
assumes $A\ B\ C$ *TriSumA* $D\ E\ F$
shows $B\ C\ A$ *TriSumA* $D\ E\ F$
⟨*proof*⟩

lemma *trisuma-perm-312*:
assumes $A\ B\ C$ *TriSumA* $D\ E\ F$
shows $C\ A\ B$ *TriSumA* $D\ E\ F$
⟨*proof*⟩

lemma *trisuma-perm-321*:
assumes $A\ B\ C$ *TriSumA* $D\ E\ F$

shows $C\ B\ A\ \mathit{TriSumA}\ D\ E\ F$
⟨*proof*⟩

lemma *trisuma-perm-213*:
assumes $A\ B\ C\ \mathit{TriSumA}\ D\ E\ F$
shows $B\ A\ C\ \mathit{TriSumA}\ D\ E\ F$
⟨*proof*⟩

lemma *trisuma-perm-132*:
assumes $A\ B\ C\ \mathit{TriSumA}\ D\ E\ F$
shows $A\ C\ B\ \mathit{TriSumA}\ D\ E\ F$
⟨*proof*⟩

lemma *conga-trisuma--trisuma*:
assumes $A\ B\ C\ \mathit{TriSumA}\ D\ E\ F$ **and**
$D\ E\ F\ \mathit{CongA}\ D'\ E'\ F'$
shows $A\ B\ C\ \mathit{TriSumA}\ D'\ E'\ F'$
⟨*proof*⟩

lemma *trisuma2--conga*:
assumes $A\ B\ C\ \mathit{TriSumA}\ D\ E\ F$ **and**
$A\ B\ C\ \mathit{TriSumA}\ D'\ E'\ F'$
shows $D\ E\ F\ \mathit{CongA}\ D'\ E'\ F'$
⟨*proof*⟩

lemma *conga3-trisuma--trisuma*:
assumes $A\ B\ C\ \mathit{TriSumA}\ D\ E\ F$ **and**
$A\ B\ C\ \mathit{CongA}\ A'\ B'\ C'$ **and**
$B\ C\ A\ \mathit{CongA}\ B'\ C'\ A'$ **and**
$C\ A\ B\ \mathit{CongA}\ C'\ A'\ B'$
shows $A'\ B'\ C'\ \mathit{TriSumA}\ D\ E\ F$
⟨*proof*⟩

lemma *col-trisuma--bet*:
assumes $\mathit{Col}\ A\ B\ C$ **and**
$A\ B\ C\ \mathit{TriSumA}\ P\ Q\ R$
shows $\mathit{Bet}\ P\ Q\ R$
⟨*proof*⟩

lemma *suma-dec*:
$A\ B\ C\ D\ E\ F\ \mathit{SumA}\ G\ H\ I \lor \neg\ A\ B\ C\ D\ E\ F\ \mathit{SumA}\ G\ H\ I$ ⟨*proof*⟩

lemma *sams-dec*:
$\mathit{SAMS}\ A\ B\ C\ D\ E\ F \lor \neg\ \mathit{SAMS}\ A\ B\ C\ D\ E\ F$ ⟨*proof*⟩

lemma *trisuma-dec*:
$A\ B\ C\ \mathit{TriSumA}\ P\ Q\ R \lor \neg\ A\ B\ C\ \mathit{TriSumA}\ P\ Q\ R$
⟨*proof*⟩

3.11 Parallelism

lemma *par-reflexivity*:
assumes $A \neq B$
shows $A\ B\ \mathit{Par}\ A\ B$
⟨*proof*⟩

lemma *par-strict-irreflexivity*:
$\neg\ A\ B\ \mathit{ParStrict}\ A\ B$
⟨*proof*⟩

lemma *not-par-strict-id*:
$\neg\ A\ B\ \mathit{ParStrict}\ A\ C$
⟨*proof*⟩

lemma *par-id*:
assumes $A\ B\ \mathit{Par}\ A\ C$

shows *Col A B C*
⟨*proof*⟩

lemma *par-strict-not-col-1*:
assumes *A B ParStrict C D*
shows ¬ *Col A B C*
⟨*proof*⟩

lemma *par-strict-not-col-2*:
assumes *A B ParStrict C D*
shows ¬ *Col B C D*
⟨*proof*⟩

lemma *par-strict-not-col-3*:
assumes *A B ParStrict C D*
shows ¬ *Col C D A*
⟨*proof*⟩

lemma *par-strict-not-col-4*:
assumes *A B ParStrict C D*
shows ¬ *Col A B D*
⟨*proof*⟩

lemma *par-id-1*:
assumes *A B Par A C*
shows *Col B A C*
⟨*proof*⟩

lemma *par-id-2*:
assumes *A B Par A C*
shows *Col B C A*
⟨*proof*⟩

lemma *par-id-3*:
assumes *A B Par A C*
shows *Col A C B*
⟨*proof*⟩

lemma *par-id-4*:
assumes *A B Par A C*
shows *Col C B A*
⟨*proof*⟩

lemma *par-id-5*:
assumes *A B Par A C*
shows *Col C A B*
⟨*proof*⟩

lemma *par-strict-symmetry*:
assumes *A B ParStrict C D*
shows *C D ParStrict A B*
⟨*proof*⟩

lemma *par-symmetry*:
assumes *A B Par C D*
shows *C D Par A B*
⟨*proof*⟩

lemma *par-left-comm*:
assumes *A B Par C D*
shows *B A Par C D*
⟨*proof*⟩

lemma *par-right-comm*:
assumes *A B Par C D*
shows *A B Par D C*

⟨*proof*⟩

lemma *par-comm*:
assumes *A B Par C D*
shows *B A Par D C*
⟨*proof*⟩

lemma *par-strict-left-comm*:
assumes *A B ParStrict C D*
shows *B A ParStrict C D*
⟨*proof*⟩

lemma *par-strict-right-comm*:
assumes *A B ParStrict C D*
shows *A B ParStrict D C*
⟨*proof*⟩

lemma *par-strict-comm*:
assumes *A B ParStrict C D*
shows *B A ParStrict D C*
⟨*proof*⟩

lemma *par-strict-neq1*:
assumes *A B ParStrict C D*
shows $A \neq B$
⟨*proof*⟩

lemma *par-strict-neq2*:
assumes *A B ParStrict C D*
shows $C \neq D$
⟨*proof*⟩

lemma *par-neq1*:
assumes *A B Par C D*
shows $A \neq B$
⟨*proof*⟩

lemma *par-neq2*:
assumes *A B Par C D*
shows $C \neq D$
⟨*proof*⟩

lemma *Par-cases*:
assumes *A B Par C D* ∨ *B A Par C D* ∨ *A B Par D C* ∨ *B A Par D C* ∨ *C D Par A B* ∨ *C D Par B A* ∨ *D C Par A B* ∨ *D C Par B A*
shows *A B Par C D*
⟨*proof*⟩

lemma *Par-perm*:
assumes *A B Par C D*
shows *A B Par C D* ∧ *B A Par C D* ∧ *A B Par D C* ∧ *B A Par D C* ∧ *C D Par A B* ∧ *C D Par B A* ∧ *D C Par A B* ∧ *D C Par B A*
⟨*proof*⟩

lemma *Par-strict-cases*:
assumes *A B ParStrict C D* ∨ *B A ParStrict C D* ∨ *A B ParStrict D C* ∨ *B A ParStrict D C* ∨ *C D ParStrict A B* ∨ *C D ParStrict B A* ∨ *D C ParStrict A B* ∨ *D C ParStrict B A*
shows *A B ParStrict C D*
⟨*proof*⟩

lemma *Par-strict-perm*:
assumes *A B ParStrict C D*
shows *A B ParStrict C D* ∧ *B A ParStrict C D* ∧ *A B ParStrict D C* ∧ *B A ParStrict D C* ∧ *C D ParStrict A B* ∧ *C D ParStrict B A* ∧ *D C ParStrict A B* ∧ *D C ParStrict B A*
⟨*proof*⟩

lemma *l12-6*:
assumes *A B ParStrict C D*
shows *A B OS C D*
⟨*proof*⟩

lemma *pars--os3412*:
assumes *A B ParStrict C D*
shows *C D OS A B*
⟨*proof*⟩

lemma *perp-dec*:
A B Perp C D ∨ ¬ *A B Perp C D*
⟨*proof*⟩

lemma *col-cop2-perp2--col*:
assumes *X1 X2 Perp A B* **and**
Y1 Y2 Perp A B **and**
Col X1 Y1 Y2 **and**
Coplanar A B X2 Y1 **and**
Coplanar A B X2 Y2
shows *Col X2 Y1 Y2*
⟨*proof*⟩

lemma *col-perp2-ncol-col*:
assumes *X1 X2 Perp A B* **and**
Y1 Y2 Perp A B **and**
Col X1 Y1 Y2 **and**
¬ *Col X1 A B*
shows *Col X2 Y1 Y2*
⟨*proof*⟩

lemma *l12-9*:
assumes
Coplanar C1 C2 A1 B1 **and**
Coplanar C1 C2 A1 B2 **and**
Coplanar C1 C2 A2 B1 **and**
Coplanar C1 C2 A2 B2 **and**
A1 A2 Perp C1 C2 **and**
B1 B2 Perp C1 C2
shows *A1 A2 Par B1 B2*
⟨*proof*⟩

lemma *parallel-existence*:
assumes $A \neq B$
shows ∃ *C D*. $C \neq D$ ∧ *A B Par C D* ∧ *Col P C D*
⟨*proof*⟩

lemma *par-col-par*:
assumes $C \neq D'$ **and**
A B Par C D **and**
Col C D D'
shows *A B Par C* D'
⟨*proof*⟩

lemma *parallel-existence1*:
assumes $A \neq B$
shows ∃ *Q*. *A B Par P Q*
⟨*proof*⟩

lemma *par-not-col*:
assumes *A B ParStrict C D* **and**
Col X A B
shows ¬ *Col X C D*
⟨*proof*⟩

lemma *not-strict-par1*:

assumes $A\ B\ Par\ C\ D$ **and**
$Col\ A\ B\ X$ **and**
$Col\ C\ D\ X$
shows $Col\ A\ B\ C$
⟨*proof*⟩

lemma *not-strict-par2*:
assumes $A\ B\ Par\ C\ D$ **and**
$Col\ A\ B\ X$ **and**
$Col\ C\ D\ X$
shows $Col\ A\ B\ D$
⟨*proof*⟩

lemma *not-strict-par*:
assumes $A\ B\ Par\ C\ D$ **and**
$Col\ A\ B\ X$ **and**
$Col\ C\ D\ X$
shows $Col\ A\ B\ C \land Col\ A\ B\ D$
⟨*proof*⟩

lemma *not-par-not-col*:
assumes $A \neq B$ **and**
$A \neq C$ **and**
$\neg\ A\ B\ Par\ A\ C$
shows $\neg\ Col\ A\ B\ C$
⟨*proof*⟩

lemma *not-par-inter-uniqueness*:
assumes $A \neq B$ **and**
$C \neq D$ **and**
$\neg\ A\ B\ Par\ C\ D$ **and**
$Col\ A\ B\ X$ **and**
$Col\ C\ D\ X$ **and**
$Col\ A\ B\ Y$ **and**
$Col\ C\ D\ Y$
shows $X = Y$
⟨*proof*⟩

lemma *inter-uniqueness-not-par*:
assumes $\neg\ Col\ A\ B\ C$ **and**
$Col\ A\ B\ P$ **and**
$Col\ C\ D\ P$
shows $\neg\ A\ B\ Par\ C\ D$
⟨*proof*⟩

lemma *col-not-col-not-par*:
assumes $\exists\ P.\ Col\ A\ B\ P \land Col\ C\ D\ P$ **and**
$\exists\ Q.\ Col\ C\ D\ Q \land \neg Col\ A\ B\ Q$
shows $\neg\ A\ B\ Par\ C\ D$
⟨*proof*⟩

lemma *par-distincts*:
assumes $A\ B\ Par\ C\ D$
shows $A\ B\ Par\ C\ D \land A \neq B \land C \neq D$
⟨*proof*⟩

lemma *par-not-col-strict*:
assumes $A\ B\ Par\ C\ D$ **and**
$Col\ C\ D\ P$ **and**
$\neg\ Col\ A\ B\ P$
shows $A\ B\ ParStrict\ C\ D$
⟨*proof*⟩

lemma *col-cop-perp2-pars*:
assumes $\neg\ Col\ A\ B\ P$ **and**
$Col\ C\ D\ P$ **and**

Coplanar A B C D **and**
A B Perp P Q **and**
C D Perp P Q
shows *A B ParStrict C D*
⟨*proof*⟩

lemma *all-one-side-par-strict*:
assumes $C \neq D$ **and**
$\forall$ *P*. *Col C D P* $\longrightarrow$ *A B OS C P*
shows *A B ParStrict C D*
⟨*proof*⟩

lemma *par-col-par-2*:
assumes $A \neq P$ **and**
Col A B P **and**
A B Par C D
shows *A P Par C D*
⟨*proof*⟩

lemma *par-col2-par*:
assumes $E \neq F$ **and**
A B Par C D **and**
Col C D E **and**
Col C D F
shows *A B Par E F*
⟨*proof*⟩

lemma *par-col2-par-bis*:
assumes $E \neq F$ **and**
A B Par C D **and**
Col E F C **and**
Col E F D
shows *A B Par E F*
⟨*proof*⟩

lemma *par-strict-col-par-strict*:
assumes $C \neq E$ **and**
A B ParStrict C D **and**
Col C D E
shows *A B ParStrict C E*
⟨*proof*⟩

lemma *par-strict-col2-par-strict*:
assumes $E \neq F$ **and**
A B ParStrict C D **and**
Col C D E **and**
Col C D F
shows *A B ParStrict E F*
⟨*proof*⟩

lemma *line-dec*:
(*Col C1 B1 B2* $\wedge$ *Col C2 B1 B2*) $\vee$ $\neg$ (*Col C1 B1 B2* $\wedge$ *Col C2 B1 B2*)
⟨*proof*⟩

lemma *par-distinct*:
assumes *A B Par C D*
shows $A \neq B \wedge C \neq D$
⟨*proof*⟩

lemma *par-col4--par*:
assumes $E \neq F$ **and**
$G \neq H$ **and**
A B Par C D **and**
Col A B E **and**
Col A B F **and**
Col C D G **and**

Col C D H
shows *E F Par G H*
⟨*proof*⟩

lemma *par-strict-col4--par-strict*:
assumes $E \neq F$ **and**
$G \neq H$ **and**
A B ParStrict C D **and**
Col A B E **and**
Col A B F **and**
Col C D G **and**
Col C D H
shows *E F ParStrict G H*
⟨*proof*⟩

lemma *par-strict-one-side*:
assumes *A B ParStrict C D* **and**
Col C D P
shows *A B OS C P*
⟨*proof*⟩

lemma *par-strict-all-one-side*:
assumes *A B ParStrict C D*
shows ∀ *P*. *Col C D P* ⟶ *A B OS C P*
⟨*proof*⟩

lemma *inter-trivial*:
assumes ¬ *Col A B X*
shows *X Inter A X B X*
⟨*proof*⟩

lemma *inter-sym*:
assumes *X Inter A B C D*
shows *X Inter C D A B*
⟨*proof*⟩

lemma *inter-left-comm*:
assumes *X Inter A B C D*
shows *X Inter B A C D*
⟨*proof*⟩

lemma *inter-right-comm*:
assumes *X Inter A B C D*
shows *X Inter A B D C*
⟨*proof*⟩

lemma *inter-comm*:
assumes *X Inter A B C D*
shows *X Inter B A D C*
⟨*proof*⟩

lemma *l12-17*:
assumes $A \neq B$ **and**
P Midpoint A C **and**
P Midpoint B D
shows *A B Par C D*
⟨*proof*⟩

lemma *l12-18-a*:
assumes *Cong A B C D* **and**
Cong B C D A **and**
¬ *Col A B C* **and**
$B \neq D$ **and**
Col A P C **and**
Col B P D
shows *A B Par C D*

⟨*proof*⟩

lemma *l12-18-b*:
assumes *Cong A B C D* **and**
Cong B C D A **and**
¬ *Col A B C* **and**
$B \neq D$ **and**
Col A P C **and**
Col B P D
shows *B C Par D A*
⟨*proof*⟩

lemma *l12-18-c*:
assumes *Cong A B C D* **and**
Cong B C D A **and**
¬ *Col A B C* **and**
$B \neq D$ **and**
Col A P C **and**
Col B P D
shows *B D TS A C*
⟨*proof*⟩

lemma *l12-18-d*:
assumes *Cong A B C D* **and**
Cong B C D A **and**
¬ *Col A B C* **and**
$B \neq D$ **and**
Col A P C **and**
Col B P D
shows *A C TS B D*
⟨*proof*⟩

lemma *l12-18*:
assumes *Cong A B C D* **and**
Cong B C D A **and**
¬ *Col A B C* **and**
$B \neq D$ **and**
Col A P C **and**
Col B P D
shows *A B Par C D* ∧ *B C Par D A* ∧ *B D TS A C* ∧ *A C TS B D*
⟨*proof*⟩

lemma *par-two-sides-two-sides*:
assumes *A B Par C D* **and**
B D TS A C
shows *A C TS B D*
⟨*proof*⟩

lemma *par-one-or-two-sides*:
assumes *A B ParStrict C D*
shows (*A C TS B D* ∧ *B D TS A C*) ∨ (*A C OS B D* ∧ *B D OS A C*)
⟨*proof*⟩

lemma *l12-21-b*:
assumes *A C TS B D* **and**
B A C CongA D C A
shows *A B Par C D*
⟨*proof*⟩

lemma *l12-22-aux*:
assumes $P \neq A$ **and**
$A \neq C$ **and**
Bet P A C **and**
P A OS B D **and**
B A P CongA D C P
shows *A B Par C D*

⟨*proof*⟩

lemma *l12-22-b*:
  assumes *P Out A C* **and**
    P A OS B D **and**
    B A P CongA D C P
  shows *A B Par C D*
⟨*proof*⟩

lemma *par-strict-par*:
  assumes *A B ParStrict C D*
  shows *A B Par C D*
  ⟨*proof*⟩

lemma *par-strict-distinct*:
  assumes *A B ParStrict C D*
  shows $A \neq B \wedge C \neq D$
  ⟨*proof*⟩

lemma *col-par*:
  assumes $A \neq B$ **and**
    $B \neq C$ **and**
    Col A B C
  shows *A B Par B C*
  ⟨*proof*⟩

lemma *acute-col-perp--out*:
  assumes *Acute A B C* **and**
    Col B C A′ **and**
    B C Perp A A′
  shows *B Out A′ C*
⟨*proof*⟩

lemma *acute-col-perp--out-1*:
  assumes *Acute A B C* **and**
    Col B C A′ **and**
    B A Perp A A′
  shows *B Out A′ C*
⟨*proof*⟩

lemma *conga-inangle-per2--inangle*:
  assumes *Per A B C* **and**
    T InAngle A B C **and**
    P B A CongA P B C **and**
    Per B P T **and**
    Coplanar A B C P
  shows *P InAngle A B C*
⟨*proof*⟩

lemma *perp-not-par*:
  assumes *A B Perp X Y*
  shows ¬ *A B Par X Y*
⟨*proof*⟩

lemma *cong-conga-perp*:
  assumes *B P TS A C* **and**
    Cong A B C B **and**
    A B P CongA C B P
  shows *A C Perp B P*
⟨*proof*⟩

lemma *perp-inter-exists*:
  assumes *A B Perp C D*
  shows ∃ *P*. *Col A B P* ∧ *Col C D P*
⟨*proof*⟩

lemma *perp-inter-perp-in*:
  assumes *A B Perp C D*
  shows ∃ *P*. *Col A B P* ∧ *Col C D P* ∧ *P PerpAt A B C D*
  ⟨*proof*⟩

end

context *Tarski-2D*

begin

lemma *l12-9-2D*:
  assumes *A1 A2 Perp C1 C2* **and**
    B1 B2 Perp C1 C2
  shows *A1 A2 Par B1 B2*
  ⟨*proof*⟩

end

context *Tarski-neutral-dimensionless*

begin

3.12 Tarski: Chapter 13

3.12.1 Introduction

lemma *per2-col-eq*:
  assumes $A \neq P$ **and**
    $A \neq P'$ **and**
    Per A P B **and**
    Per A P′ B **and**
    Col P A P′
  shows $P = P'$
  ⟨*proof*⟩

lemma *per2-preserves-diff*:
  assumes $PO \neq A'$ **and**
    $PO \neq B'$ **and**
    Col PO A′ B′ **and**
    Per PO A′ A **and**
    Per PO B′ B **and**
    $A' \neq B'$
  shows $A \neq B$
  ⟨*proof*⟩

lemma *per23-preserves-bet*:
  assumes *Bet A B C* **and**
    $A \neq B'$ **and** $A \neq C'$ **and**
    Col A B′ C′ **and**
    Per A B′ B **and**
    Per A C′ C
  shows *Bet A B′ C′*
⟨*proof*⟩

lemma *per23-preserves-bet-inv*:
  assumes *Bet A B′ C′* **and**
    $A \neq B'$ **and**
    Col A B C **and**
    Per A B′ B **and**
    Per A C′ C
  shows *Bet A B C*
⟨*proof*⟩

lemma *per13-preserves-bet*:
  assumes *Bet A B C* **and**
    $B \neq A'$ **and**

$B \neq C'$ **and**
$Col\ A'\ B\ C'$ **and**
$Per\ B\ A'\ A$ **and**
$Per\ B\ C'\ C$
shows $Bet\ A'\ B\ C'$
⟨*proof*⟩

lemma *per13-preserves-bet-inv*:
assumes $Bet\ A'\ B\ C'$ **and**
$B \neq A'$ **and**
$B \neq C'$ **and**
$Col\ A\ B\ C$ **and**
$Per\ B\ A'\ A$ **and**
$Per\ B\ C'\ C$
shows $Bet\ A\ B\ C$
⟨*proof*⟩

lemma *per3-preserves-bet1*:
assumes $Col\ PO\ A\ B$ **and**
$Bet\ A\ B\ C$ **and**
$PO \neq A'$ **and**
$PO \neq B'$ **and**
$PO \neq C'$ **and**
$Per\ PO\ A'\ A$ **and**
$Per\ PO\ B'\ B$ **and**
$Per\ PO\ C'\ C$ **and**
$Col\ A'\ B'\ C'$ **and**
$Col\ PO\ A'\ B'$
shows $Bet\ A'\ B'\ C'$
⟨*proof*⟩

lemma *per3-preserves-bet2-aux*:
assumes $Col\ PO\ A\ C$ **and**
$A \neq C'$ **and**
$Bet\ A\ B'\ C'$ **and**
$PO \neq A$ **and**
$PO \neq B'$ **and**
$PO \neq C'$ **and**
$Per\ PO\ B'\ B$ **and**
$Per\ PO\ C'\ C$ **and**
$Col\ A\ B\ C$ **and**
$Col\ PO\ A\ C'$
shows $Bet\ A\ B\ C$
⟨*proof*⟩

lemma *per3-preserves-bet2*:
assumes $Col\ PO\ A\ C$ **and**
$A' \neq C'$ **and**
$Bet\ A'\ B'\ C'$ **and**
$PO \neq A'$ **and**
$PO \neq B'$ **and**
$PO \neq C'$ **and**
$Per\ PO\ A'\ A$ **and**
$Per\ PO\ B'\ B$ **and**
$Per\ PO\ C'\ C$ **and**
$Col\ A\ B\ C$ **and**
$Col\ PO\ A'\ C'$
shows $Bet\ A\ B\ C$
⟨*proof*⟩

lemma *symmetry-preserves-per*:
assumes $Per\ B\ P\ A$ **and**
$B\ Midpoint\ A\ A'$ **and**
$B\ Midpoint\ P\ P'$
shows $Per\ B\ P'\ A'$
⟨*proof*⟩

lemma *l13-1-aux*:
assumes ¬ *Col A B C* **and**
P Midpoint B C **and**
Q Midpoint A C **and**
R Midpoint A B
shows
$\exists\ X\ Y.\ (R\ PerpAt\ X\ Y\ A\ B \wedge X\ Y\ Perp\ P\ Q \wedge Coplanar\ A\ B\ C\ X \wedge Coplanar\ A\ B\ C\ Y)$
⟨*proof*⟩

lemma *l13-1*:
assumes ¬ *Col A B C* **and**
P Midpoint B C **and**
Q Midpoint A C **and**
R Midpoint A B
shows
$\exists\ X\ Y.(R\ PerpAt\ X\ Y\ A\ B \wedge X\ Y\ Perp\ P\ Q)$
⟨*proof*⟩

lemma *per-lt*:
assumes $A \neq B$ **and**
$C \neq B$ **and**
Per A B C
shows $A\ B\ Lt\ A\ C \wedge C\ B\ Lt\ A\ C$
⟨*proof*⟩

lemma *cong-perp-conga*:
assumes *Cong A B C B* **and**
A C Perp B P
shows $A\ B\ P\ CongA\ C\ B\ P \wedge B\ P\ TS\ A\ C$
⟨*proof*⟩

lemma *perp-per-bet*:
assumes ¬ *Col A B C* **and**

Per A B C **and**
P PerpAt P B A C
shows *Bet A P C*
⟨*proof*⟩

lemma *ts-per-per-ts*:
assumes *A B TS C D* **and**
Per B C A **and**
Per B D A
shows *C D TS A B*
⟨*proof*⟩

lemma *l13-2-1*:
assumes *A B TS C D* **and**
Per B C A **and**
Per B D A **and**
Col C D E **and**
A E Perp C D **and**
C A B CongA D A B
shows $B\ A\ C\ CongA\ D\ A\ E \wedge B\ A\ D\ CongA\ C\ A\ E \wedge Bet\ C\ E\ D$
⟨*proof*⟩

lemma *triangle-mid-par*:
assumes ¬ *Col A B C* **and**
P Midpoint B C **and**
Q Midpoint A C
shows *A B ParStrict Q P*
⟨*proof*⟩

lemma *cop4-perp-in2--col*:
assumes *Coplanar X Y A A′* **and**

Coplanar X Y A B' **and**
Coplanar X Y B A' **and**
Coplanar X Y B B' **and**
P PerpAt A B X Y **and**
P PerpAt A' B' X Y
shows *Col A B A'*
⟨*proof*⟩

lemma *l13-2*:
assumes *A B TS C D* **and**
Per B C A **and**
Per B D A **and**
Col C D E **and**
A E Perp C D
shows *B A C CongA D A E* $\wedge$ *B A D CongA C A E* $\wedge$ *Bet C E D*
⟨*proof*⟩

lemma *perp2-refl*:
assumes $A \neq B$
shows *P Perp2 A B A B*
⟨*proof*⟩

lemma *perp2-sym*:
assumes *P Perp2 A B C D*
shows *P Perp2 C D A B*
⟨*proof*⟩

lemma *perp2-left-comm*:
assumes *P Perp2 A B C D*
shows *P Perp2 B A C D*
⟨*proof*⟩

lemma *perp2-right-comm*:
assumes *P Perp2 A B C D*
shows *P Perp2 A B D C*
⟨*proof*⟩

lemma *perp2-comm*:
assumes *P Perp2 A B C D*
shows *P Perp2 B A D C*
⟨*proof*⟩

lemma *perp2-pseudo-trans*:
assumes *P Perp2 A B C D* **and**
P Perp2 C D E F **and**
¬ *Col C D P*
shows *P Perp2 A B E F*
⟨*proof*⟩

lemma *col-cop-perp2--pars-bis*:
assumes ¬ *Col A B P* **and**
Col C D P **and**
Coplanar A B C D **and**
P Perp2 A B C D
shows *A B ParStrict C D*
⟨*proof*⟩

lemma *perp2-preserves-bet23*:
assumes *Bet PO A B* **and**
Col PO A' B' **and**
¬ *Col PO A A'* **and**
PO Perp2 A A' B B'
shows *Bet PO A' B'*
⟨*proof*⟩

lemma *perp2-preserves-bet13*:

assumes *Bet B PO C* **and**
Col PO B′ C′ **and**
¬ *Col PO B B′* **and**
PO Perp2 B C′ C B′
shows *Bet B′ PO C′*
⟨*proof*⟩

lemma *is-image-perp-in*:
assumes $A \neq A'$ **and**
$X \neq Y$ **and**
A A′ Reflect X Y
shows ∃ *P. P PerpAt A A′ X Y*
⟨*proof*⟩

lemma *perp-inter-perp-in-n*:
assumes *A B Perp C D*
shows ∃ *P. Col A B P* ∧ *Col C D P* ∧ *P PerpAt A B C D*
⟨*proof*⟩

lemma *perp2-perp-in*:
assumes *PO Perp2 A B C D* **and**
¬ *Col PO A B* **and**
¬ *Col PO C D*
shows ∃ *P Q. Col A B P* ∧ *Col C D Q* ∧ *Col PO P Q* ∧ *P PerpAt PO P A B* ∧ *Q PerpAt PO Q C D*
⟨*proof*⟩

lemma *l13-8*:
assumes $U \neq PO$ **and**
$V \neq PO$ **and**
Col PO P Q **and**
Col PO U V **and**
Per P U PO **and**
Per Q V PO
shows *PO Out P Q* ⟷ *PO Out U V*
⟨*proof*⟩

lemma *perp-in-rewrite*:
assumes *P PerpAt A B C D*
shows *P PerpAt A P P C* ∨ *P PerpAt A P P D* ∨ *P PerpAt B P P C* ∨ *P PerpAt B P P D*
⟨*proof*⟩

lemma *perp-out-acute*:
assumes *B Out A C′* **and**
A B Perp C C′
shows *Acute A B C*
⟨*proof*⟩

lemma *perp-bet-obtuse*:
assumes $B \neq C'$ **and**
A B Perp C C′ **and**
Bet A B C′
shows *Obtuse A B C*
⟨*proof*⟩

end

3.12.2 Part 1: 2D

context *Tarski-2D*
begin

lemma *perp-in2--col*:
assumes *P PerpAt A B X Y* **and**
P PerpAt A′ B′ X Y
shows *Col A B A′*
⟨*proof*⟩

lemma *perp2-trans*:
assumes *P Perp2 A B C D* **and**
P Perp2 C D E F
shows *P Perp2 A B E F*
⟨*proof*⟩

lemma *perp2-par*:
assumes *PO Perp2 A B C D*
shows *A B Par C D*
⟨*proof*⟩

end

3.12.3 Part 2: length

context *Tarski-neutral-dimensionless*

begin

lemma *lg-exists*:
∃ *l*. (*QCong l* ∧ *l A B*)
⟨*proof*⟩

lemma *lg-cong*:
assumes *QCong l* **and**
l A B **and**
l C D
shows *Cong A B C D*
⟨*proof*⟩

lemma *lg-cong-lg*:
assumes *QCong l* **and**
l A B **and**
Cong A B C D
shows *l C D*
⟨*proof*⟩

lemma *lg-sym*:
assumes *QCong l*
and *l A B*
shows *l B A*
⟨*proof*⟩

lemma *ex-points-lg*:
assumes *QCong l*
shows ∃ *A B*. *l A B*
⟨*proof*⟩

lemma *is-len-cong*:
assumes *TarskiLen A B l* **and**
TarskiLen C D l
shows *Cong A B C D*
⟨*proof*⟩

lemma *is-len-cong-is-len*:
assumes *TarskiLen A B l* **and**
Cong A B C D
shows *TarskiLen C D l*
⟨*proof*⟩

lemma *not-cong-is-len*:
assumes ¬ *Cong A B C D* **and**
TarskiLen A B l
shows ¬ *l C D*
⟨*proof*⟩

lemma *not-cong-is-len1*:
assumes ¬ *Cong A B C D*
and *TarskiLen A B l*
shows ¬ *TarskiLen C D l*
⟨*proof*⟩

lemma *lg-null-instance*:
assumes *QCongNull l*
shows *l A A*
⟨*proof*⟩

lemma *lg-null-trivial*:
assumes *QCong l*
and *l A A*
shows *QCongNull l*
⟨*proof*⟩

lemma *lg-null-dec*:

shows *QCongNull l* ∨ ¬ *QCongNull l*
⟨*proof*⟩

lemma *ex-point-lg*:
assumes *QCong l*
shows ∃ *B*. *l A B*
⟨*proof*⟩

lemma *ex-point-lg-out*:
assumes *A* ≠ *P* **and**
QCong l **and**
¬ *QCongNull l*
shows ∃ *B*. (*l A B* ∧ *A Out B P*)
⟨*proof*⟩

lemma *ex-point-lg-bet*:
assumes *QCong l*
shows ∃ *B*. (*l M B* ∧ *Bet A M B*)
⟨*proof*⟩

lemma *ex-points-lg-not-col*:
assumes *QCong l*
and ¬ *QCongNull l*
shows ∃ *A B*. (*l A B* ∧ ¬ *Col A B P*)
⟨*proof*⟩

lemma *ex-eql*:
assumes ∃ *A B*. (*TarskiLen A B l1* ∧ *TarskiLen A B l2*)
shows *l1* = *l2*
⟨*proof*⟩

lemma *all-eql*:
assumes *TarskiLen A B l1* **and**
TarskiLen A B l2
shows *l1* = *l2*
⟨*proof*⟩

lemma *null-len*:
assumes *TarskiLen A A la* **and**
TarskiLen B B lb
shows *la* = *lb*
⟨*proof*⟩

lemma *eqL-equivalence*:
assumes *QCong la* **and**
QCong lb **and**

QCong lc
shows $la = la \wedge (la = lb \longrightarrow lb = la) \wedge (la = lb \wedge lb = lc \longrightarrow la = lc)$
⟨*proof*⟩

lemma *ex-lg*:
$\exists\ l.\ (QCong\ l \wedge l\ A\ B)$
⟨*proof*⟩

lemma *lg-eql-lg*:
assumes *QCong l1* **and**
$l1 = l2$
shows *QCong l2*
⟨*proof*⟩

lemma *ex-eqL*:
assumes *QCong l1* **and**
QCong l2 **and**
$\exists\ A\ B.\ (l1\ A\ B \wedge l2\ A\ B)$
shows $l1 = l2$
⟨*proof*⟩

3.12.4 Part 3 : angles

lemma *ang-exists*:
assumes $A \neq B$ **and**
$C \neq B$
shows $\exists\ a.\ (QCongA\ a \wedge a\ A\ B\ C)$
⟨*proof*⟩

lemma *ex-points-eng*:
assumes *QCongA a*
shows $\exists\ A\ B\ C.\ (a\ A\ B\ C)$
⟨*proof*⟩

lemma *ang-conga*:
assumes *QCongA a* **and**
$a\ A\ B\ C$ **and**
$a\ A'\ B'\ C'$
shows $A\ B\ C\ CongA\ A'\ B'\ C'$
⟨*proof*⟩

lemma *is-ang-conga*:
assumes $A\ B\ C\ Ang\ a$ **and**
$A'\ B'\ C'\ Ang\ a$
shows $A\ B\ C\ CongA\ A'\ B'\ C'$
⟨*proof*⟩

lemma *is-ang-conga-is-ang*:
assumes $A\ B\ C\ Ang\ a$ **and**
$A\ B\ C\ CongA\ A'\ B'\ C'$
shows $A'\ B'\ C'\ Ang\ a$
⟨*proof*⟩

lemma *not-conga-not-ang*:
assumes *QCongA a* **and**
$\neg\ A\ B\ C\ CongA\ A'\ B'\ C'$ **and**
$a\ A\ B\ C$
shows $\neg\ a\ A'\ B'\ C'$
⟨*proof*⟩

lemma *not-conga-is-ang*:
assumes $\neg\ A\ B\ C\ CongA\ A'\ B'\ C'$ **and**
$A\ B\ C\ Ang\ a$
shows $\neg\ a\ A'\ B'\ C'$
⟨*proof*⟩

lemma *not-cong-is-ang1*:
  assumes $\neg$ *A B C CongA A' B' C'* **and**
    A B C Ang a
  shows $\neg$ *A' B' C' Ang a*
  ⟨*proof*⟩

lemma *ex-eqa*:
  assumes $\exists$ *A B C*.(*A B C Ang a1* $\wedge$ *A B C Ang a2*)
  shows *a1* = *a2*
⟨*proof*⟩

lemma *all-eqa*:
  assumes *A B C Ang a1* **and**
    A B C Ang a2
  shows *a1* = *a2*
  ⟨*proof*⟩

lemma *is-ang-distinct*:
  assumes *A B C Ang a*
  shows $A \neq B \wedge C \neq B$
  ⟨*proof*⟩

lemma *null-ang*:
  assumes *A B A Ang a1* **and**
    C D C Ang a2
  shows *a1* = *a2*
  ⟨*proof*⟩

lemma *flat-ang*:
  assumes *Bet A B C* **and**
    Bet A' B' C' **and**
    A B C Ang a1 **and**
    A' B' C' Ang a2
  shows *a1* = *a2*
⟨*proof*⟩

lemma *ang-distinct*:
  assumes *QCongA a* **and**
    a A B C
  shows $A \neq B \wedge C \neq B$
⟨*proof*⟩

lemma *ex-ang*:
  assumes $B \neq A$ **and**
    $B \neq C$
  shows $\exists$ *a*. (*QCongA a* $\wedge$ *a A B C*)
  ⟨*proof*⟩

lemma *anga-exists*:
  assumes $A \neq B$ **and**
    $C \neq B$ **and**
    Acute A B C
  shows $\exists$ *a*. (*QCongAAcute a* $\wedge$ *a A B C*)
⟨*proof*⟩

lemma *anga-is-ang*:
  assumes *QCongAAcute a*
  shows *QCongA a*
⟨*proof*⟩

lemma *ex-points-anga*:
  assumes *QCongAAcute a*
  shows $\exists$ *A B C*. *a A B C*
  ⟨*proof*⟩

lemma *anga-conga*:

assumes *QCongAAcute a* **and**
a A B C **and**
a A′ B′ C′
shows *A B C CongA A′ B′ C′*
⟨*proof*⟩

lemma *is-anga-to-is-ang*:
assumes *A B C AngAcute a*
shows *A B C Ang a*
⟨*proof*⟩

lemma *is-anga-conga*:
assumes *A B C AngAcute a* **and**
A′ B′ C′ AngAcute a
shows *A B C CongA A′ B′ C′*
⟨*proof*⟩

lemma *is-anga-conga-is-anga*:
assumes *A B C AngAcute a* **and**
A B C CongA A′ B′ C′
shows *A′ B′ C′ AngAcute a*
⟨*proof*⟩

lemma *not-conga-is-anga*:
assumes ¬ *A B C CongA A′ B′ C′* **and**
A B C AngAcute a
shows ¬ *a A′ B′ C′*
⟨*proof*⟩

lemma *not-cong-is-anga1*:
assumes ¬ *A B C CongA A′ B′ C′* **and**
A B C AngAcute a
shows ¬ *A′ B′ C′ AngAcute a*
⟨*proof*⟩

lemma *ex-eqaa*:
assumes ∃ *A B C*. (*A B C AngAcute a1* ∧ *A B C AngAcute a2*)
shows *a1* = *a2*
⟨*proof*⟩

lemma *all-eqaa*:
assumes *A B C AngAcute a1* **and**
A B C AngAcute a2
shows *a1* = *a2*
⟨*proof*⟩

lemma *is-anga-distinct*:
assumes *A B C AngAcute a*
shows $A \neq B \wedge C \neq B$
⟨*proof*⟩

lemma *null-anga*:
assumes *A B A AngAcute a1* **and**
C D C AngAcute a2
shows *a1* = *a2*
⟨*proof*⟩

lemma *anga-distinct*:
assumes *QCongAAcute a* **and**
a A B C
shows $A \neq B \wedge C \neq B$
⟨*proof*⟩

lemma *out-is-len-eq*:
assumes *A Out B C* **and**
TarskiLen A B l **and**

TarskiLen A C l
shows $B = C$
⟨*proof*⟩

lemma *out-len-eq*:
assumes *QCong l* **and**
A Out B C **and**
l A B **and**
l A C
shows $B = C$ ⟨*proof*⟩

lemma *ex-anga*:
assumes *Acute A B C*
shows $\exists\ a.\ (QCongAAcute\ a \wedge a\ A\ B\ C)$
⟨*proof*⟩

lemma *not-null-ang-ang*:
assumes *QCongAnNull a*
shows *QCongA a*
⟨*proof*⟩

lemma *not-null-ang-def-equiv*:
$QCongAnNull\ a \longleftrightarrow (QCongA\ a \wedge (\exists\ A\ B\ C.\ (a\ A\ B\ C \wedge\ \neg\ B\ Out\ A\ C)))$
⟨*proof*⟩

lemma *not-flat-ang-def-equiv*:
$QCongAnFlat\ a \longleftrightarrow (QCongA\ a \wedge (\exists\ A\ B\ C.\ (a\ A\ B\ C \wedge \neg\ Bet\ A\ B\ C)))$
⟨*proof*⟩

lemma *ang-const*:
assumes *QCongA a* **and**
$A \neq B$
shows $\exists\ C.\ a\ A\ B\ C$
⟨*proof*⟩

lemma *ang-sym*:
assumes *QCongA a* **and**
a A B C
shows *a C B A*
⟨*proof*⟩

lemma *ang-not-null-lg*:
assumes *QCongA a* **and**
QCong l **and**
a A B C **and**
l A B
shows ¬ *QCongNull l*
⟨*proof*⟩

lemma *ang-distincts*:
assumes *QCongA a* **and**
a A B C
shows $A \neq B \wedge C \neq B$
⟨*proof*⟩

lemma *anga-sym*:
assumes *QCongAAcute a* **and**
a A B C
shows *a C B A*
⟨*proof*⟩

lemma *anga-not-null-lg*:
assumes *QCongAAcute a* **and**
QCong l **and**
a A B C **and**
l A B

shows ¬ *QCongNull l*
⟨*proof*⟩

lemma *anga-distincts*:
assumes *QCongAAcute a* **and**
a A B C
shows $A \neq B \wedge C \neq B$
⟨*proof*⟩

lemma *ang-const-o*:
assumes ¬ *Col A B P* **and**
QCongA a **and**
QCongAnNull a **and**
QCongAnFlat a
shows ∃ *C*. *a A B C* ∧ *A B OS C P*
⟨*proof*⟩

lemma *anga-const*:
assumes *QCongAAcute a* **and**
$A \neq B$
shows ∃ *C*. *a A B C*
⟨*proof*⟩

lemma *null-anga-null-angaP*:
QCongANullAcute a ⟷ *IsNullAngaP a*
⟨*proof*⟩

lemma *is-null-anga-out*:
assumes
a A B C **and**
QCongANullAcute a
shows *B Out A C*
⟨*proof*⟩

lemma *acute-not-bet*:
assumes *Acute A B C*
shows ¬ *Bet A B C*
⟨*proof*⟩

lemma *anga-acute*:
assumes *QCongAAcute a* **and**
a A B C
shows *Acute A B C*
⟨*proof*⟩

lemma *not-null-not-col*:
assumes *QCongAAcute a* **and**
¬ *QCongANullAcute a* **and**
a A B C
shows ¬ *Col A B C*
⟨*proof*⟩

lemma *ang-cong-ang*:
assumes *QCongA a* **and**
a A B C **and**
A B C CongA A' B' C'
shows *a* A' B' C'
⟨*proof*⟩

lemma *is-null-ang-out*:
assumes
a A B C **and**
QCongANull a
shows *B Out A C*
⟨*proof*⟩

lemma *out-null-ang*:
assumes *QCongA a* **and**
a A B C **and**
B Out A C
shows *QCongANull a*
⟨*proof*⟩

lemma *bet-flat-ang*:
assumes *QCongA a* **and**
a A B C **and**
Bet A B C
shows *AngFlat a*
⟨*proof*⟩

lemma *out-null-anga*:
assumes *QCongAAcute a* **and**
a A B C **and**
B Out A C
shows *QCongANullAcute a*
⟨*proof*⟩

lemma *anga-not-flat*:
assumes *QCongAAcute a*
shows *QCongAnFlat a*
⟨*proof*⟩

lemma *anga-const-o*:
assumes ¬ *Col A B P* **and**
¬ *QCongANullAcute a* **and**
QCongAAcute a
shows ∃ *C*. (*a A B C* ∧ *A B OS C P*)
⟨*proof*⟩

lemma *anga-conga-anga*:
assumes *QCongAAcute a* **and**
a A B C **and**
A B C CongA A′ B′ C′
shows *a A′ B′ C′*
⟨*proof*⟩

lemma *anga-out-anga*:
assumes *QCongAAcute a* **and**
a A B C **and**
B Out A A′ **and**
B Out C C′
shows *a A′ B C′*
⟨*proof*⟩

lemma *out-out-anga*:
assumes *QCongAAcute a* **and**
B Out A C **and**
B′ Out A′ C′ **and**
a A B C
shows *a A′ B′ C′*
⟨*proof*⟩

lemma *is-null-all*:
assumes *A* ≠ *B* **and**
QCongANullAcute a
shows *a A B A*
⟨*proof*⟩

lemma *anga-col-out*:
assumes *QCongAAcute a* **and**
a A B C **and**
Col A B C

shows *B Out A C*
⟨*proof*⟩

lemma *ang-not-lg-null*:
assumes *QCong la* **and**
QCong lc **and**
QCongA a **and**
la A B **and**
lc C B **and**
a A B C
shows ¬ *QCongNull la* ∧ ¬ *QCongNull lc*
⟨*proof*⟩

lemma *anga-not-lg-null*:
assumes
QCongAAcute a **and**
la A B **and**
lc C B **and**
a A B C
shows ¬ *QCongNull la* ∧ ¬ *QCongNull lc*
⟨*proof*⟩

lemma *anga-col-null*:
assumes *QCongAAcute a* **and**
a A B C **and**
Col A B C
shows *B Out A C* ∧ *QCongANullAcute a*
⟨*proof*⟩

lemma *eqA-preserves-ang*:
assumes *QCongA a* **and**
$a = b$
shows *QCongA b*
⟨*proof*⟩

lemma *eqA-preserves-anga*:
assumes *QCongAAcute a* **and**

$a = b$
shows *QCongAAcute b*
⟨*proof*⟩

4 Some postulates of the parallels

lemma *euclid-5--original-euclid*:
assumes *Euclid5*
shows *EuclidSParallelPostulate*
⟨*proof*⟩

lemma *tarski-s-euclid-implies-euclid-5*:
assumes *TarskiSParallelPostulate*
shows *Euclid5*
⟨*proof*⟩

lemma *tarski-s-implies-euclid-s-parallel-postulate*:
assumes *TarskiSParallelPostulate*
shows *EuclidSParallelPostulate*
⟨*proof*⟩

theorem *tarski-s-euclid-implies-playfair-s-postulate*:
assumes *TarskiSParallelPostulate*
shows *PlayfairSPostulate*
⟨*proof*⟩

end

end

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