

Irrational numbers from THE BOOK

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September 13, 2023

Abstract

An elementary proof is formalised: that $\exp r$ is irrational for every nonzero rational number r . The mathematical development comes from the well-known volume *Proofs from THE BOOK* [1, pp. 51–2], by Aigner and Ziegler, who credit the idea to Hermite. The development illustrates a number of basic Isabelle techniques: the manipulation of summations, the calculation of quite complicated derivatives and the estimation of integrals. We also see how to import another AFP entry (Stirling’s formula) [2].

As for the theorem itself, note that a much stronger and more general result (the Hermite–Lindemann–Weierstraß transcendence theorem) is already available in the AFP [3].

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Acknowledgements The author was supported by the ERC Advanced Grant ALEXANDRIA (Project 742178) funded by the European Research Council.

1 Some irrational numbers

From Aigner and Ziegler, *Proofs from THE BOOK* (Springer, 2018), Chapter 8, pp. 50–51.

theory *Irrationals-From-THEBOOK* **imports** *Stirling-Formula.Stirling-Formula*
begin

1.1 Basic definitions and their consequences

definition *hf* **where** $hf \equiv \lambda n. \lambda x::real. (x \hat{\ } n * (1-x) \hat{\ } n) / fact\ n$

definition *cf* **where** $cf \equiv \lambda n\ i. \text{if } i < n \text{ then } 0 \text{ else } (n \text{ choose } (i-n)) * (-1) \hat{\ } (i-n)$

Mere knowledge that the coefficients are integers is not enough later on.

lemma *hf-int-poly*:

fixes $x::real$

shows $hf\ n = (\lambda x. (1 / fact\ n) * (\sum\ i=0..2*n. real-of-int\ (cf\ n\ i) * x \hat{\ } i))$

<proof>

Lemma (ii) in the text has strict inequalities, but that's more work and is less useful.

lemma

assumes $0 \leq x \leq 1$

shows *hf-nonneg*: $0 \leq hf\ n\ x$ **and** *hf-le-inverse-fact*: $hf\ n\ x \leq 1 / fact\ n$

<proof>

lemma *hf-differt* [*iff*]: *hf* n differentiable at x

<proof>

lemma *deriv-sum-int*:

$deriv\ (\lambda x. \sum\ i=0..n. real-of-int\ (c\ i) * x \hat{\ } i)\ x$

$= (\text{if } n=0 \text{ then } 0 \text{ else } (\sum\ i=0..n-1. of-int((i+1) * c(Suc\ i)) * x \hat{\ } i))$

(**is** $deriv\ ?f\ x = (\text{if } n=0 \text{ then } 0 \text{ else } ?g)$)

<proof>

We calculate the coefficients of the k th derivative precisely.

lemma *hf-deriv-int-poly*:

$(deriv \hat{\ } k)\ (hf\ n) = (\lambda x. (1 / fact\ n) * (\sum\ i=0..2*n-k. of-int\ (int(\prod\ \{i <..i+k\}) * cf\ n\ (i+k)) * x \hat{\ } i))$

<proof>

lemma *hf-deriv-0*: $(deriv \hat{\ } k)\ (hf\ n)\ 0 \in \mathbf{Z}$

<proof>

lemma *deriv-hf-minus*: $deriv\ (hf\ n) = (\lambda x. - deriv\ (hf\ n)\ (1-x))$

<proof>

lemma *deriv-n-hf-diff* [*iff*]: $(deriv \hat{\ } k)\ (hf\ n)$ field-differentiable at x

<proof>

lemma *deriv-n-hf-minus*: $(\text{deriv}^k) (hf\ n) = (\lambda x. (-1)^k * (\text{deriv}^k) (hf\ n) (1-x))$
<proof>

1.2 Towards the main result

lemma *hf-deriv-1*: $(\text{deriv}^k) (hf\ n) \ 1 \in \mathbb{Z}$
<proof>

lemma *hf-deriv-eq-0*: $k > 2*n \implies (\text{deriv}^k) (hf\ n) = (\lambda x. 0)$
<proof>

The case for positive integers

lemma *exp-nat-irrational*:
assumes $s > 0$ **shows** $\text{exp} (\text{real-of-int } s) \notin \mathbb{Q}$
<proof>

theorem *exp-irrational*:
fixes $q::\text{real}$ **assumes** $q \in \mathbb{Q}$ $q \neq 0$ **shows** $\text{exp } q \notin \mathbb{Q}$
<proof>

corollary *ln-irrational*:
fixes $q::\text{real}$ **assumes** $q \in \mathbb{Q}$ $q > 0$ $q \neq 1$ **shows** $\ln q \notin \mathbb{Q}$
<proof>

end

References

- [1] M. Aigner and G. M. Ziegler. *Proofs from THE BOOK*. Springer, 6th edition, 2018.
- [2] M. Eberl. Stirling’s formula. *Archive of Formal Proofs*, Sept. 2016. https://isa-afp.org/entries/Stirling_Formula.html, Formal proof development.
- [3] M. Eberl. The Hermite–Lindemann–Weierstraß transcendence theorem. *Archive of Formal Proofs*, Mar. 2021. https://isa-afp.org/entries/Hermite_Lindemann.html, Formal proof development.