# IsaGeoCoq: Partial porting of GeoCoq 2.4.0. Case studies: Tarski's postulate of parallels implies the 5th postulate of Euclid, the postulate of Playfair and the original postulate of Euclid.

Roland Coghetto

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**Abstract**

The GeoCoq library contains a formalization of geometry using the Coq proof assistant. It contains both proofs about the foundations of geometry [20, 15, 6, 16] and high-level proofs in the same style as in high-school. [1](Code Repository https://github.com/GeoCoq/GeoCoq).

Some theorems also inspired by [20] are also formalized with others ITP(Metamath, Mizar) or ATP [24, 25, 3, 23, 4, 2, 17, 5, 11, 19, 8, 9, 10].

We port a part of the GeoCoq 2.4.0 library within the Isabelle/Hol proof assistant: more precisely, the files Chap02.v to Chap13_3.v, suma.v as well as the associated definitions and some useful files for the demonstration of certain parallel postulates.

While the demonstrations in Coq are written in procedural language [26], the transcript is done in declarative language Isar[18].

The synthetic approach of the demonstrations are directly inspired by those contained in GeoCoq. Some demonstrations are credited to G.E Martin(«lemma bet_le_lt:» in Ch11_angles.thy, proved by Martin as Theorem 18.17 in [14]) or Gupta H.N (Krippen Lemma, proved by Gupta in its PhD in 1965 as Theorem 3.45). (See [12]).

In this work, the proofs are not contructive. The sledeghammer tool being used to find some demonstrations.

The names of the lemmas and theorems used are kept as far as possible as well as the definitions. A different translation has been proposed when the name was already used in Isabel/Hol ("Len" is translated as "TarskiLen") or that characters were not allowed in Isabel/Hol ("anga'" in Ch13_angles.v is translated as "angaP"). For some definitions the highlighting of a variable has changed the order or the position of the variables (Midpoint, Out, Inter,...).

All the lemmas are valid in absolute/neutral space defined with Tarski's axioms.

It should be noted that T.J.M. Makarios [13] has begun some demonstrations of certain proposals mainly those corresponding to SST chapters 2 and 3. It uses a definition that does not quite coincide with the definition used in Geocoq and here. As an example, Makarios introduces the axiom A11 (Axiom of continuity) in the definition of the locale "Tarski_absolute_space".

Furthermore, the definition of the locale "TarskiAbsolute" [22, 21] is not not identical to the one defined in the "Tarski_neutral_dimensionless" class of GeoCoq. Indeed this one does not contain the axiom "upper_dimension". In some cases particular, it is nevertheless to use the axiom "upper_dimension". The addition of the word "_2D" in the file indicates its presence.

In the last part, it is formalized that, in the neutral/absolute space, the axiom of the parallels of the system of Tarski implies the Playfair axiom, the 5th postulate of euclide and the postulate original from Euclid. These proofs, which are not constructive, are directly inspired by [12, 7].

# Contents

**theory** *Tarski-Neutral*

**imports**
*Main*

**begin**

# 1 Tarski's axiom system for neutral geometry

## 1.1 Tarski's axiom system for neutral geometry: dimensionless

**locale** *Tarski-neutral-dimensionless* =
**fixes** $Bet\ :: \ 'p \Rightarrow\ 'p \Rightarrow\ 'p \Rightarrow bool$
**fixes** $Cong :: \ 'p \Rightarrow\ 'p \Rightarrow\ 'p \Rightarrow\ 'p \Rightarrow bool$
**assumes** *cong-pseudo-reflexivity*: $\forall\ a\ b.$
$Cong\ a\ b\ b\ a$
**and** *cong-inner-transitivity*: $\forall\ a\ b\ p\ q\ r\ s.$
$Cong\ a\ b\ p\ q\ \wedge$
$Cong\ a\ b\ r\ s$
$\longrightarrow$
$Cong\ p\ q\ r\ s$
**and** *cong-identity*: $\forall\ a\ b\ c.$
$Cong\ a\ b\ c\ c$
$\longrightarrow$
$a = b$
**and** *segment-construction*: $\forall\ a\ b\ c\ q.$
$\exists x.\ (Bet\ q\ a\ x \wedge Cong\ a\ x\ b\ c)$
**and** *five-segment*: $\forall\ a\ b\ c\ a'\ b'\ c'.$
$a \neq b\ \wedge$
$Bet\ a\ b\ c\ \wedge$
$Bet\ a'\ b'\ c' \wedge$
$Cong\ a\ b\ a'\ b'\ \wedge$
$Cong\ b\ c\ b'\ c'\ \wedge$
$Cong\ a\ d\ a'\ d'\ \wedge$
$Cong\ b\ d\ b'\ d'$
$\longrightarrow$
$Cong\ c\ d\ c'\ d'$
**and** *between-identity*: $\forall\ a\ b.$
$Bet\ a\ b\ a$
$\longrightarrow$
$a = b$
**and** *inner-pasch*: $\forall\ a\ b\ c\ p\ q.$
$Bet\ a\ p\ c\ \wedge$
$Bet\ b\ q\ c$
$\longrightarrow$
$(\exists\ x.\ Bet\ p\ x\ b \wedge Bet\ q\ x\ a)$
**and** *lower-dim*: $\exists\ a\ b\ c.\ (\neg\ Bet\ a\ b\ c \wedge \neg\ Bet\ b\ c\ a \wedge \neg\ Bet\ c\ a\ b)$

## 1.2 Tarski's axiom system for neutral geometry: 2D

**locale** *Tarski-2D* = *Tarski-neutral-dimensionless* +
**assumes** *upper-dim*: $\forall\ a\ b\ c\ p\ q.$
$p \neq q\ \wedge$
$Cong\ a\ p\ a\ q\ \wedge$
$Cong\ b\ p\ b\ q\ \wedge$
$Cong\ c\ p\ c\ q$
$\longrightarrow$
$(Bet\ a\ b\ c \vee Bet\ b\ c\ a \vee Bet\ c\ a\ b)$

# 2 Definitions

## 2.1 Tarski's axiom system for neutral geometry: dimensionless

**context** *Tarski-neutral-dimensionless*
**begin**

### 2.1.1 Congruence

**definition** *OFSC* ::
$[{}'p,{}'p,{}'p,{}'p,{}'p,{}'p,{}'p,{}'p] \Rightarrow bool$
(- - - - *OFSC* - - - - [*99*,*99*,*99*,*99*,*99*,*99*,*99*,*99*] *50*)
**where**
$A\ B\ C\ D\ OFSC\ A'\ B'\ C'\ D' \equiv$

$Bet\ A\ B\ C\ \wedge$
$Bet\ A'\ B'\ C'\ \wedge$
$Cong\ A\ B\ A'\ B'\ \wedge$
$Cong\ B\ C\ B'\ C'\ \wedge$
$Cong\ A\ D\ A'\ D'\ \wedge$
$Cong\ B\ D\ B'\ D'$

**definition** *Cong3* ::
$[{}'p,{}'p,{}'p,{}'p,{}'p,{}'p] \Rightarrow bool$
(- - - *Cong3* - - - [*99*,*99*,*99*,*99*,*99*,*99*] *50*)
**where**
$A\ B\ C\ Cong3\ A'\ B'\ C' \equiv$

$Cong\ A\ B\ A'\ B'\ \wedge$
$Cong\ A\ C\ A'\ C'\ \wedge$
$Cong\ B\ C\ B'\ C'$

### 2.1.2 Betweenness

**definition** *Col* ::
$[{}'p,{}'p,{}'p] \Rightarrow bool$
(*Col* - - - [*99*,*99*,*99*] *50*)
**where**
$Col\ A\ B\ C \equiv$

$Bet\ A\ B\ C \vee Bet\ B\ C\ A \vee Bet\ C\ A\ B$

**definition** *Bet4* ::
$[{}'p,{}'p,{}'p,{}'p] \Rightarrow bool$
(*Bet4* - - - - [*99*,*99*,*99*,*99*] *50*)
**where**
$Bet4\ A1\ A2\ A3\ A4 \equiv$

$Bet\ A1\ A2\ A3\ \wedge$
$Bet\ A2\ A3\ A4\ \wedge$
$Bet\ A1\ A3\ A4\ \wedge$
$Bet\ A1\ A2\ A4$

**definition** *BetS* ::
$[{}'p,{}'p,{}'p] \Rightarrow bool$ (*BetS* - - - [*99*,*99*,*99*] *50*)
**where**
$BetS\ A\ B\ C \equiv$

$Bet\ A\ B\ C\ \wedge$
$A \neq B\ \wedge$
$B \neq C$

### 2.1.3 Collinearity

**definition** *FSC* ::
$[{}'p,{}'p,{}'p,{}'p,{}'p,{}'p,{}'p,{}'p] \Rightarrow bool$
(- - - - *FSC* - - - - [*99*,*99*,*99*,*99*,*99*,*99*,*99*,*99*] *50*)

**where**
$A\ B\ C\ D\ FSC\ A'\ B'\ C'\ D' \equiv$

$Col\ A\ B\ C \wedge$
$A\ B\ C\ Cong3\ A'\ B'\ C' \wedge$
$Cong\ A\ D\ A'\ D' \wedge$
$Cong\ B\ D\ B'\ D'$

### 2.1.4 Congruence and Betweenness

**definition** $IFSC$ ::
$['p,'p,'p,'p,'p,'p,'p,'p] \Rightarrow bool$
$(\text{- - - - } IFSC \text{ - - - - } [99,99,99,99,99,99,99,99]\ 50)$
**where**
$A\ B\ C\ D\ IFSC\ A'\ B'\ C'\ D' \equiv$

$Bet\ A\ B\ C \wedge$
$Bet\ A'\ B'\ C' \wedge$
$Cong\ A\ C\ A'\ C' \wedge$
$Cong\ B\ C\ B'\ C' \wedge$
$Cong\ A\ D\ A'\ D' \wedge$
$Cong\ C\ D\ C'\ D'$

### 2.1.5 Between transivitity LE

**definition** $Le$ ::
$['p,'p,'p,'p] \Rightarrow bool\ (\text{- - } Le \text{ - - } [99,99,99,99]\ 50)$
**where** $A\ B\ Le\ C\ D \equiv$

$\exists\ E.\ (Bet\ C\ E\ D \wedge Cong\ A\ B\ C\ E)$

**definition** $Lt$ ::
$['p,'p,'p,'p] \Rightarrow bool\ (\text{- - } Lt \text{ - - } [99,99,99,99]\ 50)$
**where** $A\ B\ Lt\ C\ D \equiv$

$A\ B\ Le\ C\ D \wedge \neg\ Cong\ A\ B\ C\ D$

**definition** $Ge$ ::
$['p,'p,'p,'p] \Rightarrow bool\ (\text{- -}Ge \text{ - - } [99,99,99,99]\ 50)$
**where** $A\ B\ Ge\ C\ D \equiv$

$C\ D\ Le\ A\ B$

**definition** $Gt$ ::
$['p,'p,'p,'p] \Rightarrow bool\ (\text{- - } Gt \text{ - - } [99,99,99,99]\ 50)$
**where** $A\ B\ Gt\ C\ D \equiv$

$C\ D\ Lt\ A\ B$

### 2.1.6 Out lines

**definition** $Out$ ::
$['p,'p,'p] \Rightarrow bool\ (\text{- } Out \text{ - - } [99,99,99]\ 50)$
**where** $P\ Out\ A\ B \equiv$

$A \neq P \wedge$
$B \neq P \wedge$
$(Bet\ P\ A\ B \vee Bet\ P\ B\ A)$

### 2.1.7 Midpoint

**definition** $Midpoint$ ::
$['p,'p,'p] \Rightarrow bool\ (\text{- } Midpoint \text{ - - } [99,99,99]\ 50)$
**where** $M\ Midpoint\ A\ B \equiv$

$Bet\ A\ M\ B \wedge$

$Cong\ A\ M\ M\ B$

### 2.1.8 Orthogonality

**definition** $Per$ ::
$['p,'p,'p] \Rightarrow bool\ (Per\ \text{- - -}\ [99,99,99]\ 50)$
**where** $Per\ A\ B\ C \equiv$

$\exists\ C'::'p.\ (B\ Midpoint\ C\ C' \wedge Cong\ A\ C\ A\ C')$

**definition** $PerpAt$ ::
$['p,'p,'p,'p,'p] \Rightarrow bool\ (\text{-}\ PerpAt\ \text{- - - -}\ [99,99,99,99,99]\ 50)$
**where** $X\ PerpAt\ A\ B\ C\ D \equiv$

$A \neq B \wedge$
$C \neq D \wedge$
$Col\ X\ A\ B \wedge$
$Col\ X\ C\ D \wedge$
$(\forall\ U\ V.\ ((Col\ U\ A\ B \wedge Col\ V\ C\ D) \longrightarrow Per\ U\ X\ V))$

**definition** $Perp$ ::
$['p,'p,'p,'p] \Rightarrow bool\ (\text{- -}\ Perp\ \text{- -}\ [99,99,99,99]\ 50)$
**where** $A\ B\ Perp\ C\ D \equiv$

$\exists\ X::'p.\ X\ PerpAt\ A\ B\ C\ D$

### 2.1.9 Coplanar

**definition** $Coplanar$ ::
$['p,'p,'p,'p] \Rightarrow bool\ (Coplanar\ \text{- - - -}\ [99,99,99,99]\ 50)$
**where** $Coplanar\ A\ B\ C\ D \equiv$
$\exists\ X.\ (Col\ A\ B\ X \wedge Col\ C\ D\ X) \vee$
$(Col\ A\ C\ X \wedge Col\ B\ D\ X) \vee$
$(Col\ A\ D\ X \wedge Col\ B\ C\ X)$

**definition** $TS$ ::
$['p,'p,'p,'p] \Rightarrow bool\ (\text{- -}\ TS\ \text{- -}\ [99,99,99,99]\ 50)$
**where** $A\ B\ TS\ P\ Q \equiv$
$\neg\ Col\ P\ A\ B \wedge \neg\ Col\ Q\ A\ B \wedge (\exists\ T::'p.\ Col\ T\ A\ B \wedge Bet\ P\ T\ Q)$

**definition** $ReflectL$ ::
$['p,'p,'p,'p] \Rightarrow bool\ (\text{- -}\ ReflectL\ \text{- -}\ [99,99,99,99]\ 50)$
**where** $P'\ P\ ReflectL\ A\ B \equiv$
$(\exists\ X.\ X\ Midpoint\ P\ P' \wedge Col\ A\ B\ X) \wedge (A\ B\ Perp\ P\ P' \vee P = P')$

**definition** $Reflect$ ::
$['p,'p,'p,'p] \Rightarrow bool\ (\text{- -}\ Reflect\ \text{- -}\ [99,99,99,99]\ 50)$
**where** $P'\ P\ Reflect\ A\ B \equiv$
$(A \neq B \wedge P'\ P\ ReflectL\ A\ B) \vee (A = B \wedge A\ Midpoint\ P\ P')$

**definition** $InAngle$ ::
$['p,'p,'p,'p] \Rightarrow bool\ (\text{-}\ InAngle\ \text{- - -}\ [99,99,99,99]\ 50)$
**where** $P\ InAngle\ A\ B\ C \equiv$
$A \neq B \wedge C \neq B \wedge P \neq B \wedge$
$(\exists\ X.\ Bet\ A\ X\ C \wedge (X = B \vee B\ Out\ X\ P))$

**definition** $ParStrict$::
$['p,'p,'p,'p] \Rightarrow bool\ (\text{- -}\ ParStrict\ \text{- -}\ [99,99,99,99]\ 50)$
**where** $A\ B\ ParStrict\ C\ D \equiv\ Coplanar\ A\ B\ C\ D \wedge \neg\ (\exists\ X.\ Col\ X\ A\ B \wedge Col\ X\ C\ D)$

**definition** $Par$::
$['p,'p,'p,'p] \Rightarrow bool\ (\text{- -}\ Par\ \text{- -}\ [99,99,99,99]\ 50)$
**where** $A\ B\ Par\ C\ D \equiv$
$A\ B\ ParStrict\ C\ D \vee (A \neq B \wedge C \neq D \wedge Col\ A\ C\ D \wedge Col\ B\ C\ D)$

**definition** $Plg$::
$['p,'p,'p,'p] \Rightarrow bool\ (Plg\ \text{- - - -}\ [99,99,99,99]\ 50)$

**where** $Plg\ \ A\ B\ C\ D \equiv$
$(A \neq C \vee B \neq D) \wedge (\exists\ M.\ M\ Midpoint\ A\ C \wedge M\ Midpoint\ B\ D)$

**definition** *ParallelogramStrict*::
$[{}'p,{}'p,{}'p,{}'p] \Rightarrow bool\ (ParallelogramStrict\ \text{- - - -}\ [99,99,99,99]\ 50)$
**where** $ParallelogramStrict\ A\ B\ A'\ B' \equiv$
$A\ A'\ TS\ B\ B' \wedge A\ B\ Par\ A'\ B' \wedge Cong\ A\ B\ A'\ B'$

**definition** *ParallelogramFlat*::
$[{}'p,{}'p,{}'p,{}'p] \Rightarrow bool\ (ParallelogramFlat\ \text{- - - -}\ [99,99,99,99]\ 50)$
**where** $ParallelogramFlat\ A\ B\ A'\ B' \equiv$
$Col\ A\ B\ A' \wedge Col\ A\ B\ B' \wedge$
$Cong\ A\ B\ A'\ B' \wedge Cong\ A\ B'\ A'\ B \wedge$
$(A \neq A' \vee B \neq B')$

**definition** *Parallelogram*::
$[{}'p,{}'p,{}'p,{}'p] \Rightarrow bool\ (Parallelogram\ \text{- - - -}\ [99,99,99,99]\ 50)$
**where** $Parallelogram\ A\ B\ A'\ B' \equiv$
$ParallelogramStrict\ A\ B\ A'\ B' \vee ParallelogramFlat\ A\ B\ A'\ B'$

**definition** *Rhombus*::
$[{}'p,{}'p,{}'p,{}'p] \Rightarrow bool\ (Rhombus\ \text{- - - -}\ [99,99,99,99]\ 50)$
**where** $Rhombus\ A\ B\ C\ D \equiv Plg\ A\ B\ C\ D \wedge Cong\ A\ B\ B\ C$

**definition** *Rectangle*::
$[{}'p,{}'p,{}'p,{}'p] \Rightarrow bool\ (Rectangle\ \text{- - - -}\ [99,99,99,99]\ 50)$
**where** $Rectangle\ A\ B\ C\ D \equiv Plg\ A\ B\ C\ D \wedge Cong\ A\ C\ B\ D$

**definition** *Square*::
$[{}'p,{}'p,{}'p,{}'p] \Rightarrow bool\ (Square\ \text{- - - -}\ [99,99,99,99]\ 50)$
**where** $Square\ A\ B\ C\ D \equiv Rectangle\ A\ B\ C\ D \wedge Cong\ A\ B\ B\ C$

**definition** *Lambert*::
$[{}'p,{}'p,{}'p,{}'p] \Rightarrow bool\ (Lambert\ \text{- - - -}\ [99,99,99,99]\ 50)$
**where** $Lambert\ A\ B\ C\ D \equiv$
$A \neq B \wedge B \neq C \wedge C \neq D \wedge$
$A \neq D \wedge Per\ B\ A\ D \wedge Per\ A\ D\ C \wedge Per\ A\ B\ C \wedge Coplanar\ A\ B\ C\ D$

### 2.1.10 Plane

**definition** *OS* ::
$[{}'p,{}'p,{}'p,{}'p] \Rightarrow bool\ (\text{- -}\ OS\ \text{- -}\ [99,99,99,99]\ 50)$
**where** $A\ B\ OS\ P\ Q \equiv$
$\exists\ R::{}'p.\ A\ B\ TS\ P\ R \wedge A\ B\ TS\ Q\ R$

**definition** *TSP* ::
$[{}'p,{}'p,{}'p,{}'p,{}'p] \Rightarrow bool\ (\text{- - -}TSP\ \text{- -}\ [99,99,99,99,99]\ 50)$
**where** $A\ B\ C\ TSP\ P\ Q \equiv$
$(\neg\ Coplanar\ A\ B\ C\ P) \wedge (\neg\ Coplanar\ A\ B\ C\ Q) \wedge$
$(\exists\ T.\ Coplanar\ A\ B\ C\ T \wedge Bet\ P\ T\ Q)$

**definition** *OSP* ::
$[{}'p,{}'p,{}'p,{}'p,{}'p] \Rightarrow bool\ (\text{- - -}\ OSP\ \text{- -}\ [99,99,99,99,99]\ 50)$
**where** $A\ B\ C\ OSP\ P\ Q \equiv$
$\exists\ R.\ ((A\ B\ C\ TSP\ P\ R) \wedge (A\ B\ C\ TSP\ Q\ R))$

**definition** *Saccheri*::
$[{}'p,{}'p,{}'p,{}'p] \Rightarrow bool\ (Saccheri\ \text{- - - -}\ [99,99,99,99]\ 50)$
**where** $Saccheri\ A\ B\ C\ D \equiv$
$Per\ B\ A\ D \wedge Per\ A\ D\ C \wedge Cong\ A\ B\ C\ D \wedge A\ D\ OS\ B\ C$

### 2.1.11 Line reflexivity 2D

**definition** *ReflectLAt* ::
$[{}'p,{}'p,{}'p,{}'p,{}'p] \Rightarrow bool\ (\text{-}\ ReflectLAt\ \text{- - - -}\ [99,99,99,99,99]\ 50)$
**where** $M\ ReflectLAt\ P'\ P\ A\ B \equiv$
$(M\ Midpoint\ P\ P' \wedge Col\ A\ B\ M) \wedge (A\ B\ Perp\ P\ P' \vee P = P')$

**definition** *ReflectAt* ::
$['p,'p,'p,'p,'p] \Rightarrow bool$ (- *ReflectAt* - - - - [99,99,99,99,99] 50)
**where** $M$ *ReflectAt* $P'$ $P$ $A$ $B$ $\equiv$
$(A \neq B \wedge M\ ReflectLAt\ P'\ P\ A\ B) \vee (A = B \wedge A = M \wedge M\ Midpoint\ P\ P')$

### 2.1.12 Line reflexivity

**definition** *upper-dim-axiom* ::
*bool* (*UpperDimAxiom* [] 50)
**where**
*upper-dim-axiom* $\equiv$

$\forall\ A\ B\ C\ P\ Q.$
$P \neq Q \wedge$
$Cong\ A\ P\ A\ Q \wedge$
$Cong\ B\ P\ B\ Q \wedge$
$Cong\ C\ P\ C\ Q$
$\longrightarrow$
$(Bet\ A\ B\ C \vee Bet\ B\ C\ A \vee Bet\ C\ A\ B)$

**definition** *all-coplanar-axiom* ::
*bool* (*AllCoplanarAxiom* [] 50)
**where**
*AllCoplanarAxiom* $\equiv$

$\forall\ A\ B\ C\ P\ Q.$
$P \neq Q \wedge$
$Cong\ A\ P\ A\ Q \wedge$
$Cong\ B\ P\ B\ Q \wedge$
$Cong\ C\ P\ C\ Q$
$\longrightarrow$
$(Bet\ A\ B\ C \vee Bet\ B\ C\ A \vee Bet\ C\ A\ B)$

### 2.1.13 Angles

**definition** *CongA* ::
$['p,'p,'p,'p,'p,'p] \Rightarrow bool$ (- - - *CongA* - - - [99,99,99,99,99,99] 50)
**where** $A\ B\ C\ CongA\ D\ E\ F \equiv$
$A \neq B \wedge C \neq B \wedge D \neq E \wedge F \neq E \wedge$
$(\exists\ A'\ C'\ D'\ F'.\ Bet\ B\ A\ A' \wedge Cong\ A\ A'\ E\ D \wedge$
$Bet\ B\ C\ C' \wedge Cong\ C\ C'\ E\ F \wedge$
$Bet\ E\ D\ D' \wedge Cong\ D\ D'\ B\ A \wedge$
$Bet\ E\ F\ F' \wedge Cong\ F\ F'\ B\ C \wedge$
$Cong\ A'\ C'\ D'\ F')$

**definition** *LeA* ::
$['p,'p,'p,'p,'p,'p] \Rightarrow bool$ (- - - *LeA* - - - [99,99,99,99,99,99] 50)
**where** $A\ B\ C\ LeA\ D\ E\ F \equiv$
$\exists\ P.\ (P\ InAngle\ D\ E\ F \wedge A\ B\ C\ CongA\ D\ E\ P)$

**definition** *LtA* ::
$['p,'p,'p,'p,'p,'p] \Rightarrow bool$ (- - - *LtA* - - - [99,99,99,99,99,99] 50)
**where** $A\ B\ C\ LtA\ D\ E\ F \equiv A\ B\ C\ LeA\ D\ E\ F \wedge \neg\ A\ B\ C\ CongA\ D\ E\ F$

**definition** *GtA* ::
$['p,'p,'p,'p,'p,'p] \Rightarrow bool$ (- - - *GtA* - - - [99,99,99,99,99,99] 50)
**where** $A\ B\ C\ GtA\ D\ E\ F \equiv D\ E\ F\ LtA\ A\ B\ C$

**definition** *Acute* ::
$['p,'p,'p] \Rightarrow bool$ (*Acute* - - - [99,99,99] 50)
**where** $Acute\ A\ B\ C \equiv$
$\exists\ A'\ B'\ C'.\ (Per\ A'\ B'\ C' \wedge A\ B\ C\ LtA\ A'\ B'\ C')$

**definition** *Obtuse* ::
$['p,'p,'p] \Rightarrow bool$ (*Obtuse* - - - [99,99,99] 50)
**where** $Obtuse\ A\ B\ C \equiv$

$\exists\ A'\ B'\ C'.\ (Per\ A'\ B'\ C' \wedge A'\ B'\ C'\ LtA\ A\ B\ C)$

**definition** *OrthAt* ::
$['p,'p,'p,'p,'p,'p] \Rightarrow bool$ (- *OrthAt* - - - - - [99,99,99,99,99,99] 50)
**where** $X\ OrthAt\ A\ B\ C\ U\ V \equiv$
$\neg\ Col\ A\ B\ C \wedge U \neq V \wedge Coplanar\ A\ B\ C\ X \wedge Col\ U\ V\ X \wedge$
$(\forall\ P\ Q.\ (Coplanar\ A\ B\ C\ P \wedge Col\ U\ V\ Q) \longrightarrow Per\ P\ X\ Q)$

**definition** *Orth* ::
$['p,'p,'p,'p,'p] \Rightarrow bool$ (- - - *Orth* - - [99,99,99,99,99] 50)
**where** $A\ B\ C\ Orth\ U\ V \equiv \exists\ X.\ X\ OrthAt\ A\ B\ C\ U\ V$

**definition** *SuppA* ::
$['p,'p,'p,'p,'p,'p] \Rightarrow bool$
(- - - *SuppA* - - - [99,99,99,99,99,99] 50)
**where**
$A\ B\ C\ SuppA\ D\ E\ F \equiv$
$A \neq B \wedge (\exists\ A'.\ Bet\ A\ B\ A' \wedge\ D\ E\ F\ CongA\ C\ B\ A')$

### 2.1.14 Sum of angles

**definition** *SumA* ::
$['p,'p,'p,'p,'p,'p,'p,'p,'p] \Rightarrow bool$ (- - - - - - *SumA* - - - [99,99,99,99,99,99,99,99,99] 50)
**where**
$A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I \equiv$

$\exists\ J.\ (C\ B\ J\ CongA\ D\ E\ F \wedge \neg\ B\ C\ OS\ A\ J \wedge Coplanar\ A\ B\ C\ J \wedge A\ B\ J\ CongA\ G\ H\ I)$

**definition** *TriSumA* ::
$['p,'p,'p,'p,'p,'p] \Rightarrow bool$ (- - - *TriSumA* - - - [99,99,99,99,99,99] 50)
**where**
$A\ B\ C\ TriSumA\ D\ E\ F \equiv$

$\exists\ G\ H\ I.\ (A\ B\ C\ B\ C\ A\ SumA\ G\ H\ I \wedge G\ H\ I\ C\ A\ B\ SumA\ D\ E\ F)$

**definition** *SAMS* ::
$['p,'p,'p,'p,'p,'p] \Rightarrow bool$ (*SAMS* - - - - - - [99,99,99,99,99,99] 50)
**where**
$SAMS\ A\ B\ C\ D\ E\ F \equiv$

$(A \neq B \wedge$
$(E\ Out\ D\ F \vee \neg\ Bet\ A\ B\ C)) \wedge$
$(\exists\ J.\ (C\ B\ J\ CongA\ D\ E\ F \wedge \neg\ (B\ C\ OS\ A\ J) \wedge \neg\ (A\ B\ TS\ C\ J) \wedge Coplanar\ A\ B\ C\ J))$

### 2.1.15 Parallelism

**definition** *Inter* ::
$['p,'p,'p,'p,'p] \Rightarrow bool$ (- *Inter* - - - - [99,99,99,99,99] 50)
**where** $X\ Inter\ A1\ A2\ B1\ B2 \equiv$

$B1 \neq B2 \wedge$
$(\exists\ P{::}'p.\ (Col\ P\ B1\ B2 \wedge \neg\ Col\ P\ A1\ A2)) \wedge$
$Col\ A1\ A2\ X \wedge Col\ B1\ B2\ X$

### 2.1.16 Perpendicularity

**definition** *Perp2* ::
$['p,'p,'p,'p,'p] \Rightarrow bool$ (- *Perp2* - - - - [99,99,99,99,99] 50)
**where**
$P\ Perp2\ A\ B\ C\ D \equiv$

$\exists\ X\ Y.\ (Col\ P\ X\ Y \wedge X\ Y\ Perp\ A\ B \wedge X\ Y\ Perp\ C\ D)$

### 2.1.17 Lentgh

**definition** *QCong*::
$(['p,'p] \Rightarrow bool) \Rightarrow bool$ (*QCong* - [99] 50)

**where**
$QCong\ l \equiv$

$\exists\ A\ B.\ (\forall\ X\ Y.\ (Cong\ A\ B\ X\ Y \longleftrightarrow l\ X\ Y))$

**definition** *TarskiLen*::
$['p,'p,(['p,'p] \Rightarrow bool)] \Rightarrow bool\ (TarskiLen\ \text{- - -}\ [99,99,99]\ 50)$
**where**
$TarskiLen\ A\ B\ l \equiv$

$QCong\ l \wedge l\ A\ B$

**definition** *QCongNull* ::
$(['p,'p] \Rightarrow bool) \Rightarrow bool\ (QCongNull\ \text{-}\ [99]\ 50)$
**where**
$QCongNull\ l \equiv$

$QCong\ l \wedge (\exists\ A.\ l\ A\ A)$

### 2.1.18 Equivalence Class of Angles

**definition** *QCongA* ::
$(['p,\ 'p,\ 'p] \Rightarrow bool) \Rightarrow bool\ (QCongA\ \text{-}\ [99]\ 50)$
**where**
$QCongA\ a \equiv$

$\exists\ A\ B\ C.\ (A \neq B \wedge C \neq B \wedge (\forall\ X\ Y\ Z.\ A\ B\ C\ CongA\ X\ Y\ Z \longleftrightarrow a\ X\ Y\ Z))$

**definition** *Ang* ::
$['p,'p,'p,\ (['p,\ 'p,\ 'p] \Rightarrow bool)\ ] \Rightarrow bool\ (\text{- - -}\ Ang\ \text{-}\ [99,99,99,99]\ 50)$
**where**
$A\ B\ C\ Ang\ a \equiv$

$QCongA\ a \wedge$
$a\ A\ B\ C$

**definition** *QCongAAcute* ::
$(['p,\ 'p,\ 'p] \Rightarrow bool) \Rightarrow bool\ (QCongAACute\ \text{-}\ [99]\ 50)$
**where**
$QCongAAcute\ a \equiv$

$\exists\ A\ B\ C.\ (Acute\ A\ B\ C \wedge (\forall\ X\ Y\ Z.\ (A\ B\ C\ CongA\ X\ Y\ Z \longleftrightarrow a\ X\ Y\ Z)))$

**definition** *AngAcute* ::
$['p,'p,'p,\ (['p,'p,'p] \Rightarrow bool)] \Rightarrow bool\ (\text{- - -}\ AngAcute\ \text{-}\ [99,99,99,99]\ 50)$
**where**
$A\ B\ C\ AngAcute\ a \equiv$

$((QCongAAcute\ a) \wedge (a\ A\ B\ C))$

**definition** *QCongANullAcute* ::
$(['p,'p,'p] \Rightarrow bool) \Rightarrow bool\ (QCongANullAcute\ \text{-}\ [99]\ 50)$
**where**
$QCongANullAcute\ a \equiv$

$QCongAAcute\ a \wedge$
$(\forall\ A\ B\ C.\ (a\ A\ B\ C \longrightarrow B\ Out\ A\ C))$

**definition** *QCongAnNull* ::
$(['p,'p,'p] \Rightarrow bool) \Rightarrow bool\ (QCongAnNull\ \text{-}\ [99]\ 50)$
**where**
$QCongAnNull\ a \equiv$

$QCongA\ a \wedge$
$(\forall\ A\ B\ C.\ (a\ A\ B\ C \longrightarrow \neg\ B\ Out\ A\ C))$

**definition** *QCongAnFlat* ::
$(['p,'p,'p] \Rightarrow bool) \Rightarrow bool$ (*QCongAnFlat* - [*99*] *50*)
**where**
*QCongAnFlat a* $\equiv$

*QCongA a* $\wedge$
$(\forall\ A\ B\ C.\ (a\ A\ B\ C \longrightarrow \neg\ Bet\ A\ B\ C))$

**definition** *IsNullAngaP* ::
$(['p,'p,'p] \Rightarrow bool) \Rightarrow bool$ (*IsNullAngaP* - [*99*] *50*)
**where**
*IsNullAngaP a* $\equiv$

*QCongAAcute a* $\wedge$
$(\exists\ A\ B\ C.\ (a\ A\ B\ C \wedge B\ Out\ A\ C))$

**definition** *QCongANull* ::
$(['p,'p,'p] \Rightarrow bool) \Rightarrow bool$ (*QCongANull* - [*99*] *50*)
**where**
*QCongANull a* $\equiv$

*QCongA a* $\wedge$
$(\forall\ A\ B\ C.\ (a\ A\ B\ C \longrightarrow B\ Out\ A\ C))$

**definition** *AngFlat* ::
$(['p,\ 'p,\ 'p] \Rightarrow bool) \Rightarrow bool$ (*AngFlat* - [*99*] *50*)
**where**
*AngFlat a* $\equiv$

*QCongA a* $\wedge$
$(\forall\ A\ B\ C.\ (a\ A\ B\ C \longrightarrow Bet\ A\ B\ C))$

## 2.2 Parallel's definition Postulate

**definition** *tarski-s-parallel-postulate* ::
*bool*
(*TarskiSParallelPostulate*)
**where**
*tarski-s-parallel-postulate* $\equiv$
$\forall\ A\ B\ C\ D\ T.\ (Bet\ A\ D\ T \wedge Bet\ B\ D\ C \wedge A \neq D) \longrightarrow$
$(\exists\ X\ Y.\ Bet\ A\ B\ X \wedge Bet\ A\ C\ Y \wedge Bet\ X\ T\ Y)$

**definition** *euclid-5* ::
*bool* (*Euclid5*)
**where**
*euclid-5* $\equiv$

$\forall\ P\ Q\ R\ S\ T\ U.$
$(BetS\ P\ T\ Q \wedge$
$BetS\ R\ T\ S \wedge$
$BetS\ Q\ U\ R \wedge$
$\neg\ Col\ P\ Q\ S \wedge$
$Cong\ P\ T\ Q\ T \wedge$
$Cong\ R\ T\ S\ T)$
$\longrightarrow$
$(\exists\ I.\ BetS\ S\ Q\ I \wedge BetS\ P\ U\ I)$

**definition** *euclid-s-parallel-postulate* ::
*bool* (*EuclidSParallelPostulate*)
**where**
*euclid-s-parallel-postulate* $\equiv$

$\forall\ A\ B\ C\ D\ P\ Q\ R.$
$(B\ C\ OS\ A\ D \wedge$
$SAMS\ A\ B\ C\ B\ C\ D \wedge$
$A\ B\ C\ B\ C\ D\ SumA\ P\ Q\ R \wedge$

$\neg$ *Bet P Q R*)
$\longrightarrow$
($\exists$ *Y*. *B Out A Y* $\wedge$ *C Out D Y*)

**definition** *playfair-s-postulate* ::
*bool*
(*PlayfairSPostulate*)
**where**
*playfair-s-postulate* $\equiv$

$\forall$ *A1 A2 B1 B2 C1 C2 P*.
(*A1 A2 Par B1 B2* $\wedge$
*Col P B1 B2* $\wedge$
*A1 A2 Par C1 C2* $\wedge$
*Col P C1 C2*)
$\longrightarrow$
(*Col C1 B1 B2* $\wedge$ *Col C2 B1 B2*)

# 3 Propositions

## 3.1 Congruence properties

**lemma** *cong-reflexivity*:
**shows** *Cong A B A B*
⟨*proof*⟩

**lemma** *cong-symmetry*:
**assumes** *Cong A B C D*
**shows** *Cong C D A B*
⟨*proof*⟩

**lemma** *cong-transitivity*:
**assumes** *Cong A B C D* **and** *Cong C D E F*
**shows** *Cong A B E F*
⟨*proof*⟩

**lemma** *cong-left-commutativity*:
**assumes** *Cong A B C D*
**shows** *Cong B A C D*
⟨*proof*⟩

**lemma** *cong-right-commutativity*:
**assumes** *Cong A B C D*
**shows** *Cong A B D C*
⟨*proof*⟩

**lemma** *cong-3421*:
**assumes** *Cong A B C D*
**shows** *Cong C D B A*
⟨*proof*⟩

**lemma** *cong-4312*:
**assumes** *Cong A B C D*
**shows** *Cong D C A B*
⟨*proof*⟩

**lemma** *cong-4321*:
**assumes** *Cong A B C D*
**shows** *Cong D C B A*
⟨*proof*⟩

**lemma** *cong-trivial-identity*:
**shows** *Cong A A B B*
⟨*proof*⟩

**lemma** *cong-reverse-identity*:

**assumes** *Cong* $A\ A\ C\ D$
**shows** $C = D$
⟨*proof*⟩

**lemma** *cong-commutativity*:
**assumes** *Cong* $A\ B\ C\ D$
**shows** *Cong* $B\ A\ D\ C$
⟨*proof*⟩

**lemma** *not-cong-2134*:
**assumes** $\neg$ *Cong* $A\ B\ C\ D$
**shows** $\neg$ *Cong* $B\ A\ C\ D$
⟨*proof*⟩

**lemma** *not-cong-1243*:
**assumes** $\neg$ *Cong* $A\ B\ C\ D$
**shows** $\neg$ *Cong* $A\ B\ D\ C$
⟨*proof*⟩

**lemma** *not-cong-2143*:
**assumes** $\neg$ *Cong* $A\ B\ C\ D$
**shows** $\neg$ *Cong* $B\ A\ D\ C$
⟨*proof*⟩

**lemma** *not-cong-3412*:
**assumes** $\neg$ *Cong* $A\ B\ C\ D$
**shows** $\neg$ *Cong* $C\ D\ A\ B$
⟨*proof*⟩

**lemma** *not-cong-4312*:
**assumes** $\neg$ *Cong* $A\ B\ C\ D$
**shows** $\neg$ *Cong* $D\ C\ A\ B$
⟨*proof*⟩

**lemma** *not-cong-3421*:
**assumes** $\neg$ *Cong* $A\ B\ C\ D$
**shows** $\neg$ *Cong* $C\ D\ B\ A$
⟨*proof*⟩

**lemma** *not-cong-4321*:
**assumes** $\neg$ *Cong* $A\ B\ C\ D$
**shows** $\neg$ *Cong* $D\ C\ B\ A$
⟨*proof*⟩

**lemma** *five-segment-with-def*:
**assumes** $A\ B\ C\ D$ *OFSC* $A'\ B'\ C'\ D'$ **and** $A \neq B$
**shows** *Cong* $C\ D\ C'\ D'$
⟨*proof*⟩

**lemma** *cong-diff*:
**assumes** $A \neq B$ **and** *Cong* $A\ B\ C\ D$
**shows** $C \neq D$
⟨*proof*⟩

**lemma** *cong-diff-2*:
**assumes** $B \neq A$ **and** *Cong* $A\ B\ C\ D$
**shows** $C \neq D$
⟨*proof*⟩

**lemma** *cong-diff-3*:
**assumes** $C \neq D$ **and** *Cong* $A\ B\ C\ D$
**shows** $A \neq B$
⟨*proof*⟩

**lemma** *cong-diff-4*:
**assumes** $D \neq C$ **and** *Cong* $A\ B\ C\ D$

**shows** $A \neq B$
⟨*proof*⟩

**lemma** *cong-3-sym*:
**assumes** $A\ B\ C\ \mathit{Cong3}\ A'\ B'\ C'$
**shows** $A'\ B'\ C'\ \mathit{Cong3}\ A\ B\ C$
⟨*proof*⟩

**lemma** *cong-3-swap*:
**assumes** $A\ B\ C\ \mathit{Cong3}\ A'\ B'\ C'$
**shows** $B\ A\ C\ \mathit{Cong3}\ B'\ A'\ C'$
⟨*proof*⟩

**lemma** *cong-3-swap-2*:
**assumes** $A\ B\ C\ \mathit{Cong3}\ A'\ B'\ C'$
**shows** $A\ C\ B\ \mathit{Cong3}\ A'\ C'\ B'$
⟨*proof*⟩

**lemma** *cong3-transitivity*:
**assumes** $A0\ B0\ C0\ \mathit{Cong3}\ A1\ B1\ C1$ **and**
$A1\ B1\ C1\ \mathit{Cong3}\ A2\ B2\ C2$
**shows** $A0\ B0\ C0\ \mathit{Cong3}\ A2\ B2\ C2$
⟨*proof*⟩

**lemma** *eq-dec-points*:
**shows** $A = B \lor \neg\ A = B$
⟨*proof*⟩

**lemma** *distinct*:
**assumes** $P \neq Q$
**shows** $R \neq P \lor R \neq Q$
⟨*proof*⟩

**lemma** *l2-11*:
**assumes** $\mathit{Bet}\ A\ B\ C$ **and**
$\mathit{Bet}\ A'\ B'\ C'$ **and**
$\mathit{Cong}\ A\ B\ A'\ B'$ **and**
$\mathit{Cong}\ B\ C\ B'\ C'$
**shows** $\mathit{Cong}\ A\ C\ A'\ C'$
⟨*proof*⟩

**lemma** *bet-cong3*:
**assumes** $\mathit{Bet}\ A\ B\ C$ **and**
$\mathit{Cong}\ A\ B\ A'\ B'$
**shows** $\exists\ C'.\ A\ B\ C\ \mathit{Cong3}\ A'\ B'\ C'$
⟨*proof*⟩

**lemma** *construction-uniqueness*:
**assumes** $Q \neq A$ **and**
$\mathit{Bet}\ Q\ A\ X$ **and**
$\mathit{Cong}\ A\ X\ B\ C$ **and**
$\mathit{Bet}\ Q\ A\ Y$ **and**
$\mathit{Cong}\ A\ Y\ B\ C$
**shows** $X = Y$
⟨*proof*⟩

**lemma** *Cong-cases*:
**assumes** $\mathit{Cong}\ A\ B\ C\ D \lor \mathit{Cong}\ A\ B\ D\ C \lor \mathit{Cong}\ B\ A\ C\ D \lor \mathit{Cong}\ B\ A\ D\ C \lor \mathit{Cong}\ C\ D\ A\ B \lor \mathit{Cong}\ C\ D\ B\ A \lor \mathit{Cong}\ D\ C\ A\ B \lor \mathit{Cong}\ D\ C\ B\ A$
**shows** $\mathit{Cong}\ A\ B\ C\ D$
⟨*proof*⟩

**lemma** *Cong-perm* :
**assumes** $\mathit{Cong}\ A\ B\ C\ D$
**shows** $\mathit{Cong}\ A\ B\ C\ D \land \mathit{Cong}\ A\ B\ D\ C \land \mathit{Cong}\ B\ A\ C\ D \land \mathit{Cong}\ B\ A\ D\ C \land \mathit{Cong}\ C\ D\ A\ B \land \mathit{Cong}\ C\ D\ B\ A \land \mathit{Cong}\ D\ C\ A\ B \land \mathit{Cong}\ D\ C\ B\ A$

⟨*proof*⟩

### 3.2 Betweeness properties

**lemma** *bet-col*:
**assumes** $Bet\ A\ B\ C$
**shows** $Col\ A\ B\ C$
⟨*proof*⟩

**lemma** *between-trivial*:
**shows** $Bet\ A\ B\ B$
⟨*proof*⟩

**lemma** *between-symmetry*:
**assumes** $Bet\ A\ B\ C$
**shows** $Bet\ C\ B\ A$
⟨*proof*⟩

**lemma** *Bet-cases*:
**assumes** $Bet\ A\ B\ C \lor Bet\ C\ B\ A$
**shows** $Bet\ A\ B\ C$
⟨*proof*⟩

**lemma** *Bet-perm*:
**assumes** $Bet\ A\ B\ C$
**shows** $Bet\ A\ B\ C \land Bet\ C\ B\ A$
⟨*proof*⟩

**lemma** *between-trivial2*:
**shows** $Bet\ A\ A\ B$
⟨*proof*⟩

**lemma** *between-equality*:
**assumes** $Bet\ A\ B\ C$ **and** $Bet\ B\ A\ C$
**shows** $A = B$
⟨*proof*⟩

**lemma** *between-equality-2*:
**assumes** $Bet\ A\ B\ C$ **and**
$Bet\ A\ C\ B$
**shows** $B = C$
⟨*proof*⟩

**lemma** *between-exchange3*:
**assumes** $Bet\ A\ B\ C$ **and**
$Bet\ A\ C\ D$
**shows** $Bet\ B\ C\ D$
⟨*proof*⟩

**lemma** *bet-neq12--neq*:
**assumes** $Bet\ A\ B\ C$ **and**
$A \neq B$
**shows** $A \neq C$
⟨*proof*⟩

**lemma** *bet-neq21--neq*:
**assumes** $Bet\ A\ B\ C$ **and**
$B \neq A$
**shows** $A \neq C$
⟨*proof*⟩

**lemma** *bet-neq23--neq*:
**assumes** $Bet\ A\ B\ C$ **and**
$B \neq C$
**shows** $A \neq C$
⟨*proof*⟩

**lemma** *bet-neq32--neq*:
  **assumes** $Bet\ A\ B\ C$ **and**
    $C \neq B$
  **shows** $A \neq C$
  ⟨*proof*⟩

**lemma** *not-bet-distincts*:
  **assumes** $\neg\ Bet\ A\ B\ C$
  **shows** $A \neq B \wedge B \neq C$
  ⟨*proof*⟩

**lemma** *between-inner-transitivity*:
  **assumes** $Bet\ A\ B\ D$ **and**
    $Bet\ B\ C\ D$
  **shows** $Bet\ A\ B\ C$
  ⟨*proof*⟩

**lemma** *outer-transitivity-between2*:
  **assumes** $Bet\ A\ B\ C$ **and**
    $Bet\ B\ C\ D$ **and**
    $B \neq C$
  **shows** $Bet\ A\ C\ D$
⟨*proof*⟩

**lemma** *between-exchange2*:
  **assumes** $Bet\ A\ B\ D$ **and**
    $Bet\ B\ C\ D$
  **shows** $Bet\ A\ C\ D$
  ⟨*proof*⟩

**lemma** *outer-transitivity-between*:
  **assumes** $Bet\ A\ B\ C$ **and**
    $Bet\ B\ C\ D$ **and**
    $B \neq C$
  **shows** $Bet\ A\ B\ D$
  ⟨*proof*⟩

**lemma** *between-exchange4*:
  **assumes** $Bet\ A\ B\ C$ **and**
    $Bet\ A\ C\ D$
  **shows** $Bet\ A\ B\ D$
  ⟨*proof*⟩

**lemma** *l3-9-4*:
  **assumes** $Bet4\ A1\ A2\ A3\ A4$
  **shows** $Bet4\ A4\ A3\ A2\ A1$
  ⟨*proof*⟩

**lemma** *l3-17*:
  **assumes** $Bet\ A\ B\ C$ **and**
    $Bet\ A'\ B'\ C$ **and**
    $Bet\ A\ P\ A'$
  **shows** $(\exists\ Q.\ Bet\ P\ Q\ C \wedge Bet\ B\ Q\ B')$
⟨*proof*⟩

**lemma** *lower-dim-ex*:
  $\exists\ A\ B\ C.\ \neg\ (Bet\ A\ B\ C \vee Bet\ B\ C\ A \vee Bet\ C\ A\ B)$
  ⟨*proof*⟩

**lemma** *two-distinct-points*:
  $\exists\ X::'p.\ \exists\ Y::'p.\ X \neq Y$
  ⟨*proof*⟩

**lemma** *point-construction-different*:
  $\exists\ C.\ Bet\ A\ B\ C \wedge B \neq C$

⟨*proof*⟩

**lemma** *another-point*:
$\exists\ B{::}'p.\ A \neq B$
⟨*proof*⟩

**lemma** *Cong-stability*:
**assumes** $\neg\ \neg\ Cong\ A\ B\ C\ D$
**shows** $Cong\ A\ B\ C\ D$
⟨*proof*⟩

**lemma** *l2-11-b*:
**assumes** $Bet\ A\ B\ C$ **and**
$Bet\ A'\ B'\ C'$ **and**
$Cong\ A\ B\ A'\ B'$ **and**
$Cong\ B\ C\ B'\ C'$
**shows** $Cong\ A\ C\ A'\ C'$
⟨*proof*⟩

**lemma** *cong-dec-eq-dec-b*:
**assumes** $\neg\ A \neq B$
**shows** $A = B$
⟨*proof*⟩

**lemma** *BetSEq*:
**assumes** $BetS\ A\ B\ C$
**shows** $Bet\ A\ B\ C \wedge A \neq B \wedge A \neq C \wedge B \neq C$
⟨*proof*⟩

## 3.3 Collinearity

### 3.3.1 Collinearity and betweenness

**lemma** *l4-2*:
**assumes** $A\ B\ C\ D\ IFSC\ A'\ B'\ C'\ D'$
**shows** $Cong\ B\ D\ B'\ D'$
⟨*proof*⟩

**lemma** *l4-3*:
**assumes** $Bet\ A\ B\ C$ **and**
$Bet\ A'\ B'\ C'$ **and**
$Cong\ A\ C\ A'\ C'$
**and** $Cong\ B\ C\ B'\ C'$
**shows** $Cong\ A\ B\ A'\ B'$
⟨*proof*⟩

**lemma** *l4-3-1*:
**assumes** $Bet\ A\ B\ C$ **and**
$Bet\ A'\ B'\ C'$ **and**
$Cong\ A\ B\ A'\ B'$ **and**
$Cong\ A\ C\ A'\ C'$
**shows** $Cong\ B\ C\ B'\ C'$
⟨*proof*⟩

**lemma** *l4-5*:
**assumes** $Bet\ A\ B\ C$ **and**
$Cong\ A\ C\ A'\ C'$
**shows** $\exists\ B'.\ (Bet\ A'\ B'\ C' \wedge A\ B\ C\ Cong3\ A'\ B'\ C')$
⟨*proof*⟩

**lemma** *l4-6*:
**assumes** $Bet\ A\ B\ C$ **and**
$A\ B\ C\ Cong3\ A'\ B'\ C'$
**shows** $Bet\ A'\ B'\ C'$
⟨*proof*⟩

**lemma** *cong3-bet-eq*:
**assumes** *Bet A B C* **and**
*A B C Cong3 A X C*
**shows** $X = B$
⟨*proof*⟩

### 3.3.2 Collinearity

**lemma** *col-permutation-1*:
**assumes** *Col A B C*
**shows** *Col B C A*
⟨*proof*⟩

**lemma** *col-permutation-2*:
**assumes** *Col A B C*
**shows** *Col C A B*
⟨*proof*⟩

**lemma** *col-permutation-3*:
**assumes** *Col A B C*
**shows** *Col C B A*
⟨*proof*⟩

**lemma** *col-permutation-4*:
**assumes** *Col A B C*
**shows** *Col B A C*
⟨*proof*⟩

**lemma** *col-permutation-5*:
**assumes** *Col A B C*
**shows** *Col A C B*
⟨*proof*⟩

**lemma** *not-col-permutation-1*:
**assumes** ¬ *Col A B C*
**shows** ¬ *Col B C A*
⟨*proof*⟩

**lemma** *not-col-permutation-2*:
**assumes** ~ *Col A B C*
**shows** ~ *Col C A B*
⟨*proof*⟩

**lemma** *not-col-permutation-3*:
**assumes** ¬ *Col A B C*
**shows** ¬ *Col C B A*
⟨*proof*⟩

**lemma** *not-col-permutation-4*:
**assumes** ¬ *Col A B C*
**shows** ¬ *Col B A C*
⟨*proof*⟩

**lemma** *not-col-permutation-5*:
**assumes** ¬ *Col A B C*
**shows** ¬ *Col A C B*
⟨*proof*⟩

**lemma** *Col-cases*:
**assumes** *Col A B C* ∨ *Col A C B* ∨ *Col B A C* ∨ *Col B C A* ∨ *Col C A B* ∨ *Col C B A*
**shows** *Col A B C*
⟨*proof*⟩

**lemma** *Col-perm*:
**assumes** *Col A B C*
**shows** *Col A B C* ∧ *Col A C B* ∧ *Col B A C* ∧ *Col B C A* ∧ *Col C A B* ∧ *Col C B A*

⟨*proof*⟩

**lemma** *col-trivial-1*:
*Col A A B*
⟨*proof*⟩

**lemma** *col-trivial-2*:
*Col A B B*
⟨*proof*⟩

**lemma** *col-trivial-3*:
*Col A B A*
⟨*proof*⟩

**lemma** *l4-13*:
**assumes** *Col A B C* **and**
$A\ B\ C\ Cong3\ A'\ B'\ C'$
**shows** $Col\ A'\ B'\ C'$
⟨*proof*⟩

**lemma** *l4-14R1*:
**assumes** *Bet A B C* **and**
$Cong\ A\ B\ A'\ B'$
**shows** $\exists\ C'.\ A\ B\ C\ Cong3\ A'\ B'\ C'$
⟨*proof*⟩

**lemma** *l4-14R2*:
**assumes** *Bet B C A* **and**
$Cong\ A\ B\ A'\ B'$
**shows** $\exists\ C'.\ A\ B\ C\ Cong3\ A'\ B'\ C'$
⟨*proof*⟩

**lemma** *l4-14R3*:
**assumes** *Bet C A B* **and**
$Cong\ A\ B\ A'\ B'$
**shows** $\exists\ C'.\ A\ B\ C\ Cong3\ A'\ B'\ C'$
⟨*proof*⟩

**lemma** *l4-14*:
**assumes** *Col A B C* **and**
$Cong\ A\ B\ A'\ B'$
**shows** $\exists\ C'.\ A\ B\ C\ Cong3\ A'\ B'\ C'$
⟨*proof*⟩

**lemma** *l4-16R1*:
**assumes** $A\ B\ C\ D\ FSC\ A'\ B'\ C'\ D'$ **and**
$A \neq B$ **and**
*Bet A B C*
**shows** $Cong\ C\ D\ C'\ D'$
⟨*proof*⟩

**lemma** *l4-16R2*:
**assumes** $A\ B\ C\ D\ FSC\ A'\ B'\ C'\ D'$
**and** *Bet B C A*
**shows** $Cong\ C\ D\ C'\ D'$
⟨*proof*⟩

**lemma** *l4-16R3*:
**assumes** $A\ B\ C\ D\ FSC\ A'\ B'\ C'\ D'$ **and** $A \neq B$
**and** *Bet C A B*
**shows** $Cong\ C\ D\ C'\ D'$
⟨*proof*⟩

**lemma** *l4-16*:
**assumes** $A\ B\ C\ D\ FSC\ A'\ B'\ C'\ D'$ **and**
$A \neq B$

**shows** $Cong\ C\ D\ C'\ D'$
⟨*proof*⟩

**lemma** *l4-17*:
**assumes** $A \neq B$ **and**
$Col\ A\ B\ C$ **and**
$Cong\ A\ P\ A\ Q$ **and**
$Cong\ B\ P\ B\ Q$
**shows** $Cong\ C\ P\ C\ Q$
⟨*proof*⟩

**lemma** *l4-18*:
**assumes** $A \neq B$ **and**
$Col\ A\ B\ C$ **and**
$Cong\ A\ C\ A\ C'$ **and**
$Cong\ B\ C\ B\ C'$
**shows** $C = C'$
⟨*proof*⟩

**lemma** *l4-19*:
**assumes** $Bet\ A\ C\ B$ **and**
$Cong\ A\ C\ A\ C'$ **and**
$Cong\ B\ C\ B\ C'$
**shows** $C = C'$
⟨*proof*⟩

**lemma** *not-col-distincts*:
**assumes** $\neg\ Col\ A\ B\ C$
**shows** $\neg\ Col\ A\ B\ C \wedge A \neq B \wedge B \neq C \wedge A \neq C$
⟨*proof*⟩

**lemma** *NCol-cases*:
**assumes** $\neg\ Col\ A\ B\ C \vee \neg\ Col\ A\ C\ B \vee \neg\ Col\ B\ A\ C \vee \neg\ Col\ B\ C\ A \vee \neg\ Col\ C\ A\ B \vee \neg\ Col\ C\ B\ A$
**shows** $\neg\ Col\ A\ B\ C$
⟨*proof*⟩

**lemma** *NCol-perm*:
**assumes** $\neg\ Col\ A\ B\ C$
**shows** $\neg\ Col\ A\ B\ C \wedge \sim\ Col\ A\ C\ B \wedge \sim\ Col\ B\ A\ C \wedge \sim\ Col\ B\ C\ A \wedge \sim\ Col\ C\ A\ B \wedge \sim\ Col\ C\ B\ A$
⟨*proof*⟩

**lemma** *col-cong-3-cong-3-eq*:
**assumes** $A \neq B$
**and** $Col\ A\ B\ C$
**and** $A\ B\ C\ Cong3\ A'\ B'\ C1$
**and** $A\ B\ C\ Cong3\ A'\ B'\ C2$
**shows** $C1 = C2$
⟨*proof*⟩

## 3.4 Between transitivity le

**lemma** *l5-1*:
**assumes** $A \neq B$ **and**
$Bet\ A\ B\ C$ **and**
$Bet\ A\ B\ D$
**shows** $Bet\ A\ C\ D \vee Bet\ A\ D\ C$
⟨*proof*⟩

**lemma** *l5-2*:
**assumes** $A \neq B$ **and**
$Bet\ A\ B\ C$ **and**
$Bet\ A\ B\ D$
**shows** $Bet\ B\ C\ D \vee Bet\ B\ D\ C$
⟨*proof*⟩

**lemma** *segment-construction-2*:
  **assumes** $A \neq Q$
  **shows** $\exists\ X.\ ((Bet\ Q\ A\ X \vee Bet\ Q\ X\ A) \wedge Cong\ Q\ X\ B\ C)$
$\langle proof \rangle$

**lemma** *l5-3*:
  **assumes** $Bet\ A\ B\ D$ **and**
    $Bet\ A\ C\ D$
  **shows** $Bet\ A\ B\ C \vee Bet\ A\ C\ B$
  $\langle proof \rangle$

**lemma** *bet3--bet*:
  **assumes** $Bet\ A\ B\ E$ **and**
    $Bet\ A\ D\ E$ **and**
    $Bet\ B\ C\ D$
  **shows** $Bet\ A\ C\ E$
  $\langle proof \rangle$

**lemma** *le-bet*:
  **assumes** $C\ D\ Le\ A\ B$
  **shows** $\exists\ X.\ (Bet\ A\ X\ B \wedge Cong\ A\ X\ C\ D)$
  $\langle proof \rangle$

**lemma** *l5-5-1*:
  **assumes** $A\ B\ Le\ C\ D$
  **shows** $\exists\ X.\ (Bet\ A\ B\ X \wedge Cong\ A\ X\ C\ D)$
$\langle proof \rangle$

**lemma** *l5-5-2*:
  **assumes** $\exists\ X.\ (Bet\ A\ B\ X \wedge Cong\ A\ X\ C\ D)$
  **shows** $A\ B\ Le\ C\ D$
$\langle proof \rangle$

**lemma** *l5-6*:
  **assumes** $A\ B\ Le\ C\ D$ **and**
    $Cong\ A\ B\ A'\ B'$ **and**
    $Cong\ C\ D\ C'\ D'$
  **shows** $A'\ B'\ Le\ C'\ D'$
  $\langle proof \rangle$

**lemma** *le-reflexivity*:
  **shows** $A\ B\ Le\ A\ B$
  $\langle proof \rangle$

**lemma** *le-transitivity*:
  **assumes** $A\ B\ Le\ C\ D$ **and**
    $C\ D\ Le\ E\ F$
  **shows** $A\ B\ Le\ E\ F$
  $\langle proof \rangle$

**lemma** *between-cong*:
  **assumes** $Bet\ A\ C\ B$ **and**
    $Cong\ A\ C\ A\ B$
  **shows** $C = B$
  $\langle proof \rangle$

**lemma** *cong3-symmetry*:
  **assumes** $A\ B\ C\ Cong3\ A'\ B'\ C'$
  **shows** $A'\ B'\ C'\ Cong3\ A\ B\ C$
  $\langle proof \rangle$

**lemma** *between-cong-2*:
  **assumes** $Bet\ A\ D\ B$ **and**
    $Bet\ A\ E\ B$
    **and** $Cong\ A\ D\ A\ E$
  **shows** $D = E$ $\langle proof \rangle$

**lemma** *between-cong-3*:
  **assumes** $A \neq B$
    **and** *Bet* $A$ $B$ $D$
    **and** *Bet* $A$ $B$ $E$
    **and** *Cong* $B$ $D$ $B$ $E$
  **shows** $D = E$
  ⟨*proof*⟩

**lemma** *le-anti-symmetry*:
  **assumes** $A$ $B$ *Le* $C$ $D$ **and**
    $C$ $D$ *Le* $A$ $B$
  **shows** *Cong* $A$ $B$ $C$ $D$
  ⟨*proof*⟩

**lemma** *cong-dec*:
  **shows** *Cong* $A$ $B$ $C$ $D$ $\vee$ $\neg$ *Cong* $A$ $B$ $C$ $D$
  ⟨*proof*⟩

**lemma** *bet-dec*:
  **shows** *Bet* $A$ $B$ $C$ $\vee$ $\neg$ *Bet* $A$ $B$ $C$
  ⟨*proof*⟩

**lemma** *col-dec*:
  **shows** *Col* $A$ $B$ $C$ $\vee$ $\neg$ *Col* $A$ $B$ $C$
  ⟨*proof*⟩

**lemma** *le-trivial*:
  **shows** $A$ $A$ *Le* $C$ $D$
  ⟨*proof*⟩

**lemma** *le-cases*:
  **shows** $A$ $B$ *Le* $C$ $D$ $\vee$ $C$ $D$ *Le* $A$ $B$
  ⟨*proof*⟩

**lemma** *le-zero*:
  **assumes** $A$ $B$ *Le* $C$ $C$
  **shows** $A = B$
  ⟨*proof*⟩

**lemma** *le-diff*:
  **assumes** $A \neq B$ **and** $A$ $B$ *Le* $C$ $D$
  **shows** $C \neq D$
  ⟨*proof*⟩

**lemma** *lt-diff*:
  **assumes** $A$ $B$ *Lt* $C$ $D$
  **shows** $C \neq D$
  ⟨*proof*⟩

**lemma** *bet-cong-eq*:
  **assumes** *Bet* $A$ $B$ $C$ **and**
    *Bet* $A$ $C$ $D$ **and**
    *Cong* $B$ $C$ $A$ $D$
  **shows** $C = D \wedge A = B$
⟨*proof*⟩

**lemma** *cong--le*:
  **assumes** *Cong* $A$ $B$ $C$ $D$
  **shows** $A$ $B$ *Le* $C$ $D$
  ⟨*proof*⟩

**lemma** *cong--le3412*:
  **assumes** *Cong* $A$ $B$ $C$ $D$
  **shows** $C$ $D$ *Le* $A$ $B$
  ⟨*proof*⟩

**lemma** *le1221*:
**shows** *A B Le B A*
⟨*proof*⟩

**lemma** *le-left-comm*:
**assumes** *A B Le C D*
**shows** *B A Le C D*
⟨*proof*⟩

**lemma** *le-right-comm*:
**assumes** *A B Le C D*
**shows** *A B Le D C*
⟨*proof*⟩

**lemma** *le-comm*:
**assumes** *A B Le C D*
**shows** *B A Le D C*
⟨*proof*⟩

**lemma** *ge-left-comm*:
**assumes** *A B Ge C D*
**shows** *B A Ge C D*
⟨*proof*⟩

**lemma** *ge-right-comm*:
**assumes** *A B Ge C D*
**shows** *A B Ge D C*
⟨*proof*⟩

**lemma** *ge-comm0*:
**assumes** *A B Ge C D*
**shows** *B A Ge D C*
⟨*proof*⟩

**lemma** *lt-right-comm*:
**assumes** *A B Lt C D*
**shows** *A B Lt D C*
⟨*proof*⟩

**lemma** *lt-left-comm*:
**assumes** *A B Lt C D*
**shows** *B A Lt C D*
⟨*proof*⟩

**lemma** *lt-comm*:
**assumes** *A B Lt C D*
**shows** *B A Lt D C*
⟨*proof*⟩

**lemma** *gt-left-comm0*:
**assumes** *A B Gt C D*
**shows** *B A Gt C D*
⟨*proof*⟩

**lemma** *gt-right-comm*:
**assumes** *A B Gt C D*
**shows** *A B Gt D C*
⟨*proof*⟩

**lemma** *gt-comm*:
**assumes** *A B Gt C D*
**shows** *B A Gt D C*
⟨*proof*⟩

**lemma** *cong2-lt--lt*:

**assumes** $A\ B\ Lt\ C\ D$ **and**
$Cong\ A\ B\ A'\ B'$ **and**
$Cong\ C\ D\ C'\ D'$
**shows** $A'\ B'\ Lt\ C'\ D'$
⟨*proof*⟩

**lemma** *fourth-point*:
**assumes** $A \neq B$ **and**
$B \neq C$ **and**
$Col\ A\ B\ P$ **and**
$Bet\ A\ B\ C$
**shows** $Bet\ P\ A\ B \vee Bet\ A\ P\ B \vee Bet\ B\ P\ C \vee Bet\ B\ C\ P$
⟨*proof*⟩

**lemma** *third-point*:
**assumes** $Col\ A\ B\ P$
**shows** $Bet\ P\ A\ B \vee Bet\ A\ P\ B \vee Bet\ A\ B\ P$
⟨*proof*⟩

**lemma** *l5-12-a*:
**assumes** $Bet\ A\ B\ C$
**shows** $A\ B\ Le\ A\ C \wedge B\ C\ Le\ A\ C$
⟨*proof*⟩

**lemma** *bet--le1213*:
**assumes** $Bet\ A\ B\ C$
**shows** $A\ B\ Le\ A\ C$
⟨*proof*⟩

**lemma** *bet--le2313*:
**assumes** $Bet\ A\ B\ C$
**shows** $B\ C\ Le\ A\ C$
⟨*proof*⟩

**lemma** *bet--lt1213*:
**assumes** $B \neq C$ **and**
$Bet\ A\ B\ C$
**shows** $A\ B\ Lt\ A\ C$
⟨*proof*⟩

**lemma** *bet--lt2313*:
**assumes** $A \neq B$ **and**
$Bet\ A\ B\ C$
**shows** $B\ C\ Lt\ A\ C$
⟨*proof*⟩

**lemma** *l5-12-b*:
**assumes** $Col\ A\ B\ C$ **and**
$A\ B\ Le\ A\ C$ **and**
$B\ C\ Le\ A\ C$
**shows** $Bet\ A\ B\ C$
⟨*proof*⟩

**lemma** *bet-le-eq*:
**assumes** $Bet\ A\ B\ C$
**and** $A\ C\ Le\ B\ C$
**shows** $A = B$
⟨*proof*⟩

**lemma** *or-lt-cong-gt*:
$A\ B\ Lt\ C\ D \vee A\ B\ Gt\ C\ D \vee Cong\ A\ B\ C\ D$
⟨*proof*⟩

**lemma** *lt--le*:
**assumes** $A\ B\ Lt\ C\ D$
**shows** $A\ B\ Le\ C\ D$

⟨*proof*⟩

**lemma** *le1234-lt--lt*:
**assumes** $A\ B\ Le\ C\ D$ **and**
$C\ D\ Lt\ E\ F$
**shows** $A\ B\ Lt\ E\ F$
⟨*proof*⟩

**lemma** *le3456-lt--lt*:
**assumes** $A\ B\ Lt\ C\ D$ **and**
$C\ D\ Le\ E\ F$
**shows** $A\ B\ Lt\ E\ F$
⟨*proof*⟩

**lemma** *lt-transitivity*:
**assumes** $A\ B\ Lt\ C\ D$ **and**
$C\ D\ Lt\ E\ F$
**shows** $A\ B\ Lt\ E\ F$
⟨*proof*⟩

**lemma** *not-and-lt*:
$\neg\ (A\ B\ Lt\ C\ D \wedge C\ D\ Lt\ A\ B)$
⟨*proof*⟩

**lemma** *nlt*:
$\neg\ A\ B\ Lt\ A\ B$
⟨*proof*⟩

**lemma** *le--nlt*:
**assumes** $A\ B\ Le\ C\ D$
**shows** $\neg\ C\ D\ Lt\ A\ B$
⟨*proof*⟩

**lemma** *cong--nlt*:
**assumes** $Cong\ A\ B\ C\ D$
**shows** $\neg\ A\ B\ Lt\ C\ D$
⟨*proof*⟩

**lemma** *nlt--le*:
**assumes** $\neg\ A\ B\ Lt\ C\ D$
**shows** $C\ D\ Le\ A\ B$
⟨*proof*⟩

**lemma** *lt--nle*:
**assumes** $A\ B\ Lt\ C\ D$
**shows** $\neg\ C\ D\ Le\ A\ B$
⟨*proof*⟩

**lemma** *nle--lt*:
**assumes** $\neg\ A\ B\ Le\ C\ D$
**shows** $C\ D\ Lt\ A\ B$
⟨*proof*⟩

**lemma** *lt1123*:
**assumes** $B \neq C$
**shows** $A\ A\ Lt\ B\ C$
⟨*proof*⟩

**lemma** *bet2-le2--le-R1*:
**assumes** $Bet\ a\ P\ b$ **and**
$Bet\ A\ Q\ B$ **and**
$P\ a\ Le\ Q\ A$ **and**
$P\ b\ Le\ Q\ B$ **and**
$B = Q$
**shows** $a\ b\ Le\ A\ B$
⟨*proof*⟩

**lemma** *bet2-le2--le-R2*:
**assumes** $Bet\ a\ Po\ b$ **and**
$Bet\ A\ PO\ B$ **and**
$Po\ a\ Le\ PO\ A$ **and**
$Po\ b\ Le\ PO\ B$ **and**
$A \neq PO$ **and**
$B \neq PO$
**shows** $a\ b\ Le\ A\ B$
⟨*proof*⟩

**lemma** *bet2-le2--le*:
**assumes** $Bet\ a\ P\ b$ **and**
$Bet\ A\ Q\ B$ **and**
$P\ a\ Le\ Q\ A$ **and**
$P\ b\ Le\ Q\ B$
**shows** $a\ b\ Le\ A\ B$
⟨*proof*⟩

**lemma** *Le-cases*:
**assumes** $A\ B\ Le\ C\ D \lor B\ A\ Le\ C\ D \lor A\ B\ Le\ D\ C \lor B\ A\ Le\ D\ C$
**shows** $A\ B\ Le\ C\ D$
⟨*proof*⟩

**lemma** *Lt-cases*:
**assumes** $A\ B\ Lt\ C\ D \lor B\ A\ Lt\ C\ D \lor A\ B\ Lt\ D\ C \lor B\ A\ Lt\ D\ C$
**shows** $A\ B\ Lt\ C\ D$
⟨*proof*⟩

## 3.5 Out lines

**lemma** *bet-out*:
**assumes** $B \neq A$ **and**
$Bet\ A\ B\ C$
**shows** $A\ Out\ B\ C$
⟨*proof*⟩

**lemma** *bet-out-1*:
**assumes** $B \neq A$ **and**
$Bet\ C\ B\ A$
**shows** $A\ Out\ B\ C$
⟨*proof*⟩

**lemma** *out-dec*:
**shows** $P\ Out\ A\ B \lor \neg\ P\ Out\ A\ B$
⟨*proof*⟩

**lemma** *out-diff1*:
**assumes** $A\ Out\ B\ C$
**shows** $B \neq A$
⟨*proof*⟩

**lemma** *out-diff2*:
**assumes** $A\ Out\ B\ C$
**shows** $C \neq A$
⟨*proof*⟩

**lemma** *out-distinct*:
**assumes** $A\ Out\ B\ C$
**shows** $B \neq A \land C \neq A$
⟨*proof*⟩

**lemma** *out-col*:
**assumes** $A\ Out\ B\ C$
**shows** $Col\ A\ B\ C$
⟨*proof*⟩

**lemma** *l6-2*:
  **assumes** $A \neq P$ **and**
    $B \neq P$ **and**
    $C \neq P$ **and**
    *Bet* $A$ $P$ $C$
  **shows** *Bet* $B$ $P$ $C$ $\longleftrightarrow$ $P$ *Out* $A$ $B$
  ⟨*proof*⟩

**lemma** *bet-out--bet*:
  **assumes** *Bet* $A$ $P$ $C$ **and**
    $P$ *Out* $A$ $B$
  **shows** *Bet* $B$ $P$ $C$
  ⟨*proof*⟩

**lemma** *l6-3-1*:
  **assumes** $P$ *Out* $A$ $B$
  **shows** $A \neq P \land B \neq P \land (\exists\ C.\ (C \neq P \land \mathit{Bet}\ A\ P\ C \land \mathit{Bet}\ B\ P\ C))$
  ⟨*proof*⟩

**lemma** *l6-3-2*:
  **assumes** $A \neq P$ **and**
    $B \neq P$ **and**
    $\exists\ C.\ (C \neq P \land \mathit{Bet}\ A\ P\ C \land \mathit{Bet}\ B\ P\ C)$
  **shows** $P$ *Out* $A$ $B$
  ⟨*proof*⟩

**lemma** *l6-4-1*:
  **assumes** $P$ *Out* $A$ $B$ **and**
    *Col* $A$ $P$ $B$
  **shows** $\neg$ *Bet* $A$ $P$ $B$
  ⟨*proof*⟩

**lemma** *l6-4-2*:
  **assumes** *Col* $A$ $P$ $B$
    **and** $\neg$ *Bet* $A$ $P$ $B$
  **shows** $P$ *Out* $A$ $B$
  ⟨*proof*⟩

**lemma** *out-trivial*:
  **assumes** $A \neq P$
  **shows** $P$ *Out* $A$ $A$
  ⟨*proof*⟩

**lemma** *l6-6*:
  **assumes** $P$ *Out* $A$ $B$
  **shows** $P$ *Out* $B$ $A$
  ⟨*proof*⟩

**lemma** *l6-7*:
  **assumes** $P$ *Out* $A$ $B$ **and**
    $P$ *Out* $B$ $C$
  **shows** $P$ *Out* $A$ $C$
  ⟨*proof*⟩

**lemma** *bet-out-out-bet*:
  **assumes** *Bet* $A$ $B$ $C$ **and**
    $B$ *Out* $A$ $A'$ **and**
    $B$ *Out* $C$ $C'$
  **shows** *Bet* $A'$ $B$ $C'$
  ⟨*proof*⟩

**lemma** *out2-bet-out*:
  **assumes** $B$ *Out* $A$ $C$ **and**
    $B$ *Out* $X$ $P$ **and**
    *Bet* $A$ $X$ $C$

**shows** $B$ *Out* $A\ P \wedge B$ *Out* $C\ P$
⟨*proof*⟩

**lemma** *l6-11-uniqueness*:
**assumes** $A$ *Out* $X\ R$ **and**
*Cong* $A\ X\ B\ C$ **and**
$A$ *Out* $Y\ R$ **and**
*Cong* $A\ Y\ B\ C$
**shows** $X = Y$
⟨*proof*⟩

**lemma** *l6-11-existence*:
**assumes** $R \neq A$ **and**
$B \neq C$
**shows** $\exists\ X.\ (A$ *Out* $X\ R \wedge$ *Cong* $A\ X\ B\ C)$
⟨*proof*⟩

**lemma** *segment-construction-3*:
**assumes** $A \neq B$ **and**
$X \neq Y$
**shows** $\exists\ C.\ (A$ *Out* $B\ C \wedge$ *Cong* $A\ C\ X\ Y)$
⟨*proof*⟩

**lemma** *l6-13-1*:
**assumes** $P$ *Out* $A\ B$ **and**
$P\ A$ *Le* $P\ B$
**shows** *Bet* $P\ A\ B$
⟨*proof*⟩

**lemma** *l6-13-2*:
**assumes** $P$ *Out* $A\ B$ **and**
*Bet* $P\ A\ B$
**shows** $P\ A$ *Le* $P\ B$
⟨*proof*⟩

**lemma** *l6-16-1*:
**assumes** $P \neq Q$ **and**
*Col* $S\ P\ Q$ **and**
*Col* $X\ P\ Q$
**shows** *Col* $X\ P\ S$
⟨*proof*⟩

**lemma** *col-transitivity-1*:
**assumes** $P \neq Q$ **and**
*Col* $P\ Q\ A$ **and**
*Col* $P\ Q\ B$
**shows** *Col* $P\ A\ B$
⟨*proof*⟩

**lemma** *col-transitivity-2*:
**assumes** $P \neq Q$ **and**
*Col* $P\ Q\ A$ **and**
*Col* $P\ Q\ B$
**shows** *Col* $Q\ A\ B$
⟨*proof*⟩

**lemma** *l6-21*:
**assumes** $\neg$ *Col* $A\ B\ C$ **and**
$C \neq D$ **and**
*Col* $A\ B\ P$ **and**
*Col* $A\ B\ Q$ **and**
*Col* $C\ D\ P$ **and**
*Col* $C\ D\ Q$
**shows** $P = Q$
⟨*proof*⟩

**lemma** *col2--eq*:
**assumes** $Col\ A\ X\ Y$ **and**
$Col\ B\ X\ Y$ **and**
$\neg\ Col\ A\ X\ B$
**shows** $X = Y$
⟨*proof*⟩

**lemma** *not-col-exists*:
**assumes** $A \neq B$
**shows** $\exists\ C.\ \neg\ Col\ A\ B\ C$
⟨*proof*⟩

**lemma** *col3*:
**assumes** $X \neq Y$ **and**
$Col\ X\ Y\ A$ **and**
$Col\ X\ Y\ B$ **and**
$Col\ X\ Y\ C$
**shows** $Col\ A\ B\ C$
⟨*proof*⟩

**lemma** *colx*:
**assumes** $A \neq B$ **and**
$Col\ X\ Y\ A$ **and**
$Col\ X\ Y\ B$ **and**
$Col\ A\ B\ C$
**shows** $Col\ X\ Y\ C$
⟨*proof*⟩

**lemma** *out2--bet*:
**assumes** $A\ Out\ B\ C$ **and**
$C\ Out\ A\ B$
**shows** $Bet\ A\ B\ C$
⟨*proof*⟩

**lemma** *bet2-le2--le1346*:
**assumes** $Bet\ A\ B\ C$ **and**
$Bet\ A'\ B'\ C'$ **and**
$A\ B\ Le\ A'\ B'$ **and**
$B\ C\ Le\ B'\ C'$
**shows** $A\ C\ Le\ A'\ C'$
⟨*proof*⟩

**lemma** *bet2-le2--le2356-R1*:
**assumes** $Bet\ A\ A\ C$ **and**
$Bet\ A'\ B'\ C'$ **and**
$A\ A\ Le\ A'\ B'$ **and**
$A'\ C'\ Le\ A\ C$
**shows** $B'\ C'\ Le\ A\ C$
⟨*proof*⟩

**lemma** *bet2-le2--le2356-R2*:
**assumes** $A \neq B$ **and**
$Bet\ A\ B\ C$ **and**
$Bet\ A'\ B'\ C'$ **and**
$A\ B\ Le\ A'\ B'$ **and**
$A'\ C'\ Le\ A\ C$
**shows** $B'\ C'\ Le\ B\ C$
⟨*proof*⟩

**lemma** *bet2-le2--le2356*:
**assumes** $Bet\ A\ B\ C$ **and**
$Bet\ A'\ B'\ C'$ **and**
$A\ B\ Le\ A'\ B'$ **and**
$A'\ C'\ Le\ A\ C$
**shows** $B'\ C'\ Le\ B\ C$

$\langle proof \rangle$

**lemma** *bet2-le2--le1245*:
  **assumes** *Bet A B C* **and**
    *Bet A′ B′ C′* **and**
    *B C Le B′ C′* **and**
    *A′ C′ Le A C*
  **shows** *A′ B′ Le A B*
  $\langle proof \rangle$

**lemma** *cong-preserves-bet*:
  **assumes** *Bet B A′ A0* **and**
    *Cong B A′ E D′* **and**
    *Cong B A0 E D0* **and**
    *E Out D′ D0*
  **shows** *Bet E D′ D0*
  $\langle proof \rangle$

**lemma** *out-cong-cong*:
  **assumes** *B Out A A0* **and**
    *E Out D D0* **and**
    *Cong B A E D* **and**
    *Cong B A0 E D0*
  **shows** *Cong A A0 D D0*
  $\langle proof \rangle$

**lemma** *not-out-bet*:
  **assumes** *Col A B C* **and**
    ¬ *B Out A C*
  **shows** *Bet A B C*
  $\langle proof \rangle$

**lemma** *or-bet-out*:
  **shows** *Bet A B C* ∨ *B Out A C* ∨ ¬ *Col A B C*
  $\langle proof \rangle$

**lemma** *not-bet-out*:
  **assumes** *Col A B C* **and**
    ¬ *Bet A B C*
  **shows** *B Out A C*
  $\langle proof \rangle$

**lemma** *not-bet-and-out*:
  **shows** ¬ (*Bet A B C* ∧ *B Out A C*)
  $\langle proof \rangle$

**lemma** *out-to-bet*:
  **assumes** *Col A′ B′ C′* **and**
    *B Out A C* ⟷ *B′ Out A′ C′* **and**
    *Bet A B C*
  **shows** *Bet A′ B′ C′*
  $\langle proof \rangle$

**lemma** *col-out2-col*:
  **assumes** *Col A B C* **and**
    *B Out A AA* **and**
    *B Out C CC*
  **shows** *Col AA B CC* $\langle proof \rangle$

**lemma** *bet2-out-out*:
  **assumes** *B* ≠ *A* **and**
    *B′* ≠ *A* **and**
    *A Out C C′* **and**
    *Bet A B C* **and**
    *Bet A B′ C′*
  **shows** *A Out B B′*

⟨*proof*⟩

**lemma** *bet2--out*:
**assumes** $A \neq B$ **and**
$A \neq B'$ **and**
*Bet A B C*
**and** *Bet A B' C*
**shows** *A Out B B'*
⟨*proof*⟩

**lemma** *out-bet-out-1*:
**assumes** *P Out A C* **and**
*Bet A B C*
**shows** *P Out A B*
⟨*proof*⟩

**lemma** *out-bet-out-2*:
**assumes** *P Out A C* **and**
*Bet A B C*
**shows** *P Out B C*
⟨*proof*⟩

**lemma** *out-bet--out*:
**assumes** *Bet P Q A* **and**
*Q Out A B*
**shows** *P Out A B*
⟨*proof*⟩

**lemma** *segment-reverse*:
**assumes** *Bet A B C*
**shows** $\exists\ B'.\ Bet\ A\ B'\ C \wedge Cong\ C\ B'\ A\ B$
⟨*proof*⟩

**lemma** *diff-col-ex*:
**shows** $\exists\ C.\ A \neq C \wedge B \neq C \wedge Col\ A\ B\ C$
⟨*proof*⟩

**lemma** *diff-bet-ex3*:
**assumes** *Bet A B C*
**shows** $\exists\ D.\ A \neq D \wedge B \neq D \wedge C \neq D \wedge Col\ A\ B\ D$
⟨*proof*⟩

**lemma** *diff-col-ex3*:
**assumes** *Col A B C*
**shows** $\exists\ D.\ A \neq D \wedge B \neq D \wedge C \neq D \wedge Col\ A\ B\ D$
⟨*proof*⟩

**lemma** *Out-cases*:
**assumes** $A\ Out\ B\ C \vee A\ Out\ C\ B$
**shows** *A Out B C*
⟨*proof*⟩

## 3.6 Midpoint

**lemma** *midpoint-dec*:
$I\ Midpoint\ A\ B \vee \neg\ I\ Midpoint\ A\ B$
⟨*proof*⟩

**lemma** *is-midpoint-id*:
**assumes** *A Midpoint A B*
**shows** $A = B$
⟨*proof*⟩

**lemma** *is-midpoint-id-2*:
**assumes** *A Midpoint B A*
**shows** $A = B$

⟨*proof*⟩

**lemma** *l7-2*:
**assumes** *M Midpoint A B*
**shows** *M Midpoint B A*
⟨*proof*⟩

**lemma** *l7-3*:
**assumes** *M Midpoint A A*
**shows** $M = A$
⟨*proof*⟩

**lemma** *l7-3-2*:
*A Midpoint A A*
⟨*proof*⟩

**lemma** *symmetric-point-construction*:
$\exists\ P'$. *A Midpoint P* $P'$
⟨*proof*⟩

**lemma** *symmetric-point-uniqueness*:
**assumes** *P Midpoint A P1* **and**
*P Midpoint A P2*
**shows** $P1 = P2$
⟨*proof*⟩

**lemma** *l7-9*:
**assumes** *A Midpoint P X* **and**
*A Midpoint Q X*
**shows** $P = Q$
⟨*proof*⟩

**lemma** *l7-9-bis*:
**assumes** *A Midpoint P X* **and**
*A Midpoint X Q*
**shows** $P = Q$
⟨*proof*⟩

**lemma** *l7-13-R1*:
**assumes** $A \neq P$ **and**
*A Midpoint* $P'$ *P* **and**
*A Midpoint* $Q'$ *Q*
**shows** *Cong P Q* $P'$ $Q'$
⟨*proof*⟩

**lemma** *l7-13*:
**assumes** *A Midpoint* $P'$ *P* **and**
*A Midpoint* $Q'$ *Q*
**shows** *Cong P Q* $P'$ $Q'$
⟨*proof*⟩

**lemma** *l7-15*:
**assumes** *A Midpoint P* $P'$ **and**
*A Midpoint Q* $Q'$ **and**
*A Midpoint R* $R'$ **and**
*Bet P Q R*
**shows** *Bet* $P'$ $Q'$ $R'$
⟨*proof*⟩

**lemma** *l7-16*:
**assumes** *A Midpoint P* $P'$ **and**
*A Midpoint Q* $Q'$ **and**
*A Midpoint R* $R'$ **and**
*A Midpoint S* $S'$ **and**
*Cong P Q R S*
**shows** *Cong* $P'$ $Q'$ $R'$ $S'$

⟨*proof*⟩

**lemma** *symmetry-preserves-midpoint*:
**assumes** *Z Midpoint A D* **and**
*Z Midpoint B E* **and**
*Z Midpoint C F* **and**
*B Midpoint A C*
**shows** *E Midpoint D F*
⟨*proof*⟩

**lemma** *Mid-cases*:
**assumes** *A Midpoint B C* ∨ *A Midpoint C B*
**shows** *A Midpoint B C*
⟨*proof*⟩

**lemma** *Mid-perm*:
**assumes** *A Midpoint B C*
**shows** *A Midpoint B C* ∧ *A Midpoint C B*
⟨*proof*⟩

**lemma** *l7-17*:
**assumes** *A Midpoint P P′* **and**
*B Midpoint P P′*
**shows** $A = B$
⟨*proof*⟩

**lemma** *l7-17-bis*:
**assumes** *A Midpoint P P′* **and**
*B Midpoint P′ P*
**shows** $A = B$
⟨*proof*⟩

**lemma** *l7-20*:
**assumes** *Col A M B* **and**
*Cong M A M B*
**shows** $A = B$ ∨ *M Midpoint A B*
⟨*proof*⟩

**lemma** *l7-20-bis*:
**assumes** $A \neq B$ **and**
*Col A M B* **and**
*Cong M A M B*
**shows** *M Midpoint A B*
⟨*proof*⟩

**lemma** *cong-col-mid*:
**assumes** $A \neq C$ **and**
*Col A B C* **and**
*Cong A B B C*
**shows** *B Midpoint A C*
⟨*proof*⟩

**lemma** *l7-21-R1*:
**assumes** ¬ *Col A B C* **and**
$B \neq D$ **and**
*Cong A B C D* **and**
*Cong B C D A* **and**
*Col A P C* **and**
*Col B P D*
**shows** *P Midpoint A C*
⟨*proof*⟩

**lemma** *l7-21*:
**assumes** ¬ *Col A B C* **and**
$B \neq D$ **and**
*Cong A B C D* **and**

*Cong B C D A* **and**
*Col A P C* **and**
*Col B P D*
**shows** *P Midpoint A C* ∧ *P Midpoint B D*
⟨*proof*⟩

**lemma** *l7-22-aux-R1*:
**assumes** *Bet A1 C C* **and**
*Bet B1 C B2* **and**
*Cong C A1 C B1* **and**
*Cong C C C B2* **and**
*M1 Midpoint A1 B1* **and**
*M2 Midpoint A2 B2***and**
*C A1 Le C C*
**shows** *Bet M1 C M2*
⟨*proof*⟩

**lemma** *l7-22-aux-R2*:
**assumes** *A2* ≠ *C* **and**
*Bet A1 C A2* **and**
*Bet B1 C B2* **and**
*Cong C A1 C B1* **and**
*Cong C A2 C B2* **and**
*M1 Midpoint A1 B1* **and**
*M2 Midpoint A2 B2* **and**
*C A1 Le C A2*
**shows** *Bet M1 C M2*
⟨*proof*⟩

**lemma** *l7-22-aux*:
**assumes** *Bet A1 C A2* **and**
*Bet B1 C B2* **and**
*Cong C A1 C B1* **and**
*Cong C A2 C B2* **and**
*M1 Midpoint A1 B1* **and**
*M2 Midpoint A2 B2* **and**
*C A1 Le C A2*
**shows** *Bet M1 C M2*
⟨*proof*⟩

**lemma** *l7-22*:
**assumes** *Bet A1 C A2* **and**
*Bet B1 C B2* **and**
*Cong C A1 C B1* **and**
*Cong C A2 C B2* **and**
*M1 Midpoint A1 B1* **and**
*M2 Midpoint A2 B2*
**shows** *Bet M1 C M2*
⟨*proof*⟩

**lemma** *bet-col1*:
**assumes** *Bet A B D* **and**
*Bet A C D*
**shows** *Col A B C*
⟨*proof*⟩

**lemma** *l7-25-R1*:
**assumes** *Cong C A C B* **and**
*Col A B C*
**shows** ∃ *X*. *X Midpoint A B*
⟨*proof*⟩

**lemma** *l7-25-R2*:
**assumes** *Cong C A C B* **and**
¬ *Col A B C*
**shows** ∃ *X*. *X Midpoint A B*

⟨*proof*⟩

**lemma** *l7-25*:
  **assumes** *Cong C A C B*
  **shows** $\exists$ *X*. *X Midpoint A B*
  ⟨*proof*⟩

**lemma** *midpoint-distinct-1*:
  **assumes** $A \neq B$ **and**
    *I Midpoint A B*
  **shows** $I \neq A \wedge I \neq B$
  ⟨*proof*⟩

**lemma** *midpoint-distinct-2*:
  **assumes** $I \neq A$ **and**
    *I Midpoint A B*
  **shows** $A \neq B \wedge I \neq B$
  ⟨*proof*⟩

**lemma** *midpoint-distinct-3*:
  **assumes** $I \neq B$ **and**
    *I Midpoint A B*
  **shows** $A \neq B \wedge I \neq A$
  ⟨*proof*⟩

**lemma** *midpoint-def*:
  **assumes** *Bet A B C* **and**
    *Cong A B B C*
  **shows** *B Midpoint A C*
  ⟨*proof*⟩

**lemma** *midpoint-bet*:
  **assumes** *B Midpoint A C*
  **shows** *Bet A B C*
  ⟨*proof*⟩

**lemma** *midpoint-col*:
  **assumes** *M Midpoint A B*
  **shows** *Col M A B*
  ⟨*proof*⟩

**lemma** *midpoint-cong*:
  **assumes** *B Midpoint A C*
  **shows** *Cong A B B C*
  ⟨*proof*⟩

**lemma** *midpoint-out*:
  **assumes** $A \neq C$ **and**
    *B Midpoint A C*
  **shows** *A Out B C*
  ⟨*proof*⟩

**lemma** *midpoint-out-1*:
  **assumes** $A \neq C$ **and**
    *B Midpoint A C*
  **shows** *C Out A B*
  ⟨*proof*⟩

**lemma** *midpoint-not-midpoint*:
  **assumes** $A \neq B$ **and**
    *I Midpoint A B*
  **shows** $\neg$ *B Midpoint A I*
  ⟨*proof*⟩

**lemma** *swap-diff*:
  **assumes** $A \neq B$

**shows** $B \neq A$
⟨*proof*⟩

**lemma** *cong-cong-half-1*:
**assumes** *M Midpoint A B* **and**
$M'$ *Midpoint* $A'$ $B'$ **and**
*Cong A B* $A'$ $B'$
**shows** *Cong A M* $A'$ $M'$
⟨*proof*⟩

**lemma** *cong-cong-half-2*:
**assumes** *M Midpoint A B* **and**
$M'$ *Midpoint* $A'$ $B'$ **and**
*Cong A B* $A'$ $B'$
**shows** *Cong B M* $B'$ $M'$
⟨*proof*⟩

**lemma** *cong-mid2--cong*:
**assumes** *M Midpoint A B* **and**
$M'$ *Midpoint* $A'$ $B'$ **and**
*Cong A M* $A'$ $M'$
**shows** *Cong A B* $A'$ $B'$
⟨*proof*⟩

**lemma** *mid--lt*:
**assumes** $A \neq B$ **and**
*M Midpoint A B*
**shows** *A M Lt A B*
⟨*proof*⟩

**lemma** *le-mid2--le13*:
**assumes** *M Midpoint A B* **and**
$M'$ *Midpoint* $A'$ $B'$ **and**
*A M Le* $A'$ $M'$
**shows** *A B Le* $A'$ $B'$
⟨*proof*⟩

**lemma** *le-mid2--le12*:
**assumes** *M Midpoint A B* **and**
$M'$ *Midpoint* $A'$ $B'$
**and** *A B Le* $A'$ $B'$
**shows** *A M Le* $A'$ $M'$
⟨*proof*⟩

**lemma** *lt-mid2--lt13*:
**assumes** *M Midpoint A B* **and**
$M'$ *Midpoint* $A'$ $B'$ **and**
*A M Lt* $A'$ $M'$
**shows** *A B Lt* $A'$ $B'$
⟨*proof*⟩

**lemma** *lt-mid2--lt12*:
**assumes** *M Midpoint A B* **and**
$M'$ *Midpoint* $A'$ $B'$ **and**
*A B Lt* $A'$ $B'$
**shows** *A M Lt* $A'$ $M'$
⟨*proof*⟩

**lemma** *midpoint-preserves-out*:
**assumes** *A Out B C* **and**
*M Midpoint A* $A'$ **and**
*M Midpoint B* $B'$ **and**
*M Midpoint C* $C'$
**shows** $A'$ *Out* $B'$ $C'$
⟨*proof*⟩

**lemma** *col-cong-bet*:
  **assumes** *Col A B D* **and**
    *Cong A B C D* **and**
    *Bet A C B*
  **shows** *Bet C A D* ∨ *Bet C B D*
  ⟨*proof*⟩

**lemma** *col-cong2-bet1*:
  **assumes** *Col A B D* **and**
    *Bet A C B* **and**
    *Cong A B C D* **and**
    *Cong A C B D*
  **shows** *Bet C B D*
  ⟨*proof*⟩

**lemma** *col-cong2-bet2*:
  **assumes** *Col A B D* **and**
    *Bet A C B* **and**
    *Cong A B C D* **and**
    *Cong A D B C*
  **shows** *Bet C A D*
  ⟨*proof*⟩

**lemma** *col-cong2-bet3*:
  **assumes** *Col A B D* **and**
    *Bet A B C* **and**
    *Cong A B C D* **and**
    *Cong A C B D*
  **shows** *Bet B C D*
  ⟨*proof*⟩

**lemma** *col-cong2-bet4*:
  **assumes** *Col A B C* **and**
    *Bet A B D* **and**
    *Cong A B C D* **and**
    *Cong A D B C*
  **shows** *Bet B D C*
  ⟨*proof*⟩

**lemma** *col-bet2-cong1*:
  **assumes** *Col A B D* **and**
    *Bet A C B* **and**
    *Cong A B C D* **and**
    *Bet C B D*
  **shows** *Cong A C D B*
  ⟨*proof*⟩

**lemma** *col-bet2-cong2*:
  **assumes** *Col A B D* **and**
    *Bet A C B* **and**
    *Cong A B C D* **and**
    *Bet C A D*
  **shows** *Cong D A B C*
  ⟨*proof*⟩

**lemma** *bet2-lt2--lt*:
  **assumes** *Bet a Po b* **and**
    *Bet A PO B* **and**
    *Po a Lt PO A* **and**
    *Po b Lt PO B*
  **shows** *a b Lt A B*
  ⟨*proof*⟩

**lemma** *bet2-lt-le--lt*:
  **assumes** *Bet a Po b* **and**
    *Bet A PO B* **and**

$Cong\ Po\ a\ PO\ A$ **and**
$Po\ b\ Lt\ PO\ B$
**shows** $a\ b\ Lt\ A\ B$
⟨*proof*⟩

## 3.7 Orthogonality

**lemma** *per-dec*:
$Per\ A\ B\ C \vee \neg\ Per\ A\ B\ C$
⟨*proof*⟩

**lemma** *l8-2*:
**assumes** $Per\ A\ B\ C$
**shows** $Per\ C\ B\ A$
⟨*proof*⟩

**lemma** *Per-cases*:
**assumes** $Per\ A\ B\ C \vee Per\ C\ B\ A$
**shows** $Per\ A\ B\ C$
⟨*proof*⟩

**lemma** *Per-perm* :
**assumes** $Per\ A\ B\ C$
**shows** $Per\ A\ B\ C \wedge Per\ C\ B\ A$
⟨*proof*⟩

**lemma** *l8-3* :
**assumes** $Per\ A\ B\ C$ **and**
$A \neq B$ **and**
$Col\ B\ A\ A'$
**shows** $Per\ A'\ B\ C$
⟨*proof*⟩

**lemma** *l8-4*:
**assumes** $Per\ A\ B\ C$ **and**
$B\ Midpoint\ C\ C'$
**shows** $Per\ A\ B\ C'$
⟨*proof*⟩

**lemma** *l8-5*:
$Per\ A\ B\ B$
⟨*proof*⟩

**lemma** *l8-6*:
**assumes** $Per\ A\ B\ C$ **and**
$Per\ A'\ B\ C$ **and**
$Bet\ A\ C\ A'$
**shows** $B = C$
⟨*proof*⟩

**lemma** *l8-7*:
**assumes** $Per\ A\ B\ C$ **and**
$Per\ A\ C\ B$
**shows** $B = C$
⟨*proof*⟩

**lemma** *l8-8*:
**assumes** $Per\ A\ B\ A$
**shows** $A = B$
⟨*proof*⟩

**lemma** *per-distinct*:
**assumes** $Per\ A\ B\ C$ **and**
$A \neq B$
**shows** $A \neq C$
⟨*proof*⟩

**lemma** *per-distinct-1*:
  **assumes** $Per\ A\ B\ C$ **and**
    $B \neq C$
  **shows** $A \neq C$
  $\langle proof \rangle$

**lemma** *l8-9*:
  **assumes** $Per\ A\ B\ C$ **and**
    $Col\ A\ B\ C$
  **shows** $A = B \vee C = B$
  $\langle proof \rangle$

**lemma** *l8-10*:
  **assumes** $Per\ A\ B\ C$ **and**
    $A\ B\ C\ Cong3\ A'\ B'\ C'$
  **shows** $Per\ A'\ B'\ C'$
$\langle proof \rangle$

**lemma** *col-col-per-per*:
  **assumes** $A \neq X$ **and**
    $C \neq X$ **and**
    $Col\ U\ A\ X$ **and**
    $Col\ V\ C\ X$ **and**
    $Per\ A\ X\ C$
  **shows** $Per\ U\ X\ V$
  $\langle proof \rangle$

**lemma** *perp-in-dec*:
  $X\ PerpAt\ A\ B\ C\ D \vee \neg\ X\ PerpAt\ A\ B\ C\ D$
  $\langle proof \rangle$

**lemma** *perp-distinct*:
  **assumes** $A\ B\ Perp\ C\ D$
  **shows** $A \neq B \wedge C \neq D$
  $\langle proof \rangle$

**lemma** *l8-12*:
  **assumes** $X\ PerpAt\ A\ B\ C\ D$
  **shows** $X\ PerpAt\ C\ D\ A\ B$
  $\langle proof \rangle$

**lemma** *per-col*:
  **assumes** $B \neq C$ **and**
    $Per\ A\ B\ C$ **and**
    $Col\ B\ C\ D$
  **shows** $Per\ A\ B\ D$
  $\langle proof \rangle$

**lemma** *l8-13-2*:
  **assumes** $A \neq B$ **and**
    $C \neq D$ **and**
    $Col\ X\ A\ B$ **and**
    $Col\ X\ C\ D$ **and**
    $\exists\ U.\ \exists\ V.\ Col\ U\ A\ B \wedge Col\ V\ C\ D \wedge U \neq X \wedge V \neq X \wedge Per\ U\ X\ V$
  **shows** $X\ PerpAt\ A\ B\ C\ D$
$\langle proof \rangle$

**lemma** *l8-14-1*:
  $\neg\ A\ B\ Perp\ A\ B$
  $\langle proof \rangle$

**lemma** *l8-14-2-1a*:
  **assumes** $X\ PerpAt\ A\ B\ C\ D$
  **shows** $A\ B\ Perp\ C\ D$
  $\langle proof \rangle$

**lemma** *perp-in-distinct*:
  **assumes** $X$ *PerpAt* $A\ B\ C\ D$
  **shows** $A \neq B \wedge C \neq D$
  ⟨*proof*⟩

**lemma** *l8-14-2-1b*:
  **assumes** $X$ *PerpAt* $A\ B\ C\ D$ **and**
    *Col* $Y\ A\ B$ **and**
    *Col* $Y\ C\ D$
  **shows** $X = Y$
  ⟨*proof*⟩

**lemma** *l8-14-2-1b-bis*:
  **assumes** $A\ B$ *Perp* $C\ D$ **and**
    *Col* $X\ A\ B$ **and**
    *Col* $X\ C\ D$
  **shows** $X$ *PerpAt* $A\ B\ C\ D$
  ⟨*proof*⟩

**lemma** *l8-14-2-2*:
  **assumes** $A\ B$ *Perp* $C\ D$ **and**
    $\forall\ Y.\ (Col\ Y\ A\ B \wedge Col\ Y\ C\ D) \longrightarrow X = Y$
  **shows** $X$ *PerpAt* $A\ B\ C\ D$
  ⟨*proof*⟩

**lemma** *l8-14-3*:
  **assumes** $X$ *PerpAt* $A\ B\ C\ D$ **and**
    $Y$ *PerpAt* $A\ B\ C\ D$
  **shows** $X = Y$
  ⟨*proof*⟩

**lemma** *l8-15-1*:
  **assumes** *Col* $A\ B\ X$ **and**
    $A\ B$ *Perp* $C\ X$
  **shows** $X$ *PerpAt* $A\ B\ C\ X$
  ⟨*proof*⟩

**lemma** *l8-15-2*:
  **assumes** *Col* $A\ B\ X$ **and**
    $X$ *PerpAt* $A\ B\ C\ X$
  **shows** $A\ B$ *Perp* $C\ X$
  ⟨*proof*⟩

**lemma** *perp-in-per*:
  **assumes** $B$ *PerpAt* $A\ B\ B\ C$
  **shows** *Per* $A\ B\ C$
  ⟨*proof*⟩

**lemma** *perp-sym*:
  **assumes** $A\ B$ *Perp* $A\ B$
  **shows** $C\ D$ *Perp* $C\ D$
  ⟨*proof*⟩

**lemma** *perp-col0*:
  **assumes** $A\ B$ *Perp* $C\ D$ **and**
    $X \neq Y$ **and**
    *Col* $A\ B\ X$ **and**
    *Col* $A\ B\ Y$
  **shows** $C\ D$ *Perp* $X\ Y$
⟨*proof*⟩

**lemma** *per-perp-in*:
  **assumes** $A \neq B$ **and**
    $B \neq C$ **and**
    *Per* $A\ B\ C$

**shows** $B$ *PerpAt* $A\ B\ B\ C$
⟨*proof*⟩

**lemma** *per-perp*:
**assumes** $A \neq B$ **and**
$B \neq C$ **and**
*Per* $A\ B\ C$
**shows** $A\ B$ *Perp* $B\ C$
⟨*proof*⟩

**lemma** *perp-left-comm*:
**assumes** $A\ B$ *Perp* $C\ D$
**shows** $B\ A$ *Perp* $C\ D$
⟨*proof*⟩

**lemma** *perp-right-comm*:
**assumes** $A\ B$ *Perp* $C\ D$
**shows** $A\ B$ *Perp* $D\ C$
⟨*proof*⟩

**lemma** *perp-comm*:
**assumes** $A\ B$ *Perp* $C\ D$
**shows** $B\ A$ *Perp* $D\ C$
⟨*proof*⟩

**lemma** *perp-in-sym*:
**assumes** $X$ *PerpAt* $A\ B\ C\ D$
**shows** $X$ *PerpAt* $C\ D\ A\ B$
⟨*proof*⟩

**lemma** *perp-in-left-comm*:
**assumes** $X$ *PerpAt* $A\ B\ C\ D$
**shows** $X$ *PerpAt* $B\ A\ C\ D$
⟨*proof*⟩

**lemma** *perp-in-right-comm*:
**assumes** $X$ *PerpAt* $A\ B\ C\ D$
**shows** $X$ *PerpAt* $A\ B\ D\ C$
⟨*proof*⟩

**lemma** *perp-in-comm*:
**assumes** $X$ *PerpAt* $A\ B\ C\ D$
**shows** $X$ *PerpAt* $B\ A\ D\ C$
⟨*proof*⟩

**lemma** *Perp-cases*:
**assumes** $A\ B$ *Perp* $C\ D \lor B\ A$ *Perp* $C\ D \lor A\ B$ *Perp* $D\ C \lor B\ A$ *Perp* $D\ C \lor C\ D$ *Perp* $A\ B \lor C\ D$ *Perp* $B\ A \lor$ $D\ C$ *Perp* $A\ B \lor D\ C$ *Perp* $B\ A$
**shows** $A\ B$ *Perp* $C\ D$
⟨*proof*⟩

**lemma** *Perp-perm* :
**assumes** $A\ B$ *Perp* $C\ D$
**shows** $A\ B$ *Perp* $C\ D \land B\ A$ *Perp* $C\ D \land A\ B$ *Perp* $D\ C \land B\ A$ *Perp* $D\ C \land C\ D$ *Perp* $A\ B \land C\ D$ *Perp* $B\ A \land D\ C$ *Perp* $A\ B \land D\ C$ *Perp* $B\ A$
⟨*proof*⟩

**lemma** *Perp-in-cases*:
**assumes** $X$ *PerpAt* $A\ B\ C\ D \lor X$ *PerpAt* $B\ A\ C\ D \lor X$ *PerpAt* $A\ B\ D\ C \lor X$ *PerpAt* $B\ A\ D\ C \lor X$ *PerpAt* $C\ D\ A$ $B \lor X$ *PerpAt* $C\ D\ B\ A \lor X$ *PerpAt* $D\ C\ A\ B \lor X$ *PerpAt* $D\ C\ B\ A$
**shows** $X$ *PerpAt* $A\ B\ C\ D$
⟨*proof*⟩

**lemma** *Perp-in-perm*:
**assumes** $X$ *PerpAt* $A\ B\ C\ D$
**shows** $X$ *PerpAt* $A\ B\ C\ D \land X$ *PerpAt* $B\ A\ C\ D \land X$ *PerpAt* $A\ B\ D\ C \land X$ *PerpAt* $B\ A\ D\ C \land X$ *PerpAt* $C\ D\ A\ B$

$\wedge$ *X PerpAt C D B A* $\wedge$ *X PerpAt D C A B* $\wedge$ *X PerpAt D C B A*
⟨*proof*⟩

**lemma** *perp-in-col*:
**assumes** *X PerpAt A B C D*
**shows** *Col A B X* $\wedge$ *Col C D X*
⟨*proof*⟩

**lemma** *perp-perp-in*:
**assumes** *A B Perp C A*
**shows** *A PerpAt A B C A*
⟨*proof*⟩

**lemma** *perp-per-1*:
**assumes** *A B Perp C A*
**shows** *Per B A C*
⟨*proof*⟩

**lemma** *perp-per-2*:
**assumes** *A B Perp A C*
**shows** *Per B A C*
⟨*proof*⟩

**lemma** *perp-col*:
**assumes** $A \neq E$ **and**
*A B Perp C D* **and**
*Col A B E*
**shows** *A E Perp C D*
⟨*proof*⟩

**lemma** *perp-col2*:
**assumes** *A B Perp X Y* **and**
$C \neq D$ **and**
*Col A B C* **and**
*Col A B D*
**shows** *C D Perp X Y*
⟨*proof*⟩

**lemma** *perp-col4*:
**assumes** $P \neq Q$ **and**
$R \neq S$ **and**
*Col A B P* **and**
*Col A B Q* **and**
*Col C D R* **and**
*Col C D S* **and**
*A B Perp C D*
**shows** *P Q Perp R S*
⟨*proof*⟩

**lemma** *perp-not-eq-1*:
**assumes** *A B Perp C D*
**shows** $A \neq B$
⟨*proof*⟩

**lemma** *perp-not-eq-2*:
**assumes** *A B Perp C D*
**shows** $C \neq D$
⟨*proof*⟩

**lemma** *diff-per-diff*:
**assumes** $A \neq B$ **and**
*Cong A P B R* **and**
*Per B A P*
**and** *Per A B R*
**shows** $P \neq R$
⟨*proof*⟩

**lemma** *per-not-colp*:
  **assumes** $A \neq B$ **and**
    $A \neq P$ **and**
    $B \neq R$ **and**
    *Per* $B$ $A$ $P$
    **and** *Per* $A$ $B$ $R$
  **shows** $\neg$ *Col* $P$ $A$ $R$
  ⟨*proof*⟩

**lemma** *per-not-col*:
  **assumes** $A \neq B$ **and**
    $B \neq C$ **and**
    *Per* $A$ $B$ $C$
  **shows** $\neg$ *Col* $A$ $B$ $C$
  ⟨*proof*⟩

**lemma** *perp-not-col2*:
  **assumes** $A$ $B$ *Perp* $C$ $D$
  **shows** $\neg$ *Col* $A$ $B$ $C$ $\vee$ $\neg$ *Col* $A$ $B$ $D$
  ⟨*proof*⟩

**lemma** *perp-not-col*:
  **assumes** $A$ $B$ *Perp* $P$ $A$
  **shows** $\neg$ *Col* $A$ $B$ $P$
⟨*proof*⟩

**lemma** *perp-in-col-perp-in*:
  **assumes** $C \neq E$ **and**
    *Col* $C$ $D$ $E$ **and**
    $P$ *PerpAt* $A$ $B$ $C$ $D$
  **shows** $P$ *PerpAt* $A$ $B$ $C$ $E$
⟨*proof*⟩

**lemma** *perp-col2-bis*:
  **assumes** $A$ $B$ *Perp* $C$ $D$ **and**
    *Col* $C$ $D$ $P$ **and**
    *Col* $C$ $D$ $Q$ **and**
    $P \neq Q$
  **shows** $A$ $B$ *Perp* $P$ $Q$
  ⟨*proof*⟩

**lemma** *perp-in-perp-bis-R1*:
  **assumes** $X \neq A$ **and**
    $X$ *PerpAt* $A$ $B$ $C$ $D$
  **shows** $X$ $B$ *Perp* $C$ $D$ $\vee$ $A$ $X$ *Perp* $C$ $D$
  ⟨*proof*⟩

**lemma** *perp-in-perp-bis*:
  **assumes** $X$ *PerpAt* $A$ $B$ $C$ $D$
  **shows** $X$ $B$ *Perp* $C$ $D$ $\vee$ $A$ $X$ *Perp* $C$ $D$
  ⟨*proof*⟩

**lemma** *col-per-perp*:
  **assumes** $A \neq B$ **and**
    $B \neq C$ **and**

    $D \neq C$ **and**
    *Col* $B$ $C$ $D$ **and**
    *Per* $A$ $B$ $C$
  **shows** $C$ $D$ *Perp* $A$ $B$
  ⟨*proof*⟩

**lemma** *per-cong-mid-R1*:
  **assumes** $B = H$ **and**

$Bet\ A\ B\ C$ **and**
$Cong\ A\ H\ C\ H$ **and**
$Per\ H\ B\ C$
**shows** $B\ Midpoint\ A\ C$
⟨*proof*⟩

**lemma** *per-cong-mid-R2*:
**assumes**
$B \neq C$ **and**
$Bet\ A\ B\ C$ **and**
$Cong\ A\ H\ C\ H$ **and**
$Per\ H\ B\ C$
**shows** $B\ Midpoint\ A\ C$
⟨*proof*⟩

**lemma** *per-cong-mid*:
**assumes** $B \neq C$ **and**
$Bet\ A\ B\ C$ **and**
$Cong\ A\ H\ C\ H$ **and**
$Per\ H\ B\ C$
**shows** $B\ Midpoint\ A\ C$
⟨*proof*⟩

**lemma** *per-double-cong*:
**assumes** $Per\ A\ B\ C$ **and**
$B\ Midpoint\ C\ C'$
**shows** $Cong\ A\ C\ A\ C'$
⟨*proof*⟩

**lemma** *cong-perp-or-mid-R1*:
**assumes** $Col\ A\ B\ X$ **and**
$A \neq B$ **and**
$M\ Midpoint\ A\ B$ **and**
$Cong\ A\ X\ B\ X$
**shows** $X = M \lor \neg\ Col\ A\ B\ X \land M\ PerpAt\ X\ M\ A\ B$
⟨*proof*⟩

**lemma** *cong-perp-or-mid-R2*:
**assumes** $\neg\ Col\ A\ B\ X$ **and**
$A \neq B$ **and**
$M\ Midpoint\ A\ B$ **and**
$Cong\ A\ X\ B\ X$
**shows** $X = M \lor \neg\ Col\ A\ B\ X \land M\ PerpAt\ X\ M\ A\ B$
⟨*proof*⟩

**lemma** *cong-perp-or-mid*:
**assumes** $A \neq B$ **and**
$M\ Midpoint\ A\ B$ **and**
$Cong\ A\ X\ B\ X$
**shows** $X = M \lor \neg\ Col\ A\ B\ X \land M\ PerpAt\ X\ M\ A\ B$
⟨*proof*⟩

**lemma** *col-per2-cases*:
**assumes** $B \neq C$ **and**
$B' \neq C$ **and**
$C \neq D$ **and**
$Col\ B\ C\ D$ **and**
$Per\ A\ B\ C$ **and**
$Per\ A\ B'\ C$
**shows** $B = B' \lor \neg\ Col\ B'\ C\ D$
⟨*proof*⟩

**lemma** *l8-16-1*:
**assumes** $Col\ A\ B\ X$ **and**
$Col\ A\ B\ U$ **and**
$A\ B\ Perp\ C\ X$

**shows** ¬ *Col A B C* ∧ *Per C X U*
⟨*proof*⟩

**lemma** *l8-16-2*:
**assumes** *Col A B X* **and**
*Col A B U*
**and** $U \neq X$ **and**
¬ *Col A B C* **and**
*Per C X U*
**shows** *A B Perp C X*
⟨*proof*⟩

**lemma** *l8-18-uniqueness*:
**assumes**
*Col A B X* **and**
*A B Perp C X* **and**
*Col A B Y* **and**
*A B Perp C Y*
**shows** $X = Y$
⟨*proof*⟩

**lemma** *midpoint-distinct*:
**assumes** ¬ *Col A B C* **and**
*Col A B X* **and**
*X Midpoint C C'*
**shows** $C \neq C'$
⟨*proof*⟩

**lemma** *l8-20-1-R1*:
**assumes** $A = B$
**shows** *Per B A P*
⟨*proof*⟩

**lemma** *l8-20-1-R2*:
**assumes** $A \neq B$ **and**
*Per A B C* **and**
*P Midpoint C' D* **and**
*A Midpoint C' C* **and**
*B Midpoint D C*
**shows** *Per B A P*
⟨*proof*⟩

**lemma** *l8-20-1*:
**assumes** *Per A B C* **and**
*P Midpoint C' D* **and**
*A Midpoint C' C* **and**
*B Midpoint D C*
**shows** *Per B A P*
⟨*proof*⟩

**lemma** *l8-20-2*:
**assumes** *P Midpoint C' D* **and**
*A Midpoint C' C* **and**
*B Midpoint D C* **and**
$B \neq C$
**shows** $A \neq P$
⟨*proof*⟩

**lemma** *perp-col1*:
**assumes** $C \neq X$ **and**
*A B Perp C D* **and**
*Col C D X*
**shows** *A B Perp C X*
⟨*proof*⟩

**lemma** *l8-18-existence*:

**assumes** ¬ *Col A B C*
**shows** ∃ *X*. *Col A B X* ∧ *A B Perp C X*
⟨*proof*⟩

**lemma** *l8-21-aux*:
**assumes** ¬ *Col A B C*
**shows** ∃ *P*. ∃ *T*. (*A B Perp P A* ∧ *Col A B T* ∧ *Bet C T P*)
⟨*proof*⟩

**lemma** *l8-21*:
**assumes** *A* ≠ *B*
**shows** ∃ *P T*. *A B Perp P A* ∧ *Col A B T* ∧ *Bet C T P*
⟨*proof*⟩

**lemma** *per-cong*:
**assumes** *A* ≠ *B* **and**
*A* ≠ *P* **and**
*Per B A P* **and**
*Per A B R* **and**
*Cong A P B R* **and**
*Col A B X* **and**
*Bet P X R*
**shows** *Cong A R P B*
⟨*proof*⟩

**lemma** *perp-cong*:
**assumes** *A* ≠ *B* **and**
*A* ≠ *P* **and**
*A B Perp P A* **and**
*A B Perp R B* **and**
*Cong A P B R* **and**
*Col A B X* **and**
*Bet P X R*
**shows** *Cong A R P B*
⟨*proof*⟩

**lemma** *perp-exists*:
**assumes** *A* ≠ *B*
**shows** ∃ *X*. *PO X Perp A B*
⟨*proof*⟩

**lemma** *perp-vector*:
**assumes** *A* ≠ *B*
**shows** ∃ *X Y*. *A B Perp X Y*
⟨*proof*⟩

**lemma** *midpoint-existence-aux*:
**assumes** *A* ≠ *B* **and**
*A B Perp Q B* **and**
*A B Perp P A* **and**
*Col A B T* **and**
*Bet Q T P* **and**
*A P Le B Q*
**shows** ∃ *X*. *X Midpoint A B*
⟨*proof*⟩

**lemma** *midpoint-existence*:
∃ *X*. *X Midpoint A B*
⟨*proof*⟩

**lemma** *perp-in-id*:
**assumes** *X PerpAt A B C A*
**shows** *X* = *A*
⟨*proof*⟩

**lemma** *l8-22*:

**assumes** $A \neq B$ **and**
$A \neq P$ **and**
*Per B A P* **and**
*Per A B R* **and**
*Cong A P B R* **and**
*Col A B X* **and**
*Bet P X R* **and**
*Cong A R P B*
**shows** *X Midpoint A B* $\wedge$ *X Midpoint P R*
⟨*proof*⟩

**lemma** *l8-22-bis*:
**assumes** $A \neq B$ **and**
$A \neq P$ **and**
*A B Perp P A* **and**
*A B Perp R B* **and**
*Cong A P B R* **and**
*Col A B X* **and**
*Bet P X R*
**shows** *Cong A R P B* $\wedge$ *X Midpoint A B* $\wedge$ *X Midpoint P R*
⟨*proof*⟩

**lemma** *perp-in-perp*:
**assumes** *X PerpAt A B C D*
**shows** *A B Perp C D*
⟨*proof*⟩

**lemma** *perp-proj*:
**assumes** *A B Perp C D* **and**
$\neg$ *Col A C D*
**shows** $\exists$ *X*. *Col A B X* $\wedge$ *A X Perp C D*
⟨*proof*⟩

**lemma** *l8-24* :
**assumes** *P A Perp A B* **and**
*Q B Perp A B* **and**
*Col A B T* **and**
*Bet P T Q* **and**
*Bet B R Q* **and**
*Cong A P B R*
**shows** $\exists$ *X*. *X Midpoint A B* $\wedge$ *X Midpoint P R*
⟨*proof*⟩

**lemma** *col-per2--per*:
**assumes** $A \neq B$ **and**
*Col A B C* **and**
*Per A X P* **and**
*Per B X P*
**shows** *Per C X P*
⟨*proof*⟩

**lemma** *perp-in-per-1*:
**assumes** *X PerpAt A B C D*
**shows** *Per A X C*
⟨*proof*⟩

**lemma** *perp-in-per-2*:
**assumes** *X PerpAt A B C D*
**shows** *Per A X D*
⟨*proof*⟩

**lemma** *perp-in-per-3*:
**assumes** *X PerpAt A B C D*
**shows** *Per B X C*
⟨*proof*⟩

**lemma** *perp-in-per-4*:
**assumes** *X PerpAt A B C D*
**shows** *Per B X D*
⟨*proof*⟩

## 3.8 Planes

### 3.8.1 Coplanar

**lemma** *coplanar-perm-1*:
**assumes** *Coplanar A B C D*
**shows** *Coplanar A B D C*
⟨*proof*⟩

**lemma** *coplanar-perm-2*:
**assumes** *Coplanar A B C D*
**shows** *Coplanar A C B D*
⟨*proof*⟩

**lemma** *coplanar-perm-3*:
**assumes** *Coplanar A B C D*
**shows** *Coplanar A C D B*
⟨*proof*⟩

**lemma** *coplanar-perm-4*:
**assumes** *Coplanar A B C D*
**shows** *Coplanar A D B C*
⟨*proof*⟩

**lemma** *coplanar-perm-5*:
**assumes** *Coplanar A B C D*
**shows** *Coplanar A D C B*
⟨*proof*⟩

**lemma** *coplanar-perm-6*:
**assumes** *Coplanar A B C D*
**shows** *Coplanar B A C D*
⟨*proof*⟩

**lemma** *coplanar-perm-7*:
**assumes** *Coplanar A B C D*
**shows** *Coplanar B A D C*
⟨*proof*⟩

**lemma** *coplanar-perm-8*:
**assumes** *Coplanar A B C D*
**shows** *Coplanar B C A D*
⟨*proof*⟩

**lemma** *coplanar-perm-9*:
**assumes** *Coplanar A B C D*
**shows** *Coplanar B C D A*
⟨*proof*⟩

**lemma** *coplanar-perm-10*:
**assumes** *Coplanar A B C D*
**shows** *Coplanar B D A C*
⟨*proof*⟩

**lemma** *coplanar-perm-11*:
**assumes** *Coplanar A B C D*
**shows** *Coplanar B D C A*
⟨*proof*⟩

**lemma** *coplanar-perm-12*:
**assumes** *Coplanar A B C D*
**shows** *Coplanar C A B D*

⟨*proof*⟩

**lemma** *coplanar-perm-13*:
**assumes** *Coplanar A B C D*
**shows** *Coplanar C A D B*
⟨*proof*⟩

**lemma** *coplanar-perm-14*:
**assumes** *Coplanar A B C D*
**shows** *Coplanar C B A D*
⟨*proof*⟩

**lemma** *coplanar-perm-15*:
**assumes** *Coplanar A B C D*
**shows** *Coplanar C B D A*
⟨*proof*⟩

**lemma** *coplanar-perm-16*:
**assumes** *Coplanar A B C D*
**shows** *Coplanar C D A B*
⟨*proof*⟩

**lemma** *coplanar-perm-17*:
**assumes** *Coplanar A B C D*
**shows** *Coplanar C D B A*
⟨*proof*⟩

**lemma** *coplanar-perm-18*:
**assumes** *Coplanar A B C D*
**shows** *Coplanar D A B C*
⟨*proof*⟩

**lemma** *coplanar-perm-19*:
**assumes** *Coplanar A B C D*
**shows** *Coplanar D A C B*
⟨*proof*⟩

**lemma** *coplanar-perm-20*:
**assumes** *Coplanar A B C D*
**shows** *Coplanar D B A C*
⟨*proof*⟩

**lemma** *coplanar-perm-21*:
**assumes** *Coplanar A B C D*
**shows** *Coplanar D B C A*
⟨*proof*⟩

**lemma** *coplanar-perm-22*:
**assumes** *Coplanar A B C D*
**shows** *Coplanar D C A B*
⟨*proof*⟩

**lemma** *coplanar-perm-23*:
**assumes** *Coplanar A B C D*
**shows** *Coplanar D C B A*
⟨*proof*⟩

**lemma** *ncoplanar-perm-1*:
**assumes** ¬ *Coplanar A B C D*
**shows** ¬ *Coplanar A B D C*
⟨*proof*⟩

**lemma** *ncoplanar-perm-2*:
**assumes** ¬ *Coplanar A B C D*
**shows** ¬ *Coplanar A C B D*
⟨*proof*⟩

**lemma** *ncoplanar-perm-3*:
**assumes** ¬ *Coplanar A B C D*
**shows** ¬ *Coplanar A C D B*
⟨*proof*⟩

**lemma** *ncoplanar-perm-4*:
**assumes** ¬ *Coplanar A B C D*
**shows** ¬ *Coplanar A D B C*
⟨*proof*⟩

**lemma** *ncoplanar-perm-5*:
**assumes** ¬ *Coplanar A B C D*
**shows** ¬ *Coplanar A D C B*
⟨*proof*⟩

**lemma** *ncoplanar-perm-6*:
**assumes** ¬ *Coplanar A B C D*
**shows** ¬ *Coplanar B A C D*
⟨*proof*⟩

**lemma** *ncoplanar-perm-7*:
**assumes** ¬ *Coplanar A B C D*
**shows** ¬ *Coplanar B A D C*
⟨*proof*⟩

**lemma** *ncoplanar-perm-8*:
**assumes** ¬ *Coplanar A B C D*
**shows** ¬ *Coplanar B C A D*
⟨*proof*⟩

**lemma** *ncoplanar-perm-9*:
**assumes** ¬ *Coplanar A B C D*
**shows** ¬ *Coplanar B C D A*
⟨*proof*⟩

**lemma** *ncoplanar-perm-10*:
**assumes** ¬ *Coplanar A B C D*
**shows** ¬ *Coplanar B D A C*
⟨*proof*⟩

**lemma** *ncoplanar-perm-11*:
**assumes** ¬ *Coplanar A B C D*
**shows** ¬ *Coplanar B D C A*
⟨*proof*⟩

**lemma** *ncoplanar-perm-12*:
**assumes** ¬ *Coplanar A B C D*
**shows** ¬ *Coplanar C A B D*
⟨*proof*⟩

**lemma** *ncoplanar-perm-13*:
**assumes** ¬ *Coplanar A B C D*
**shows** ¬ *Coplanar C A D B*
⟨*proof*⟩

**lemma** *ncoplanar-perm-14*:
**assumes** ¬ *Coplanar A B C D*
**shows** ¬ *Coplanar C B A D*
⟨*proof*⟩

**lemma** *ncoplanar-perm-15*:
**assumes** ¬ *Coplanar A B C D*
**shows** ¬ *Coplanar C B D A*
⟨*proof*⟩

**lemma** *ncoplanar-perm-16*:
**assumes** ¬ *Coplanar A B C D*
**shows** ¬ *Coplanar C D A B*
⟨*proof*⟩

**lemma** *ncoplanar-perm-17*:
**assumes** ¬ *Coplanar A B C D*
**shows** ¬ *Coplanar C D B A*
⟨*proof*⟩

**lemma** *ncoplanar-perm-18*:
**assumes** ¬ *Coplanar A B C D*
**shows** ¬ *Coplanar D A B C*
⟨*proof*⟩

**lemma** *ncoplanar-perm-19*:
**assumes** ¬ *Coplanar A B C D*
**shows** ¬ *Coplanar D A C B*
⟨*proof*⟩

**lemma** *ncoplanar-perm-20*:
**assumes** ¬ *Coplanar A B C D*
**shows** ¬ *Coplanar D B A C*
⟨*proof*⟩

**lemma** *ncoplanar-perm-21*:
**assumes** ¬ *Coplanar A B C D*
**shows** ¬ *Coplanar D B C A*
⟨*proof*⟩

**lemma** *ncoplanar-perm-22*:
**assumes** ¬ *Coplanar A B C D*
**shows** ¬ *Coplanar D C A B*
⟨*proof*⟩

**lemma** *ncoplanar-perm-23*:
**assumes** ¬ *Coplanar A B C D*
**shows** ¬ *Coplanar D C B A*
⟨*proof*⟩

**lemma** *coplanar-trivial*:
**shows** *Coplanar A A B C*
⟨*proof*⟩

**lemma** *col--coplanar*:
**assumes** *Col A B C*
**shows** *Coplanar A B C D*
⟨*proof*⟩

**lemma** *ncop--ncol*:
**assumes** ¬ *Coplanar A B C D*
**shows** ¬ *Col A B C*
⟨*proof*⟩

**lemma** *ncop--ncols*:
**assumes** ¬ *Coplanar A B C D*
**shows** ¬ *Col A B C* ∧ ¬ *Col A B D* ∧ ¬ *Col A C D* ∧ ¬ *Col B C D*
⟨*proof*⟩

**lemma** *bet--coplanar*:
**assumes** *Bet A B C*
**shows** *Coplanar A B C D*
⟨*proof*⟩

**lemma** *out--coplanar*:
**assumes** *A Out B C*

**shows** *Coplanar A B C D*
⟨*proof*⟩

**lemma** *midpoint--coplanar*:
**assumes** *A Midpoint B C*
**shows** *Coplanar A B C D*
⟨*proof*⟩

**lemma** *perp--coplanar*:
**assumes** *A B Perp C D*
**shows** *Coplanar A B C D*
⟨*proof*⟩

**lemma** *ts--coplanar*:
**assumes** *A B TS C D*
**shows** *Coplanar A B C D*
⟨*proof*⟩

**lemma** *reflectl--coplanar*:
**assumes** *A B ReflectL C D*
**shows** *Coplanar A B C D*
⟨*proof*⟩

**lemma** *reflect--coplanar*:
**assumes** *A B Reflect C D*
**shows** *Coplanar A B C D*
⟨*proof*⟩

**lemma** *inangle--coplanar*:
**assumes** *A InAngle B C D*
**shows** *Coplanar A B C D*
⟨*proof*⟩

**lemma** *pars--coplanar*:
**assumes** *A B ParStrict C D*
**shows** *Coplanar A B C D*
⟨*proof*⟩

**lemma** *par--coplanar*:
**assumes** *A B Par C D*
**shows** *Coplanar A B C D*
⟨*proof*⟩

**lemma** *plg--coplanar*:
**assumes** *Plg A B C D*
**shows** *Coplanar A B C D*
⟨*proof*⟩

**lemma** *plgs--coplanar*:
**assumes** *ParallelogramStrict A B C D*
**shows** *Coplanar A B C D*
⟨*proof*⟩

**lemma** *plgf--coplanar*:
**assumes** *ParallelogramFlat A B C D*
**shows** *Coplanar A B C D*
⟨*proof*⟩

**lemma** *parallelogram--coplanar*:
**assumes** *Parallelogram A B C D*
**shows** *Coplanar A B C D*
⟨*proof*⟩

**lemma** *rhombus--coplanar*:
**assumes** *Rhombus A B C D*
**shows** *Coplanar A B C D*

⟨*proof*⟩

**lemma** *rectangle--coplanar*:
**assumes** *Rectangle A B C D*
**shows** *Coplanar A B C D*
⟨*proof*⟩

**lemma** *square--coplanar*:
**assumes** *Square A B C D*
**shows** *Coplanar A B C D*
⟨*proof*⟩

**lemma** *lambert--coplanar*:
**assumes** *Lambert A B C D*
**shows** *Coplanar A B C D*
⟨*proof*⟩

### 3.8.2 Planes

**lemma** *ts-distincts*:
**assumes** *A B TS P Q*
**shows** $A \neq B \wedge A \neq P \wedge A \neq Q \wedge B \neq P \wedge B \neq Q \wedge P \neq Q$
⟨*proof*⟩

**lemma** *l9-2*:
**assumes** *A B TS P Q*
**shows** *A B TS Q P*
⟨*proof*⟩

**lemma** *invert-two-sides*:
**assumes** *A B TS P Q*
**shows** *B A TS P Q*
⟨*proof*⟩

**lemma** *l9-3*:
**assumes** *P Q TS A C* **and**
*Col M P Q* **and**
*M Midpoint A C* **and**
*Col R P Q* **and**
*R Out A B*
**shows** *P Q TS B C*
⟨*proof*⟩

**lemma** *mid-preserves-col*:
**assumes** *Col A B C* **and**
*M Midpoint A* $A'$ **and**
*M Midpoint B* $B'$ **and**
*M Midpoint C* $C'$
**shows** *Col* $A'$ $B'$ $C'$
⟨*proof*⟩

**lemma** *per-mid-per*:
**assumes**
*Per X A B* **and**
*M Midpoint A B* **and**
*M Midpoint X Y*
**shows** *Cong A X B Y* $\wedge$ *Per Y B A*
⟨*proof*⟩

**lemma** *sym-preserve-diff*:
**assumes** $A \neq B$ **and**
*M Midpoint A* $A'$ **and**
*M Midpoint B* $B'$
**shows** $A' \neq B'$
⟨*proof*⟩

**lemma** *l9-4-1-aux-R1*:
**assumes** $R = S$ **and**
$S\ C\ Le\ R\ A$ **and**
$P\ Q\ TS\ A\ C$ **and**
$Col\ R\ P\ Q$ **and**
$P\ Q\ Perp\ A\ R$ **and**
$Col\ S\ P\ Q$ **and**
$P\ Q\ Perp\ C\ S$ **and**
$M\ Midpoint\ R\ S$
**shows** $\forall\ U\ C'.\ M\ Midpoint\ U\ C' \longrightarrow (R\ Out\ U\ A \longleftrightarrow S\ Out\ C\ C')$
⟨*proof*⟩

**lemma** *l9-4-1-aux-R21*:
**assumes** $R \neq S$ **and**
$S\ C\ Le\ R\ A$ **and**
$P\ Q\ TS\ A\ C$ **and**
$Col\ R\ P\ Q$ **and**
$P\ Q\ Perp\ A\ R$ **and**
$Col\ S\ P\ Q$ **and**
$P\ Q\ Perp\ C\ S$ **and**
$M\ Midpoint\ R\ S$
**shows** $\forall\ U\ C'.\ M\ Midpoint\ U\ C' \longrightarrow (R\ Out\ U\ A \longleftrightarrow S\ Out\ C\ C')$
⟨*proof*⟩

**lemma** *l9-4-1-aux*:
**assumes** $S\ C\ Le\ R\ A$ **and**
$P\ Q\ TS\ A\ C$ **and**
$Col\ R\ P\ Q$ **and**
$P\ Q\ Perp\ A\ R$ **and**
$Col\ S\ P\ Q$ **and**
$P\ Q\ Perp\ C\ S$ **and**
$M\ Midpoint\ R\ S$
**shows** $\forall\ U\ C'.\ (M\ Midpoint\ U\ C' \longrightarrow (R\ Out\ U\ A \longleftrightarrow S\ Out\ C\ C'))$
⟨*proof*⟩

**lemma** *per-col-eq*:
**assumes** $Per\ A\ B\ C$ **and**
$Col\ A\ B\ C$ **and**
$B \neq C$
**shows** $A = B$
⟨*proof*⟩

**lemma** *l9-4-1*:
**assumes** $P\ Q\ TS\ A\ C$ **and**
$Col\ R\ P\ Q$ **and**
$P\ Q\ Perp\ A\ R$ **and**
$Col\ S\ P\ Q$ **and**
$P\ Q\ Perp\ C\ S$ **and**
$M\ Midpoint\ R\ S$
**shows** $\forall\ U\ C'.\ M\ Midpoint\ U\ C' \longrightarrow (R\ Out\ U\ A \longleftrightarrow S\ Out\ C\ C')$
⟨*proof*⟩

**lemma** *mid-two-sides*:
**assumes** $M\ Midpoint\ A\ B$ **and**
$\neg\ Col\ A\ B\ X$ **and**
$M\ Midpoint\ X\ Y$
**shows** $A\ B\ TS\ X\ Y$
⟨*proof*⟩

**lemma** *col-preserves-two-sides*:
**assumes** $C \neq D$ **and**
$Col\ A\ B\ C$ **and**
$Col\ A\ B\ D$ **and**
$A\ B\ TS\ X\ Y$
**shows** $C\ D\ TS\ X\ Y$

⟨*proof*⟩

**lemma** *out-out-two-sides*:
**assumes** $A \neq B$ **and**
*A B TS X Y* **and**
*Col I A B* **and**
*Col I X Y* **and**
*I Out X U* **and**
*I Out Y V*
**shows** *A B TS U V*
⟨*proof*⟩

**lemma** *l9-4-2-aux-R1*:
**assumes** $R = S$ **and**
*S C Le R A* **and**
*P Q TS A C* **and**
*Col R P Q* **and**
*P Q Perp A R* **and**
*Col S P Q* **and**
*P Q Perp C S* **and**
*R Out U A* **and**
*S Out V C*
**shows** *P Q TS U V*
⟨*proof*⟩

**lemma** *l9-4-2-aux-R2*:
**assumes** $R \neq S$ **and**
*S C Le R A* **and**
*P Q TS A C* **and**
*Col R P Q* **and**
*P Q Perp A R* **and**
*Col S P Q* **and**
*P Q Perp C S* **and**
*R Out U A* **and**
*S Out V C*
**shows** *P Q TS U V*
⟨*proof*⟩

**lemma** *l9-4-2-aux*:
**assumes** *S C Le R A* **and**
*P Q TS A C* **and**
*Col R P Q* **and**
*P Q Perp A R* **and**
*Col S P Q* **and**
*P Q Perp C S* **and**
*R Out U A* **and**
*S Out V C*
**shows** *P Q TS U V*
⟨*proof*⟩

**lemma** *l9-4-2*:
**assumes** *P Q TS A C* **and**
*Col R P Q* **and**
*P Q Perp A R* **and**
*Col S P Q* **and**
*P Q Perp C S* **and**
*R Out U A* **and**
*S Out V C*
**shows** *P Q TS U V*
⟨*proof*⟩

**lemma** *l9-5*:
**assumes** *P Q TS A C* **and**
*Col R P Q* **and**
*R Out A B*
**shows** *P Q TS B C*

⟨*proof*⟩

**lemma** *outer-pasch-R1*:
  **assumes** *Col P Q C* **and**
    *Bet A C P* **and**
    *Bet B Q C*
  **shows** $\exists$ *X*. *Bet A X B* $\wedge$ *Bet P Q X*
  ⟨*proof*⟩

**lemma** *outer-pasch-R2*:
  **assumes** $\neg$ *Col P Q C* **and**
    *Bet A C P* **and**
    *Bet B Q C*
  **shows** $\exists$ *X*. *Bet A X B* $\wedge$ *Bet P Q X*
⟨*proof*⟩

**lemma** *outer-pasch*:
  **assumes** *Bet A C P* **and**
    *Bet B Q C*
  **shows** $\exists$ *X*. *Bet A X B* $\wedge$ *Bet P Q X*
  ⟨*proof*⟩

**lemma** *os-distincts*:
  **assumes** *A B OS X Y*
  **shows** $A \neq B \wedge A \neq X \wedge A \neq Y \wedge B \neq X \wedge B \neq Y$
  ⟨*proof*⟩

**lemma** *invert-one-side*:
  **assumes** *A B OS P Q*
  **shows** *B A OS P Q*
⟨*proof*⟩

**lemma** *l9-8-1*:
  **assumes** *P Q TS A C* **and**
    *P Q TS B C*
  **shows** *P Q OS A B*
⟨*proof*⟩

**lemma** *not-two-sides-id*:
  **shows** $\neg$ *P Q TS A A*
  ⟨*proof*⟩

**lemma** *l9-8-2*:
  **assumes** *P Q TS A C* **and**
    *P Q OS A B*
  **shows** *P Q TS B C*
⟨*proof*⟩

**lemma** *l9-9*:
  **assumes** *P Q TS A B*
  **shows** $\neg$ *P Q OS A B*
  ⟨*proof*⟩

**lemma** *l9-9-bis*:
  **assumes** *P Q OS A B*
  **shows** $\neg$ *P Q TS A B*
  ⟨*proof*⟩

**lemma** *one-side-chara*:
  **assumes** *P Q OS A B*
  **shows** $\forall$ *X*. *Col X P Q* $\longrightarrow$ $\neg$ *Bet A X B*
⟨*proof*⟩

**lemma** *l9-10*:
  **assumes** $\neg$ *Col A P Q*
  **shows** $\exists$ *C*. *P Q TS A C*

⟨*proof*⟩

**lemma** *one-side-reflexivity*:
**assumes** ¬ *Col A P Q*
**shows** *P Q OS A A*
⟨*proof*⟩

**lemma** *one-side-symmetry*:
**assumes** *P Q OS A B*
**shows** *P Q OS B A*
⟨*proof*⟩

**lemma** *one-side-transitivity*:
**assumes** *P Q OS A B* **and**
*P Q OS B C*
**shows** *P Q OS A C*
⟨*proof*⟩

**lemma** *l9-17*:
**assumes** *P Q OS A C* **and**
*Bet A B C*
**shows** *P Q OS A B*
⟨*proof*⟩

**lemma** *l9-18-R1*:
**assumes** *Col X Y P* **and**
*Col A B P*
**and** *X Y TS A B*
**shows** *Bet A P B* ∧ ¬ *Col X Y A* ∧ ¬ *Col X Y B*
⟨*proof*⟩

**lemma** *l9-18-R2*:
**assumes** *Col X Y P* **and**
*Col A B P* **and**
*Bet A P B* **and**
¬ *Col X Y A* **and**
¬ *Col X Y B*
**shows** *X Y TS A B*
⟨*proof*⟩

**lemma** *l9-18*:
**assumes** *Col X Y P* **and**
*Col A B P*
**shows** *X Y TS A B* ⟷ (*Bet A P B* ∧ ¬ *Col X Y A* ∧ ¬ *Col X Y B*)
⟨*proof*⟩

**lemma** *l9-19-R1*:
**assumes** *Col X Y P* **and**
*Col A B P* **and**
*X Y OS A B*
**shows** *P Out A B* ∧ ¬ *Col X Y A*
⟨*proof*⟩

**lemma** *l9-19-R2*:
**assumes** *Col X Y P* **and**

*P Out A B* **and**
¬ *Col X Y A*
**shows** *X Y OS A B*
⟨*proof*⟩

**lemma** *l9-19*:
**assumes** *Col X Y P* **and**
*Col A B P*
**shows** *X Y OS A B* ⟷ (*P Out A B* ∧ ¬ *Col X Y A*)
⟨*proof*⟩

**lemma** *one-side-not-col123*:
  **assumes** *A B OS X Y*
  **shows** ¬ *Col A B X*
  ⟨*proof*⟩

**lemma** *one-side-not-col124*:
  **assumes** *A B OS X Y*
  **shows** ¬ *Col A B Y*
  ⟨*proof*⟩

**lemma** *col-two-sides*:
  **assumes** *Col A B C* **and**
    $A \neq C$ **and**
    *A B TS P Q*
  **shows** *A C TS P Q*
  ⟨*proof*⟩

**lemma** *col-one-side*:
  **assumes** *Col A B C* **and**
    $A \neq C$ **and**
    *A B OS P Q*
  **shows** *A C OS P Q*
⟨*proof*⟩

**lemma** *out-out-one-side*:
  **assumes** *A B OS X Y* **and**
    *A Out Y Z*
  **shows** *A B OS X Z*
  ⟨*proof*⟩

**lemma** *out-one-side*:
  **assumes** ¬ *Col A B X* ∨ ¬ *Col A B Y* **and**
    *A Out X Y*
  **shows** *A B OS X Y*
  ⟨*proof*⟩

**lemma** *bet--ts*:
  **assumes** $A \neq Y$ **and**
    ¬ *Col A B X* **and**
    *Bet X A Y*
  **shows** *A B TS X Y*
⟨*proof*⟩

**lemma** *bet-ts--ts*:
  **assumes** *A B TS X Y* **and**
    *Bet X Y Z*
  **shows** *A B TS X Z*
⟨*proof*⟩

**lemma** *bet-ts--os*:
  **assumes** *A B TS X Y* **and**
    *Bet X Y Z*
  **shows** *A B OS Y Z*
  ⟨*proof*⟩

**lemma** *l9-31* :
  **assumes** *A X OS Y Z* **and**
    *A Z OS Y X*
  **shows** *A Y TS X Z*
⟨*proof*⟩

**lemma** *col123--nos*:
  **assumes** *Col P Q A*
  **shows** ¬ *P Q OS A B*

⟨*proof*⟩

**lemma** *col124--nos*:
**assumes** *Col P Q B*
**shows** ¬ *P Q OS A B*
⟨*proof*⟩

**lemma** *col2-os--os*:
**assumes** $C \neq D$ **and**
*Col A B C* **and**
*Col A B D* **and**
*A B OS X Y*
**shows** *C D OS X Y*
⟨*proof*⟩

**lemma** *os-out-os*:
**assumes** *Col A B P* **and**
*A B OS C D* **and**
*P Out C C'*
**shows** *A B OS C' D*
⟨*proof*⟩

**lemma** *ts-ts-os*:
**assumes** *A B TS C D* **and**
*C D TS A B*
**shows** *A C OS B D*
⟨*proof*⟩

**lemma** *col-one-side-out*:
**assumes** *Col A X Y* **and**
*A B OS X Y*
**shows** *A Out X Y*
⟨*proof*⟩

**lemma** *col-two-sides-bet*:
**assumes** *Col A X Y* **and**
*A B TS X Y*
**shows** *Bet X A Y*
⟨*proof*⟩

**lemma** *os-ts1324--os*:
**assumes** *A X OS Y Z* **and**
*A Y TS X Z*
**shows** *A Z OS X Y*
⟨*proof*⟩

**lemma** *ts2--ex-bet2*:
**assumes** *A C TS B D* **and**
*B D TS A C*
**shows** ∃ *X*. *Bet A X C* ∧ *Bet B X D*
⟨*proof*⟩

**lemma** *out-one-side-1*:
**assumes** ¬ *Col A B C* **and**
*Col A B X* **and**
*X Out C D*
**shows** *A B OS C D*
⟨*proof*⟩

**lemma** *out-two-sides-two-sides*:
**assumes**
*Col A B PX* **and**
*PX Out X P* **and**
*A B TS P Y*
**shows** *A B TS X Y*
⟨*proof*⟩

**lemma** *l8-21-bis*:
  **assumes** $X \neq Y$ **and**
    $\neg$ *Col C A B*
  **shows** $\exists$ *P*. *Cong A P X Y* $\wedge$ *A B Perp P A* $\wedge$ *A B TS C P*
⟨*proof*⟩

**lemma** *ts--ncol*:
  **assumes** *A B TS X Y*
  **shows** $\neg$ *Col A X Y* $\vee$ $\neg$ *Col B X Y*
  ⟨*proof*⟩

**lemma** *one-or-two-sides-aux*:
  **assumes** $\neg$ *Col C A B* **and**
    $\neg$ *Col D A B* **and**
    *Col A C X*
    **and** *Col B D X*
  **shows** *A B TS C D* $\vee$ *A B OS C D*
⟨*proof*⟩

**lemma** *cop--one-or-two-sides*:
  **assumes** *Coplanar A B C D* **and**
    $\neg$ *Col C A B* **and**
    $\neg$ *Col D A B*
  **shows** *A B TS C D* $\vee$ *A B OS C D*
⟨*proof*⟩

**lemma** *os--coplanar*:
  **assumes** *A B OS C D*
  **shows** *Coplanar A B C D*
⟨*proof*⟩

**lemma** *coplanar-trans-1*:
  **assumes** $\neg$ *Col P Q R* **and**
    *Coplanar P Q R A* **and**
    *Coplanar P Q R B*
  **shows** *Coplanar Q R A B*
⟨*proof*⟩

**lemma** *col-cop--cop*:
  **assumes** *Coplanar A B C D* **and**
    $C \neq D$ **and**
    *Col C D E*
  **shows** *Coplanar A B C E*
⟨*proof*⟩

**lemma** *bet-cop--cop*:
  **assumes** *Coplanar A B C E* **and**
    *Bet C D E*
  **shows** *Coplanar A B C D*
  ⟨*proof*⟩

**lemma** *col2-cop--cop*:
  **assumes** *Coplanar A B C D* **and**
    $C \neq D$ **and**
    *Col C D E* **and**
    *Col C D F*
  **shows** *Coplanar A B E F*
⟨*proof*⟩

**lemma** *col-cop2--cop*:
  **assumes** $U \neq V$ **and**
    *Coplanar A B C U* **and**
    *Coplanar A B C V* **and**
    *Col U V P*
  **shows** *Coplanar A B C P*

⟨*proof*⟩

**lemma** *bet-cop2--cop*:
  **assumes** *Coplanar A B C U* **and**
    *Coplanar A B C W* **and**
    *Bet U V W*
  **shows** *Coplanar A B C V*
⟨*proof*⟩

**lemma** *coplanar-pseudo-trans*:
  **assumes** ¬ *Col P Q R* **and**
    *Coplanar P Q R A* **and**
    *Coplanar P Q R B* **and**
    *Coplanar P Q R C* **and**
    *Coplanar P Q R D*
  **shows** *Coplanar A B C D*
⟨*proof*⟩

**lemma** *l9-30*:
  **assumes** ¬ *Coplanar A B C P* **and**
    ¬ *Col D E F* **and**
    *Coplanar D E F P* **and**
    *Coplanar A B C X* **and**
    *Coplanar A B C Y* **and**
    *Coplanar A B C Z* **and**
    *Coplanar D E F X* **and**
    *Coplanar D E F Y* **and**
    *Coplanar D E F Z*
  **shows** *Col X Y Z*
⟨*proof*⟩

**lemma** *cop-per2--col*:
  **assumes** *Coplanar A X Y Z* **and**
    $A \neq Z$ **and**
    *Per X Z A* **and**
    *Per Y Z A*
  **shows** *Col X Y Z*
⟨*proof*⟩

**lemma** *cop-perp2--col*:
  **assumes** *Coplanar A B Y Z* **and**
    *X Y Perp A B* **and**
    *X Z Perp A B*
  **shows** *Col X Y Z*
⟨*proof*⟩

**lemma** *two-sides-dec*:
  **shows** *A B TS C D* ∨ ¬ *A B TS C D*
  ⟨*proof*⟩

**lemma** *cop-nts--os*:
  **assumes** *Coplanar A B C D* **and**
    ¬ *Col C A B* **and**
    ¬ *Col D A B* **and**
    ¬ *A B TS C D*
  **shows** *A B OS C D*
  ⟨*proof*⟩

**lemma** *cop-nos--ts*:
  **assumes** *Coplanar A B C D* **and**
    ¬ *Col C A B* **and**
    ¬ *Col D A B* **and**
    ¬ *A B OS C D*
  **shows** *A B TS C D*
  ⟨*proof*⟩

**lemma** *one-side-dec*:
$A\ B\ OS\ C\ D \lor \neg\ A\ B\ OS\ C\ D$
⟨*proof*⟩

**lemma** *cop-dec*:
*Coplanar* $A\ B\ C\ D \lor \neg$ *Coplanar* $A\ B\ C\ D$
⟨*proof*⟩

**lemma** *ex-diff-cop*:
$\exists\ E.$ *Coplanar* $A\ B\ C\ E \land D \neq E$
⟨*proof*⟩

**lemma** *ex-ncol-cop*:
**assumes** $D \neq E$
**shows** $\exists\ F.$ *Coplanar* $A\ B\ C\ F \land \neg$ *Col* $D\ E\ F$
⟨*proof*⟩

**lemma** *ex-ncol-cop2*:
$\exists\ E\ F.$ (*Coplanar* $A\ B\ C\ E \land$ *Coplanar* $A\ B\ C\ F \land \neg$ *Col* $D\ E\ F$)
⟨*proof*⟩

**lemma** *col2-cop2--eq*:
**assumes** $\neg$ *Coplanar* $A\ B\ C\ U$ **and**
$U \neq V$ **and**
*Coplanar* $A\ B\ C\ P$ **and**
*Coplanar* $A\ B\ C\ Q$ **and**
*Col* $U\ V\ P$ **and**
*Col* $U\ V\ Q$
**shows** $P = Q$
⟨*proof*⟩

**lemma** *cong3-cop2--col*:
**assumes** *Coplanar* $A\ B\ C\ P$ **and**
*Coplanar* $A\ B\ C\ Q$ **and**
$P \neq Q$ **and**
*Cong* $A\ P\ A\ Q$ **and**
*Cong* $B\ P\ B\ Q$ **and**
*Cong* $C\ P\ C\ Q$
**shows** *Col* $A\ B\ C$
⟨*proof*⟩

**lemma** *l9-38*:
**assumes** $A\ B\ C$ *TSP* $P\ Q$
**shows** $A\ B\ C$ *TSP* $Q\ P$
⟨*proof*⟩

**lemma** *l9-39*:
**assumes** $A\ B\ C$ *TSP* $P\ R$ **and**
*Coplanar* $A\ B\ C\ D$ **and**
$D$ *Out* $P\ Q$
**shows** $A\ B\ C$ *TSP* $Q\ R$
⟨*proof*⟩

**lemma** *l9-41-1*:
**assumes** $A\ B\ C$ *TSP* $P\ R$ **and**
$A\ B\ C$ *TSP* $Q\ R$
**shows** $A\ B\ C$ *OSP* $P\ Q$
⟨*proof*⟩

**lemma** *l9-41-2*:
**assumes** $A\ B\ C$ *TSP* $P\ R$ **and**
$A\ B\ C$ *OSP* $P\ Q$
**shows** $A\ B\ C$ *TSP* $Q\ R$
⟨*proof*⟩

**lemma** *tsp-exists*:

**assumes** $\neg$ *Coplanar A B C P*
**shows** $\exists$ *Q*. *A B C TSP P Q*
⟨*proof*⟩

**lemma** *osp-reflexivity*:
**assumes** $\neg$ *Coplanar A B C P*
**shows** *A B C OSP P P*
⟨*proof*⟩

**lemma** *osp-symmetry*:
**assumes** *A B C OSP P Q*
**shows** *A B C OSP Q P*
⟨*proof*⟩

**lemma** *osp-transitivity*:
**assumes** *A B C OSP P Q* **and**
*A B C OSP Q R*
**shows** *A B C OSP P R*
⟨*proof*⟩

**lemma** *cop3-tsp--tsp*:
**assumes** $\neg$ *Col D E F* **and**
*Coplanar A B C D* **and**
*Coplanar A B C E* **and**
*Coplanar A B C F* **and**
*A B C TSP P Q*
**shows** *D E F TSP P Q*
⟨*proof*⟩

**lemma** *cop3-osp--osp*:
**assumes** $\neg$ *Col D E F* **and**
*Coplanar A B C D* **and**
*Coplanar A B C E* **and**
*Coplanar A B C F* **and**
*A B C OSP P Q*
**shows** *D E F OSP P Q*
⟨*proof*⟩

**lemma** *ncop-distincts*:
**assumes** $\neg$ *Coplanar A B C D*
**shows** $A \neq B \wedge A \neq C \wedge A \neq D \wedge B \neq C \wedge B \neq D \wedge C \neq D$
⟨*proof*⟩

**lemma** *tsp-distincts*:
**assumes** *A B C TSP P Q*
**shows** $A \neq B \wedge A \neq C \wedge B \neq C \wedge A \neq P \wedge B \neq P \wedge C \neq P \wedge A \neq Q \wedge B \neq Q \wedge C \neq Q \wedge P \neq Q$
⟨*proof*⟩

**lemma** *osp-distincts*:
**assumes** *A B C OSP P Q*
**shows** $A \neq B \wedge A \neq C \wedge B \neq C \wedge A \neq P \wedge B \neq P \wedge C \neq P \wedge A \neq Q \wedge B \neq Q \wedge C \neq Q$
⟨*proof*⟩

**lemma** *tsp--ncop1*:
**assumes** *A B C TSP P Q*
**shows** $\neg$ *Coplanar A B C P*
⟨*proof*⟩

**lemma** *tsp--ncop2*:
**assumes** *A B C TSP P Q*
**shows** $\neg$ *Coplanar A B C Q*
⟨*proof*⟩

**lemma** *osp--ncop1*:
**assumes** *A B C OSP P Q*

**shows** ¬ *Coplanar A B C P*
⟨*proof*⟩

**lemma** *osp--ncop2*:
**assumes** *A B C OSP P Q*
**shows** ¬ *Coplanar A B C Q*
⟨*proof*⟩

**lemma** *tsp--nosp*:
**assumes** *A B C TSP P Q*
**shows** ¬ *A B C OSP P Q*
⟨*proof*⟩

**lemma** *osp--ntsp*:
**assumes** *A B C OSP P Q*
**shows** ¬ *A B C TSP P Q*
⟨*proof*⟩

**lemma** *osp-bet--osp*:
**assumes** *A B C OSP P R* **and**
*Bet P Q R*
**shows** *A B C OSP P Q*
⟨*proof*⟩

**lemma** *l9-18-3*:
**assumes** *Coplanar A B C P* **and**
*Col X Y P*
**shows** *A B C TSP X Y* ⟷ (*Bet X P Y* ∧ ¬ *Coplanar A B C X* ∧ ¬ *Coplanar A B C Y*)
⟨*proof*⟩

**lemma** *bet-cop--tsp*:
**assumes** ¬ *Coplanar A B C X* **and**
$P \neq Y$ **and**
*Coplanar A B C P* **and**
*Bet X P Y*
**shows** *A B C TSP X Y*
⟨*proof*⟩

**lemma** *cop-out--osp*:
**assumes** ¬ *Coplanar A B C X* **and**
*Coplanar A B C P* **and**
*P Out X Y*
**shows** *A B C OSP X Y*
⟨*proof*⟩

**lemma** *l9-19-3*:
**assumes** *Coplanar A B C P* **and**
*Col X Y P*
**shows** *A B C OSP X Y* ⟷ (*P Out X Y* ∧ ¬ *Coplanar A B C X*)
⟨*proof*⟩

**lemma** *cop2-ts--tsp*:
**assumes** ¬ *Coplanar A B C X* **and** *Coplanar A B C D* **and**
*Coplanar A B C E* **and** *D E TS X Y*
**shows** *A B C TSP X Y*
⟨*proof*⟩

**lemma** *cop2-os--osp*:
**assumes** ¬ *Coplanar A B C X* **and**
*Coplanar A B C D* **and**
*Coplanar A B C E* **and**
*D E OS X Y*
**shows** *A B C OSP X Y*
⟨*proof*⟩

**lemma** *cop3-tsp--ts*:

**assumes** $D \neq E$ **and**
*Coplanar* $A$ $B$ $C$ $D$ **and**
*Coplanar* $A$ $B$ $C$ $E$ **and**
*Coplanar* $D$ $E$ $X$ $Y$ **and**
$A$ $B$ $C$ *TSP* $X$ $Y$
**shows** $D$ $E$ *TS* $X$ $Y$
⟨*proof*⟩

**lemma** *cop3-osp--os*:
**assumes** $D \neq E$ **and**
*Coplanar* $A$ $B$ $C$ $D$ **and**
*Coplanar* $A$ $B$ $C$ $E$ **and**
*Coplanar* $D$ $E$ $X$ $Y$ **and**
$A$ $B$ $C$ *OSP* $X$ $Y$
**shows** $D$ $E$ *OS* $X$ $Y$
⟨*proof*⟩

**lemma** *cop-tsp--ex-cop2*:
**assumes**
$A$ $B$ $C$ *TSP* $D$ $E$
**shows** $\exists$ $Q$. (*Coplanar* $A$ $B$ $C$ $Q$ $\wedge$ *Coplanar* $D$ $E$ $P$ $Q$ $\wedge$ $P \neq Q$)
⟨*proof*⟩

**lemma** *cop-osp--ex-cop2*:
**assumes** *Coplanar* $A$ $B$ $C$ $P$ **and**
$A$ $B$ $C$ *OSP* $D$ $E$
**shows** $\exists$ $Q$. *Coplanar* $A$ $B$ $C$ $Q$ $\wedge$ *Coplanar* $D$ $E$ $P$ $Q$ $\wedge$ $P \neq Q$
⟨*proof*⟩

**lemma** *sac--coplanar*:
**assumes** *Saccheri* $A$ $B$ $C$ $D$
**shows** *Coplanar* $A$ $B$ $C$ $D$
⟨*proof*⟩

## 3.9 Line reflexivity

### 3.9.1 Dimensionless

**lemma** *Ch10-Goal1*:
**assumes** $\neg$ *Coplanar* $D$ $C$ $B$ $A$
**shows** $\neg$ *Coplanar* $A$ $B$ $C$ $D$
⟨*proof*⟩

**lemma** *ex-sym*:
$\exists$ $Y$. ($A$ $B$ *Perp* $X$ $Y$ $\vee$ $X = Y$) $\wedge$ ($\exists$ $M$. *Col* $A$ $B$ $M$ $\wedge$ $M$ *Midpoint* $X$ $Y$)
⟨*proof*⟩

**lemma** *is-image-is-image-spec*:
**assumes** $A \neq B$
**shows** $P'$ $P$ *Reflect* $A$ $B$ $\longleftrightarrow$ $P'$ $P$ *ReflectL* $A$ $B$
⟨*proof*⟩

**lemma** *ex-sym1*:
**assumes** $A \neq B$
**shows** $\exists$ $Y$. ($A$ $B$ *Perp* $X$ $Y$ $\vee$ $X = Y$) $\wedge$ ($\exists$ $M$. *Col* $A$ $B$ $M$ $\wedge$ $M$ *Midpoint* $X$ $Y$ $\wedge$ $X$ $Y$ *Reflect* $A$ $B$)
⟨*proof*⟩

**lemma** *l10-2-uniqueness*:
**assumes** *P1* $P$ *Reflect* $A$ $B$ **and**
*P2* $P$ *Reflect* $A$ $B$
**shows** $P1 = P2$
⟨*proof*⟩

**lemma** *l10-2-uniqueness-spec*:
**assumes** *P1* $P$ *ReflectL* $A$ $B$ **and**
*P2* $P$ *ReflectL* $A$ $B$
**shows** $P1 = P2$

⟨*proof*⟩

**lemma** *l10-2-existence-spec*:
$\exists$ *P'*. *P' P ReflectL A B*
⟨*proof*⟩

**lemma** *l10-2-existence*:
$\exists$ *P'*. *P' P Reflect A B*
⟨*proof*⟩

**lemma** *l10-4-spec*:
**assumes** *P P' ReflectL A B*
**shows** *P' P ReflectL A B*
⟨*proof*⟩

**lemma** *l10-4*:
**assumes** *P P' Reflect A B*
**shows** *P' P Reflect A B*
⟨*proof*⟩

**lemma** *l10-5*:
**assumes** *P' P Reflect A B* **and**
*P'' P' Reflect A B*
**shows** $P = P''$
⟨*proof*⟩

**lemma** *l10-6-uniqueness*:
**assumes** *P P1 Reflect A B* **and**
*P P2 Reflect A B*
**shows** *P1* = *P2*
⟨*proof*⟩

**lemma** *l10-6-uniqueness-spec*:
**assumes** *P P1 ReflectL A B* **and**
*P P2 ReflectL A B*
**shows** *P1* = *P2*
⟨*proof*⟩

**lemma** *l10-6-existence-spec*:
**assumes** $A \neq B$
**shows** $\exists$ *P*. *P' P ReflectL A B*
⟨*proof*⟩

**lemma** *l10-6-existence*:
$\exists$ *P*. *P' P Reflect A B*
⟨*proof*⟩

**lemma** *l10-7*:
**assumes** *P' P Reflect A B* **and**
*Q' Q Reflect A B* **and**
$P' = Q'$
**shows** $P = Q$
⟨*proof*⟩

**lemma** *l10-8*:
**assumes** *P P Reflect A B*
**shows** *Col P A B*
⟨*proof*⟩

**lemma** *col--refl*:
**assumes** *Col P A B*
**shows** *P P ReflectL A B*
⟨*proof*⟩

**lemma** *is-image-col-cong*:
**assumes** $A \neq B$ **and**

$P\ P'$ *Reflect* $A\ B$ **and**
*Col* $A\ B\ X$
**shows** *Cong* $P\ X\ P'\ X$
⟨*proof*⟩

**lemma** *is-image-spec-col-cong*:
**assumes** $P\ P'$ *ReflectL* $A\ B$ **and**
*Col* $A\ B\ X$
**shows** *Cong* $P\ X\ P'\ X$
⟨*proof*⟩

**lemma** *image-id*:
**assumes** $A \neq B$ **and**
*Col* $A\ B\ T$ **and**
$T\ T'$ *Reflect* $A\ B$
**shows** $T = T'$
⟨*proof*⟩

**lemma** *osym-not-col*:
**assumes** $P\ P'$ *Reflect* $A\ B$ **and**
$\neg$ *Col* $A\ B\ P$
**shows** $\neg$ *Col* $A\ B\ P'$
⟨*proof*⟩

**lemma** *midpoint-preserves-image*:
**assumes** $A \neq B$ **and**
*Col* $A\ B\ M$ **and**
$P\ P'$ *Reflect* $A\ B$ **and**
$M$ *Midpoint* $P\ Q$ **and**
$M$ *Midpoint* $P'\ Q'$
**shows** $Q\ Q'$ *Reflect* $A\ B$
⟨*proof*⟩

**lemma** *image-in-is-image-spec*:
**assumes** $M$ *ReflectLAt* $P\ P'\ A\ B$
**shows** $P\ P'$ *ReflectL* $A\ B$
⟨*proof*⟩

**lemma** *image-in-gen-is-image*:
**assumes** $M$ *ReflectAt* $P\ P'\ A\ B$
**shows** $P\ P'$ *Reflect* $A\ B$
⟨*proof*⟩

**lemma** *image-image-in*:
**assumes** $P \neq P'$ **and**
$P\ P'$ *ReflectL* $A\ B$ **and**
*Col* $A\ B\ M$ **and**
*Col* $P\ M\ P'$
**shows** $M$ *ReflectLAt* $P\ P'\ A\ B$
⟨*proof*⟩

**lemma** *image-in-col*:
**assumes** $Y$ *ReflectLAt* $P\ P'\ A\ B$
**shows** *Col* $P\ P'\ Y$
⟨*proof*⟩

**lemma** *is-image-spec-rev*:
**assumes** $P\ P'$ *ReflectL* $A\ B$
**shows** $P\ P'$ *ReflectL* $B\ A$
⟨*proof*⟩

**lemma** *is-image-rev*:
**assumes** $P\ P'$ *Reflect* $A\ B$
**shows** $P\ P'$ *Reflect* $B\ A$
⟨*proof*⟩

**lemma** *midpoint-preserves-per*:
  **assumes** *Per A B C* **and**
    *M Midpoint A A1* **and**
    *M Midpoint B B1* **and**
    *M Midpoint C C1*
  **shows** *Per A1 B1 C1*
⟨*proof*⟩

**lemma** *col--image-spec*:
  **assumes** *Col A B X*
  **shows** *X X ReflectL A B*
  ⟨*proof*⟩

**lemma** *image-triv*:
  *A A Reflect A B*
  ⟨*proof*⟩

**lemma** *cong-midpoint--image*:
  **assumes** *Cong A X A Y* **and**
    *B Midpoint X Y*
  **shows** *Y X Reflect A B*
⟨*proof*⟩

**lemma** *col-image-spec--eq*:
  **assumes** *Col A B P* **and**
    $P\ P'$ *ReflectL A B*
  **shows** $P = P'$
  ⟨*proof*⟩

**lemma** *image-spec-triv*:
  *A A ReflectL B B*
  ⟨*proof*⟩

**lemma** *image-spec--eq*:
  **assumes** $P\ P'$ *ReflectL A A*
  **shows** $P = P'$
  ⟨*proof*⟩

**lemma** *image--midpoint*:
  **assumes** $P\ P'$ *Reflect A A*
  **shows** *A Midpoint* $P'\ P$
  ⟨*proof*⟩

**lemma** *is-image-spec-dec*:
  *A B ReflectL C D* ∨ ¬ *A B ReflectL C D*
  ⟨*proof*⟩

**lemma** *l10-14*:
  **assumes** $P \neq P'$ **and**
    $A \neq B$ **and**
    $P\ P'$ *Reflect A B*
  **shows** *A B TS* $P\ P'$
⟨*proof*⟩

**lemma** *l10-15*:
  **assumes** *Col A B C* **and**
    ¬ *Col A B P*
  **shows** ∃ *Q. A B Perp Q C* ∧ *A B OS P Q*
⟨*proof*⟩

**lemma** *ex-per-cong*:
  **assumes** $A \neq B$ **and**
    $X \neq Y$ **and**
    *Col A B C* **and**
    ¬ *Col A B D*

**shows** $\exists$ *P*. *Per P C A* $\wedge$ *Cong P C X Y* $\wedge$ *A B OS P D*
⟨*proof*⟩

**lemma** *exists-cong-per*:
$\exists$ *C*. *Per A B C* $\wedge$ *Cong B C X Y*
⟨*proof*⟩

### 3.9.2 Upper dim 2

**lemma** *upper-dim-implies-per2--col*:
**assumes** *upper-dim-axiom*
**shows** $\forall$ *A B C X*. (*Per A X C* $\wedge$ *X* $\neq$ *C* $\wedge$ *Per B X C*) $\longrightarrow$ *Col A B X*
⟨*proof*⟩

**lemma** *upper-dim-implies-col-perp2--col*:
**assumes** *upper-dim-axiom*
**shows** $\forall$ *A B X Y P*. (*Col A B P* $\wedge$ *A B Perp X P* $\wedge$ *P A Perp Y P*) $\longrightarrow$ *Col Y X P*
⟨*proof*⟩

**lemma** *upper-dim-implies-perp2--col*:
**assumes** *upper-dim-axiom*
**shows** $\forall$ *X Y Z A B*. (*X Y Perp A B* $\wedge$ *X Z Perp A B*) $\longrightarrow$ *Col X Y Z*
⟨*proof*⟩

**lemma** *upper-dim-implies-not-two-sides-one-side-aux*:
**assumes** *upper-dim-axiom*
**shows** $\forall$ *A B X Y PX*. (*A* $\neq$ *B* $\wedge$ *PX* $\neq$ *A* $\wedge$ *A B Perp X PX* $\wedge$ *Col A B PX* $\wedge$ $\neg$ *Col X A B* $\wedge$ $\neg$ *Col Y A B* $\wedge$ $\neg$ *A B TS X Y*) $\longrightarrow$ *A B OS X Y*
⟨*proof*⟩

**lemma** *upper-dim-implies-not-two-sides-one-side*:
**assumes** *upper-dim-axiom*
**shows** $\forall$ *A B X Y*. ($\neg$ *Col X A B* $\wedge$ $\neg$ *Col Y A B* $\wedge$ $\neg$ *A B TS X Y*) $\longrightarrow$ *A B OS X Y*
⟨*proof*⟩

**lemma** *upper-dim-implies-not-one-side-two-sides*:
**assumes** *upper-dim-axiom*
**shows** $\forall$ *A B X Y*. ($\neg$ *Col X A B* $\wedge$ $\neg$ *Col Y A B* $\wedge$ $\neg$ *A B OS X Y*) $\longrightarrow$ *A B TS X Y*
⟨*proof*⟩

**lemma** *upper-dim-implies-one-or-two-sides*:
**assumes** *upper-dim-axiom*
**shows** $\forall$ *A B X Y*. ($\neg$ *Col X A B* $\wedge$ $\neg$ *Col Y A B*) $\longrightarrow$ (*A B TS X Y* $\vee$ *A B OS X Y*)
⟨*proof*⟩

**lemma** *upper-dim-implies-all-coplanar*:
**assumes** *upper-dim-axiom*
**shows** *all-coplanar-axiom*
⟨*proof*⟩

**lemma** *all-coplanar-implies-upper-dim*:
**assumes** *all-coplanar-axiom*
**shows** *upper-dim-axiom*
⟨*proof*⟩

**lemma** *all-coplanar-upper-dim*:
**shows** *all-coplanar-axiom* $\longleftrightarrow$ *upper-dim-axiom*
⟨*proof*⟩

**lemma** *upper-dim-stab*:
**shows** $\neg$ $\neg$ *upper-dim-axiom* $\longrightarrow$ *upper-dim-axiom* ⟨*proof*⟩

**lemma** *cop--cong-on-bissect*:
**assumes** *Coplanar A B X P* **and**
*M Midpoint A B* **and**
*M PerpAt A B P M* **and**

*Cong X A X B*
**shows** *Col M P X*
⟨*proof*⟩

**lemma** *cong-cop-mid-perp--col*:
**assumes** *Coplanar A B X P* **and**
*Cong A X B X* **and**
*M Midpoint A B* **and**
*A B Perp P M*
**shows** *Col M P X*
⟨*proof*⟩

**lemma** *cop-image-in2--col*:
**assumes** *Coplanar A B P Q* **and**
*M ReflectLAt P P′ A B* **and**
*M ReflectLAt Q Q′ A B*
**shows** *Col M P Q*
⟨*proof*⟩

**lemma** *l10-10-spec*:
**assumes** *P′ P ReflectL A B* **and**
*Q′ Q ReflectL A B*
**shows** *Cong P Q P′ Q′*
⟨*proof*⟩

**lemma** *l10-10*:
**assumes** *P′ P Reflect A B* **and**
*Q′ Q Reflect A B*
**shows** *Cong P Q P′ Q′*
⟨*proof*⟩

**lemma** *image-preserves-bet*:
**assumes** *A A′ ReflectL X Y* **and**
*B B′ ReflectL X Y* **and**
*C C′ ReflectL X Y* **and**
*Bet A B C*
**shows** *Bet A′ B′ C′*
⟨*proof*⟩

**lemma** *image-gen-preserves-bet*:
**assumes** *A A′ Reflect X Y* **and**
*B B′ Reflect X Y* **and**
*C C′ Reflect X Y* **and**
*Bet A B C*
**shows** *Bet A′ B′ C′*
⟨*proof*⟩

**lemma** *image-preserves-col*:
**assumes** *A A′ ReflectL X Y* **and**
*B B′ ReflectL X Y* **and**
*C C′ ReflectL X Y* **and**
*Col A B C*
**shows** *Col A′ B′ C′* ⟨*proof*⟩

**lemma** *image-gen-preserves-col*:
**assumes** *A A′ Reflect X Y* **and**
*B B′ Reflect X Y* **and**
*C C′ Reflect X Y* **and**
*Col A B C*
**shows** *Col A′ B′ C′*
⟨*proof*⟩

**lemma** *image-gen-preserves-ncol*:
**assumes** *A A′ Reflect X Y* **and**
*B B′ Reflect X Y* **and**
*C C′ Reflect X Y* **and**

$\neg$ *Col A B C*
**shows** $\neg$ *Col* $A'$ $B'$ $C'$
⟨*proof*⟩

**lemma** *image-gen-preserves-inter*:
**assumes** $A$ $A'$ *Reflect X Y* **and**
$B$ $B'$ *Reflect X Y* **and**
$C$ $C'$ *Reflect X Y* **and**
$D$ $D'$ *Reflect X Y* **and**
$\neg$ *Col A B C* **and**
$C \neq D$ **and**
*Col A B I* **and**
*Col C D I* **and**
*Col* $A'$ $B'$ $I'$ **and**
*Col* $C'$ $D'$ $I'$
**shows** $I$ $I'$ *Reflect X Y*
⟨*proof*⟩

**lemma** *intersection-with-image-gen*:
**assumes** $A$ $A'$ *Reflect X Y* **and**
$B$ $B'$ *Reflect X Y* **and**
$\neg$ *Col A B* $A'$ **and**
*Col A B C* **and**
*Col* $A'$ $B'$ $C$
**shows** *Col C X Y*
⟨*proof*⟩

**lemma** *image-preserves-midpoint* :
**assumes** $A$ $A'$ *ReflectL X Y* **and**
$B$ $B'$ *ReflectL X Y* **and**
$C$ $C'$ *ReflectL X Y* **and**
*A Midpoint B C*
**shows** $A'$ *Midpoint* $B'$ $C'$
⟨*proof*⟩

**lemma** *image-spec-preserves-per*:
**assumes** $A$ $A'$ *ReflectL X Y* **and**
$B$ $B'$ *ReflectL X Y* **and**
$C$ $C'$ *ReflectL X Y* **and**
*Per A B C*
**shows** *Per* $A'$ $B'$ $C'$
⟨*proof*⟩

**lemma** *image-preserves-per*:
**assumes** $A$ $A'$ *Reflect X Y* **and**
$B$ $B'$ *Reflect X Y***and**
$C$ $C'$ *Reflect X Y* **and**
*Per A B C*
**shows** *Per* $A'$ $B'$ $C'$
⟨*proof*⟩

**lemma** *l10-12*:
**assumes** *Per A B C* **and**
*Per* $A'$ $B'$ $C'$ **and**
*Cong A B* $A'$ $B'$ **and**
*Cong B C* $B'$ $C'$
**shows** *Cong A C* $A'$ $C'$
⟨*proof*⟩

**lemma** *l10-16*:
**assumes** $\neg$ *Col A B C* **and**
$\neg$ *Col* $A'$ $B'$ $P$ **and**
*Cong A B* $A'$ $B'$
**shows** $\exists$ $C'$. *A B C Cong3* $A'$ $B'$ $C'$ $\wedge$ $A'$ $B'$ *OS P* $C'$
⟨*proof*⟩

**lemma** *cong-cop-image--col*:
**assumes** $P \neq P'$ **and**
$P\ P'$ *Reflect* $A\ B$ **and**
*Cong* $P\ X\ P'\ X$ **and**
*Coplanar* $A\ B\ P\ X$
**shows** *Col* $A\ B\ X$
⟨*proof*⟩

**lemma** *cong-cop-per2-1*:
**assumes** $A \neq B$ **and**
*Per* $A\ B\ X$ **and**
*Per* $A\ B\ Y$ **and**
*Cong* $B\ X\ B\ Y$ **and**
*Coplanar* $A\ B\ X\ Y$
**shows** $X = Y \lor B$ *Midpoint* $X\ Y$
⟨*proof*⟩

**lemma** *cong-cop-per2*:
**assumes** $A \neq B$ **and**
*Per* $A\ B\ X$ **and**
*Per* $A\ B\ Y$ **and**
*Cong* $B\ X\ B\ Y$ **and**
*Coplanar* $A\ B\ X\ Y$
**shows** $X = Y \lor X\ Y$ *ReflectL* $A\ B$
⟨*proof*⟩

**lemma** *cong-cop-per2-gen*:
**assumes** $A \neq B$ **and**
*Per* $A\ B\ X$ **and**
*Per* $A\ B\ Y$ **and**
*Cong* $B\ X\ B\ Y$ **and**
*Coplanar* $A\ B\ X\ Y$
**shows** $X = Y \lor X\ Y$ *Reflect* $A\ B$
⟨*proof*⟩

**lemma** *ex-perp-cop*:
**assumes** $A \neq B$
**shows** $\exists\ Q.\ A\ B$ *Perp* $Q\ C \land$ *Coplanar* $A\ B\ P\ Q$
⟨*proof*⟩

**lemma** *hilbert-s-version-of-pasch-aux*:
**assumes** *Coplanar* $A\ B\ C\ P$ **and**
$\neg$ *Col* $A\ I\ P$ **and**
$\neg$ *Col* $B\ C\ P$ **and**
*Bet* $B\ I\ C$ **and**
$B \neq I$ **and**
$I \neq C$ **and**
$B \neq C$
**shows** $\exists\ X.$ *Col* $I\ P\ X \land (($*Bet* $A\ X\ B \land A \neq X \land X \neq B \land A \neq B) \lor ($*Bet* $A\ X\ C \land A \neq X \land X \neq C \land A \neq C))$
⟨*proof*⟩

**lemma** *hilbert-s-version-of-pasch*:
**assumes** *Coplanar* $A\ B\ C\ P$ **and**
$\neg$ *Col* $C\ Q\ P$ **and**
$\neg$ *Col* $A\ B\ P$ **and**
*BetS* $A\ Q\ B$
**shows** $\exists\ X.$ *Col* $P\ Q\ X \land ($*BetS* $A\ X\ C \lor$ *BetS* $B\ X\ C)$
⟨*proof*⟩

**lemma** *two-sides-cases*:
**assumes** $\neg$ *Col* $PO\ A\ B$ **and**
$PO\ P$ *OS* $A\ B$
**shows** $PO\ A$ *TS* $P\ B \lor PO\ B$ *TS* $P\ A$
⟨*proof*⟩

**lemma** *not-par-two-sides*:

**assumes** $C \neq D$ **and**
$Col\ A\ B\ I$ **and**
$Col\ C\ D\ I$ **and**
$\neg\ Col\ A\ B\ C$
**shows** $\exists\ X\ Y.\ Col\ C\ D\ X \wedge Col\ C\ D\ Y \wedge A\ B\ TS\ X\ Y$
⟨*proof*⟩

**lemma** *cop-not-par-other-side*:
**assumes** $C \neq D$ **and**
$Col\ A\ B\ I$ **and**
$Col\ C\ D\ I$ **and**
$\neg\ Col\ A\ B\ C$ **and**
$\neg\ Col\ A\ B\ P$ **and**
$Coplanar\ A\ B\ C\ P$
**shows** $\exists\ Q.\ Col\ C\ D\ Q \wedge A\ B\ TS\ P\ Q$
⟨*proof*⟩

**lemma** *cop-not-par-same-side*:
**assumes** $C \neq D$ **and**
$Col\ A\ B\ I$ **and**
$Col\ C\ D\ I$ **and**
$\neg\ Col\ A\ B\ C$ **and**
$\neg\ Col\ A\ B\ P$ **and**
$Coplanar\ A\ B\ C\ P$
**shows** $\exists\ Q.\ Col\ C\ D\ Q \wedge A\ B\ OS\ P\ Q$
⟨*proof*⟩

**end**

### 3.9.3 Line reflexivity: 2D

**context** *Tarski-2D*

**begin**

**lemma** *all-coplanar*:
$Coplanar\ A\ B\ C\ D$
⟨*proof*⟩

**lemma** *per2--col*:
**assumes** $Per\ A\ X\ C$ **and**
$X \neq C$ **and**
$Per\ B\ X\ C$
**shows** $Col\ A\ B\ X$
⟨*proof*⟩

**lemma** *perp2--col*:
**assumes** $X\ Y\ Perp\ A\ B$ **and**
$X\ Z\ Perp\ A\ B$
**shows** $Col\ X\ Y\ Z$
⟨*proof*⟩
**end**

## 3.10 Angles

### 3.10.1 Some generalites

**context** *Tarski-neutral-dimensionless*

**begin**

**lemma** *l11-3*:
**assumes** $A\ B\ C\ CongA\ D\ E\ F$
**shows** $\exists\ A'\ C'\ D'\ F'.\ B\ Out\ A'\ A \wedge B\ Out\ C\ C' \wedge E\ Out\ D'\ D \wedge E\ Out\ F\ F' \wedge A'\ B\ C'\ Cong3\ D'\ E\ F'$
⟨*proof*⟩

**lemma** *l11-aux*:
  **assumes** $B$ *Out* $A$ $A'$ **and**
    $E$ *Out* $D$ $D'$ **and**
    *Cong* $B$ $A'$ $E$ $D'$ **and**
    *Bet* $B$ $A$ *A0* **and**
    *Bet* $E$ $D$ *D0* **and**
    *Cong* $A$ *A0* $E$ $D$ **and**
    *Cong* $D$ *D0* $B$ $A$
  **shows** *Cong* $B$ *A0* $E$ *D0* $\wedge$ *Cong* $A'$ *A0* $D'$ *D0*
⟨*proof*⟩

**lemma** *l11-3-bis*:
  **assumes** $\exists$ $A'$ $C'$ $D'$ $F'$. ($B$ *Out* $A'$ $A$ $\wedge$ $B$ *Out* $C'$ $C$ $\wedge$ $E$ *Out* $D'$ $D$ $\wedge$ $E$ *Out* $F'$ $F$ $\wedge$ $A'$ $B$ $C'$ *Cong3* $D'$ $E$ $F'$)
  **shows** $A$ $B$ $C$ *CongA* $D$ $E$ $F$
⟨*proof*⟩

**lemma** *l11-4-1*:
  **assumes** $A$ $B$ $C$ *CongA* $D$ $E$ $F$ **and**

    $B$ *Out* $A'$ $A$ **and**
    $B$ *Out* $C'$ $C$ **and**
    $E$ *Out* $D'$ $D$ **and**
    $E$ *Out* $F'$ $F$ **and**
    *Cong* $B$ $A'$ $E$ $D'$ **and** *Cong* $B$ $C'$ $E$ $F'$
  **shows** *Cong* $A'$ $C'$ $D'$ $F'$
⟨*proof*⟩

**lemma** *l11-4-2*:
  **assumes** $A \neq B$ **and**
    $C \neq B$ **and**
    $D \neq E$ **and**
    $F \neq E$ **and**
    $\forall$ $A'$ $C'$ $D'$ $F'$. ($B$ *Out* $A'$ $A$ $\wedge$ $B$ *Out* $C'$ $C$ $\wedge$ $E$ *Out* $D'$ $D$ $\wedge$ $E$ *Out* $F'$ $F$ $\wedge$ *Cong* $B$ $A'$ $E$ $D'$ $\wedge$ *Cong* $B$ $C'$ $E$ $F'$ $\longrightarrow$
*Cong* $A'$ $C'$ $D'$ $F'$)
  **shows** $A$ $B$ $C$ *CongA* $D$ $E$ $F$
⟨*proof*⟩

**lemma** *conga-refl*:
  **assumes** $A \neq B$ **and**
    $C \neq B$
  **shows** $A$ $B$ $C$ *CongA* $A$ $B$ $C$
  ⟨*proof*⟩

**lemma** *conga-sym*:
  **assumes** $A$ $B$ $C$ *CongA* $A'$ $B'$ $C'$
  **shows** $A'$ $B'$ $C'$ *CongA* $A$ $B$ $C$
⟨*proof*⟩

**lemma** *l11-10*:
  **assumes** $A$ $B$ $C$ *CongA* $D$ $E$ $F$ **and**
    $B$ *Out* $A'$ $A$ **and**
    $B$ *Out* $C'$ $C$ **and**
    $E$ *Out* $D'$ $D$ **and**
    $E$ *Out* $F'$ $F$
  **shows** $A'$ $B$ $C'$ *CongA* $D'$ $E$ $F'$
⟨*proof*⟩

**lemma** *out2--conga*:
  **assumes** $B$ *Out* $A'$ $A$ **and**
    $B$ *Out* $C'$ $C$
  **shows** $A$ $B$ $C$ *CongA* $A'$ $B$ $C'$
  ⟨*proof*⟩

**lemma** *cong3-diff*:
  **assumes** $A \neq B$ **and**
    $A$ $B$ $C$ *Cong3* $A'$ $B'$ $C'$

**shows** $A' \neq B'$
⟨*proof*⟩

**lemma** *cong3-diff2*:
**assumes** $B \neq C$ **and**
$A\ B\ C\ Cong3\ A'\ B'\ C'$
**shows** $B' \neq C'$
⟨*proof*⟩

**lemma** *cong3-conga*:
**assumes** $A \neq B$ **and**
$C \neq B$ **and**
$A\ B\ C\ Cong3\ A'\ B'\ C'$
**shows** $A\ B\ C\ CongA\ A'\ B'\ C'$
⟨*proof*⟩

**lemma** *cong3-conga2*:
**assumes** $A\ B\ C\ Cong3\ A'\ B'\ C'$ **and**
$A\ B\ C\ CongA\ A''\ B''\ C''$
**shows** $A'\ B'\ C'\ CongA\ A''\ B''\ C''$
⟨*proof*⟩

**lemma** *conga-diff1*:
**assumes** $A\ B\ C\ CongA\ A'\ B'\ C'$
**shows** $A \neq B$
⟨*proof*⟩

**lemma** *conga-diff2*:
**assumes** $A\ B\ C\ CongA\ A'\ B'\ C'$
**shows** $C \neq B$
⟨*proof*⟩

**lemma** *conga-diff45*:
**assumes** $A\ B\ C\ CongA\ A'\ B'\ C'$
**shows** $A' \neq B'$
⟨*proof*⟩

**lemma** *conga-diff56*:
**assumes** $A\ B\ C\ CongA\ A'\ B'\ C'$
**shows** $C' \neq B'$
⟨*proof*⟩

**lemma** *conga-trans*:
**assumes** $A\ B\ C\ CongA\ A'\ B'\ C'$ **and**
$A'\ B'\ C'\ CongA\ A''\ B''\ C''$
**shows** $A\ B\ C\ CongA\ A''\ B''\ C''$
⟨*proof*⟩

**lemma** *conga-pseudo-refl*:
**assumes** $A \neq B$ **and**
$C \neq B$
**shows** $A\ B\ C\ CongA\ C\ B\ A$
⟨*proof*⟩

**lemma** *conga-trivial-1*:
**assumes** $A \neq B$ **and**
$C \neq D$
**shows** $A\ B\ A\ CongA\ C\ D\ C$
⟨*proof*⟩

**lemma** *l11-13*:
**assumes** $A\ B\ C\ CongA\ D\ E\ F$ **and**
$Bet\ A\ B\ A'$ **and**
$A' \neq B$ **and**
$Bet\ D\ E\ D'$ **and**
$D' \neq E$

**shows** $A'$ $B$ $C$ $CongA$ $D'$ $E$ $F$
⟨*proof*⟩

**lemma** *conga-right-comm*:
**assumes** $A$ $B$ $C$ $CongA$ $D$ $E$ $F$
**shows** $A$ $B$ $C$ $CongA$ $F$ $E$ $D$
⟨*proof*⟩

**lemma** *conga-left-comm*:
**assumes** $A$ $B$ $C$ $CongA$ $D$ $E$ $F$
**shows** $C$ $B$ $A$ $CongA$ $D$ $E$ $F$
⟨*proof*⟩

**lemma** *conga-comm*:
**assumes** $A$ $B$ $C$ $CongA$ $D$ $E$ $F$
**shows** $C$ $B$ $A$ $CongA$ $F$ $E$ $D$
⟨*proof*⟩

**lemma** *conga-line*:
**assumes** $A \neq B$ **and**
$B \neq C$ **and**
$A' \neq B'$ **and**
$B' \neq C'$
**and** $Bet$ $A$ $B$ $C$ **and**
$Bet$ $A'$ $B'$ $C'$
**shows** $A$ $B$ $C$ $CongA$ $A'$ $B'$ $C'$
⟨*proof*⟩

**lemma** *l11-14*:
**assumes** $Bet$ $A$ $B$ $A'$ **and**
$A \neq B$ **and**
$A' \neq B$ **and**
$Bet$ $C$ $B$ $C'$ **and**
$B \neq C$ **and**
$B \neq C'$
**shows** $A$ $B$ $C$ $CongA$ $A'$ $B$ $C'$
⟨*proof*⟩

**lemma** *l11-16*:
**assumes** $Per$ $A$ $B$ $C$ **and**
$A \neq B$ **and**
$C \neq B$ **and**
$Per$ $A'$ $B'$ $C'$ **and**
$A' \neq B'$ **and**
$C' \neq B'$
**shows** $A$ $B$ $C$ $CongA$ $A'$ $B'$ $C'$
⟨*proof*⟩

**lemma** *l11-17*:
**assumes** $Per$ $A$ $B$ $C$ **and**
$A$ $B$ $C$ $CongA$ $A'$ $B'$ $C'$
**shows** $Per$ $A'$ $B'$ $C'$
⟨*proof*⟩

**lemma** *l11-18-1*:
**assumes** $Bet$ $C$ $B$ $D$ **and**
$B \neq C$ **and**
$B \neq D$ **and**
$A \neq B$ **and**
$Per$ $A$ $B$ $C$
**shows** $A$ $B$ $C$ $CongA$ $A$ $B$ $D$
⟨*proof*⟩

**lemma** *l11-18-2*:
**assumes** $Bet$ $C$ $B$ $D$ **and**
$A$ $B$ $C$ $CongA$ $A$ $B$ $D$

**shows** *Per A B C*
⟨*proof*⟩

**lemma** *cong3-preserves-out*:
**assumes** *A Out B C* **and**
*A B C Cong3 A′ B′ C′*
**shows** *A′ Out B′ C′*
⟨*proof*⟩

**lemma** *l11-21-a*:
**assumes** *B Out A C* **and**
*A B C CongA A′ B′ C′*
**shows** *B′ Out A′ C′*
⟨*proof*⟩

**lemma** *l11-21-b*:
**assumes** *B Out A C* **and**
*B′ Out A′ C′*
**shows** *A B C CongA A′ B′ C′*
⟨*proof*⟩

**lemma** *conga-cop--or-out-ts*:
**assumes** *Coplanar A B C C′* **and**
*A B C CongA A B C′*
**shows** *B Out C C′* ∨ *A B TS C C′*
⟨*proof*⟩

**lemma** *conga-os--out*:
**assumes** *A B C CongA A B C′* **and**
*A B OS C C′*
**shows** *B Out C C′*
⟨*proof*⟩

**lemma** *cong2-conga-cong*:
**assumes** *A B C CongA A′ B′ C′* **and**
*Cong A B A′ B′* **and**
*Cong B C B′ C′*
**shows** *Cong A C A′ C′*
⟨*proof*⟩

**lemma** *angle-construction-1*:
**assumes** ¬ *Col A B C* **and**
¬ *Col A′ B′ P*
**shows** ∃ *C′*. (*A B C CongA A′ B′ C′* ∧ *A′ B′ OS C′ P*)
⟨*proof*⟩

**lemma** *angle-construction-2*:
**assumes** *A* ≠ *B* **and**
*B* ≠ *C* **and**
¬ *Col A′ B′ P*
**shows** ∃ *C′*. (*A B C CongA A′ B′ C′* ∧ (*A′ B′ OS C′ P* ∨ *Col A′ B′ C′*))
⟨*proof*⟩

**lemma** *ex-conga-ts*:
**assumes** ¬ *Col A B C* **and**
¬ *Col A′ B′ P*
**shows** ∃ *C′*. *A B C CongA A′ B′ C′* ∧ *A′ B′ TS C′ P*
⟨*proof*⟩

**lemma** *l11-15*:
**assumes** ¬ *Col A B C* **and**
¬ *Col D E P*
**shows**
∃ *F*. (*A B C CongA D E F* ∧ *E D OS F P*) ∧
(∀ *F1 F2*. ((*A B C CongA D E F1* ∧ *E D OS F1 P*) ∧
(*A B C CongA D E F2* ∧ *E D OS F2 P*))

$\longrightarrow$ *E Out F1 F2*)
⟨*proof*⟩

**lemma** *l11-19*:
**assumes** *Per A B P1* **and**
*Per A B P2* **and**
*A B OS P1 P2*
**shows** *B Out P1 P2*
⟨*proof*⟩

**lemma** *l11-22-bet*:
**assumes** *Bet A B C* **and**
$P'$ $B'$ *TS* $A'$ $C'$ **and**
*A B P CongA* $A'$ $B'$ $P'$ **and**
*P B C CongA* $P'$ $B'$ $C'$
**shows** *Bet* $A'$ $B'$ $C'$
⟨*proof*⟩

**lemma** *l11-22a*:
**assumes** *B P TS A C* **and**
$B'$ $P'$ *TS* $A'$ $C'$ **and**
*A B P CongA* $A'$ $B'$ $P'$ **and**
*P B C CongA* $P'$ $B'$ $C'$
**shows** *A B C CongA* $A'$ $B'$ $C'$
⟨*proof*⟩

**lemma** *l11-22b*:
**assumes** *B P OS A C* **and**
$B'$ $P'$ *OS* $A'$ $C'$ **and**
*A B P CongA* $A'$ $B'$ $P'$ **and**
*P B C CongA* $P'$ $B'$ $C'$
**shows** *A B C CongA* $A'$ $B'$ $C'$
⟨*proof*⟩

**lemma** *l11-22*:
**assumes** ((*B P TS A C* $\wedge$ $B'$ $P'$ *TS* $A'$ $C'$)$\vee$(*B P OS A C* $\wedge$ $B'$ $P'$ *OS* $A'$ $C'$)) **and**
*A B P CongA* $A'$ $B'$ $P'$ **and**
*P B C CongA* $P'$ $B'$ $C'$
**shows** *A B C CongA* $A'$ $B'$ $C'$
⟨*proof*⟩

**lemma** *l11-24*:
**assumes** *P InAngle A B C*
**shows** *P InAngle C B A*
⟨*proof*⟩

**lemma** *col-in-angle*:
**assumes** $A \neq B$ **and**
$C \neq B$ **and**
$P \neq B$ **and**
*B Out A P* $\vee$ *B Out C P*
**shows** *P InAngle A B C*
⟨*proof*⟩

**lemma** *out321--inangle*:
**assumes** $C \neq B$ **and**
*B Out A P*
**shows** *P InAngle A B C*
⟨*proof*⟩

**lemma** *inangle1123*:
**assumes** $A \neq B$ **and**
$C \neq B$
**shows** *A InAngle A B C*
⟨*proof*⟩

**lemma** *out341--inangle*:
**assumes** $A \neq B$ **and**
$B$ *Out* $C$ $P$
**shows** $P$ *InAngle* $A$ $B$ $C$
⟨*proof*⟩

**lemma** *inangle3123*:
**assumes** $A \neq B$ **and**
$C \neq B$
**shows** $C$ *InAngle* $A$ $B$ $C$
⟨*proof*⟩

**lemma** *in-angle-two-sides*:
**assumes** $\neg$ *Col* $B$ $A$ $P$ **and**
$\neg$ *Col* $B$ $C$ $P$ **and**
$P$ *InAngle* $A$ $B$ $C$
**shows** $P$ $B$ *TS* $A$ $C$
⟨*proof*⟩

**lemma** *in-angle-out*:
**assumes** $B$ *Out* $A$ $C$ **and**
$P$ *InAngle* $A$ $B$ $C$
**shows** $B$ *Out* $A$ $P$
⟨*proof*⟩

**lemma** *col-in-angle-out*:
**assumes** *Col* $B$ $A$ $P$ **and**
$\neg$ *Bet* $A$ $B$ $C$ **and**
$P$ *InAngle* $A$ $B$ $C$
**shows** $B$ *Out* $A$ $P$
⟨*proof*⟩

**lemma** *l11-25-aux*:
**assumes** $P$ *InAngle* $A$ $B$ $C$ **and**
$\neg$ *Bet* $A$ $B$ $C$ **and**
$B$ *Out* $A'$ $A$
**shows** $P$ *InAngle* $A'$ $B$ $C$
⟨*proof*⟩

**lemma** *l11-25*:
**assumes** $P$ *InAngle* $A$ $B$ $C$ **and**
$B$ *Out* $A'$ $A$ **and**
$B$ *Out* $C'$ $C$ **and**
$B$ *Out* $P'$ $P$
**shows** $P'$ *InAngle* $A'$ $B$ $C'$
⟨*proof*⟩

**lemma** *inangle-distincts*:
**assumes** $P$ *InAngle* $A$ $B$ $C$
**shows** $A \neq B \wedge C \neq B \wedge P \neq B$
⟨*proof*⟩

**lemma** *segment-construction-0*:
**shows** $\exists$ $B'$. *Cong* $A'$ $B'$ $A$ $B$
⟨*proof*⟩

**lemma** *angle-construction-3*:
**assumes** $A \neq B$ **and**
$C \neq B$ **and**
$A' \neq B'$
**shows** $\exists$ $C'$. $A$ $B$ $C$ *CongA* $A'$ $B'$ $C'$
⟨*proof*⟩

**lemma** *l11-28*:
**assumes** $A$ $B$ $C$ *Cong3* $A'$ $B'$ $C'$ **and**
*Col* $A$ $C$ $D$

**shows** $\exists\ D'.\ (Cong\ A\ D\ A'\ D' \wedge Cong\ B\ D\ B'\ D' \wedge Cong\ C\ D\ C'\ D')$
⟨*proof*⟩

**lemma** *bet-conga--bet*:
**assumes** $Bet\ A\ B\ C$ **and**
$A\ B\ C\ CongA\ A'\ B'\ C'$
**shows** $Bet\ A'\ B'\ C'$
⟨*proof*⟩

**lemma** *in-angle-one-side*:
**assumes** $\neg\ Col\ A\ B\ C$ **and**
$\neg\ Col\ B\ A\ P$ **and**
$P\ InAngle\ A\ B\ C$
**shows** $A\ B\ OS\ P\ C$
⟨*proof*⟩

**lemma** *inangle-one-side*:
**assumes** $\neg\ Col\ A\ B\ C$ **and**
$\neg\ Col\ A\ B\ P$ **and**
$\neg\ Col\ A\ B\ Q$ **and**
$P\ InAngle\ A\ B\ C$ **and**
$Q\ InAngle\ A\ B\ C$
**shows** $A\ B\ OS\ P\ Q$
⟨*proof*⟩

**lemma** *inangle-one-side2*:
**assumes** $\neg\ Col\ A\ B\ C$ **and**
$\neg\ Col\ A\ B\ P$ **and**
$\neg\ Col\ A\ B\ Q$ **and**
$\neg\ Col\ C\ B\ P$ **and**
$\neg\ Col\ C\ B\ Q$ **and**
$P\ InAngle\ A\ B\ C$ **and**
$Q\ InAngle\ A\ B\ C$
**shows** $A\ B\ OS\ P\ Q \wedge C\ B\ OS\ P\ Q$
⟨*proof*⟩

**lemma** *col-conga-col*:
**assumes** $Col\ A\ B\ C$ **and**
$A\ B\ C\ CongA\ D\ E\ F$
**shows** $Col\ D\ E\ F$
⟨*proof*⟩

**lemma** *ncol-conga-ncol*:
**assumes** $\neg\ Col\ A\ B\ C$ **and**
$A\ B\ C\ CongA\ D\ E\ F$
**shows** $\neg\ Col\ D\ E\ F$
⟨*proof*⟩

**lemma** *angle-construction-4*:
**assumes** $A \neq B$ **and**
$C \neq B$ **and**
$A' \neq B'$
**shows** $\exists\, C'.\ (A\ B\ C\ CongA\ A'\ B'\ C' \wedge Coplanar\ A'\ B'\ C'\ P)$
⟨*proof*⟩

**lemma** *lea-distincts*:
**assumes** $A\ B\ C\ LeA\ D\ E\ F$
**shows** $A{\neq}B \wedge C{\neq}B \wedge D{\neq}E \wedge F{\neq}E$
⟨*proof*⟩

**lemma** *l11-29-a*:
**assumes** $A\ B\ C\ LeA\ D\ E\ F$
**shows** $\exists\ Q.\ (C\ InAngle\ A\ B\ Q \wedge A\ B\ Q\ CongA\ D\ E\ F)$
⟨*proof*⟩

**lemma** *in-angle-line*:

**assumes** $P \neq B$ **and**
$A \neq B$ **and**
$C \neq B$ **and**
*Bet A B C*
**shows** *P InAngle A B C*
⟨*proof*⟩

**lemma** *l11-29-b*:
**assumes** $\exists\ Q.\ (C\ \mathit{InAngle}\ A\ B\ Q \wedge A\ B\ Q\ \mathit{CongA}\ D\ E\ F)$
**shows** *A B C LeA D E F*
⟨*proof*⟩

**lemma** *bet-in-angle-bet*:
**assumes** *Bet A B P* **and**
*P InAngle A B C*
**shows** *Bet A B C*
⟨*proof*⟩

**lemma** *lea-line*:
**assumes** *Bet A B P* **and**
*A B P LeA A B C*
**shows** *Bet A B C*
⟨*proof*⟩

**lemma** *eq-conga-out*:
**assumes** *A B A CongA D E F*
**shows** *E Out D F*
⟨*proof*⟩

**lemma** *out-conga-out*:
**assumes** *B Out A C* **and**
*A B C CongA D E F*
**shows** *E Out D F*
⟨*proof*⟩

**lemma** *conga-ex-cong3*:
**assumes** $A\ B\ C\ \mathit{CongA}\ A'\ B'\ C'$
**shows** $\exists\ AA\ CC.\ ((B\ \mathit{Out}\ A\ AA \wedge B\ \mathit{Out}\ C\ CC) \longrightarrow AA\ B\ CC\ \mathit{Cong3}\ A'\ B'\ C')$
⟨*proof*⟩

**lemma** *conga-preserves-in-angle*:
**assumes** $A\ B\ C\ \mathit{CongA}\ A'\ B'\ C'$ **and**
$A\ B\ I\ \mathit{CongA}\ A'\ B'\ I'$ **and**
*I InAngle A B C* **and** $A'\ B'\ \mathit{OS}\ I'\ C'$
**shows** $I'\ \mathit{InAngle}\ A'\ B'\ C'$
⟨*proof*⟩

**lemma** *l11-30*:
**assumes** *A B C LeA D E F* **and**
$A\ B\ C\ \mathit{CongA}\ A'\ B'\ C'$ **and**
$D\ E\ F\ \mathit{CongA}\ D'\ E'\ F'$
**shows** $A'\ B'\ C'\ \mathit{LeA}\ D'\ E'\ F'$
⟨*proof*⟩

**lemma** *l11-31-1*:
**assumes** *B Out A C* **and**
$D \neq E$ **and**
$F \neq E$
**shows** *A B C LeA D E F*
⟨*proof*⟩

**lemma** *l11-31-2*:
**assumes** $A \neq B$ **and**
$C \neq B$ **and**
$D \neq E$ **and**
$F \neq E$ **and**

*Bet D E F*
**shows** *A B C LeA D E F*
⟨*proof*⟩

**lemma** *lea-refl*:
**assumes** $A \neq B$ **and**
$C \neq B$
**shows** *A B C LeA A B C*
⟨*proof*⟩

**lemma** *conga--lea*:
**assumes** *A B C CongA D E F*
**shows** *A B C LeA D E F*
⟨*proof*⟩

**lemma** *conga--lea456123*:
**assumes** *A B C CongA D E F*
**shows** *D E F LeA A B C*
⟨*proof*⟩

**lemma** *lea-left-comm*:
**assumes** *A B C LeA D E F*
**shows** *C B A LeA D E F*
⟨*proof*⟩

**lemma** *lea-right-comm*:
**assumes** *A B C LeA D E F*
**shows** *A B C LeA F E D*
⟨*proof*⟩

**lemma** *lea-comm*:
**assumes***A B C LeA D E F*
**shows** *C B A LeA F E D*
⟨*proof*⟩

**lemma** *lta-left-comm*:
**assumes** *A B C LtA D E F*
**shows** *C B A LtA D E F*
⟨*proof*⟩

**lemma** *lta-right-comm*:
**assumes** *A B C LtA D E F*
**shows** *A B C LtA F E D*
⟨*proof*⟩

**lemma** *lta-comm*:
**assumes** *A B C LtA D E F*
**shows** *C B A LtA F E D*
⟨*proof*⟩

**lemma** *lea-out4--lea*:
**assumes** *A B C LeA D E F* **and**
*B Out A* $A'$ **and**
*B Out C* $C'$ **and**
*E Out D* $D'$ **and**
*E Out F* $F'$
**shows** $A'$ *B* $C'$ *LeA* $D'$ *E* $F'$
⟨*proof*⟩

**lemma** *lea121345*:
**assumes** $A \neq B$ **and**
$C \neq D$ **and**
$D \neq E$
**shows** *A B A LeA C D E*
⟨*proof*⟩

**lemma** *inangle--lea*:
  **assumes** *P InAngle A B C*
  **shows** *A B P LeA A B C*
  ⟨*proof*⟩

**lemma** *inangle--lea-1*:
  **assumes** *P InAngle A B C*
  **shows** *P B C LeA A B C*
  ⟨*proof*⟩

**lemma** *inangle--lta*:
  **assumes** ¬ *Col P B C* **and**
    *P InAngle A B C*
  **shows** *A B P LtA A B C*
  ⟨*proof*⟩

**lemma** *in-angle-trans*:
  **assumes** *C InAngle A B D* **and**
    *D InAngle A B E*
  **shows** *C InAngle A B E*
⟨*proof*⟩

**lemma** *lea-trans*:
  **assumes** *A B C LeA A1 B1 C1* **and**
    *A1 B1 C1 LeA A2 B2 C2*
  **shows** *A B C LeA A2 B2 C2*
⟨*proof*⟩

**lemma** *in-angle-asym*:
  **assumes** *D InAngle A B C* **and**
    *C InAngle A B D*
  **shows** *A B C CongA A B D*
⟨*proof*⟩

**lemma** *lea-asym*:
  **assumes** *A B C LeA D E F* **and**
    *D E F LeA A B C*
  **shows** *A B C CongA D E F*
⟨*proof*⟩

**lemma** *col-lta--bet*:
  **assumes** *Col X Y Z* **and**
    *A B C LtA X Y Z*
  **shows** *Bet X Y Z*
⟨*proof*⟩

**lemma** *col-lta--out*:
  **assumes** *Col A B C* **and**
    *A B C LtA X Y Z*
  **shows** *B Out A C*
⟨*proof*⟩

**lemma** *lta-distincts*:
  **assumes** *A B C LtA D E F*
  **shows** $A{\neq}B \wedge C{\neq}B \wedge D{\neq}E \wedge F{\neq}E \wedge D \neq F$
  ⟨*proof*⟩

**lemma** *gta-distincts*:
  **assumes** *A B C GtA D E F*
  **shows** $A{\neq}B \wedge C{\neq}B \wedge D{\neq}E \wedge F{\neq}E \wedge A \neq C$
  ⟨*proof*⟩

**lemma** *acute-distincts*:
  **assumes** *Acute A B C*
  **shows** $A{\neq}B \wedge C{\neq}B$

⟨*proof*⟩

**lemma** *obtuse-distincts*:
**assumes** *Obtuse A B C*
**shows** $A \neq B \wedge C \neq B \wedge A \neq C$
⟨*proof*⟩

**lemma** *two-sides-in-angle*:
**assumes** $B \neq P'$ **and**
*B P TS A C* **and**
*Bet P B* $P'$
**shows** *P InAngle A B C* $\vee$ $P'$ *InAngle A B C*
⟨*proof*⟩

**lemma** *in-angle-reverse*:
**assumes** $A' \neq B$ **and**
*Bet A B* $A'$ **and**
*C InAngle A B D*
**shows** *D InAngle* $A'$ *B C*
⟨*proof*⟩

**lemma** *in-angle-trans2*:
**assumes** *C InAngle A B D* **and**
*D InAngle A B E*
**shows** *D InAngle C B E*
⟨*proof*⟩

**lemma** *l11-36-aux1*:
**assumes** $A \neq B$ **and**
$A' \neq B$ **and**
$D \neq E$ **and**
$D' \neq E$ **and**
*Bet A B* $A'$ **and**
*Bet D E* $D'$ **and**
*A B C LeA D E F*
**shows** $D'$ *E F LeA* $A'$ *B C*
⟨*proof*⟩

**lemma** *l11-36-aux2*:
**assumes** $A \neq B$ **and**
$A' \neq B$ **and**
$D \neq E$ **and**
$D' \neq E$ **and**
*Bet A B* $A'$ **and**
*Bet D E* $D'$ **and**
$D'$ *E F LeA* $A'$ *B C*
**shows** *A B C LeA D E F*
⟨*proof*⟩

**lemma** *l11-36*:
**assumes** $A \neq B$ **and**
$A' \neq B$ **and**
$D \neq E$ **and**
$D' \neq E$ **and**
*Bet A B* $A'$ **and**
*Bet D E* $D'$
**shows** *A B C LeA D E F* $\longleftrightarrow$ $D'$ *E F LeA* $A'$ *B C*
⟨*proof*⟩

**lemma** *l11-41-aux*:
**assumes** $\neg$ *Col A B C* **and**
*Bet B A D* **and**
$A \neq D$
**shows** *A C B LtA C A D*
⟨*proof*⟩

**lemma** *l11-41*:
  **assumes** $\neg$ *Col A B C* **and**
    *Bet B A D* **and**
    $A \neq D$
  **shows** *A C B LtA C A D* $\wedge$ *A B C LtA C A D*
⟨*proof*⟩

**lemma** *not-conga*:
  **assumes** *A B C CongA A′ B′ C′* **and**
    $\neg$ *A B C CongA D E F*
  **shows** $\neg$ *A′ B′ C′ CongA D E F*
  ⟨*proof*⟩

**lemma** *not-conga-sym*:
  **assumes** $\neg$ *A B C CongA D E F*
  **shows** $\neg$ *D E F CongA A B C*
  ⟨*proof*⟩

**lemma** *not-and-lta*:
  **shows** $\neg$ (*A B C LtA D E F* $\wedge$ *D E F LtA A B C*)
⟨*proof*⟩

**lemma** *conga-preserves-lta*:
  **assumes** *A B C CongA A′ B′ C′* **and**
    *D E F CongA D′ E′ F′* **and**
    *A B C LtA D E F*
  **shows** *A′ B′ C′ LtA D′ E′ F′*
  ⟨*proof*⟩

**lemma** *lta-trans*:
  **assumes** *A B C LtA A1 B1 C1* **and**
    *A1 B1 C1 LtA A2 B2 C2*
  **shows** *A B C LtA A2 B2 C2*
⟨*proof*⟩

**lemma** *obtuse-sym*:
  **assumes** *Obtuse A B C*
  **shows** *Obtuse C B A*
  ⟨*proof*⟩

**lemma** *acute-sym*:
  **assumes** *Acute A B C*
  **shows** *Acute C B A*
  ⟨*proof*⟩

**lemma** *acute-col--out*:
  **assumes** *Col A B C* **and**
    *Acute A B C*
  **shows** *B Out A C*
  ⟨*proof*⟩

**lemma** *col-obtuse--bet*:
  **assumes** *Col A B C* **and**
    *Obtuse A B C*
  **shows** *Bet A B C*
  ⟨*proof*⟩

**lemma** *out--acute*:
  **assumes** *B Out A C*
  **shows** *Acute A B C*
⟨*proof*⟩

**lemma** *bet--obtuse*:
  **assumes** *Bet A B C* **and**
    $A \neq B$ **and** $B \neq C$
  **shows** *Obtuse A B C*

⟨*proof*⟩

**lemma** *l11-43-aux*:
**assumes** $A \neq B$ **and**
$A \neq C$ **and**
*Per* $B\ A\ C$ ∨ *Obtuse* $B\ A\ C$
**shows** *Acute* $A\ B\ C$
⟨*proof*⟩

**lemma** *l11-43*:
**assumes** $A \neq B$ **and**
$A \neq C$ **and**
*Per* $B\ A\ C$ ∨ *Obtuse* $B\ A\ C$
**shows** *Acute* $A\ B\ C$ ∧ *Acute* $A\ C\ B$
⟨*proof*⟩

**lemma** *acute-lea-acute*:
**assumes** *Acute* $D\ E\ F$ **and**
$A\ B\ C$ *LeA* $D\ E\ F$
**shows** *Acute* $A\ B\ C$
⟨*proof*⟩

**lemma** *lea-obtuse-obtuse*:
**assumes** *Obtuse* $D\ E\ F$ **and**
$D\ E\ F$ *LeA* $A\ B\ C$
**shows** *Obtuse* $A\ B\ C$
⟨*proof*⟩

**lemma** *l11-44-1-a*:
**assumes** $A \neq B$ **and**
$A \neq C$ **and**
*Cong* $B\ A\ B\ C$
**shows** $B\ A\ C$ *CongA* $B\ C\ A$
⟨*proof*⟩

**lemma** *l11-44-2-a*:
**assumes** ¬ *Col* $A\ B\ C$ **and**
$B\ A$ *Lt* $B\ C$
**shows** $B\ C\ A$ *LtA* $B\ A\ C$
⟨*proof*⟩

**lemma** *not-lta-and-conga*:
¬ ( $A\ B\ C$ *LtA* $D\ E\ F$ ∧ $A\ B\ C$ *CongA* $D\ E\ F$)
⟨*proof*⟩

**lemma** *conga-sym-equiv*:
$A\ B\ C$ *CongA* $A'\ B'\ C' \longleftrightarrow A'\ B'\ C'$ *CongA* $A\ B\ C$
⟨*proof*⟩

**lemma** *conga-dec*:
$A\ B\ C$ *CongA* $D\ E\ F$ ∨ ¬ $A\ B\ C$ *CongA* $D\ E\ F$
⟨*proof*⟩

**lemma** *lta-not-conga*:
**assumes** $A\ B\ C$ *LtA* $D\ E\ F$
**shows** ¬ $A\ B\ C$ *CongA* $D\ E\ F$
⟨*proof*⟩

**lemma** *lta--lea*:
**assumes** $A\ B\ C$ *LtA* $D\ E\ F$
**shows** $A\ B\ C$ *LeA* $D\ E\ F$
⟨*proof*⟩

**lemma** *nlta*:
¬ $A\ B\ C$ *LtA* $A\ B\ C$
⟨*proof*⟩

**lemma** *lea--nlta*:
  **assumes** $A\ B\ C\ \mathit{LeA}\ D\ E\ F$
  **shows** $\neg\ D\ E\ F\ \mathit{LtA}\ A\ B\ C$
  ⟨*proof*⟩

**lemma** *lta--nlea*:
  **assumes** $A\ B\ C\ \mathit{LtA}\ D\ E\ F$
  **shows** $\neg\ D\ E\ F\ \mathit{LeA}\ A\ B\ C$
  ⟨*proof*⟩

**lemma** *l11-44-1-b*:
  **assumes** $\neg\ \mathit{Col}\ A\ B\ C$ **and**
    $B\ A\ C\ \mathit{CongA}\ B\ C\ A$
  **shows** $\mathit{Cong}\ B\ A\ B\ C$
⟨*proof*⟩

**lemma** *l11-44-2-b*:
  **assumes** $B\ A\ C\ \mathit{LtA}\ B\ C\ A$
  **shows** $B\ C\ \mathit{Lt}\ B\ A$
⟨*proof*⟩

**lemma** *l11-44-1*:
  **assumes** $\neg\ \mathit{Col}\ A\ B\ C$
  **shows** $B\ A\ C\ \mathit{CongA}\ B\ C\ A \longleftrightarrow \mathit{Cong}\ B\ A\ B\ C$
  ⟨*proof*⟩

**lemma** *l11-44-2*:
  **assumes** $\neg\ \mathit{Col}\ A\ B\ C$
  **shows** $B\ A\ C\ \mathit{LtA}\ B\ C\ A \longleftrightarrow B\ C\ \mathit{Lt}\ B\ A$
  ⟨*proof*⟩

**lemma** *l11-44-2bis*:
  **assumes** $\neg\ \mathit{Col}\ A\ B\ C$
  **shows** $B\ A\ C\ \mathit{LeA}\ B\ C\ A \longleftrightarrow B\ C\ \mathit{Le}\ B\ A$
⟨*proof*⟩

**lemma** *l11-46*:
  **assumes** $A \neq B$ **and**
    $B \neq C$ **and**
    $\mathit{Per}\ A\ B\ C \lor \mathit{Obtuse}\ A\ B\ C$
  **shows** $B\ A\ \mathit{Lt}\ A\ C \land B\ C\ \mathit{Lt}\ A\ C$
⟨*proof*⟩

**lemma** *l11-47*:
  **assumes** $\mathit{Per}\ A\ C\ B$ **and**
    $H\ \mathit{PerpAt}\ C\ H\ A\ B$
  **shows** $\mathit{Bet}\ A\ H\ B \land A \neq H \land B \neq H$
⟨*proof*⟩

**lemma** *l11-49*:
  **assumes** $A\ B\ C\ \mathit{CongA}\ A'\ B'\ C'$ **and**
    $\mathit{Cong}\ B\ A\ B'\ A'$ **and**
    $\mathit{Cong}\ B\ C\ B'\ C'$
  **shows** $\mathit{Cong}\ A\ C\ A'\ C' \land (A \neq C \longrightarrow (B\ A\ C\ \mathit{CongA}\ B'\ A'\ C' \land B\ C\ A\ \mathit{CongA}\ B'\ C'\ A'))$
⟨*proof*⟩

**lemma** *l11-50-1*:
  **assumes** $\neg\ \mathit{Col}\ A\ B\ C$ **and**
    $B\ A\ C\ \mathit{CongA}\ B'\ A'\ C'$ **and**
    $A\ B\ C\ \mathit{CongA}\ A'\ B'\ C'$ **and**
    $\mathit{Cong}\ A\ B\ A'\ B'$
  **shows** $\mathit{Cong}\ A\ C\ A'\ C' \land \mathit{Cong}\ B\ C\ B'\ C' \land A\ C\ B\ \mathit{CongA}\ A'\ C'\ B'$
⟨*proof*⟩

**lemma** *l11-50-2*:

**assumes** $\neg$ *Col A B C* **and**
*B C A CongA B′ C′ A′* **and**
*A B C CongA A′ B′ C′* **and**
*Cong A B A′ B′*
**shows** *Cong A C A′ C′* $\wedge$ *Cong B C B′ C′* $\wedge$ *C A B CongA C′ A′ B′*
⟨*proof*⟩

**lemma** *l11-51*:
**assumes** $A \neq B$ **and**
$A \neq C$ **and**
$B \neq C$ **and**
*Cong A B A′ B′* **and**
*Cong A C A′ C′* **and**
*Cong B C B′ C′*
**shows**
*B A C CongA B′ A′ C′* $\wedge$ *A B C CongA A′ B′ C′* $\wedge$ *B C A CongA B′ C′ A′*
⟨*proof*⟩

**lemma** *conga-distinct*:
**assumes** *A B C CongA D E F*
**shows** $A \neq B \wedge C \neq B \wedge D \neq E \wedge F \neq E$
⟨*proof*⟩

**lemma** *l11-52*:
**assumes** *A B C CongA A′ B′ C′* **and**
*Cong A C A′ C′* **and**
*Cong B C B′ C′* **and**
*B C Le A C*
**shows** *Cong B A B′ A′* $\wedge$ *B A C CongA B′ A′ C′* $\wedge$ *B C A CongA B′ C′ A′*
⟨*proof*⟩

**lemma** *l11-53*:
**assumes** *Per D C B* **and**
$C \neq D$ **and**
$A \neq B$ **and**
$B \neq C$ **and**
*Bet A B C*
**shows** *C A D LtA C B D* $\wedge$ *B D Lt A D*
⟨*proof*⟩

**lemma** *cong2-conga-obtuse--cong-conga2*:
**assumes** *Obtuse A B C* **and**
*A B C CongA A′ B′ C′* **and**
*Cong A C A′ C′* **and**
*Cong B C B′ C′*
**shows** *Cong B A B′ A′* $\wedge$ *B A C CongA B′ A′ C′* $\wedge$
*B C A CongA B′ C′ A′*
⟨*proof*⟩

**lemma** *cong2-per2--cong-conga2*:
**assumes** $A \neq B$ **and**
$B \neq C$ **and**
*Per A B C* **and**
*Per A′ B′ C′* **and**
*Cong A C A′ C′* **and**
*Cong B C B′ C′*
**shows** *Cong B A B′ A′* $\wedge$ *B A C CongA B′ A′ C′* $\wedge$
*B C A CongA B′ C′ A′*
⟨*proof*⟩

**lemma** *cong2-per2--cong*:
**assumes** *Per A B C* **and**
*Per A′ B′ C′* **and**
*Cong A C A′ C′* **and**
*Cong B C B′ C′*
**shows** *Cong B A B′ A′*

⟨*proof*⟩

**lemma** *cong2-per2--cong-3*:
  **assumes** *Per A B C*
    *Per A' B' C'* **and**
    *Cong A C A' C'* **and**
    *Cong B C B' C'*
  **shows** *A B C Cong3 A' B' C'*
  ⟨*proof*⟩

**lemma** *cong-lt-per2--lt*:
  **assumes** *Per A B C* **and**
    *Per A' B' C'* **and**
    *Cong A B A' B'* **and**
    *B C Lt B' C'*
  **shows** *A C Lt A' C'*
⟨*proof*⟩

**lemma** *cong-le-per2--le*:
  **assumes** *Per A B C* **and**
    *Per A' B' C'* **and**
    *Cong A B A' B'* **and**
    *B C Le B' C'*
  **shows** *A C Le A' C'*
⟨*proof*⟩

**lemma** *lt2-per2--lt*:
  **assumes** *Per A B C* **and**
    *Per A' B' C'* **and**
    *A B Lt A' B'* **and**
    *B C Lt B' C'*
  **shows** *A C Lt A' C'*
⟨*proof*⟩

**lemma** *le-lt-per2--lt*:
  **assumes** *Per A B C* **and**
    *Per A' B' C'* **and**
    *A B Le A' B'* **and**
    *B C Lt B' C'*
  **shows** *A C Lt A' C'*
  ⟨*proof*⟩

**lemma** *le2-per2--le*:
  **assumes** *Per A B C* **and**
    *Per A' B' C'* **and**
    *A B Le A' B'* **and**
    *B C Le B' C'*
  **shows** *A C Le A' C'*
⟨*proof*⟩

**lemma** *cong-lt-per2--lt-1*:
  **assumes** *Per A B C* **and**
    *Per A' B' C'* **and**
    *A B Lt A' B'* **and**
    *Cong A C A' C'*
  **shows** *B' C' Lt B C*
  ⟨*proof*⟩

**lemma** *symmetry-preserves-conga*:
  **assumes** $A \neq B$ **and** $C \neq B$ **and**
    *M Midpoint A A'* **and**
    *M Midpoint B B'* **and**
    *M Midpoint C C'*
  **shows** *A B C CongA A' B' C'*
  ⟨*proof*⟩

**lemma** *l11-57*:
  **assumes** $A\ A'\ OS\ B\ B'$ **and**
    $Per\ B\ A\ A'$ **and**
    $Per\ B'\ A'\ A$ **and**
    $A\ A'\ OS\ C\ C'$ **and**
    $Per\ C\ A\ A'$ **and**
    $Per\ C'\ A'\ A$
  **shows** $B\ A\ C\ CongA\ B'\ A'\ C'$
⟨*proof*⟩

**lemma** *cop3-orth-at--orth-at*:
  **assumes** $\neg\ Col\ D\ E\ F$ **and**
    $Coplanar\ A\ B\ C\ D$ **and**
    $Coplanar\ A\ B\ C\ E$ **and**
    $Coplanar\ A\ B\ C\ F$ **and**
    $X\ OrthAt\ A\ B\ C\ U\ V$
  **shows** $X\ OrthAt\ D\ E\ F\ U\ V$
⟨*proof*⟩

**lemma** *col2-orth-at--orth-at*:
  **assumes** $U \neq V$ **and**
    $Col\ P\ Q\ U$ **and**
    $Col\ P\ Q\ V$ **and**
    $X\ OrthAt\ A\ B\ C\ P\ Q$
  **shows** $X\ OrthAt\ A\ B\ C\ U\ V$
⟨*proof*⟩

**lemma** *col-orth-at--orth-at*:
  **assumes** $U \neq W$ **and**
    $Col\ U\ V\ W$ **and**
    $X\ OrthAt\ A\ B\ C\ U\ V$
  **shows** $X\ OrthAt\ A\ B\ C\ U\ W$
  ⟨*proof*⟩

**lemma** *orth-at-symmetry*:
  **assumes** $X\ OrthAt\ A\ B\ C\ U\ V$
  **shows** $X\ OrthAt\ A\ B\ C\ V\ U$
  ⟨*proof*⟩

**lemma** *orth-at-distincts*:
  **assumes** $X\ OrthAt\ A\ B\ C\ U\ V$
  **shows** $A \neq B \wedge B \neq C \wedge A \neq C \wedge U \neq V$
  ⟨*proof*⟩

**lemma** *orth-at-chara*:
  $X\ OrthAt\ A\ B\ C\ X\ P \longleftrightarrow$
  $(\neg\ Col\ A\ B\ C \wedge X \neq P \wedge Coplanar\ A\ B\ C\ X \wedge (\forall\ D.(Coplanar\ A\ B\ C\ D \longrightarrow Per\ D\ X\ P)))$
⟨*proof*⟩

**lemma** *cop3-orth--orth*:
  **assumes** $\neg\ Col\ D\ E\ F$ **and**
    $Coplanar\ A\ B\ C\ D$ **and**
    $Coplanar\ A\ B\ C\ E$ **and**
    $Coplanar\ A\ B\ C\ F$ **and**
    $A\ B\ C\ Orth\ U\ V$
  **shows** $D\ E\ F\ Orth\ U\ V$
  ⟨*proof*⟩

**lemma** *col2-orth--orth*:
  **assumes** $U \neq V$ **and**
    $Col\ P\ Q\ U$ **and**
    $Col\ P\ Q\ V$ **and**
    $A\ B\ C\ Orth\ P\ Q$
  **shows** $A\ B\ C\ Orth\ U\ V$
  ⟨*proof*⟩

**lemma** *col-orth--orth*:
  **assumes** $U \neq W$ **and**
    *Col* $U$ $V$ $W$ **and**
    $A$ $B$ $C$ *Orth* $U$ $V$
  **shows** $A$ $B$ $C$ *Orth* $U$ $W$
  ⟨*proof*⟩

**lemma** *orth-symmetry*:
  **assumes** $A$ $B$ $C$ *Orth* $U$ $V$
  **shows** $A$ $B$ $C$ *Orth* $V$ $U$
  ⟨*proof*⟩

**lemma** *orth-distincts*:
  **assumes** $A$ $B$ $C$ *Orth* $U$ $V$
  **shows** $A \neq B \wedge B \neq C \wedge A \neq C \wedge U \neq V$
  ⟨*proof*⟩

**lemma** *col-cop-orth--orth-at*:
  **assumes** $A$ $B$ $C$ *Orth* $U$ $V$ **and**
    *Coplanar* $A$ $B$ $C$ $X$ **and**
    *Col* $U$ $V$ $X$
  **shows** $X$ *OrthAt* $A$ $B$ $C$ $U$ $V$
⟨*proof*⟩

**lemma** *l11-60-aux*:
  **assumes** $\neg$ *Col* $A$ $B$ $C$ **and**
    *Cong* $A$ $P$ $A$ $Q$ **and**
    *Cong* $B$ $P$ $B$ $Q$ **and**
    *Cong* $C$ $P$ $C$ $Q$ **and**
    *Coplanar* $A$ $B$ $C$ $D$
  **shows** *Cong* $D$ $P$ $D$ $Q$
⟨*proof*⟩

**lemma** *l11-60*:
  **assumes** $\neg$ *Col* $A$ $B$ $C$ **and**
    *Per* $A$ $D$ $P$ **and**
    *Per* $B$ $D$ $P$ **and**
    *Per* $C$ $D$ $P$ **and**
    *Coplanar* $A$ $B$ $C$ $E$
  **shows** *Per* $E$ $D$ $P$
  ⟨*proof*⟩

**lemma** *l11-60-bis*:
  **assumes** $\neg$ *Col* $A$ $B$ $C$ **and**
    $D \neq P$ **and**
    *Coplanar* $A$ $B$ $C$ $D$ **and**
    *Per* $A$ $D$ $P$ **and**
    *Per* $B$ $D$ $P$ **and**
    *Per* $C$ $D$ $P$
  **shows** $D$ *OrthAt* $A$ $B$ $C$ $D$ $P$
  ⟨*proof*⟩

**lemma** *l11-61*:
  **assumes** $A \neq A'$ **and**
    $A \neq B$ **and**
    $A \neq C$ **and**
    *Coplanar* $A$ $A'$ $B$ $B'$ **and**
    *Per* $B$ $A$ $A'$ **and**
    *Per* $B'$ $A'$ $A$ **and**
    *Coplanar* $A$ $A'$ $C$ $C'$ **and**
    *Per* $C$ $A$ $A'$ **and**
    *Per* $B$ $A$ $C$
  **shows** *Per* $B'$ $A'$ $C'$
⟨*proof*⟩

**lemma** *l11-61-bis*:

**assumes** *D OrthAt A B C D P* **and**
*D E Perp E Q* **and**
*Coplanar A B C E* **and**
*Coplanar D E P Q*
**shows** *E OrthAt A B C E Q*
⟨*proof*⟩

**lemma** *l11-62-unicity*:
**assumes** *Coplanar A B C D* **and**
*Coplanar A B C D'* **and**
$\forall$ *E. Coplanar A B C E* $\longrightarrow$ *Per E D P* **and**
$\forall$ *E. Coplanar A B C E* $\longrightarrow$ *Per E D' P*
**shows** $D = D'$
⟨*proof*⟩

**lemma** *l11-62-unicity-bis*:
**assumes** *X OrthAt A B C X U* **and**
*Y OrthAt A B C Y U*
**shows** $X = Y$
⟨*proof*⟩

**lemma** *orth-at2--eq*:
**assumes** *X OrthAt A B C U V* **and**
*Y OrthAt A B C U V*
**shows** $X = Y$
⟨*proof*⟩

**lemma** *col-cop-orth-at--eq*:
**assumes** *X OrthAt A B C U V* **and**
*Coplanar A B C Y* **and**
*Col U V Y*
**shows** $X = Y$
⟨*proof*⟩

**lemma** *orth-at--ncop1*:
**assumes** $U \neq X$ **and**
*X OrthAt A B C U V*
**shows** $\neg$ *Coplanar A B C U*
⟨*proof*⟩

**lemma** *orth-at--ncop2*:
**assumes** $V \neq X$ **and**
*X OrthAt A B C U V*
**shows** $\neg$ *Coplanar A B C V*
⟨*proof*⟩

**lemma** *orth-at--ncop*:
**assumes** *X OrthAt A B C X P*
**shows** $\neg$ *Coplanar A B C P*
⟨*proof*⟩

**lemma** *l11-62-existence*:
$\exists$ *D*. (*Coplanar A B C D* $\wedge$ ($\forall$ *E*. (*Coplanar A B C E* $\longrightarrow$ *Per E D P*)))
⟨*proof*⟩

**lemma** *l11-62-existence-bis*:
**assumes** $\neg$ *Coplanar A B C P*
**shows** $\exists$ *X. X OrthAt A B C X P*
⟨*proof*⟩

**lemma** *l11-63-aux*:
**assumes** *Coplanar A B C D* **and**
$D \neq E$ **and**
*E OrthAt A B C E P*
**shows** $\exists$ *Q*. (*D E OS P Q* $\wedge$ *A B C Orth D Q*)
⟨*proof*⟩

**lemma** *l11-63-existence*:
  **assumes** *Coplanar A B C D* **and**
    ¬ *Coplanar A B C P*
  **shows** ∃ *Q*. *A B C Orth D Q*
  ⟨*proof*⟩

**lemma** *l8-21-3*:
  **assumes** *Coplanar A B C D* **and**
    ¬ *Coplanar A B C X*
  **shows**
    ∃ *P T*. (*A B C Orth D P* ∧ *Coplanar A B C T* ∧ *Bet X T P*)
⟨*proof*⟩

**lemma** *mid2-orth-at2--cong*:
  **assumes** *X OrthAt A B C X P* **and**
    *Y OrthAt A B C Y Q* **and**
    *X Midpoint P P′* **and**
    *Y Midpoint Q Q′*
  **shows** *Cong P Q P′ Q′*
⟨*proof*⟩

**lemma** *orth-at2-tsp--ts*:
  **assumes** *P* ≠ *Q* **and**
    *P OrthAt A B C P X* **and**
    *Q OrthAt A B C Q Y* **and**
    *A B C TSP X Y*
  **shows** *P Q TS X Y*
⟨*proof*⟩

**lemma** *orth-dec*:
  **shows** *A B C Orth U V* ∨ ¬ *A B C Orth U V* ⟨*proof*⟩

**lemma** *orth-at-dec*:
  **shows** *X OrthAt A B C U V* ∨ ¬ *X OrthAt A B C U V* ⟨*proof*⟩

**lemma** *tsp-dec*:
  **shows** *A B C TSP X Y* ∨ ¬ *A B C TSP X Y* ⟨*proof*⟩

**lemma** *osp-dec*:
  **shows** *A B C OSP X Y* ∨ ¬ *A B C OSP X Y* ⟨*proof*⟩

**lemma** *ts2--inangle*:
  **assumes** *A C TS B P* **and**
    *B P TS A C*
  **shows** *P InAngle A B C*
  ⟨*proof*⟩

**lemma** *os-ts--inangle*:
  **assumes** *B P TS A C* **and**
    *B A OS C P*
  **shows** *P InAngle A B C*
⟨*proof*⟩

**lemma** *os2--inangle*:
  **assumes** *B A OS C P* **and**
    *B C OS A P*
  **shows** *P InAngle A B C*
  ⟨*proof*⟩

**lemma** *acute-conga--acute*:
  **assumes** *Acute A B C* **and**
    *A B C CongA D E F*
  **shows** *Acute D E F*
⟨*proof*⟩

**lemma** *acute-out2--acute*:
**assumes** $B$ *Out* $A'$ $A$ **and**
$B$ *Out* $C'$ $C$ **and**
*Acute* $A$ $B$ $C$
**shows** *Acute* $A'$ $B$ $C'$
⟨*proof*⟩

**lemma** *conga-obtuse--obtuse*:
**assumes** *Obtuse* $A$ $B$ $C$ **and**
$A$ $B$ $C$ *CongA* $D$ $E$ $F$
**shows** *Obtuse* $D$ $E$ $F$
⟨*proof*⟩

**lemma** *obtuse-out2--obtuse*:
**assumes** $B$ *Out* $A'$ $A$ **and**
$B$ *Out* $C'$ $C$ **and**
*Obtuse* $A$ $B$ $C$
**shows** *Obtuse* $A'$ $B$ $C'$
⟨*proof*⟩

**lemma** *bet-lea--bet*:
**assumes** *Bet* $A$ $B$ $C$ **and**
$A$ $B$ $C$ *LeA* $D$ $E$ $F$
**shows** *Bet* $D$ $E$ $F$
⟨*proof*⟩

**lemma** *out-lea--out*:
**assumes** $E$ *Out* $D$ $F$ **and**
$A$ $B$ $C$ *LeA* $D$ $E$ $F$
**shows** $B$ *Out* $A$ $C$
⟨*proof*⟩

**lemma** *bet2-lta--lta*:
**assumes** $A$ $B$ $C$ *LtA* $D$ $E$ $F$ **and**
*Bet* $A$ $B$ $A'$ **and**
$A' \neq B$ **and**
*Bet* $D$ $E$ $D'$ **and**
$D' \neq E$
**shows** $D'$ $E$ $F$ *LtA* $A'$ $B$ $C$
⟨*proof*⟩

**lemma** *lea123456-lta--lta*:
**assumes** $A$ $B$ $C$ *LeA* $D$ $E$ $F$ **and**
$D$ $E$ $F$ *LtA* $G$ $H$ $I$
**shows** $A$ $B$ $C$ *LtA* $G$ $H$ $I$
⟨*proof*⟩

**lemma** *lea456789-lta--lta*:
**assumes** $A$ $B$ $C$ *LtA* $D$ $E$ $F$ **and**
$D$ $E$ $F$ *LeA* $G$ $H$ $I$
**shows** $A$ $B$ $C$ *LtA* $G$ $H$ $I$
⟨*proof*⟩

**lemma** *acute-per--lta*:
**assumes** *Acute* $A$ $B$ $C$ **and**
$D \neq E$ **and**
$E \neq F$ **and**
*Per* $D$ $E$ $F$
**shows** $A$ $B$ $C$ *LtA* $D$ $E$ $F$
⟨*proof*⟩

**lemma** *obtuse-per--lta*:
**assumes** *Obtuse* $A$ $B$ $C$ **and**
$D \neq E$ **and**
$E \neq F$ **and**
*Per* $D$ $E$ $F$

**shows** $D\ E\ F\ LtA\ A\ B\ C$
⟨proof⟩

**lemma** *acute-obtuse--lta*:
**assumes** $Acute\ A\ B\ C$ **and**
$Obtuse\ D\ E\ F$
**shows** $A\ B\ C\ LtA\ D\ E\ F$
⟨proof⟩

**lemma** *lea-in-angle*:
**assumes** $A\ B\ P\ LeA\ A\ B\ C$ **and**
$A\ B\ OS\ C\ P$
**shows** $P\ InAngle\ A\ B\ C$
⟨proof⟩

**lemma** *acute-bet--obtuse*:
**assumes** $Bet\ A\ B\ A'$ **and**
$A' \neq B$ **and**
$Acute\ A\ B\ C$
**shows** $Obtuse\ A'\ B\ C$
⟨proof⟩

**lemma** *bet-obtuse--acute*:
**assumes** $Bet\ A\ B\ A'$ **and**
$A' \neq B$ **and**
$Obtuse\ A\ B\ C$
**shows** $Acute\ A'\ B\ C$
⟨proof⟩

**lemma** *inangle-dec*:
$P\ InAngle\ A\ B\ C \vee \neg\ P\ InAngle\ A\ B\ C$ ⟨proof⟩

**lemma** *lea-dec*:
$A\ B\ C\ LeA\ D\ E\ F \vee \neg\ A\ B\ C\ LeA\ D\ E\ F$ ⟨proof⟩

**lemma** *lta-dec*:
$A\ B\ C\ LtA\ D\ E\ F \vee \neg\ A\ B\ C\ LtA\ D\ E\ F$ ⟨proof⟩

**lemma** *lea-total*:
**assumes** $A \neq B$ **and**
$B \neq C$ **and**
$D \neq E$ **and**
$E \neq F$
**shows** $A\ B\ C\ LeA\ D\ E\ F \vee D\ E\ F\ LeA\ A\ B\ C$
⟨proof⟩

**lemma** *or-lta2-conga*:
**assumes** $A \neq B$ **and**
$C \neq B$ **and**
$D \neq E$ **and**
$F \neq E$
**shows** $A\ B\ C\ LtA\ D\ E\ F \vee D\ E\ F\ LtA\ A\ B\ C \vee A\ B\ C\ CongA\ D\ E\ F$
⟨proof⟩

**lemma** *angle-partition*:
**assumes** $A \neq B$ **and**
$B \neq C$
**shows** $Acute\ A\ B\ C \vee Per\ A\ B\ C \vee Obtuse\ A\ B\ C$
⟨proof⟩

**lemma** *acute-chara-1*:
**assumes** $Bet\ A\ B\ A'$ **and**
$B \neq A'$ **and**
$Acute\ A\ B\ C$
**shows** $A\ B\ C\ LtA\ A'\ B\ C$
⟨proof⟩

**lemma** *acute-chara-2*:
  **assumes** *Bet A B A′* **and**
    *A B C LtA A′ B C*
  **shows** *Acute A B C*
  ⟨*proof*⟩

**lemma** *acute-chara*:
  **assumes** *Bet A B A′* **and**
    $B \neq A'$
  **shows** *Acute A B C* $\longleftrightarrow$ *A B C LtA A′ B C*
  ⟨*proof*⟩

**lemma** *obtuse-chara*:
  **assumes** *Bet A B A′* **and**
    $B \neq A'$
  **shows** *Obtuse A B C* $\longleftrightarrow$ *A′ B C LtA A B C*
  ⟨*proof*⟩

**lemma** *conga--acute*:
  **assumes** *A B C CongA A C B*
  **shows** *Acute A B C*
  ⟨*proof*⟩

**lemma** *cong--acute*:
  **assumes** $A \neq B$ **and**
    $B \neq C$ **and**
    *Cong A B A C*
  **shows** *Acute A B C*
  ⟨*proof*⟩

**lemma** *nlta--lea*:
  **assumes** ¬ *A B C LtA D E F* **and**
    $A \neq B$ **and**
    $B \neq C$ **and**
    $D \neq E$ **and**
    $E \neq F$
  **shows** *D E F LeA A B C*
  ⟨*proof*⟩

**lemma** *nlea--lta*:
  **assumes** ¬ *A B C LeA D E F* **and**
    $A \neq B$ **and**
    $B \neq C$ **and**
    $D \neq E$ **and**
    $E \neq F$
  **shows** *D E F LtA A B C*
  ⟨*proof*⟩

**lemma** *triangle-strict-inequality*:
  **assumes** *Bet A B D* **and**
    *Cong B C B D* **and**
    ¬ *Bet A B C*
  **shows** *A C Lt A D*
⟨*proof*⟩

**lemma** *triangle-inequality*:
  **assumes** *Bet A B D* **and**
    *Cong B C B D*
  **shows** *A C Le A D*
⟨*proof*⟩

**lemma** *triangle-strict-inequality-2*:
  **assumes** *Bet A′ B′ C′* **and**
    *Cong A B A′ B′* **and**
    *Cong B C B′ C′* **and**

$\neg$ *Bet A B C*
**shows** *A C Lt A' C'*
⟨*proof*⟩

**lemma** *triangle-inequality-2*:
**assumes** *Bet A' B' C'* **and**
*Cong A B A' B'* **and**
*Cong B C B' C'*
**shows** *A C Le A' C'*
⟨*proof*⟩

**lemma** *triangle-strict-reverse-inequality*:
**assumes** *A Out B D* **and**
*Cong A C A D* **and**
$\neg$ *A Out B C*
**shows** *B D Lt B C*
⟨*proof*⟩

**lemma** *triangle-reverse-inequality*:
**assumes** *A Out B D* **and**
*Cong A C A D*
**shows** *B D Le B C*
⟨*proof*⟩

**lemma** *os3--lta*:
**assumes** *A B OS C D* **and**
*B C OS A D* **and**
*A C OS B D*
**shows** *B A C LtA B D C*
⟨*proof*⟩

**lemma** *bet-le--lt*:
**assumes** *Bet A D B* **and**
$A \neq D$ **and**
$D \neq B$ **and**
*A C Le B C*
**shows** *D C Lt B C*
⟨*proof*⟩

**lemma** *cong2--ncol*:
**assumes** $A \neq B$ **and**
$B \neq C$ **and**
$A \neq C$ **and**
*Cong A P B P* **and**
*Cong A P C P*
**shows** $\neg$ *Col A B C*
⟨*proof*⟩

**lemma** *cong4-cop2--eq*:
**assumes** $A \neq B$ **and**
$B \neq C$ **and**
$A \neq C$ **and**
*Cong A P B P* **and**
*Cong A P C P* **and**
*Coplanar A B C P* **and**
*Cong A Q B Q* **and**
*Cong A Q C Q* **and**
*Coplanar A B C Q*
**shows** $P = Q$
⟨*proof*⟩

**lemma** *t18-18-aux*:
**assumes** *Cong A B D E* **and**
*Cong A C D F* **and**
*F D E LtA C A B* **and**
$\neg$ *Col A B C* **and**

$\neg$ *Col D E F* **and**
*D F Le D E*
**shows** *E F Lt B C*
⟨*proof*⟩

**lemma** *t18-18*:
**assumes** *Cong A B D E* **and**
*Cong A C D F* **and**
*F D E LtA C A B*
**shows** *E F Lt B C*
⟨*proof*⟩

**lemma** *t18-19*:
**assumes** $A \neq B$ **and**
$A \neq C$ **and**
*Cong A B D E* **and**
*Cong A C D F* **and**
*E F Lt B C*
**shows** *F D E LtA C A B*
⟨*proof*⟩

**lemma** *acute-trivial*:
**assumes** $A \neq B$
**shows** *Acute A B A*
⟨*proof*⟩

**lemma** *acute-not-per*:
**assumes** *Acute A B C*
**shows** $\neg$ *Per A B C*
⟨*proof*⟩

**lemma** *angle-bisector*:
**assumes** $A \neq B$ **and**
$C \neq B$
**shows** $\exists$ *P*. (*P InAngle A B C* $\wedge$ *P B A CongA P B C*)
⟨*proof*⟩

**lemma** *reflectl--conga*:
**assumes** $A \neq B$ **and**
$B \neq P$ **and**
*P P′ ReflectL A B*
**shows** *A B P CongA A B P′*
⟨*proof*⟩

**lemma** *conga-cop-out-reflectl--out*:
**assumes** $\neg$ *B Out A C* **and**
*Coplanar A B C P* **and**
*P B A CongA P B C* **and**
*B Out A T* **and**
*T T′ ReflectL B P*
**shows** *B Out C T′*
⟨*proof*⟩

**lemma** *col-conga-cop-reflectl--col*:
**assumes** $\neg$ *B Out A C* **and**
*Coplanar A B C P* **and**
*P B A CongA P B C* **and**
*Col B A T* **and**
*T T′ ReflectL B P*
**shows** *Col B C T′*
⟨*proof*⟩

**lemma** *conga2-cop2--col*:
**assumes** $\neg$ *B Out A C* **and**
*P B A CongA P B C* **and**
*P′ B A CongA P′ B C* **and**

*Coplanar A B P P'* **and**
*Coplanar B C P P'*
**shows** *Col B P P'*
⟨*proof*⟩

**lemma** *conga2-cop2--col-1*:
**assumes** $\neg$ *Col A B C* **and**
*P B A CongA P B C* **and**
*P' B A CongA P' B C* **and**
*Coplanar A B C P* **and**
*Coplanar A B C P'*
**shows** *Col B P P'*
⟨*proof*⟩

**lemma** *col-conga--conga*:
**assumes** *P B A CongA P B C* **and**
*Col B P P'* **and**
$B \neq P'$
**shows** *P' B A CongA P' B C*
⟨*proof*⟩

**lemma** *cop-inangle--ex-col-inangle*:
**assumes** $\neg$ *B Out A C* **and**
*P InAngle A B C* **and**
*Coplanar A B C Q*
**shows** $\exists$ *R*. (*R InAngle A B C* $\wedge$ $P \neq R$ $\wedge$ *Col P Q R*)
⟨*proof*⟩

**lemma** *col-inangle2--out*:
**assumes** $\neg$ *Bet A B C* **and**
*P InAngle A B C* **and**
*Q InAngle A B C* **and**
*Col B P Q*
**shows** *B Out P Q*
⟨*proof*⟩

**lemma** *inangle2--lea*:
**assumes** *P InAngle A B C* **and**
*Q InAngle A B C*
**shows** *P B Q LeA A B C*
⟨*proof*⟩

**lemma** *conga-inangle-per--acute*:
**assumes** *Per A B C* **and**
*P InAngle A B C* **and**
*P B A CongA P B C*
**shows** *Acute A B P*
⟨*proof*⟩

**lemma** *conga-inangle2-per--acute*:
**assumes** *Per A B C* **and**
*P InAngle A B C* **and**
*P B A CongA P B C* **and**
*Q InAngle A B C*
**shows** *Acute P B Q*
⟨*proof*⟩

**lemma** *lta-os--ts*:
**assumes**
*A O1 P LtA A O1 B* **and**
*O1 A OS B P*
**shows** *O1 P TS A B*
⟨*proof*⟩

**lemma** *bet--suppa*:
**assumes** $A \neq B$ **and**

$B \neq C$ **and**
$B \neq A'$ **and**
$Bet\ A\ B\ A'$
**shows** $A\ B\ C\ SuppA\ C\ B\ A'$
⟨*proof*⟩

**lemma** *ex-suppa*:
**assumes** $A \neq B$ **and**
$B \neq C$
**shows** $\exists\ D\ E\ F.\ A\ B\ C\ SuppA\ D\ E\ F$
⟨*proof*⟩

**lemma** *suppa-distincts*:
**assumes** $A\ B\ C\ SuppA\ D\ E\ F$
**shows** $A \neq B \wedge B \neq C \wedge D \neq E \wedge E \neq F$
⟨*proof*⟩

**lemma** *suppa-right-comm*:
**assumes** $A\ B\ C\ SuppA\ D\ E\ F$
**shows** $A\ B\ C\ SuppA\ F\ E\ D$
⟨*proof*⟩

**lemma** *suppa-left-comm*:
**assumes** $A\ B\ C\ SuppA\ D\ E\ F$
**shows** $C\ B\ A\ SuppA\ D\ E\ F$
⟨*proof*⟩

**lemma** *suppa-comm*:
**assumes** $A\ B\ C\ SuppA\ D\ E\ F$
**shows** $C\ B\ A\ SuppA\ F\ E\ D$
⟨*proof*⟩

**lemma** *suppa-sym*:
**assumes** $A\ B\ C\ SuppA\ D\ E\ F$
**shows** $D\ E\ F\ SuppA\ A\ B\ C$
⟨*proof*⟩

**lemma** *conga2-suppa--suppa*:
**assumes** $A\ B\ C\ CongA\ A'\ B'\ C'$ **and**
$D\ E\ F\ CongA\ D'\ E'\ F'$ **and**
$A\ B\ C\ SuppA\ D\ E\ F$
**shows** $A'\ B'\ C'\ SuppA\ D'\ E'\ F'$
⟨*proof*⟩

**lemma** *suppa2--conga456*:
**assumes** $A\ B\ C\ SuppA\ D\ E\ F$ **and**
$A\ B\ C\ SuppA\ D'\ E'\ F'$
**shows** $D\ E\ F\ CongA\ D'\ E'\ F'$
⟨*proof*⟩

**lemma** *suppa2--conga123*:
**assumes** $A\ B\ C\ SuppA\ D\ E\ F$ **and**
$A'\ B'\ C'\ SuppA\ D\ E\ F$
**shows** $A\ B\ C\ CongA\ A'\ B'\ C'$
⟨*proof*⟩

**lemma** *bet-out--suppa*:
**assumes** $A \neq B$ **and**
$B \neq C$ **and**
$Bet\ A\ B\ C$ **and**
$E\ Out\ D\ F$
**shows** $A\ B\ C\ SuppA\ D\ E\ F$
⟨*proof*⟩

**lemma** *bet-suppa--out*:
**assumes** $Bet\ A\ B\ C$ **and**

$A\ B\ C\ SuppA\ D\ E\ F$
**shows** $E\ Out\ D\ F$
⟨*proof*⟩

**lemma** *out-suppa--bet*:
**assumes** $B\ Out\ A\ C$ **and**
$A\ B\ C\ SuppA\ D\ E\ F$
**shows** $Bet\ D\ E\ F$
⟨*proof*⟩

**lemma** *per-suppa--per*:
**assumes** $Per\ A\ B\ C$ **and**
$A\ B\ C\ SuppA\ D\ E\ F$
**shows** $Per\ D\ E\ F$
⟨*proof*⟩

**lemma** *per2--suppa*:
**assumes** $A \neq B$ **and**
$B \neq C$ **and**
$D \neq E$ **and**
$E \neq F$ **and**
$Per\ A\ B\ C$ **and**
$Per\ D\ E\ F$
**shows** $A\ B\ C\ SuppA\ D\ E\ F$
⟨*proof*⟩

**lemma** *suppa--per*:
**assumes** $A\ B\ C\ SuppA\ A\ B\ C$
**shows** $Per\ A\ B\ C$
⟨*proof*⟩

**lemma** *acute-suppa--obtuse*:
**assumes** $Acute\ A\ B\ C$ **and**
$A\ B\ C\ SuppA\ D\ E\ F$
**shows** $Obtuse\ D\ E\ F$
⟨*proof*⟩

**lemma** *obtuse-suppa--acute*:
**assumes** $Obtuse\ A\ B\ C$ **and**
$A\ B\ C\ SuppA\ D\ E\ F$
**shows** $Acute\ D\ E\ F$
⟨*proof*⟩

**lemma** *lea-suppa2--lea*:
**assumes** $A\ B\ C\ SuppA\ A'\ B'\ C'$ **and**
$D\ E\ F\ SuppA\ D'\ E'\ F'$
$A\ B\ C\ LeA\ D\ E\ F$
**shows** $D'\ E'\ F'\ LeA\ A'\ B'\ C'$
⟨*proof*⟩

**lemma** *lta-suppa2--lta*:
**assumes** $A\ B\ C\ SuppA\ A'\ B'\ C'$
**and** $D\ E\ F\ SuppA\ D'\ E'\ F'$
**and** $A\ B\ C\ LtA\ D\ E\ F$
**shows** $D'\ E'\ F'\ LtA\ A'\ B'\ C'$
⟨*proof*⟩

**lemma** *suppa-dec*:
$A\ B\ C\ SuppA\ D\ E\ F \vee \neg\ A\ B\ C\ SuppA\ D\ E\ F$
⟨*proof*⟩

**lemma** *acute-one-side-aux*:
**assumes** $C\ A\ OS\ P\ B$ **and**
$Acute\ A\ C\ P$ **and**
$C\ A\ Perp\ B\ C$
**shows** $C\ B\ OS\ A\ P$

⟨*proof*⟩

**lemma** *acute-one-side-aux0*:
**assumes** *Col A C P* **and**
*Acute A C P* **and**
*C A Perp B C*
**shows** *C B OS A P*
⟨*proof*⟩

**lemma** *acute-cop-perp--one-side*:
**assumes** *Acute A C P* **and**
*C A Perp B C* **and**
*Coplanar A B C P*
**shows** *C B OS A P*
⟨*proof*⟩

**lemma** *acute--not-obtuse*:
**assumes** *Acute A B C*
**shows** ¬ *Obtuse A B C*
⟨*proof*⟩

### 3.10.2 Sum of angles

**lemma** *suma-distincts*:
**assumes** *A B C D E F SumA G H I*
**shows** $A \neq B \wedge B \neq C \wedge D \neq E \wedge E \neq F \wedge G \neq H \wedge H \neq I$
⟨*proof*⟩

**lemma** *trisuma-distincts*:
**assumes** *A B C TriSumA D E F*
**shows** $A \neq B \wedge B \neq C \wedge A \neq C \wedge D \neq E \wedge E \neq F$
⟨*proof*⟩

**lemma** *ex-suma*:
**assumes** $A \neq B$ **and**
$B \neq C$ **and**
$D \neq E$ **and**
$E \neq F$
**shows** ∃ *G H I. A B C D E F SumA G H I*
⟨*proof*⟩

**lemma** *suma2--conga*:
**assumes** *A B C D E F SumA G H I* **and**
*A B C D E F SumA G′ H′ I′*
**shows** *G H I CongA G′ H′ I′*
⟨*proof*⟩

**lemma** *suma-sym*:
**assumes** *A B C D E F SumA G H I*
**shows** *D E F A B C SumA G H I*
⟨*proof*⟩

**lemma** *conga3-suma--suma*:
**assumes** *A B C D E F SumA G H I* **and**
*A B C CongA A′ B′ C′* **and**
*D E F CongA D′ E′ F′* **and**
*G H I CongA G′ H′ I′*
**shows** *A′ B′ C′ D′ E′ F′ SumA G′ H′ I′*
⟨*proof*⟩

**lemma** *out6-suma--suma*:
**assumes** *A B C D E F SumA G H I* **and**
*B Out A A′* **and**
*B Out C C′* **and**
*E Out D D′* **and**
*E Out F F′* **and**

$H$ *Out* $G$ $G'$ **and**
$H$ *Out* $I$ $I'$
**shows** $A'$ $B$ $C'$ $D'$ $E$ $F'$ *SumA* $G'$ $H$ $I'$
⟨*proof*⟩

**lemma** *out546-suma--conga*:
**assumes** $A$ $B$ $C$ $D$ $E$ $F$ *SumA* $G$ $H$ $I$ **and**
$E$ *Out* $D$ $F$
**shows** $A$ $B$ $C$ *CongA* $G$ $H$ $I$
⟨*proof*⟩

**lemma** *out546--suma*:
**assumes** $A \neq B$ **and**
$B \neq C$ **and**
$E$ *Out* $D$ $F$
**shows** $A$ $B$ $C$ $D$ $E$ $F$ *SumA* $A$ $B$ $C$
⟨*proof*⟩

**lemma** *out213-suma--conga*:
**assumes** $A$ $B$ $C$ $D$ $E$ $F$ *SumA* $G$ $H$ $I$ **and**
$B$ *Out* $A$ $C$
**shows** $D$ $E$ $F$ *CongA* $G$ $H$ $I$
⟨*proof*⟩

**lemma** *out213--suma*:
**assumes** $D \neq E$ **and**
$E \neq F$ **and**
$B$ *Out* $A$ $C$
**shows** $A$ $B$ $C$ $D$ $E$ $F$ *SumA* $D$ $E$ $F$
⟨*proof*⟩

**lemma** *suma-left-comm*:
**assumes** $A$ $B$ $C$ $D$ $E$ $F$ *SumA* $G$ $H$ $I$
**shows** $C$ $B$ $A$ $D$ $E$ $F$ *SumA* $G$ $H$ $I$
⟨*proof*⟩

**lemma** *suma-middle-comm*:
**assumes** $A$ $B$ $C$ $D$ $E$ $F$ *SumA* $G$ $H$ $I$
**shows** $A$ $B$ $C$ $F$ $E$ $D$ *SumA* $G$ $H$ $I$
⟨*proof*⟩

**lemma** *suma-right-comm*:
**assumes** $A$ $B$ $C$ $D$ $E$ $F$ *SumA* $G$ $H$ $I$
**shows** $A$ $B$ $C$ $D$ $E$ $F$ *SumA* $I$ $H$ $G$
⟨*proof*⟩

**lemma** *suma-comm*:
**assumes** $A$ $B$ $C$ $D$ $E$ $F$ *SumA* $G$ $H$ $I$
**shows** $C$ $B$ $A$ $F$ $E$ $D$ *SumA* $I$ $H$ $G$
⟨*proof*⟩

**lemma** *ts--suma*:
**assumes** $A$ $B$ *TS* $C$ $D$
**shows** $C$ $B$ $A$ $A$ $B$ $D$ *SumA* $C$ $B$ $D$
⟨*proof*⟩

**lemma** *ts--suma-1*:
**assumes** $A$ $B$ *TS* $C$ $D$
**shows** $C$ $A$ $B$ $B$ $A$ $D$ *SumA* $C$ $A$ $D$
⟨*proof*⟩

**lemma** *inangle--suma*:
**assumes** $P$ *InAngle* $A$ $B$ $C$
**shows** $A$ $B$ $P$ $P$ $B$ $C$ *SumA* $A$ $B$ $C$
⟨*proof*⟩

**lemma** *bet--suma*:
**assumes** $A \neq B$ **and**
$B \neq C$ **and**
$P \neq B$ **and** *Bet* $A$ $B$ $C$
**shows** $A$ $B$ $P$ $P$ $B$ $C$ *SumA* $A$ $B$ $C$
⟨*proof*⟩

**lemma** *sams-chara*:
**assumes** $A \neq B$ **and**
$A' \neq B$ **and**
*Bet* $A$ $B$ $A'$
**shows** *SAMS* $A$ $B$ $C$ $D$ $E$ $F$ $\longleftrightarrow$ $D$ $E$ $F$ *LeA* $C$ $B$ $A'$
⟨*proof*⟩

**lemma** *sams-distincts*:
**assumes** *SAMS* $A$ $B$ $C$ $D$ $E$ $F$
**shows** $A \neq B \wedge B \neq C \wedge D \neq E \wedge E \neq F$
⟨*proof*⟩

**lemma** *sams-sym*:
**assumes** *SAMS* $A$ $B$ $C$ $D$ $E$ $F$
**shows** *SAMS* $D$ $E$ $F$ $A$ $B$ $C$
⟨*proof*⟩

**lemma** *sams-right-comm*:
**assumes** *SAMS* $A$ $B$ $C$ $D$ $E$ $F$
**shows** *SAMS* $A$ $B$ $C$ $F$ $E$ $D$
⟨*proof*⟩

**lemma** *sams-left-comm*:
**assumes** *SAMS* $A$ $B$ $C$ $D$ $E$ $F$
**shows** *SAMS* $C$ $B$ $A$ $D$ $E$ $F$
⟨*proof*⟩

**lemma** *sams-comm*:
**assumes** *SAMS* $A$ $B$ $C$ $D$ $E$ $F$
**shows** *SAMS* $C$ $B$ $A$ $F$ $E$ $D$
⟨*proof*⟩

**lemma** *conga2-sams--sams*:
**assumes** $A$ $B$ $C$ *CongA* $A'$ $B'$ $C'$ **and**
$D$ $E$ $F$ *CongA* $D'$ $E'$ $F'$ **and**
*SAMS* $A$ $B$ $C$ $D$ $E$ $F$
**shows** *SAMS* $A'$ $B'$ $C'$ $D'$ $E'$ $F'$
⟨*proof*⟩

**lemma** *out546--sams*:
**assumes** $A \neq B$ **and**
$B \neq C$ **and**
$E$ *Out* $D$ $F$
**shows** *SAMS* $A$ $B$ $C$ $D$ $E$ $F$
⟨*proof*⟩

**lemma** *out213--sams*:
**assumes** $D \neq E$ **and**
$E \neq F$ **and**
$B$ *Out* $A$ $C$
**shows** *SAMS* $A$ $B$ $C$ $D$ $E$ $F$
⟨*proof*⟩

**lemma** *bet-suma--sams*:
**assumes** $A$ $B$ $C$ $D$ $E$ $F$ *SumA* $G$ $H$ $I$ **and**
*Bet* $G$ $H$ $I$
**shows** *SAMS* $A$ $B$ $C$ $D$ $E$ $F$
⟨*proof*⟩

**lemma** *bet--sams*:
  **assumes** $A \neq B$ **and**
    $B \neq C$ **and**
    $P \neq B$ **and**
    *Bet A B C*
  **shows** *SAMS A B P P B C*
  ⟨*proof*⟩

**lemma** *suppa--sams*:
  **assumes** *A B C SuppA D E F*
  **shows** *SAMS A B C D E F*
⟨*proof*⟩

**lemma** *os-ts--sams*:
  **assumes** *B P TS A C* **and**
    *A B OS P C*
  **shows** *SAMS A B P P B C*
⟨*proof*⟩

**lemma** *os2--sams*:
  **assumes** *A B OS P C* **and**
    *C B OS P A*
  **shows** *SAMS A B P P B C*
  ⟨*proof*⟩

**lemma** *inangle--sams*:
  **assumes** *P InAngle A B C*
  **shows** *SAMS A B P P B C*
⟨*proof*⟩

**lemma** *sams-suma--lea123789*:
  **assumes** *A B C D E F SumA G H I* **and**
    *SAMS A B C D E F*
  **shows** *A B C LeA G H I*
⟨*proof*⟩

**lemma** *sams-suma--lea456789*:
  **assumes** *A B C D E F SumA G H I* **and**
    *SAMS A B C D E F*
  **shows** *D E F LeA G H I*
⟨*proof*⟩

**lemma** *sams-lea2--sams*:
  **assumes** *SAMS* $A'$ $B'$ $C'$ $D'$ $E'$ $F'$ **and**
    *A B C LeA* $A'$ $B'$ $C'$ **and**
    *D E F LeA* $D'$ $E'$ $F'$
  **shows** *SAMS A B C D E F*
⟨*proof*⟩

**lemma** *sams-lea456-suma2--lea*:
  **assumes** *D E F LeA* $D'$ $E'$ $F'$ **and**
    *SAMS A B C* $D'$ $E'$ $F'$ **and**
    *A B C D E F SumA G H I* **and**
    *A B C* $D'$ $E'$ $F'$ *SumA* $G'$ $H'$ $I'$
  **shows** *G H I LeA* $G'$ $H'$ $I'$
⟨*proof*⟩

**lemma** *sams-lea123-suma2--lea*:
  **assumes** *A B C LeA* $A'$ $B'$ $C'$ **and**
    *SAMS* $A'$ $B'$ $C'$ *D E F* **and**
    *A B C D E F SumA G H I* **and**
    $A'$ $B'$ $C'$ *D E F SumA* $G'$ $H'$ $I'$
  **shows** *G H I LeA* $G'$ $H'$ $I'$
  ⟨*proof*⟩

**lemma** *sams-lea2-suma2--lea*:

**assumes** $A\ B\ C\ LeA\ A'\ B'\ C'$ **and**
$D\ E\ F\ LeA\ D'\ E'\ F'$ **and**
$SAMS\ A'\ B'\ C'\ D'\ E'\ F'$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
$A'\ B'\ C'\ D'\ E'\ F'\ SumA\ G'\ H'\ I'$
**shows** $G\ H\ I\ LeA\ G'\ H'\ I'$
⟨*proof*⟩

**lemma** *sams2-suma2--conga456*:
**assumes** $SAMS\ A\ B\ C\ D\ E\ F$ **and**
$SAMS\ A\ B\ C\ D'\ E'\ F'$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
$A\ B\ C\ D'\ E'\ F'\ SumA\ G\ H\ I$
**shows** $D\ E\ F\ CongA\ D'\ E'\ F'$
⟨*proof*⟩

**lemma** *sams2-suma2--conga123*:
**assumes** $SAMS\ A\ B\ C\ D\ E\ F$ **and**
$SAMS\ A'\ B'\ C'\ D\ E\ F$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
$A'\ B'\ C'\ D\ E\ F\ SumA\ G\ H\ I$
**shows** $A\ B\ C\ CongA\ A'\ B'\ C'$
⟨*proof*⟩

**lemma** *suma-assoc-1*:
**assumes** $SAMS\ A\ B\ C\ D\ E\ F$ **and**
$SAMS\ D\ E\ F\ G\ H\ I$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ A'\ B'\ C'$ **and**
$D\ E\ F\ G\ H\ I\ SumA\ D'\ E'\ F'$ **and**
$A'\ B'\ C'\ G\ H\ I\ SumA\ K\ L\ M$
**shows** $A\ B\ C\ D'\ E'\ F'\ SumA\ K\ L\ M$
⟨*proof*⟩

**lemma** *suma-assoc-2*:
**assumes** $SAMS\ A\ B\ C\ D\ E\ F$ **and**
$SAMS\ D\ E\ F\ G\ H\ I$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ A'\ B'\ C'$ **and**
$D\ E\ F\ G\ H\ I\ SumA\ D'\ E'\ F'$ **and**
$A\ B\ C\ D'\ E'\ F'\ SumA\ K\ L\ M$
**shows** $A'\ B'\ C'\ G\ H\ I\ SumA\ K\ L\ M$
⟨*proof*⟩

**lemma** *suma-assoc*:
**assumes** $SAMS\ A\ B\ C\ D\ E\ F$ **and**
$SAMS\ D\ E\ F\ G\ H\ I$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ A'\ B'\ C'$ **and**
$D\ E\ F\ G\ H\ I\ SumA\ D'\ E'\ F'$
**shows**
$A'\ B'\ C'\ G\ H\ I\ SumA\ K\ L\ M \longleftrightarrow A\ B\ C\ D'\ E'\ F'\ SumA\ K\ L\ M$
⟨*proof*⟩

**lemma** *sams-assoc-1*:
**assumes** $SAMS\ A\ B\ C\ D\ E\ F$ **and**
$SAMS\ D\ E\ F\ G\ H\ I$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ A'\ B'\ C'$ **and**
$D\ E\ F\ G\ H\ I\ SumA\ D'\ E'\ F'$ **and**
$SAMS\ A'\ B'\ C'\ G\ H\ I$
**shows** $SAMS\ A\ B\ C\ D'\ E'\ F'$
⟨*proof*⟩

**lemma** *sams-assoc-2*:
**assumes** $SAMS\ A\ B\ C\ D\ E\ F$ **and**
$SAMS\ D\ E\ F\ G\ H\ I$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ A'\ B'\ C'$ **and**
$D\ E\ F\ G\ H\ I\ SumA\ D'\ E'\ F'$ **and**
$SAMS\ A\ B\ C\ D'\ E'\ F'$

**shows** $SAMS\ A'\ B'\ C'\ G\ H\ I$
⟨*proof*⟩

**lemma** *sams-assoc*:
**assumes** $SAMS\ A\ B\ C\ D\ E\ F$ **and**
$SAMS\ D\ E\ F\ G\ H\ I$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ A'\ B'\ C'$ **and**
$D\ E\ F\ G\ H\ I\ SumA\ D'\ E'\ F'$
**shows** $(SAMS\ A'\ B'\ C'\ G\ H\ I) \longleftrightarrow (SAMS\ A\ B\ C\ D'\ E'\ F')$
⟨*proof*⟩

**lemma** *sams-nos--nts*:
**assumes** $SAMS\ A\ B\ C\ C\ B\ J$ **and**
$\neg\ B\ C\ OS\ A\ J$
**shows** $\neg\ \ A\ B\ TS\ C\ J$
⟨*proof*⟩

**lemma** *conga-sams-nos--nts*:
**assumes** $SAMS\ A\ B\ C\ D\ E\ F$ **and**
$C\ B\ J\ CongA\ D\ E\ F$ **and**
$\neg\ B\ C\ OS\ A\ J$
**shows** $\neg\ A\ B\ TS\ C\ J$
⟨*proof*⟩

**lemma** *sams-lea2-suma2--conga123*:
**assumes** $A\ B\ C\ LeA\ A'\ B'\ C'$ **and**
$D\ E\ F\ LeA\ D'\ E'\ F'$ **and**
$SAMS\ A'\ B'\ C'\ D'\ E'\ F'$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
$A'\ B'\ C'\ D'\ E'\ F'\ SumA\ G\ H\ I$
**shows** $A\ B\ C\ CongA\ A'\ B'\ C'$
⟨*proof*⟩

**lemma** *sams-lea2-suma2--conga456*:
**assumes** $A\ B\ C\ LeA\ A'\ B'\ C'$ **and**
$D\ E\ F\ LeA\ D'\ E'\ F'$ **and**
$SAMS\ A'\ B'\ C'\ D'\ E'\ F'$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
$A'\ B'\ C'\ D'\ E'\ F'\ SumA\ G\ H\ I$
**shows** $D\ E\ F\ CongA\ D'\ E'\ F'$
⟨*proof*⟩

**lemma** *sams-suma--out213*:
**assumes** $A\ B\ C\ D\ E\ F\ SumA\ D\ E\ F$ **and**
$SAMS\ A\ B\ C\ D\ E\ F$
**shows** $B\ Out\ A\ C$
⟨*proof*⟩

**lemma** *sams-suma--out546*:
**assumes** $A\ B\ C\ D\ E\ F\ SumA\ A\ B\ C$ **and**
$SAMS\ A\ B\ C\ D\ E\ F$
**shows** $E\ Out\ D\ F$
⟨*proof*⟩

**lemma** *sams-lea-lta123-suma2--lta*:
**assumes** $A\ B\ C\ LtA\ A'\ B'\ C'$ **and**
$D\ E\ F\ LeA\ D'\ E'\ F'$ **and**
$SAMS\ A'\ B'\ C'\ D'\ E'\ F'$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
$A'\ B'\ C'\ D'\ E'\ F'\ SumA\ G'\ H'\ I'$
**shows** $G\ H\ I\ LtA\ G'\ H'\ I'$
⟨*proof*⟩

**lemma** *sams-lea-lta456-suma2--lta*:
**assumes** $A\ B\ C\ LeA\ A'\ B'\ C'$ **and**
$D\ E\ F\ LtA\ D'\ E'\ F'$ **and**

$SAMS\ A'\ B'\ C'\ D'\ E'\ F'$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
$A'\ B'\ C'\ D'\ E'\ F'\ SumA\ G'\ H'\ I'$
**shows** $G\ H\ I\ LtA\ G'\ H'\ I'$
⟨*proof*⟩

**lemma** *sams-lta2-suma2--lta*:
**assumes** $A\ B\ C\ LtA\ A'\ B'\ C'$ **and**
$D\ E\ F\ LtA\ D'\ E'\ F'$ **and**
$SAMS\ A'\ B'\ C'\ D'\ E'\ F'$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
$A'\ B'\ C'\ D'\ E'\ F'\ SumA\ G'\ H'\ I'$
**shows** $G\ H\ I\ LtA\ G'\ H'\ I'$
⟨*proof*⟩

**lemma** *sams-lea2-suma2--lea123*:
**assumes** $D'\ E'\ F'\ LeA\ D\ E\ F$ **and**
$G\ H\ I\ LeA\ G'\ H'\ I'$ **and**
$SAMS\ A\ B\ C\ D\ E\ F$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
$A'\ B'\ C'\ D'\ E'\ F'\ SumA\ G'\ H'\ I'$
**shows** $A\ B\ C\ LeA\ A'\ B'\ C'$
⟨*proof*⟩

**lemma** *sams-lea2-suma2--lea456*:
**assumes** $A'\ B'\ C'\ LeA\ A\ B\ C$ **and**
$G\ H\ I\ LeA\ G'\ H'\ I'$ **and**
$SAMS\ A\ B\ C\ D\ E\ F$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
$A'\ B'\ C'\ D'\ E'\ F'\ SumA\ G'\ H'\ I'$
**shows** $D\ E\ F\ LeA\ D'\ E'\ F'$
⟨*proof*⟩

**lemma** *sams-lea-lta456-suma2--lta123*:
**assumes** $D'\ E'\ F'\ LtA\ D\ E\ F$ **and**
$G\ H\ I\ LeA\ G'\ H'\ I'$ **and**
$SAMS\ A\ B\ C\ D\ E\ F$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
$A'\ B'\ C'\ D'\ E'\ F'\ SumA\ G'\ H'\ I'$
**shows** $A\ B\ C\ LtA\ A'\ B'\ C'$
⟨*proof*⟩

**lemma** *sams-lea-lta123-suma2--lta456*:
**assumes** $A'\ B'\ C'\ LtA\ A\ B\ C$ **and**
$G\ H\ I\ LeA\ G'\ H'\ I'$ **and**
$SAMS\ A\ B\ C\ D\ E\ F$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
$A'\ B'\ C'\ D'\ E'\ F'\ SumA\ G'\ H'\ I'$
**shows** $D\ E\ F\ LtA\ D'\ E'\ F'$
⟨*proof*⟩

**lemma** *sams-lea-lta789-suma2--lta123*:
**assumes** $D'\ E'\ F'\ LeA\ D\ E\ F$ **and**
$G\ H\ I\ LtA\ G'\ H'\ I'$ **and**
$SAMS\ A\ B\ C\ D\ E\ F$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
$A'\ B'\ C'\ D'\ E'\ F'\ SumA\ G'\ H'\ I'$
**shows** $A\ B\ C\ LtA\ A'\ B'\ C'$
⟨*proof*⟩

**lemma** *sams-lea-lta789-suma2--lta456*:
**assumes** $A'\ B'\ C'\ LeA\ A\ B\ C$ **and**
$G\ H\ I\ LtA\ G'\ H'\ I'$ **and**
$SAMS\ A\ B\ C\ D\ E\ F$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
$A'\ B'\ C'\ D'\ E'\ F'\ SumA\ G'\ H'\ I'$

**shows** $D\ E\ F\ LtA\ D'\ E'\ F'$
⟨proof⟩

**lemma** *sams-lta2-suma2--lta123*:
**assumes** $D'\ E'\ F'\ LtA\ D\ E\ F$ **and**
$G\ H\ I\ LtA\ G'\ H'\ I'$ **and**
$SAMS\ A\ B\ C\ D\ E\ F$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
$A'\ B'\ C'\ D'\ E'\ F'\ SumA\ G'\ H'\ I'$
**shows** $A\ B\ C\ LtA\ A'\ B'\ C'$
⟨proof⟩

**lemma** *sams-lta2-suma2--lta456*:
**assumes** $A'\ B'\ C'\ LtA\ A\ B\ C$ **and**
$G\ H\ I\ LtA\ G'\ H'\ I'$ **and**
$SAMS\ A\ B\ C\ D\ E\ F$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
$A'\ B'\ C'\ D'\ E'\ F'\ SumA\ G'\ H'\ I'$
**shows** $D\ E\ F\ LtA\ D'\ E'\ F'$
⟨proof⟩

**lemma** *sams123231*:
**assumes** $A \neq B$ **and**
$A \neq C$ **and**
$B \neq C$
**shows** $SAMS\ A\ B\ C\ B\ C\ A$
⟨proof⟩

**lemma** *col-suma--col*:
**assumes** $Col\ D\ E\ F$ **and**
$A\ B\ C\ B\ C\ A\ SumA\ D\ E\ F$
**shows** $Col\ A\ B\ C$
⟨proof⟩

**lemma** *ncol-suma--ncol*:
**assumes** $\neg\ Col\ A\ B\ C$ **and**
$A\ B\ C\ B\ C\ A\ SumA\ D\ E\ F$
**shows** $\neg\ Col\ D\ E\ F$
⟨proof⟩

**lemma** *per2-suma--bet*:
**assumes** $Per\ A\ B\ C$ **and**
$Per\ D\ E\ F$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$
**shows** $Bet\ G\ H\ I$
⟨proof⟩

**lemma** *bet-per2--suma*:
**assumes** $A \neq B$ **and**
$B \neq C$ **and**
$D \neq E$ **and**
$E \neq F$ **and**
$G \neq H$ **and**
$H \neq I$ **and**
$Per\ A\ B\ C$ **and**
$Per\ D\ E\ F$ **and**
$Bet\ G\ H\ I$
**shows** $A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$
⟨proof⟩

**lemma** *per2--sams*:
**assumes** $A \neq B$ **and**
$B \neq C$ **and**
$D \neq E$ **and**
$E \neq F$ **and**
$Per\ A\ B\ C$ **and**

*Per D E F*
**shows** *SAMS A B C D E F*
⟨*proof*⟩

**lemma** *bet-per-suma--per456*:
**assumes** *Per A B C* **and**
*Bet G H I* **and**
*A B C D E F SumA G H I*
**shows** *Per D E F*
⟨*proof*⟩

**lemma** *bet-per-suma--per123*:
**assumes** *Per D E F* **and**
*Bet G H I* **and**
*A B C D E F SumA G H I*
**shows** *Per A B C*
⟨*proof*⟩

**lemma** *bet-suma--per*:
**assumes** *Bet D E F* **and**
*A B C A B C SumA D E F*
**shows** *Per A B C*
⟨*proof*⟩

**lemma** *acute--sams*:
**assumes** *Acute A B C*
**shows** *SAMS A B C A B C*
⟨*proof*⟩

**lemma** *acute-suma--nbet*:
**assumes** *Acute A B C* **and**
*A B C A B C SumA D E F*
**shows** ¬ *Bet D E F*
⟨*proof*⟩

**lemma** *acute2--sams*:
**assumes** *Acute A B C* **and**
*Acute D E F*
**shows** *SAMS A B C D E F*
⟨*proof*⟩

**lemma** *acute2-suma--nbet-a*:
**assumes** *Acute A B C* **and**
*D E F LeA A B C* **and**
*A B C D E F SumA G H I*
**shows** ¬ *Bet G H I*
⟨*proof*⟩

**lemma** *acute2-suma--nbet*:
**assumes** *Acute A B C* **and**
*Acute D E F* **and**
*A B C D E F SumA G H I*
**shows** ¬ *Bet G H I*
⟨*proof*⟩

**lemma** *acute-per--sams*:
**assumes** $A \neq B$ **and**
$B \neq C$ **and**
*Per A B C* **and**
*Acute D E F*
**shows** *SAMS A B C D E F*
⟨*proof*⟩

**lemma** *acute-per-suma--nbet*:
**assumes** $A \neq B$ **and**
$B \neq C$ **and**

*Per A B C* **and**
*Acute D E F* **and**
*A B C D E F SumA G H I*
**shows** ¬ *Bet G H I*
⟨*proof*⟩

**lemma** *obtuse--nsams*:
**assumes** *Obtuse A B C*
**shows** ¬ *SAMS A B C A B C*
⟨*proof*⟩

**lemma** *nbet-sams-suma--acute*:
**assumes** ¬ *Bet D E F* **and**
*SAMS A B C A B C* **and**
*A B C A B C SumA D E F*
**shows** *Acute A B C*
⟨*proof*⟩

**lemma** *nsams--obtuse*:
**assumes** $A \neq B$ **and**
$B \neq C$ **and**
¬ *SAMS A B C A B C*
**shows** *Obtuse A B C*
⟨*proof*⟩

**lemma** *sams2-suma2--conga*:
**assumes** *SAMS A B C A B C* **and**
*A B C A B C SumA D E F* **and**
*SAMS* $A'$ $B'$ $C'$ $A'$ $B'$ $C'$ **and**
$A'$ $B'$ $C'$ $A'$ $B'$ $C'$ *SumA D E F*
**shows** *A B C CongA* $A'$ $B'$ $C'$
⟨*proof*⟩

**lemma** *acute2-suma2--conga*:
**assumes** *Acute A B C* **and**
*A B C A B C SumA D E F* **and**
*Acute* $A'$ $B'$ $C'$ **and**
$A'$ $B'$ $C'$ $A'$ $B'$ $C'$ *SumA D E F*
**shows** *A B C CongA* $A'$ $B'$ $C'$
⟨*proof*⟩

**lemma** *bet2-suma--out*:
**assumes** *Bet A B C* **and**
*Bet D E F* **and**
*A B C D E F SumA G H I*
**shows** *H Out G I*
⟨*proof*⟩

**lemma** *col2-suma--col*:
**assumes** *Col A B C* **and**
*Col D E F* **and**
*A B C D E F SumA G H I*
**shows** *Col G H I*
⟨*proof*⟩

**lemma** *suma-suppa--bet*:
**assumes** *A B C SuppA D E F* **and**
*A B C D E F SumA G H I*
**shows** *Bet G H I*
⟨*proof*⟩

**lemma** *bet-suppa--suma*:
**assumes** $G \neq H$ **and**
$H \neq I$ **and**
*A B C SuppA D E F* **and**
*Bet G H I*

**shows** $A\ B\ C\ D\ E\ F$ *SumA* $G\ H\ I$
⟨*proof*⟩

**lemma** *bet-suma--suppa*:
**assumes** $A\ B\ C\ D\ E\ F$ *SumA* $G\ H\ I$ **and**
*Bet* $G\ H\ I$
**shows** $A\ B\ C$ *SuppA* $D\ E\ F$
⟨*proof*⟩

**lemma** *bet2-suma--suma*:
**assumes** $A' \neq B$ **and**
$D' \neq E$ **and**
*Bet* $A\ B\ A'$ **and**
*Bet* $D\ E\ D'$ **and**
$A\ B\ C\ D\ E\ F$ *SumA* $G\ H\ I$
**shows** $A'\ B\ C\ D'\ E\ F$ *SumA* $G\ H\ I$
⟨*proof*⟩

**lemma** *suma-suppa2--suma*:
**assumes** $A\ B\ C$ *SuppA* $A'\ B'\ C'$ **and**
$D\ E\ F$ *SuppA* $D'\ E'\ F'$ **and**
$A\ B\ C\ D\ E\ F$ *SumA* $G\ H\ I$
**shows** $A'\ B'\ C'\ D'\ E'\ F'$ *SumA* $G\ H\ I$
⟨*proof*⟩

**lemma** *suma2-obtuse2--conga*:
**assumes** *Obtuse* $A\ B\ C$ **and**
$A\ B\ C\ A\ B\ C$ *SumA* $D\ E\ F$ **and**
*Obtuse* $A'\ B'\ C'$ **and**
$A'\ B'\ C'\ A'\ B'\ C'$ *SumA* $D\ E\ F$
**shows** $A\ B\ C$ *CongA* $A'\ B'\ C'$
⟨*proof*⟩

**lemma** *bet-suma2--or-conga*:
**assumes** *A0* $\neq B$ **and**
*Bet* $A\ B$ *A0* **and**
$A\ B\ C\ A\ B\ C$ *SumA* $D\ E\ F$ **and**
$A'\ B'\ C'\ A'\ B'\ C'$ *SumA* $D\ E\ F$
**shows** $A\ B\ C$ *CongA* $A'\ B'\ C' \vee$ *A0* $B\ C$ *CongA* $A'\ B'\ C'$
⟨*proof*⟩

**lemma** *suma2--or-conga-suppa*:
**assumes** $A\ B\ C\ A\ B\ C$ *SumA* $D\ E\ F$ **and**
$A'\ B'\ C'\ A'\ B'\ C'$ *SumA* $D\ E\ F$
**shows** $A\ B\ C$ *CongA* $A'\ B'\ C' \vee A\ B\ C$ *SuppA* $A'\ B'\ C'$
⟨*proof*⟩

**lemma** *ex-trisuma*:
**assumes** $A \neq B$ **and**
$B \neq C$ **and**
$A \neq C$
**shows** $\exists\ D\ E\ F.\ A\ B\ C$ *TriSumA* $D\ E\ F$
⟨*proof*⟩

**lemma** *trisuma-perm-231*:
**assumes** $A\ B\ C$ *TriSumA* $D\ E\ F$
**shows** $B\ C\ A$ *TriSumA* $D\ E\ F$
⟨*proof*⟩

**lemma** *trisuma-perm-312*:
**assumes** $A\ B\ C$ *TriSumA* $D\ E\ F$
**shows** $C\ A\ B$ *TriSumA* $D\ E\ F$
⟨*proof*⟩

**lemma** *trisuma-perm-321*:
**assumes** $A\ B\ C$ *TriSumA* $D\ E\ F$

**shows** $C\ B\ A\ TriSumA\ D\ E\ F$
⟨*proof*⟩

**lemma** *trisuma-perm-213*:
**assumes** $A\ B\ C\ TriSumA\ D\ E\ F$
**shows** $B\ A\ C\ TriSumA\ D\ E\ F$
⟨*proof*⟩

**lemma** *trisuma-perm-132*:
**assumes** $A\ B\ C\ TriSumA\ D\ E\ F$
**shows** $A\ C\ B\ TriSumA\ D\ E\ F$
⟨*proof*⟩

**lemma** *conga-trisuma--trisuma*:
**assumes** $A\ B\ C\ TriSumA\ D\ E\ F$ **and**
$D\ E\ F\ CongA\ D'\ E'\ F'$
**shows** $A\ B\ C\ TriSumA\ D'\ E'\ F'$
⟨*proof*⟩

**lemma** *trisuma2--conga*:
**assumes** $A\ B\ C\ TriSumA\ D\ E\ F$ **and**
$A\ B\ C\ TriSumA\ D'\ E'\ F'$
**shows** $D\ E\ F\ CongA\ D'\ E'\ F'$
⟨*proof*⟩

**lemma** *conga3-trisuma--trisuma*:
**assumes** $A\ B\ C\ TriSumA\ D\ E\ F$ **and**
$A\ B\ C\ CongA\ A'\ B'\ C'$ **and**
$B\ C\ A\ CongA\ B'\ C'\ A'$ **and**
$C\ A\ B\ CongA\ C'\ A'\ B'$
**shows** $A'\ B'\ C'\ TriSumA\ D\ E\ F$
⟨*proof*⟩

**lemma** *col-trisuma--bet*:
**assumes** $Col\ A\ B\ C$ **and**
$A\ B\ C\ TriSumA\ P\ Q\ R$
**shows** $Bet\ P\ Q\ R$
⟨*proof*⟩

**lemma** *suma-dec*:
$A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I \vee \neg\ A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ ⟨*proof*⟩

**lemma** *sams-dec*:
$SAMS\ A\ B\ C\ D\ E\ F \vee \neg\ SAMS\ A\ B\ C\ D\ E\ F$ ⟨*proof*⟩

**lemma** *trisuma-dec*:
$A\ B\ C\ TriSumA\ P\ Q\ R \vee \neg\ A\ B\ C\ TriSumA\ P\ Q\ R$
⟨*proof*⟩

## 3.11 Parallelism

**lemma** *par-reflexivity*:
**assumes** $A \neq B$
**shows** $A\ B\ Par\ A\ B$
⟨*proof*⟩

**lemma** *par-strict-irreflexivity*:
$\neg\ A\ B\ ParStrict\ A\ B$
⟨*proof*⟩

**lemma** *not-par-strict-id*:
$\neg\ A\ B\ ParStrict\ A\ C$
⟨*proof*⟩

**lemma** *par-id*:
**assumes** $A\ B\ Par\ A\ C$

**shows** *Col A B C*
⟨*proof*⟩

**lemma** *par-strict-not-col-1*:
**assumes** *A B ParStrict C D*
**shows** ¬ *Col A B C*
⟨*proof*⟩

**lemma** *par-strict-not-col-2*:
**assumes** *A B ParStrict C D*
**shows** ¬ *Col B C D*
⟨*proof*⟩

**lemma** *par-strict-not-col-3*:
**assumes** *A B ParStrict C D*
**shows** ¬ *Col C D A*
⟨*proof*⟩

**lemma** *par-strict-not-col-4*:
**assumes** *A B ParStrict C D*
**shows** ¬ *Col A B D*
⟨*proof*⟩

**lemma** *par-id-1*:
**assumes** *A B Par A C*
**shows** *Col B A C*
⟨*proof*⟩

**lemma** *par-id-2*:
**assumes** *A B Par A C*
**shows** *Col B C A*
⟨*proof*⟩

**lemma** *par-id-3*:
**assumes** *A B Par A C*
**shows** *Col A C B*
⟨*proof*⟩

**lemma** *par-id-4*:
**assumes** *A B Par A C*
**shows** *Col C B A*
⟨*proof*⟩

**lemma** *par-id-5*:
**assumes** *A B Par A C*
**shows** *Col C A B*
⟨*proof*⟩

**lemma** *par-strict-symmetry*:
**assumes** *A B ParStrict C D*
**shows** *C D ParStrict A B*
⟨*proof*⟩

**lemma** *par-symmetry*:
**assumes** *A B Par C D*
**shows** *C D Par A B*
⟨*proof*⟩

**lemma** *par-left-comm*:
**assumes** *A B Par C D*
**shows** *B A Par C D*
⟨*proof*⟩

**lemma** *par-right-comm*:
**assumes** *A B Par C D*
**shows** *A B Par D C*

⟨*proof*⟩

**lemma** *par-comm*:
**assumes** *A B Par C D*
**shows** *B A Par D C*
⟨*proof*⟩

**lemma** *par-strict-left-comm*:
**assumes** *A B ParStrict C D*
**shows** *B A ParStrict C D*
⟨*proof*⟩

**lemma** *par-strict-right-comm*:
**assumes** *A B ParStrict C D*
**shows** *A B ParStrict D C*
⟨*proof*⟩

**lemma** *par-strict-comm*:
**assumes** *A B ParStrict C D*
**shows** *B A ParStrict D C*
⟨*proof*⟩

**lemma** *par-strict-neq1*:
**assumes** *A B ParStrict C D*
**shows** $A \neq B$
⟨*proof*⟩

**lemma** *par-strict-neq2*:
**assumes** *A B ParStrict C D*
**shows** $C \neq D$
⟨*proof*⟩

**lemma** *par-neq1*:
**assumes** *A B Par C D*
**shows** $A \neq B$
⟨*proof*⟩

**lemma** *par-neq2*:
**assumes** *A B Par C D*
**shows** $C \neq D$
⟨*proof*⟩

**lemma** *Par-cases*:
**assumes** *A B Par C D* ∨ *B A Par C D* ∨ *A B Par D C* ∨ *B A Par D C* ∨ *C D Par A B* ∨ *C D Par B A* ∨ *D C Par A B* ∨ *D C Par B A*
**shows** *A B Par C D*
⟨*proof*⟩

**lemma** *Par-perm*:
**assumes** *A B Par C D*
**shows** *A B Par C D* ∧ *B A Par C D* ∧ *A B Par D C* ∧ *B A Par D C* ∧ *C D Par A B* ∧ *C D Par B A* ∧ *D C Par A B* ∧ *D C Par B A*
⟨*proof*⟩

**lemma** *Par-strict-cases*:
**assumes** *A B ParStrict C D* ∨ *B A ParStrict C D* ∨ *A B ParStrict D C* ∨ *B A ParStrict D C* ∨ *C D ParStrict A B* ∨ *C D ParStrict B A* ∨ *D C ParStrict A B* ∨ *D C ParStrict B A*
**shows** *A B ParStrict C D*
⟨*proof*⟩

**lemma** *Par-strict-perm*:
**assumes** *A B ParStrict C D*
**shows** *A B ParStrict C D* ∧ *B A ParStrict C D* ∧ *A B ParStrict D C* ∧ *B A ParStrict D C* ∧ *C D ParStrict A B* ∧ *C D ParStrict B A* ∧ *D C ParStrict A B* ∧ *D C ParStrict B A*
⟨*proof*⟩

**lemma** *l12-6*:
  **assumes** *A B ParStrict C D*
  **shows** *A B OS C D*
  ⟨*proof*⟩

**lemma** *pars--os3412*:
  **assumes** *A B ParStrict C D*
  **shows** *C D OS A B*
  ⟨*proof*⟩

**lemma** *perp-dec*:
  *A B Perp C D* ∨ ¬ *A B Perp C D*
  ⟨*proof*⟩

**lemma** *col-cop2-perp2--col*:
  **assumes** *X1 X2 Perp A B* **and**
    *Y1 Y2 Perp A B* **and**
    *Col X1 Y1 Y2* **and**
    *Coplanar A B X2 Y1* **and**
    *Coplanar A B X2 Y2*
  **shows** *Col X2 Y1 Y2*
⟨*proof*⟩

**lemma** *col-perp2-ncol-col*:
  **assumes** *X1 X2 Perp A B* **and**
    *Y1 Y2 Perp A B* **and**
    *Col X1 Y1 Y2* **and**
    ¬ *Col X1 A B*
  **shows** *Col X2 Y1 Y2*
⟨*proof*⟩

**lemma** *l12-9*:
  **assumes**
    *Coplanar C1 C2 A1 B1* **and**
    *Coplanar C1 C2 A1 B2* **and**
    *Coplanar C1 C2 A2 B1* **and**
    *Coplanar C1 C2 A2 B2* **and**
    *A1 A2 Perp C1 C2* **and**
    *B1 B2 Perp C1 C2*
  **shows** *A1 A2 Par B1 B2*
⟨*proof*⟩

**lemma** *parallel-existence*:
  **assumes** $A \neq B$
  **shows** $\exists\ C\ D.\ C \neq D \wedge A\ B\ Par\ C\ D \wedge Col\ P\ C\ D$
⟨*proof*⟩

**lemma** *par-col-par*:
  **assumes** $C \neq D'$ **and**
    *A B Par C D* **and**
    $Col\ C\ D\ D'$
  **shows** $A\ B\ Par\ C\ D'$
⟨*proof*⟩

**lemma** *parallel-existence1*:
  **assumes** $A \neq B$
  **shows** $\exists\ Q.\ A\ B\ Par\ P\ Q$
⟨*proof*⟩

**lemma** *par-not-col*:
  **assumes** *A B ParStrict C D* **and**
    *Col X A B*
  **shows** ¬ *Col X C D*
  ⟨*proof*⟩

**lemma** *not-strict-par1*:

**assumes** $A\ B\ Par\ C\ D$ **and**
$Col\ A\ B\ X$ **and**
$Col\ C\ D\ X$
**shows** $Col\ A\ B\ C$
⟨*proof*⟩

**lemma** *not-strict-par2*:
**assumes** $A\ B\ Par\ C\ D$ **and**
$Col\ A\ B\ X$ **and**
$Col\ C\ D\ X$
**shows** $Col\ A\ B\ D$
⟨*proof*⟩

**lemma** *not-strict-par*:
**assumes** $A\ B\ Par\ C\ D$ **and**
$Col\ A\ B\ X$ **and**
$Col\ C\ D\ X$
**shows** $Col\ A\ B\ C \wedge Col\ A\ B\ D$
⟨*proof*⟩

**lemma** *not-par-not-col*:
**assumes** $A \neq B$ **and**
$A \neq C$ **and**
$\neg\ A\ B\ Par\ A\ C$
**shows** $\neg\ Col\ A\ B\ C$
⟨*proof*⟩

**lemma** *not-par-inter-uniqueness*:
**assumes** $A \neq B$ **and**
$C \neq D$ **and**
$\neg\ A\ B\ Par\ C\ D$ **and**
$Col\ A\ B\ X$ **and**
$Col\ C\ D\ X$ **and**
$Col\ A\ B\ Y$ **and**
$Col\ C\ D\ Y$
**shows** $X = Y$
⟨*proof*⟩

**lemma** *inter-uniqueness-not-par*:
**assumes** $\neg\ Col\ A\ B\ C$ **and**
$Col\ A\ B\ P$ **and**
$Col\ C\ D\ P$
**shows** $\neg\ A\ B\ Par\ C\ D$
⟨*proof*⟩

**lemma** *col-not-col-not-par*:
**assumes** $\exists\ P.\ Col\ A\ B\ P \wedge Col\ C\ D\ P$ **and**
$\exists\ Q.\ Col\ C\ D\ Q \wedge \neg Col\ A\ B\ Q$
**shows** $\neg\ A\ B\ Par\ C\ D$
⟨*proof*⟩

**lemma** *par-distincts*:
**assumes** $A\ B\ Par\ C\ D$
**shows** $A\ B\ Par\ C\ D \wedge A \neq B \wedge C \neq D$
⟨*proof*⟩

**lemma** *par-not-col-strict*:
**assumes** $A\ B\ Par\ C\ D$ **and**
$Col\ C\ D\ P$ **and**
$\neg\ Col\ A\ B\ P$
**shows** $A\ B\ ParStrict\ C\ D$
⟨*proof*⟩

**lemma** *col-cop-perp2-pars*:
**assumes** $\neg\ Col\ A\ B\ P$ **and**
$Col\ C\ D\ P$ **and**

*Coplanar A B C D* **and**
*A B Perp P Q* **and**
*C D Perp P Q*
**shows** *A B ParStrict C D*
⟨*proof*⟩

**lemma** *all-one-side-par-strict*:
**assumes** $C \neq D$ **and**
$\forall$ *P*. *Col C D P* $\longrightarrow$ *A B OS C P*
**shows** *A B ParStrict C D*
⟨*proof*⟩

**lemma** *par-col-par-2*:
**assumes** $A \neq P$ **and**
*Col A B P* **and**
*A B Par C D*
**shows** *A P Par C D*
⟨*proof*⟩

**lemma** *par-col2-par*:
**assumes** $E \neq F$ **and**
*A B Par C D* **and**
*Col C D E* **and**
*Col C D F*
**shows** *A B Par E F*
⟨*proof*⟩

**lemma** *par-col2-par-bis*:
**assumes** $E \neq F$ **and**
*A B Par C D* **and**
*Col E F C* **and**
*Col E F D*
**shows** *A B Par E F*
⟨*proof*⟩

**lemma** *par-strict-col-par-strict*:
**assumes** $C \neq E$ **and**
*A B ParStrict C D* **and**
*Col C D E*
**shows** *A B ParStrict C E*
⟨*proof*⟩

**lemma** *par-strict-col2-par-strict*:
**assumes** $E \neq F$ **and**
*A B ParStrict C D* **and**
*Col C D E* **and**
*Col C D F*
**shows** *A B ParStrict E F*
⟨*proof*⟩

**lemma** *line-dec*:
(*Col C1 B1 B2* $\wedge$ *Col C2 B1 B2*) $\vee$ $\neg$ (*Col C1 B1 B2* $\wedge$ *Col C2 B1 B2*)
⟨*proof*⟩

**lemma** *par-distinct*:
**assumes** *A B Par C D*
**shows** $A \neq B \wedge C \neq D$
⟨*proof*⟩

**lemma** *par-col4--par*:
**assumes** $E \neq F$ **and**
$G \neq H$ **and**
*A B Par C D* **and**
*Col A B E* **and**
*Col A B F* **and**
*Col C D G* **and**

*Col C D H*
**shows** *E F Par G H*
⟨*proof*⟩

**lemma** *par-strict-col4--par-strict*:
**assumes** $E \neq F$ **and**
$G \neq H$ **and**
*A B ParStrict C D* **and**
*Col A B E* **and**
*Col A B F* **and**
*Col C D G* **and**
*Col C D H*
**shows** *E F ParStrict G H*
⟨*proof*⟩

**lemma** *par-strict-one-side*:
**assumes** *A B ParStrict C D* **and**
*Col C D P*
**shows** *A B OS C P*
⟨*proof*⟩

**lemma** *par-strict-all-one-side*:
**assumes** *A B ParStrict C D*
**shows** $\forall$ *P*. *Col C D P* $\longrightarrow$ *A B OS C P*
⟨*proof*⟩

**lemma** *inter-trivial*:
**assumes** $\neg$ *Col A B X*
**shows** *X Inter A X B X*
⟨*proof*⟩

**lemma** *inter-sym*:
**assumes** *X Inter A B C D*
**shows** *X Inter C D A B*
⟨*proof*⟩

**lemma** *inter-left-comm*:
**assumes** *X Inter A B C D*
**shows** *X Inter B A C D*
⟨*proof*⟩

**lemma** *inter-right-comm*:
**assumes** *X Inter A B C D*
**shows** *X Inter A B D C*
⟨*proof*⟩

**lemma** *inter-comm*:
**assumes** *X Inter A B C D*
**shows** *X Inter B A D C*
⟨*proof*⟩

**lemma** *l12-17*:
**assumes** $A \neq B$ **and**
*P Midpoint A C* **and**
*P Midpoint B D*
**shows** *A B Par C D*
⟨*proof*⟩

**lemma** *l12-18-a*:
**assumes** *Cong A B C D* **and**
*Cong B C D A* **and**
$\neg$ *Col A B C* **and**
$B \neq D$ **and**
*Col A P C* **and**
*Col B P D*
**shows** *A B Par C D*

⟨*proof*⟩

**lemma** *l12-18-b*:
**assumes** *Cong A B C D* **and**
*Cong B C D A* **and**
¬ *Col A B C* **and**
$B \neq D$ **and**
*Col A P C* **and**
*Col B P D*
**shows** *B C Par D A*
⟨*proof*⟩

**lemma** *l12-18-c*:
**assumes** *Cong A B C D* **and**
*Cong B C D A* **and**
¬ *Col A B C* **and**
$B \neq D$ **and**
*Col A P C* **and**
*Col B P D*
**shows** *B D TS A C*
⟨*proof*⟩

**lemma** *l12-18-d*:
**assumes** *Cong A B C D* **and**
*Cong B C D A* **and**
¬ *Col A B C* **and**
$B \neq D$ **and**
*Col A P C* **and**
*Col B P D*
**shows** *A C TS B D*
⟨*proof*⟩

**lemma** *l12-18*:
**assumes** *Cong A B C D* **and**
*Cong B C D A* **and**
¬ *Col A B C* **and**
$B \neq D$ **and**
*Col A P C* **and**
*Col B P D*
**shows** *A B Par C D* ∧ *B C Par D A* ∧ *B D TS A C* ∧ *A C TS B D*
⟨*proof*⟩

**lemma** *par-two-sides-two-sides*:
**assumes** *A B Par C D* **and**
*B D TS A C*
**shows** *A C TS B D*
⟨*proof*⟩

**lemma** *par-one-or-two-sides*:
**assumes** *A B ParStrict C D*
**shows** (*A C TS B D* ∧ *B D TS A C*) ∨ (*A C OS B D* ∧ *B D OS A C*)
⟨*proof*⟩

**lemma** *l12-21-b*:
**assumes** *A C TS B D* **and**
*B A C CongA D C A*
**shows** *A B Par C D*
⟨*proof*⟩

**lemma** *l12-22-aux*:
**assumes** $P \neq A$ **and**
$A \neq C$ **and**
*Bet P A C* **and**
*P A OS B D* **and**
*B A P CongA D C P*
**shows** *A B Par C D*

⟨*proof*⟩

**lemma** *l12-22-b*:
**assumes** $P$ *Out* $A$ $C$ **and**
$P$ $A$ *OS* $B$ $D$ **and**
$B$ $A$ $P$ *CongA* $D$ $C$ $P$
**shows** $A$ $B$ *Par* $C$ $D$
⟨*proof*⟩

**lemma** *par-strict-par*:
**assumes** $A$ $B$ *ParStrict* $C$ $D$
**shows** $A$ $B$ *Par* $C$ $D$
⟨*proof*⟩

**lemma** *par-strict-distinct*:
**assumes** $A$ $B$ *ParStrict* $C$ $D$
**shows** $A \neq B \wedge C \neq D$
⟨*proof*⟩

**lemma** *col-par*:
**assumes** $A \neq B$ **and**
$B \neq C$ **and**
*Col* $A$ $B$ $C$
**shows** $A$ $B$ *Par* $B$ $C$
⟨*proof*⟩

**lemma** *acute-col-perp--out*:
**assumes** *Acute* $A$ $B$ $C$ **and**
*Col* $B$ $C$ $A'$ **and**
$B$ $C$ *Perp* $A$ $A'$
**shows** $B$ *Out* $A'$ $C$
⟨*proof*⟩

**lemma** *acute-col-perp--out-1*:
**assumes** *Acute* $A$ $B$ $C$ **and**
*Col* $B$ $C$ $A'$ **and**
$B$ $A$ *Perp* $A$ $A'$
**shows** $B$ *Out* $A'$ $C$
⟨*proof*⟩

**lemma** *conga-inangle-per2--inangle*:
**assumes** *Per* $A$ $B$ $C$ **and**
$T$ *InAngle* $A$ $B$ $C$ **and**
$P$ $B$ $A$ *CongA* $P$ $B$ $C$ **and**
*Per* $B$ $P$ $T$ **and**
*Coplanar* $A$ $B$ $C$ $P$
**shows** $P$ *InAngle* $A$ $B$ $C$
⟨*proof*⟩

**lemma** *perp-not-par*:
**assumes** $A$ $B$ *Perp* $X$ $Y$
**shows** $\neg$ $A$ $B$ *Par* $X$ $Y$
⟨*proof*⟩

**lemma** *cong-conga-perp*:
**assumes** $B$ $P$ *TS* $A$ $C$ **and**
*Cong* $A$ $B$ $C$ $B$ **and**
$A$ $B$ $P$ *CongA* $C$ $B$ $P$
**shows** $A$ $C$ *Perp* $B$ $P$
⟨*proof*⟩

**lemma** *perp-inter-exists*:
**assumes** $A$ $B$ *Perp* $C$ $D$
**shows** $\exists$ $P$. *Col* $A$ $B$ $P$ $\wedge$ *Col* $C$ $D$ $P$
⟨*proof*⟩

**lemma** *perp-inter-perp-in*:
  **assumes** $A\ B\ Perp\ C\ D$
  **shows** $\exists\ P.\ Col\ A\ B\ P \wedge Col\ C\ D\ P \wedge P\ PerpAt\ A\ B\ C\ D$
  ⟨*proof*⟩

**end**

**context** *Tarski-2D*

**begin**

**lemma** *l12-9-2D*:
  **assumes** $A1\ A2\ Perp\ C1\ C2$ **and**
    $B1\ B2\ Perp\ C1\ C2$
  **shows** $A1\ A2\ Par\ B1\ B2$
  ⟨*proof*⟩

**end**

**context** *Tarski-neutral-dimensionless*

**begin**

## 3.12 Tarski: Chapter 13

### 3.12.1 Introduction

**lemma** *per2-col-eq*:
  **assumes** $A \neq P$ **and**
    $A \neq P'$ **and**
    $Per\ A\ P\ B$ **and**
    $Per\ A\ P'\ B$ **and**
    $Col\ P\ A\ P'$
  **shows** $P = P'$
  ⟨*proof*⟩

**lemma** *per2-preserves-diff*:
  **assumes** $PO \neq A'$ **and**
    $PO \neq B'$ **and**
    $Col\ PO\ A'\ B'$ **and**
    $Per\ PO\ A'\ A$ **and**
    $Per\ PO\ B'\ B$ **and**
    $A' \neq B'$
  **shows** $A \neq B$
  ⟨*proof*⟩

**lemma** *per23-preserves-bet*:
  **assumes** $Bet\ A\ B\ C$ **and**
    $A \neq B'$ **and** $A \neq C'$ **and**
    $Col\ A\ B'\ C'$ **and**
    $Per\ A\ B'\ B$ **and**
    $Per\ A\ C'\ C$
  **shows** $Bet\ A\ B'\ C'$
⟨*proof*⟩

**lemma** *per23-preserves-bet-inv*:
  **assumes** $Bet\ A\ B'\ C'$ **and**
    $A \neq B'$ **and**
    $Col\ A\ B\ C$ **and**
    $Per\ A\ B'\ B$ **and**
    $Per\ A\ C'\ C$
  **shows** $Bet\ A\ B\ C$
⟨*proof*⟩

**lemma** *per13-preserves-bet*:
  **assumes** $Bet\ A\ B\ C$ **and**
    $B \neq A'$ **and**

$B \neq C'$ **and**
$Col\ A'\ B\ C'$ **and**
$Per\ B\ A'\ A$ **and**
$Per\ B\ C'\ C$
**shows** $Bet\ A'\ B\ C'$
⟨*proof*⟩

**lemma** *per13-preserves-bet-inv*:
**assumes** $Bet\ A'\ B\ C'$ **and**
$B \neq A'$ **and**
$B \neq C'$ **and**
$Col\ A\ B\ C$ **and**
$Per\ B\ A'\ A$ **and**
$Per\ B\ C'\ C$
**shows** $Bet\ A\ B\ C$
⟨*proof*⟩

**lemma** *per3-preserves-bet1*:
**assumes** $Col\ PO\ A\ B$ **and**
$Bet\ A\ B\ C$ **and**
$PO \neq A'$ **and**
$PO \neq B'$ **and**
$PO \neq C'$ **and**
$Per\ PO\ A'\ A$ **and**
$Per\ PO\ B'\ B$ **and**
$Per\ PO\ C'\ C$ **and**
$Col\ A'\ B'\ C'$ **and**
$Col\ PO\ A'\ B'$
**shows** $Bet\ A'\ B'\ C'$
⟨*proof*⟩

**lemma** *per3-preserves-bet2-aux*:
**assumes** $Col\ PO\ A\ C$ **and**
$A \neq C'$ **and**
$Bet\ A\ B'\ C'$ **and**
$PO \neq A$ **and**
$PO \neq B'$ **and**
$PO \neq C'$ **and**
$Per\ PO\ B'\ B$ **and**
$Per\ PO\ C'\ C$ **and**
$Col\ A\ B\ C$ **and**
$Col\ PO\ A\ C'$
**shows** $Bet\ A\ B\ C$
⟨*proof*⟩

**lemma** *per3-preserves-bet2*:
**assumes** $Col\ PO\ A\ C$ **and**
$A' \neq C'$ **and**
$Bet\ A'\ B'\ C'$ **and**
$PO \neq A'$ **and**
$PO \neq B'$ **and**
$PO \neq C'$ **and**
$Per\ PO\ A'\ A$ **and**
$Per\ PO\ B'\ B$ **and**
$Per\ PO\ C'\ C$ **and**
$Col\ A\ B\ C$ **and**
$Col\ PO\ A'\ C'$
**shows** $Bet\ A\ B\ C$
⟨*proof*⟩

**lemma** *symmetry-preserves-per*:
**assumes** $Per\ B\ P\ A$ **and**
$B\ Midpoint\ A\ A'$ **and**
$B\ Midpoint\ P\ P'$
**shows** $Per\ B\ P'\ A'$
⟨*proof*⟩

**lemma** *l13-1-aux*:
  **assumes** $\neg$ *Col A B C* **and**
    *P Midpoint B C* **and**
    *Q Midpoint A C* **and**
    *R Midpoint A B*
  **shows**
    $\exists\ X\ Y.\ (R\ PerpAt\ X\ Y\ A\ B \wedge X\ Y\ Perp\ P\ Q \wedge Coplanar\ A\ B\ C\ X \wedge Coplanar\ A\ B\ C\ Y)$
$\langle proof \rangle$

**lemma** *l13-1*:
  **assumes** $\neg$ *Col A B C* **and**
    *P Midpoint B C* **and**
    *Q Midpoint A C* **and**
    *R Midpoint A B*
  **shows**
    $\exists\ X\ Y.(R\ PerpAt\ X\ Y\ A\ B \wedge X\ Y\ Perp\ P\ Q)$
$\langle proof \rangle$

**lemma** *per-lt*:
  **assumes** $A \neq B$ **and**
    $C \neq B$ **and**
    *Per A B C*
  **shows** $A\ B\ Lt\ A\ C \wedge C\ B\ Lt\ A\ C$
$\langle proof \rangle$

**lemma** *cong-perp-conga*:
  **assumes** *Cong A B C B* **and**
    *A C Perp B P*
  **shows** $A\ B\ P\ CongA\ C\ B\ P \wedge B\ P\ TS\ A\ C$
$\langle proof \rangle$

**lemma** *perp-per-bet*:
  **assumes** $\neg$ *Col A B C* **and**

    *Per A B C* **and**
    *P PerpAt P B A C*
  **shows** *Bet A P C*
$\langle proof \rangle$

**lemma** *ts-per-per-ts*:
  **assumes** *A B TS C D* **and**
    *Per B C A* **and**
    *Per B D A*
  **shows** *C D TS A B*
$\langle proof \rangle$

**lemma** *l13-2-1*:
  **assumes** *A B TS C D* **and**
    *Per B C A* **and**
    *Per B D A* **and**
    *Col C D E* **and**
    *A E Perp C D* **and**
    *C A B CongA D A B*
  **shows** $B\ A\ C\ CongA\ D\ A\ E \wedge B\ A\ D\ CongA\ C\ A\ E \wedge Bet\ C\ E\ D$
$\langle proof \rangle$

**lemma** *triangle-mid-par*:
  **assumes** $\neg$ *Col A B C* **and**
    *P Midpoint B C* **and**
    *Q Midpoint A C*
  **shows** *A B ParStrict Q P*
$\langle proof \rangle$

**lemma** *cop4-perp-in2--col*:
  **assumes** *Coplanar X Y A A′* **and**

*Coplanar X Y A B′* **and**
*Coplanar X Y B A′* **and**
*Coplanar X Y B B′* **and**
*P PerpAt A B X Y* **and**
*P PerpAt A′ B′ X Y*
**shows** *Col A B A′*
⟨*proof*⟩

**lemma** *l13-2*:
**assumes** *A B TS C D* **and**
*Per B C A* **and**
*Per B D A* **and**
*Col C D E* **and**
*A E Perp C D*
**shows** *B A C CongA D A E* ∧ *B A D CongA C A E* ∧ *Bet C E D*
⟨*proof*⟩

**lemma** *perp2-refl*:
**assumes** *A* ≠ *B*
**shows** *P Perp2 A B A B*
⟨*proof*⟩

**lemma** *perp2-sym*:
**assumes** *P Perp2 A B C D*
**shows** *P Perp2 C D A B*
⟨*proof*⟩

**lemma** *perp2-left-comm*:
**assumes** *P Perp2 A B C D*
**shows** *P Perp2 B A C D*
⟨*proof*⟩

**lemma** *perp2-right-comm*:
**assumes** *P Perp2 A B C D*
**shows** *P Perp2 A B D C*
⟨*proof*⟩

**lemma** *perp2-comm*:
**assumes** *P Perp2 A B C D*
**shows** *P Perp2 B A D C*
⟨*proof*⟩

**lemma** *perp2-pseudo-trans*:
**assumes** *P Perp2 A B C D* **and**
*P Perp2 C D E F* **and**
¬ *Col C D P*
**shows** *P Perp2 A B E F*
⟨*proof*⟩

**lemma** *col-cop-perp2--pars-bis*:
**assumes** ¬ *Col A B P* **and**
*Col C D P* **and**
*Coplanar A B C D* **and**
*P Perp2 A B C D*
**shows** *A B ParStrict C D*
⟨*proof*⟩

**lemma** *perp2-preserves-bet23*:
**assumes** *Bet PO A B* **and**
*Col PO A′ B′* **and**
¬ *Col PO A A′* **and**
*PO Perp2 A A′ B B′*
**shows** *Bet PO A′ B′*
⟨*proof*⟩

**lemma** *perp2-preserves-bet13*:

**assumes** *Bet B PO C* **and**
*Col PO B' C'* **and**
$\neg$ *Col PO B B'* **and**
*PO Perp2 B C' C B'*
**shows** *Bet B' PO C'*
⟨*proof*⟩

**lemma** *is-image-perp-in*:
**assumes** $A \neq A'$ **and**
$X \neq Y$ **and**
*A A' Reflect X Y*
**shows** $\exists$ *P. P PerpAt A A' X Y*
⟨*proof*⟩

**lemma** *perp-inter-perp-in-n*:
**assumes** *A B Perp C D*
**shows** $\exists$ *P. Col A B P* $\wedge$ *Col C D P* $\wedge$ *P PerpAt A B C D*
⟨*proof*⟩

**lemma** *perp2-perp-in*:
**assumes** *PO Perp2 A B C D* **and**
$\neg$ *Col PO A B* **and**
$\neg$ *Col PO C D*
**shows** $\exists$ *P Q. Col A B P* $\wedge$ *Col C D Q* $\wedge$ *Col PO P Q* $\wedge$ *P PerpAt PO P A B* $\wedge$ *Q PerpAt PO Q C D*
⟨*proof*⟩

**lemma** *l13-8*:
**assumes** $U \neq PO$ **and**
$V \neq PO$ **and**
*Col PO P Q* **and**
*Col PO U V* **and**
*Per P U PO* **and**
*Per Q V PO*
**shows** *PO Out P Q* $\longleftrightarrow$ *PO Out U V*
⟨*proof*⟩

**lemma** *perp-in-rewrite*:
**assumes** *P PerpAt A B C D*
**shows** *P PerpAt A P P C* $\vee$ *P PerpAt A P P D* $\vee$ *P PerpAt B P P C* $\vee$ *P PerpAt B P P D*
⟨*proof*⟩

**lemma** *perp-out-acute*:
**assumes** *B Out A C'* **and**
*A B Perp C C'*
**shows** *Acute A B C*
⟨*proof*⟩

**lemma** *perp-bet-obtuse*:
**assumes** $B \neq C'$ **and**
*A B Perp C C'* **and**
*Bet A B C'*
**shows** *Obtuse A B C*
⟨*proof*⟩

**end**

### 3.12.2 Part 1: 2D

**context** *Tarski-2D*
**begin**

**lemma** *perp-in2--col*:
**assumes** *P PerpAt A B X Y* **and**
*P PerpAt A' B' X Y*
**shows** *Col A B A'*
⟨*proof*⟩

**lemma** *perp2-trans*:
  **assumes** *P Perp2 A B C D* **and**
    *P Perp2 C D E F*
  **shows** *P Perp2 A B E F*
⟨*proof*⟩

**lemma** *perp2-par*:
  **assumes** *PO Perp2 A B C D*
  **shows** *A B Par C D*
  ⟨*proof*⟩

**end**

### 3.12.3 Part 2: length

**context** *Tarski-neutral-dimensionless*

**begin**

**lemma** *lg-exists*:
  $\exists$ *l*. (*QCong l* $\wedge$ *l A B*)
  ⟨*proof*⟩

**lemma** *lg-cong*:
  **assumes** *QCong l* **and**
    *l A B* **and**
    *l C D*
  **shows** *Cong A B C D*
  ⟨*proof*⟩

**lemma** *lg-cong-lg*:
  **assumes** *QCong l* **and**
    *l A B* **and**
    *Cong A B C D*
  **shows** *l C D*
  ⟨*proof*⟩

**lemma** *lg-sym*:
  **assumes** *QCong l*
    **and** *l A B*
  **shows** *l B A*
  ⟨*proof*⟩

**lemma** *ex-points-lg*:
  **assumes** *QCong l*
  **shows** $\exists$ *A B*. *l A B*
  ⟨*proof*⟩

**lemma** *is-len-cong*:
  **assumes** *TarskiLen A B l* **and**
    *TarskiLen C D l*
  **shows** *Cong A B C D*
  ⟨*proof*⟩

**lemma** *is-len-cong-is-len*:
  **assumes** *TarskiLen A B l* **and**
    *Cong A B C D*
  **shows** *TarskiLen C D l*
  ⟨*proof*⟩

**lemma** *not-cong-is-len*:
  **assumes** $\neg$ *Cong A B C D* **and**
    *TarskiLen A B l*
  **shows** $\neg$ *l C D*
  ⟨*proof*⟩

**lemma** *not-cong-is-len1*:
  **assumes** ¬ *Cong A B C D*
    **and** *TarskiLen A B l*
  **shows** ¬ *TarskiLen C D l*
  ⟨*proof*⟩

**lemma** *lg-null-instance*:
  **assumes** *QCongNull l*
  **shows** *l A A*
  ⟨*proof*⟩

**lemma** *lg-null-trivial*:
  **assumes** *QCong l*
    **and** *l A A*
  **shows** *QCongNull l*
  ⟨*proof*⟩

**lemma** *lg-null-dec*:

  **shows** *QCongNull l* ∨ ¬ *QCongNull l*
  ⟨*proof*⟩

**lemma** *ex-point-lg*:
  **assumes** *QCong l*
  **shows** ∃ *B*. *l A B*
  ⟨*proof*⟩

**lemma** *ex-point-lg-out*:
  **assumes** *A* ≠ *P* **and**
    *QCong l* **and**
    ¬ *QCongNull l*
  **shows** ∃ *B*. (*l A B* ∧ *A Out B P*)
⟨*proof*⟩

**lemma** *ex-point-lg-bet*:
  **assumes** *QCong l*
  **shows** ∃ *B*. (*l M B* ∧ *Bet A M B*)
⟨*proof*⟩

**lemma** *ex-points-lg-not-col*:
  **assumes** *QCong l*
    **and** ¬ *QCongNull l*
  **shows** ∃ *A B*. (*l A B* ∧ ¬ *Col A B P*)
⟨*proof*⟩

**lemma** *ex-eql*:
  **assumes** ∃ *A B*. (*TarskiLen A B l1* ∧ *TarskiLen A B l2*)
  **shows** *l1* = *l2*
⟨*proof*⟩

**lemma** *all-eql*:
  **assumes** *TarskiLen A B l1* **and**
    *TarskiLen A B l2*
  **shows** *l1* = *l2*
  ⟨*proof*⟩

**lemma** *null-len*:
  **assumes** *TarskiLen A A la* **and**
    *TarskiLen B B lb*
  **shows** *la* = *lb*
  ⟨*proof*⟩

**lemma** *eqL-equivalence*:
  **assumes** *QCong la* **and**
    *QCong lb* **and**

$QCong\ lc$
**shows** $la = la \wedge (la = lb \longrightarrow lb = la) \wedge (la = lb \wedge lb = lc \longrightarrow la = lc)$
⟨*proof*⟩

**lemma** *ex-lg*:
$\exists\ l.\ (QCong\ l \wedge l\ A\ B)$
⟨*proof*⟩

**lemma** *lg-eql-lg*:
**assumes** $QCong\ l1$ **and**
$l1 = l2$
**shows** $QCong\ l2$
⟨*proof*⟩

**lemma** *ex-eqL*:
**assumes** $QCong\ l1$ **and**
$QCong\ l2$ **and**
$\exists\ A\ B.\ (l1\ A\ B \wedge l2\ A\ B)$
**shows** $l1 = l2$
⟨*proof*⟩

### 3.12.4 Part 3 : angles

**lemma** *ang-exists*:
**assumes** $A \neq B$ **and**
$C \neq B$
**shows** $\exists\ a.\ (QCongA\ a \wedge a\ A\ B\ C)$
⟨*proof*⟩

**lemma** *ex-points-eng*:
**assumes** $QCongA\ a$
**shows** $\exists\ A\ B\ C.\ (a\ A\ B\ C)$
⟨*proof*⟩

**lemma** *ang-conga*:
**assumes** $QCongA\ a$ **and**
$a\ A\ B\ C$ **and**
$a\ A'\ B'\ C'$
**shows** $A\ B\ C\ CongA\ A'\ B'\ C'$
⟨*proof*⟩

**lemma** *is-ang-conga*:
**assumes** $A\ B\ C\ Ang\ a$ **and**
$A'\ B'\ C'\ Ang\ a$
**shows** $A\ B\ C\ CongA\ A'\ B'\ C'$
⟨*proof*⟩

**lemma** *is-ang-conga-is-ang*:
**assumes** $A\ B\ C\ Ang\ a$ **and**
$A\ B\ C\ CongA\ A'\ B'\ C'$
**shows** $A'\ B'\ C'\ Ang\ a$
⟨*proof*⟩

**lemma** *not-conga-not-ang*:
**assumes** $QCongA\ a$ **and**
$\neg\ A\ B\ C\ CongA\ A'\ B'\ C'$ **and**
$a\ A\ B\ C$
**shows** $\neg\ a\ A'\ B'\ C'$
⟨*proof*⟩

**lemma** *not-conga-is-ang*:
**assumes** $\neg\ A\ B\ C\ CongA\ A'\ B'\ C'$ **and**
$A\ B\ C\ Ang\ a$
**shows** $\neg\ a\ A'\ B'\ C'$
⟨*proof*⟩

**lemma** *not-cong-is-ang1*:
  **assumes** $\neg$ *A B C CongA A' B' C'* **and**
    *A B C Ang a*
  **shows** $\neg$ *A' B' C' Ang a*
  ⟨*proof*⟩

**lemma** *ex-eqa*:
  **assumes** $\exists$ *A B C*.(*A B C Ang a1* $\wedge$ *A B C Ang a2*)
  **shows** *a1* = *a2*
⟨*proof*⟩

**lemma** *all-eqa*:
  **assumes** *A B C Ang a1* **and**
    *A B C Ang a2*
  **shows** *a1* = *a2*
  ⟨*proof*⟩

**lemma** *is-ang-distinct*:
  **assumes** *A B C Ang a*
  **shows** $A \neq B \wedge C \neq B$
  ⟨*proof*⟩

**lemma** *null-ang*:
  **assumes** *A B A Ang a1* **and**
    *C D C Ang a2*
  **shows** *a1* = *a2*
  ⟨*proof*⟩

**lemma** *flat-ang*:
  **assumes** *Bet A B C* **and**
    *Bet A' B' C'* **and**
    *A B C Ang a1* **and**
    *A' B' C' Ang a2*
  **shows** *a1* = *a2*
⟨*proof*⟩

**lemma** *ang-distinct*:
  **assumes** *QCongA a* **and**
    *a A B C*
  **shows** $A \neq B \wedge C \neq B$
⟨*proof*⟩

**lemma** *ex-ang*:
  **assumes** $B \neq A$ **and**
    $B \neq C$
  **shows** $\exists$ *a*. (*QCongA a* $\wedge$ *a A B C*)
  ⟨*proof*⟩

**lemma** *anga-exists*:
  **assumes** $A \neq B$ **and**
    $C \neq B$ **and**
    *Acute A B C*
  **shows** $\exists$ *a*. (*QCongAAcute a* $\wedge$ *a A B C*)
⟨*proof*⟩

**lemma** *anga-is-ang*:
  **assumes** *QCongAAcute a*
  **shows** *QCongA a*
⟨*proof*⟩

**lemma** *ex-points-anga*:
  **assumes** *QCongAAcute a*
  **shows** $\exists$ *A B C*. *a A B C*
  ⟨*proof*⟩

**lemma** *anga-conga*:

**assumes** *QCongAAcute a* **and**
*a A B C* **and**
*a A′ B′ C′*
**shows** *A B C CongA A′ B′ C′*
⟨*proof*⟩

**lemma** *is-anga-to-is-ang*:
**assumes** *A B C AngAcute a*
**shows** *A B C Ang a*
⟨*proof*⟩

**lemma** *is-anga-conga*:
**assumes** *A B C AngAcute a* **and**
*A′ B′ C′ AngAcute a*
**shows** *A B C CongA A′ B′ C′*
⟨*proof*⟩

**lemma** *is-anga-conga-is-anga*:
**assumes** *A B C AngAcute a* **and**
*A B C CongA A′ B′ C′*
**shows** *A′ B′ C′ AngAcute a*
⟨*proof*⟩

**lemma** *not-conga-is-anga*:
**assumes** ¬ *A B C CongA A′ B′ C′* **and**
*A B C AngAcute a*
**shows** ¬ *a A′ B′ C′*
⟨*proof*⟩

**lemma** *not-cong-is-anga1*:
**assumes** ¬ *A B C CongA A′ B′ C′* **and**
*A B C AngAcute a*
**shows** ¬ *A′ B′ C′ AngAcute a*
⟨*proof*⟩

**lemma** *ex-eqaa*:
**assumes** ∃ *A B C*. (*A B C AngAcute a1* ∧ *A B C AngAcute a2*)
**shows** *a1* = *a2*
⟨*proof*⟩

**lemma** *all-eqaa*:
**assumes** *A B C AngAcute a1* **and**
*A B C AngAcute a2*
**shows** *a1* = *a2*
⟨*proof*⟩

**lemma** *is-anga-distinct*:
**assumes** *A B C AngAcute a*
**shows** *A* ≠ *B* ∧ *C* ≠ *B*
⟨*proof*⟩

**lemma** *null-anga*:
**assumes** *A B A AngAcute a1* **and**
*C D C AngAcute a2*
**shows** *a1* = *a2*
⟨*proof*⟩

**lemma** *anga-distinct*:
**assumes** *QCongAAcute a* **and**
*a A B C*
**shows** *A* ≠ *B* ∧ *C* ≠ *B*
⟨*proof*⟩

**lemma** *out-is-len-eq*:
**assumes** *A Out B C* **and**
*TarskiLen A B l* **and**

*TarskiLen A C l*
**shows** $B = C$
⟨*proof*⟩

**lemma** *out-len-eq*:
**assumes** *QCong l* **and**
*A Out B C* **and**
*l A B* **and**
*l A C*
**shows** $B = C$ ⟨*proof*⟩

**lemma** *ex-anga*:
**assumes** *Acute A B C*
**shows** $\exists\ a.\ (QCongAAcute\ a \wedge a\ A\ B\ C)$
⟨*proof*⟩

**lemma** *not-null-ang-ang*:
**assumes** *QCongAnNull a*
**shows** *QCongA a*
⟨*proof*⟩

**lemma** *not-null-ang-def-equiv*:
$QCongAnNull\ a \longleftrightarrow (QCongA\ a \wedge (\exists\ A\ B\ C.\ (a\ A\ B\ C \wedge\ \neg\ B\ Out\ A\ C)))$
⟨*proof*⟩

**lemma** *not-flat-ang-def-equiv*:
$QCongAnFlat\ a \longleftrightarrow (QCongA\ a \wedge (\exists\ A\ B\ C.\ (a\ A\ B\ C \wedge \neg\ Bet\ A\ B\ C)))$
⟨*proof*⟩

**lemma** *ang-const*:
**assumes** *QCongA a* **and**
$A \neq B$
**shows** $\exists\ C.\ a\ A\ B\ C$
⟨*proof*⟩

**lemma** *ang-sym*:
**assumes** *QCongA a* **and**
*a A B C*
**shows** *a C B A*
⟨*proof*⟩

**lemma** *ang-not-null-lg*:
**assumes** *QCongA a* **and**
*QCong l* **and**
*a A B C* **and**
*l A B*
**shows** $\neg$ *QCongNull l*
⟨*proof*⟩

**lemma** *ang-distincts*:
**assumes** *QCongA a* **and**
*a A B C*
**shows** $A \neq B \wedge C \neq B$
⟨*proof*⟩

**lemma** *anga-sym*:
**assumes** *QCongAAcute a* **and**
*a A B C*
**shows** *a C B A*
⟨*proof*⟩

**lemma** *anga-not-null-lg*:
**assumes** *QCongAAcute a* **and**
*QCong l* **and**
*a A B C* **and**
*l A B*

**shows** ¬ *QCongNull l*
⟨*proof*⟩

**lemma** *anga-distincts*:
**assumes** *QCongAAcute a* **and**
*a A B C*
**shows** $A \neq B \wedge C \neq B$
⟨*proof*⟩

**lemma** *ang-const-o*:
**assumes** ¬ *Col A B P* **and**
*QCongA a* **and**
*QCongAnNull a* **and**
*QCongAnFlat a*
**shows** ∃ *C*. *a A B C* ∧ *A B OS C P*
⟨*proof*⟩

**lemma** *anga-const*:
**assumes** *QCongAAcute a* **and**
$A \neq B$
**shows** ∃ *C*. *a A B C*
⟨*proof*⟩

**lemma** *null-anga-null-angaP*:
*QCongANullAcute a* ⟷ *IsNullAngaP a*
⟨*proof*⟩

**lemma** *is-null-anga-out*:
**assumes**
*a A B C* **and**
*QCongANullAcute a*
**shows** *B Out A C*
⟨*proof*⟩

**lemma** *acute-not-bet*:
**assumes** *Acute A B C*
**shows** ¬ *Bet A B C*
⟨*proof*⟩

**lemma** *anga-acute*:
**assumes** *QCongAAcute a* **and**
*a A B C*
**shows** *Acute A B C*
⟨*proof*⟩

**lemma** *not-null-not-col*:
**assumes** *QCongAAcute a* **and**
¬ *QCongANullAcute a* **and**
*a A B C*
**shows** ¬ *Col A B C*
⟨*proof*⟩

**lemma** *ang-cong-ang*:
**assumes** *QCongA a* **and**
*a A B C* **and**
*A B C CongA* $A'$ $B'$ $C'$
**shows** *a* $A'$ $B'$ $C'$
⟨*proof*⟩

**lemma** *is-null-ang-out*:
**assumes**
*a A B C* **and**
*QCongANull a*
**shows** *B Out A C*
⟨*proof*⟩

**lemma** *out-null-ang*:
**assumes** *QCongA a* **and**
*a A B C* **and**
*B Out A C*
**shows** *QCongANull a*
⟨*proof*⟩

**lemma** *bet-flat-ang*:
**assumes** *QCongA a* **and**
*a A B C* **and**
*Bet A B C*
**shows** *AngFlat a*
⟨*proof*⟩

**lemma** *out-null-anga*:
**assumes** *QCongAAcute a* **and**
*a A B C* **and**
*B Out A C*
**shows** *QCongANullAcute a*
⟨*proof*⟩

**lemma** *anga-not-flat*:
**assumes** *QCongAAcute a*
**shows** *QCongAnFlat a*
⟨*proof*⟩

**lemma** *anga-const-o*:
**assumes** ¬ *Col A B P* **and**
¬ *QCongANullAcute a* **and**
*QCongAAcute a*
**shows** $\exists$ *C*. (*a A B C* $\wedge$ *A B OS C P*)
⟨*proof*⟩

**lemma** *anga-conga-anga*:
**assumes** *QCongAAcute a* **and**
*a A B C* **and**
*A B C CongA A′ B′ C′*
**shows** *a A′ B′ C′*
⟨*proof*⟩

**lemma** *anga-out-anga*:
**assumes** *QCongAAcute a* **and**
*a A B C* **and**
*B Out A A′* **and**
*B Out C C′*
**shows** *a A′ B C′*
⟨*proof*⟩

**lemma** *out-out-anga*:
**assumes** *QCongAAcute a* **and**
*B Out A C* **and**
*B′ Out A′ C′* **and**
*a A B C*
**shows** *a A′ B′ C′*
⟨*proof*⟩

**lemma** *is-null-all*:
**assumes** $A \neq B$ **and**
*QCongANullAcute a*
**shows** *a A B A*
⟨*proof*⟩

**lemma** *anga-col-out*:
**assumes** *QCongAAcute a* **and**
*a A B C* **and**
*Col A B C*

**shows** *B Out A C*
⟨*proof*⟩

**lemma** *ang-not-lg-null*:
**assumes** *QCong la* **and**
*QCong lc* **and**
*QCongA a* **and**
*la A B* **and**
*lc C B* **and**
*a A B C*
**shows** ¬ *QCongNull la* ∧ ¬ *QCongNull lc*
⟨*proof*⟩

**lemma** *anga-not-lg-null*:
**assumes**
*QCongAAcute a* **and**
*la A B* **and**
*lc C B* **and**
*a A B C*
**shows** ¬ *QCongNull la* ∧ ¬ *QCongNull lc*
⟨*proof*⟩

**lemma** *anga-col-null*:
**assumes** *QCongAAcute a* **and**
*a A B C* **and**
*Col A B C*
**shows** *B Out A C* ∧ *QCongANullAcute a*
⟨*proof*⟩

**lemma** *eqA-preserves-ang*:
**assumes** *QCongA a* **and**
$a = b$
**shows** *QCongA b*
⟨*proof*⟩

**lemma** *eqA-preserves-anga*:
**assumes** *QCongAAcute a* **and**

$a = b$
**shows** *QCongAAcute b*
⟨*proof*⟩

# 4 Some postulates of the parallels

**lemma** *euclid-5--original-euclid*:
**assumes** *Euclid5*
**shows** *EuclidSParallelPostulate*
⟨*proof*⟩

**lemma** *tarski-s-euclid-implies-euclid-5*:
**assumes** *TarskiSParallelPostulate*
**shows** *Euclid5*
⟨*proof*⟩

**lemma** *tarski-s-implies-euclid-s-parallel-postulate*:
**assumes** *TarskiSParallelPostulate*
**shows** *EuclidSParallelPostulate*
⟨*proof*⟩

**theorem** *tarski-s-euclid-implies-playfair-s-postulate*:
**assumes** *TarskiSParallelPostulate*
**shows** *PlayfairSPostulate*
⟨*proof*⟩

**end**

**end**

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