

# IsaGeoCoq: Partial porting of GeoCoq 2.4.0. Case studies: Tarski's postulate of parallels implies the 5th postulate of Euclid, the postulate of Playfair and the original postulate of Euclid.

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## Abstract

The GeoCoq library contains a formalization of geometry using the Coq proof assistant. It contains both proofs about the foundations of geometry [20, 15, 6, 16] and high-level proofs in the same style as in high-school. [1](Code Repository <https://github.com/GeoCoq/GeoCoq>).

Some theorems also inspired by [20] are also formalized with others ITP(Metamath, Mizar) or ATP [24, 25, 3, 23, 4, 2, 17, 5, 11, 19, 8, 9, 10].

We port a part of the GeoCoq 2.4.0 library within the Isabelle/Hol proof assistant: more precisely, the files Chap02.v to Chap13\_3.v, suma.v as well as the associated definitions and some useful files for the demonstration of certain parallel postulates.

While the demonstrations in Coq are written in procedural language [26], the transcript is done in declarative language Isar[18].

The synthetic approach of the demonstrations are directly inspired by those contained in GeoCoq. Some demonstrations are credited to G.E Martin («lemma bet\_le\_lt:» in Ch11\_angles.thy, proved by Martin as Theorem 18.17 in [14]) or Gupta H.N (Krippen Lemma, proved by Gupta in its PhD in 1965 as Theorem 3.45). (See [12]).

In this work, the proofs are not constructive. The sledgehammer tool being used to find some demonstrations.

The names of the lemmas and theorems used are kept as far as possible as well as the definitions. A different translation has been proposed when the name was already used in Isabel/Hol ("Len" is translated as "TarskiLen") or that characters were not allowed in Isabel/Hol ("anga" in Ch13\_angles.v is translated as "angaP"). For some definitions the highlighting of a variable has changed the order or the position of the variables (Midpoint, Out, Inter,...).

All the lemmas are valid in absolute/neutral space defined with Tarski's axioms.

It should be noted that T.J.M. Makarios [13] has begun some demonstrations of certain proposals mainly those corresponding to SST chapters 2 and 3. It uses a definition that does not quite coincide with the definition used in Geocoq and here. As an example, Makarios introduces the axiom A11 (Axiom of continuity) in the definition of the locale "Tarski\_absolute\_space".

Furthermore, the definition of the locale "TarskiAbsolute" [22, 21] is not not identical to the one defined in the "Tarski\_neutral\_dimensionless" class of GeoCoq. Indeed this one does not contain the axiom "upper\_dimension". In some cases particular, it is nevertheless to use the axiom "upper\_dimension". The addition of the word "\_2D" in the file indicates its presence.

In the last part, it is formalized that, in the neutral/absolute space, the axiom of the parallels of the system of Tarski implies the Playfair axiom, the 5th postulate of euclide and the postulate original from Euclid. These proofs, which are not constructive, are directly inspired by [12, 7].

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theory *Tarski-Neutral*

imports

*Main*

begin

## 1 Tarski's axiom system for neutral geometry

### 1.1 Tarski's axiom system for neutral geometry: dimensionless

locale *Tarski-neutral-dimensionless* =

fixes *Bet* :: 'p ⇒ 'p ⇒ 'p ⇒ bool

fixes *Cong* :: 'p ⇒ 'p ⇒ 'p ⇒ 'p ⇒ bool

assumes *cong-pseudo-reflexivity*: ∀ a b.

$Cong\ a\ b\ b\ a$

and *cong-inner-transitivity*: ∀ a b p q r s.

$Cong\ a\ b\ p\ q \wedge$

$Cong\ a\ b\ r\ s$

→

$Cong\ p\ q\ r\ s$

and *cong-identity*: ∀ a b c.

$Cong\ a\ b\ c\ c$

→

$a = b$

and *segment-construction*: ∀ a b c q.

$\exists x. (Bet\ q\ a\ x \wedge Cong\ a\ x\ b\ c)$

and *five-segment*: ∀ a b c a' b' c'.

$a \neq b \wedge$

$Bet\ a\ b\ c \wedge$

$Bet\ a'\ b'\ c' \wedge$

$Cong\ a\ b\ a'\ b' \wedge$

$Cong\ b\ c\ b'\ c' \wedge$

$Cong\ a\ d\ a'\ d' \wedge$

$Cong\ b\ d\ b'\ d'$

→

$Cong\ c\ d\ c'\ d'$

and *between-identity*: ∀ a b.

$Bet\ a\ b\ a$

→

$a = b$

and *inner-pasch*: ∀ a b c p q.

$Bet\ a\ p\ c \wedge$

$Bet\ b\ q\ c$

→

$(\exists x. Bet\ p\ x\ b \wedge Bet\ q\ x\ a)$

and *lower-dim*: ∃ a b c. (¬  $Bet\ a\ b\ c \wedge \neg Bet\ b\ c\ a \wedge \neg Bet\ c\ a\ b$ )

### 1.2 Tarski's axiom system for neutral geometry: 2D

locale *Tarski-2D* = *Tarski-neutral-dimensionless* +

assumes *upper-dim*: ∀ a b c p q.

$p \neq q \wedge$

$Cong\ a\ p\ a\ q \wedge$

$Cong\ b\ p\ b\ q \wedge$

$Cong\ c\ p\ c\ q$

→

$(Bet\ a\ b\ c \vee Bet\ b\ c\ a \vee Bet\ c\ a\ b)$

## 2 Definitions

### 2.1 Tarski's axiom system for neutral geometry: dimensionless

context *Tarski-neutral-dimensionless*

begin

#### 2.1.1 Congruence

definition *OFSC* ::

$[p, p, p, p, p, p, p, p] \Rightarrow \text{bool}$   
(*OFSC* - - - [99,99,99,99,99,99,99,99] 50)

where

$A B C D \text{ OFSC } A' B' C' D' \equiv$

$Bet A B C \wedge$   
 $Bet A' B' C' \wedge$   
 $Cong A B A' B' \wedge$   
 $Cong B C B' C' \wedge$   
 $Cong A D A' D' \wedge$   
 $Cong B D B' D'$

definition *Cong3* ::

$[p, p, p, p, p, p] \Rightarrow \text{bool}$   
(*Cong3* - - - [99,99,99,99,99,99] 50)

where

$A B C \text{ Cong3 } A' B' C' \equiv$

$Cong A B A' B' \wedge$   
 $Cong A C A' C' \wedge$   
 $Cong B C B' C'$

#### 2.1.2 Betweenness

definition *Col* ::

$[p, p, p] \Rightarrow \text{bool}$   
(*Col* - - - [99,99,99] 50)

where

$Col A B C \equiv$

$Bet A B C \vee Bet B C A \vee Bet C A B$

definition *Bet4* ::

$[p, p, p, p] \Rightarrow \text{bool}$   
(*Bet4* - - - [99,99,99,99] 50)

where

$Bet4 A1 A2 A3 A4 \equiv$

$Bet A1 A2 A3 \wedge$   
 $Bet A2 A3 A4 \wedge$   
 $Bet A1 A3 A4 \wedge$   
 $Bet A1 A2 A4$

definition *BetS* ::

$[p, p, p] \Rightarrow \text{bool}$  (*BetS* - - - [99,99,99] 50)

where

$BetS A B C \equiv$

$Bet A B C \wedge$   
 $A \neq B \wedge$   
 $B \neq C$

#### 2.1.3 Collinearity

definition *FSC* ::

$[p, p, p, p, p, p, p, p] \Rightarrow \text{bool}$   
(*FSC* - - - [99,99,99,99,99,99,99,99] 50)

**where**

$A B C D FSC A' B' C' D' \equiv$

$Col A B C \wedge$   
 $A B C Cong3 A' B' C' \wedge$   
 $Cong A D A' D' \wedge$   
 $Cong B D B' D'$

### 2.1.4 Congruence and Betweenness

**definition** *IFSC* ::

$[ 'p, 'p, 'p, 'p, 'p, 'p, 'p ] \Rightarrow bool$   
 $( - - - - IFSC - - - - [ 99, 99, 99, 99, 99, 99, 99 ] 50 )$

**where**

$A B C D IFSC A' B' C' D' \equiv$

$Bet A B C \wedge$   
 $Bet A' B' C' \wedge$   
 $Cong A C A' C' \wedge$   
 $Cong B C B' C' \wedge$   
 $Cong A D A' D' \wedge$   
 $Cong C D C' D'$

### 2.1.5 Between transitivity LE

**definition** *Le* ::

$[ 'p, 'p, 'p, 'p ] \Rightarrow bool ( - - Le - - [ 99, 99, 99, 99 ] 50 )$

**where**  $A B Le C D \equiv$

$\exists E. ( Bet C E D \wedge Cong A B C E )$

**definition** *Lt* ::

$[ 'p, 'p, 'p, 'p ] \Rightarrow bool ( - - Lt - - [ 99, 99, 99, 99 ] 50 )$

**where**  $A B Lt C D \equiv$

$A B Le C D \wedge \neg Cong A B C D$

**definition** *Ge* ::

$[ 'p, 'p, 'p, 'p ] \Rightarrow bool ( - - Ge - - [ 99, 99, 99, 99 ] 50 )$

**where**  $A B Ge C D \equiv$

$C D Le A B$

**definition** *Gt* ::

$[ 'p, 'p, 'p, 'p ] \Rightarrow bool ( - - Gt - - [ 99, 99, 99, 99 ] 50 )$

**where**  $A B Gt C D \equiv$

$C D Lt A B$

### 2.1.6 Out lines

**definition** *Out* ::

$[ 'p, 'p, 'p ] \Rightarrow bool ( - Out - - [ 99, 99, 99 ] 50 )$

**where**  $P Out A B \equiv$

$A \neq P \wedge$   
 $B \neq P \wedge$   
 $( Bet P A B \vee Bet P B A )$

### 2.1.7 Midpoint

**definition** *Midpoint* ::

$[ 'p, 'p, 'p ] \Rightarrow bool ( - Midpoint - - [ 99, 99, 99 ] 50 )$

**where**  $M Midpoint A B \equiv$

$Bet A M B \wedge$

$Cong\ A\ M\ M\ B$

### 2.1.8 Orthogonality

**definition**  $Per ::$

$[p, p, p] \Rightarrow bool\ (Per\ -\ -\ -\ [99,99,99]\ 50)$   
**where**  $Per\ A\ B\ C \equiv$

$\exists\ C'::'p.\ (B\ Midpoint\ C\ C' \wedge Cong\ A\ C\ A\ C')$

**definition**  $PerpAt ::$

$[p, p, p, p, p] \Rightarrow bool\ (-\ PerpAt\ -\ -\ -\ -\ [99,99,99,99,99]\ 50)$   
**where**  $X\ PerpAt\ A\ B\ C\ D \equiv$

$A \neq B \wedge$   
 $C \neq D \wedge$   
 $Col\ X\ A\ B \wedge$   
 $Col\ X\ C\ D \wedge$   
 $(\forall\ U\ V.\ ((Col\ U\ A\ B \wedge Col\ V\ C\ D) \longrightarrow Per\ U\ X\ V))$

**definition**  $Perp ::$

$[p, p, p, p] \Rightarrow bool\ (-\ -\ Perp\ -\ -\ [99,99,99,99]\ 50)$   
**where**  $A\ B\ Perp\ C\ D \equiv$

$\exists\ X::'p.\ X\ PerpAt\ A\ B\ C\ D$

### 2.1.9 Coplanar

**definition**  $Coplanar ::$

$[p, p, p, p] \Rightarrow bool\ (Coplanar\ -\ -\ -\ -\ [99,99,99,99]\ 50)$   
**where**  $Coplanar\ A\ B\ C\ D \equiv$

$\exists\ X.\ (Col\ A\ B\ X \wedge Col\ C\ D\ X) \vee$   
 $(Col\ A\ C\ X \wedge Col\ B\ D\ X) \vee$   
 $(Col\ A\ D\ X \wedge Col\ B\ C\ X)$

**definition**  $TS ::$

$[p, p, p, p, p] \Rightarrow bool\ (-\ -\ TS\ -\ -\ [99,99,99,99]\ 50)$   
**where**  $A\ B\ TS\ P\ Q \equiv$   
 $\neg\ Col\ P\ A\ B \wedge \neg\ Col\ Q\ A\ B \wedge (\exists\ T::'p.\ Col\ T\ A\ B \wedge Bet\ P\ T\ Q)$

**definition**  $ReflectL ::$

$[p, p, p, p, p] \Rightarrow bool\ (-\ -\ ReflectL\ -\ -\ [99,99,99,99]\ 50)$   
**where**  $P'\ P\ ReflectL\ A\ B \equiv$   
 $(\exists\ X.\ X\ Midpoint\ P\ P' \wedge Col\ A\ B\ X) \wedge (A\ B\ Perp\ P\ P' \vee P = P')$

**definition**  $Reflect ::$

$[p, p, p, p, p] \Rightarrow bool\ (-\ -\ Reflect\ -\ -\ [99,99,99,99]\ 50)$   
**where**  $P'\ P\ Reflect\ A\ B \equiv$   
 $(A \neq B \wedge P'\ P\ ReflectL\ A\ B) \vee (A = B \wedge A\ Midpoint\ P\ P')$

**definition**  $InAngle ::$

$[p, p, p, p, p] \Rightarrow bool\ (-\ InAngle\ -\ -\ -\ [99,99,99,99]\ 50)$   
**where**  $P\ InAngle\ A\ B\ C \equiv$   
 $A \neq B \wedge C \neq B \wedge P \neq B \wedge$   
 $(\exists\ X.\ Bet\ A\ X\ C \wedge (X = B \vee B\ Out\ X\ P))$

**definition**  $ParStrict::$

$[p, p, p, p, p] \Rightarrow bool\ (-\ -\ ParStrict\ -\ -\ [99,99,99,99]\ 50)$   
**where**  $A\ B\ ParStrict\ C\ D \equiv Coplanar\ A\ B\ C\ D \wedge \neg (\exists\ X.\ Col\ X\ A\ B \wedge Col\ X\ C\ D)$

**definition**  $Par::$

$[p, p, p, p, p] \Rightarrow bool\ (-\ -\ Par\ -\ -\ [99,99,99,99]\ 50)$   
**where**  $A\ B\ Par\ C\ D \equiv$   
 $A\ B\ ParStrict\ C\ D \vee (A \neq B \wedge C \neq D \wedge Col\ A\ C\ D \wedge Col\ B\ C\ D)$

**definition**  $Plg::$

$[p, p, p, p, p] \Rightarrow bool\ (Plg\ -\ -\ -\ -\ [99,99,99,99]\ 50)$

**where**  $Plg\ A\ B\ C\ D \equiv$   
 $(A \neq C \vee B \neq D) \wedge (\exists M. M\ Midpoint\ A\ C \wedge M\ Midpoint\ B\ D)$

**definition** *ParallelogramStrict*::

$[p, p, p, p] \Rightarrow bool\ (ParallelogramStrict\ -\ -\ -\ [99,99,99,99]\ 50)$   
**where**  $ParallelogramStrict\ A\ B\ A'\ B' \equiv$   
 $A\ A'\ TS\ B\ B' \wedge A\ B\ Par\ A'\ B' \wedge Cong\ A\ B\ A'\ B'$

**definition** *ParallelogramFlat*::

$[p, p, p, p] \Rightarrow bool\ (ParallelogramFlat\ -\ -\ -\ [99,99,99,99]\ 50)$   
**where**  $ParallelogramFlat\ A\ B\ A'\ B' \equiv$   
 $Col\ A\ B\ A' \wedge Col\ A\ B\ B' \wedge$   
 $Cong\ A\ B\ A'\ B' \wedge Cong\ A\ B'\ A'\ B \wedge$   
 $(A \neq A' \vee B \neq B')$

**definition** *Parallelogram*::

$[p, p, p, p] \Rightarrow bool\ (Parallelogram\ -\ -\ -\ [99,99,99,99]\ 50)$   
**where**  $Parallelogram\ A\ B\ A'\ B' \equiv$   
 $ParallelogramStrict\ A\ B\ A'\ B' \vee ParallelogramFlat\ A\ B\ A'\ B'$

**definition** *Rhombus*::

$[p, p, p, p] \Rightarrow bool\ (Rhombus\ -\ -\ -\ [99,99,99,99]\ 50)$   
**where**  $Rhombus\ A\ B\ C\ D \equiv Plg\ A\ B\ C\ D \wedge Cong\ A\ B\ B\ C$

**definition** *Rectangle*::

$[p, p, p, p] \Rightarrow bool\ (Rectangle\ -\ -\ -\ [99,99,99,99]\ 50)$   
**where**  $Rectangle\ A\ B\ C\ D \equiv Plg\ A\ B\ C\ D \wedge Cong\ A\ C\ B\ D$

**definition** *Square*::

$[p, p, p, p] \Rightarrow bool\ (Square\ -\ -\ -\ [99,99,99,99]\ 50)$   
**where**  $Square\ A\ B\ C\ D \equiv Rectangle\ A\ B\ C\ D \wedge Cong\ A\ B\ B\ C$

**definition** *Lambert*::

$[p, p, p, p] \Rightarrow bool\ (Lambert\ -\ -\ -\ [99,99,99,99]\ 50)$   
**where**  $Lambert\ A\ B\ C\ D \equiv$   
 $A \neq B \wedge B \neq C \wedge C \neq D \wedge$   
 $A \neq D \wedge Per\ B\ A\ D \wedge Per\ A\ D\ C \wedge Per\ A\ B\ C \wedge Coplanar\ A\ B\ C\ D$

## 2.1.10 Plane

**definition** *OS* ::

$[p, p, p, p] \Rightarrow bool\ (-\ -\ OS\ -\ -\ [99,99,99,99]\ 50)$   
**where**  $A\ B\ OS\ P\ Q \equiv$   
 $\exists R::p. A\ B\ TS\ P\ R \wedge A\ B\ TS\ Q\ R$

**definition** *TSP* ::

$[p, p, p, p, p] \Rightarrow bool\ (-\ -\ -\ TSP\ -\ -\ [99,99,99,99,99]\ 50)$   
**where**  $A\ B\ C\ TSP\ P\ Q \equiv$   
 $(\neg\ Coplanar\ A\ B\ C\ P) \wedge (\neg\ Coplanar\ A\ B\ C\ Q) \wedge$   
 $(\exists T. Coplanar\ A\ B\ C\ T \wedge Bet\ P\ T\ Q)$

**definition** *OSP* ::

$[p, p, p, p, p] \Rightarrow bool\ (-\ -\ -\ OSP\ -\ -\ [99,99,99,99,99]\ 50)$   
**where**  $A\ B\ C\ OSP\ P\ Q \equiv$   
 $\exists R. ((A\ B\ C\ TSP\ P\ R) \wedge (A\ B\ C\ TSP\ Q\ R))$

**definition** *Saccheri*::

$[p, p, p, p] \Rightarrow bool\ (Saccheri\ -\ -\ -\ [99,99,99,99]\ 50)$   
**where**  $Saccheri\ A\ B\ C\ D \equiv$   
 $Per\ B\ A\ D \wedge Per\ A\ D\ C \wedge Cong\ A\ B\ C\ D \wedge A\ D\ OS\ B\ C$

## 2.1.11 Line reflexivity 2D

**definition** *ReflectLAt* ::

$[p, p, p, p, p] \Rightarrow bool\ (-\ ReflectLAt\ -\ -\ -\ [99,99,99,99,99]\ 50)$   
**where**  $M\ ReflectLAt\ P'\ P\ A\ B \equiv$   
 $(M\ Midpoint\ P\ P' \wedge Col\ A\ B\ M) \wedge (A\ B\ Perp\ P\ P' \vee P = P')$

**definition** *ReflectAt* ::  
 $[p, p, p, p, p] \Rightarrow \text{bool} (- \text{ReflectAt} - - - [99, 99, 99, 99, 99] 50)$   
**where**  $M \text{ReflectAt} P' P A B \equiv$   
 $(A \neq B \wedge M \text{ReflectLAt} P' P A B) \vee (A = B \wedge A = M \wedge M \text{Midpoint} P P')$

### 2.1.12 Line reflexivity

**definition** *upper-dim-axiom* ::  
 $\text{bool} (\text{UpperDimAxiom} [] 50)$   
**where**  
 $\text{upper-dim-axiom} \equiv$

$\forall A B C P Q.$   
 $P \neq Q \wedge$   
 $\text{Cong} A P A Q \wedge$   
 $\text{Cong} B P B Q \wedge$   
 $\text{Cong} C P C Q$   
 $\longrightarrow$   
 $(\text{Bet} A B C \vee \text{Bet} B C A \vee \text{Bet} C A B)$

**definition** *all-coplanar-axiom* ::  
 $\text{bool} (\text{AllCoplanarAxiom} [] 50)$   
**where**  
 $\text{AllCoplanarAxiom} \equiv$

$\forall A B C P Q.$   
 $P \neq Q \wedge$   
 $\text{Cong} A P A Q \wedge$   
 $\text{Cong} B P B Q \wedge$   
 $\text{Cong} C P C Q$   
 $\longrightarrow$   
 $(\text{Bet} A B C \vee \text{Bet} B C A \vee \text{Bet} C A B)$

### 2.1.13 Angles

**definition** *CongA* ::  
 $[p, p, p, p, p, p] \Rightarrow \text{bool} (- - - \text{CongA} - - - [99, 99, 99, 99, 99, 99] 50)$   
**where**  $A B C \text{CongA} D E F \equiv$   
 $A \neq B \wedge C \neq B \wedge D \neq E \wedge F \neq E \wedge$   
 $(\exists A' C' D' F'. \text{Bet} B A A' \wedge \text{Cong} A A' E D \wedge$   
 $\text{Bet} B C C' \wedge \text{Cong} C C' E F \wedge$   
 $\text{Bet} E D D' \wedge \text{Cong} D D' B A \wedge$   
 $\text{Bet} E F F' \wedge \text{Cong} F F' B C \wedge$   
 $\text{Cong} A' C' D' F')$

**definition** *LeA* ::  
 $[p, p, p, p, p, p] \Rightarrow \text{bool} (- - - \text{LeA} - - - [99, 99, 99, 99, 99, 99] 50)$   
**where**  $A B C \text{LeA} D E F \equiv$   
 $\exists P. (P \text{InAngle} D E F \wedge A B C \text{CongA} D E P)$

**definition** *LtA* ::  
 $[p, p, p, p, p, p] \Rightarrow \text{bool} (- - - \text{LtA} - - - [99, 99, 99, 99, 99, 99] 50)$   
**where**  $A B C \text{LtA} D E F \equiv A B C \text{LeA} D E F \wedge \neg A B C \text{CongA} D E F$

**definition** *GtA* ::  
 $[p, p, p, p, p, p] \Rightarrow \text{bool} (- - - \text{GtA} - - - [99, 99, 99, 99, 99, 99] 50)$   
**where**  $A B C \text{GtA} D E F \equiv D E F \text{LtA} A B C$

**definition** *Acute* ::  
 $[p, p, p] \Rightarrow \text{bool} (\text{Acute} - - - [99, 99, 99] 50)$   
**where**  $\text{Acute} A B C \equiv$   
 $\exists A' B' C'. (\text{Per} A' B' C' \wedge A B C \text{LtA} A' B' C')$

**definition** *Obtuse* ::  
 $[p, p, p] \Rightarrow \text{bool} (\text{Obtuse} - - - [99, 99, 99] 50)$   
**where**  $\text{Obtuse} A B C \equiv$

$\exists A' B' C'. (Per A' B' C' \wedge A' B' C' LtA A B C)$

**definition** *OrthAt* ::

$[p, p, p, p, p, p] \Rightarrow bool (- OrthAt - - - - [99,99,99,99,99,99] 50)$   
**where**  $X OrthAt A B C U V \equiv$   
 $\neg Col A B C \wedge U \neq V \wedge Coplanar A B C X \wedge Col U V X \wedge$   
 $(\forall P Q. (Coplanar A B C P \wedge Col U V Q) \longrightarrow Per P X Q)$

**definition** *Orth* ::

$[p, p, p, p, p] \Rightarrow bool (- - - Orth - - [99,99,99,99,99] 50)$   
**where**  $A B C Orth U V \equiv \exists X. X OrthAt A B C U V$

**definition** *SuppA* ::

$[p, p, p, p, p, p] \Rightarrow bool$   
 $(- - - SuppA - - - [99,99,99,99,99,99] 50)$   
**where**  
 $A B C SuppA D E F \equiv$   
 $A \neq B \wedge (\exists A'. Bet A B A' \wedge D E F CongA C B A')$

### 2.1.14 Sum of angles

**definition** *SumA* ::

$[p, p, p, p, p, p, p, p] \Rightarrow bool (- - - - - SumA - - - [99,99,99,99,99,99,99,99] 50)$   
**where**  
 $A B C D E F SumA G H I \equiv$

$\exists J. (C B J CongA D E F \wedge \neg B C OS A J \wedge Coplanar A B C J \wedge A B J CongA G H I)$

**definition** *TriSumA* ::

$[p, p, p, p, p, p] \Rightarrow bool (- - - TriSumA - - - [99,99,99,99,99,99] 50)$   
**where**  
 $A B C TriSumA D E F \equiv$

$\exists G H I. (A B C B C A SumA G H I \wedge G H I C A B SumA D E F)$

**definition** *SAMS* ::

$[p, p, p, p, p, p] \Rightarrow bool (SAMS - - - - - [99,99,99,99,99,99] 50)$   
**where**  
 $SAMS A B C D E F \equiv$

$(A \neq B \wedge$   
 $(E Out D F \vee \neg Bet A B C)) \wedge$   
 $(\exists J. (C B J CongA D E F \wedge \neg (B C OS A J) \wedge \neg (A B TS C J) \wedge Coplanar A B C J))$

### 2.1.15 Parallelism

**definition** *Inter* ::

$[p, p, p, p, p, p] \Rightarrow bool (- Inter - - - - [99,99,99,99,99] 50)$   
**where**  $X Inter A1 A2 B1 B2 \equiv$

$B1 \neq B2 \wedge$   
 $(\exists P::'p. (Col P B1 B2 \wedge \neg Col P A1 A2)) \wedge$   
 $Col A1 A2 X \wedge Col B1 B2 X$

### 2.1.16 Perpendicularity

**definition** *Perp2* ::

$[p, p, p, p, p, p] \Rightarrow bool (- Perp2 - - - - [99,99,99,99,99] 50)$   
**where**  
 $P Perp2 A B C D \equiv$

$\exists X Y. (Col P X Y \wedge X Y Perp A B \wedge X Y Perp C D)$

### 2.1.17 Lentgh

**definition** *QCong*::

$([p, p] \Rightarrow bool) \Rightarrow bool (QCong - [99] 50)$

**where**

$QCong\ l \equiv$

$\exists A\ B. (\forall X\ Y. (Cong\ A\ B\ X\ Y \longleftrightarrow l\ X\ Y))$

**definition** *TarskiLen*::

$['p, 'p, (['p, 'p] \Rightarrow bool)] \Rightarrow bool\ (TarskiLen\ -\ -\ -\ [99, 99, 99]\ 50)$

**where**

$TarskiLen\ A\ B\ l \equiv$

$QCong\ l \wedge l\ A\ B$

**definition** *QCongNull* ::

$(['p, 'p] \Rightarrow bool) \Rightarrow bool\ (QCongNull\ -\ [99]\ 50)$

**where**

$QCongNull\ l \equiv$

$QCong\ l \wedge (\exists A. l\ A\ A)$

### 2.1.18 Equivalence Class of Angles

**definition** *QCongA* ::

$(['p, 'p, 'p] \Rightarrow bool) \Rightarrow bool\ (QCongA\ -\ [99]\ 50)$

**where**

$QCongA\ a \equiv$

$\exists A\ B\ C. (A \neq B \wedge C \neq B \wedge (\forall X\ Y\ Z. A\ B\ C\ CongA\ X\ Y\ Z \longleftrightarrow a\ X\ Y\ Z))$

**definition** *Ang* ::

$['p, 'p, 'p, (['p, 'p, 'p] \Rightarrow bool)] \Rightarrow bool\ (-\ -\ -\ Ang\ -\ [99, 99, 99, 99]\ 50)$

**where**

$A\ B\ C\ Ang\ a \equiv$

$QCongA\ a \wedge$

$a\ A\ B\ C$

**definition** *QCongAAcute* ::

$(['p, 'p, 'p] \Rightarrow bool) \Rightarrow bool\ (QCongAAcute\ -\ [99]\ 50)$

**where**

$QCongAAcute\ a \equiv$

$\exists A\ B\ C. (Acute\ A\ B\ C \wedge (\forall X\ Y\ Z. (A\ B\ C\ CongA\ X\ Y\ Z \longleftrightarrow a\ X\ Y\ Z)))$

**definition** *AngAcute* ::

$['p, 'p, 'p, (['p, 'p, 'p] \Rightarrow bool)] \Rightarrow bool\ (-\ -\ -\ AngAcute\ -\ [99, 99, 99, 99]\ 50)$

**where**

$A\ B\ C\ AngAcute\ a \equiv$

$((QCongAAcute\ a) \wedge (a\ A\ B\ C))$

**definition** *QCongANullAcute* ::

$(['p, 'p, 'p] \Rightarrow bool) \Rightarrow bool\ (QCongANullAcute\ -\ [99]\ 50)$

**where**

$QCongANullAcute\ a \equiv$

$QCongAAcute\ a \wedge$

$(\forall A\ B\ C. (a\ A\ B\ C \longrightarrow B\ Out\ A\ C))$

**definition** *QCongAnNull* ::

$(['p, 'p, 'p] \Rightarrow bool) \Rightarrow bool\ (QCongAnNull\ -\ [99]\ 50)$

**where**

$QCongAnNull\ a \equiv$

$QCongA\ a \wedge$

$(\forall A\ B\ C. (a\ A\ B\ C \longrightarrow \neg B\ Out\ A\ C))$

**definition** *QCongAnFlat* ::  
 ( $[p, p, p] \Rightarrow \text{bool}$ )  $\Rightarrow$  *bool* (*QCongAnFlat* - [99] 50)  
**where**  
*QCongAnFlat* *a*  $\equiv$   
  
*QCongA* *a*  $\wedge$   
 $(\forall A B C. (a A B C \longrightarrow \neg \text{Bet } A B C))$

**definition** *IsNullAngaP* ::  
 ( $[p, p, p] \Rightarrow \text{bool}$ )  $\Rightarrow$  *bool* (*IsNullAngaP* - [99] 50)  
**where**  
*IsNullAngaP* *a*  $\equiv$   
  
*QCongAAcute* *a*  $\wedge$   
 $(\exists A B C. (a A B C \wedge B \text{ Out } A C))$

**definition** *QCongANull* ::  
 ( $[p, p, p] \Rightarrow \text{bool}$ )  $\Rightarrow$  *bool* (*QCongANull* - [99] 50)  
**where**  
*QCongANull* *a*  $\equiv$   
  
*QCongA* *a*  $\wedge$   
 $(\forall A B C. (a A B C \longrightarrow B \text{ Out } A C))$

**definition** *AngFlat* ::  
 ( $[p, p, p] \Rightarrow \text{bool}$ )  $\Rightarrow$  *bool* (*AngFlat* - [99] 50)  
**where**  
*AngFlat* *a*  $\equiv$   
  
*QCongA* *a*  $\wedge$   
 $(\forall A B C. (a A B C \longrightarrow \text{Bet } A B C))$

## 2.2 Parallel's definition Postulate

**definition** *tarski-s-parallel-postulate* ::  
*bool*  
 (*TarskiSParallelPostulate*)  
**where**  
*tarski-s-parallel-postulate*  $\equiv$   
 $\forall A B C D T. (\text{Bet } A D T \wedge \text{Bet } B D C \wedge A \neq D) \longrightarrow$   
 $(\exists X Y. \text{Bet } A B X \wedge \text{Bet } A C Y \wedge \text{Bet } X T Y)$

**definition** *euclid-5* ::  
*bool* (*Euclid5*)  
**where**  
*euclid-5*  $\equiv$   
  
 $\forall P Q R S T U.$   
 $(\text{Bet } S P T Q \wedge$   
 $\text{Bet } S R T S \wedge$   
 $\text{Bet } S Q U R \wedge$   
 $\neg \text{Col } P Q S \wedge$   
 $\text{Cong } P T Q T \wedge$   
 $\text{Cong } R T S T)$   
 $\longrightarrow$   
 $(\exists I. \text{Bet } S S Q I \wedge \text{Bet } S P U I)$

**definition** *euclid-s-parallel-postulate* ::  
*bool* (*EuclidSParallelPostulate*)  
**where**  
*euclid-s-parallel-postulate*  $\equiv$   
  
 $\forall A B C D P Q R.$   
 $(B C O S A D \wedge$   
 $S A M S A B C B C D \wedge$   
 $A B C B C D \text{ Sum } A P Q R \wedge$

$\neg \text{Bet } P \ Q \ R)$   
 $\longrightarrow$   
 $(\exists Y. B \text{ Out } A \ Y \wedge C \text{ Out } D \ Y)$

**definition** *playfair-s-postulate* ::

*bool*  
*(PlayfairSPostulate)*

**where**

*playfair-s-postulate*  $\equiv$

$\forall A1 \ A2 \ B1 \ B2 \ C1 \ C2 \ P.$

$(A1 \ A2 \ \text{Par} \ B1 \ B2 \ \wedge$

$\text{Col} \ P \ B1 \ B2 \ \wedge$

$A1 \ A2 \ \text{Par} \ C1 \ C2 \ \wedge$

$\text{Col} \ P \ C1 \ C2)$

$\longrightarrow$

$(\text{Col} \ C1 \ B1 \ B2 \ \wedge \ \text{Col} \ C2 \ B1 \ B2)$

### 3 Propositions

#### 3.1 Congruence properties

**lemma** *cong-reflexivity*:

**shows** *Cong*  $A \ B \ A \ B$

*<proof>*

**lemma** *cong-symmetry*:

**assumes** *Cong*  $A \ B \ C \ D$

**shows** *Cong*  $C \ D \ A \ B$

*<proof>*

**lemma** *cong-transitivity*:

**assumes** *Cong*  $A \ B \ C \ D$  **and** *Cong*  $C \ D \ E \ F$

**shows** *Cong*  $A \ B \ E \ F$

*<proof>*

**lemma** *cong-left-commutativity*:

**assumes** *Cong*  $A \ B \ C \ D$

**shows** *Cong*  $B \ A \ C \ D$

*<proof>*

**lemma** *cong-right-commutativity*:

**assumes** *Cong*  $A \ B \ C \ D$

**shows** *Cong*  $A \ B \ D \ C$

*<proof>*

**lemma** *cong-3421*:

**assumes** *Cong*  $A \ B \ C \ D$

**shows** *Cong*  $C \ D \ B \ A$

*<proof>*

**lemma** *cong-4312*:

**assumes** *Cong*  $A \ B \ C \ D$

**shows** *Cong*  $D \ C \ A \ B$

*<proof>*

**lemma** *cong-4321*:

**assumes** *Cong*  $A \ B \ C \ D$

**shows** *Cong*  $D \ C \ B \ A$

*<proof>*

**lemma** *cong-trivial-identity*:

**shows** *Cong*  $A \ A \ B \ B$

*<proof>*

**lemma** *cong-reverse-identity*:

**assumes**  $Cong\ A\ A\ C\ D$   
**shows**  $C = D$   
 $\langle proof \rangle$

**lemma** *cong-commutativity*:  
**assumes**  $Cong\ A\ B\ C\ D$   
**shows**  $Cong\ B\ A\ D\ C$   
 $\langle proof \rangle$

**lemma** *not-cong-2134*:  
**assumes**  $\neg\ Cong\ A\ B\ C\ D$   
**shows**  $\neg\ Cong\ B\ A\ C\ D$   
 $\langle proof \rangle$

**lemma** *not-cong-1243*:  
**assumes**  $\neg\ Cong\ A\ B\ C\ D$   
**shows**  $\neg\ Cong\ A\ B\ D\ C$   
 $\langle proof \rangle$

**lemma** *not-cong-2143*:  
**assumes**  $\neg\ Cong\ A\ B\ C\ D$   
**shows**  $\neg\ Cong\ B\ A\ D\ C$   
 $\langle proof \rangle$

**lemma** *not-cong-3412*:  
**assumes**  $\neg\ Cong\ A\ B\ C\ D$   
**shows**  $\neg\ Cong\ C\ D\ A\ B$   
 $\langle proof \rangle$

**lemma** *not-cong-4312*:  
**assumes**  $\neg\ Cong\ A\ B\ C\ D$   
**shows**  $\neg\ Cong\ D\ C\ A\ B$   
 $\langle proof \rangle$

**lemma** *not-cong-3421*:  
**assumes**  $\neg\ Cong\ A\ B\ C\ D$   
**shows**  $\neg\ Cong\ C\ D\ B\ A$   
 $\langle proof \rangle$

**lemma** *not-cong-4321*:  
**assumes**  $\neg\ Cong\ A\ B\ C\ D$   
**shows**  $\neg\ Cong\ D\ C\ B\ A$   
 $\langle proof \rangle$

**lemma** *five-segment-with-def*:  
**assumes**  $A\ B\ C\ D\ OFSC\ A'\ B'\ C'\ D'$  **and**  $A \neq B$   
**shows**  $Cong\ C\ D\ C'\ D'$   
 $\langle proof \rangle$

**lemma** *cong-diff*:  
**assumes**  $A \neq B$  **and**  $Cong\ A\ B\ C\ D$   
**shows**  $C \neq D$   
 $\langle proof \rangle$

**lemma** *cong-diff-2*:  
**assumes**  $B \neq A$  **and**  $Cong\ A\ B\ C\ D$   
**shows**  $C \neq D$   
 $\langle proof \rangle$

**lemma** *cong-diff-3*:  
**assumes**  $C \neq D$  **and**  $Cong\ A\ B\ C\ D$   
**shows**  $A \neq B$   
 $\langle proof \rangle$

**lemma** *cong-diff-4*:  
**assumes**  $D \neq C$  **and**  $Cong\ A\ B\ C\ D$

**shows**  $A \neq B$   
*<proof>*

**lemma** *cong-3-sym*:  
**assumes**  $A B C \text{ Cong3 } A' B' C'$   
**shows**  $A' B' C' \text{ Cong3 } A B C$   
*<proof>*

**lemma** *cong-3-swap*:  
**assumes**  $A B C \text{ Cong3 } A' B' C'$   
**shows**  $B A C \text{ Cong3 } B' A' C'$   
*<proof>*

**lemma** *cong-3-swap-2*:  
**assumes**  $A B C \text{ Cong3 } A' B' C'$   
**shows**  $A C B \text{ Cong3 } A' C' B'$   
*<proof>*

**lemma** *cong3-transitivity*:  
**assumes**  $A0 B0 C0 \text{ Cong3 } A1 B1 C1$  **and**  
 $A1 B1 C1 \text{ Cong3 } A2 B2 C2$   
**shows**  $A0 B0 C0 \text{ Cong3 } A2 B2 C2$   
*<proof>*

**lemma** *eq-dec-points*:  
**shows**  $A = B \vee \neg A = B$   
*<proof>*

**lemma** *distinct*:  
**assumes**  $P \neq Q$   
**shows**  $R \neq P \vee R \neq Q$   
*<proof>*

**lemma** *l2-11*:  
**assumes**  $Bet A B C$  **and**  
 $Bet A' B' C'$  **and**  
 $Cong A B A' B'$  **and**  
 $Cong B C B' C'$   
**shows**  $Cong A C A' C'$   
*<proof>*

**lemma** *bet-cong3*:  
**assumes**  $Bet A B C$  **and**  
 $Cong A B A' B'$   
**shows**  $\exists C'. A B C \text{ Cong3 } A' B' C'$   
*<proof>*

**lemma** *construction-uniqueness*:  
**assumes**  $Q \neq A$  **and**  
 $Bet Q A X$  **and**  
 $Cong A X B C$  **and**  
 $Bet Q A Y$  **and**  
 $Cong A Y B C$   
**shows**  $X = Y$   
*<proof>*

**lemma** *Cong-cases*:  
**assumes**  $Cong A B C D \vee Cong A B D C \vee Cong B A C D \vee Cong B A D C \vee Cong C D A B \vee Cong C D B A$   
 $\vee Cong D C A B \vee Cong D C B A$   
**shows**  $Cong A B C D$   
*<proof>*

**lemma** *Cong-perm* :  
**assumes**  $Cong A B C D$   
**shows**  $Cong A B C D \wedge Cong A B D C \wedge Cong B A C D \wedge Cong B A D C \wedge Cong C D A B \wedge Cong C D B A \wedge$   
 $Cong D C A B \wedge Cong D C B A$

*<proof>*

### 3.2 Betweenness properties

**lemma** *bet-col:*

**assumes**  $Bet\ A\ B\ C$

**shows**  $Col\ A\ B\ C$

*<proof>*

**lemma** *between-trivial:*

**shows**  $Bet\ A\ B\ B$

*<proof>*

**lemma** *between-symmetry:*

**assumes**  $Bet\ A\ B\ C$

**shows**  $Bet\ C\ B\ A$

*<proof>*

**lemma** *Bet-cases:*

**assumes**  $Bet\ A\ B\ C \vee Bet\ C\ B\ A$

**shows**  $Bet\ A\ B\ C$

*<proof>*

**lemma** *Bet-perm:*

**assumes**  $Bet\ A\ B\ C$

**shows**  $Bet\ A\ B\ C \wedge Bet\ C\ B\ A$

*<proof>*

**lemma** *between-trivial2:*

**shows**  $Bet\ A\ A\ B$

*<proof>*

**lemma** *between-equality:*

**assumes**  $Bet\ A\ B\ C$  and  $Bet\ B\ A\ C$

**shows**  $A = B$

*<proof>*

**lemma** *between-equality-2:*

**assumes**  $Bet\ A\ B\ C$  and

$Bet\ A\ C\ B$

**shows**  $B = C$

*<proof>*

**lemma** *between-exchange3:*

**assumes**  $Bet\ A\ B\ C$  and

$Bet\ A\ C\ D$

**shows**  $Bet\ B\ C\ D$

*<proof>*

**lemma** *bet-neq12--neq:*

**assumes**  $Bet\ A\ B\ C$  and

$A \neq B$

**shows**  $A \neq C$

*<proof>*

**lemma** *bet-neq21--neq:*

**assumes**  $Bet\ A\ B\ C$  and

$B \neq A$

**shows**  $A \neq C$

*<proof>*

**lemma** *bet-neq23--neq:*

**assumes**  $Bet\ A\ B\ C$  and

$B \neq C$

**shows**  $A \neq C$

*<proof>*

**lemma** *bet-neq32--neq*:  
**assumes**  $Bet\ A\ B\ C$  **and**  
 $C \neq B$   
**shows**  $A \neq C$   
 $\langle proof \rangle$

**lemma** *not-bet-distincts*:  
**assumes**  $\neg\ Bet\ A\ B\ C$   
**shows**  $A \neq B \wedge B \neq C$   
 $\langle proof \rangle$

**lemma** *between-inner-transitivity*:  
**assumes**  $Bet\ A\ B\ D$  **and**  
 $Bet\ B\ C\ D$   
**shows**  $Bet\ A\ B\ C$   
 $\langle proof \rangle$

**lemma** *outer-transitivity-between2*:  
**assumes**  $Bet\ A\ B\ C$  **and**  
 $Bet\ B\ C\ D$  **and**  
 $B \neq C$   
**shows**  $Bet\ A\ C\ D$   
 $\langle proof \rangle$

**lemma** *between-exchange2*:  
**assumes**  $Bet\ A\ B\ D$  **and**  
 $Bet\ B\ C\ D$   
**shows**  $Bet\ A\ C\ D$   
 $\langle proof \rangle$

**lemma** *outer-transitivity-between*:  
**assumes**  $Bet\ A\ B\ C$  **and**  
 $Bet\ B\ C\ D$  **and**  
 $B \neq C$   
**shows**  $Bet\ A\ B\ D$   
 $\langle proof \rangle$

**lemma** *between-exchange4*:  
**assumes**  $Bet\ A\ B\ C$  **and**  
 $Bet\ A\ C\ D$   
**shows**  $Bet\ A\ B\ D$   
 $\langle proof \rangle$

**lemma** *l3-9-4*:  
**assumes**  $Bet4\ A1\ A2\ A3\ A4$   
**shows**  $Bet4\ A4\ A3\ A2\ A1$   
 $\langle proof \rangle$

**lemma** *l3-17*:  
**assumes**  $Bet\ A\ B\ C$  **and**  
 $Bet\ A'\ B'\ C$  **and**  
 $Bet\ A\ P\ A'$   
**shows**  $(\exists\ Q. Bet\ P\ Q\ C \wedge Bet\ B\ Q\ B')$   
 $\langle proof \rangle$

**lemma** *lower-dim-ex*:  
 $\exists\ A\ B\ C. \neg (Bet\ A\ B\ C \vee Bet\ B\ C\ A \vee Bet\ C\ A\ B)$   
 $\langle proof \rangle$

**lemma** *two-distinct-points*:  
 $\exists\ X::'p. \exists\ Y::'p. X \neq Y$   
 $\langle proof \rangle$

**lemma** *point-construction-different*:  
 $\exists\ C. Bet\ A\ B\ C \wedge B \neq C$

$\langle proof \rangle$

**lemma** *another-point*:

$\exists B: \neg p. A \neq B$

$\langle proof \rangle$

**lemma** *Cong-stability*:

**assumes**  $\neg \neg Cong A B C D$

**shows**  $Cong A B C D$

$\langle proof \rangle$

**lemma** *l2-11-b*:

**assumes**  $Bet A B C$  **and**

$Bet A' B' C'$  **and**

$Cong A B A' B'$  **and**

$Cong B C B' C'$

**shows**  $Cong A C A' C'$

$\langle proof \rangle$

**lemma** *cong-dec-eq-dec-b*:

**assumes**  $\neg A \neq B$

**shows**  $A = B$

$\langle proof \rangle$

**lemma** *BetSEq*:

**assumes**  $BetS A B C$

**shows**  $Bet A B C \wedge A \neq B \wedge A \neq C \wedge B \neq C$

$\langle proof \rangle$

### 3.3 Collinearity

#### 3.3.1 Collinearity and betweenness

**lemma** *l4-2*:

**assumes**  $A B C D$  *IFSC*  $A' B' C' D'$

**shows**  $Cong B D B' D'$

$\langle proof \rangle$

**lemma** *l4-3*:

**assumes**  $Bet A B C$  **and**

$Bet A' B' C'$  **and**

$Cong A C A' C'$

**and**  $Cong B C B' C'$

**shows**  $Cong A B A' B'$

$\langle proof \rangle$

**lemma** *l4-3-1*:

**assumes**  $Bet A B C$  **and**

$Bet A' B' C'$  **and**

$Cong A B A' B'$  **and**

$Cong A C A' C'$

**shows**  $Cong B C B' C'$

$\langle proof \rangle$

**lemma** *l4-5*:

**assumes**  $Bet A B C$  **and**

$Cong A C A' C'$

**shows**  $\exists B'. (Bet A' B' C' \wedge A B C Cong3 A' B' C')$

$\langle proof \rangle$

**lemma** *l4-6*:

**assumes**  $Bet A B C$  **and**

$A B C Cong3 A' B' C'$

**shows**  $Bet A' B' C'$

$\langle proof \rangle$

**lemma** *cong3-bet-eq*:  
**assumes**  $Bet\ A\ B\ C$  **and**  
 $A\ B\ C\ Cong3\ A\ X\ C$   
**shows**  $X = B$   
 $\langle proof \rangle$

### 3.3.2 Collinearity

**lemma** *col-permutation-1*:  
**assumes**  $Col\ A\ B\ C$   
**shows**  $Col\ B\ C\ A$   
 $\langle proof \rangle$

**lemma** *col-permutation-2*:  
**assumes**  $Col\ A\ B\ C$   
**shows**  $Col\ C\ A\ B$   
 $\langle proof \rangle$

**lemma** *col-permutation-3*:  
**assumes**  $Col\ A\ B\ C$   
**shows**  $Col\ C\ B\ A$   
 $\langle proof \rangle$

**lemma** *col-permutation-4*:  
**assumes**  $Col\ A\ B\ C$   
**shows**  $Col\ B\ A\ C$   
 $\langle proof \rangle$

**lemma** *col-permutation-5*:  
**assumes**  $Col\ A\ B\ C$   
**shows**  $Col\ A\ C\ B$   
 $\langle proof \rangle$

**lemma** *not-col-permutation-1*:  
**assumes**  $\neg\ Col\ A\ B\ C$   
**shows**  $\neg\ Col\ B\ C\ A$   
 $\langle proof \rangle$

**lemma** *not-col-permutation-2*:  
**assumes**  $\sim\ Col\ A\ B\ C$   
**shows**  $\sim\ Col\ C\ A\ B$   
 $\langle proof \rangle$

**lemma** *not-col-permutation-3*:  
**assumes**  $\neg\ Col\ A\ B\ C$   
**shows**  $\neg\ Col\ C\ B\ A$   
 $\langle proof \rangle$

**lemma** *not-col-permutation-4*:  
**assumes**  $\neg\ Col\ A\ B\ C$   
**shows**  $\neg\ Col\ B\ A\ C$   
 $\langle proof \rangle$

**lemma** *not-col-permutation-5*:  
**assumes**  $\neg\ Col\ A\ B\ C$   
**shows**  $\neg\ Col\ A\ C\ B$   
 $\langle proof \rangle$

**lemma** *Col-cases*:  
**assumes**  $Col\ A\ B\ C \vee Col\ A\ C\ B \vee Col\ B\ A\ C \vee Col\ B\ C\ A \vee Col\ C\ A\ B \vee Col\ C\ B\ A$   
**shows**  $Col\ A\ B\ C$   
 $\langle proof \rangle$

**lemma** *Col-perm*:  
**assumes**  $Col\ A\ B\ C$   
**shows**  $Col\ A\ B\ C \wedge Col\ A\ C\ B \wedge Col\ B\ A\ C \wedge Col\ B\ C\ A \wedge Col\ C\ A\ B \wedge Col\ C\ B\ A$

*<proof>*

**lemma** *col-trivial-1:*

*Col A A B*

*<proof>*

**lemma** *col-trivial-2:*

*Col A B B*

*<proof>*

**lemma** *col-trivial-3:*

*Col A B A*

*<proof>*

**lemma** *l4-13:*

**assumes** *Col A B C and*

*A B C Cong3 A' B' C'*

**shows** *Col A' B' C'*

*<proof>*

**lemma** *l4-14R1:*

**assumes** *Bet A B C and*

*Cong A B A' B'*

**shows**  $\exists C'. A B C \text{ Cong3 } A' B' C'$

*<proof>*

**lemma** *l4-14R2:*

**assumes** *Bet B C A and*

*Cong A B A' B'*

**shows**  $\exists C'. A B C \text{ Cong3 } A' B' C'$

*<proof>*

**lemma** *l4-14R3:*

**assumes** *Bet C A B and*

*Cong A B A' B'*

**shows**  $\exists C'. A B C \text{ Cong3 } A' B' C'$

*<proof>*

**lemma** *l4-14:*

**assumes** *Col A B C and*

*Cong A B A' B'*

**shows**  $\exists C'. A B C \text{ Cong3 } A' B' C'$

*<proof>*

**lemma** *l4-16R1:*

**assumes** *A B C D FSC A' B' C' D' and*

*A  $\neq$  B and*

*Bet A B C*

**shows** *Cong C D C' D'*

*<proof>*

**lemma** *l4-16R2:*

**assumes** *A B C D FSC A' B' C' D'*

**and** *Bet B C A*

**shows** *Cong C D C' D'*

*<proof>*

**lemma** *l4-16R3:*

**assumes** *A B C D FSC A' B' C' D' and A  $\neq$  B*

**and** *Bet C A B*

**shows** *Cong C D C' D'*

*<proof>*

**lemma** *l4-16:*

**assumes** *A B C D FSC A' B' C' D' and*

*A  $\neq$  B*

**shows**  $Cong\ C\ D\ C'\ D'$   
 $\langle proof \rangle$

**lemma**  $l_4-17$ :  
**assumes**  $A \neq B$  **and**  
     $Col\ A\ B\ C$  **and**  
     $Cong\ A\ P\ A\ Q$  **and**  
     $Cong\ B\ P\ B\ Q$   
**shows**  $Cong\ C\ P\ C\ Q$   
 $\langle proof \rangle$

**lemma**  $l_4-18$ :  
**assumes**  $A \neq B$  **and**  
     $Col\ A\ B\ C$  **and**  
     $Cong\ A\ C\ A\ C'$  **and**  
     $Cong\ B\ C\ B\ C'$   
**shows**  $C = C'$   
 $\langle proof \rangle$

**lemma**  $l_4-19$ :  
**assumes**  $Bet\ A\ C\ B$  **and**  
     $Cong\ A\ C\ A\ C'$  **and**  
     $Cong\ B\ C\ B\ C'$   
**shows**  $C = C'$   
 $\langle proof \rangle$

**lemma**  $not-col-distincts$ :  
**assumes**  $\neg Col\ A\ B\ C$   
**shows**  $\neg Col\ A\ B\ C \wedge A \neq B \wedge B \neq C \wedge A \neq C$   
 $\langle proof \rangle$

**lemma**  $NCol-cases$ :  
**assumes**  $\neg Col\ A\ B\ C \vee \neg Col\ A\ C\ B \vee \neg Col\ B\ A\ C \vee \neg Col\ B\ C\ A \vee \neg Col\ C\ A\ B \vee \neg Col\ C\ B\ A$   
**shows**  $\neg Col\ A\ B\ C$   
 $\langle proof \rangle$

**lemma**  $NCol-perm$ :  
**assumes**  $\neg Col\ A\ B\ C$   
**shows**  $\neg Col\ A\ B\ C \wedge \sim Col\ A\ C\ B \wedge \sim Col\ B\ A\ C \wedge \sim Col\ B\ C\ A \wedge \sim Col\ C\ A\ B \wedge \sim Col\ C\ B\ A$   
 $\langle proof \rangle$

**lemma**  $col-cong-3-cong-3-eq$ :  
**assumes**  $A \neq B$   
    **and**  $Col\ A\ B\ C$   
    **and**  $A\ B\ C\ Cong^3\ A'\ B'\ C_1$   
    **and**  $A\ B\ C\ Cong^3\ A'\ B'\ C_2$   
**shows**  $C_1 = C_2$   
 $\langle proof \rangle$

### 3.4 Between transitivity le

**lemma**  $l_5-1$ :  
**assumes**  $A \neq B$  **and**  
     $Bet\ A\ B\ C$  **and**  
     $Bet\ A\ B\ D$   
**shows**  $Bet\ A\ C\ D \vee Bet\ A\ D\ C$   
 $\langle proof \rangle$

**lemma**  $l_5-2$ :  
**assumes**  $A \neq B$  **and**  
     $Bet\ A\ B\ C$  **and**  
     $Bet\ A\ B\ D$   
**shows**  $Bet\ B\ C\ D \vee Bet\ B\ D\ C$   
 $\langle proof \rangle$

**lemma** *segment-construction-2*:

**assumes**  $A \neq Q$

**shows**  $\exists X. ((\text{Bet } Q \ A \ X \vee \text{Bet } Q \ X \ A) \wedge \text{Cong } Q \ X \ B \ C)$   
 $\langle \text{proof} \rangle$

**lemma** *l5-3*:

**assumes**  $\text{Bet } A \ B \ D$  **and**

$\text{Bet } A \ C \ D$

**shows**  $\text{Bet } A \ B \ C \vee \text{Bet } A \ C \ B$

$\langle \text{proof} \rangle$

**lemma** *bet3--bet*:

**assumes**  $\text{Bet } A \ B \ E$  **and**

$\text{Bet } A \ D \ E$  **and**

$\text{Bet } B \ C \ D$

**shows**  $\text{Bet } A \ C \ E$

$\langle \text{proof} \rangle$

**lemma** *le-bet*:

**assumes**  $C \ D \ \text{Le } A \ B$

**shows**  $\exists X. (\text{Bet } A \ X \ B \wedge \text{Cong } A \ X \ C \ D)$

$\langle \text{proof} \rangle$

**lemma** *l5-5-1*:

**assumes**  $A \ B \ \text{Le } C \ D$

**shows**  $\exists X. (\text{Bet } A \ B \ X \wedge \text{Cong } A \ X \ C \ D)$

$\langle \text{proof} \rangle$

**lemma** *l5-5-2*:

**assumes**  $\exists X. (\text{Bet } A \ B \ X \wedge \text{Cong } A \ X \ C \ D)$

**shows**  $A \ B \ \text{Le } C \ D$

$\langle \text{proof} \rangle$

**lemma** *l5-6*:

**assumes**  $A \ B \ \text{Le } C \ D$  **and**

$\text{Cong } A \ B \ A' \ B'$  **and**

$\text{Cong } C \ D \ C' \ D'$

**shows**  $A' \ B' \ \text{Le } C' \ D'$

$\langle \text{proof} \rangle$

**lemma** *le-reflexivity*:

**shows**  $A \ B \ \text{Le } A \ B$

$\langle \text{proof} \rangle$

**lemma** *le-transitivity*:

**assumes**  $A \ B \ \text{Le } C \ D$  **and**

$C \ D \ \text{Le } E \ F$

**shows**  $A \ B \ \text{Le } E \ F$

$\langle \text{proof} \rangle$

**lemma** *between-cong*:

**assumes**  $\text{Bet } A \ C \ B$  **and**

$\text{Cong } A \ C \ A \ B$

**shows**  $C = B$

$\langle \text{proof} \rangle$

**lemma** *cong3-symmetry*:

**assumes**  $A \ B \ C \ \text{Cong3 } A' \ B' \ C'$

**shows**  $A' \ B' \ C' \ \text{Cong3 } A \ B \ C$

$\langle \text{proof} \rangle$

**lemma** *between-cong-2*:

**assumes**  $\text{Bet } A \ D \ B$  **and**

$\text{Bet } A \ E \ B$

**and**  $\text{Cong } A \ D \ A \ E$

**shows**  $D = E$   $\langle \text{proof} \rangle$

**lemma** *between-cong-3*:

**assumes**  $A \neq B$   
**and**  $Bet\ A\ B\ D$   
**and**  $Bet\ A\ B\ E$   
**and**  $Cong\ B\ D\ B\ E$   
**shows**  $D = E$   
*<proof>*

**lemma** *le-anti-symmetry*:

**assumes**  $A\ B\ Le\ C\ D$  **and**  
 $C\ D\ Le\ A\ B$   
**shows**  $Cong\ A\ B\ C\ D$   
*<proof>*

**lemma** *cong-dec*:

**shows**  $Cong\ A\ B\ C\ D \vee \neg\ Cong\ A\ B\ C\ D$   
*<proof>*

**lemma** *bet-dec*:

**shows**  $Bet\ A\ B\ C \vee \neg\ Bet\ A\ B\ C$   
*<proof>*

**lemma** *col-dec*:

**shows**  $Col\ A\ B\ C \vee \neg\ Col\ A\ B\ C$   
*<proof>*

**lemma** *le-trivial*:

**shows**  $A\ A\ Le\ C\ D$   
*<proof>*

**lemma** *le-cases*:

**shows**  $A\ B\ Le\ C\ D \vee C\ D\ Le\ A\ B$   
*<proof>*

**lemma** *le-zero*:

**assumes**  $A\ B\ Le\ C\ C$   
**shows**  $A = B$   
*<proof>*

**lemma** *le-diff*:

**assumes**  $A \neq B$  **and**  $A\ B\ Le\ C\ D$   
**shows**  $C \neq D$   
*<proof>*

**lemma** *lt-diff*:

**assumes**  $A\ B\ Lt\ C\ D$   
**shows**  $C \neq D$   
*<proof>*

**lemma** *bet-cong-eq*:

**assumes**  $Bet\ A\ B\ C$  **and**  
 $Bet\ A\ C\ D$  **and**  
 $Cong\ B\ C\ A\ D$   
**shows**  $C = D \wedge A = B$   
*<proof>*

**lemma** *cong--le*:

**assumes**  $Cong\ A\ B\ C\ D$   
**shows**  $A\ B\ Le\ C\ D$   
*<proof>*

**lemma** *cong--le3412*:

**assumes**  $Cong\ A\ B\ C\ D$   
**shows**  $C\ D\ Le\ A\ B$   
*<proof>*

**lemma** *le1221*:  
**shows**  $A B Le B A$   
 $\langle proof \rangle$

**lemma** *le-left-comm*:  
**assumes**  $A B Le C D$   
**shows**  $B A Le C D$   
 $\langle proof \rangle$

**lemma** *le-right-comm*:  
**assumes**  $A B Le C D$   
**shows**  $A B Le D C$   
 $\langle proof \rangle$

**lemma** *le-comm*:  
**assumes**  $A B Le C D$   
**shows**  $B A Le D C$   
 $\langle proof \rangle$

**lemma** *ge-left-comm*:  
**assumes**  $A B Ge C D$   
**shows**  $B A Ge C D$   
 $\langle proof \rangle$

**lemma** *ge-right-comm*:  
**assumes**  $A B Ge C D$   
**shows**  $A B Ge D C$   
 $\langle proof \rangle$

**lemma** *ge-comm0*:  
**assumes**  $A B Ge C D$   
**shows**  $B A Ge D C$   
 $\langle proof \rangle$

**lemma** *lt-right-comm*:  
**assumes**  $A B Lt C D$   
**shows**  $A B Lt D C$   
 $\langle proof \rangle$

**lemma** *lt-left-comm*:  
**assumes**  $A B Lt C D$   
**shows**  $B A Lt C D$   
 $\langle proof \rangle$

**lemma** *lt-comm*:  
**assumes**  $A B Lt C D$   
**shows**  $B A Lt D C$   
 $\langle proof \rangle$

**lemma** *gt-left-comm0*:  
**assumes**  $A B Gt C D$   
**shows**  $B A Gt C D$   
 $\langle proof \rangle$

**lemma** *gt-right-comm*:  
**assumes**  $A B Gt C D$   
**shows**  $A B Gt D C$   
 $\langle proof \rangle$

**lemma** *gt-comm*:  
**assumes**  $A B Gt C D$   
**shows**  $B A Gt D C$   
 $\langle proof \rangle$

**lemma** *cong2-lt--lt*:

**assumes**  $A B Lt C D$  **and**  
*Cong*  $A B A' B'$  **and**  
*Cong*  $C D C' D'$   
**shows**  $A' B' Lt C' D'$   
 ⟨*proof*⟩

**lemma** *fourth-point*:  
**assumes**  $A \neq B$  **and**  
 $B \neq C$  **and**  
*Col*  $A B P$  **and**  
*Bet*  $A B C$   
**shows**  $Bet P A B \vee Bet A P B \vee Bet B P C \vee Bet B C P$   
 ⟨*proof*⟩

**lemma** *third-point*:  
**assumes** *Col*  $A B P$   
**shows**  $Bet P A B \vee Bet A P B \vee Bet A B P$   
 ⟨*proof*⟩

**lemma** *l5-12-a*:  
**assumes** *Bet*  $A B C$   
**shows**  $A B Le A C \wedge B C Le A C$   
 ⟨*proof*⟩

**lemma** *bet--le1213*:  
**assumes** *Bet*  $A B C$   
**shows**  $A B Le A C$   
 ⟨*proof*⟩

**lemma** *bet--le2313*:  
**assumes** *Bet*  $A B C$   
**shows**  $B C Le A C$   
 ⟨*proof*⟩

**lemma** *bet--lt1213*:  
**assumes**  $B \neq C$  **and**  
*Bet*  $A B C$   
**shows**  $A B Lt A C$   
 ⟨*proof*⟩

**lemma** *bet--lt2313*:  
**assumes**  $A \neq B$  **and**  
*Bet*  $A B C$   
**shows**  $B C Lt A C$   
 ⟨*proof*⟩

**lemma** *l5-12-b*:  
**assumes** *Col*  $A B C$  **and**  
 $A B Le A C$  **and**  
 $B C Le A C$   
**shows** *Bet*  $A B C$   
 ⟨*proof*⟩

**lemma** *bet-le-eq*:  
**assumes** *Bet*  $A B C$   
**and**  $A C Le B C$   
**shows**  $A = B$   
 ⟨*proof*⟩

**lemma** *or-lt-cong-gt*:  
 $A B Lt C D \vee A B Gt C D \vee Cong A B C D$   
 ⟨*proof*⟩

**lemma** *lt--le*:  
**assumes**  $A B Lt C D$   
**shows**  $A B Le C D$

*<proof>*

**lemma** *le1234-lt--lt:*  
**assumes**  $A B Le C D$  **and**  
 $C D Lt E F$   
**shows**  $A B Lt E F$   
*<proof>*

**lemma** *le3456-lt--lt:*  
**assumes**  $A B Lt C D$  **and**  
 $C D Le E F$   
**shows**  $A B Lt E F$   
*<proof>*

**lemma** *lt-transitivity:*  
**assumes**  $A B Lt C D$  **and**  
 $C D Lt E F$   
**shows**  $A B Lt E F$   
*<proof>*

**lemma** *not-and-lt:*  
 $\neg (A B Lt C D \wedge C D Lt A B)$   
*<proof>*

**lemma** *nlt:*  
 $\neg A B Lt A B$   
*<proof>*

**lemma** *le--nlt:*  
**assumes**  $A B Le C D$   
**shows**  $\neg C D Lt A B$   
*<proof>*

**lemma** *cong--nlt:*  
**assumes**  $Cong A B C D$   
**shows**  $\neg A B Lt C D$   
*<proof>*

**lemma** *nlt--le:*  
**assumes**  $\neg A B Lt C D$   
**shows**  $C D Le A B$   
*<proof>*

**lemma** *lt--nle:*  
**assumes**  $A B Lt C D$   
**shows**  $\neg C D Le A B$   
*<proof>*

**lemma** *nle--lt:*  
**assumes**  $\neg A B Le C D$   
**shows**  $C D Lt A B$   
*<proof>*

**lemma** *lt1123:*  
**assumes**  $B \neq C$   
**shows**  $A A Lt B C$   
*<proof>*

**lemma** *bet2-le2--le-R1:*  
**assumes**  $Bet a P b$  **and**  
 $Bet A Q B$  **and**  
 $P a Le Q A$  **and**  
 $P b Le Q B$  **and**  
 $B = Q$   
**shows**  $a b Le A B$   
*<proof>*

**lemma** *bet2-le2--le-R2*:  
**assumes** *Bet a Po b* **and**  
*Bet A PO B* **and**  
*Po a Le PO A* **and**  
*Po b Le PO B* **and**  
*A ≠ PO* **and**  
*B ≠ PO*  
**shows** *a b Le A B*  
⟨*proof*⟩

**lemma** *bet2-le2--le*:  
**assumes** *Bet a P b* **and**  
*Bet A Q B* **and**  
*P a Le Q A* **and**  
*P b Le Q B*  
**shows** *a b Le A B*  
⟨*proof*⟩

**lemma** *Le-cases*:  
**assumes** *A B Le C D ∨ B A Le C D ∨ A B Le D C ∨ B A Le D C*  
**shows** *A B Le C D*  
⟨*proof*⟩

**lemma** *Lt-cases*:  
**assumes** *A B Lt C D ∨ B A Lt C D ∨ A B Lt D C ∨ B A Lt D C*  
**shows** *A B Lt C D*  
⟨*proof*⟩

### 3.5 Out lines

**lemma** *bet-out*:  
**assumes** *B ≠ A* **and**  
*Bet A B C*  
**shows** *A Out B C*  
⟨*proof*⟩

**lemma** *bet-out-1*:  
**assumes** *B ≠ A* **and**  
*Bet C B A*  
**shows** *A Out B C*  
⟨*proof*⟩

**lemma** *out-dec*:  
**shows** *P Out A B ∨ ¬ P Out A B*  
⟨*proof*⟩

**lemma** *out-diff1*:  
**assumes** *A Out B C*  
**shows** *B ≠ A*  
⟨*proof*⟩

**lemma** *out-diff2*:  
**assumes** *A Out B C*  
**shows** *C ≠ A*  
⟨*proof*⟩

**lemma** *out-distinct*:  
**assumes** *A Out B C*  
**shows** *B ≠ A ∧ C ≠ A*  
⟨*proof*⟩

**lemma** *out-col*:  
**assumes** *A Out B C*  
**shows** *Col A B C*  
⟨*proof*⟩

**lemma l6-2:**

**assumes**  $A \neq P$  **and**

$B \neq P$  **and**

$C \neq P$  **and**

$Bet\ A\ P\ C$

**shows**  $Bet\ B\ P\ C \longleftrightarrow P\ Out\ A\ B$

$\langle proof \rangle$

**lemma bet-out--bet:**

**assumes**  $Bet\ A\ P\ C$  **and**

$P\ Out\ A\ B$

**shows**  $Bet\ B\ P\ C$

$\langle proof \rangle$

**lemma l6-3-1:**

**assumes**  $P\ Out\ A\ B$

**shows**  $A \neq P \wedge B \neq P \wedge (\exists\ C. (C \neq P \wedge Bet\ A\ P\ C \wedge Bet\ B\ P\ C))$

$\langle proof \rangle$

**lemma l6-3-2:**

**assumes**  $A \neq P$  **and**

$B \neq P$  **and**

$\exists\ C. (C \neq P \wedge Bet\ A\ P\ C \wedge Bet\ B\ P\ C)$

**shows**  $P\ Out\ A\ B$

$\langle proof \rangle$

**lemma l6-4-1:**

**assumes**  $P\ Out\ A\ B$  **and**

$Col\ A\ P\ B$

**shows**  $\neg\ Bet\ A\ P\ B$

$\langle proof \rangle$

**lemma l6-4-2:**

**assumes**  $Col\ A\ P\ B$

**and**  $\neg\ Bet\ A\ P\ B$

**shows**  $P\ Out\ A\ B$

$\langle proof \rangle$

**lemma out-trivial:**

**assumes**  $A \neq P$

**shows**  $P\ Out\ A\ A$

$\langle proof \rangle$

**lemma l6-6:**

**assumes**  $P\ Out\ A\ B$

**shows**  $P\ Out\ B\ A$

$\langle proof \rangle$

**lemma l6-7:**

**assumes**  $P\ Out\ A\ B$  **and**

$P\ Out\ B\ C$

**shows**  $P\ Out\ A\ C$

$\langle proof \rangle$

**lemma bet-out-out-bet:**

**assumes**  $Bet\ A\ B\ C$  **and**

$B\ Out\ A\ A'$  **and**

$B\ Out\ C\ C'$

**shows**  $Bet\ A'\ B\ C'$

$\langle proof \rangle$

**lemma out2-bet-out:**

**assumes**  $B\ Out\ A\ C$  **and**

$B\ Out\ X\ P$  **and**

$Bet\ A\ X\ C$

**shows**  $B \text{ Out } A \text{ P} \wedge B \text{ Out } C \text{ P}$   
 $\langle \text{proof} \rangle$

**lemma** *l6-11-uniqueness*:  
**assumes**  $A \text{ Out } X \text{ R}$  **and**  
     $\text{Cong } A \text{ X } B \text{ C}$  **and**  
     $A \text{ Out } Y \text{ R}$  **and**  
     $\text{Cong } A \text{ Y } B \text{ C}$   
**shows**  $X = Y$   
 $\langle \text{proof} \rangle$

**lemma** *l6-11-existence*:  
**assumes**  $R \neq A$  **and**  
     $B \neq C$   
**shows**  $\exists X. (A \text{ Out } X \text{ R} \wedge \text{Cong } A \text{ X } B \text{ C})$   
 $\langle \text{proof} \rangle$

**lemma** *segment-construction-3*:  
**assumes**  $A \neq B$  **and**  
     $X \neq Y$   
**shows**  $\exists C. (A \text{ Out } B \text{ C} \wedge \text{Cong } A \text{ C } X \text{ Y})$   
 $\langle \text{proof} \rangle$

**lemma** *l6-13-1*:  
**assumes**  $P \text{ Out } A \text{ B}$  **and**  
     $P \text{ A } \text{Le } P \text{ B}$   
**shows**  $\text{Bet } P \text{ A } B$   
 $\langle \text{proof} \rangle$

**lemma** *l6-13-2*:  
**assumes**  $P \text{ Out } A \text{ B}$  **and**  
     $\text{Bet } P \text{ A } B$   
**shows**  $P \text{ A } \text{Le } P \text{ B}$   
 $\langle \text{proof} \rangle$

**lemma** *l6-16-1*:  
**assumes**  $P \neq Q$  **and**  
     $\text{Col } S \text{ P } Q$  **and**  
     $\text{Col } X \text{ P } Q$   
**shows**  $\text{Col } X \text{ P } S$   
 $\langle \text{proof} \rangle$

**lemma** *col-transitivity-1*:  
**assumes**  $P \neq Q$  **and**  
     $\text{Col } P \text{ Q } A$  **and**  
     $\text{Col } P \text{ Q } B$   
**shows**  $\text{Col } P \text{ A } B$   
 $\langle \text{proof} \rangle$

**lemma** *col-transitivity-2*:  
**assumes**  $P \neq Q$  **and**  
     $\text{Col } P \text{ Q } A$  **and**  
     $\text{Col } P \text{ Q } B$   
**shows**  $\text{Col } Q \text{ A } B$   
 $\langle \text{proof} \rangle$

**lemma** *l6-21*:  
**assumes**  $\neg \text{Col } A \text{ B } C$  **and**  
     $C \neq D$  **and**  
     $\text{Col } A \text{ B } P$  **and**  
     $\text{Col } A \text{ B } Q$  **and**  
     $\text{Col } C \text{ D } P$  **and**  
     $\text{Col } C \text{ D } Q$   
**shows**  $P = Q$   
 $\langle \text{proof} \rangle$

**lemma col2--eq:**  
**assumes**  $Col\ A\ X\ Y$  **and**  
 $Col\ B\ X\ Y$  **and**  
 $\neg\ Col\ A\ X\ B$   
**shows**  $X = Y$   
 $\langle proof \rangle$

**lemma not-col-exists:**  
**assumes**  $A \neq B$   
**shows**  $\exists\ C.\ \neg\ Col\ A\ B\ C$   
 $\langle proof \rangle$

**lemma col3:**  
**assumes**  $X \neq Y$  **and**  
 $Col\ X\ Y\ A$  **and**  
 $Col\ X\ Y\ B$  **and**  
 $Col\ X\ Y\ C$   
**shows**  $Col\ A\ B\ C$   
 $\langle proof \rangle$

**lemma colx:**  
**assumes**  $A \neq B$  **and**  
 $Col\ X\ Y\ A$  **and**  
 $Col\ X\ Y\ B$  **and**  
 $Col\ A\ B\ C$   
**shows**  $Col\ X\ Y\ C$   
 $\langle proof \rangle$

**lemma out2--bet:**  
**assumes**  $A\ Out\ B\ C$  **and**  
 $C\ Out\ A\ B$   
**shows**  $Bet\ A\ B\ C$   
 $\langle proof \rangle$

**lemma bet2-le2--le1346:**  
**assumes**  $Bet\ A\ B\ C$  **and**  
 $Bet\ A'\ B'\ C'$  **and**  
 $A\ B\ Le\ A'\ B'$  **and**  
 $B\ C\ Le\ B'\ C'$   
**shows**  $A\ C\ Le\ A'\ C'$   
 $\langle proof \rangle$

**lemma bet2-le2--le2356-R1:**  
**assumes**  $Bet\ A\ A\ C$  **and**  
 $Bet\ A'\ B'\ C'$  **and**  
 $A\ A\ Le\ A'\ B'$  **and**  
 $A'\ C'\ Le\ A\ C$   
**shows**  $B'\ C'\ Le\ A\ C$   
 $\langle proof \rangle$

**lemma bet2-le2--le2356-R2:**  
**assumes**  $A \neq B$  **and**  
 $Bet\ A\ B\ C$  **and**  
 $Bet\ A'\ B'\ C'$  **and**  
 $A\ B\ Le\ A'\ B'$  **and**  
 $A'\ C'\ Le\ A\ C$   
**shows**  $B'\ C'\ Le\ B\ C$   
 $\langle proof \rangle$

**lemma bet2-le2--le2356:**  
**assumes**  $Bet\ A\ B\ C$  **and**  
 $Bet\ A'\ B'\ C'$  **and**  
 $A\ B\ Le\ A'\ B'$  **and**  
 $A'\ C'\ Le\ A\ C$   
**shows**  $B'\ C'\ Le\ B\ C$

$\langle \text{proof} \rangle$

**lemma** *bet2-le2--le1245*:

**assumes**  $Bet\ A\ B\ C$  **and**

$Bet\ A'\ B'\ C'$  **and**

$B\ C\ Le\ B'\ C'$  **and**

$A'\ C'\ Le\ A\ C$

**shows**  $A'\ B'\ Le\ A\ B$

$\langle \text{proof} \rangle$

**lemma** *cong-preserves-bet*:

**assumes**  $Bet\ B\ A'\ A0$  **and**

$Cong\ B\ A'\ E\ D'$  **and**

$Cong\ B\ A0\ E\ D0$  **and**

$E\ Out\ D'\ D0$

**shows**  $Bet\ E\ D'\ D0$

$\langle \text{proof} \rangle$

**lemma** *out-cong-cong*:

**assumes**  $B\ Out\ A\ A0$  **and**

$E\ Out\ D\ D0$  **and**

$Cong\ B\ A\ E\ D$  **and**

$Cong\ B\ A0\ E\ D0$

**shows**  $Cong\ A\ A0\ D\ D0$

$\langle \text{proof} \rangle$

**lemma** *not-out-bet*:

**assumes**  $Col\ A\ B\ C$  **and**

$\neg\ B\ Out\ A\ C$

**shows**  $Bet\ A\ B\ C$

$\langle \text{proof} \rangle$

**lemma** *or-bet-out*:

**shows**  $Bet\ A\ B\ C \vee B\ Out\ A\ C \vee \neg\ Col\ A\ B\ C$

$\langle \text{proof} \rangle$

**lemma** *not-bet-out*:

**assumes**  $Col\ A\ B\ C$  **and**

$\neg\ Bet\ A\ B\ C$

**shows**  $B\ Out\ A\ C$

$\langle \text{proof} \rangle$

**lemma** *not-bet-and-out*:

**shows**  $\neg\ (Bet\ A\ B\ C \wedge B\ Out\ A\ C)$

$\langle \text{proof} \rangle$

**lemma** *out-to-bet*:

**assumes**  $Col\ A'\ B'\ C'$  **and**

$B\ Out\ A\ C \longleftrightarrow B'\ Out\ A'\ C'$  **and**

$Bet\ A\ B\ C$

**shows**  $Bet\ A'\ B'\ C'$

$\langle \text{proof} \rangle$

**lemma** *col-out2-col*:

**assumes**  $Col\ A\ B\ C$  **and**

$B\ Out\ A\ AA$  **and**

$B\ Out\ C\ CC$

**shows**  $Col\ AA\ B\ CC$   $\langle \text{proof} \rangle$

**lemma** *bet2-out-out*:

**assumes**  $B \neq A$  **and**

$B' \neq A$  **and**

$A\ Out\ C\ C'$  **and**

$Bet\ A\ B\ C$  **and**

$Bet\ A\ B'\ C'$

**shows**  $A\ Out\ B\ B'$

*<proof>*

**lemma** *bet2--out*:  
**assumes**  $A \neq B$  **and**  
     $A \neq B'$  **and**  
    *Bet*  $A B C$   
    **and** *Bet*  $A B' C$   
**shows**  $A \text{ Out } B B'$   
*<proof>*

**lemma** *out-bet-out-1*:  
**assumes**  $P \text{ Out } A C$  **and**  
    *Bet*  $A B C$   
**shows**  $P \text{ Out } A B$   
*<proof>*

**lemma** *out-bet-out-2*:  
**assumes**  $P \text{ Out } A C$  **and**  
    *Bet*  $A B C$   
**shows**  $P \text{ Out } B C$   
*<proof>*

**lemma** *out-bet--out*:  
**assumes** *Bet*  $P Q A$  **and**  
     $Q \text{ Out } A B$   
**shows**  $P \text{ Out } A B$   
*<proof>*

**lemma** *segment-reverse*:  
**assumes** *Bet*  $A B C$   
**shows**  $\exists B'. \text{Bet } A B' C \wedge \text{Cong } C B' A B$   
*<proof>*

**lemma** *diff-col-ex*:  
**shows**  $\exists C. A \neq C \wedge B \neq C \wedge \text{Col } A B C$   
*<proof>*

**lemma** *diff-bet-ex3*:  
**assumes** *Bet*  $A B C$   
**shows**  $\exists D. A \neq D \wedge B \neq D \wedge C \neq D \wedge \text{Col } A B D$   
*<proof>*

**lemma** *diff-col-ex3*:  
**assumes** *Col*  $A B C$   
**shows**  $\exists D. A \neq D \wedge B \neq D \wedge C \neq D \wedge \text{Col } A B D$   
*<proof>*

**lemma** *Out-cases*:  
**assumes**  $A \text{ Out } B C \vee A \text{ Out } C B$   
**shows**  $A \text{ Out } B C$   
*<proof>*

### 3.6 Midpoint

**lemma** *midpoint-dec*:  
 $I \text{ Midpoint } A B \vee \neg I \text{ Midpoint } A B$   
*<proof>*

**lemma** *is-midpoint-id*:  
**assumes**  $A \text{ Midpoint } A B$   
**shows**  $A = B$   
*<proof>*

**lemma** *is-midpoint-id-2*:  
**assumes**  $A \text{ Midpoint } B A$   
**shows**  $A = B$

*<proof>*

**lemma l7-2:**

**assumes**  $M$  *Midpoint*  $A$   $B$

**shows**  $M$  *Midpoint*  $B$   $A$

*<proof>*

**lemma l7-3:**

**assumes**  $M$  *Midpoint*  $A$   $A$

**shows**  $M = A$

*<proof>*

**lemma l7-3-2:**

$A$  *Midpoint*  $A$   $A$

*<proof>*

**lemma symmetric-point-construction:**

$\exists P'. A$  *Midpoint*  $P$   $P'$

*<proof>*

**lemma symmetric-point-uniqueness:**

**assumes**  $P$  *Midpoint*  $A$   $P1$  **and**

$P$  *Midpoint*  $A$   $P2$

**shows**  $P1 = P2$

*<proof>*

**lemma l7-9:**

**assumes**  $A$  *Midpoint*  $P$   $X$  **and**

$A$  *Midpoint*  $Q$   $X$

**shows**  $P = Q$

*<proof>*

**lemma l7-9-bis:**

**assumes**  $A$  *Midpoint*  $P$   $X$  **and**

$A$  *Midpoint*  $X$   $Q$

**shows**  $P = Q$

*<proof>*

**lemma l7-13-R1:**

**assumes**  $A \neq P$  **and**

$A$  *Midpoint*  $P'$   $P$  **and**

$A$  *Midpoint*  $Q'$   $Q$

**shows** *Cong*  $P$   $Q$   $P'$   $Q'$

*<proof>*

**lemma l7-13:**

**assumes**  $A$  *Midpoint*  $P'$   $P$  **and**

$A$  *Midpoint*  $Q'$   $Q$

**shows** *Cong*  $P$   $Q$   $P'$   $Q'$

*<proof>*

**lemma l7-15:**

**assumes**  $A$  *Midpoint*  $P$   $P'$  **and**

$A$  *Midpoint*  $Q$   $Q'$  **and**

$A$  *Midpoint*  $R$   $R'$  **and**

*Bet*  $P$   $Q$   $R$

**shows** *Bet*  $P'$   $Q'$   $R'$

*<proof>*

**lemma l7-16:**

**assumes**  $A$  *Midpoint*  $P$   $P'$  **and**

$A$  *Midpoint*  $Q$   $Q'$  **and**

$A$  *Midpoint*  $R$   $R'$  **and**

$A$  *Midpoint*  $S$   $S'$  **and**

*Cong*  $P$   $Q$   $R$   $S$

**shows** *Cong*  $P'$   $Q'$   $R'$   $S'$

*<proof>*

**lemma** *symmetry-preserves-midpoint*:

**assumes**  $Z$  Midpoint  $A D$  **and**

$Z$  Midpoint  $B E$  **and**

$Z$  Midpoint  $C F$  **and**

$B$  Midpoint  $A C$

**shows**  $E$  Midpoint  $D F$

*<proof>*

**lemma** *Mid-cases*:

**assumes**  $A$  Midpoint  $B C \vee A$  Midpoint  $C B$

**shows**  $A$  Midpoint  $B C$

*<proof>*

**lemma** *Mid-perm*:

**assumes**  $A$  Midpoint  $B C$

**shows**  $A$  Midpoint  $B C \wedge A$  Midpoint  $C B$

*<proof>*

**lemma** *l7-17*:

**assumes**  $A$  Midpoint  $P P'$  **and**

$B$  Midpoint  $P P'$

**shows**  $A = B$

*<proof>*

**lemma** *l7-17-bis*:

**assumes**  $A$  Midpoint  $P P'$  **and**

$B$  Midpoint  $P' P$

**shows**  $A = B$

*<proof>*

**lemma** *l7-20*:

**assumes**  $Col A M B$  **and**

$Cong M A M B$

**shows**  $A = B \vee M$  Midpoint  $A B$

*<proof>*

**lemma** *l7-20-bis*:

**assumes**  $A \neq B$  **and**

$Col A M B$  **and**

$Cong M A M B$

**shows**  $M$  Midpoint  $A B$

*<proof>*

**lemma** *cong-col-mid*:

**assumes**  $A \neq C$  **and**

$Col A B C$  **and**

$Cong A B B C$

**shows**  $B$  Midpoint  $A C$

*<proof>*

**lemma** *l7-21-R1*:

**assumes**  $\neg Col A B C$  **and**

$B \neq D$  **and**

$Cong A B C D$  **and**

$Cong B C D A$  **and**

$Col A P C$  **and**

$Col B P D$

**shows**  $P$  Midpoint  $A C$

*<proof>*

**lemma** *l7-21*:

**assumes**  $\neg Col A B C$  **and**

$B \neq D$  **and**

$Cong A B C D$  **and**

*Cong B C D A and*

*Col A P C and*

*Col B P D*

**shows**  $P \text{ Midpoint } A C \wedge P \text{ Midpoint } B D$

*<proof>*

**lemma** *l7-22-aux-R1:*

**assumes** *Bet A1 C C and*

*Bet B1 C B2 and*

*Cong C A1 C B1 and*

*Cong C C C B2 and*

*M1 Midpoint A1 B1 and*

*M2 Midpoint A2 B2 and*

*C A1 Le C C*

**shows** *Bet M1 C M2*

*<proof>*

**lemma** *l7-22-aux-R2:*

**assumes**  $A2 \neq C$  **and**

*Bet A1 C A2 and*

*Bet B1 C B2 and*

*Cong C A1 C B1 and*

*Cong C A2 C B2 and*

*M1 Midpoint A1 B1 and*

*M2 Midpoint A2 B2 and*

*C A1 Le C A2*

**shows** *Bet M1 C M2*

*<proof>*

**lemma** *l7-22-aux:*

**assumes** *Bet A1 C A2 and*

*Bet B1 C B2 and*

*Cong C A1 C B1 and*

*Cong C A2 C B2 and*

*M1 Midpoint A1 B1 and*

*M2 Midpoint A2 B2 and*

*C A1 Le C A2*

**shows** *Bet M1 C M2*

*<proof>*

**lemma** *l7-22:*

**assumes** *Bet A1 C A2 and*

*Bet B1 C B2 and*

*Cong C A1 C B1 and*

*Cong C A2 C B2 and*

*M1 Midpoint A1 B1 and*

*M2 Midpoint A2 B2*

**shows** *Bet M1 C M2*

*<proof>*

**lemma** *bet-col1:*

**assumes** *Bet A B D and*

*Bet A C D*

**shows** *Col A B C*

*<proof>*

**lemma** *l7-25-R1:*

**assumes** *Cong C A C B and*

*Col A B C*

**shows**  $\exists X. X \text{ Midpoint } A B$

*<proof>*

**lemma** *l7-25-R2:*

**assumes** *Cong C A C B and*

$\neg \text{Col } A B C$

**shows**  $\exists X. X \text{ Midpoint } A B$

$\langle proof \rangle$

**lemma** *l7-25*:

**assumes**  $Cong\ C\ A\ C\ B$

**shows**  $\exists X. X\ Midpoint\ A\ B$

$\langle proof \rangle$

**lemma** *midpoint-distinct-1*:

**assumes**  $A \neq B$  **and**

$I\ Midpoint\ A\ B$

**shows**  $I \neq A \wedge I \neq B$

$\langle proof \rangle$

**lemma** *midpoint-distinct-2*:

**assumes**  $I \neq A$  **and**

$I\ Midpoint\ A\ B$

**shows**  $A \neq B \wedge I \neq B$

$\langle proof \rangle$

**lemma** *midpoint-distinct-3*:

**assumes**  $I \neq B$  **and**

$I\ Midpoint\ A\ B$

**shows**  $A \neq B \wedge I \neq A$

$\langle proof \rangle$

**lemma** *midpoint-def*:

**assumes**  $Bet\ A\ B\ C$  **and**

$Cong\ A\ B\ B\ C$

**shows**  $B\ Midpoint\ A\ C$

$\langle proof \rangle$

**lemma** *midpoint-bet*:

**assumes**  $B\ Midpoint\ A\ C$

**shows**  $Bet\ A\ B\ C$

$\langle proof \rangle$

**lemma** *midpoint-col*:

**assumes**  $M\ Midpoint\ A\ B$

**shows**  $Col\ M\ A\ B$

$\langle proof \rangle$

**lemma** *midpoint-cong*:

**assumes**  $B\ Midpoint\ A\ C$

**shows**  $Cong\ A\ B\ B\ C$

$\langle proof \rangle$

**lemma** *midpoint-out*:

**assumes**  $A \neq C$  **and**

$B\ Midpoint\ A\ C$

**shows**  $A\ Out\ B\ C$

$\langle proof \rangle$

**lemma** *midpoint-out-1*:

**assumes**  $A \neq C$  **and**

$B\ Midpoint\ A\ C$

**shows**  $C\ Out\ A\ B$

$\langle proof \rangle$

**lemma** *midpoint-not-midpoint*:

**assumes**  $A \neq B$  **and**

$I\ Midpoint\ A\ B$

**shows**  $\neg B\ Midpoint\ A\ I$

$\langle proof \rangle$

**lemma** *swap-diff*:

**assumes**  $A \neq B$

shows  $B \neq A$   
(proof)

lemma *cong-cong-half-1*:  
assumes  $M$  Midpoint  $A B$  and  
 $M'$  Midpoint  $A' B'$  and  
 $Cong A B A' B'$   
shows  $Cong A M A' M'$   
(proof)

lemma *cong-cong-half-2*:  
assumes  $M$  Midpoint  $A B$  and  
 $M'$  Midpoint  $A' B'$  and  
 $Cong A B A' B'$   
shows  $Cong B M B' M'$   
(proof)

lemma *cong-mid2--cong*:  
assumes  $M$  Midpoint  $A B$  and  
 $M'$  Midpoint  $A' B'$  and  
 $Cong A M A' M'$   
shows  $Cong A B A' B'$   
(proof)

lemma *mid--lt*:  
assumes  $A \neq B$  and  
 $M$  Midpoint  $A B$   
shows  $A M Lt A B$   
(proof)

lemma *le-mid2--le13*:  
assumes  $M$  Midpoint  $A B$  and  
 $M'$  Midpoint  $A' B'$  and  
 $A M Le A' M'$   
shows  $A B Le A' B'$   
(proof)

lemma *le-mid2--le12*:  
assumes  $M$  Midpoint  $A B$  and  
 $M'$  Midpoint  $A' B'$   
and  $A B Le A' B'$   
shows  $A M Le A' M'$   
(proof)

lemma *lt-mid2--lt13*:  
assumes  $M$  Midpoint  $A B$  and  
 $M'$  Midpoint  $A' B'$  and  
 $A M Lt A' M'$   
shows  $A B Lt A' B'$   
(proof)

lemma *lt-mid2--lt12*:  
assumes  $M$  Midpoint  $A B$  and  
 $M'$  Midpoint  $A' B'$  and  
 $A B Lt A' B'$   
shows  $A M Lt A' M'$   
(proof)

lemma *midpoint-preserves-out*:  
assumes  $A$  Out  $B C$  and  
 $M$  Midpoint  $A A'$  and  
 $M$  Midpoint  $B B'$  and  
 $M$  Midpoint  $C C'$   
shows  $A' Out B' C'$   
(proof)

**lemma** *col-cong-bet*:  
**assumes** *Col A B D* **and**  
  *Cong A B C D* **and**  
  *Bet A C B*  
**shows** *Bet C A D*  $\vee$  *Bet C B D*  
*<proof>*

**lemma** *col-cong2-bet1*:  
**assumes** *Col A B D* **and**  
  *Bet A C B* **and**  
  *Cong A B C D* **and**  
  *Cong A C B D*  
**shows** *Bet C B D*  
*<proof>*

**lemma** *col-cong2-bet2*:  
**assumes** *Col A B D* **and**  
  *Bet A C B* **and**  
  *Cong A B C D* **and**  
  *Cong A D B C*  
**shows** *Bet C A D*  
*<proof>*

**lemma** *col-cong2-bet3*:  
**assumes** *Col A B D* **and**  
  *Bet A B C* **and**  
  *Cong A B C D* **and**  
  *Cong A C B D*  
**shows** *Bet B C D*  
*<proof>*

**lemma** *col-cong2-bet4*:  
**assumes** *Col A B C* **and**  
  *Bet A B D* **and**  
  *Cong A B C D* **and**  
  *Cong A D B C*  
**shows** *Bet B D C*  
*<proof>*

**lemma** *col-bet2-cong1*:  
**assumes** *Col A B D* **and**  
  *Bet A C B* **and**  
  *Cong A B C D* **and**  
  *Bet C B D*  
**shows** *Cong A C D B*  
*<proof>*

**lemma** *col-bet2-cong2*:  
**assumes** *Col A B D* **and**  
  *Bet A C B* **and**  
  *Cong A B C D* **and**  
  *Bet C A D*  
**shows** *Cong D A B C*  
*<proof>*

**lemma** *bet2-lt2--lt*:  
**assumes** *Bet a Po b* **and**  
  *Bet A PO B* **and**  
  *Po a Lt PO A* **and**  
  *Po b Lt PO B*  
**shows** *a b Lt A B*  
*<proof>*

**lemma** *bet2-lt-le--lt*:  
**assumes** *Bet a Po b* **and**  
  *Bet A PO B* **and**

*Cong Po a PO A and*  
*Po b Lt PO B*  
**shows** *a b Lt A B*  
 ⟨*proof*⟩

### 3.7 Orthogonality

**lemma** *per-dec*:  
*Per A B C  $\vee$   $\neg$  Per A B C*  
 ⟨*proof*⟩

**lemma** *l8-2*:  
**assumes** *Per A B C*  
**shows** *Per C B A*  
 ⟨*proof*⟩

**lemma** *Per-cases*:  
**assumes** *Per A B C  $\vee$  Per C B A*  
**shows** *Per A B C*  
 ⟨*proof*⟩

**lemma** *Per-perm* :  
**assumes** *Per A B C*  
**shows** *Per A B C  $\wedge$  Per C B A*  
 ⟨*proof*⟩

**lemma** *l8-3* :  
**assumes** *Per A B C and*  
*A  $\neq$  B and*  
*Col B A A'*  
**shows** *Per A' B C*  
 ⟨*proof*⟩

**lemma** *l8-4*:  
**assumes** *Per A B C and*  
*B Midpoint C C'*  
**shows** *Per A B C'*  
 ⟨*proof*⟩

**lemma** *l8-5*:  
*Per A B B*  
 ⟨*proof*⟩

**lemma** *l8-6*:  
**assumes** *Per A B C and*  
*Per A' B C and*  
*Bet A C A'*  
**shows** *B = C*  
 ⟨*proof*⟩

**lemma** *l8-7*:  
**assumes** *Per A B C and*  
*Per A C B*  
**shows** *B = C*  
 ⟨*proof*⟩

**lemma** *l8-8*:  
**assumes** *Per A B A*  
**shows** *A = B*  
 ⟨*proof*⟩

**lemma** *per-distinct*:  
**assumes** *Per A B C and*  
*A  $\neq$  B*  
**shows** *A  $\neq$  C*  
 ⟨*proof*⟩

**lemma** *per-distinct-1*:  
**assumes**  $Per\ A\ B\ C$  **and**  
 $B \neq C$   
**shows**  $A \neq C$   
 $\langle proof \rangle$

**lemma** *l8-9*:  
**assumes**  $Per\ A\ B\ C$  **and**  
 $Col\ A\ B\ C$   
**shows**  $A = B \vee C = B$   
 $\langle proof \rangle$

**lemma** *l8-10*:  
**assumes**  $Per\ A\ B\ C$  **and**  
 $A\ B\ C\ Cong^3\ A'\ B'\ C'$   
**shows**  $Per\ A'\ B'\ C'$   
 $\langle proof \rangle$

**lemma** *col-col-per-per*:  
**assumes**  $A \neq X$  **and**  
 $C \neq X$  **and**  
 $Col\ U\ A\ X$  **and**  
 $Col\ V\ C\ X$  **and**  
 $Per\ A\ X\ C$   
**shows**  $Per\ U\ X\ V$   
 $\langle proof \rangle$

**lemma** *perp-in-dec*:  
 $X\ PerpAt\ A\ B\ C\ D \vee \neg X\ PerpAt\ A\ B\ C\ D$   
 $\langle proof \rangle$

**lemma** *perp-distinct*:  
**assumes**  $A\ B\ Perp\ C\ D$   
**shows**  $A \neq B \wedge C \neq D$   
 $\langle proof \rangle$

**lemma** *l8-12*:  
**assumes**  $X\ PerpAt\ A\ B\ C\ D$   
**shows**  $X\ PerpAt\ C\ D\ A\ B$   
 $\langle proof \rangle$

**lemma** *per-col*:  
**assumes**  $B \neq C$  **and**  
 $Per\ A\ B\ C$  **and**  
 $Col\ B\ C\ D$   
**shows**  $Per\ A\ B\ D$   
 $\langle proof \rangle$

**lemma** *l8-13-2*:  
**assumes**  $A \neq B$  **and**  
 $C \neq D$  **and**  
 $Col\ X\ A\ B$  **and**  
 $Col\ X\ C\ D$  **and**  
 $\exists U. \exists V. Col\ U\ A\ B \wedge Col\ V\ C\ D \wedge U \neq X \wedge V \neq X \wedge Per\ U\ X\ V$   
**shows**  $X\ PerpAt\ A\ B\ C\ D$   
 $\langle proof \rangle$

**lemma** *l8-14-1*:  
 $\neg A\ B\ Perp\ A\ B$   
 $\langle proof \rangle$

**lemma** *l8-14-2-1a*:  
**assumes**  $X\ PerpAt\ A\ B\ C\ D$   
**shows**  $A\ B\ Perp\ C\ D$   
 $\langle proof \rangle$

**lemma** *perp-in-distinct*:

**assumes**  $X \text{ PerpAt } A \ B \ C \ D$

**shows**  $A \neq B \wedge C \neq D$

$\langle$ *proof* $\rangle$

**lemma** *l8-14-2-1b*:

**assumes**  $X \text{ PerpAt } A \ B \ C \ D$  **and**

$\text{Col } Y \ A \ B$  **and**

$\text{Col } Y \ C \ D$

**shows**  $X = Y$

$\langle$ *proof* $\rangle$

**lemma** *l8-14-2-1b-bis*:

**assumes**  $A \ B \ \text{Perp} \ C \ D$  **and**

$\text{Col } X \ A \ B$  **and**

$\text{Col } X \ C \ D$

**shows**  $X \text{ PerpAt } A \ B \ C \ D$

$\langle$ *proof* $\rangle$

**lemma** *l8-14-2-2*:

**assumes**  $A \ B \ \text{Perp} \ C \ D$  **and**

$\forall Y. (\text{Col } Y \ A \ B \wedge \text{Col } Y \ C \ D) \longrightarrow X = Y$

**shows**  $X \text{ PerpAt } A \ B \ C \ D$

$\langle$ *proof* $\rangle$

**lemma** *l8-14-3*:

**assumes**  $X \text{ PerpAt } A \ B \ C \ D$  **and**

$Y \text{ PerpAt } A \ B \ C \ D$

**shows**  $X = Y$

$\langle$ *proof* $\rangle$

**lemma** *l8-15-1*:

**assumes**  $\text{Col } A \ B \ X$  **and**

$A \ B \ \text{Perp} \ C \ X$

**shows**  $X \text{ PerpAt } A \ B \ C \ X$

$\langle$ *proof* $\rangle$

**lemma** *l8-15-2*:

**assumes**  $\text{Col } A \ B \ X$  **and**

$X \text{ PerpAt } A \ B \ C \ X$

**shows**  $A \ B \ \text{Perp} \ C \ X$

$\langle$ *proof* $\rangle$

**lemma** *perp-in-per*:

**assumes**  $B \text{ PerpAt } A \ B \ B \ C$

**shows**  $\text{Per } A \ B \ C$

$\langle$ *proof* $\rangle$

**lemma** *perp-sym*:

**assumes**  $A \ B \ \text{Perp} \ A \ B$

**shows**  $C \ D \ \text{Perp} \ C \ D$

$\langle$ *proof* $\rangle$

**lemma** *perp-col0*:

**assumes**  $A \ B \ \text{Perp} \ C \ D$  **and**

$X \neq Y$  **and**

$\text{Col } A \ B \ X$  **and**

$\text{Col } A \ B \ Y$

**shows**  $C \ D \ \text{Perp} \ X \ Y$

$\langle$ *proof* $\rangle$

**lemma** *per-perp-in*:

**assumes**  $A \neq B$  **and**

$B \neq C$  **and**

$\text{Per } A \ B \ C$

**shows**  $B \text{ PerpAt } A B B C$   
*<proof>*

**lemma** *per-perp*:  
**assumes**  $A \neq B$  **and**  
 $B \neq C$  **and**  
 $\text{Per } A B C$   
**shows**  $A B \text{ Perp } B C$   
*<proof>*

**lemma** *perp-left-comm*:  
**assumes**  $A B \text{ Perp } C D$   
**shows**  $B A \text{ Perp } C D$   
*<proof>*

**lemma** *perp-right-comm*:  
**assumes**  $A B \text{ Perp } C D$   
**shows**  $A B \text{ Perp } D C$   
*<proof>*

**lemma** *perp-comm*:  
**assumes**  $A B \text{ Perp } C D$   
**shows**  $B A \text{ Perp } D C$   
*<proof>*

**lemma** *perp-in-sym*:  
**assumes**  $X \text{ PerpAt } A B C D$   
**shows**  $X \text{ PerpAt } C D A B$   
*<proof>*

**lemma** *perp-in-left-comm*:  
**assumes**  $X \text{ PerpAt } A B C D$   
**shows**  $X \text{ PerpAt } B A C D$   
*<proof>*

**lemma** *perp-in-right-comm*:  
**assumes**  $X \text{ PerpAt } A B C D$   
**shows**  $X \text{ PerpAt } A B D C$   
*<proof>*

**lemma** *perp-in-comm*:  
**assumes**  $X \text{ PerpAt } A B C D$   
**shows**  $X \text{ PerpAt } B A D C$   
*<proof>*

**lemma** *Perp-cases*:  
**assumes**  $A B \text{ Perp } C D \vee B A \text{ Perp } C D \vee A B \text{ Perp } D C \vee B A \text{ Perp } D C \vee C D \text{ Perp } A B \vee C D \text{ Perp } B A \vee$   
 $D C \text{ Perp } A B \vee D C \text{ Perp } B A$   
**shows**  $A B \text{ Perp } C D$   
*<proof>*

**lemma** *Perp-perm* :  
**assumes**  $A B \text{ Perp } C D$   
**shows**  $A B \text{ Perp } C D \wedge B A \text{ Perp } C D \wedge A B \text{ Perp } D C \wedge B A \text{ Perp } D C \wedge C D \text{ Perp } A B \wedge C D \text{ Perp } B A \wedge D C$   
 $\text{Perp } A B \wedge D C \text{ Perp } B A$   
*<proof>*

**lemma** *Perp-in-cases*:  
**assumes**  $X \text{ PerpAt } A B C D \vee X \text{ PerpAt } B A C D \vee X \text{ PerpAt } A B D C \vee X \text{ PerpAt } B A D C \vee X \text{ PerpAt } C D A$   
 $B \vee X \text{ PerpAt } C D B A \vee X \text{ PerpAt } D C A B \vee X \text{ PerpAt } D C B A$   
**shows**  $X \text{ PerpAt } A B C D$   
*<proof>*

**lemma** *Perp-in-perm*:  
**assumes**  $X \text{ PerpAt } A B C D$   
**shows**  $X \text{ PerpAt } A B C D \wedge X \text{ PerpAt } B A C D \wedge X \text{ PerpAt } A B D C \wedge X \text{ PerpAt } B A D C \wedge X \text{ PerpAt } C D A B$

$\wedge X \text{ PerpAt } C D B A \wedge X \text{ PerpAt } D C A B \wedge X \text{ PerpAt } D C B A$   
 $\langle \text{proof} \rangle$

**lemma** *perp-in-col*:  
**assumes**  $X \text{ PerpAt } A B C D$   
**shows**  $\text{Col } A B X \wedge \text{Col } C D X$   
 $\langle \text{proof} \rangle$

**lemma** *perp-perp-in*:  
**assumes**  $A B \text{ Perp } C A$   
**shows**  $A \text{ PerpAt } A B C A$   
 $\langle \text{proof} \rangle$

**lemma** *perp-per-1*:  
**assumes**  $A B \text{ Perp } C A$   
**shows**  $\text{Per } B A C$   
 $\langle \text{proof} \rangle$

**lemma** *perp-per-2*:  
**assumes**  $A B \text{ Perp } A C$   
**shows**  $\text{Per } B A C$   
 $\langle \text{proof} \rangle$

**lemma** *perp-col*:  
**assumes**  $A \neq E$  **and**  
 $A B \text{ Perp } C D$  **and**  
 $\text{Col } A B E$   
**shows**  $A E \text{ Perp } C D$   
 $\langle \text{proof} \rangle$

**lemma** *perp-col2*:  
**assumes**  $A B \text{ Perp } X Y$  **and**  
 $C \neq D$  **and**  
 $\text{Col } A B C$  **and**  
 $\text{Col } A B D$   
**shows**  $C D \text{ Perp } X Y$   
 $\langle \text{proof} \rangle$

**lemma** *perp-col4*:  
**assumes**  $P \neq Q$  **and**  
 $R \neq S$  **and**  
 $\text{Col } A B P$  **and**  
 $\text{Col } A B Q$  **and**  
 $\text{Col } C D R$  **and**  
 $\text{Col } C D S$  **and**  
 $A B \text{ Perp } C D$   
**shows**  $P Q \text{ Perp } R S$   
 $\langle \text{proof} \rangle$

**lemma** *perp-not-eq-1*:  
**assumes**  $A B \text{ Perp } C D$   
**shows**  $A \neq B$   
 $\langle \text{proof} \rangle$

**lemma** *perp-not-eq-2*:  
**assumes**  $A B \text{ Perp } C D$   
**shows**  $C \neq D$   
 $\langle \text{proof} \rangle$

**lemma** *diff-per-diff*:  
**assumes**  $A \neq B$  **and**  
 $\text{Cong } A P B R$  **and**  
 $\text{Per } B A P$   
**and**  $\text{Per } A B R$   
**shows**  $P \neq R$   
 $\langle \text{proof} \rangle$

**lemma** *per-not-colp*:  
**assumes**  $A \neq B$  **and**  
 $A \neq P$  **and**  
 $B \neq R$  **and**  
 $Per\ B\ A\ P$   
**and**  $Per\ A\ B\ R$   
**shows**  $\neg\ Col\ P\ A\ R$   
 $\langle proof \rangle$

**lemma** *per-not-col*:  
**assumes**  $A \neq B$  **and**  
 $B \neq C$  **and**  
 $Per\ A\ B\ C$   
**shows**  $\neg\ Col\ A\ B\ C$   
 $\langle proof \rangle$

**lemma** *perp-not-col2*:  
**assumes**  $A\ B\ Perp\ C\ D$   
**shows**  $\neg\ Col\ A\ B\ C \vee \neg\ Col\ A\ B\ D$   
 $\langle proof \rangle$

**lemma** *perp-not-col*:  
**assumes**  $A\ B\ Perp\ P\ A$   
**shows**  $\neg\ Col\ A\ B\ P$   
 $\langle proof \rangle$

**lemma** *perp-in-col-perp-in*:  
**assumes**  $C \neq E$  **and**  
 $Col\ C\ D\ E$  **and**  
 $P\ PerpAt\ A\ B\ C\ D$   
**shows**  $P\ PerpAt\ A\ B\ C\ E$   
 $\langle proof \rangle$

**lemma** *perp-col2-bis*:  
**assumes**  $A\ B\ Perp\ C\ D$  **and**  
 $Col\ C\ D\ P$  **and**  
 $Col\ C\ D\ Q$  **and**  
 $P \neq Q$   
**shows**  $A\ B\ Perp\ P\ Q$   
 $\langle proof \rangle$

**lemma** *perp-in-perp-bis-R1*:  
**assumes**  $X \neq A$  **and**  
 $X\ PerpAt\ A\ B\ C\ D$   
**shows**  $X\ B\ Perp\ C\ D \vee A\ X\ Perp\ C\ D$   
 $\langle proof \rangle$

**lemma** *perp-in-perp-bis*:  
**assumes**  $X\ PerpAt\ A\ B\ C\ D$   
**shows**  $X\ B\ Perp\ C\ D \vee A\ X\ Perp\ C\ D$   
 $\langle proof \rangle$

**lemma** *col-per-perp*:  
**assumes**  $A \neq B$  **and**  
 $B \neq C$  **and**  
  
 $D \neq C$  **and**  
 $Col\ B\ C\ D$  **and**  
 $Per\ A\ B\ C$   
**shows**  $C\ D\ Perp\ A\ B$   
 $\langle proof \rangle$

**lemma** *per-cong-mid-R1*:  
**assumes**  $B = H$  **and**

*Bet A B C and*  
*Cong A H C H and*  
*Per H B C*  
**shows** *B Midpoint A C*  
 ⟨proof⟩

**lemma** *per-cong-mid-R2:*

**assumes**  
*B ≠ C and*  
*Bet A B C and*  
*Cong A H C H and*  
*Per H B C*  
**shows** *B Midpoint A C*  
 ⟨proof⟩

**lemma** *per-cong-mid:*

**assumes** *B ≠ C and*  
*Bet A B C and*  
*Cong A H C H and*  
*Per H B C*  
**shows** *B Midpoint A C*  
 ⟨proof⟩

**lemma** *per-double-cong:*

**assumes** *Per A B C and*  
*B Midpoint C C'*  
**shows** *Cong A C A C'*  
 ⟨proof⟩

**lemma** *cong-perp-or-mid-R1:*

**assumes** *Col A B X and*  
*A ≠ B and*  
*M Midpoint A B and*  
*Cong A X B X*  
**shows**  $X = M \vee \neg \text{Col } A B X \wedge M \text{ PerpAt } X M A B$   
 ⟨proof⟩

**lemma** *cong-perp-or-mid-R2:*

**assumes**  $\neg \text{Col } A B X$  **and**  
*A ≠ B and*  
*M Midpoint A B and*  
*Cong A X B X*  
**shows**  $X = M \vee \neg \text{Col } A B X \wedge M \text{ PerpAt } X M A B$   
 ⟨proof⟩

**lemma** *cong-perp-or-mid:*

**assumes** *A ≠ B and*  
*M Midpoint A B and*  
*Cong A X B X*  
**shows**  $X = M \vee \neg \text{Col } A B X \wedge M \text{ PerpAt } X M A B$   
 ⟨proof⟩

**lemma** *col-per2-cases:*

**assumes** *B ≠ C and*  
*B' ≠ C and*  
*C ≠ D and*  
*Col B C D and*  
*Per A B C and*  
*Per A B' C*  
**shows**  $B = B' \vee \neg \text{Col } B' C D$   
 ⟨proof⟩

**lemma** *l8-16-1:*

**assumes** *Col A B X and*  
*Col A B U and*  
*A B Perp C X*

**shows**  $\neg \text{Col } A B C \wedge \text{Per } C X U$   
*<proof>*

**lemma** *l8-16-2*:  
**assumes**  $\text{Col } A B X$  **and**  
 $\text{Col } A B U$   
**and**  $U \neq X$  **and**  
 $\neg \text{Col } A B C$  **and**  
 $\text{Per } C X U$   
**shows**  $A B \text{Perp } C X$   
*<proof>*

**lemma** *l8-18-uniqueness*:  
**assumes**  
 $\text{Col } A B X$  **and**  
 $A B \text{Perp } C X$  **and**  
 $\text{Col } A B Y$  **and**  
 $A B \text{Perp } C Y$   
**shows**  $X = Y$   
*<proof>*

**lemma** *midpoint-distinct*:  
**assumes**  $\neg \text{Col } A B C$  **and**  
 $\text{Col } A B X$  **and**  
 $X \text{Midpoint } C C'$   
**shows**  $C \neq C'$   
*<proof>*

**lemma** *l8-20-1-R1*:  
**assumes**  $A = B$   
**shows**  $\text{Per } B A P$   
*<proof>*

**lemma** *l8-20-1-R2*:  
**assumes**  $A \neq B$  **and**  
 $\text{Per } A B C$  **and**  
 $P \text{Midpoint } C' D$  **and**  
 $A \text{Midpoint } C' C$  **and**  
 $B \text{Midpoint } D C$   
**shows**  $\text{Per } B A P$   
*<proof>*

**lemma** *l8-20-1*:  
**assumes**  $\text{Per } A B C$  **and**  
 $P \text{Midpoint } C' D$  **and**  
 $A \text{Midpoint } C' C$  **and**  
 $B \text{Midpoint } D C$   
**shows**  $\text{Per } B A P$   
*<proof>*

**lemma** *l8-20-2*:  
**assumes**  $P \text{Midpoint } C' D$  **and**  
 $A \text{Midpoint } C' C$  **and**  
 $B \text{Midpoint } D C$  **and**  
 $B \neq C$   
**shows**  $A \neq P$   
*<proof>*

**lemma** *perp-col1*:  
**assumes**  $C \neq X$  **and**  
 $A B \text{Perp } C D$  **and**  
 $\text{Col } C D X$   
**shows**  $A B \text{Perp } C X$   
*<proof>*

**lemma** *l8-18-existence*:

**assumes**  $\neg \text{Col } A \ B \ C$   
**shows**  $\exists X. \text{Col } A \ B \ X \wedge A \ B \ \text{Perp } C \ X$   
 $\langle \text{proof} \rangle$

**lemma** *l8-21-aux*:  
**assumes**  $\neg \text{Col } A \ B \ C$   
**shows**  $\exists P. \exists T. (A \ B \ \text{Perp } P \ A \wedge \text{Col } A \ B \ T \wedge \text{Bet } C \ T \ P)$   
 $\langle \text{proof} \rangle$

**lemma** *l8-21*:  
**assumes**  $A \neq B$   
**shows**  $\exists P \ T. A \ B \ \text{Perp } P \ A \wedge \text{Col } A \ B \ T \wedge \text{Bet } C \ T \ P$   
 $\langle \text{proof} \rangle$

**lemma** *per-cong*:  
**assumes**  $A \neq B$  **and**  
 $A \neq P$  **and**  
 $\text{Per } B \ A \ P$  **and**  
 $\text{Per } A \ B \ R$  **and**  
 $\text{Cong } A \ P \ B \ R$  **and**  
 $\text{Col } A \ B \ X$  **and**  
 $\text{Bet } P \ X \ R$   
**shows**  $\text{Cong } A \ R \ P \ B$   
 $\langle \text{proof} \rangle$

**lemma** *perp-cong*:  
**assumes**  $A \neq B$  **and**  
 $A \neq P$  **and**  
 $A \ B \ \text{Perp } P \ A$  **and**  
 $A \ B \ \text{Perp } R \ B$  **and**  
 $\text{Cong } A \ P \ B \ R$  **and**  
 $\text{Col } A \ B \ X$  **and**  
 $\text{Bet } P \ X \ R$   
**shows**  $\text{Cong } A \ R \ P \ B$   
 $\langle \text{proof} \rangle$

**lemma** *perp-exists*:  
**assumes**  $A \neq B$   
**shows**  $\exists X. \text{PO } X \ \text{Perp } A \ B$   
 $\langle \text{proof} \rangle$

**lemma** *perp-vector*:  
**assumes**  $A \neq B$   
**shows**  $\exists X \ Y. A \ B \ \text{Perp } X \ Y$   
 $\langle \text{proof} \rangle$

**lemma** *midpoint-existence-aux*:  
**assumes**  $A \neq B$  **and**  
 $A \ B \ \text{Perp } Q \ B$  **and**  
 $A \ B \ \text{Perp } P \ A$  **and**  
 $\text{Col } A \ B \ T$  **and**  
 $\text{Bet } Q \ T \ P$  **and**  
 $A \ P \ \text{Le } B \ Q$   
**shows**  $\exists X. X \ \text{Midpoint } A \ B$   
 $\langle \text{proof} \rangle$

**lemma** *midpoint-existence*:  
 $\exists X. X \ \text{Midpoint } A \ B$   
 $\langle \text{proof} \rangle$

**lemma** *perp-in-id*:  
**assumes**  $X \ \text{PerpAt } A \ B \ C \ A$   
**shows**  $X = A$   
 $\langle \text{proof} \rangle$

**lemma** *l8-22*:

**assumes**  $A \neq B$  **and**  
 $A \neq P$  **and**  
 $Per\ B\ A\ P$  **and**  
 $Per\ A\ B\ R$  **and**  
 $Cong\ A\ P\ B\ R$  **and**  
 $Col\ A\ B\ X$  **and**  
 $Bet\ P\ X\ R$  **and**  
 $Cong\ A\ R\ P\ B$   
**shows**  $X\ Midpoint\ A\ B \wedge X\ Midpoint\ P\ R$   
 $\langle proof \rangle$

**lemma** *l8-22-bis*:  
**assumes**  $A \neq B$  **and**  
 $A \neq P$  **and**  
 $A\ B\ Perp\ P\ A$  **and**  
 $A\ B\ Perp\ R\ B$  **and**  
 $Cong\ A\ P\ B\ R$  **and**  
 $Col\ A\ B\ X$  **and**  
 $Bet\ P\ X\ R$   
**shows**  $Cong\ A\ R\ P\ B \wedge X\ Midpoint\ A\ B \wedge X\ Midpoint\ P\ R$   
 $\langle proof \rangle$

**lemma** *perp-in-perp*:  
**assumes**  $X\ PerpAt\ A\ B\ C\ D$   
**shows**  $A\ B\ Perp\ C\ D$   
 $\langle proof \rangle$

**lemma** *perp-proj*:  
**assumes**  $A\ B\ Perp\ C\ D$  **and**  
 $\neg\ Col\ A\ C\ D$   
**shows**  $\exists\ X.\ Col\ A\ B\ X \wedge A\ X\ Perp\ C\ D$   
 $\langle proof \rangle$

**lemma** *l8-24* :  
**assumes**  $P\ A\ Perp\ A\ B$  **and**  
 $Q\ B\ Perp\ A\ B$  **and**  
 $Col\ A\ B\ T$  **and**  
 $Bet\ P\ T\ Q$  **and**  
 $Bet\ B\ R\ Q$  **and**  
 $Cong\ A\ P\ B\ R$   
**shows**  $\exists\ X.\ X\ Midpoint\ A\ B \wedge X\ Midpoint\ P\ R$   
 $\langle proof \rangle$

**lemma** *col-per2--per*:  
**assumes**  $A \neq B$  **and**  
 $Col\ A\ B\ C$  **and**  
 $Per\ A\ X\ P$  **and**  
 $Per\ B\ X\ P$   
**shows**  $Per\ C\ X\ P$   
 $\langle proof \rangle$

**lemma** *perp-in-per-1*:  
**assumes**  $X\ PerpAt\ A\ B\ C\ D$   
**shows**  $Per\ A\ X\ C$   
 $\langle proof \rangle$

**lemma** *perp-in-per-2*:  
**assumes**  $X\ PerpAt\ A\ B\ C\ D$   
**shows**  $Per\ A\ X\ D$   
 $\langle proof \rangle$

**lemma** *perp-in-per-3*:  
**assumes**  $X\ PerpAt\ A\ B\ C\ D$   
**shows**  $Per\ B\ X\ C$   
 $\langle proof \rangle$

**lemma** *perp-in-per-4*:  
  **assumes**  $X \text{ PerpAt } A B C D$   
  **shows**  $\text{Per } B X D$   
  ⟨*proof*⟩

## 3.8 Planes

### 3.8.1 Coplanar

**lemma** *coplanar-perm-1*:  
  **assumes**  $\text{Coplanar } A B C D$   
  **shows**  $\text{Coplanar } A B D C$   
  ⟨*proof*⟩

**lemma** *coplanar-perm-2*:  
  **assumes**  $\text{Coplanar } A B C D$   
  **shows**  $\text{Coplanar } A C B D$   
  ⟨*proof*⟩

**lemma** *coplanar-perm-3*:  
  **assumes**  $\text{Coplanar } A B C D$   
  **shows**  $\text{Coplanar } A C D B$   
  ⟨*proof*⟩

**lemma** *coplanar-perm-4*:  
  **assumes**  $\text{Coplanar } A B C D$   
  **shows**  $\text{Coplanar } A D B C$   
  ⟨*proof*⟩

**lemma** *coplanar-perm-5*:  
  **assumes**  $\text{Coplanar } A B C D$   
  **shows**  $\text{Coplanar } A D C B$   
  ⟨*proof*⟩

**lemma** *coplanar-perm-6*:  
  **assumes**  $\text{Coplanar } A B C D$   
  **shows**  $\text{Coplanar } B A C D$   
  ⟨*proof*⟩

**lemma** *coplanar-perm-7*:  
  **assumes**  $\text{Coplanar } A B C D$   
  **shows**  $\text{Coplanar } B A D C$   
  ⟨*proof*⟩

**lemma** *coplanar-perm-8*:  
  **assumes**  $\text{Coplanar } A B C D$   
  **shows**  $\text{Coplanar } B C A D$   
  ⟨*proof*⟩

**lemma** *coplanar-perm-9*:  
  **assumes**  $\text{Coplanar } A B C D$   
  **shows**  $\text{Coplanar } B C D A$   
  ⟨*proof*⟩

**lemma** *coplanar-perm-10*:  
  **assumes**  $\text{Coplanar } A B C D$   
  **shows**  $\text{Coplanar } B D A C$   
  ⟨*proof*⟩

**lemma** *coplanar-perm-11*:  
  **assumes**  $\text{Coplanar } A B C D$   
  **shows**  $\text{Coplanar } B D C A$   
  ⟨*proof*⟩

**lemma** *coplanar-perm-12*:  
  **assumes**  $\text{Coplanar } A B C D$   
  **shows**  $\text{Coplanar } C A B D$

$\langle proof \rangle$

**lemma** *coplanar-perm-13*:  
  **assumes** *Coplanar A B C D*  
  **shows** *Coplanar C A D B*  
 $\langle proof \rangle$

**lemma** *coplanar-perm-14*:  
  **assumes** *Coplanar A B C D*  
  **shows** *Coplanar C B A D*  
 $\langle proof \rangle$

**lemma** *coplanar-perm-15*:  
  **assumes** *Coplanar A B C D*  
  **shows** *Coplanar C B D A*  
 $\langle proof \rangle$

**lemma** *coplanar-perm-16*:  
  **assumes** *Coplanar A B C D*  
  **shows** *Coplanar C D A B*  
 $\langle proof \rangle$

**lemma** *coplanar-perm-17*:  
  **assumes** *Coplanar A B C D*  
  **shows** *Coplanar C D B A*  
 $\langle proof \rangle$

**lemma** *coplanar-perm-18*:  
  **assumes** *Coplanar A B C D*  
  **shows** *Coplanar D A B C*  
 $\langle proof \rangle$

**lemma** *coplanar-perm-19*:  
  **assumes** *Coplanar A B C D*  
  **shows** *Coplanar D A C B*  
 $\langle proof \rangle$

**lemma** *coplanar-perm-20*:  
  **assumes** *Coplanar A B C D*  
  **shows** *Coplanar D B A C*  
 $\langle proof \rangle$

**lemma** *coplanar-perm-21*:  
  **assumes** *Coplanar A B C D*  
  **shows** *Coplanar D B C A*  
 $\langle proof \rangle$

**lemma** *coplanar-perm-22*:  
  **assumes** *Coplanar A B C D*  
  **shows** *Coplanar D C A B*  
 $\langle proof \rangle$

**lemma** *coplanar-perm-23*:  
  **assumes** *Coplanar A B C D*  
  **shows** *Coplanar D C B A*  
 $\langle proof \rangle$

**lemma** *ncoplanar-perm-1*:  
  **assumes**  $\neg$  *Coplanar A B C D*  
  **shows**  $\neg$  *Coplanar A B D C*  
 $\langle proof \rangle$

**lemma** *ncoplanar-perm-2*:  
  **assumes**  $\neg$  *Coplanar A B C D*  
  **shows**  $\neg$  *Coplanar A C B D*  
 $\langle proof \rangle$

**lemma** *ncoplanar-perm-3*:  
 **assumes**  $\neg \text{Coplanar } A B C D$   
 **shows**  $\neg \text{Coplanar } A C D B$   
  $\langle \text{proof} \rangle$

**lemma** *ncoplanar-perm-4*:  
 **assumes**  $\neg \text{Coplanar } A B C D$   
 **shows**  $\neg \text{Coplanar } A D B C$   
  $\langle \text{proof} \rangle$

**lemma** *ncoplanar-perm-5*:  
 **assumes**  $\neg \text{Coplanar } A B C D$   
 **shows**  $\neg \text{Coplanar } A D C B$   
  $\langle \text{proof} \rangle$

**lemma** *ncoplanar-perm-6*:  
 **assumes**  $\neg \text{Coplanar } A B C D$   
 **shows**  $\neg \text{Coplanar } B A C D$   
  $\langle \text{proof} \rangle$

**lemma** *ncoplanar-perm-7*:  
 **assumes**  $\neg \text{Coplanar } A B C D$   
 **shows**  $\neg \text{Coplanar } B A D C$   
  $\langle \text{proof} \rangle$

**lemma** *ncoplanar-perm-8*:  
 **assumes**  $\neg \text{Coplanar } A B C D$   
 **shows**  $\neg \text{Coplanar } B C A D$   
  $\langle \text{proof} \rangle$

**lemma** *ncoplanar-perm-9*:  
 **assumes**  $\neg \text{Coplanar } A B C D$   
 **shows**  $\neg \text{Coplanar } B C D A$   
  $\langle \text{proof} \rangle$

**lemma** *ncoplanar-perm-10*:  
 **assumes**  $\neg \text{Coplanar } A B C D$   
 **shows**  $\neg \text{Coplanar } B D A C$   
  $\langle \text{proof} \rangle$

**lemma** *ncoplanar-perm-11*:  
 **assumes**  $\neg \text{Coplanar } A B C D$   
 **shows**  $\neg \text{Coplanar } B D C A$   
  $\langle \text{proof} \rangle$

**lemma** *ncoplanar-perm-12*:  
 **assumes**  $\neg \text{Coplanar } A B C D$   
 **shows**  $\neg \text{Coplanar } C A B D$   
  $\langle \text{proof} \rangle$

**lemma** *ncoplanar-perm-13*:  
 **assumes**  $\neg \text{Coplanar } A B C D$   
 **shows**  $\neg \text{Coplanar } C A D B$   
  $\langle \text{proof} \rangle$

**lemma** *ncoplanar-perm-14*:  
 **assumes**  $\neg \text{Coplanar } A B C D$   
 **shows**  $\neg \text{Coplanar } C B A D$   
  $\langle \text{proof} \rangle$

**lemma** *ncoplanar-perm-15*:  
 **assumes**  $\neg \text{Coplanar } A B C D$   
 **shows**  $\neg \text{Coplanar } C B D A$   
  $\langle \text{proof} \rangle$

**lemma** *ncoplanar-perm-16*:  
**assumes**  $\neg \text{Coplanar } A B C D$   
**shows**  $\neg \text{Coplanar } C D A B$   
*<proof>*

**lemma** *ncoplanar-perm-17*:  
**assumes**  $\neg \text{Coplanar } A B C D$   
**shows**  $\neg \text{Coplanar } C D B A$   
*<proof>*

**lemma** *ncoplanar-perm-18*:  
**assumes**  $\neg \text{Coplanar } A B C D$   
**shows**  $\neg \text{Coplanar } D A B C$   
*<proof>*

**lemma** *ncoplanar-perm-19*:  
**assumes**  $\neg \text{Coplanar } A B C D$   
**shows**  $\neg \text{Coplanar } D A C B$   
*<proof>*

**lemma** *ncoplanar-perm-20*:  
**assumes**  $\neg \text{Coplanar } A B C D$   
**shows**  $\neg \text{Coplanar } D B A C$   
*<proof>*

**lemma** *ncoplanar-perm-21*:  
**assumes**  $\neg \text{Coplanar } A B C D$   
**shows**  $\neg \text{Coplanar } D B C A$   
*<proof>*

**lemma** *ncoplanar-perm-22*:  
**assumes**  $\neg \text{Coplanar } A B C D$   
**shows**  $\neg \text{Coplanar } D C A B$   
*<proof>*

**lemma** *ncoplanar-perm-23*:  
**assumes**  $\neg \text{Coplanar } A B C D$   
**shows**  $\neg \text{Coplanar } D C B A$   
*<proof>*

**lemma** *coplanar-trivial*:  
**shows**  $\text{Coplanar } A A B C$   
*<proof>*

**lemma** *col--coplanar*:  
**assumes**  $\text{Col } A B C$   
**shows**  $\text{Coplanar } A B C D$   
*<proof>*

**lemma** *ncop--ncol*:  
**assumes**  $\neg \text{Coplanar } A B C D$   
**shows**  $\neg \text{Col } A B C$   
*<proof>*

**lemma** *ncop--ncols*:  
**assumes**  $\neg \text{Coplanar } A B C D$   
**shows**  $\neg \text{Col } A B C \wedge \neg \text{Col } A B D \wedge \neg \text{Col } A C D \wedge \neg \text{Col } B C D$   
*<proof>*

**lemma** *bet--coplanar*:  
**assumes**  $\text{Bet } A B C$   
**shows**  $\text{Coplanar } A B C D$   
*<proof>*

**lemma** *out--coplanar*:  
**assumes**  $A \text{ Out } B C$

**shows** *Coplanar A B C D*  
*<proof>*

**lemma** *midpoint--coplanar*:  
**assumes** *A Midpoint B C*  
**shows** *Coplanar A B C D*  
*<proof>*

**lemma** *perp--coplanar*:  
**assumes** *A B Perp C D*  
**shows** *Coplanar A B C D*  
*<proof>*

**lemma** *ts--coplanar*:  
**assumes** *A B TS C D*  
**shows** *Coplanar A B C D*  
*<proof>*

**lemma** *reflectl--coplanar*:  
**assumes** *A B ReflectL C D*  
**shows** *Coplanar A B C D*  
*<proof>*

**lemma** *reflect--coplanar*:  
**assumes** *A B Reflect C D*  
**shows** *Coplanar A B C D*  
*<proof>*

**lemma** *inangle--coplanar*:  
**assumes** *A InAngle B C D*  
**shows** *Coplanar A B C D*  
*<proof>*

**lemma** *pars--coplanar*:  
**assumes** *A B ParStrict C D*  
**shows** *Coplanar A B C D*  
*<proof>*

**lemma** *par--coplanar*:  
**assumes** *A B Par C D*  
**shows** *Coplanar A B C D*  
*<proof>*

**lemma** *plg--coplanar*:  
**assumes** *Plg A B C D*  
**shows** *Coplanar A B C D*  
*<proof>*

**lemma** *plgs--coplanar*:  
**assumes** *ParallelogramStrict A B C D*  
**shows** *Coplanar A B C D*  
*<proof>*

**lemma** *plgf--coplanar*:  
**assumes** *ParallelogramFlat A B C D*  
**shows** *Coplanar A B C D*  
*<proof>*

**lemma** *parallelogram--coplanar*:  
**assumes** *Parallelogram A B C D*  
**shows** *Coplanar A B C D*  
*<proof>*

**lemma** *rhombus--coplanar*:  
**assumes** *Rhombus A B C D*  
**shows** *Coplanar A B C D*

*<proof>*

**lemma** *rectangle--coplanar*:  
**assumes** *Rectangle A B C D*  
**shows** *Coplanar A B C D*  
*<proof>*

**lemma** *square--coplanar*:  
**assumes** *Square A B C D*  
**shows** *Coplanar A B C D*  
*<proof>*

**lemma** *lambert--coplanar*:  
**assumes** *Lambert A B C D*  
**shows** *Coplanar A B C D*  
*<proof>*

### 3.8.2 Planes

**lemma** *ts-distincts*:  
**assumes** *A B TS P Q*  
**shows**  $A \neq B \wedge A \neq P \wedge A \neq Q \wedge B \neq P \wedge B \neq Q \wedge P \neq Q$   
*<proof>*

**lemma** *l9-2*:  
**assumes** *A B TS P Q*  
**shows** *A B TS Q P*  
*<proof>*

**lemma** *invert-two-sides*:  
**assumes** *A B TS P Q*  
**shows** *B A TS P Q*  
*<proof>*

**lemma** *l9-3*:  
**assumes** *P Q TS A C* **and**  
*Col M P Q* **and**  
*M Midpoint A C* **and**  
*Col R P Q* **and**  
*R Out A B*  
**shows** *P Q TS B C*  
*<proof>*

**lemma** *mid-preserves-col*:  
**assumes** *Col A B C* **and**  
*M Midpoint A A'* **and**  
*M Midpoint B B'* **and**  
*M Midpoint C C'*  
**shows** *Col A' B' C'*  
*<proof>*

**lemma** *per-mid-per*:  
**assumes**  
*Per X A B* **and**  
*M Midpoint A B* **and**  
*M Midpoint X Y*  
**shows**  $Cong A X B Y \wedge Per Y B A$   
*<proof>*

**lemma** *sym-preserve-diff*:  
**assumes**  $A \neq B$  **and**  
*M Midpoint A A'* **and**  
*M Midpoint B B'*  
**shows**  $A' \neq B'$   
*<proof>*

**lemma l9-4-1-aux-R1:**  
**assumes**  $R = S$  **and**  
 $S C Le R A$  **and**  
 $P Q TS A C$  **and**  
 $Col R P Q$  **and**  
 $P Q Perp A R$  **and**  
 $Col S P Q$  **and**  
 $P Q Perp C S$  **and**  
 $M Midpoint R S$   
**shows**  $\forall U C'. M Midpoint U C' \longrightarrow (R Out U A \longleftrightarrow S Out C C')$   
⟨proof⟩

**lemma l9-4-1-aux-R21:**  
**assumes**  $R \neq S$  **and**  
 $S C Le R A$  **and**  
 $P Q TS A C$  **and**  
 $Col R P Q$  **and**  
 $P Q Perp A R$  **and**  
 $Col S P Q$  **and**  
 $P Q Perp C S$  **and**  
 $M Midpoint R S$   
**shows**  $\forall U C'. M Midpoint U C' \longrightarrow (R Out U A \longleftrightarrow S Out C C')$   
⟨proof⟩

**lemma l9-4-1-aux:**  
**assumes**  $S C Le R A$  **and**  
 $P Q TS A C$  **and**  
 $Col R P Q$  **and**  
 $P Q Perp A R$  **and**  
 $Col S P Q$  **and**  
 $P Q Perp C S$  **and**  
 $M Midpoint R S$   
**shows**  $\forall U C'. (M Midpoint U C' \longrightarrow (R Out U A \longleftrightarrow S Out C C'))$   
⟨proof⟩

**lemma per-col-eq:**  
**assumes**  $Per A B C$  **and**  
 $Col A B C$  **and**  
 $B \neq C$   
**shows**  $A = B$   
⟨proof⟩

**lemma l9-4-1:**  
**assumes**  $P Q TS A C$  **and**  
 $Col R P Q$  **and**  
 $P Q Perp A R$  **and**  
 $Col S P Q$  **and**  
 $P Q Perp C S$  **and**  
 $M Midpoint R S$   
**shows**  $\forall U C'. M Midpoint U C' \longrightarrow (R Out U A \longleftrightarrow S Out C C')$   
⟨proof⟩

**lemma mid-two-sides:**  
**assumes**  $M Midpoint A B$  **and**  
 $\neg Col A B X$  **and**  
 $M Midpoint X Y$   
**shows**  $A B TS X Y$   
⟨proof⟩

**lemma col-preserves-two-sides:**  
**assumes**  $C \neq D$  **and**  
 $Col A B C$  **and**  
 $Col A B D$  **and**  
 $A B TS X Y$   
**shows**  $C D TS X Y$

$\langle \text{proof} \rangle$

**lemma** *out-out-two-sides*:

**assumes**  $A \neq B$  **and**  
 $A B TS X Y$  **and**  
 $Col I A B$  **and**  
 $Col I X Y$  **and**  
 $I Out X U$  **and**  
 $I Out Y V$

**shows**  $A B TS U V$

$\langle \text{proof} \rangle$

**lemma** *l9-4-2-aux-R1*:

**assumes**  $R = S$  **and**  
 $S C Le R A$  **and**  
 $P Q TS A C$  **and**  
 $Col R P Q$  **and**  
 $P Q Perp A R$  **and**  
 $Col S P Q$  **and**  
 $P Q Perp C S$  **and**  
 $R Out U A$  **and**  
 $S Out V C$

**shows**  $P Q TS U V$

$\langle \text{proof} \rangle$

**lemma** *l9-4-2-aux-R2*:

**assumes**  $R \neq S$  **and**  
 $S C Le R A$  **and**  
 $P Q TS A C$  **and**  
 $Col R P Q$  **and**  
 $P Q Perp A R$  **and**  
 $Col S P Q$  **and**  
 $P Q Perp C S$  **and**  
 $R Out U A$  **and**  
 $S Out V C$

**shows**  $P Q TS U V$

$\langle \text{proof} \rangle$

**lemma** *l9-4-2-aux*:

**assumes**  $S C Le R A$  **and**  
 $P Q TS A C$  **and**  
 $Col R P Q$  **and**  
 $P Q Perp A R$  **and**  
 $Col S P Q$  **and**  
 $P Q Perp C S$  **and**  
 $R Out U A$  **and**  
 $S Out V C$

**shows**  $P Q TS U V$

$\langle \text{proof} \rangle$

**lemma** *l9-4-2*:

**assumes**  $P Q TS A C$  **and**  
 $Col R P Q$  **and**  
 $P Q Perp A R$  **and**  
 $Col S P Q$  **and**  
 $P Q Perp C S$  **and**  
 $R Out U A$  **and**  
 $S Out V C$

**shows**  $P Q TS U V$

$\langle \text{proof} \rangle$

**lemma** *l9-5*:

**assumes**  $P Q TS A C$  **and**  
 $Col R P Q$  **and**  
 $R Out A B$

**shows**  $P Q TS B C$

$\langle proof \rangle$

**lemma** *outer-pasch-R1*:

**assumes**  $Col\ P\ Q\ C$  **and**

$Bet\ A\ C\ P$  **and**

$Bet\ B\ Q\ C$

**shows**  $\exists\ X.\ Bet\ A\ X\ B \wedge Bet\ P\ Q\ X$

$\langle proof \rangle$

**lemma** *outer-pasch-R2*:

**assumes**  $\neg\ Col\ P\ Q\ C$  **and**

$Bet\ A\ C\ P$  **and**

$Bet\ B\ Q\ C$

**shows**  $\exists\ X.\ Bet\ A\ X\ B \wedge Bet\ P\ Q\ X$

$\langle proof \rangle$

**lemma** *outer-pasch*:

**assumes**  $Bet\ A\ C\ P$  **and**

$Bet\ B\ Q\ C$

**shows**  $\exists\ X.\ Bet\ A\ X\ B \wedge Bet\ P\ Q\ X$

$\langle proof \rangle$

**lemma** *os-distincts*:

**assumes**  $A\ B\ OS\ X\ Y$

**shows**  $A \neq B \wedge A \neq X \wedge A \neq Y \wedge B \neq X \wedge B \neq Y$

$\langle proof \rangle$

**lemma** *invert-one-side*:

**assumes**  $A\ B\ OS\ P\ Q$

**shows**  $B\ A\ OS\ P\ Q$

$\langle proof \rangle$

**lemma** *l9-8-1*:

**assumes**  $P\ Q\ TS\ A\ C$  **and**

$P\ Q\ TS\ B\ C$

**shows**  $P\ Q\ OS\ A\ B$

$\langle proof \rangle$

**lemma** *not-two-sides-id*:

**shows**  $\neg\ P\ Q\ TS\ A\ A$

$\langle proof \rangle$

**lemma** *l9-8-2*:

**assumes**  $P\ Q\ TS\ A\ C$  **and**

$P\ Q\ OS\ A\ B$

**shows**  $P\ Q\ TS\ B\ C$

$\langle proof \rangle$

**lemma** *l9-9*:

**assumes**  $P\ Q\ TS\ A\ B$

**shows**  $\neg\ P\ Q\ OS\ A\ B$

$\langle proof \rangle$

**lemma** *l9-9-bis*:

**assumes**  $P\ Q\ OS\ A\ B$

**shows**  $\neg\ P\ Q\ TS\ A\ B$

$\langle proof \rangle$

**lemma** *one-side-chara*:

**assumes**  $P\ Q\ OS\ A\ B$

**shows**  $\forall\ X.\ Col\ X\ P\ Q \longrightarrow \neg\ Bet\ A\ X\ B$

$\langle proof \rangle$

**lemma** *l9-10*:

**assumes**  $\neg\ Col\ A\ P\ Q$

**shows**  $\exists\ C.\ P\ Q\ TS\ A\ C$

*<proof>*

**lemma** *one-side-reflexivity*:

**assumes**  $\neg Col\ A\ P\ Q$

**shows**  $P\ Q\ OS\ A\ A$

*<proof>*

**lemma** *one-side-symmetry*:

**assumes**  $P\ Q\ OS\ A\ B$

**shows**  $P\ Q\ OS\ B\ A$

*<proof>*

**lemma** *one-side-transitivity*:

**assumes**  $P\ Q\ OS\ A\ B$  **and**

$P\ Q\ OS\ B\ C$

**shows**  $P\ Q\ OS\ A\ C$

*<proof>*

**lemma** *l9-17*:

**assumes**  $P\ Q\ OS\ A\ C$  **and**

$Bet\ A\ B\ C$

**shows**  $P\ Q\ OS\ A\ B$

*<proof>*

**lemma** *l9-18-R1*:

**assumes**  $Col\ X\ Y\ P$  **and**

$Col\ A\ B\ P$

**and**  $X\ Y\ TS\ A\ B$

**shows**  $Bet\ A\ P\ B \wedge \neg Col\ X\ Y\ A \wedge \neg Col\ X\ Y\ B$

*<proof>*

**lemma** *l9-18-R2*:

**assumes**  $Col\ X\ Y\ P$  **and**

$Col\ A\ B\ P$  **and**

$Bet\ A\ P\ B$  **and**

$\neg Col\ X\ Y\ A$  **and**

$\neg Col\ X\ Y\ B$

**shows**  $X\ Y\ TS\ A\ B$

*<proof>*

**lemma** *l9-18*:

**assumes**  $Col\ X\ Y\ P$  **and**

$Col\ A\ B\ P$

**shows**  $X\ Y\ TS\ A\ B \longleftrightarrow (Bet\ A\ P\ B \wedge \neg Col\ X\ Y\ A \wedge \neg Col\ X\ Y\ B)$

*<proof>*

**lemma** *l9-19-R1*:

**assumes**  $Col\ X\ Y\ P$  **and**

$Col\ A\ B\ P$  **and**

$X\ Y\ OS\ A\ B$

**shows**  $P\ Out\ A\ B \wedge \neg Col\ X\ Y\ A$

*<proof>*

**lemma** *l9-19-R2*:

**assumes**  $Col\ X\ Y\ P$  **and**

$P\ Out\ A\ B$  **and**

$\neg Col\ X\ Y\ A$

**shows**  $X\ Y\ OS\ A\ B$

*<proof>*

**lemma** *l9-19*:

**assumes**  $Col\ X\ Y\ P$  **and**

$Col\ A\ B\ P$

**shows**  $X\ Y\ OS\ A\ B \longleftrightarrow (P\ Out\ A\ B \wedge \neg Col\ X\ Y\ A)$

*<proof>*

**lemma** *one-side-not-col123*:

**assumes**  $A B OS X Y$

**shows**  $\neg Col A B X$

$\langle proof \rangle$

**lemma** *one-side-not-col124*:

**assumes**  $A B OS X Y$

**shows**  $\neg Col A B Y$

$\langle proof \rangle$

**lemma** *col-two-sides*:

**assumes**  $Col A B C$  **and**

$A \neq C$  **and**

$A B TS P Q$

**shows**  $A C TS P Q$

$\langle proof \rangle$

**lemma** *col-one-side*:

**assumes**  $Col A B C$  **and**

$A \neq C$  **and**

$A B OS P Q$

**shows**  $A C OS P Q$

$\langle proof \rangle$

**lemma** *out-out-one-side*:

**assumes**  $A B OS X Y$  **and**

$A Out Y Z$

**shows**  $A B OS X Z$

$\langle proof \rangle$

**lemma** *out-one-side*:

**assumes**  $\neg Col A B X \vee \neg Col A B Y$  **and**

$A Out X Y$

**shows**  $A B OS X Y$

$\langle proof \rangle$

**lemma** *bet--ts*:

**assumes**  $A \neq Y$  **and**

$\neg Col A B X$  **and**

$Bet X A Y$

**shows**  $A B TS X Y$

$\langle proof \rangle$

**lemma** *bet-ts--ts*:

**assumes**  $A B TS X Y$  **and**

$Bet X Y Z$

**shows**  $A B TS X Z$

$\langle proof \rangle$

**lemma** *bet-ts--os*:

**assumes**  $A B TS X Y$  **and**

$Bet X Y Z$

**shows**  $A B OS Y Z$

$\langle proof \rangle$

**lemma** *l9-31* :

**assumes**  $A X OS Y Z$  **and**

$A Z OS Y X$

**shows**  $A Y TS X Z$

$\langle proof \rangle$

**lemma** *col123--nos*:

**assumes**  $Col P Q A$

**shows**  $\neg P Q OS A B$

*<proof>*

**lemma** *col124--nos*:  
**assumes** *Col P Q B*  
**shows**  $\neg P Q OS A B$   
*<proof>*

**lemma** *col2-os--os*:  
**assumes** *C  $\neq$  D and*  
*Col A B C and*  
*Col A B D and*  
*A B OS X Y*  
**shows** *C D OS X Y*  
*<proof>*

**lemma** *os-out-os*:  
**assumes** *Col A B P and*  
*A B OS C D and*  
*P Out C C'*  
**shows** *A B OS C' D*  
*<proof>*

**lemma** *ts-ts-os*:  
**assumes** *A B TS C D and*  
*C D TS A B*  
**shows** *A C OS B D*  
*<proof>*

**lemma** *col-one-side-out*:  
**assumes** *Col A X Y and*  
*A B OS X Y*  
**shows** *A Out X Y*  
*<proof>*

**lemma** *col-two-sides-bet*:  
**assumes** *Col A X Y and*  
*A B TS X Y*  
**shows** *Bet X A Y*  
*<proof>*

**lemma** *os-ts1324--os*:  
**assumes** *A X OS Y Z and*  
*A Y TS X Z*  
**shows** *A Z OS X Y*  
*<proof>*

**lemma** *ts2--ex-bet2*:  
**assumes** *A C TS B D and*  
*B D TS A C*  
**shows**  $\exists X. Bet A X C \wedge Bet B X D$   
*<proof>*

**lemma** *out-one-side-1*:  
**assumes**  $\neg Col A B C$  **and**  
*Col A B X and*  
*X Out C D*  
**shows** *A B OS C D*  
*<proof>*

**lemma** *out-two-sides-two-sides*:  
**assumes**  
*Col A B PX and*  
*PX Out X P and*  
*A B TS P Y*  
**shows** *A B TS X Y*  
*<proof>*

**lemma** *l8-21-bis*:  
**assumes**  $X \neq Y$  **and**  
 $\neg \text{Col } C \ A \ B$   
**shows**  $\exists P. \text{Cong } A \ P \ X \ Y \wedge A \ B \ \text{Perp } P \ A \wedge A \ B \ \text{TS } C \ P$   
 $\langle \text{proof} \rangle$

**lemma** *ts--ncol*:  
**assumes**  $A \ B \ \text{TS } X \ Y$   
**shows**  $\neg \text{Col } A \ X \ Y \vee \neg \text{Col } B \ X \ Y$   
 $\langle \text{proof} \rangle$

**lemma** *one-or-two-sides-aux*:  
**assumes**  $\neg \text{Col } C \ A \ B$  **and**  
 $\neg \text{Col } D \ A \ B$  **and**  
 $\text{Col } A \ C \ X$   
**and**  $\text{Col } B \ D \ X$   
**shows**  $A \ B \ \text{TS } C \ D \vee A \ B \ \text{OS } C \ D$   
 $\langle \text{proof} \rangle$

**lemma** *cop--one-or-two-sides*:  
**assumes**  $\text{Coplanar } A \ B \ C \ D$  **and**  
 $\neg \text{Col } C \ A \ B$  **and**  
 $\neg \text{Col } D \ A \ B$   
**shows**  $A \ B \ \text{TS } C \ D \vee A \ B \ \text{OS } C \ D$   
 $\langle \text{proof} \rangle$

**lemma** *os--coplanar*:  
**assumes**  $A \ B \ \text{OS } C \ D$   
**shows**  $\text{Coplanar } A \ B \ C \ D$   
 $\langle \text{proof} \rangle$

**lemma** *coplanar-trans-1*:  
**assumes**  $\neg \text{Col } P \ Q \ R$  **and**  
 $\text{Coplanar } P \ Q \ R \ A$  **and**  
 $\text{Coplanar } P \ Q \ R \ B$   
**shows**  $\text{Coplanar } Q \ R \ A \ B$   
 $\langle \text{proof} \rangle$

**lemma** *col-cop--cop*:  
**assumes**  $\text{Coplanar } A \ B \ C \ D$  **and**  
 $C \neq D$  **and**  
 $\text{Col } C \ D \ E$   
**shows**  $\text{Coplanar } A \ B \ C \ E$   
 $\langle \text{proof} \rangle$

**lemma** *bet-cop--cop*:  
**assumes**  $\text{Coplanar } A \ B \ C \ E$  **and**  
 $\text{Bet } C \ D \ E$   
**shows**  $\text{Coplanar } A \ B \ C \ D$   
 $\langle \text{proof} \rangle$

**lemma** *col2-cop--cop*:  
**assumes**  $\text{Coplanar } A \ B \ C \ D$  **and**  
 $C \neq D$  **and**  
 $\text{Col } C \ D \ E$  **and**  
 $\text{Col } C \ D \ F$   
**shows**  $\text{Coplanar } A \ B \ E \ F$   
 $\langle \text{proof} \rangle$

**lemma** *col-cop2--cop*:  
**assumes**  $U \neq V$  **and**  
 $\text{Coplanar } A \ B \ C \ U$  **and**  
 $\text{Coplanar } A \ B \ C \ V$  **and**  
 $\text{Col } U \ V \ P$   
**shows**  $\text{Coplanar } A \ B \ C \ P$

$\langle \text{proof} \rangle$

**lemma** *bet-cop2--cop*:

**assumes** *Coplanar A B C U* **and**

*Coplanar A B C W* **and**

*Bet U V W*

**shows** *Coplanar A B C V*

$\langle \text{proof} \rangle$

**lemma** *coplanar-pseudo-trans*:

**assumes**  $\neg$  *Col P Q R* **and**

*Coplanar P Q R A* **and**

*Coplanar P Q R B* **and**

*Coplanar P Q R C* **and**

*Coplanar P Q R D*

**shows** *Coplanar A B C D*

$\langle \text{proof} \rangle$

**lemma** *l9-30*:

**assumes**  $\neg$  *Coplanar A B C P* **and**

$\neg$  *Col D E F* **and**

*Coplanar D E F P* **and**

*Coplanar A B C X* **and**

*Coplanar A B C Y* **and**

*Coplanar A B C Z* **and**

*Coplanar D E F X* **and**

*Coplanar D E F Y* **and**

*Coplanar D E F Z*

**shows** *Col X Y Z*

$\langle \text{proof} \rangle$

**lemma** *cop-per2--col*:

**assumes** *Coplanar A X Y Z* **and**

$A \neq Z$  **and**

*Per X Z A* **and**

*Per Y Z A*

**shows** *Col X Y Z*

$\langle \text{proof} \rangle$

**lemma** *cop-perp2--col*:

**assumes** *Coplanar A B Y Z* **and**

$X Y \text{ Perp } A B$  **and**

$X Z \text{ Perp } A B$

**shows** *Col X Y Z*

$\langle \text{proof} \rangle$

**lemma** *two-sides-dec*:

**shows**  $A B \text{ TS } C D \vee \neg A B \text{ TS } C D$

$\langle \text{proof} \rangle$

**lemma** *cop-nts--os*:

**assumes** *Coplanar A B C D* **and**

$\neg$  *Col C A B* **and**

$\neg$  *Col D A B* **and**

$\neg A B \text{ TS } C D$

**shows**  $A B \text{ OS } C D$

$\langle \text{proof} \rangle$

**lemma** *cop-nos--ts*:

**assumes** *Coplanar A B C D* **and**

$\neg$  *Col C A B* **and**

$\neg$  *Col D A B* **and**

$\neg A B \text{ OS } C D$

**shows**  $A B \text{ TS } C D$

$\langle \text{proof} \rangle$

**lemma one-side-dec:**

$A B O S C D \vee \neg A B O S C D$

$\langle proof \rangle$

**lemma cop-dec:**

$Coplanar A B C D \vee \neg Coplanar A B C D$

$\langle proof \rangle$

**lemma ex-diff-cop:**

$\exists E. Coplanar A B C E \wedge D \neq E$

$\langle proof \rangle$

**lemma ex-ncol-cop:**

**assumes**  $D \neq E$

**shows**  $\exists F. Coplanar A B C F \wedge \neg Col D E F$

$\langle proof \rangle$

**lemma ex-ncol-cop2:**

$\exists E F. (Coplanar A B C E \wedge Coplanar A B C F \wedge \neg Col D E F)$

$\langle proof \rangle$

**lemma col2-cop2--eq:**

**assumes**  $\neg Coplanar A B C U$  **and**

$U \neq V$  **and**

$Coplanar A B C P$  **and**

$Coplanar A B C Q$  **and**

$Col U V P$  **and**

$Col U V Q$

**shows**  $P = Q$

$\langle proof \rangle$

**lemma cong3-cop2--col:**

**assumes**  $Coplanar A B C P$  **and**

$Coplanar A B C Q$  **and**

$P \neq Q$  **and**

$Cong A P A Q$  **and**

$Cong B P B Q$  **and**

$Cong C P C Q$

**shows**  $Col A B C$

$\langle proof \rangle$

**lemma l9-38:**

**assumes**  $A B C TSP P Q$

**shows**  $A B C TSP Q P$

$\langle proof \rangle$

**lemma l9-39:**

**assumes**  $A B C TSP P R$  **and**

$Coplanar A B C D$  **and**

$D Out P Q$

**shows**  $A B C TSP Q R$

$\langle proof \rangle$

**lemma l9-41-1:**

**assumes**  $A B C TSP P R$  **and**

$A B C TSP Q R$

**shows**  $A B C OSP P Q$

$\langle proof \rangle$

**lemma l9-41-2:**

**assumes**  $A B C TSP P R$  **and**

$A B C OSP P Q$

**shows**  $A B C TSP Q R$

$\langle proof \rangle$

**lemma tsp-exists:**

**assumes**  $\neg \text{Coplanar } A B C P$   
**shows**  $\exists Q. A B C TSP P Q$   
 ⟨proof⟩

**lemma osp-reflexivity:**  
**assumes**  $\neg \text{Coplanar } A B C P$   
**shows**  $A B C OSP P P$   
 ⟨proof⟩

**lemma osp-symmetry:**  
**assumes**  $A B C OSP P Q$   
**shows**  $A B C OSP Q P$   
 ⟨proof⟩

**lemma osp-transitivity:**  
**assumes**  $A B C OSP P Q$  and  
 $A B C OSP Q R$   
**shows**  $A B C OSP P R$   
 ⟨proof⟩

**lemma cop3-tsp--tsp:**  
**assumes**  $\neg \text{Col } D E F$  and  
 $\text{Coplanar } A B C D$  and  
 $\text{Coplanar } A B C E$  and  
 $\text{Coplanar } A B C F$  and  
 $A B C TSP P Q$   
**shows**  $D E F TSP P Q$   
 ⟨proof⟩

**lemma cop3-osp--osp:**  
**assumes**  $\neg \text{Col } D E F$  and  
 $\text{Coplanar } A B C D$  and  
 $\text{Coplanar } A B C E$  and  
 $\text{Coplanar } A B C F$  and  
 $A B C OSP P Q$   
**shows**  $D E F OSP P Q$   
 ⟨proof⟩

**lemma ncop-distincts:**  
**assumes**  $\neg \text{Coplanar } A B C D$   
**shows**  $A \neq B \wedge A \neq C \wedge A \neq D \wedge B \neq C \wedge B \neq D \wedge C \neq D$   
 ⟨proof⟩

**lemma tsp-distincts:**  
**assumes**  $A B C TSP P Q$   
**shows**  $A \neq B \wedge A \neq C \wedge B \neq C \wedge A \neq P \wedge B \neq P \wedge C \neq P \wedge A \neq Q \wedge B \neq Q \wedge C \neq Q \wedge P \neq Q$   
 ⟨proof⟩

**lemma osp-distincts:**  
**assumes**  $A B C OSP P Q$   
**shows**  $A \neq B \wedge A \neq C \wedge B \neq C \wedge A \neq P \wedge B \neq P \wedge C \neq P \wedge A \neq Q \wedge B \neq Q \wedge C \neq Q$   
 ⟨proof⟩

**lemma tsp--ncop1:**  
**assumes**  $A B C TSP P Q$   
**shows**  $\neg \text{Coplanar } A B C P$   
 ⟨proof⟩

**lemma tsp--ncop2:**  
**assumes**  $A B C TSP P Q$   
**shows**  $\neg \text{Coplanar } A B C Q$   
 ⟨proof⟩

**lemma osp--ncop1:**  
**assumes**  $A B C OSP P Q$

**shows**  $\neg$  Coplanar  $A B C P$   
(proof)

**lemma** *osp--ncop2*:  
**assumes**  $A B C OSP P Q$   
**shows**  $\neg$  Coplanar  $A B C Q$   
(proof)

**lemma** *tsp--nosp*:  
**assumes**  $A B C TSP P Q$   
**shows**  $\neg$   $A B C OSP P Q$   
(proof)

**lemma** *osp--ntsp*:  
**assumes**  $A B C OSP P Q$   
**shows**  $\neg$   $A B C TSP P Q$   
(proof)

**lemma** *osp-bet--osp*:  
**assumes**  $A B C OSP P R$  and  
 $Bet P Q R$   
**shows**  $A B C OSP P Q$   
(proof)

**lemma** *l9-18-3*:  
**assumes** Coplanar  $A B C P$  and  
 $Col X Y P$   
**shows**  $A B C TSP X Y \longleftrightarrow (Bet X P Y \wedge \neg Coplanar A B C X \wedge \neg Coplanar A B C Y)$   
(proof)

**lemma** *bet-cop--tsp*:  
**assumes**  $\neg$  Coplanar  $A B C X$  and  
 $P \neq Y$  and  
Coplanar  $A B C P$  and  
 $Bet X P Y$   
**shows**  $A B C TSP X Y$   
(proof)

**lemma** *cop-out--osp*:  
**assumes**  $\neg$  Coplanar  $A B C X$  and  
Coplanar  $A B C P$  and  
 $P Out X Y$   
**shows**  $A B C OSP X Y$   
(proof)

**lemma** *l9-19-3*:  
**assumes** Coplanar  $A B C P$  and  
 $Col X Y P$   
**shows**  $A B C OSP X Y \longleftrightarrow (P Out X Y \wedge \neg Coplanar A B C X)$   
(proof)

**lemma** *cop2-ts--tsp*:  
**assumes**  $\neg$  Coplanar  $A B C X$  and Coplanar  $A B C D$  and  
Coplanar  $A B C E$  and  $D E TS X Y$   
**shows**  $A B C TSP X Y$   
(proof)

**lemma** *cop2-os--osp*:  
**assumes**  $\neg$  Coplanar  $A B C X$  and  
Coplanar  $A B C D$  and  
Coplanar  $A B C E$  and  
 $D E OS X Y$   
**shows**  $A B C OSP X Y$   
(proof)

**lemma** *cop3-tsp--ts*:

**assumes**  $D \neq E$  **and**  
*Coplanar*  $A B C D$  **and**  
*Coplanar*  $A B C E$  **and**  
*Coplanar*  $D E X Y$  **and**  
 $A B C TSP X Y$   
**shows**  $D E TS X Y$   
 $\langle proof \rangle$

**lemma** *cop3-osp--os*:  
**assumes**  $D \neq E$  **and**  
*Coplanar*  $A B C D$  **and**  
*Coplanar*  $A B C E$  **and**  
*Coplanar*  $D E X Y$  **and**  
 $A B C OSP X Y$   
**shows**  $D E OS X Y$   
 $\langle proof \rangle$

**lemma** *cop-tsp--ex-cop2*:  
**assumes**  
 $A B C TSP D E$   
**shows**  $\exists Q. (Coplanar A B C Q \wedge Coplanar D E P Q \wedge P \neq Q)$   
 $\langle proof \rangle$

**lemma** *cop-osp--ex-cop2*:  
**assumes** *Coplanar*  $A B C P$  **and**  
 $A B C OSP D E$   
**shows**  $\exists Q. Coplanar A B C Q \wedge Coplanar D E P Q \wedge P \neq Q$   
 $\langle proof \rangle$

**lemma** *sac--coplanar*:  
**assumes** *Saccheri*  $A B C D$   
**shows** *Coplanar*  $A B C D$   
 $\langle proof \rangle$

## 3.9 Line reflexivity

### 3.9.1 Dimensionless

**lemma** *Ch10-Goal1*:  
**assumes**  $\neg Coplanar D C B A$   
**shows**  $\neg Coplanar A B C D$   
 $\langle proof \rangle$

**lemma** *ex-sym*:  
 $\exists Y. (A B Perp X Y \vee X = Y) \wedge (\exists M. Col A B M \wedge M Midpoint X Y)$   
 $\langle proof \rangle$

**lemma** *is-image-is-image-spec*:  
**assumes**  $A \neq B$   
**shows**  $P' P Reflect A B \longleftrightarrow P' P ReflectL A B$   
 $\langle proof \rangle$

**lemma** *ex-sym1*:  
**assumes**  $A \neq B$   
**shows**  $\exists Y. (A B Perp X Y \vee X = Y) \wedge (\exists M. Col A B M \wedge M Midpoint X Y \wedge X Y Reflect A B)$   
 $\langle proof \rangle$

**lemma** *l10-2-uniqueness*:  
**assumes**  $P1 P Reflect A B$  **and**  
 $P2 P Reflect A B$   
**shows**  $P1 = P2$   
 $\langle proof \rangle$

**lemma** *l10-2-uniqueness-spec*:  
**assumes**  $P1 P ReflectL A B$  **and**  
 $P2 P ReflectL A B$   
**shows**  $P1 = P2$

$\langle \text{proof} \rangle$

**lemma** *l10-2-existence-spec*:

$\exists P'. P' P \text{ ReflectL } A B$

$\langle \text{proof} \rangle$

**lemma** *l10-2-existence*:

$\exists P'. P' P \text{ Reflect } A B$

$\langle \text{proof} \rangle$

**lemma** *l10-4-spec*:

**assumes**  $P P' \text{ ReflectL } A B$

**shows**  $P' P \text{ ReflectL } A B$

$\langle \text{proof} \rangle$

**lemma** *l10-4*:

**assumes**  $P P' \text{ Reflect } A B$

**shows**  $P' P \text{ Reflect } A B$

$\langle \text{proof} \rangle$

**lemma** *l10-5*:

**assumes**  $P' P \text{ Reflect } A B$  **and**

$P'' P' \text{ Reflect } A B$

**shows**  $P = P''$

$\langle \text{proof} \rangle$

**lemma** *l10-6-uniqueness*:

**assumes**  $P P1 \text{ Reflect } A B$  **and**

$P P2 \text{ Reflect } A B$

**shows**  $P1 = P2$

$\langle \text{proof} \rangle$

**lemma** *l10-6-uniqueness-spec*:

**assumes**  $P P1 \text{ ReflectL } A B$  **and**

$P P2 \text{ ReflectL } A B$

**shows**  $P1 = P2$

$\langle \text{proof} \rangle$

**lemma** *l10-6-existence-spec*:

**assumes**  $A \neq B$

**shows**  $\exists P. P' P \text{ ReflectL } A B$

$\langle \text{proof} \rangle$

**lemma** *l10-6-existence*:

$\exists P. P' P \text{ Reflect } A B$

$\langle \text{proof} \rangle$

**lemma** *l10-7*:

**assumes**  $P' P \text{ Reflect } A B$  **and**

$Q' Q \text{ Reflect } A B$  **and**

$P' = Q'$

**shows**  $P = Q$

$\langle \text{proof} \rangle$

**lemma** *l10-8*:

**assumes**  $P P \text{ Reflect } A B$

**shows**  $\text{Col } P A B$

$\langle \text{proof} \rangle$

**lemma** *col--refl*:

**assumes**  $\text{Col } P A B$

**shows**  $P P \text{ ReflectL } A B$

$\langle \text{proof} \rangle$

**lemma** *is-image-col-cong*:

**assumes**  $A \neq B$  **and**

$P P' \text{ Reflect } A B$  **and**  
 $\text{Col } A B X$   
**shows**  $\text{Cong } P X P' X$   
 $\langle \text{proof} \rangle$

**lemma** *is-image-spec-col-cong*:  
**assumes**  $P P' \text{ ReflectL } A B$  **and**  
 $\text{Col } A B X$   
**shows**  $\text{Cong } P X P' X$   
 $\langle \text{proof} \rangle$

**lemma** *image-id*:  
**assumes**  $A \neq B$  **and**  
 $\text{Col } A B T$  **and**  
 $T T' \text{ Reflect } A B$   
**shows**  $T = T'$   
 $\langle \text{proof} \rangle$

**lemma** *osym-not-col*:  
**assumes**  $P P' \text{ Reflect } A B$  **and**  
 $\neg \text{Col } A B P$   
**shows**  $\neg \text{Col } A B P'$   
 $\langle \text{proof} \rangle$

**lemma** *midpoint-preserves-image*:  
**assumes**  $A \neq B$  **and**  
 $\text{Col } A B M$  **and**  
 $P P' \text{ Reflect } A B$  **and**  
 $M \text{ Midpoint } P Q$  **and**  
 $M \text{ Midpoint } P' Q'$   
**shows**  $Q Q' \text{ Reflect } A B$   
 $\langle \text{proof} \rangle$

**lemma** *image-in-is-image-spec*:  
**assumes**  $M \text{ ReflectLAt } P P' A B$   
**shows**  $P P' \text{ ReflectL } A B$   
 $\langle \text{proof} \rangle$

**lemma** *image-in-gen-is-image*:  
**assumes**  $M \text{ ReflectAt } P P' A B$   
**shows**  $P P' \text{ Reflect } A B$   
 $\langle \text{proof} \rangle$

**lemma** *image-image-in*:  
**assumes**  $P \neq P'$  **and**  
 $P P' \text{ ReflectL } A B$  **and**  
 $\text{Col } A B M$  **and**  
 $\text{Col } P M P'$   
**shows**  $M \text{ ReflectLAt } P P' A B$   
 $\langle \text{proof} \rangle$

**lemma** *image-in-col*:  
**assumes**  $Y \text{ ReflectLAt } P P' A B$   
**shows**  $\text{Col } P P' Y$   
 $\langle \text{proof} \rangle$

**lemma** *is-image-spec-rev*:  
**assumes**  $P P' \text{ ReflectL } A B$   
**shows**  $P P' \text{ ReflectL } B A$   
 $\langle \text{proof} \rangle$

**lemma** *is-image-rev*:  
**assumes**  $P P' \text{ Reflect } A B$   
**shows**  $P P' \text{ Reflect } B A$   
 $\langle \text{proof} \rangle$

**lemma** *midpoint-preserves-per*:  
**assumes**  $Per\ A\ B\ C$  **and**  
 $M\ Midpoint\ A\ A1$  **and**  
 $M\ Midpoint\ B\ B1$  **and**  
 $M\ Midpoint\ C\ C1$   
**shows**  $Per\ A1\ B1\ C1$   
 $\langle proof \rangle$

**lemma** *col--image-spec*:  
**assumes**  $Col\ A\ B\ X$   
**shows**  $X\ X\ ReflectL\ A\ B$   
 $\langle proof \rangle$

**lemma** *image-triv*:  
 $A\ A\ Reflect\ A\ B$   
 $\langle proof \rangle$

**lemma** *cong-midpoint--image*:  
**assumes**  $Cong\ A\ X\ A\ Y$  **and**  
 $B\ Midpoint\ X\ Y$   
**shows**  $Y\ X\ Reflect\ A\ B$   
 $\langle proof \rangle$

**lemma** *col-image-spec--eq*:  
**assumes**  $Col\ A\ B\ P$  **and**  
 $P\ P'\ ReflectL\ A\ B$   
**shows**  $P = P'$   
 $\langle proof \rangle$

**lemma** *image-spec-triv*:  
 $A\ A\ ReflectL\ B\ B$   
 $\langle proof \rangle$

**lemma** *image-spec--eq*:  
**assumes**  $P\ P'\ ReflectL\ A\ A$   
**shows**  $P = P'$   
 $\langle proof \rangle$

**lemma** *image--midpoint*:  
**assumes**  $P\ P'\ Reflect\ A\ A$   
**shows**  $A\ Midpoint\ P'\ P$   
 $\langle proof \rangle$

**lemma** *is-image-spec-dec*:  
 $A\ B\ ReflectL\ C\ D \vee \neg A\ B\ ReflectL\ C\ D$   
 $\langle proof \rangle$

**lemma** *l10-14*:  
**assumes**  $P \neq P'$  **and**  
 $A \neq B$  **and**  
 $P\ P'\ Reflect\ A\ B$   
**shows**  $A\ B\ TS\ P\ P'$   
 $\langle proof \rangle$

**lemma** *l10-15*:  
**assumes**  $Col\ A\ B\ C$  **and**  
 $\neg Col\ A\ B\ P$   
**shows**  $\exists Q. A\ B\ Perp\ Q\ C \wedge A\ B\ OS\ P\ Q$   
 $\langle proof \rangle$

**lemma** *ex-per-cong*:  
**assumes**  $A \neq B$  **and**  
 $X \neq Y$  **and**  
 $Col\ A\ B\ C$  **and**  
 $\neg Col\ A\ B\ D$

**shows**  $\exists P. \text{Per } P C A \wedge \text{Cong } P C X Y \wedge A B \text{ OS } P D$   
 ⟨proof⟩

**lemma** *exists-cong-per*:  
 $\exists C. \text{Per } A B C \wedge \text{Cong } B C X Y$   
 ⟨proof⟩

### 3.9.2 Upper dim 2

**lemma** *upper-dim-implies-per2--col*:  
**assumes** *upper-dim-axiom*  
**shows**  $\forall A B C X. (\text{Per } A X C \wedge X \neq C \wedge \text{Per } B X C) \longrightarrow \text{Col } A B X$   
 ⟨proof⟩

**lemma** *upper-dim-implies-col-perp2--col*:  
**assumes** *upper-dim-axiom*  
**shows**  $\forall A B X Y P. (\text{Col } A B P \wedge A B \text{ Perp } X P \wedge P A \text{ Perp } Y P) \longrightarrow \text{Col } Y X P$   
 ⟨proof⟩

**lemma** *upper-dim-implies-perp2--col*:  
**assumes** *upper-dim-axiom*  
**shows**  $\forall X Y Z A B. (X Y \text{ Perp } A B \wedge X Z \text{ Perp } A B) \longrightarrow \text{Col } X Y Z$   
 ⟨proof⟩

**lemma** *upper-dim-implies-not-two-sides-one-side-aux*:  
**assumes** *upper-dim-axiom*  
**shows**  $\forall A B X Y P X. (A \neq B \wedge P X \neq A \wedge A B \text{ Perp } X P X \wedge \text{Col } A B P X \wedge \neg \text{Col } X A B \wedge \neg \text{Col } Y A B \wedge \neg A B \text{ TS } X Y) \longrightarrow A B \text{ OS } X Y$   
 ⟨proof⟩

**lemma** *upper-dim-implies-not-two-sides-one-side*:  
**assumes** *upper-dim-axiom*  
**shows**  $\forall A B X Y. (\neg \text{Col } X A B \wedge \neg \text{Col } Y A B \wedge \neg A B \text{ TS } X Y) \longrightarrow A B \text{ OS } X Y$   
 ⟨proof⟩

**lemma** *upper-dim-implies-not-one-side-two-sides*:  
**assumes** *upper-dim-axiom*  
**shows**  $\forall A B X Y. (\neg \text{Col } X A B \wedge \neg \text{Col } Y A B \wedge \neg A B \text{ OS } X Y) \longrightarrow A B \text{ TS } X Y$   
 ⟨proof⟩

**lemma** *upper-dim-implies-one-or-two-sides*:  
**assumes** *upper-dim-axiom*  
**shows**  $\forall A B X Y. (\neg \text{Col } X A B \wedge \neg \text{Col } Y A B) \longrightarrow (A B \text{ TS } X Y \vee A B \text{ OS } X Y)$   
 ⟨proof⟩

**lemma** *upper-dim-implies-all-coplanar*:  
**assumes** *upper-dim-axiom*  
**shows** *all-coplanar-axiom*  
 ⟨proof⟩

**lemma** *all-coplanar-implies-upper-dim*:  
**assumes** *all-coplanar-axiom*  
**shows** *upper-dim-axiom*  
 ⟨proof⟩

**lemma** *all-coplanar-upper-dim*:  
**shows** *all-coplanar-axiom*  $\longleftrightarrow$  *upper-dim-axiom*  
 ⟨proof⟩

**lemma** *upper-dim-stab*:  
**shows**  $\neg \neg$  *upper-dim-axiom*  $\longrightarrow$  *upper-dim-axiom* ⟨proof⟩

**lemma** *cop--cong-on-bisect*:  
**assumes** *Coplanar*  $A B X P$  **and**  
 $M$  *Midpoint*  $A B$  **and**  
 $M$  *PerpAt*  $A B P M$  **and**

*Cong X A X B*  
**shows** *Col M P X*  
<proof>

**lemma** *cong-cop-mid-perp--col:*  
**assumes** *Coplanar A B X P* **and**  
*Cong A X B X* **and**  
*M Midpoint A B* **and**  
*A B Perp P M*  
**shows** *Col M P X*  
<proof>

**lemma** *cop-image-in2--col:*  
**assumes** *Coplanar A B P Q* **and**  
*M ReflectLAt P P' A B* **and**  
*M ReflectLAt Q Q' A B*  
**shows** *Col M P Q*  
<proof>

**lemma** *l10-10-spec:*  
**assumes** *P' P ReflectL A B* **and**  
*Q' Q ReflectL A B*  
**shows** *Cong P Q P' Q'*  
<proof>

**lemma** *l10-10:*  
**assumes** *P' P Reflect A B* **and**  
*Q' Q Reflect A B*  
**shows** *Cong P Q P' Q'*  
<proof>

**lemma** *image-preserves-bet:*  
**assumes** *A A' ReflectL X Y* **and**  
*B B' ReflectL X Y* **and**  
*C C' ReflectL X Y* **and**  
*Bet A B C*  
**shows** *Bet A' B' C'*  
<proof>

**lemma** *image-gen-preserves-bet:*  
**assumes** *A A' Reflect X Y* **and**  
*B B' Reflect X Y* **and**  
*C C' Reflect X Y* **and**  
*Bet A B C*  
**shows** *Bet A' B' C'*  
<proof>

**lemma** *image-preserves-col:*  
**assumes** *A A' ReflectL X Y* **and**  
*B B' ReflectL X Y* **and**  
*C C' ReflectL X Y* **and**  
*Col A B C*  
**shows** *Col A' B' C'* <proof>

**lemma** *image-gen-preserves-col:*  
**assumes** *A A' Reflect X Y* **and**  
*B B' Reflect X Y* **and**  
*C C' Reflect X Y* **and**  
*Col A B C*  
**shows** *Col A' B' C'*  
<proof>

**lemma** *image-gen-preserves-ncol:*  
**assumes** *A A' Reflect X Y* **and**  
*B B' Reflect X Y* **and**  
*C C' Reflect X Y* **and**

$\neg \text{Col } A B C$   
**shows**  $\neg \text{Col } A' B' C'$   
 ⟨proof⟩

**lemma** *image-gen-preserves-inter:*  
**assumes**  $A A' \text{ Reflect } X Y$  **and**  
 $B B' \text{ Reflect } X Y$  **and**  
 $C C' \text{ Reflect } X Y$  **and**  
 $D D' \text{ Reflect } X Y$  **and**  
 $\neg \text{Col } A B C$  **and**  
 $C \neq D$  **and**  
 $\text{Col } A B I$  **and**  
 $\text{Col } C D I$  **and**  
 $\text{Col } A' B' I'$  **and**  
 $\text{Col } C' D' I'$   
**shows**  $I I' \text{ Reflect } X Y$   
 ⟨proof⟩

**lemma** *intersection-with-image-gen:*  
**assumes**  $A A' \text{ Reflect } X Y$  **and**  
 $B B' \text{ Reflect } X Y$  **and**  
 $\neg \text{Col } A B A'$  **and**  
 $\text{Col } A B C$  **and**  
 $\text{Col } A' B' C$   
**shows**  $\text{Col } C X Y$   
 ⟨proof⟩

**lemma** *image-preserves-midpoint :*  
**assumes**  $A A' \text{ ReflectL } X Y$  **and**  
 $B B' \text{ ReflectL } X Y$  **and**  
 $C C' \text{ ReflectL } X Y$  **and**  
 $A \text{ Midpoint } B C$   
**shows**  $A' \text{ Midpoint } B' C'$   
 ⟨proof⟩

**lemma** *image-spec-preserves-per:*  
**assumes**  $A A' \text{ ReflectL } X Y$  **and**  
 $B B' \text{ ReflectL } X Y$  **and**  
 $C C' \text{ ReflectL } X Y$  **and**  
 $\text{Per } A B C$   
**shows**  $\text{Per } A' B' C'$   
 ⟨proof⟩

**lemma** *image-preserves-per:*  
**assumes**  $A A' \text{ Reflect } X Y$  **and**  
 $B B' \text{ Reflect } X Y$  **and**  
 $C C' \text{ Reflect } X Y$  **and**  
 $\text{Per } A B C$   
**shows**  $\text{Per } A' B' C'$   
 ⟨proof⟩

**lemma** *l10-12:*  
**assumes**  $\text{Per } A B C$  **and**  
 $\text{Per } A' B' C'$  **and**  
 $\text{Cong } A B A' B'$  **and**  
 $\text{Cong } B C B' C'$   
**shows**  $\text{Cong } A C A' C'$   
 ⟨proof⟩

**lemma** *l10-16:*  
**assumes**  $\neg \text{Col } A B C$  **and**  
 $\neg \text{Col } A' B' P$  **and**  
 $\text{Cong } A B A' B'$   
**shows**  $\exists C'. A B C \text{ Cong3 } A' B' C' \wedge A' B' \text{ OS } P C'$   
 ⟨proof⟩

**lemma** *cong-cop-image--col:*

**assumes**  $P \neq P'$  **and**  
 $P P' \text{ Reflect } A B$  **and**  
 $\text{Cong } P X P' X$  **and**  
 $\text{Coplanar } A B P X$

**shows**  $\text{Col } A B X$

*<proof>*

**lemma** *cong-cop-per2-1:*

**assumes**  $A \neq B$  **and**  
 $\text{Per } A B X$  **and**  
 $\text{Per } A B Y$  **and**  
 $\text{Cong } B X B Y$  **and**  
 $\text{Coplanar } A B X Y$

**shows**  $X = Y \vee B \text{ Midpoint } X Y$

*<proof>*

**lemma** *cong-cop-per2:*

**assumes**  $A \neq B$  **and**  
 $\text{Per } A B X$  **and**  
 $\text{Per } A B Y$  **and**  
 $\text{Cong } B X B Y$  **and**  
 $\text{Coplanar } A B X Y$

**shows**  $X = Y \vee X Y \text{ ReflectL } A B$

*<proof>*

**lemma** *cong-cop-per2-gen:*

**assumes**  $A \neq B$  **and**  
 $\text{Per } A B X$  **and**  
 $\text{Per } A B Y$  **and**  
 $\text{Cong } B X B Y$  **and**  
 $\text{Coplanar } A B X Y$

**shows**  $X = Y \vee X Y \text{ Reflect } A B$

*<proof>*

**lemma** *ex-perp-cop:*

**assumes**  $A \neq B$   
**shows**  $\exists Q. A B \text{ Perp } Q C \wedge \text{Coplanar } A B P Q$

*<proof>*

**lemma** *hilbert-s-version-of-pasch-aux:*

**assumes**  $\text{Coplanar } A B C P$  **and**  
 $\neg \text{Col } A I P$  **and**  
 $\neg \text{Col } B C P$  **and**  
 $\text{Bet } B I C$  **and**  
 $B \neq I$  **and**  
 $I \neq C$  **and**  
 $B \neq C$

**shows**  $\exists X. \text{Col } I P X \wedge ((\text{Bet } A X B \wedge A \neq X \wedge X \neq B \wedge A \neq B) \vee (\text{Bet } A X C \wedge A \neq X \wedge X \neq C \wedge A \neq C))$

*<proof>*

**lemma** *hilbert-s-version-of-pasch:*

**assumes**  $\text{Coplanar } A B C P$  **and**  
 $\neg \text{Col } C Q P$  **and**  
 $\neg \text{Col } A B P$  **and**  
 $\text{BetS } A Q B$

**shows**  $\exists X. \text{Col } P Q X \wedge (\text{BetS } A X C \vee \text{BetS } B X C)$

*<proof>*

**lemma** *two-sides-cases:*

**assumes**  $\neg \text{Col } P O A B$  **and**  
 $P O P O S A B$

**shows**  $P O A T S P B \vee P O B T S P A$

*<proof>*

**lemma** *not-par-two-sides:*

**assumes**  $C \neq D$  **and**  
 $Col\ A\ B\ I$  **and**  
 $Col\ C\ D\ I$  **and**  
 $\neg\ Col\ A\ B\ C$   
**shows**  $\exists\ X\ Y. Col\ C\ D\ X \wedge Col\ C\ D\ Y \wedge A\ B\ TS\ X\ Y$   
 ⟨proof⟩

**lemma** *cop-not-par-other-side:*

**assumes**  $C \neq D$  **and**  
 $Col\ A\ B\ I$  **and**  
 $Col\ C\ D\ I$  **and**  
 $\neg\ Col\ A\ B\ C$  **and**  
 $\neg\ Col\ A\ B\ P$  **and**  
 $Coplanar\ A\ B\ C\ P$   
**shows**  $\exists\ Q. Col\ C\ D\ Q \wedge A\ B\ TS\ P\ Q$   
 ⟨proof⟩

**lemma** *cop-not-par-same-side:*

**assumes**  $C \neq D$  **and**  
 $Col\ A\ B\ I$  **and**  
 $Col\ C\ D\ I$  **and**  
 $\neg\ Col\ A\ B\ C$  **and**  
 $\neg\ Col\ A\ B\ P$  **and**  
 $Coplanar\ A\ B\ C\ P$   
**shows**  $\exists\ Q. Col\ C\ D\ Q \wedge A\ B\ OS\ P\ Q$   
 ⟨proof⟩

**end**

### 3.9.3 Line reflexivity: 2D

**context** *Tarski-2D*

**begin**

**lemma** *all-coplanar:*

$Coplanar\ A\ B\ C\ D$   
 ⟨proof⟩

**lemma** *per2--col:*

**assumes**  $Per\ A\ X\ C$  **and**  
 $X \neq C$  **and**  
 $Per\ B\ X\ C$   
**shows**  $Col\ A\ B\ X$   
 ⟨proof⟩

**lemma** *perp2--col:*

**assumes**  $X\ Y\ Perp\ A\ B$  **and**  
 $X\ Z\ Perp\ A\ B$   
**shows**  $Col\ X\ Y\ Z$   
 ⟨proof⟩

**end**

## 3.10 Angles

### 3.10.1 Some generalites

**context** *Tarski-neutral-dimensionless*

**begin**

**lemma** *l11-3:*

**assumes**  $A\ B\ C\ Cong\ A\ D\ E\ F$   
**shows**  $\exists\ A'\ C' D' F'. B\ Out\ A'\ A \wedge B\ Out\ C'\ C' \wedge E\ Out\ D'\ D \wedge E\ Out\ F'\ F' \wedge A'\ B\ C'\ Cong\ D'\ E\ F'$   
 ⟨proof⟩

**lemma l11-aux:**

**assumes**  $B \text{ Out } A \ A'$  **and**

$E \text{ Out } D \ D'$  **and**

$\text{Cong } B \ A' \ E \ D'$  **and**

$\text{Bet } B \ A \ A0$  **and**

$\text{Bet } E \ D \ D0$  **and**

$\text{Cong } A \ A0 \ E \ D$  **and**

$\text{Cong } D \ D0 \ B \ A$

**shows**  $\text{Cong } B \ A0 \ E \ D0 \ \wedge \ \text{Cong } A' \ A0 \ D' \ D0$

$\langle \text{proof} \rangle$

**lemma l11-3-bis:**

**assumes**  $\exists A' \ C' \ D' \ F'. (B \text{ Out } A' \ A \ \wedge \ B \text{ Out } C' \ C \ \wedge \ E \text{ Out } D' \ D \ \wedge \ E \text{ Out } F' \ F \ \wedge \ A' \ B \ C' \ \text{Cong3 } D' \ E \ F')$

**shows**  $A \ B \ C \ \text{Cong} A \ D \ E \ F$

$\langle \text{proof} \rangle$

**lemma l11-4-1:**

**assumes**  $A \ B \ C \ \text{Cong} A \ D \ E \ F$  **and**

$B \text{ Out } A' \ A$  **and**

$B \text{ Out } C' \ C$  **and**

$E \text{ Out } D' \ D$  **and**

$E \text{ Out } F' \ F$  **and**

$\text{Cong } B \ A' \ E \ D'$  **and**  $\text{Cong } B \ C' \ E \ F'$

**shows**  $\text{Cong } A' \ C' \ D' \ F'$

$\langle \text{proof} \rangle$

**lemma l11-4-2:**

**assumes**  $A \neq B$  **and**

$C \neq B$  **and**

$D \neq E$  **and**

$F \neq E$  **and**

$\forall A' \ C' \ D' \ F'. (B \text{ Out } A' \ A \ \wedge \ B \text{ Out } C' \ C \ \wedge \ E \text{ Out } D' \ D \ \wedge \ E \text{ Out } F' \ F \ \wedge \ \text{Cong } B \ A' \ E \ D' \ \wedge \ \text{Cong } B \ C' \ E \ F' \ \longrightarrow \ \text{Cong } A' \ C' \ D' \ F')$

**shows**  $A \ B \ C \ \text{Cong} A \ D \ E \ F$

$\langle \text{proof} \rangle$

**lemma conga-refl:**

**assumes**  $A \neq B$  **and**

$C \neq B$

**shows**  $A \ B \ C \ \text{Cong} A \ A \ B \ C$

$\langle \text{proof} \rangle$

**lemma conga-sym:**

**assumes**  $A \ B \ C \ \text{Cong} A \ A' \ B' \ C'$

**shows**  $A' \ B' \ C' \ \text{Cong} A \ A \ B \ C$

$\langle \text{proof} \rangle$

**lemma l11-10:**

**assumes**  $A \ B \ C \ \text{Cong} A \ D \ E \ F$  **and**

$B \text{ Out } A' \ A$  **and**

$B \text{ Out } C' \ C$  **and**

$E \text{ Out } D' \ D$  **and**

$E \text{ Out } F' \ F$

**shows**  $A' \ B \ C' \ \text{Cong} A \ D' \ E \ F'$

$\langle \text{proof} \rangle$

**lemma out2--conga:**

**assumes**  $B \text{ Out } A' \ A$  **and**

$B \text{ Out } C' \ C$

**shows**  $A \ B \ C \ \text{Cong} A \ A' \ B \ C'$

$\langle \text{proof} \rangle$

**lemma cong3-diff:**

**assumes**  $A \neq B$  **and**

$A \ B \ C \ \text{Cong3 } A' \ B' \ C'$

shows  $A' \neq B'$   
(proof)

lemma *cong3-diff2*:  
assumes  $B \neq C$  and  
 $A B C \text{ Cong3 } A' B' C'$   
shows  $B' \neq C'$   
(proof)

lemma *cong3-conga*:  
assumes  $A \neq B$  and  
 $C \neq B$  and  
 $A B C \text{ Cong3 } A' B' C'$   
shows  $A B C \text{ CongA } A' B' C'$   
(proof)

lemma *cong3-conga2*:  
assumes  $A B C \text{ Cong3 } A' B' C'$  and  
 $A B C \text{ CongA } A'' B'' C''$   
shows  $A' B' C' \text{ CongA } A'' B'' C''$   
(proof)

lemma *conga-diff1*:  
assumes  $A B C \text{ CongA } A' B' C'$   
shows  $A \neq B$   
(proof)

lemma *conga-diff2*:  
assumes  $A B C \text{ CongA } A' B' C'$   
shows  $C \neq B$   
(proof)

lemma *conga-diff45*:  
assumes  $A B C \text{ CongA } A' B' C'$   
shows  $A' \neq B'$   
(proof)

lemma *conga-diff56*:  
assumes  $A B C \text{ CongA } A' B' C'$   
shows  $C' \neq B'$   
(proof)

lemma *conga-trans*:  
assumes  $A B C \text{ CongA } A' B' C'$  and  
 $A' B' C' \text{ CongA } A'' B'' C''$   
shows  $A B C \text{ CongA } A'' B'' C''$   
(proof)

lemma *conga-pseudo-refl*:  
assumes  $A \neq B$  and  
 $C \neq B$   
shows  $A B C \text{ CongA } C B A$   
(proof)

lemma *conga-trivial-1*:  
assumes  $A \neq B$  and  
 $C \neq D$   
shows  $A B A \text{ CongA } C D C$   
(proof)

lemma *l11-13*:  
assumes  $A B C \text{ CongA } D E F$  and  
 $B \neq A B A'$  and  
 $A' \neq B$  and  
 $B \neq D E D'$  and  
 $D' \neq E$

**shows**  $A' B C \text{ Cong} A D' E F$   
 $\langle \text{proof} \rangle$

**lemma** *conga-right-comm*:  
**assumes**  $A B C \text{ Cong} A D E F$   
**shows**  $A B C \text{ Cong} A F E D$   
 $\langle \text{proof} \rangle$

**lemma** *conga-left-comm*:  
**assumes**  $A B C \text{ Cong} A D E F$   
**shows**  $C B A \text{ Cong} A D E F$   
 $\langle \text{proof} \rangle$

**lemma** *conga-comm*:  
**assumes**  $A B C \text{ Cong} A D E F$   
**shows**  $C B A \text{ Cong} A F E D$   
 $\langle \text{proof} \rangle$

**lemma** *conga-line*:  
**assumes**  $A \neq B$  **and**  
     $B \neq C$  **and**  
     $A' \neq B'$  **and**  
     $B' \neq C'$   
**and**  $\text{Bet } A B C$  **and**  
     $\text{Bet } A' B' C'$   
**shows**  $A B C \text{ Cong} A A' B' C'$   
 $\langle \text{proof} \rangle$

**lemma** *l11-14*:  
**assumes**  $\text{Bet } A B A'$  **and**  
     $A \neq B$  **and**  
     $A' \neq B$  **and**  
     $\text{Bet } C B C'$  **and**  
     $B \neq C$  **and**  
     $B \neq C'$   
**shows**  $A B C \text{ Cong} A A' B C'$   
 $\langle \text{proof} \rangle$

**lemma** *l11-16*:  
**assumes**  $\text{Per } A B C$  **and**  
     $A \neq B$  **and**  
     $C \neq B$  **and**  
     $\text{Per } A' B' C'$  **and**  
     $A' \neq B'$  **and**  
     $C' \neq B'$   
**shows**  $A B C \text{ Cong} A A' B' C'$   
 $\langle \text{proof} \rangle$

**lemma** *l11-17*:  
**assumes**  $\text{Per } A B C$  **and**  
     $A B C \text{ Cong} A A' B' C'$   
**shows**  $\text{Per } A' B' C'$   
 $\langle \text{proof} \rangle$

**lemma** *l11-18-1*:  
**assumes**  $\text{Bet } C B D$  **and**  
     $B \neq C$  **and**  
     $B \neq D$  **and**  
     $A \neq B$  **and**  
     $\text{Per } A B C$   
**shows**  $A B C \text{ Cong} A A B D$   
 $\langle \text{proof} \rangle$

**lemma** *l11-18-2*:  
**assumes**  $\text{Bet } C B D$  **and**  
     $A B C \text{ Cong} A A B D$

**shows**  $Per\ A\ B\ C$   
 $\langle proof \rangle$

**lemma** *cong3-preserves-out*:  
**assumes**  $A\ Out\ B\ C$  **and**  
 $A\ B\ C\ Cong3\ A'\ B'\ C'$   
**shows**  $A'\ Out\ B'\ C'$   
 $\langle proof \rangle$

**lemma** *l11-21-a*:  
**assumes**  $B\ Out\ A\ C$  **and**  
 $A\ B\ C\ CongA\ A'\ B'\ C'$   
**shows**  $B'\ Out\ A'\ C'$   
 $\langle proof \rangle$

**lemma** *l11-21-b*:  
**assumes**  $B\ Out\ A\ C$  **and**  
 $B'\ Out\ A'\ C'$   
**shows**  $A\ B\ C\ CongA\ A'\ B'\ C'$   
 $\langle proof \rangle$

**lemma** *conga-cop--or-out-ts*:  
**assumes**  $Coplanar\ A\ B\ C\ C'$  **and**  
 $A\ B\ C\ CongA\ A\ B\ C'$   
**shows**  $B\ Out\ C\ C' \vee A\ B\ TS\ C\ C'$   
 $\langle proof \rangle$

**lemma** *conga-os--out*:  
**assumes**  $A\ B\ C\ CongA\ A\ B\ C'$  **and**  
 $A\ B\ OS\ C\ C'$   
**shows**  $B\ Out\ C\ C'$   
 $\langle proof \rangle$

**lemma** *cong2-conga-cong*:  
**assumes**  $A\ B\ C\ CongA\ A'\ B'\ C'$  **and**  
 $Cong\ A\ B\ A'\ B'$  **and**  
 $Cong\ B\ C\ B'\ C'$   
**shows**  $Cong\ A\ C\ A'\ C'$   
 $\langle proof \rangle$

**lemma** *angle-construction-1*:  
**assumes**  $\neg\ Col\ A\ B\ C$  **and**  
 $\neg\ Col\ A'\ B'\ P$   
**shows**  $\exists\ C'. (A\ B\ C\ CongA\ A'\ B'\ C' \wedge A'\ B'\ OS\ C'\ P)$   
 $\langle proof \rangle$

**lemma** *angle-construction-2*:  
**assumes**  $A \neq B$  **and**  
 $B \neq C$  **and**  
 $\neg\ Col\ A'\ B'\ P$   
**shows**  $\exists\ C'. (A\ B\ C\ CongA\ A'\ B'\ C' \wedge (A'\ B'\ OS\ C'\ P \vee Col\ A'\ B'\ C'))$   
 $\langle proof \rangle$

**lemma** *ex-conga-ts*:  
**assumes**  $\neg\ Col\ A\ B\ C$  **and**  
 $\neg\ Col\ A'\ B'\ P$   
**shows**  $\exists\ C'. A\ B\ C\ CongA\ A'\ B'\ C' \wedge A'\ B'\ TS\ C'\ P$   
 $\langle proof \rangle$

**lemma** *l11-15*:  
**assumes**  $\neg\ Col\ A\ B\ C$  **and**  
 $\neg\ Col\ D\ E\ P$   
**shows**  
 $\exists\ F. (A\ B\ C\ CongA\ D\ E\ F \wedge E\ D\ OS\ F\ P) \wedge$   
 $(\forall\ F1\ F2. ((A\ B\ C\ CongA\ D\ E\ F1 \wedge E\ D\ OS\ F1\ P) \wedge$   
 $(A\ B\ C\ CongA\ D\ E\ F2 \wedge E\ D\ OS\ F2\ P)))$

$\rightarrow E$  Out  $F1 F2$ )  
(proof)

**lemma l11-19:**  
assumes  $Per A B P1$  and  
     $Per A B P2$  and  
     $A B OS P1 P2$   
shows  $B$  Out  $P1 P2$   
(proof)

**lemma l11-22-bet:**  
assumes  $Bet A B C$  and  
     $P' B' TS A' C'$  and  
     $A B P CongA A' B' P'$  and  
     $P B C CongA P' B' C'$   
shows  $Bet A' B' C'$   
(proof)

**lemma l11-22a:**  
assumes  $B P TS A C$  and  
     $B' P' TS A' C'$  and  
     $A B P CongA A' B' P'$  and  
     $P B C CongA P' B' C'$   
shows  $A B C CongA A' B' C'$   
(proof)

**lemma l11-22b:**  
assumes  $B P OS A C$  and  
     $B' P' OS A' C'$  and  
     $A B P CongA A' B' P'$  and  
     $P B C CongA P' B' C'$   
shows  $A B C CongA A' B' C'$   
(proof)

**lemma l11-22:**  
assumes  $((B P TS A C \wedge B' P' TS A' C') \vee (B P OS A C \wedge B' P' OS A' C'))$  and  
     $A B P CongA A' B' P'$  and  
     $P B C CongA P' B' C'$   
shows  $A B C CongA A' B' C'$   
(proof)

**lemma l11-24:**  
assumes  $P InAngle A B C$   
shows  $P InAngle C B A$   
(proof)

**lemma col-in-angle:**  
assumes  $A \neq B$  and  
     $C \neq B$  and  
     $P \neq B$  and  
     $B$  Out  $A P \vee B$  Out  $C P$   
shows  $P InAngle A B C$   
(proof)

**lemma out321--inangle:**  
assumes  $C \neq B$  and  
     $B$  Out  $A P$   
shows  $P InAngle A B C$   
(proof)

**lemma inangle1123:**  
assumes  $A \neq B$  and  
     $C \neq B$   
shows  $A InAngle A B C$   
(proof)

**lemma** *out341--inangle:*

**assumes**  $A \neq B$  **and**

$B \text{ Out } C P$

**shows**  $P \text{ InAngle } A B C$

*<proof>*

**lemma** *inangle3123:*

**assumes**  $A \neq B$  **and**

$C \neq B$

**shows**  $C \text{ InAngle } A B C$

*<proof>*

**lemma** *in-angle-two-sides:*

**assumes**  $\neg \text{Col } B A P$  **and**

$\neg \text{Col } B C P$  **and**

$P \text{ InAngle } A B C$

**shows**  $P B \text{ TS } A C$

*<proof>*

**lemma** *in-angle-out:*

**assumes**  $B \text{ Out } A C$  **and**

$P \text{ InAngle } A B C$

**shows**  $B \text{ Out } A P$

*<proof>*

**lemma** *col-in-angle-out:*

**assumes**  $\text{Col } B A P$  **and**

$\neg \text{Bet } A B C$  **and**

$P \text{ InAngle } A B C$

**shows**  $B \text{ Out } A P$

*<proof>*

**lemma** *l11-25-aux:*

**assumes**  $P \text{ InAngle } A B C$  **and**

$\neg \text{Bet } A B C$  **and**

$B \text{ Out } A' A$

**shows**  $P \text{ InAngle } A' B C$

*<proof>*

**lemma** *l11-25:*

**assumes**  $P \text{ InAngle } A B C$  **and**

$B \text{ Out } A' A$  **and**

$B \text{ Out } C' C$  **and**

$B \text{ Out } P' P$

**shows**  $P' \text{ InAngle } A' B C'$

*<proof>*

**lemma** *inangle-distincts:*

**assumes**  $P \text{ InAngle } A B C$

**shows**  $A \neq B \wedge C \neq B \wedge P \neq B$

*<proof>*

**lemma** *segment-construction-0:*

**shows**  $\exists B'. \text{Cong } A' B' A B$

*<proof>*

**lemma** *angle-construction-3:*

**assumes**  $A \neq B$  **and**

$C \neq B$  **and**

$A' \neq B'$

**shows**  $\exists C'. A B C \text{ Cong } A' B' C'$

*<proof>*

**lemma** *l11-28:*

**assumes**  $A B C \text{ Cong } A' B' C'$  **and**

$\text{Col } A C D$

**shows**  $\exists D'. (Cong A D A' D' \wedge Cong B D B' D' \wedge Cong C D C' D')$   
(proof)

**lemma** *bet-conga--bet*:  
**assumes** *Bet A B C* **and**  
  *A B C CongA A' B' C'*  
**shows** *Bet A' B' C'*  
(proof)

**lemma** *in-angle-one-side*:  
**assumes**  $\neg Col A B C$  **and**  
   $\neg Col B A P$  **and**  
  *P InAngle A B C*  
**shows** *A B OS P C*  
(proof)

**lemma** *inangle-one-side*:  
**assumes**  $\neg Col A B C$  **and**  
   $\neg Col A B P$  **and**  
   $\neg Col A B Q$  **and**  
  *P InAngle A B C* **and**  
  *Q InAngle A B C*  
**shows** *A B OS P Q*  
(proof)

**lemma** *inangle-one-side2*:  
**assumes**  $\neg Col A B C$  **and**  
   $\neg Col A B P$  **and**  
   $\neg Col A B Q$  **and**  
   $\neg Col C B P$  **and**  
   $\neg Col C B Q$  **and**  
  *P InAngle A B C* **and**  
  *Q InAngle A B C*  
**shows** *A B OS P Q*  $\wedge$  *C B OS P Q*  
(proof)

**lemma** *col-conga-col*:  
**assumes** *Col A B C* **and**  
  *A B C CongA D E F*  
**shows** *Col D E F*  
(proof)

**lemma** *ncol-conga-ncol*:  
**assumes**  $\neg Col A B C$  **and**  
  *A B C CongA D E F*  
**shows**  $\neg Col D E F$   
(proof)

**lemma** *angle-construction-4*:  
**assumes** *A*  $\neq$  *B* **and**  
  *C*  $\neq$  *B* **and**  
  *A'*  $\neq$  *B'*  
**shows**  $\exists C'. (A B C CongA A' B' C' \wedge Coplanar A' B' C' P)$   
(proof)

**lemma** *lea-distincts*:  
**assumes** *A B C LeA D E F*  
**shows** *A*  $\neq$  *B*  $\wedge$  *C*  $\neq$  *B*  $\wedge$  *D*  $\neq$  *E*  $\wedge$  *F*  $\neq$  *E*  
(proof)

**lemma** *l11-29-a*:  
**assumes** *A B C LeA D E F*  
**shows**  $\exists Q. (C InAngle A B Q \wedge A B Q CongA D E F)$   
(proof)

**lemma** *in-angle-line*:

**assumes**  $P \neq B$  **and**  
 $A \neq B$  **and**  
 $C \neq B$  **and**  
 $Bet\ A\ B\ C$   
**shows**  $P\ InAngle\ A\ B\ C$   
 $\langle proof \rangle$

**lemma** *l11-29-b*:  
**assumes**  $\exists\ Q. (C\ InAngle\ A\ B\ Q \wedge A\ B\ Q\ CongA\ D\ E\ F)$   
**shows**  $A\ B\ C\ LeA\ D\ E\ F$   
 $\langle proof \rangle$

**lemma** *bet-in-angle-bet*:  
**assumes**  $Bet\ A\ B\ P$  **and**  
 $P\ InAngle\ A\ B\ C$   
**shows**  $Bet\ A\ B\ C$   
 $\langle proof \rangle$

**lemma** *lea-line*:  
**assumes**  $Bet\ A\ B\ P$  **and**  
 $A\ B\ P\ LeA\ A\ B\ C$   
**shows**  $Bet\ A\ B\ C$   
 $\langle proof \rangle$

**lemma** *eq-conga-out*:  
**assumes**  $A\ B\ A\ CongA\ D\ E\ F$   
**shows**  $E\ Out\ D\ F$   
 $\langle proof \rangle$

**lemma** *out-conga-out*:  
**assumes**  $B\ Out\ A\ C$  **and**  
 $A\ B\ C\ CongA\ D\ E\ F$   
**shows**  $E\ Out\ D\ F$   
 $\langle proof \rangle$

**lemma** *conga-ex-cong3*:  
**assumes**  $A\ B\ C\ CongA\ A'\ B'\ C'$   
**shows**  $\exists\ AA\ CC. ((B\ Out\ A\ AA \wedge B\ Out\ C\ CC) \longrightarrow AA\ B\ CC\ Cong3\ A'\ B'\ C')$   
 $\langle proof \rangle$

**lemma** *conga-preserves-in-angle*:  
**assumes**  $A\ B\ C\ CongA\ A'\ B'\ C'$  **and**  
 $A\ B\ I\ CongA\ A'\ B'\ I'$  **and**  
 $I\ InAngle\ A\ B\ C$  **and**  $A'\ B'\ OS\ I'\ C'$   
**shows**  $I'\ InAngle\ A'\ B'\ C'$   
 $\langle proof \rangle$

**lemma** *l11-30*:  
**assumes**  $A\ B\ C\ LeA\ D\ E\ F$  **and**  
 $A\ B\ C\ CongA\ A'\ B'\ C'$  **and**  
 $D\ E\ F\ CongA\ D'\ E'\ F'$   
**shows**  $A'\ B'\ C'\ LeA\ D'\ E'\ F'$   
 $\langle proof \rangle$

**lemma** *l11-31-1*:  
**assumes**  $B\ Out\ A\ C$  **and**  
 $D \neq E$  **and**  
 $F \neq E$   
**shows**  $A\ B\ C\ LeA\ D\ E\ F$   
 $\langle proof \rangle$

**lemma** *l11-31-2*:  
**assumes**  $A \neq B$  **and**  
 $C \neq B$  **and**  
 $D \neq E$  **and**  
 $F \neq E$  **and**

*Bet D E F*  
**shows**  $A B C LeA D E F$   
*<proof>*

**lemma** *lea-refl*:  
**assumes**  $A \neq B$  **and**  
 $C \neq B$   
**shows**  $A B C LeA A B C$   
*<proof>*

**lemma** *conga--lea*:  
**assumes**  $A B C CongA D E F$   
**shows**  $A B C LeA D E F$   
*<proof>*

**lemma** *conga--lea456123*:  
**assumes**  $A B C CongA D E F$   
**shows**  $D E F LeA A B C$   
*<proof>*

**lemma** *lea-left-comm*:  
**assumes**  $A B C LeA D E F$   
**shows**  $C B A LeA D E F$   
*<proof>*

**lemma** *lea-right-comm*:  
**assumes**  $A B C LeA D E F$   
**shows**  $A B C LeA F E D$   
*<proof>*

**lemma** *lea-comm*:  
**assumes**  $A B C LeA D E F$   
**shows**  $C B A LeA F E D$   
*<proof>*

**lemma** *lta-left-comm*:  
**assumes**  $A B C LtA D E F$   
**shows**  $C B A LtA D E F$   
*<proof>*

**lemma** *lta-right-comm*:  
**assumes**  $A B C LtA D E F$   
**shows**  $A B C LtA F E D$   
*<proof>*

**lemma** *lta-comm*:  
**assumes**  $A B C LtA D E F$   
**shows**  $C B A LtA F E D$   
*<proof>*

**lemma** *lea-out4--lea*:  
**assumes**  $A B C LeA D E F$  **and**  
 $B Out A A'$  **and**  
 $B Out C C'$  **and**  
 $E Out D D'$  **and**  
 $E Out F F'$   
**shows**  $A' B C' LeA D' E F'$   
*<proof>*

**lemma** *lea121345*:  
**assumes**  $A \neq B$  **and**  
 $C \neq D$  **and**  
 $D \neq E$   
**shows**  $A B A LeA C D E$   
*<proof>*

**lemma** *inangle--lea:*

**assumes**  $P$  *InAngle*  $A B C$

**shows**  $A B P$  *LeA*  $A B C$

*<proof>*

**lemma** *inangle--lea-1:*

**assumes**  $P$  *InAngle*  $A B C$

**shows**  $P B C$  *LeA*  $A B C$

*<proof>*

**lemma** *inangle--lta:*

**assumes**  $\neg$  *Col*  $P B C$  **and**

$P$  *InAngle*  $A B C$

**shows**  $A B P$  *LtA*  $A B C$

*<proof>*

**lemma** *in-angle-trans:*

**assumes**  $C$  *InAngle*  $A B D$  **and**

$D$  *InAngle*  $A B E$

**shows**  $C$  *InAngle*  $A B E$

*<proof>*

**lemma** *lea-trans:*

**assumes**  $A B C$  *LeA*  $A_1 B_1 C_1$  **and**

$A_1 B_1 C_1$  *LeA*  $A_2 B_2 C_2$

**shows**  $A B C$  *LeA*  $A_2 B_2 C_2$

*<proof>*

**lemma** *in-angle-asy:*

**assumes**  $D$  *InAngle*  $A B C$  **and**

$C$  *InAngle*  $A B D$

**shows**  $A B C$  *CongA*  $A B D$

*<proof>*

**lemma** *lea-asy:*

**assumes**  $A B C$  *LeA*  $D E F$  **and**

$D E F$  *LeA*  $A B C$

**shows**  $A B C$  *CongA*  $D E F$

*<proof>*

**lemma** *col-lta--bet:*

**assumes** *Col*  $X Y Z$  **and**

$A B C$  *LtA*  $X Y Z$

**shows** *Bet*  $X Y Z$

*<proof>*

**lemma** *col-lta--out:*

**assumes** *Col*  $A B C$  **and**

$A B C$  *LtA*  $X Y Z$

**shows** *B Out*  $A C$

*<proof>*

**lemma** *lta-distincts:*

**assumes**  $A B C$  *LtA*  $D E F$

**shows**  $A \neq B \wedge C \neq B \wedge D \neq E \wedge F \neq E \wedge D \neq F$

*<proof>*

**lemma** *gta-distincts:*

**assumes**  $A B C$  *GtA*  $D E F$

**shows**  $A \neq B \wedge C \neq B \wedge D \neq E \wedge F \neq E \wedge A \neq C$

*<proof>*

**lemma** *acute-distincts:*

**assumes** *Acute*  $A B C$

**shows**  $A \neq B \wedge C \neq B$

$\langle \text{proof} \rangle$

**lemma** *obtuse-distincts*:

**assumes** *Obtuse*  $A B C$

**shows**  $A \neq B \wedge C \neq B \wedge A \neq C$

$\langle \text{proof} \rangle$

**lemma** *two-sides-in-angle*:

**assumes**  $B \neq P'$  **and**

$B P T S A C$  **and**

*Bet*  $P B P'$

**shows**  $P \text{ InAngle } A B C \vee P' \text{ InAngle } A B C$

$\langle \text{proof} \rangle$

**lemma** *in-angle-reverse*:

**assumes**  $A' \neq B$  **and**

*Bet*  $A B A'$  **and**

$C \text{ InAngle } A B D$

**shows**  $D \text{ InAngle } A' B C$

$\langle \text{proof} \rangle$

**lemma** *in-angle-trans2*:

**assumes**  $C \text{ InAngle } A B D$  **and**

$D \text{ InAngle } A B E$

**shows**  $D \text{ InAngle } C B E$

$\langle \text{proof} \rangle$

**lemma** *l11-36-aux1*:

**assumes**  $A \neq B$  **and**

$A' \neq B$  **and**

$D \neq E$  **and**

$D' \neq E$  **and**

*Bet*  $A B A'$  **and**

*Bet*  $D E D'$  **and**

$A B C \text{ LeA } D E F$

**shows**  $D' E F \text{ LeA } A' B C$

$\langle \text{proof} \rangle$

**lemma** *l11-36-aux2*:

**assumes**  $A \neq B$  **and**

$A' \neq B$  **and**

$D \neq E$  **and**

$D' \neq E$  **and**

*Bet*  $A B A'$  **and**

*Bet*  $D E D'$  **and**

$D' E F \text{ LeA } A' B C$

**shows**  $A B C \text{ LeA } D E F$

$\langle \text{proof} \rangle$

**lemma** *l11-36*:

**assumes**  $A \neq B$  **and**

$A' \neq B$  **and**

$D \neq E$  **and**

$D' \neq E$  **and**

*Bet*  $A B A'$  **and**

*Bet*  $D E D'$

**shows**  $A B C \text{ LeA } D E F \longleftrightarrow D' E F \text{ LeA } A' B C$

$\langle \text{proof} \rangle$

**lemma** *l11-41-aux*:

**assumes**  $\neg \text{Col } A B C$  **and**

*Bet*  $B A D$  **and**

$A \neq D$

**shows**  $A C B \text{ LtA } C A D$

$\langle \text{proof} \rangle$

**lemma l11-41:**

**assumes**  $\neg \text{Col } A \ B \ C$  **and**

$\text{Bet } B \ A \ D$  **and**

$A \neq D$

**shows**  $A \ C \ B \ \text{Lt} \ A \ C \ A \ D \wedge A \ B \ C \ \text{Lt} \ A \ C \ A \ D$

$\langle \text{proof} \rangle$

**lemma not-conga:**

**assumes**  $A \ B \ C \ \text{Cong} \ A \ A' \ B' \ C'$  **and**

$\neg A \ B \ C \ \text{Cong} \ A \ D \ E \ F$

**shows**  $\neg A' \ B' \ C' \ \text{Cong} \ A \ D \ E \ F$

$\langle \text{proof} \rangle$

**lemma not-conga-sym:**

**assumes**  $\neg A \ B \ C \ \text{Cong} \ A \ D \ E \ F$

**shows**  $\neg D \ E \ F \ \text{Cong} \ A \ A \ B \ C$

$\langle \text{proof} \rangle$

**lemma not-and-lta:**

**shows**  $\neg (A \ B \ C \ \text{Lt} \ A \ D \ E \ F \wedge D \ E \ F \ \text{Lt} \ A \ B \ C)$

$\langle \text{proof} \rangle$

**lemma conga-preserves-lta:**

**assumes**  $A \ B \ C \ \text{Cong} \ A \ A' \ B' \ C'$  **and**

$D \ E \ F \ \text{Cong} \ A \ D' \ E' \ F'$  **and**

$A \ B \ C \ \text{Lt} \ A \ D \ E \ F$

**shows**  $A' \ B' \ C' \ \text{Lt} \ A \ D' \ E' \ F'$

$\langle \text{proof} \rangle$

**lemma lta-trans:**

**assumes**  $A \ B \ C \ \text{Lt} \ A \ 1 \ B \ 1 \ C \ 1$  **and**

$A \ 1 \ B \ 1 \ C \ 1 \ \text{Lt} \ A \ 2 \ B \ 2 \ C \ 2$

**shows**  $A \ B \ C \ \text{Lt} \ A \ 2 \ B \ 2 \ C \ 2$

$\langle \text{proof} \rangle$

**lemma obtuse-sym:**

**assumes**  $\text{Obtuse } A \ B \ C$

**shows**  $\text{Obtuse } C \ B \ A$

$\langle \text{proof} \rangle$

**lemma acute-sym:**

**assumes**  $\text{Acute } A \ B \ C$

**shows**  $\text{Acute } C \ B \ A$

$\langle \text{proof} \rangle$

**lemma acute-col--out:**

**assumes**  $\text{Col } A \ B \ C$  **and**

$\text{Acute } A \ B \ C$

**shows**  $B \ \text{Out} \ A \ C$

$\langle \text{proof} \rangle$

**lemma col-obtuse--bet:**

**assumes**  $\text{Col } A \ B \ C$  **and**

$\text{Obtuse } A \ B \ C$

**shows**  $\text{Bet } A \ B \ C$

$\langle \text{proof} \rangle$

**lemma out--acute:**

**assumes**  $B \ \text{Out} \ A \ C$

**shows**  $\text{Acute } A \ B \ C$

$\langle \text{proof} \rangle$

**lemma bet--obtuse:**

**assumes**  $\text{Bet } A \ B \ C$  **and**

$A \neq B$  **and**  $B \neq C$

**shows**  $\text{Obtuse } A \ B \ C$

$\langle \text{proof} \rangle$

**lemma** *l11-43-aux:*

**assumes**  $A \neq B$  **and**

$A \neq C$  **and**

$\text{Per } B A C \vee \text{Obtuse } B A C$

**shows**  $\text{Acute } A B C$

$\langle \text{proof} \rangle$

**lemma** *l11-43:*

**assumes**  $A \neq B$  **and**

$A \neq C$  **and**

$\text{Per } B A C \vee \text{Obtuse } B A C$

**shows**  $\text{Acute } A B C \wedge \text{Acute } A C B$

$\langle \text{proof} \rangle$

**lemma** *acute-lea-acute:*

**assumes**  $\text{Acute } D E F$  **and**

$A B C \text{ Le } A D E F$

**shows**  $\text{Acute } A B C$

$\langle \text{proof} \rangle$

**lemma** *lea-obtuse-obtuse:*

**assumes**  $\text{Obtuse } D E F$  **and**

$D E F \text{ Le } A A B C$

**shows**  $\text{Obtuse } A B C$

$\langle \text{proof} \rangle$

**lemma** *l11-44-1-a:*

**assumes**  $A \neq B$  **and**

$A \neq C$  **and**

$\text{Cong } B A B C$

**shows**  $B A C \text{ Cong } A B C A$

$\langle \text{proof} \rangle$

**lemma** *l11-44-2-a:*

**assumes**  $\neg \text{Col } A B C$  **and**

$B A \text{ Lt } B C$

**shows**  $B C A \text{ Lt } A B A C$

$\langle \text{proof} \rangle$

**lemma** *not-lta-and-conga:*

$\neg (A B C \text{ Lt } A D E F \wedge A B C \text{ Cong } A D E F)$

$\langle \text{proof} \rangle$

**lemma** *conga-sym-equiv:*

$A B C \text{ Cong } A A' B' C' \longleftrightarrow A' B' C' \text{ Cong } A B C$

$\langle \text{proof} \rangle$

**lemma** *conga-dec:*

$A B C \text{ Cong } A D E F \vee \neg A B C \text{ Cong } A D E F$

$\langle \text{proof} \rangle$

**lemma** *lta-not-conga:*

**assumes**  $A B C \text{ Lt } A D E F$

**shows**  $\neg A B C \text{ Cong } A D E F$

$\langle \text{proof} \rangle$

**lemma** *lta--lea:*

**assumes**  $A B C \text{ Lt } A D E F$

**shows**  $A B C \text{ Le } A D E F$

$\langle \text{proof} \rangle$

**lemma** *nlta:*

$\neg A B C \text{ Lt } A B C$

$\langle \text{proof} \rangle$

**lemma** *lea--nlta*:

**assumes**  $A B C LeA D E F$   
**shows**  $\neg D E F LtA A B C$   
*<proof>*

**lemma** *lta--nlea*:

**assumes**  $A B C LtA D E F$   
**shows**  $\neg D E F LeA A B C$   
*<proof>*

**lemma** *l11-44-1-b*:

**assumes**  $\neg Col A B C$  **and**  
 $B A C CongA B C A$   
**shows**  $Cong B A B C$   
*<proof>*

**lemma** *l11-44-2-b*:

**assumes**  $B A C LtA B C A$   
**shows**  $B C Lt B A$   
*<proof>*

**lemma** *l11-44-1*:

**assumes**  $\neg Col A B C$   
**shows**  $B A C CongA B C A \longleftrightarrow Cong B A B C$   
*<proof>*

**lemma** *l11-44-2*:

**assumes**  $\neg Col A B C$   
**shows**  $B A C LtA B C A \longleftrightarrow B C Lt B A$   
*<proof>*

**lemma** *l11-44-2bis*:

**assumes**  $\neg Col A B C$   
**shows**  $B A C LeA B C A \longleftrightarrow B C Le B A$   
*<proof>*

**lemma** *l11-46*:

**assumes**  $A \neq B$  **and**  
 $B \neq C$  **and**  
 $Per A B C \vee Obtuse A B C$   
**shows**  $B A Lt A C \wedge B C Lt A C$   
*<proof>*

**lemma** *l11-47*:

**assumes**  $Per A C B$  **and**  
 $H PerpAt C H A B$   
**shows**  $Bet A H B \wedge A \neq H \wedge B \neq H$   
*<proof>*

**lemma** *l11-49*:

**assumes**  $A B C CongA A' B' C'$  **and**  
 $Cong B A B' A'$  **and**  
 $Cong B C B' C'$   
**shows**  $Cong A C A' C' \wedge (A \neq C \longrightarrow (B A C CongA B' A' C' \wedge B C A CongA B' C' A'))$   
*<proof>*

**lemma** *l11-50-1*:

**assumes**  $\neg Col A B C$  **and**  
 $B A C CongA B' A' C'$  **and**  
 $A B C CongA A' B' C'$  **and**  
 $Cong A B A' B'$   
**shows**  $Cong A C A' C' \wedge Cong B C B' C' \wedge A C B CongA A' C' B'$   
*<proof>*

**lemma** *l11-50-2*:

**assumes**  $\neg \text{Col } A B C$  **and**  
 $B C A \text{ Cong} A B' C' A'$  **and**  
 $A B C \text{ Cong} A A' B' C'$  **and**  
 $\text{Cong } A B A' B'$   
**shows**  $\text{Cong } A C A' C' \wedge \text{Cong } B C B' C' \wedge C A B \text{ Cong} A C' A' B'$   
⟨proof⟩

**lemma l11-51:**  
**assumes**  $A \neq B$  **and**  
 $A \neq C$  **and**  
 $B \neq C$  **and**  
 $\text{Cong } A B A' B'$  **and**  
 $\text{Cong } A C A' C'$  **and**  
 $\text{Cong } B C B' C'$   
**shows**  
 $B A C \text{ Cong} A B' A' C' \wedge A B C \text{ Cong} A A' B' C' \wedge B C A \text{ Cong} A B' C' A'$   
⟨proof⟩

**lemma conga-distinct:**  
**assumes**  $A B C \text{ Cong} A D E F$   
**shows**  $A \neq B \wedge C \neq B \wedge D \neq E \wedge F \neq E$   
⟨proof⟩

**lemma l11-52:**  
**assumes**  $A B C \text{ Cong} A A' B' C'$  **and**  
 $\text{Cong } A C A' C'$  **and**  
 $\text{Cong } B C B' C'$  **and**  
 $B C \text{ Le } A C$   
**shows**  $\text{Cong } B A B' A' \wedge B A C \text{ Cong} A B' A' C' \wedge B C A \text{ Cong} A B' C' A'$   
⟨proof⟩

**lemma l11-53:**  
**assumes**  $\text{Per } D C B$  **and**  
 $C \neq D$  **and**  
 $A \neq B$  **and**  
 $B \neq C$  **and**  
 $\text{Bet } A B C$   
**shows**  $C A D \text{ Lt} A C B D \wedge B D \text{ Lt } A D$   
⟨proof⟩

**lemma cong2-conga-obtuse--cong-conga2:**  
**assumes**  $\text{Obtuse } A B C$  **and**  
 $A B C \text{ Cong} A A' B' C'$  **and**  
 $\text{Cong } A C A' C'$  **and**  
 $\text{Cong } B C B' C'$   
**shows**  $\text{Cong } B A B' A' \wedge B A C \text{ Cong} A B' A' C' \wedge$   
 $B C A \text{ Cong} A B' C' A'$   
⟨proof⟩

**lemma cong2-per2--cong-conga2:**  
**assumes**  $A \neq B$  **and**  
 $B \neq C$  **and**  
 $\text{Per } A B C$  **and**  
 $\text{Per } A' B' C'$  **and**  
 $\text{Cong } A C A' C'$  **and**  
 $\text{Cong } B C B' C'$   
**shows**  $\text{Cong } B A B' A' \wedge B A C \text{ Cong} A B' A' C' \wedge$   
 $B C A \text{ Cong} A B' C' A'$   
⟨proof⟩

**lemma cong2-per2--cong:**  
**assumes**  $\text{Per } A B C$  **and**  
 $\text{Per } A' B' C'$  **and**  
 $\text{Cong } A C A' C'$  **and**  
 $\text{Cong } B C B' C'$   
**shows**  $\text{Cong } B A B' A'$

$\langle \text{proof} \rangle$

**lemma** *cong2-per2--cong-3*:

**assumes**  $\text{Per } A \ B \ C$

$\text{Per } A' \ B' \ C'$  **and**

$\text{Cong } A \ C \ A' \ C'$  **and**

$\text{Cong } B \ C \ B' \ C'$

**shows**  $A \ B \ C \ \text{Cong}^3 \ A' \ B' \ C'$

$\langle \text{proof} \rangle$

**lemma** *cong-lt-per2--lt*:

**assumes**  $\text{Per } A \ B \ C$  **and**

$\text{Per } A' \ B' \ C'$  **and**

$\text{Cong } A \ B \ A' \ B'$  **and**

$B \ C \ \text{Lt } B' \ C'$

**shows**  $A \ C \ \text{Lt } A' \ C'$

$\langle \text{proof} \rangle$

**lemma** *cong-le-per2--le*:

**assumes**  $\text{Per } A \ B \ C$  **and**

$\text{Per } A' \ B' \ C'$  **and**

$\text{Cong } A \ B \ A' \ B'$  **and**

$B \ C \ \text{Le } B' \ C'$

**shows**  $A \ C \ \text{Le } A' \ C'$

$\langle \text{proof} \rangle$

**lemma** *lt2-per2--lt*:

**assumes**  $\text{Per } A \ B \ C$  **and**

$\text{Per } A' \ B' \ C'$  **and**

$A \ B \ \text{Lt } A' \ B'$  **and**

$B \ C \ \text{Lt } B' \ C'$

**shows**  $A \ C \ \text{Lt } A' \ C'$

$\langle \text{proof} \rangle$

**lemma** *le-lt-per2--lt*:

**assumes**  $\text{Per } A \ B \ C$  **and**

$\text{Per } A' \ B' \ C'$  **and**

$A \ B \ \text{Le } A' \ B'$  **and**

$B \ C \ \text{Lt } B' \ C'$

**shows**  $A \ C \ \text{Lt } A' \ C'$

$\langle \text{proof} \rangle$

**lemma** *le2-per2--le*:

**assumes**  $\text{Per } A \ B \ C$  **and**

$\text{Per } A' \ B' \ C'$  **and**

$A \ B \ \text{Le } A' \ B'$  **and**

$B \ C \ \text{Le } B' \ C'$

**shows**  $A \ C \ \text{Le } A' \ C'$

$\langle \text{proof} \rangle$

**lemma** *cong-lt-per2--lt-1*:

**assumes**  $\text{Per } A \ B \ C$  **and**

$\text{Per } A' \ B' \ C'$  **and**

$A \ B \ \text{Lt } A' \ B'$  **and**

$\text{Cong } A \ C \ A' \ C'$

**shows**  $B' \ C' \ \text{Lt } B \ C$

$\langle \text{proof} \rangle$

**lemma** *symmetry-preserves-conga*:

**assumes**  $A \neq B$  **and**  $C \neq B$  **and**

$M \ \text{Midpoint } A \ A'$  **and**

$M \ \text{Midpoint } B \ B'$  **and**

$M \ \text{Midpoint } C \ C'$

**shows**  $A \ B \ C \ \text{Cong} \ A' \ B' \ C'$

$\langle \text{proof} \rangle$

**lemma l11-57:**  
**assumes**  $A A' OS B B'$  **and**  
 $Per B A A'$  **and**  
 $Per B' A' A$  **and**  
 $A A' OS C C'$  **and**  
 $Per C A A'$  **and**  
 $Per C' A' A$   
**shows**  $B A C Cong A B' A' C'$   
⟨proof⟩

**lemma cop3-orth-at--orth-at:**  
**assumes**  $\neg Col D E F$  **and**  
 $Coplanar A B C D$  **and**  
 $Coplanar A B C E$  **and**  
 $Coplanar A B C F$  **and**  
 $X OrthAt A B C U V$   
**shows**  $X OrthAt D E F U V$   
⟨proof⟩

**lemma col2-orth-at--orth-at:**  
**assumes**  $U \neq V$  **and**  
 $Col P Q U$  **and**  
 $Col P Q V$  **and**  
 $X OrthAt A B C P Q$   
**shows**  $X OrthAt A B C U V$   
⟨proof⟩

**lemma col-orth-at--orth-at:**  
**assumes**  $U \neq W$  **and**  
 $Col U V W$  **and**  
 $X OrthAt A B C U V$   
**shows**  $X OrthAt A B C U W$   
⟨proof⟩

**lemma orth-at-symmetry:**  
**assumes**  $X OrthAt A B C U V$   
**shows**  $X OrthAt A B C V U$   
⟨proof⟩

**lemma orth-at-distincts:**  
**assumes**  $X OrthAt A B C U V$   
**shows**  $A \neq B \wedge B \neq C \wedge A \neq C \wedge U \neq V$   
⟨proof⟩

**lemma orth-at-chara:**  
 $X OrthAt A B C X P \longleftrightarrow$   
 $(\neg Col A B C \wedge X \neq P \wedge Coplanar A B C X \wedge (\forall D. (Coplanar A B C D \longrightarrow Per D X P)))$   
⟨proof⟩

**lemma cop3-orth--orth:**  
**assumes**  $\neg Col D E F$  **and**  
 $Coplanar A B C D$  **and**  
 $Coplanar A B C E$  **and**  
 $Coplanar A B C F$  **and**  
 $A B C Orth U V$   
**shows**  $D E F Orth U V$   
⟨proof⟩

**lemma col2-orth--orth:**  
**assumes**  $U \neq V$  **and**  
 $Col P Q U$  **and**  
 $Col P Q V$  **and**  
 $A B C Orth P Q$   
**shows**  $A B C Orth U V$   
⟨proof⟩

**lemma** *col-orth--orth*:

**assumes**  $U \neq W$  **and**

*Col*  $U V W$  **and**

$A B C$  *Orth*  $U V$

**shows**  $A B C$  *Orth*  $U W$

*<proof>*

**lemma** *orth-symmetry*:

**assumes**  $A B C$  *Orth*  $U V$

**shows**  $A B C$  *Orth*  $V U$

*<proof>*

**lemma** *orth-distincts*:

**assumes**  $A B C$  *Orth*  $U V$

**shows**  $A \neq B \wedge B \neq C \wedge A \neq C \wedge U \neq V$

*<proof>*

**lemma** *col-cop-orth--orth-at*:

**assumes**  $A B C$  *Orth*  $U V$  **and**

*Coplanar*  $A B C X$  **and**

*Col*  $U V X$

**shows**  $X$  *OrthAt*  $A B C U V$

*<proof>*

**lemma** *l11-60-aux*:

**assumes**  $\neg$  *Col*  $A B C$  **and**

*Cong*  $A P A Q$  **and**

*Cong*  $B P B Q$  **and**

*Cong*  $C P C Q$  **and**

*Coplanar*  $A B C D$

**shows** *Cong*  $D P D Q$

*<proof>*

**lemma** *l11-60*:

**assumes**  $\neg$  *Col*  $A B C$  **and**

*Per*  $A D P$  **and**

*Per*  $B D P$  **and**

*Per*  $C D P$  **and**

*Coplanar*  $A B C E$

**shows** *Per*  $E D P$

*<proof>*

**lemma** *l11-60-bis*:

**assumes**  $\neg$  *Col*  $A B C$  **and**

$D \neq P$  **and**

*Coplanar*  $A B C D$  **and**

*Per*  $A D P$  **and**

*Per*  $B D P$  **and**

*Per*  $C D P$

**shows**  $D$  *OrthAt*  $A B C D P$

*<proof>*

**lemma** *l11-61*:

**assumes**  $A \neq A'$  **and**

$A \neq B$  **and**

$A \neq C$  **and**

*Coplanar*  $A A' B B'$  **and**

*Per*  $B A A'$  **and**

*Per*  $B' A' A$  **and**

*Coplanar*  $A A' C C'$  **and**

*Per*  $C A A'$  **and**

*Per*  $B A C$

**shows** *Per*  $B' A' C'$

*<proof>*

**lemma** *l11-61-bis*:

**assumes**  $D \text{ OrthAt } A B C D P$  **and**  
 $D E \text{ Perp } E Q$  **and**  
 $\text{Coplanar } A B C E$  **and**  
 $\text{Coplanar } D E P Q$   
**shows**  $E \text{ OrthAt } A B C E Q$   
⟨proof⟩

**lemma** *l11-62-unicity*:  
**assumes**  $\text{Coplanar } A B C D$  **and**  
 $\text{Coplanar } A B C D'$  **and**  
 $\forall E. \text{Coplanar } A B C E \longrightarrow \text{Per } E D P$  **and**  
 $\forall E. \text{Coplanar } A B C E \longrightarrow \text{Per } E D' P$   
**shows**  $D = D'$   
⟨proof⟩

**lemma** *l11-62-unicity-bis*:  
**assumes**  $X \text{ OrthAt } A B C X U$  **and**  
 $Y \text{ OrthAt } A B C Y U$   
**shows**  $X = Y$   
⟨proof⟩

**lemma** *orth-at2--eq*:  
**assumes**  $X \text{ OrthAt } A B C U V$  **and**  
 $Y \text{ OrthAt } A B C U V$   
**shows**  $X = Y$   
⟨proof⟩

**lemma** *col-cop-orth-at--eq*:  
**assumes**  $X \text{ OrthAt } A B C U V$  **and**  
 $\text{Coplanar } A B C Y$  **and**  
 $\text{Col } U V Y$   
**shows**  $X = Y$   
⟨proof⟩

**lemma** *orth-at--ncop1*:  
**assumes**  $U \neq X$  **and**  
 $X \text{ OrthAt } A B C U V$   
**shows**  $\neg \text{Coplanar } A B C U$   
⟨proof⟩

**lemma** *orth-at--ncop2*:  
**assumes**  $V \neq X$  **and**  
 $X \text{ OrthAt } A B C U V$   
**shows**  $\neg \text{Coplanar } A B C V$   
⟨proof⟩

**lemma** *orth-at--ncop*:  
**assumes**  $X \text{ OrthAt } A B C X P$   
**shows**  $\neg \text{Coplanar } A B C P$   
⟨proof⟩

**lemma** *l11-62-existence*:  
 $\exists D. (\text{Coplanar } A B C D \wedge (\forall E. (\text{Coplanar } A B C E \longrightarrow \text{Per } E D P)))$   
⟨proof⟩

**lemma** *l11-62-existence-bis*:  
**assumes**  $\neg \text{Coplanar } A B C P$   
**shows**  $\exists X. X \text{ OrthAt } A B C X P$   
⟨proof⟩

**lemma** *l11-63-aux*:  
**assumes**  $\text{Coplanar } A B C D$  **and**  
 $D \neq E$  **and**  
 $E \text{ OrthAt } A B C E P$   
**shows**  $\exists Q. (D E \text{ OS } P Q \wedge A B C \text{ Orth } D Q)$   
⟨proof⟩

**lemma** *l11-63-existence*:

**assumes** *Coplanar A B C D* **and**

$\neg$  *Coplanar A B C P*

**shows**  $\exists Q. A B C \text{ Orth } D Q$

*<proof>*

**lemma** *l8-21-3*:

**assumes** *Coplanar A B C D* **and**

$\neg$  *Coplanar A B C X*

**shows**

$\exists P T. (A B C \text{ Orth } D P \wedge \text{Coplanar } A B C T \wedge \text{Bet } X T P)$

*<proof>*

**lemma** *mid2-orth-at2--cong*:

**assumes** *X OrthAt A B C X P* **and**

*Y OrthAt A B C Y Q* **and**

*X Midpoint P P'* **and**

*Y Midpoint Q Q'*

**shows** *Cong P Q P' Q'*

*<proof>*

**lemma** *orth-at2-tsp--ts*:

**assumes**  $P \neq Q$  **and**

*P OrthAt A B C P X* **and**

*Q OrthAt A B C Q Y* **and**

*A B C TSP X Y*

**shows** *P Q TS X Y*

*<proof>*

**lemma** *orth-dec*:

**shows**  $A B C \text{ Orth } U V \vee \neg A B C \text{ Orth } U V$  *<proof>*

**lemma** *orth-at-dec*:

**shows**  $X \text{ OrthAt } A B C U V \vee \neg X \text{ OrthAt } A B C U V$  *<proof>*

**lemma** *tsp-dec*:

**shows**  $A B C \text{ TSP } X Y \vee \neg A B C \text{ TSP } X Y$  *<proof>*

**lemma** *osp-dec*:

**shows**  $A B C \text{ OSP } X Y \vee \neg A B C \text{ OSP } X Y$  *<proof>*

**lemma** *ts2--inangle*:

**assumes** *A C TS B P* **and**

*B P TS A C*

**shows** *P InAngle A B C*

*<proof>*

**lemma** *os-ts--inangle*:

**assumes** *B P TS A C* **and**

*B A OS C P*

**shows** *P InAngle A B C*

*<proof>*

**lemma** *os2--inangle*:

**assumes** *B A OS C P* **and**

*B C OS A P*

**shows** *P InAngle A B C*

*<proof>*

**lemma** *acute-conga--acute*:

**assumes** *Acute A B C* **and**

*A B C Cong A D E F*

**shows** *Acute D E F*

*<proof>*

**lemma** *acute-out2--acute:*

**assumes** *B Out A' A and*

*B Out C' C and*

*Acute A B C*

**shows** *Acute A' B C'*

*<proof>*

**lemma** *conga-obtuse--obtuse:*

**assumes** *Obtuse A B C and*

*A B C CongA D E F*

**shows** *Obtuse D E F*

*<proof>*

**lemma** *obtuse-out2--obtuse:*

**assumes** *B Out A' A and*

*B Out C' C and*

*Obtuse A B C*

**shows** *Obtuse A' B C'*

*<proof>*

**lemma** *bet-lea--bet:*

**assumes** *Bet A B C and*

*A B C LeA D E F*

**shows** *Bet D E F*

*<proof>*

**lemma** *out-lea--out:*

**assumes** *E Out D F and*

*A B C LeA D E F*

**shows** *B Out A C*

*<proof>*

**lemma** *bet2-lta--lta:*

**assumes** *A B C LtA D E F and*

*Bet A B A' and*

*A' ≠ B and*

*Bet D E D' and*

*D' ≠ E*

**shows** *D' E F LtA A' B C*

*<proof>*

**lemma** *lea123456-lta--lta:*

**assumes** *A B C LeA D E F and*

*D E F LtA G H I*

**shows** *A B C LtA G H I*

*<proof>*

**lemma** *lea456789-lta--lta:*

**assumes** *A B C LtA D E F and*

*D E F LeA G H I*

**shows** *A B C LtA G H I*

*<proof>*

**lemma** *acute-per--lta:*

**assumes** *Acute A B C and*

*D ≠ E and*

*E ≠ F and*

*Per D E F*

**shows** *A B C LtA D E F*

*<proof>*

**lemma** *obtuse-per--lta:*

**assumes** *Obtuse A B C and*

*D ≠ E and*

*E ≠ F and*

*Per D E F*

**shows**  $D E F LtA A B C$   
 $\langle proof \rangle$

**lemma** *acute-obtuse--lta*:  
**assumes**  $Acute A B C$  **and**  
 $Obtuse D E F$   
**shows**  $A B C LtA D E F$   
 $\langle proof \rangle$

**lemma** *lea-in-angle*:  
**assumes**  $A B P LeA A B C$  **and**  
 $A B OS C P$   
**shows**  $P InAngle A B C$   
 $\langle proof \rangle$

**lemma** *acute-bet--obtuse*:  
**assumes**  $Bet A B A'$  **and**  
 $A' \neq B$  **and**  
 $Acute A B C$   
**shows**  $Obtuse A' B C$   
 $\langle proof \rangle$

**lemma** *bet-obtuse--acute*:  
**assumes**  $Bet A B A'$  **and**  
 $A' \neq B$  **and**  
 $Obtuse A B C$   
**shows**  $Acute A' B C$   
 $\langle proof \rangle$

**lemma** *inangle-dec*:  
 $P InAngle A B C \vee \neg P InAngle A B C$   $\langle proof \rangle$

**lemma** *lea-dec*:  
 $A B C LeA D E F \vee \neg A B C LeA D E F$   $\langle proof \rangle$

**lemma** *lta-dec*:  
 $A B C LtA D E F \vee \neg A B C LtA D E F$   $\langle proof \rangle$

**lemma** *lea-total*:  
**assumes**  $A \neq B$  **and**  
 $B \neq C$  **and**  
 $D \neq E$  **and**  
 $E \neq F$   
**shows**  $A B C LeA D E F \vee D E F LeA A B C$   
 $\langle proof \rangle$

**lemma** *or-lta2-conga*:  
**assumes**  $A \neq B$  **and**  
 $C \neq B$  **and**  
 $D \neq E$  **and**  
 $F \neq E$   
**shows**  $A B C LtA D E F \vee D E F LtA A B C \vee A B C CongA D E F$   
 $\langle proof \rangle$

**lemma** *angle-partition*:  
**assumes**  $A \neq B$  **and**  
 $B \neq C$   
**shows**  $Acute A B C \vee Per A B C \vee Obtuse A B C$   
 $\langle proof \rangle$

**lemma** *acute-chara-1*:  
**assumes**  $Bet A B A'$  **and**  
 $B \neq A'$  **and**  
 $Acute A B C$   
**shows**  $A B C LtA A' B C$   
 $\langle proof \rangle$

**lemma acute-chara-2:**  
**assumes**  $Bet\ A\ B\ A'$  **and**  
 $A\ B\ C\ Lt\ A'\ B\ C$   
**shows**  $Acute\ A\ B\ C$   
 $\langle proof \rangle$

**lemma acute-chara:**  
**assumes**  $Bet\ A\ B\ A'$  **and**  
 $B \neq A'$   
**shows**  $Acute\ A\ B\ C \longleftrightarrow A\ B\ C\ Lt\ A'\ B\ C$   
 $\langle proof \rangle$

**lemma obtuse-chara:**  
**assumes**  $Bet\ A\ B\ A'$  **and**  
 $B \neq A'$   
**shows**  $Obtuse\ A\ B\ C \longleftrightarrow A'\ B\ C\ Lt\ A\ B\ C$   
 $\langle proof \rangle$

**lemma conga--acute:**  
**assumes**  $A\ B\ C\ Cong\ A\ A\ C\ B$   
**shows**  $Acute\ A\ B\ C$   
 $\langle proof \rangle$

**lemma cong--acute:**  
**assumes**  $A \neq B$  **and**  
 $B \neq C$  **and**  
 $Cong\ A\ B\ A\ C$   
**shows**  $Acute\ A\ B\ C$   
 $\langle proof \rangle$

**lemma nltA--lea:**  
**assumes**  $\neg\ A\ B\ C\ Lt\ A\ D\ E\ F$  **and**  
 $A \neq B$  **and**  
 $B \neq C$  **and**  
 $D \neq E$  **and**  
 $E \neq F$   
**shows**  $D\ E\ F\ Le\ A\ A\ B\ C$   
 $\langle proof \rangle$

**lemma nlea--lta:**  
**assumes**  $\neg\ A\ B\ C\ Le\ A\ D\ E\ F$  **and**  
 $A \neq B$  **and**  
 $B \neq C$  **and**  
 $D \neq E$  **and**  
 $E \neq F$   
**shows**  $D\ E\ F\ Lt\ A\ A\ B\ C$   
 $\langle proof \rangle$

**lemma triangle-strict-inequality:**  
**assumes**  $Bet\ A\ B\ D$  **and**  
 $Cong\ B\ C\ B\ D$  **and**  
 $\neg\ Bet\ A\ B\ C$   
**shows**  $A\ C\ Lt\ A\ D$   
 $\langle proof \rangle$

**lemma triangle-inequality:**  
**assumes**  $Bet\ A\ B\ D$  **and**  
 $Cong\ B\ C\ B\ D$   
**shows**  $A\ C\ Le\ A\ D$   
 $\langle proof \rangle$

**lemma triangle-strict-inequality-2:**  
**assumes**  $Bet\ A'\ B'\ C'$  **and**  
 $Cong\ A\ B\ A'\ B'$  **and**  
 $Cong\ B\ C\ B'\ C'$  **and**

$\neg$  *Bet*  $A B C$   
**shows**  $A C$  *Lt*  $A' C'$   
⟨*proof*⟩

**lemma** *triangle-inequality-2*:  
**assumes** *Bet*  $A' B' C'$  **and**  
*Cong*  $A B A' B'$  **and**  
*Cong*  $B C B' C'$   
**shows**  $A C$  *Le*  $A' C'$   
⟨*proof*⟩

**lemma** *triangle-strict-reverse-inequality*:  
**assumes**  $A$  *Out*  $B D$  **and**  
*Cong*  $A C A D$  **and**  
 $\neg$   $A$  *Out*  $B C$   
**shows**  $B D$  *Lt*  $B C$   
⟨*proof*⟩

**lemma** *triangle-reverse-inequality*:  
**assumes**  $A$  *Out*  $B D$  **and**  
*Cong*  $A C A D$   
**shows**  $B D$  *Le*  $B C$   
⟨*proof*⟩

**lemma** *os3--lta*:  
**assumes**  $A B$  *OS*  $C D$  **and**  
 $B C$  *OS*  $A D$  **and**  
 $A C$  *OS*  $B D$   
**shows**  $B A C$  *Lt*  $A B D C$   
⟨*proof*⟩

**lemma** *bet-le--lt*:  
**assumes** *Bet*  $A D B$  **and**  
 $A \neq D$  **and**  
 $D \neq B$  **and**  
 $A C$  *Le*  $B C$   
**shows**  $D C$  *Lt*  $B C$   
⟨*proof*⟩

**lemma** *cong2--ncol*:  
**assumes**  $A \neq B$  **and**  
 $B \neq C$  **and**  
 $A \neq C$  **and**  
*Cong*  $A P B P$  **and**  
*Cong*  $A P C P$   
**shows**  $\neg$  *Col*  $A B C$   
⟨*proof*⟩

**lemma** *cong4-cop2--eq*:  
**assumes**  $A \neq B$  **and**  
 $B \neq C$  **and**  
 $A \neq C$  **and**  
*Cong*  $A P B P$  **and**  
*Cong*  $A P C P$  **and**  
*Coplanar*  $A B C P$  **and**  
*Cong*  $A Q B Q$  **and**  
*Cong*  $A Q C Q$  **and**  
*Coplanar*  $A B C Q$   
**shows**  $P = Q$   
⟨*proof*⟩

**lemma** *t18-18-aux*:  
**assumes** *Cong*  $A B D E$  **and**  
*Cong*  $A C D F$  **and**  
 $F D E$  *Lt*  $C A B$  **and**  
 $\neg$  *Col*  $A B C$  **and**

$\neg$  Col D E F and  
 D F Le D E  
 shows E F Lt B C  
 ⟨proof⟩

**lemma t18-18:**  
 assumes Cong A B D E and  
 Cong A C D F and  
 F D E LtA C A B  
 shows E F Lt B C  
 ⟨proof⟩

**lemma t18-19:**  
 assumes  $A \neq B$  and  
 $A \neq C$  and  
 Cong A B D E and  
 Cong A C D F and  
 E F Lt B C  
 shows F D E LtA C A B  
 ⟨proof⟩

**lemma acute-trivial:**  
 assumes  $A \neq B$   
 shows Acute A B A  
 ⟨proof⟩

**lemma acute-not-per:**  
 assumes Acute A B C  
 shows  $\neg$  Per A B C  
 ⟨proof⟩

**lemma angle-bisector:**  
 assumes  $A \neq B$  and  
 $C \neq B$   
 shows  $\exists P. (P \text{ InAngle } A B C \wedge P B A \text{ Cong } A P B C)$   
 ⟨proof⟩

**lemma reflectl--conga:**  
 assumes  $A \neq B$  and  
 $B \neq P$  and  
 $P P' \text{ ReflectL } A B$   
 shows  $A B P \text{ Cong } A A B P'$   
 ⟨proof⟩

**lemma conga-cop-out-reflectl--out:**  
 assumes  $\neg B \text{ Out } A C$  and  
 Coplanar A B C P and  
 $P B A \text{ Cong } A P B C$  and  
 $B \text{ Out } A T$  and  
 $T T' \text{ ReflectL } B P$   
 shows  $B \text{ Out } C T'$   
 ⟨proof⟩

**lemma col-conga-cop-reflectl--col:**  
 assumes  $\neg B \text{ Out } A C$  and  
 Coplanar A B C P and  
 $P B A \text{ Cong } A P B C$  and  
 Col B A T and  
 $T T' \text{ ReflectL } B P$   
 shows  $\text{Col } B C T'$   
 ⟨proof⟩

**lemma conga2-cop2--col:**  
 assumes  $\neg B \text{ Out } A C$  and  
 $P B A \text{ Cong } A P B C$  and  
 $P' B A \text{ Cong } A P' B C$  and

*Coplanar A B P P'* and  
*Coplanar B C P P'*  
**shows** *Col B P P'*  
 ⟨proof⟩

**lemma** *conga2-cop2--col-1*:  
**assumes**  $\neg$  *Col A B C* and  
*P B A CongA P B C* and  
*P' B A CongA P' B C* and  
*Coplanar A B C P* and  
*Coplanar A B C P'*  
**shows** *Col B P P'*  
 ⟨proof⟩

**lemma** *col-conga--conga*:  
**assumes** *P B A CongA P B C* and  
*Col B P P'* and  
*B  $\neq$  P'*  
**shows** *P' B A CongA P' B C*  
 ⟨proof⟩

**lemma** *cop-inangle--ex-col-inangle*:  
**assumes**  $\neg$  *B Out A C* and  
*P InAngle A B C* and  
*Coplanar A B C Q*  
**shows**  $\exists$  *R. (R InAngle A B C  $\wedge$  P  $\neq$  R  $\wedge$  Col P Q R)*  
 ⟨proof⟩

**lemma** *col-inangle2--out*:  
**assumes**  $\neg$  *Bet A B C* and  
*P InAngle A B C* and  
*Q InAngle A B C* and  
*Col B P Q*  
**shows** *B Out P Q*  
 ⟨proof⟩

**lemma** *inangle2--lea*:  
**assumes** *P InAngle A B C* and  
*Q InAngle A B C*  
**shows** *P B Q LeA A B C*  
 ⟨proof⟩

**lemma** *conga-inangle-per--acute*:  
**assumes** *Per A B C* and  
*P InAngle A B C* and  
*P B A CongA P B C*  
**shows** *Acute A B P*  
 ⟨proof⟩

**lemma** *conga-inangle2-per--acute*:  
**assumes** *Per A B C* and  
*P InAngle A B C* and  
*P B A CongA P B C* and  
*Q InAngle A B C*  
**shows** *Acute P B Q*  
 ⟨proof⟩

**lemma** *lta-os--ts*:  
**assumes**  
*A O1 P LtA A O1 B* and  
*O1 A OS B P*  
**shows** *O1 P TS A B*  
 ⟨proof⟩

**lemma** *bet--suppa*:  
**assumes** *A  $\neq$  B* and

$B \neq C$  and  
 $B \neq A'$  and  
 $Bet A B A'$   
**shows**  $A B C SuppA C B A'$   
 ⟨proof⟩

**lemma** *ex-suppa*:  
**assumes**  $A \neq B$  and  
 $B \neq C$   
**shows**  $\exists D E F. A B C SuppA D E F$   
 ⟨proof⟩

**lemma** *suppa-distincts*:  
**assumes**  $A B C SuppA D E F$   
**shows**  $A \neq B \wedge B \neq C \wedge D \neq E \wedge E \neq F$   
 ⟨proof⟩

**lemma** *suppa-right-comm*:  
**assumes**  $A B C SuppA D E F$   
**shows**  $A B C SuppA F E D$   
 ⟨proof⟩

**lemma** *suppa-left-comm*:  
**assumes**  $A B C SuppA D E F$   
**shows**  $C B A SuppA D E F$   
 ⟨proof⟩

**lemma** *suppa-comm*:  
**assumes**  $A B C SuppA D E F$   
**shows**  $C B A SuppA F E D$   
 ⟨proof⟩

**lemma** *suppa-sym*:  
**assumes**  $A B C SuppA D E F$   
**shows**  $D E F SuppA A B C$   
 ⟨proof⟩

**lemma** *conga2-suppa--suppa*:  
**assumes**  $A B C CongA A' B' C'$  and  
 $D E F CongA D' E' F'$  and  
 $A B C SuppA D E F$   
**shows**  $A' B' C' SuppA D' E' F'$   
 ⟨proof⟩

**lemma** *suppa2--conga456*:  
**assumes**  $A B C SuppA D E F$  and  
 $A B C SuppA D' E' F'$   
**shows**  $D E F CongA D' E' F'$   
 ⟨proof⟩

**lemma** *suppa2--conga123*:  
**assumes**  $A B C SuppA D E F$  and  
 $A' B' C' SuppA D E F$   
**shows**  $A B C CongA A' B' C'$   
 ⟨proof⟩

**lemma** *bet-out--suppa*:  
**assumes**  $A \neq B$  and  
 $B \neq C$  and  
 $Bet A B C$  and  
 $E Out D F$   
**shows**  $A B C SuppA D E F$   
 ⟨proof⟩

**lemma** *bet-suppa--out*:  
**assumes**  $Bet A B C$  and

$A B C \text{ Supp} A D E F$   
**shows**  $E \text{ Out} D F$   
(proof)

**lemma** *out-suppa--bet*:  
**assumes**  $B \text{ Out} A C$  **and**  
 $A B C \text{ Supp} A D E F$   
**shows**  $B \text{ et} D E F$   
(proof)

**lemma** *per-suppa--per*:  
**assumes**  $Per A B C$  **and**  
 $A B C \text{ Supp} A D E F$   
**shows**  $Per D E F$   
(proof)

**lemma** *per2--suppa*:  
**assumes**  $A \neq B$  **and**  
 $B \neq C$  **and**  
 $D \neq E$  **and**  
 $E \neq F$  **and**  
 $Per A B C$  **and**  
 $Per D E F$   
**shows**  $A B C \text{ Supp} A D E F$   
(proof)

**lemma** *suppa--per*:  
**assumes**  $A B C \text{ Supp} A A B C$   
**shows**  $Per A B C$   
(proof)

**lemma** *acute-suppa--obtuse*:  
**assumes**  $Acute A B C$  **and**  
 $A B C \text{ Supp} A D E F$   
**shows**  $Obtuse D E F$   
(proof)

**lemma** *obtuse-suppa--acute*:  
**assumes**  $Obtuse A B C$  **and**  
 $A B C \text{ Supp} A D E F$   
**shows**  $Acute D E F$   
(proof)

**lemma** *lea-suppa2--lea*:  
**assumes**  $A B C \text{ Supp} A A' B' C'$  **and**  
 $D E F \text{ Supp} A D' E' F'$   
 $A B C \text{ Le} A D E F$   
**shows**  $D' E' F' \text{ Le} A A' B' C'$   
(proof)

**lemma** *lta-suppa2--lta*:  
**assumes**  $A B C \text{ Supp} A A' B' C'$   
**and**  $D E F \text{ Supp} A D' E' F'$   
**and**  $A B C \text{ Lt} A D E F$   
**shows**  $D' E' F' \text{ Lt} A A' B' C'$   
(proof)

**lemma** *suppa-dec*:  
 $A B C \text{ Supp} A D E F \vee \neg A B C \text{ Supp} A D E F$   
(proof)

**lemma** *acute-one-side-aux*:  
**assumes**  $C A \text{ OS} P B$  **and**  
 $Acute A C P$  **and**  
 $C A \text{ Perp} B C$   
**shows**  $C B \text{ OS} A P$

$\langle \text{proof} \rangle$

**lemma** *acute-one-side-aux0*:

**assumes** *Col A C P* **and**

*Acute A C P* **and**

*C A Perp B C*

**shows** *C B OS A P*

$\langle \text{proof} \rangle$

**lemma** *acute-cop-perp--one-side*:

**assumes** *Acute A C P* **and**

*C A Perp B C* **and**

*Coplanar A B C P*

**shows** *C B OS A P*

$\langle \text{proof} \rangle$

**lemma** *acute--not-obtuse*:

**assumes** *Acute A B C*

**shows**  $\neg$  *Obtuse A B C*

$\langle \text{proof} \rangle$

### 3.10.2 Sum of angles

**lemma** *suma-distincts*:

**assumes** *A B C D E F SumA G H I*

**shows**  $A \neq B \wedge B \neq C \wedge D \neq E \wedge E \neq F \wedge G \neq H \wedge H \neq I$

$\langle \text{proof} \rangle$

**lemma** *trisuma-distincts*:

**assumes** *A B C TriSumA D E F*

**shows**  $A \neq B \wedge B \neq C \wedge A \neq C \wedge D \neq E \wedge E \neq F$

$\langle \text{proof} \rangle$

**lemma** *ex-suma*:

**assumes**  $A \neq B$  **and**

$B \neq C$  **and**

$D \neq E$  **and**

$E \neq F$

**shows**  $\exists G H I. A B C D E F SumA G H I$

$\langle \text{proof} \rangle$

**lemma** *suma2--conga*:

**assumes** *A B C D E F SumA G H I* **and**

*A B C D E F SumA G' H' I'*

**shows** *G H I CongA G' H' I'*

$\langle \text{proof} \rangle$

**lemma** *suma-sym*:

**assumes** *A B C D E F SumA G H I*

**shows** *D E F A B C SumA G H I*

$\langle \text{proof} \rangle$

**lemma** *conga3-suma--suma*:

**assumes** *A B C D E F SumA G H I* **and**

*A B C CongA A' B' C'* **and**

*D E F CongA D' E' F'* **and**

*G H I CongA G' H' I'*

**shows** *A' B' C' D' E' F' SumA G' H' I'*

$\langle \text{proof} \rangle$

**lemma** *out6-suma--suma*:

**assumes** *A B C D E F SumA G H I* **and**

*B Out A A'* **and**

*B Out C C'* **and**

*E Out D D'* **and**

*E Out F F'* **and**

$H$  Out  $G G'$  and  
 $H$  Out  $I I'$   
**shows**  $A' B C' D' E F' SumA G' H I'$   
 ⟨proof⟩

**lemma** *out546-suma--conga*:  
**assumes**  $A B C D E F SumA G H I$  and  
 $E$  Out  $D F$   
**shows**  $A B C CongA G H I$   
 ⟨proof⟩

**lemma** *out546--suma*:  
**assumes**  $A \neq B$  and  
 $B \neq C$  and  
 $E$  Out  $D F$   
**shows**  $A B C D E F SumA A B C$   
 ⟨proof⟩

**lemma** *out213-suma--conga*:  
**assumes**  $A B C D E F SumA G H I$  and  
 $B$  Out  $A C$   
**shows**  $D E F CongA G H I$   
 ⟨proof⟩

**lemma** *out213--suma*:  
**assumes**  $D \neq E$  and  
 $E \neq F$  and  
 $B$  Out  $A C$   
**shows**  $A B C D E F SumA D E F$   
 ⟨proof⟩

**lemma** *suma-left-comm*:  
**assumes**  $A B C D E F SumA G H I$   
**shows**  $C B A D E F SumA G H I$   
 ⟨proof⟩

**lemma** *suma-middle-comm*:  
**assumes**  $A B C D E F SumA G H I$   
**shows**  $A B C F E D SumA G H I$   
 ⟨proof⟩

**lemma** *suma-right-comm*:  
**assumes**  $A B C D E F SumA G H I$   
**shows**  $A B C D E F SumA I H G$   
 ⟨proof⟩

**lemma** *suma-comm*:  
**assumes**  $A B C D E F SumA G H I$   
**shows**  $C B A F E D SumA I H G$   
 ⟨proof⟩

**lemma** *ts--suma*:  
**assumes**  $A B TS C D$   
**shows**  $C B A A B D SumA C B D$   
 ⟨proof⟩

**lemma** *ts--suma-1*:  
**assumes**  $A B TS C D$   
**shows**  $C A B B A D SumA C A D$   
 ⟨proof⟩

**lemma** *inangle--suma*:  
**assumes**  $P$  InAngle  $A B C$   
**shows**  $A B P P B C SumA A B C$   
 ⟨proof⟩

**lemma** *bet--suma*:

**assumes**  $A \neq B$  **and**

$B \neq C$  **and**

$P \neq B$  **and**  $Bet\ A\ B\ C$

**shows**  $A\ B\ P\ P\ B\ C\ SumA\ A\ B\ C$

*<proof>*

**lemma** *sams-chara*:

**assumes**  $A \neq B$  **and**

$A' \neq B$  **and**

$Bet\ A\ B\ A'$

**shows**  $SAMS\ A\ B\ C\ D\ E\ F \longleftrightarrow D\ E\ F\ LeA\ C\ B\ A'$

*<proof>*

**lemma** *sams-distincts*:

**assumes**  $SAMS\ A\ B\ C\ D\ E\ F$

**shows**  $A \neq B \wedge B \neq C \wedge D \neq E \wedge E \neq F$

*<proof>*

**lemma** *sams-sym*:

**assumes**  $SAMS\ A\ B\ C\ D\ E\ F$

**shows**  $SAMS\ D\ E\ F\ A\ B\ C$

*<proof>*

**lemma** *sams-right-comm*:

**assumes**  $SAMS\ A\ B\ C\ D\ E\ F$

**shows**  $SAMS\ A\ B\ C\ F\ E\ D$

*<proof>*

**lemma** *sams-left-comm*:

**assumes**  $SAMS\ A\ B\ C\ D\ E\ F$

**shows**  $SAMS\ C\ B\ A\ D\ E\ F$

*<proof>*

**lemma** *sams-comm*:

**assumes**  $SAMS\ A\ B\ C\ D\ E\ F$

**shows**  $SAMS\ C\ B\ A\ F\ E\ D$

*<proof>*

**lemma** *conga2-sams--sams*:

**assumes**  $A\ B\ C\ CongA\ A'\ B'\ C'$  **and**

$D\ E\ F\ CongA\ D'\ E'\ F'$  **and**

$SAMS\ A\ B\ C\ D\ E\ F$

**shows**  $SAMS\ A'\ B'\ C'\ D'\ E'\ F'$

*<proof>*

**lemma** *out546--sams*:

**assumes**  $A \neq B$  **and**

$B \neq C$  **and**

$E\ Out\ D\ F$

**shows**  $SAMS\ A\ B\ C\ D\ E\ F$

*<proof>*

**lemma** *out213--sams*:

**assumes**  $D \neq E$  **and**

$E \neq F$  **and**

$B\ Out\ A\ C$

**shows**  $SAMS\ A\ B\ C\ D\ E\ F$

*<proof>*

**lemma** *bet-suma--sams*:

**assumes**  $A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$  **and**

$Bet\ G\ H\ I$

**shows**  $SAMS\ A\ B\ C\ D\ E\ F$

*<proof>*

**lemma** *bet--sams*:

**assumes**  $A \neq B$  **and**

$B \neq C$  **and**

$P \neq B$  **and**

$Bet\ A\ B\ C$

**shows**  $SAMS\ A\ B\ P\ P\ B\ C$

*<proof>*

**lemma** *suppa--sams*:

**assumes**  $A\ B\ C\ SuppA\ D\ E\ F$

**shows**  $SAMS\ A\ B\ C\ D\ E\ F$

*<proof>*

**lemma** *os-ts--sams*:

**assumes**  $B\ P\ TS\ A\ C$  **and**

$A\ B\ OS\ P\ C$

**shows**  $SAMS\ A\ B\ P\ P\ B\ C$

*<proof>*

**lemma** *os2--sams*:

**assumes**  $A\ B\ OS\ P\ C$  **and**

$C\ B\ OS\ P\ A$

**shows**  $SAMS\ A\ B\ P\ P\ B\ C$

*<proof>*

**lemma** *inangle--sams*:

**assumes**  $P\ InAngle\ A\ B\ C$

**shows**  $SAMS\ A\ B\ P\ P\ B\ C$

*<proof>*

**lemma** *sams-suma--lea123789*:

**assumes**  $A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$  **and**

$SAMS\ A\ B\ C\ D\ E\ F$

**shows**  $A\ B\ C\ LeA\ G\ H\ I$

*<proof>*

**lemma** *sams-suma--lea456789*:

**assumes**  $A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$  **and**

$SAMS\ A\ B\ C\ D\ E\ F$

**shows**  $D\ E\ F\ LeA\ G\ H\ I$

*<proof>*

**lemma** *sams-lea2--sams*:

**assumes**  $SAMS\ A'\ B'\ C'\ D'\ E'\ F'$  **and**

$A\ B\ C\ LeA\ A'\ B'\ C'$  **and**

$D\ E\ F\ LeA\ D'\ E'\ F'$

**shows**  $SAMS\ A\ B\ C\ D\ E\ F$

*<proof>*

**lemma** *sams-lea456-suma2--lea*:

**assumes**  $D\ E\ F\ LeA\ D'\ E'\ F'$  **and**

$SAMS\ A\ B\ C\ D'\ E'\ F'$  **and**

$A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$  **and**

$A\ B\ C\ D'\ E'\ F'\ SumA\ G'\ H'\ I'$

**shows**  $G\ H\ I\ LeA\ G'\ H'\ I'$

*<proof>*

**lemma** *sams-lea123-suma2--lea*:

**assumes**  $A\ B\ C\ LeA\ A'\ B'\ C'$  **and**

$SAMS\ A'\ B'\ C'\ D\ E\ F$  **and**

$A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$  **and**

$A'\ B'\ C'\ D\ E\ F\ SumA\ G'\ H'\ I'$

**shows**  $G\ H\ I\ LeA\ G'\ H'\ I'$

*<proof>*

**lemma** *sams-lea2-suma2--lea*:

**assumes**  $A B C \text{ Le}A A' B' C'$  **and**  
 $D E F \text{ Le}A D' E' F'$  **and**  
 $SAMS A' B' C' D' E' F'$  **and**  
 $A B C D E F \text{ Sum}A G H I$  **and**  
 $A' B' C' D' E' F' \text{ Sum}A G' H' I'$   
**shows**  $G H I \text{ Le}A G' H' I'$   
⟨proof⟩

**lemma** *sams2-suma2--conga456*:  
**assumes**  $SAMS A B C D E F$  **and**  
 $SAMS A B C D' E' F'$  **and**  
 $A B C D E F \text{ Sum}A G H I$  **and**  
 $A B C D' E' F' \text{ Sum}A G H I$   
**shows**  $D E F \text{ Cong}A D' E' F'$   
⟨proof⟩

**lemma** *sams2-suma2--conga123*:  
**assumes**  $SAMS A B C D E F$  **and**  
 $SAMS A' B' C' D E F$  **and**  
 $A B C D E F \text{ Sum}A G H I$  **and**  
 $A' B' C' D E F \text{ Sum}A G H I$   
**shows**  $A B C \text{ Cong}A A' B' C'$   
⟨proof⟩

**lemma** *suma-assoc-1*:  
**assumes**  $SAMS A B C D E F$  **and**  
 $SAMS D E F G H I$  **and**  
 $A B C D E F \text{ Sum}A A' B' C'$  **and**  
 $D E F G H I \text{ Sum}A D' E' F'$  **and**  
 $A' B' C' G H I \text{ Sum}A K L M$   
**shows**  $A B C D' E' F' \text{ Sum}A K L M$   
⟨proof⟩

**lemma** *suma-assoc-2*:  
**assumes**  $SAMS A B C D E F$  **and**  
 $SAMS D E F G H I$  **and**  
 $A B C D E F \text{ Sum}A A' B' C'$  **and**  
 $D E F G H I \text{ Sum}A D' E' F'$  **and**  
 $A B C D' E' F' \text{ Sum}A K L M$   
**shows**  $A' B' C' G H I \text{ Sum}A K L M$   
⟨proof⟩

**lemma** *suma-assoc*:  
**assumes**  $SAMS A B C D E F$  **and**  
 $SAMS D E F G H I$  **and**  
 $A B C D E F \text{ Sum}A A' B' C'$  **and**  
 $D E F G H I \text{ Sum}A D' E' F'$   
**shows**  
 $A' B' C' G H I \text{ Sum}A K L M \longleftrightarrow A B C D' E' F' \text{ Sum}A K L M$   
⟨proof⟩

**lemma** *sams-assoc-1*:  
**assumes**  $SAMS A B C D E F$  **and**  
 $SAMS D E F G H I$  **and**  
 $A B C D E F \text{ Sum}A A' B' C'$  **and**  
 $D E F G H I \text{ Sum}A D' E' F'$  **and**  
 $SAMS A' B' C' G H I$   
**shows**  $SAMS A B C D' E' F'$   
⟨proof⟩

**lemma** *sams-assoc-2*:  
**assumes**  $SAMS A B C D E F$  **and**  
 $SAMS D E F G H I$  **and**  
 $A B C D E F \text{ Sum}A A' B' C'$  **and**  
 $D E F G H I \text{ Sum}A D' E' F'$  **and**  
 $SAMS A B C D' E' F'$

**shows**  $SAMS A' B' C' G H I$   
(proof)

**lemma** *sams-assoc*:

**assumes**  $SAMS A B C D E F$  **and**  
 $SAMS D E F G H I$  **and**  
 $A B C D E F SumA A' B' C'$  **and**  
 $D E F G H I SumA D' E' F'$   
**shows**  $(SAMS A' B' C' G H I) \longleftrightarrow (SAMS A B C D' E' F')$   
(proof)

**lemma** *sams-nos--nts*:

**assumes**  $SAMS A B C C B J$  **and**  
 $\neg B C OS A J$   
**shows**  $\neg A B TS C J$   
(proof)

**lemma** *conga-sams-nos--nts*:

**assumes**  $SAMS A B C D E F$  **and**  
 $C B J CongA D E F$  **and**  
 $\neg B C OS A J$   
**shows**  $\neg A B TS C J$   
(proof)

**lemma** *sams-lea2-suma2--conga123*:

**assumes**  $A B C LeA A' B' C'$  **and**  
 $D E F LeA D' E' F'$  **and**  
 $SAMS A' B' C' D' E' F'$  **and**  
 $A B C D E F SumA G H I$  **and**  
 $A' B' C' D' E' F' SumA G H I$   
**shows**  $A B C CongA A' B' C'$   
(proof)

**lemma** *sams-lea2-suma2--conga456*:

**assumes**  $A B C LeA A' B' C'$  **and**  
 $D E F LeA D' E' F'$  **and**  
 $SAMS A' B' C' D' E' F'$  **and**  
 $A B C D E F SumA G H I$  **and**  
 $A' B' C' D' E' F' SumA G H I$   
**shows**  $D E F CongA D' E' F'$   
(proof)

**lemma** *sams-suma--out213*:

**assumes**  $A B C D E F SumA D E F$  **and**  
 $SAMS A B C D E F$   
**shows**  $B Out A C$   
(proof)

**lemma** *sams-suma--out546*:

**assumes**  $A B C D E F SumA A B C$  **and**  
 $SAMS A B C D E F$   
**shows**  $E Out D F$   
(proof)

**lemma** *sams-lea-lta123-suma2--lta*:

**assumes**  $A B C LtA A' B' C'$  **and**  
 $D E F LeA D' E' F'$  **and**  
 $SAMS A' B' C' D' E' F'$  **and**  
 $A B C D E F SumA G H I$  **and**  
 $A' B' C' D' E' F' SumA G' H' I'$   
**shows**  $G H I LtA G' H' I'$   
(proof)

**lemma** *sams-lea-lta456-suma2--lta*:

**assumes**  $A B C LeA A' B' C'$  **and**  
 $D E F LtA D' E' F'$  **and**

*SAMS A' B' C' D' E' F' and*  
*A B C D E F SumA G H I and*  
*A' B' C' D' E' F' SumA G' H' I'*  
**shows** *G H I LtA G' H' I'*  
 ⟨*proof*⟩

**lemma** *sams-lta2-suma2--lta:*  
**assumes** *A B C LtA A' B' C' and*  
*D E F LtA D' E' F' and*  
*SAMS A' B' C' D' E' F' and*  
*A B C D E F SumA G H I and*  
*A' B' C' D' E' F' SumA G' H' I'*  
**shows** *G H I LtA G' H' I'*  
 ⟨*proof*⟩

**lemma** *sams-lea2-suma2--lea123:*  
**assumes** *D' E' F' LeA D E F and*  
*G H I LeA G' H' I' and*  
*SAMS A B C D E F and*  
*A B C D E F SumA G H I and*  
*A' B' C' D' E' F' SumA G' H' I'*  
**shows** *A B C LeA A' B' C'*  
 ⟨*proof*⟩

**lemma** *sams-lea2-suma2--lea456:*  
**assumes** *A' B' C' LeA A B C and*  
*G H I LeA G' H' I' and*  
*SAMS A B C D E F and*  
*A B C D E F SumA G H I and*  
*A' B' C' D' E' F' SumA G' H' I'*  
**shows** *D E F LeA D' E' F'*  
 ⟨*proof*⟩

**lemma** *sams-lea-lta456-suma2--lta123:*  
**assumes** *D' E' F' LtA D E F and*  
*G H I LeA G' H' I' and*  
*SAMS A B C D E F and*  
*A B C D E F SumA G H I and*  
*A' B' C' D' E' F' SumA G' H' I'*  
**shows** *A B C LtA A' B' C'*  
 ⟨*proof*⟩

**lemma** *sams-lea-lta123-suma2--lta456:*  
**assumes** *A' B' C' LtA A B C and*  
*G H I LeA G' H' I' and*  
*SAMS A B C D E F and*  
*A B C D E F SumA G H I and*  
*A' B' C' D' E' F' SumA G' H' I'*  
**shows** *D E F LtA D' E' F'*  
 ⟨*proof*⟩

**lemma** *sams-lea-lta789-suma2--lta123:*  
**assumes** *D' E' F' LeA D E F and*  
*G H I LtA G' H' I' and*  
*SAMS A B C D E F and*  
*A B C D E F SumA G H I and*  
*A' B' C' D' E' F' SumA G' H' I'*  
**shows** *A B C LtA A' B' C'*  
 ⟨*proof*⟩

**lemma** *sams-lea-lta789-suma2--lta456:*  
**assumes** *A' B' C' LeA A B C and*  
*G H I LtA G' H' I' and*  
*SAMS A B C D E F and*  
*A B C D E F SumA G H I and*  
*A' B' C' D' E' F' SumA G' H' I'*

**shows**  $D E F LtA D' E' F'$   
(proof)

**lemma** *sams-lta2-suma2--lta123*:  
**assumes**  $D' E' F' LtA D E F$  **and**  
 $G H I LtA G' H' I'$  **and**  
 $SAMS A B C D E F$  **and**  
 $A B C D E F SumA G H I$  **and**  
 $A' B' C' D' E' F' SumA G' H' I'$   
**shows**  $A B C LtA A' B' C'$   
(proof)

**lemma** *sams-lta2-suma2--lta456*:  
**assumes**  $A' B' C' LtA A B C$  **and**  
 $G H I LtA G' H' I'$  **and**  
 $SAMS A B C D E F$  **and**  
 $A B C D E F SumA G H I$  **and**  
 $A' B' C' D' E' F' SumA G' H' I'$   
**shows**  $D E F LtA D' E' F'$   
(proof)

**lemma** *sams123231*:  
**assumes**  $A \neq B$  **and**  
 $A \neq C$  **and**  
 $B \neq C$   
**shows**  $SAMS A B C B C A$   
(proof)

**lemma** *col-suma--col*:  
**assumes**  $Col D E F$  **and**  
 $A B C B C A SumA D E F$   
**shows**  $Col A B C$   
(proof)

**lemma** *ncol-suma--ncol*:  
**assumes**  $\neg Col A B C$  **and**  
 $A B C B C A SumA D E F$   
**shows**  $\neg Col D E F$   
(proof)

**lemma** *per2-suma--bet*:  
**assumes**  $Per A B C$  **and**  
 $Per D E F$  **and**  
 $A B C D E F SumA G H I$   
**shows**  $Bet G H I$   
(proof)

**lemma** *bet-per2--suma*:  
**assumes**  $A \neq B$  **and**  
 $B \neq C$  **and**  
 $D \neq E$  **and**  
 $E \neq F$  **and**  
 $G \neq H$  **and**  
 $H \neq I$  **and**  
 $Per A B C$  **and**  
 $Per D E F$  **and**  
 $Bet G H I$   
**shows**  $A B C D E F SumA G H I$   
(proof)

**lemma** *per2--sams*:  
**assumes**  $A \neq B$  **and**  
 $B \neq C$  **and**  
 $D \neq E$  **and**  
 $E \neq F$  **and**  
 $Per A B C$  **and**

*Per D E F*  
**shows** *SAMS A B C D E F*  
(proof)

**lemma** *bet-per-suma--per456:*  
**assumes** *Per A B C and*  
*Bet G H I and*  
*A B C D E F SumA G H I*  
**shows** *Per D E F*  
(proof)

**lemma** *bet-per-suma--per123:*  
**assumes** *Per D E F and*  
*Bet G H I and*  
*A B C D E F SumA G H I*  
**shows** *Per A B C*  
(proof)

**lemma** *bet-suma--per:*  
**assumes** *Bet D E F and*  
*A B C A B C SumA D E F*  
**shows** *Per A B C*  
(proof)

**lemma** *acute--sams:*  
**assumes** *Acute A B C*  
**shows** *SAMS A B C A B C*  
(proof)

**lemma** *acute-suma--nbet:*  
**assumes** *Acute A B C and*  
*A B C A B C SumA D E F*  
**shows**  $\neg$  *Bet D E F*  
(proof)

**lemma** *acute2--sams:*  
**assumes** *Acute A B C and*  
*Acute D E F*  
**shows** *SAMS A B C D E F*  
(proof)

**lemma** *acute2-suma--nbet-a:*  
**assumes** *Acute A B C and*  
*D E F LeA A B C and*  
*A B C D E F SumA G H I*  
**shows**  $\neg$  *Bet G H I*  
(proof)

**lemma** *acute2-suma--nbet:*  
**assumes** *Acute A B C and*  
*Acute D E F and*  
*A B C D E F SumA G H I*  
**shows**  $\neg$  *Bet G H I*  
(proof)

**lemma** *acute-per--sams:*  
**assumes**  $A \neq B$  **and**  
 $B \neq C$  **and**  
*Per A B C and*  
*Acute D E F*  
**shows** *SAMS A B C D E F*  
(proof)

**lemma** *acute-per-suma--nbet:*  
**assumes**  $A \neq B$  **and**  
 $B \neq C$  **and**

*Per A B C and*  
*Acute D E F and*  
*A B C D E F SumA G H I*  
**shows**  $\neg$  *Bet G H I*  
 ⟨proof⟩

**lemma** *obtuse--nsams:*  
**assumes** *Obtuse A B C*  
**shows**  $\neg$  *SAMS A B C A B C*  
 ⟨proof⟩

**lemma** *nbet-sams-suma--acute:*  
**assumes**  $\neg$  *Bet D E F and*  
*SAMS A B C A B C and*  
*A B C A B C SumA D E F*  
**shows** *Acute A B C*  
 ⟨proof⟩

**lemma** *nsams--obtuse:*  
**assumes**  $A \neq B$  **and**  
 $B \neq C$  **and**  
 $\neg$  *SAMS A B C A B C*  
**shows** *Obtuse A B C*  
 ⟨proof⟩

**lemma** *sams2-suma2--conga:*  
**assumes** *SAMS A B C A B C and*  
*A B C A B C SumA D E F and*  
*SAMS A' B' C' A' B' C' and*  
*A' B' C' A' B' C' SumA D E F*  
**shows** *A B C CongA A' B' C'*  
 ⟨proof⟩

**lemma** *acute2-suma2--conga:*  
**assumes** *Acute A B C and*  
*A B C A B C SumA D E F and*  
*Acute A' B' C' and*  
*A' B' C' A' B' C' SumA D E F*  
**shows** *A B C CongA A' B' C'*  
 ⟨proof⟩

**lemma** *bet2-suma--out:*  
**assumes** *Bet A B C and*  
*Bet D E F and*  
*A B C D E F SumA G H I*  
**shows** *H Out G I*  
 ⟨proof⟩

**lemma** *col2-suma--col:*  
**assumes** *Col A B C and*  
*Col D E F and*  
*A B C D E F SumA G H I*  
**shows** *Col G H I*  
 ⟨proof⟩

**lemma** *suma-suppa--bet:*  
**assumes** *A B C SuppA D E F and*  
*A B C D E F SumA G H I*  
**shows** *Bet G H I*  
 ⟨proof⟩

**lemma** *bet-suppa--suma:*  
**assumes**  $G \neq H$  **and**  
 $H \neq I$  **and**  
*A B C SuppA D E F and*  
*Bet G H I*

**shows**  $A B C D E F \text{ Sum}A G H I$   
(proof)

**lemma** *bet-suma--suppa*:  
**assumes**  $A B C D E F \text{ Sum}A G H I$  **and**  
 $Bet G H I$   
**shows**  $A B C \text{ Supp}A D E F$   
(proof)

**lemma** *bet2-suma--suma*:  
**assumes**  $A' \neq B$  **and**  
 $D' \neq E$  **and**  
 $Bet A B A'$  **and**  
 $Bet D E D'$  **and**  
 $A B C D E F \text{ Sum}A G H I$   
**shows**  $A' B C D' E F \text{ Sum}A G H I$   
(proof)

**lemma** *suma-suppa2--suma*:  
**assumes**  $A B C \text{ Supp}A A' B' C'$  **and**  
 $D E F \text{ Supp}A D' E' F'$  **and**  
 $A B C D E F \text{ Sum}A G H I$   
**shows**  $A' B' C' D' E' F' \text{ Sum}A G H I$   
(proof)

**lemma** *suma2-obtuse2--conga*:  
**assumes**  $Obtuse A B C$  **and**  
 $A B C A B C \text{ Sum}A D E F$  **and**  
 $Obtuse A' B' C'$  **and**  
 $A' B' C' A' B' C' \text{ Sum}A D E F$   
**shows**  $A B C \text{ Cong}A A' B' C'$   
(proof)

**lemma** *bet-suma2--or-conga*:  
**assumes**  $A0 \neq B$  **and**  
 $Bet A B A0$  **and**  
 $A B C A B C \text{ Sum}A D E F$  **and**  
 $A' B' C' A' B' C' \text{ Sum}A D E F$   
**shows**  $A B C \text{ Cong}A A' B' C' \vee A0 B C \text{ Cong}A A' B' C'$   
(proof)

**lemma** *suma2--or-conga-suppa*:  
**assumes**  $A B C A B C \text{ Sum}A D E F$  **and**  
 $A' B' C' A' B' C' \text{ Sum}A D E F$   
**shows**  $A B C \text{ Cong}A A' B' C' \vee A B C \text{ Supp}A A' B' C'$   
(proof)

**lemma** *ex-trisuma*:  
**assumes**  $A \neq B$  **and**  
 $B \neq C$  **and**  
 $A \neq C$   
**shows**  $\exists D E F. A B C \text{ TriSum}A D E F$   
(proof)

**lemma** *trisuma-perm-231*:  
**assumes**  $A B C \text{ TriSum}A D E F$   
**shows**  $B C A \text{ TriSum}A D E F$   
(proof)

**lemma** *trisuma-perm-312*:  
**assumes**  $A B C \text{ TriSum}A D E F$   
**shows**  $C A B \text{ TriSum}A D E F$   
(proof)

**lemma** *trisuma-perm-321*:  
**assumes**  $A B C \text{ TriSum}A D E F$

**shows**  $C B A \text{ TriSum} A D E F$   
 $\langle \text{proof} \rangle$

**lemma** *trisuma-perm-213*:  
**assumes**  $A B C \text{ TriSum} A D E F$   
**shows**  $B A C \text{ TriSum} A D E F$   
 $\langle \text{proof} \rangle$

**lemma** *trisuma-perm-132*:  
**assumes**  $A B C \text{ TriSum} A D E F$   
**shows**  $A C B \text{ TriSum} A D E F$   
 $\langle \text{proof} \rangle$

**lemma** *conga-trisuma--trisuma*:  
**assumes**  $A B C \text{ TriSum} A D E F$  **and**  
 $D E F \text{ Cong} A D' E' F'$   
**shows**  $A B C \text{ TriSum} A D' E' F'$   
 $\langle \text{proof} \rangle$

**lemma** *trisuma2--conga*:  
**assumes**  $A B C \text{ TriSum} A D E F$  **and**  
 $A B C \text{ TriSum} A D' E' F'$   
**shows**  $D E F \text{ Cong} A D' E' F'$   
 $\langle \text{proof} \rangle$

**lemma** *conga3-trisuma--trisuma*:  
**assumes**  $A B C \text{ TriSum} A D E F$  **and**  
 $A B C \text{ Cong} A' B' C'$  **and**  
 $B C A \text{ Cong} A' B' C'$  **and**  
 $C A B \text{ Cong} A' B' C'$   
**shows**  $A' B' C' \text{ TriSum} A D E F$   
 $\langle \text{proof} \rangle$

**lemma** *col-trisuma--bet*:  
**assumes**  $\text{Col } A B C$  **and**  
 $A B C \text{ TriSum} A P Q R$   
**shows**  $\text{Bet } P Q R$   
 $\langle \text{proof} \rangle$

**lemma** *suma-dec*:  
 $A B C D E F \text{ Sum} A G H I \vee \neg A B C D E F \text{ Sum} A G H I$   $\langle \text{proof} \rangle$

**lemma** *sams-dec*:  
 $\text{SAMS } A B C D E F \vee \neg \text{SAMS } A B C D E F$   $\langle \text{proof} \rangle$

**lemma** *trisuma-dec*:  
 $A B C \text{ TriSum} A P Q R \vee \neg A B C \text{ TriSum} A P Q R$   
 $\langle \text{proof} \rangle$

### 3.11 Parallelism

**lemma** *par-reflexivity*:  
**assumes**  $A \neq B$   
**shows**  $A B \text{ Par} A B$   
 $\langle \text{proof} \rangle$

**lemma** *par-strict-irreflexivity*:  
 $\neg A B \text{ ParStrict } A B$   
 $\langle \text{proof} \rangle$

**lemma** *not-par-strict-id*:  
 $\neg A B \text{ ParStrict } A C$   
 $\langle \text{proof} \rangle$

**lemma** *par-id*:  
**assumes**  $A B \text{ Par} A C$

**shows**  $Col\ A\ B\ C$   
 $\langle proof \rangle$

**lemma** *par-strict-not-col-1*:  
**assumes**  $A\ B\ ParStrict\ C\ D$   
**shows**  $\neg\ Col\ A\ B\ C$   
 $\langle proof \rangle$

**lemma** *par-strict-not-col-2*:  
**assumes**  $A\ B\ ParStrict\ C\ D$   
**shows**  $\neg\ Col\ B\ C\ D$   
 $\langle proof \rangle$

**lemma** *par-strict-not-col-3*:  
**assumes**  $A\ B\ ParStrict\ C\ D$   
**shows**  $\neg\ Col\ C\ D\ A$   
 $\langle proof \rangle$

**lemma** *par-strict-not-col-4*:  
**assumes**  $A\ B\ ParStrict\ C\ D$   
**shows**  $\neg\ Col\ A\ B\ D$   
 $\langle proof \rangle$

**lemma** *par-id-1*:  
**assumes**  $A\ B\ Par\ A\ C$   
**shows**  $Col\ B\ A\ C$   
 $\langle proof \rangle$

**lemma** *par-id-2*:  
**assumes**  $A\ B\ Par\ A\ C$   
**shows**  $Col\ B\ C\ A$   
 $\langle proof \rangle$

**lemma** *par-id-3*:  
**assumes**  $A\ B\ Par\ A\ C$   
**shows**  $Col\ A\ C\ B$   
 $\langle proof \rangle$

**lemma** *par-id-4*:  
**assumes**  $A\ B\ Par\ A\ C$   
**shows**  $Col\ C\ B\ A$   
 $\langle proof \rangle$

**lemma** *par-id-5*:  
**assumes**  $A\ B\ Par\ A\ C$   
**shows**  $Col\ C\ A\ B$   
 $\langle proof \rangle$

**lemma** *par-strict-symmetry*:  
**assumes**  $A\ B\ ParStrict\ C\ D$   
**shows**  $C\ D\ ParStrict\ A\ B$   
 $\langle proof \rangle$

**lemma** *par-symmetry*:  
**assumes**  $A\ B\ Par\ C\ D$   
**shows**  $C\ D\ Par\ A\ B$   
 $\langle proof \rangle$

**lemma** *par-left-comm*:  
**assumes**  $A\ B\ Par\ C\ D$   
**shows**  $B\ A\ Par\ C\ D$   
 $\langle proof \rangle$

**lemma** *par-right-comm*:  
**assumes**  $A\ B\ Par\ C\ D$   
**shows**  $A\ B\ Par\ D\ C$

*<proof>*

**lemma** *par-comm:*

**assumes**  $A B Par C D$

**shows**  $B A Par D C$

*<proof>*

**lemma** *par-strict-left-comm:*

**assumes**  $A B ParStrict C D$

**shows**  $B A ParStrict C D$

*<proof>*

**lemma** *par-strict-right-comm:*

**assumes**  $A B ParStrict C D$

**shows**  $A B ParStrict D C$

*<proof>*

**lemma** *par-strict-comm:*

**assumes**  $A B ParStrict C D$

**shows**  $B A ParStrict D C$

*<proof>*

**lemma** *par-strict-neq1:*

**assumes**  $A B ParStrict C D$

**shows**  $A \neq B$

*<proof>*

**lemma** *par-strict-neq2:*

**assumes**  $A B ParStrict C D$

**shows**  $C \neq D$

*<proof>*

**lemma** *par-neq1:*

**assumes**  $A B Par C D$

**shows**  $A \neq B$

*<proof>*

**lemma** *par-neq2:*

**assumes**  $A B Par C D$

**shows**  $C \neq D$

*<proof>*

**lemma** *Par-cases:*

**assumes**  $A B Par C D \vee B A Par C D \vee A B Par D C \vee B A Par D C \vee C D Par A B \vee C D Par B A \vee D C Par A B \vee D C Par B A$

**shows**  $A B Par C D$

*<proof>*

**lemma** *Par-perm:*

**assumes**  $A B Par C D$

**shows**  $A B Par C D \wedge B A Par C D \wedge A B Par D C \wedge B A Par D C \wedge C D Par A B \wedge C D Par B A \wedge D C Par A B \wedge D C Par B A$

*<proof>*

**lemma** *Par-strict-cases:*

**assumes**  $A B ParStrict C D \vee B A ParStrict C D \vee A B ParStrict D C \vee B A ParStrict D C \vee C D ParStrict A B \vee C D ParStrict B A \vee D C ParStrict A B \vee D C ParStrict B A$

**shows**  $A B ParStrict C D$

*<proof>*

**lemma** *Par-strict-perm:*

**assumes**  $A B ParStrict C D$

**shows**  $A B ParStrict C D \wedge B A ParStrict C D \wedge A B ParStrict D C \wedge B A ParStrict D C \wedge C D ParStrict A B \wedge C D ParStrict B A \wedge D C ParStrict A B \wedge D C ParStrict B A$

*<proof>*

**lemma l12-6:**

**assumes**  $A B \text{ ParStrict } C D$   
**shows**  $A B \text{ OS } C D$   
(proof)

**lemma pars--os3412:**

**assumes**  $A B \text{ ParStrict } C D$   
**shows**  $C D \text{ OS } A B$   
(proof)

**lemma perp-dec:**

$A B \text{ Perp } C D \vee \neg A B \text{ Perp } C D$   
(proof)

**lemma col-cop2-perp2--col:**

**assumes**  $X1 X2 \text{ Perp } A B$  **and**  
 $Y1 Y2 \text{ Perp } A B$  **and**  
 $\text{Col } X1 Y1 Y2$  **and**  
 $\text{Coplanar } A B X2 Y1$  **and**  
 $\text{Coplanar } A B X2 Y2$   
**shows**  $\text{Col } X2 Y1 Y2$   
(proof)

**lemma col-perp2-ncol-col:**

**assumes**  $X1 X2 \text{ Perp } A B$  **and**  
 $Y1 Y2 \text{ Perp } A B$  **and**  
 $\text{Col } X1 Y1 Y2$  **and**  
 $\neg \text{Col } X1 A B$   
**shows**  $\text{Col } X2 Y1 Y2$   
(proof)

**lemma l12-9:**

**assumes**  
 $\text{Coplanar } C1 C2 A1 B1$  **and**  
 $\text{Coplanar } C1 C2 A1 B2$  **and**  
 $\text{Coplanar } C1 C2 A2 B1$  **and**  
 $\text{Coplanar } C1 C2 A2 B2$  **and**  
 $A1 A2 \text{ Perp } C1 C2$  **and**  
 $B1 B2 \text{ Perp } C1 C2$   
**shows**  $A1 A2 \text{ Par } B1 B2$   
(proof)

**lemma parallel-existence:**

**assumes**  $A \neq B$   
**shows**  $\exists C D. C \neq D \wedge A B \text{ Par } C D \wedge \text{Col } P C D$   
(proof)

**lemma par-col-par:**

**assumes**  $C \neq D'$  **and**  
 $A B \text{ Par } C D$  **and**  
 $\text{Col } C D D'$   
**shows**  $A B \text{ Par } C D'$   
(proof)

**lemma parallel-existence1:**

**assumes**  $A \neq B$   
**shows**  $\exists Q. A B \text{ Par } P Q$   
(proof)

**lemma par-not-col:**

**assumes**  $A B \text{ ParStrict } C D$  **and**  
 $\text{Col } X A B$   
**shows**  $\neg \text{Col } X C D$   
(proof)

**lemma not-strict-par1:**

**assumes**  $A B \text{ Par } C D$  **and**  
 $\text{Col } A B X$  **and**  
 $\text{Col } C D X$   
**shows**  $\text{Col } A B C$   
 $\langle \text{proof} \rangle$

**lemma** *not-strict-par2*:  
**assumes**  $A B \text{ Par } C D$  **and**  
 $\text{Col } A B X$  **and**  
 $\text{Col } C D X$   
**shows**  $\text{Col } A B D$   
 $\langle \text{proof} \rangle$

**lemma** *not-strict-par*:  
**assumes**  $A B \text{ Par } C D$  **and**  
 $\text{Col } A B X$  **and**  
 $\text{Col } C D X$   
**shows**  $\text{Col } A B C \wedge \text{Col } A B D$   
 $\langle \text{proof} \rangle$

**lemma** *not-par-not-col*:  
**assumes**  $A \neq B$  **and**  
 $A \neq C$  **and**  
 $\neg A B \text{ Par } A C$   
**shows**  $\neg \text{Col } A B C$   
 $\langle \text{proof} \rangle$

**lemma** *not-par-inter-uniqueness*:  
**assumes**  $A \neq B$  **and**  
 $C \neq D$  **and**  
 $\neg A B \text{ Par } C D$  **and**  
 $\text{Col } A B X$  **and**  
 $\text{Col } C D X$  **and**  
 $\text{Col } A B Y$  **and**  
 $\text{Col } C D Y$   
**shows**  $X = Y$   
 $\langle \text{proof} \rangle$

**lemma** *inter-uniqueness-not-par*:  
**assumes**  $\neg \text{Col } A B C$  **and**  
 $\text{Col } A B P$  **and**  
 $\text{Col } C D P$   
**shows**  $\neg A B \text{ Par } C D$   
 $\langle \text{proof} \rangle$

**lemma** *col-not-col-not-par*:  
**assumes**  $\exists P. \text{Col } A B P \wedge \text{Col } C D P$  **and**  
 $\exists Q. \text{Col } C D Q \wedge \neg \text{Col } A B Q$   
**shows**  $\neg A B \text{ Par } C D$   
 $\langle \text{proof} \rangle$

**lemma** *par-distincts*:  
**assumes**  $A B \text{ Par } C D$   
**shows**  $A B \text{ Par } C D \wedge A \neq B \wedge C \neq D$   
 $\langle \text{proof} \rangle$

**lemma** *par-not-col-strict*:  
**assumes**  $A B \text{ Par } C D$  **and**  
 $\text{Col } C D P$  **and**  
 $\neg \text{Col } A B P$   
**shows**  $A B \text{ ParStrict } C D$   
 $\langle \text{proof} \rangle$

**lemma** *col-cop-perp2-pars*:  
**assumes**  $\neg \text{Col } A B P$  **and**  
 $\text{Col } C D P$  **and**

*Coplanar A B C D and*  
*A B Perp P Q and*  
*C D Perp P Q*  
**shows** *A B ParStrict C D*  
 ⟨proof⟩

**lemma** *all-one-side-par-strict:*  
**assumes** *C ≠ D and*  
*∀ P. Col C D P → A B OS C P*  
**shows** *A B ParStrict C D*  
 ⟨proof⟩

**lemma** *par-col-par-2:*  
**assumes** *A ≠ P and*  
*Col A B P and*  
*A B Par C D*  
**shows** *A P Par C D*  
 ⟨proof⟩

**lemma** *par-col2-par:*  
**assumes** *E ≠ F and*  
*A B Par C D and*  
*Col C D E and*  
*Col C D F*  
**shows** *A B Par E F*  
 ⟨proof⟩

**lemma** *par-col2-par-bis:*  
**assumes** *E ≠ F and*  
*A B Par C D and*  
*Col E F C and*  
*Col E F D*  
**shows** *A B Par E F*  
 ⟨proof⟩

**lemma** *par-strict-col-par-strict:*  
**assumes** *C ≠ E and*  
*A B ParStrict C D and*  
*Col C D E*  
**shows** *A B ParStrict C E*  
 ⟨proof⟩

**lemma** *par-strict-col2-par-strict:*  
**assumes** *E ≠ F and*  
*A B ParStrict C D and*  
*Col C D E and*  
*Col C D F*  
**shows** *A B ParStrict E F*  
 ⟨proof⟩

**lemma** *line-dec:*  
*(Col C1 B1 B2 ∧ Col C2 B1 B2) ∨ ¬ (Col C1 B1 B2 ∧ Col C2 B1 B2)*  
 ⟨proof⟩

**lemma** *par-distinct:*  
**assumes** *A B Par C D*  
**shows** *A ≠ B ∧ C ≠ D*  
 ⟨proof⟩

**lemma** *par-col4--par:*  
**assumes** *E ≠ F and*  
*G ≠ H and*  
*A B Par C D and*  
*Col A B E and*  
*Col A B F and*  
*Col C D G and*

*Col C D H*  
**shows** *E F Par G H*  
{proof}

**lemma** *par-strict-col4--par-strict:*  
**assumes** *E ≠ F and*  
*G ≠ H and*  
*A B ParStrict C D and*  
*Col A B E and*  
*Col A B F and*  
*Col C D G and*  
*Col C D H*  
**shows** *E F ParStrict G H*  
{proof}

**lemma** *par-strict-one-side:*  
**assumes** *A B ParStrict C D and*  
*Col C D P*  
**shows** *A B OS C P*  
{proof}

**lemma** *par-strict-all-one-side:*  
**assumes** *A B ParStrict C D*  
**shows**  $\forall P. \text{Col } C D P \longrightarrow A B OS C P$   
{proof}

**lemma** *inter-trivial:*  
**assumes**  $\neg \text{Col } A B X$   
**shows** *X Inter A X B X*  
{proof}

**lemma** *inter-sym:*  
**assumes** *X Inter A B C D*  
**shows** *X Inter C D A B*  
{proof}

**lemma** *inter-left-comm:*  
**assumes** *X Inter A B C D*  
**shows** *X Inter B A C D*  
{proof}

**lemma** *inter-right-comm:*  
**assumes** *X Inter A B C D*  
**shows** *X Inter A B D C*  
{proof}

**lemma** *inter-comm:*  
**assumes** *X Inter A B C D*  
**shows** *X Inter B A D C*  
{proof}

**lemma** *l12-17:*  
**assumes** *A ≠ B and*  
*P Midpoint A C and*  
*P Midpoint B D*  
**shows** *A B Par C D*  
{proof}

**lemma** *l12-18-a:*  
**assumes** *Cong A B C D and*  
*Cong B C D A and*  
 $\neg \text{Col } A B C$  **and**  
*B ≠ D and*  
*Col A P C and*  
*Col B P D*  
**shows** *A B Par C D*

$\langle \text{proof} \rangle$

**lemma l12-18-b:**

**assumes**  $\text{Cong } A B C D$  **and**

$\text{Cong } B C D A$  **and**

$\neg \text{Col } A B C$  **and**

$B \neq D$  **and**

$\text{Col } A P C$  **and**

$\text{Col } B P D$

**shows**  $B C \text{ Par } D A$

$\langle \text{proof} \rangle$

**lemma l12-18-c:**

**assumes**  $\text{Cong } A B C D$  **and**

$\text{Cong } B C D A$  **and**

$\neg \text{Col } A B C$  **and**

$B \neq D$  **and**

$\text{Col } A P C$  **and**

$\text{Col } B P D$

**shows**  $B D \text{ TS } A C$

$\langle \text{proof} \rangle$

**lemma l12-18-d:**

**assumes**  $\text{Cong } A B C D$  **and**

$\text{Cong } B C D A$  **and**

$\neg \text{Col } A B C$  **and**

$B \neq D$  **and**

$\text{Col } A P C$  **and**

$\text{Col } B P D$

**shows**  $A C \text{ TS } B D$

$\langle \text{proof} \rangle$

**lemma l12-18:**

**assumes**  $\text{Cong } A B C D$  **and**

$\text{Cong } B C D A$  **and**

$\neg \text{Col } A B C$  **and**

$B \neq D$  **and**

$\text{Col } A P C$  **and**

$\text{Col } B P D$

**shows**  $A B \text{ Par } C D \wedge B C \text{ Par } D A \wedge B D \text{ TS } A C \wedge A C \text{ TS } B D$

$\langle \text{proof} \rangle$

**lemma par-two-sides-two-sides:**

**assumes**  $A B \text{ Par } C D$  **and**

$B D \text{ TS } A C$

**shows**  $A C \text{ TS } B D$

$\langle \text{proof} \rangle$

**lemma par-one-or-two-sides:**

**assumes**  $A B \text{ ParStrict } C D$

**shows**  $(A C \text{ TS } B D \wedge B D \text{ TS } A C) \vee (A C \text{ OS } B D \wedge B D \text{ OS } A C)$

$\langle \text{proof} \rangle$

**lemma l12-21-b:**

**assumes**  $A C \text{ TS } B D$  **and**

$B A C \text{ Cong } A D C A$

**shows**  $A B \text{ Par } C D$

$\langle \text{proof} \rangle$

**lemma l12-22-aux:**

**assumes**  $P \neq A$  **and**

$A \neq C$  **and**

$\text{Bet } P A C$  **and**

$P A \text{ OS } B D$  **and**

$B A P \text{ Cong } A D C P$

**shows**  $A B \text{ Par } C D$

$\langle \text{proof} \rangle$

**lemma** *l12-22-b*:

**assumes**  $P \text{ Out } A \ C$  **and**

$P \ A \ OS \ B \ D$  **and**

$B \ A \ P \ CongA \ D \ C \ P$

**shows**  $A \ B \ Par \ C \ D$

$\langle \text{proof} \rangle$

**lemma** *par-strict-par*:

**assumes**  $A \ B \ ParStrict \ C \ D$

**shows**  $A \ B \ Par \ C \ D$

$\langle \text{proof} \rangle$

**lemma** *par-strict-distinct*:

**assumes**  $A \ B \ ParStrict \ C \ D$

**shows**  $A \neq B \wedge C \neq D$

$\langle \text{proof} \rangle$

**lemma** *col-par*:

**assumes**  $A \neq B$  **and**

$B \neq C$  **and**

$Col \ A \ B \ C$

**shows**  $A \ B \ Par \ B \ C$

$\langle \text{proof} \rangle$

**lemma** *acute-col-perp--out*:

**assumes**  $Acute \ A \ B \ C$  **and**

$Col \ B \ C \ A'$  **and**

$B \ C \ Perp \ A \ A'$

**shows**  $B \ Out \ A' \ C$

$\langle \text{proof} \rangle$

**lemma** *acute-col-perp--out-1*:

**assumes**  $Acute \ A \ B \ C$  **and**

$Col \ B \ C \ A'$  **and**

$B \ A \ Perp \ A \ A'$

**shows**  $B \ Out \ A' \ C$

$\langle \text{proof} \rangle$

**lemma** *conga-inangle-per2--inangle*:

**assumes**  $Per \ A \ B \ C$  **and**

$T \ InAngle \ A \ B \ C$  **and**

$P \ B \ A \ CongA \ P \ B \ C$  **and**

$Per \ B \ P \ T$  **and**

$Coplanar \ A \ B \ C \ P$

**shows**  $P \ InAngle \ A \ B \ C$

$\langle \text{proof} \rangle$

**lemma** *perp-not-par*:

**assumes**  $A \ B \ Perp \ X \ Y$

**shows**  $\neg \ A \ B \ Par \ X \ Y$

$\langle \text{proof} \rangle$

**lemma** *cong-conga-perp*:

**assumes**  $B \ P \ TS \ A \ C$  **and**

$Cong \ A \ B \ C \ B$  **and**

$A \ B \ P \ CongA \ C \ B \ P$

**shows**  $A \ C \ Perp \ B \ P$

$\langle \text{proof} \rangle$

**lemma** *perp-inter-exists*:

**assumes**  $A \ B \ Perp \ C \ D$

**shows**  $\exists \ P. \ Col \ A \ B \ P \wedge \ Col \ C \ D \ P$

$\langle \text{proof} \rangle$

**lemma** *perp-inter-perp-in*:  
**assumes**  $A B \text{ Perp } C D$   
**shows**  $\exists P. \text{Col } A B P \wedge \text{Col } C D P \wedge P \text{ PerpAt } A B C D$   
 $\langle \text{proof} \rangle$

**end**

**context** *Tarski-2D*

**begin**

**lemma** *l12-9-2D*:  
**assumes**  $A1 A2 \text{ Perp } C1 C2$  **and**  
 $B1 B2 \text{ Perp } C1 C2$   
**shows**  $A1 A2 \text{ Par } B1 B2$   
 $\langle \text{proof} \rangle$

**end**

**context** *Tarski-neutral-dimensionless*

**begin**

## 3.12 Tarski: Chapter 13

### 3.12.1 Introduction

**lemma** *per2-col-eq*:  
**assumes**  $A \neq P$  **and**  
 $A \neq P'$  **and**  
 $\text{Per } A P B$  **and**  
 $\text{Per } A P' B$  **and**  
 $\text{Col } P A P'$   
**shows**  $P = P'$   
 $\langle \text{proof} \rangle$

**lemma** *per2-preserves-diff*:  
**assumes**  $PO \neq A'$  **and**  
 $PO \neq B'$  **and**  
 $\text{Col } PO A' B'$  **and**  
 $\text{Per } PO A' A$  **and**  
 $\text{Per } PO B' B$  **and**  
 $A' \neq B'$   
**shows**  $A \neq B$   
 $\langle \text{proof} \rangle$

**lemma** *per23-preserves-bet*:  
**assumes**  $\text{Bet } A B C$  **and**  
 $A \neq B'$  **and**  $A \neq C'$  **and**  
 $\text{Col } A B' C'$  **and**  
 $\text{Per } A B' B$  **and**  
 $\text{Per } A C' C$   
**shows**  $\text{Bet } A B' C'$   
 $\langle \text{proof} \rangle$

**lemma** *per23-preserves-bet-inv*:  
**assumes**  $\text{Bet } A B' C'$  **and**  
 $A \neq B'$  **and**  
 $\text{Col } A B C$  **and**  
 $\text{Per } A B' B$  **and**  
 $\text{Per } A C' C$   
**shows**  $\text{Bet } A B C$   
 $\langle \text{proof} \rangle$

**lemma** *per13-preserves-bet*:  
**assumes**  $\text{Bet } A B C$  **and**  
 $B \neq A'$  **and**

$B \neq C'$  and  
 $Col A' B C'$  and  
 $Per B A' A$  and  
 $Per B C' C$   
**shows**  $Bet A' B C'$   
 ⟨proof⟩

**lemma** *per13-preserves-bet-inv*:  
**assumes**  $Bet A' B C'$  and  
 $B \neq A'$  and  
 $B \neq C'$  and  
 $Col A B C$  and  
 $Per B A' A$  and  
 $Per B C' C$   
**shows**  $Bet A B C$   
 ⟨proof⟩

**lemma** *per3-preserves-bet1*:  
**assumes**  $Col PO A B$  and  
 $Bet A B C$  and  
 $PO \neq A'$  and  
 $PO \neq B'$  and  
 $PO \neq C'$  and  
 $Per PO A' A$  and  
 $Per PO B' B$  and  
 $Per PO C' C$  and  
 $Col A' B' C'$  and  
 $Col PO A' B'$   
**shows**  $Bet A' B' C'$   
 ⟨proof⟩

**lemma** *per3-preserves-bet2-aux*:  
**assumes**  $Col PO A C$  and  
 $A \neq C'$  and  
 $Bet A B' C'$  and  
 $PO \neq A$  and  
 $PO \neq B'$  and  
 $PO \neq C'$  and  
 $Per PO B' B$  and  
 $Per PO C' C$  and  
 $Col A B C$  and  
 $Col PO A C'$   
**shows**  $Bet A B C$   
 ⟨proof⟩

**lemma** *per3-preserves-bet2*:  
**assumes**  $Col PO A C$  and  
 $A' \neq C'$  and  
 $Bet A' B' C'$  and  
 $PO \neq A'$  and  
 $PO \neq B'$  and  
 $PO \neq C'$  and  
 $Per PO A' A$  and  
 $Per PO B' B$  and  
 $Per PO C' C$  and  
 $Col A B C$  and  
 $Col PO A' C'$   
**shows**  $Bet A B C$   
 ⟨proof⟩

**lemma** *symmetry-preserves-per*:  
**assumes**  $Per B P A$  and  
 $B$  Midpoint  $A A'$  and  
 $B$  Midpoint  $P P'$   
**shows**  $Per B P' A'$   
 ⟨proof⟩

**lemma** *l13-1-aux*:

**assumes**  $\neg \text{Col } A B C$  **and**

*P* *Midpoint* *B C* **and**

*Q* *Midpoint* *A C* **and**

*R* *Midpoint* *A B*

**shows**

$\exists X Y. (R \text{ PerpAt } X Y A B \wedge X Y \text{ Perp } P Q \wedge \text{Coplanar } A B C X \wedge \text{Coplanar } A B C Y)$

*<proof>*

**lemma** *l13-1*:

**assumes**  $\neg \text{Col } A B C$  **and**

*P* *Midpoint* *B C* **and**

*Q* *Midpoint* *A C* **and**

*R* *Midpoint* *A B*

**shows**

$\exists X Y. (R \text{ PerpAt } X Y A B \wedge X Y \text{ Perp } P Q)$

*<proof>*

**lemma** *per-lt*:

**assumes**  $A \neq B$  **and**

$C \neq B$  **and**

*Per* *A B C*

**shows**  $A B \text{ Lt } A C \wedge C B \text{ Lt } A C$

*<proof>*

**lemma** *cong-perp-conga*:

**assumes**  $\text{Cong } A B C B$  **and**

*A C* *Perp* *B P*

**shows**  $A B P \text{ Cong } A C B P \wedge B P \text{ TS } A C$

*<proof>*

**lemma** *perp-per-bet*:

**assumes**  $\neg \text{Col } A B C$  **and**

*Per* *A B C* **and**

*P* *PerpAt* *P B A C*

**shows** *Bet* *A P C*

*<proof>*

**lemma** *ts-per-per-ts*:

**assumes**  $A B \text{ TS } C D$  **and**

*Per* *B C A* **and**

*Per* *B D A*

**shows**  $C D \text{ TS } A B$

*<proof>*

**lemma** *l13-2-1*:

**assumes**  $A B \text{ TS } C D$  **and**

*Per* *B C A* **and**

*Per* *B D A* **and**

*Col* *C D E* **and**

*A E* *Perp* *C D* **and**

*C A B* *Cong* *A D A B*

**shows**  $B A C \text{ Cong } A D A E \wedge B A D \text{ Cong } A C A E \wedge \text{Bet } C E D$

*<proof>*

**lemma** *triangle-mid-par*:

**assumes**  $\neg \text{Col } A B C$  **and**

*P* *Midpoint* *B C* **and**

*Q* *Midpoint* *A C*

**shows**  $A B \text{ ParStrict } Q P$

*<proof>*

**lemma** *cop4-perp-in2--col*:

**assumes**  $\text{Coplanar } X Y A A'$  **and**

*Coplanar X Y A B' and*  
*Coplanar X Y B A' and*  
*Coplanar X Y B B' and*  
*P PerpAt A B X Y and*  
*P PerpAt A' B' X Y*  
**shows Col A B A'**  
 ⟨proof⟩

**lemma l13-2:**  
**assumes A B TS C D and**  
*Per B C A and*  
*Per B D A and*  
*Col C D E and*  
*A E Perp C D*  
**shows B A C CongA D A E  $\wedge$  B A D CongA C A E  $\wedge$  Bet C E D**  
 ⟨proof⟩

**lemma perp2-refl:**  
**assumes A  $\neq$  B**  
**shows P Perp2 A B A B**  
 ⟨proof⟩

**lemma perp2-sym:**  
**assumes P Perp2 A B C D**  
**shows P Perp2 C D A B**  
 ⟨proof⟩

**lemma perp2-left-comm:**  
**assumes P Perp2 A B C D**  
**shows P Perp2 B A C D**  
 ⟨proof⟩

**lemma perp2-right-comm:**  
**assumes P Perp2 A B C D**  
**shows P Perp2 A B D C**  
 ⟨proof⟩

**lemma perp2-comm:**  
**assumes P Perp2 A B C D**  
**shows P Perp2 B A D C**  
 ⟨proof⟩

**lemma perp2-pseudo-trans:**  
**assumes P Perp2 A B C D and**  
*P Perp2 C D E F and*  
 *$\neg$  Col C D P*  
**shows P Perp2 A B E F**  
 ⟨proof⟩

**lemma col-cop-perp2--pars-bis:**  
**assumes  $\neg$  Col A B P and**  
*Col C D P and*  
*Coplanar A B C D and*  
*P Perp2 A B C D*  
**shows A B ParStrict C D**  
 ⟨proof⟩

**lemma perp2-preserves-bet23:**  
**assumes Bet PO A B and**  
*Col PO A' B' and*  
 *$\neg$  Col PO A A' and*  
*PO Perp2 A A' B B'*  
**shows Bet PO A' B'**  
 ⟨proof⟩

**lemma perp2-preserves-bet13:**

**assumes**  $Bet\ B\ PO\ C$  **and**  
 $Col\ PO\ B'\ C'$  **and**  
 $\neg\ Col\ PO\ B\ B'$  **and**  
 $PO\ Perp2\ B\ C'\ C\ B'$   
**shows**  $Bet\ B'\ PO\ C'$   
 ⟨proof⟩

**lemma** *is-image-perp-in*:  
**assumes**  $A \neq A'$  **and**  
 $X \neq Y$  **and**  
 $A\ A'\ Reflect\ X\ Y$   
**shows**  $\exists\ P.\ P\ PerpAt\ A\ A'\ X\ Y$   
 ⟨proof⟩

**lemma** *perp-inter-perp-in-n*:  
**assumes**  $A\ B\ Perp\ C\ D$   
**shows**  $\exists\ P.\ Col\ A\ B\ P \wedge Col\ C\ D\ P \wedge P\ PerpAt\ A\ B\ C\ D$   
 ⟨proof⟩

**lemma** *perp2-perp-in*:  
**assumes**  $PO\ Perp2\ A\ B\ C\ D$  **and**  
 $\neg\ Col\ PO\ A\ B$  **and**  
 $\neg\ Col\ PO\ C\ D$   
**shows**  $\exists\ P\ Q.\ Col\ A\ B\ P \wedge Col\ C\ D\ Q \wedge Col\ PO\ P\ Q \wedge P\ PerpAt\ PO\ P\ A\ B \wedge Q\ PerpAt\ PO\ Q\ C\ D$   
 ⟨proof⟩

**lemma** *l13-8*:  
**assumes**  $U \neq PO$  **and**  
 $V \neq PO$  **and**  
 $Col\ PO\ P\ Q$  **and**  
 $Col\ PO\ U\ V$  **and**  
 $Per\ P\ U\ PO$  **and**  
 $Per\ Q\ V\ PO$   
**shows**  $PO\ Out\ P\ Q \longleftrightarrow PO\ Out\ U\ V$   
 ⟨proof⟩

**lemma** *perp-in-rewrite*:  
**assumes**  $P\ PerpAt\ A\ B\ C\ D$   
**shows**  $P\ PerpAt\ A\ P\ P\ C \vee P\ PerpAt\ A\ P\ P\ D \vee P\ PerpAt\ B\ P\ P\ C \vee P\ PerpAt\ B\ P\ P\ D$   
 ⟨proof⟩

**lemma** *perp-out-acute*:  
**assumes**  $B\ Out\ A\ C'$  **and**  
 $A\ B\ Perp\ C\ C'$   
**shows**  $Acute\ A\ B\ C$   
 ⟨proof⟩

**lemma** *perp-bet-obtuse*:  
**assumes**  $B \neq C'$  **and**  
 $A\ B\ Perp\ C\ C'$  **and**  
 $Bet\ A\ B\ C'$   
**shows**  $Obtuse\ A\ B\ C$   
 ⟨proof⟩

end

### 3.12.2 Part 1: 2D

**context** *Tarski-2D*  
**begin**

**lemma** *perp-in2--col*:  
**assumes**  $P\ PerpAt\ A\ B\ X\ Y$  **and**  
 $P\ PerpAt\ A'\ B'\ X\ Y$   
**shows**  $Col\ A\ B\ A'$   
 ⟨proof⟩

**lemma** *perp2-trans*:  
**assumes**  $P$  *Perp2*  $A B C D$  **and**  
 $P$  *Perp2*  $C D E F$   
**shows**  $P$  *Perp2*  $A B E F$   
 $\langle$ *proof* $\rangle$

**lemma** *perp2-par*:  
**assumes**  $PO$  *Perp2*  $A B C D$   
**shows**  $A B$  *Par*  $C D$   
 $\langle$ *proof* $\rangle$

**end**

### 3.12.3 Part 2: length

**context** *Tarski-neutral-dimensionless*

**begin**

**lemma** *lg-exists*:  
 $\exists l. (QCong\ l \wedge l\ A\ B)$   
 $\langle$ *proof* $\rangle$

**lemma** *lg-cong*:  
**assumes** *QCong*  $l$  **and**  
 $l\ A\ B$  **and**  
 $l\ C\ D$   
**shows** *Cong*  $A\ B\ C\ D$   
 $\langle$ *proof* $\rangle$

**lemma** *lg-cong-lg*:  
**assumes** *QCong*  $l$  **and**  
 $l\ A\ B$  **and**  
*Cong*  $A\ B\ C\ D$   
**shows**  $l\ C\ D$   
 $\langle$ *proof* $\rangle$

**lemma** *lg-sym*:  
**assumes** *QCong*  $l$   
**and**  $l\ A\ B$   
**shows**  $l\ B\ A$   
 $\langle$ *proof* $\rangle$

**lemma** *ex-points-lg*:  
**assumes** *QCong*  $l$   
**shows**  $\exists A\ B. l\ A\ B$   
 $\langle$ *proof* $\rangle$

**lemma** *is-len-cong*:  
**assumes** *TarskiLen*  $A\ B\ l$  **and**  
*TarskiLen*  $C\ D\ l$   
**shows** *Cong*  $A\ B\ C\ D$   
 $\langle$ *proof* $\rangle$

**lemma** *is-len-cong-is-len*:  
**assumes** *TarskiLen*  $A\ B\ l$  **and**  
*Cong*  $A\ B\ C\ D$   
**shows** *TarskiLen*  $C\ D\ l$   
 $\langle$ *proof* $\rangle$

**lemma** *not-cong-is-len*:  
**assumes**  $\neg$  *Cong*  $A\ B\ C\ D$  **and**  
*TarskiLen*  $A\ B\ l$   
**shows**  $\neg$   $l\ C\ D$   
 $\langle$ *proof* $\rangle$

**lemma** *not-cong-is-len1*:  
**assumes**  $\neg \text{Cong } A B C D$   
**and**  $\text{TarskiLen } A B l$   
**shows**  $\neg \text{TarskiLen } C D l$   
 $\langle \text{proof} \rangle$

**lemma** *lg-null-instance*:  
**assumes**  $\text{QCongNull } l$   
**shows**  $l A A$   
 $\langle \text{proof} \rangle$

**lemma** *lg-null-trivial*:  
**assumes**  $\text{QCong } l$   
**and**  $l A A$   
**shows**  $\text{QCongNull } l$   
 $\langle \text{proof} \rangle$

**lemma** *lg-null-dec*:  
  
**shows**  $\text{QCongNull } l \vee \neg \text{QCongNull } l$   
 $\langle \text{proof} \rangle$

**lemma** *ex-point-lg*:  
**assumes**  $\text{QCong } l$   
**shows**  $\exists B. l A B$   
 $\langle \text{proof} \rangle$

**lemma** *ex-point-lg-out*:  
**assumes**  $A \neq P$  **and**  
 $\text{QCong } l$  **and**  
 $\neg \text{QCongNull } l$   
**shows**  $\exists B. (l A B \wedge A \text{ Out } B P)$   
 $\langle \text{proof} \rangle$

**lemma** *ex-point-lg-bet*:  
**assumes**  $\text{QCong } l$   
**shows**  $\exists B. (l M B \wedge \text{Bet } A M B)$   
 $\langle \text{proof} \rangle$

**lemma** *ex-points-lg-not-col*:  
**assumes**  $\text{QCong } l$   
**and**  $\neg \text{QCongNull } l$   
**shows**  $\exists A B. (l A B \wedge \neg \text{Col } A B P)$   
 $\langle \text{proof} \rangle$

**lemma** *ex-eql*:  
**assumes**  $\exists A B. (\text{TarskiLen } A B l1 \wedge \text{TarskiLen } A B l2)$   
**shows**  $l1 = l2$   
 $\langle \text{proof} \rangle$

**lemma** *all-eql*:  
**assumes**  $\text{TarskiLen } A B l1$  **and**  
 $\text{TarskiLen } A B l2$   
**shows**  $l1 = l2$   
 $\langle \text{proof} \rangle$

**lemma** *null-len*:  
**assumes**  $\text{TarskiLen } A A la$  **and**  
 $\text{TarskiLen } B B lb$   
**shows**  $la = lb$   
 $\langle \text{proof} \rangle$

**lemma** *eqL-equivalence*:  
**assumes**  $\text{QCong } la$  **and**  
 $\text{QCong } lb$  **and**

*QCong lc*  
**shows**  $la = la \wedge (la = lb \longrightarrow lb = la) \wedge (la = lb \wedge lb = lc \longrightarrow la = lc)$   
 ⟨proof⟩

**lemma** *ex-lg*:  
 $\exists l. (QCong l \wedge l A B)$   
 ⟨proof⟩

**lemma** *lg-eql-lg*:  
**assumes** *QCong l1* and  
 $l1 = l2$   
**shows** *QCong l2*  
 ⟨proof⟩

**lemma** *ex-eqL*:  
**assumes** *QCong l1* and  
*QCong l2* and  
 $\exists A B. (l1 A B \wedge l2 A B)$   
**shows**  $l1 = l2$   
 ⟨proof⟩

### 3.12.4 Part 3 : angles

**lemma** *ang-exists*:  
**assumes**  $A \neq B$  and  
 $C \neq B$   
**shows**  $\exists a. (QCongA a \wedge a A B C)$   
 ⟨proof⟩

**lemma** *ex-points-eng*:  
**assumes** *QCongA a*  
**shows**  $\exists A B C. (a A B C)$   
 ⟨proof⟩

**lemma** *ang-conga*:  
**assumes** *QCongA a* and  
 $a A B C$  and  
 $a A' B' C'$   
**shows**  $A B C CongA A' B' C'$   
 ⟨proof⟩

**lemma** *is-ang-conga*:  
**assumes**  $A B C Ang a$  and  
 $A' B' C' Ang a$   
**shows**  $A B C CongA A' B' C'$   
 ⟨proof⟩

**lemma** *is-ang-conga-is-ang*:  
**assumes**  $A B C Ang a$  and  
 $A B C CongA A' B' C'$   
**shows**  $A' B' C' Ang a$   
 ⟨proof⟩

**lemma** *not-conga-not-ang*:  
**assumes** *QCongA a* and  
 $\neg A B C CongA A' B' C'$  and  
 $a A B C$   
**shows**  $\neg a A' B' C'$   
 ⟨proof⟩

**lemma** *not-conga-is-ang*:  
**assumes**  $\neg A B C CongA A' B' C'$  and  
 $A B C Ang a$   
**shows**  $\neg a A' B' C'$   
 ⟨proof⟩

**lemma** *not-cong-is-ang1*:

**assumes**  $\neg A B C \text{ Cong} A A' B' C'$  **and**

$A B C \text{ Ang } a$

**shows**  $\neg A' B' C' \text{ Ang } a$

$\langle \text{proof} \rangle$

**lemma** *ex-ega*:

**assumes**  $\exists A B C. (A B C \text{ Ang } a1 \wedge A B C \text{ Ang } a2)$

**shows**  $a1 = a2$

$\langle \text{proof} \rangle$

**lemma** *all-ega*:

**assumes**  $A B C \text{ Ang } a1$  **and**

$A B C \text{ Ang } a2$

**shows**  $a1 = a2$

$\langle \text{proof} \rangle$

**lemma** *is-ang-distinct*:

**assumes**  $A B C \text{ Ang } a$

**shows**  $A \neq B \wedge C \neq B$

$\langle \text{proof} \rangle$

**lemma** *null-ang*:

**assumes**  $A B A \text{ Ang } a1$  **and**

$C D C \text{ Ang } a2$

**shows**  $a1 = a2$

$\langle \text{proof} \rangle$

**lemma** *flat-ang*:

**assumes**  $B \text{ et } A B C$  **and**

$B \text{ et } A' B' C'$  **and**

$A B C \text{ Ang } a1$  **and**

$A' B' C' \text{ Ang } a2$

**shows**  $a1 = a2$

$\langle \text{proof} \rangle$

**lemma** *ang-distinct*:

**assumes**  $Q \text{ Cong} A a$  **and**

$a A B C$

**shows**  $A \neq B \wedge C \neq B$

$\langle \text{proof} \rangle$

**lemma** *ex-ang*:

**assumes**  $B \neq A$  **and**

$B \neq C$

**shows**  $\exists a. (Q \text{ Cong} A a \wedge a A B C)$

$\langle \text{proof} \rangle$

**lemma** *anga-exists*:

**assumes**  $A \neq B$  **and**

$C \neq B$  **and**

$\text{Acute } A B C$

**shows**  $\exists a. (Q \text{ Cong} A \text{Acute } a \wedge a A B C)$

$\langle \text{proof} \rangle$

**lemma** *anga-is-ang*:

**assumes**  $Q \text{ Cong} A \text{Acute } a$

**shows**  $Q \text{ Cong} A a$

$\langle \text{proof} \rangle$

**lemma** *ex-points-anga*:

**assumes**  $Q \text{ Cong} A \text{Acute } a$

**shows**  $\exists A B C. a A B C$

$\langle \text{proof} \rangle$

**lemma** *anga-conga*:

**assumes**  $QCongA$   $a$  **and**  
 $a A B C$  **and**  
 $a A' B' C'$   
**shows**  $A B C CongA A' B' C'$   
 $\langle proof \rangle$

**lemma** *is-anga-to-is-ang*:  
**assumes**  $A B C AngAcute a$   
**shows**  $A B C Ang a$   
 $\langle proof \rangle$

**lemma** *is-anga-conga*:  
**assumes**  $A B C AngAcute a$  **and**  
 $A' B' C' AngAcute a$   
**shows**  $A B C CongA A' B' C'$   
 $\langle proof \rangle$

**lemma** *is-anga-conga-is-anga*:  
**assumes**  $A B C AngAcute a$  **and**  
 $A B C CongA A' B' C'$   
**shows**  $A' B' C' AngAcute a$   
 $\langle proof \rangle$

**lemma** *not-conga-is-anga*:  
**assumes**  $\neg A B C CongA A' B' C'$  **and**  
 $A B C AngAcute a$   
**shows**  $\neg a A' B' C'$   
 $\langle proof \rangle$

**lemma** *not-cong-is-anga1*:  
**assumes**  $\neg A B C CongA A' B' C'$  **and**  
 $A B C AngAcute a$   
**shows**  $\neg A' B' C' AngAcute a$   
 $\langle proof \rangle$

**lemma** *ex-eqaa*:  
**assumes**  $\exists A B C. (A B C AngAcute a1 \wedge A B C AngAcute a2)$   
**shows**  $a1 = a2$   
 $\langle proof \rangle$

**lemma** *all-eqaa*:  
**assumes**  $A B C AngAcute a1$  **and**  
 $A B C AngAcute a2$   
**shows**  $a1 = a2$   
 $\langle proof \rangle$

**lemma** *is-anga-distinct*:  
**assumes**  $A B C AngAcute a$   
**shows**  $A \neq B \wedge C \neq B$   
 $\langle proof \rangle$

**lemma** *null-anga*:  
**assumes**  $A B A AngAcute a1$  **and**  
 $C D C AngAcute a2$   
**shows**  $a1 = a2$   
 $\langle proof \rangle$

**lemma** *anga-distinct*:  
**assumes**  $QCongA$   $a$  **and**  
 $a A B C$   
**shows**  $A \neq B \wedge C \neq B$   
 $\langle proof \rangle$

**lemma** *out-is-len-eq*:  
**assumes**  $A Out B C$  **and**  
 $TarskiLen A B l$  **and**

*TarskiLen A C l*  
**shows**  $B = C$   
 ⟨proof⟩

**lemma** *out-len-eq*:  
**assumes**  $QCong\ l$  **and**  
 $A\ Out\ B\ C$  **and**  
 $l\ A\ B$  **and**  
 $l\ A\ C$   
**shows**  $B = C$  ⟨proof⟩

**lemma** *ex-anga*:  
**assumes**  $Acute\ A\ B\ C$   
**shows**  $\exists a. (QCong\ A\ Acute\ a \wedge a\ A\ B\ C)$   
 ⟨proof⟩

**lemma** *not-null-ang-ang*:  
**assumes**  $QCong\ An\ Null\ a$   
**shows**  $QCong\ A\ a$   
 ⟨proof⟩

**lemma** *not-null-ang-def-equiv*:  
 $QCong\ An\ Null\ a \longleftrightarrow (QCong\ A\ a \wedge (\exists A\ B\ C. (a\ A\ B\ C \wedge \neg B\ Out\ A\ C)))$   
 ⟨proof⟩

**lemma** *not-flat-ang-def-equiv*:  
 $QCong\ An\ Flat\ a \longleftrightarrow (QCong\ A\ a \wedge (\exists A\ B\ C. (a\ A\ B\ C \wedge \neg B\ et\ A\ B\ C)))$   
 ⟨proof⟩

**lemma** *ang-const*:  
**assumes**  $QCong\ A\ a$  **and**  
 $A \neq B$   
**shows**  $\exists C. a\ A\ B\ C$   
 ⟨proof⟩

**lemma** *ang-sym*:  
**assumes**  $QCong\ A\ a$  **and**  
 $a\ A\ B\ C$   
**shows**  $a\ C\ B\ A$   
 ⟨proof⟩

**lemma** *ang-not-null-lg*:  
**assumes**  $QCong\ A\ a$  **and**  
 $QCong\ l$  **and**  
 $a\ A\ B\ C$  **and**  
 $l\ A\ B$   
**shows**  $\neg QCong\ Null\ l$   
 ⟨proof⟩

**lemma** *ang-distincts*:  
**assumes**  $QCong\ A\ a$  **and**  
 $a\ A\ B\ C$   
**shows**  $A \neq B \wedge C \neq B$   
 ⟨proof⟩

**lemma** *anga-sym*:  
**assumes**  $QCong\ A\ Acute\ a$  **and**  
 $a\ A\ B\ C$   
**shows**  $a\ C\ B\ A$   
 ⟨proof⟩

**lemma** *anga-not-null-lg*:  
**assumes**  $QCong\ A\ Acute\ a$  **and**  
 $QCong\ l$  **and**  
 $a\ A\ B\ C$  **and**  
 $l\ A\ B$

**shows**  $\neg QCongNull\ l$   
*<proof>*

**lemma** *anga-distincts*:  
**assumes**  $QCongAAcute\ a$  **and**  
 $a\ A\ B\ C$   
**shows**  $A \neq B \wedge C \neq B$   
*<proof>*

**lemma** *ang-const-o*:  
**assumes**  $\neg Col\ A\ B\ P$  **and**  
 $QCongA\ a$  **and**  
 $QCongAnNull\ a$  **and**  
 $QCongAnFlat\ a$   
**shows**  $\exists\ C.\ a\ A\ B\ C \wedge A\ B\ OS\ C\ P$   
*<proof>*

**lemma** *anga-const*:  
**assumes**  $QCongAAcute\ a$  **and**  
 $A \neq B$   
**shows**  $\exists\ C.\ a\ A\ B\ C$   
*<proof>*

**lemma** *null-anga-null-angaP*:  
 $QCongANullAcute\ a \longleftrightarrow IsNullAngaP\ a$   
*<proof>*

**lemma** *is-null-anga-out*:  
**assumes**  
 $a\ A\ B\ C$  **and**  
 $QCongANullAcute\ a$   
**shows**  $B\ Out\ A\ C$   
*<proof>*

**lemma** *acute-not-bet*:  
**assumes**  $Acute\ A\ B\ C$   
**shows**  $\neg Bet\ A\ B\ C$   
*<proof>*

**lemma** *anga-acute*:  
**assumes**  $QCongAAcute\ a$  **and**  
 $a\ A\ B\ C$   
**shows**  $Acute\ A\ B\ C$   
*<proof>*

**lemma** *not-null-not-col*:  
**assumes**  $QCongAAcute\ a$  **and**  
 $\neg QCongANullAcute\ a$  **and**  
 $a\ A\ B\ C$   
**shows**  $\neg Col\ A\ B\ C$   
*<proof>*

**lemma** *ang-cong-ang*:  
**assumes**  $QCongA\ a$  **and**  
 $a\ A\ B\ C$  **and**  
 $A\ B\ C\ CongA\ A'\ B'\ C'$   
**shows**  $a\ A'\ B'\ C'$   
*<proof>*

**lemma** *is-null-ang-out*:  
**assumes**  
 $a\ A\ B\ C$  **and**  
 $QCongANull\ a$   
**shows**  $B\ Out\ A\ C$   
*<proof>*

**lemma** *out-null-ang*:  
**assumes**  $QCongA$   $a$  **and**  
 $a$   $A$   $B$   $C$  **and**  
 $B$  *Out*  $A$   $C$   
**shows**  $QCongANull$   $a$   
 $\langle$ *proof* $\rangle$

**lemma** *bet-flat-ang*:  
**assumes**  $QCongA$   $a$  **and**  
 $a$   $A$   $B$   $C$  **and**  
 $Bet$   $A$   $B$   $C$   
**shows**  $AngFlat$   $a$   
 $\langle$ *proof* $\rangle$

**lemma** *out-null-anga*:  
**assumes**  $QCongAAcute$   $a$  **and**  
 $a$   $A$   $B$   $C$  **and**  
 $B$  *Out*  $A$   $C$   
**shows**  $QCongANullAcute$   $a$   
 $\langle$ *proof* $\rangle$

**lemma** *anga-not-flat*:  
**assumes**  $QCongAAcute$   $a$   
**shows**  $QCongAnFlat$   $a$   
 $\langle$ *proof* $\rangle$

**lemma** *anga-const-o*:  
**assumes**  $\neg$   $Col$   $A$   $B$   $P$  **and**  
 $\neg$   $QCongANullAcute$   $a$  **and**  
 $QCongAAcute$   $a$   
**shows**  $\exists$   $C$ . ( $a$   $A$   $B$   $C$   $\wedge$   $A$   $B$  *OS*  $C$   $P$ )  
 $\langle$ *proof* $\rangle$

**lemma** *anga-conga-anga*:  
**assumes**  $QCongAAcute$   $a$  **and**  
 $a$   $A$   $B$   $C$  **and**  
 $A$   $B$   $C$  *CongA*  $A'$   $B'$   $C'$   
**shows**  $a$   $A'$   $B'$   $C'$   
 $\langle$ *proof* $\rangle$

**lemma** *anga-out-anga*:  
**assumes**  $QCongAAcute$   $a$  **and**  
 $a$   $A$   $B$   $C$  **and**  
 $B$  *Out*  $A$   $A'$  **and**  
 $B$  *Out*  $C$   $C'$   
**shows**  $a$   $A'$   $B$   $C'$   
 $\langle$ *proof* $\rangle$

**lemma** *out-out-anga*:  
**assumes**  $QCongAAcute$   $a$  **and**  
 $B$  *Out*  $A$   $C$  **and**  
 $B'$  *Out*  $A'$   $C'$  **and**  
 $a$   $A$   $B$   $C$   
**shows**  $a$   $A'$   $B'$   $C'$   
 $\langle$ *proof* $\rangle$

**lemma** *is-null-all*:  
**assumes**  $A \neq B$  **and**  
 $QCongANullAcute$   $a$   
**shows**  $a$   $A$   $B$   $A$   
 $\langle$ *proof* $\rangle$

**lemma** *anga-col-out*:  
**assumes**  $QCongAAcute$   $a$  **and**  
 $a$   $A$   $B$   $C$  **and**  
 $Col$   $A$   $B$   $C$

**shows**  $B \text{ Out } A \ C$   
*<proof>*

**lemma** *ang-not-lg-null:*

**assumes**  $QCong \ la$  **and**

$QCong \ lc$  **and**

$QCongA \ a$  **and**

$la \ A \ B$  **and**

$lc \ C \ B$  **and**

$a \ A \ B \ C$

**shows**  $\neg \ QCongNull \ la \ \wedge \ \neg \ QCongNull \ lc$

*<proof>*

**lemma** *anga-not-lg-null:*

**assumes**

$QCongAAcute \ a$  **and**

$la \ A \ B$  **and**

$lc \ C \ B$  **and**

$a \ A \ B \ C$

**shows**  $\neg \ QCongNull \ la \ \wedge \ \neg \ QCongNull \ lc$

*<proof>*

**lemma** *anga-col-null:*

**assumes**  $QCongAAcute \ a$  **and**

$a \ A \ B \ C$  **and**

$Col \ A \ B \ C$

**shows**  $B \text{ Out } A \ C \ \wedge \ QCongANullAcute \ a$

*<proof>*

**lemma** *eqA-preserves-ang:*

**assumes**  $QCongA \ a$  **and**

$a = b$

**shows**  $QCongA \ b$

*<proof>*

**lemma** *eqA-preserves-anga:*

**assumes**  $QCongAAcute \ a$  **and**

$a = b$

**shows**  $QCongAAcute \ b$

*<proof>*

## 4 Some postulates of the parallels

**lemma** *euclid-5--original-euclid:*

**assumes**  $Euclid5$

**shows**  $EuclidSParallelPostulate$

*<proof>*

**lemma** *tarski-s-euclid-implies-euclid-5:*

**assumes**  $TarskiSParallelPostulate$

**shows**  $Euclid5$

*<proof>*

**lemma** *tarski-s-implies-euclid-s-parallel-postulate:*

**assumes**  $TarskiSParallelPostulate$

**shows**  $EuclidSParallelPostulate$

*<proof>*

**theorem** *tarski-s-euclid-implies-playfair-s-postulate:*

**assumes**  $TarskiSParallelPostulate$

**shows**  $PlayfairSPostulate$

*<proof>*

**end**

end

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