Irrational numbers from THE BOOK

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Abstract

An elementary proof is formalised: that $\exp r$ is irrational for every nonzero rational number r. The mathematical development comes from the well-known volume *Proofs from THE BOOK* [1, pp. 51–2], by Aigner and Ziegler, who credit the idea to Hermite. The development illustrates a number of basic Isabelle techniques: the manipulation of summations, the calculation of quite complicated derivatives and the estimation of integrals. We also see how to import another AFP entry (Stirling's formula) [2].

As for the theorem itself, note that a much stronger and more general result (the Hermite–Lindemann–Weierstraß transcendence theorem) is already available in the AFP [3].

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1 Some irrational numbers

From Aigner and Ziegler, *Proofs from THE BOOK* (Springer, 2018), Chapter 8, pp. 50–51.

theory *Irrationals-From-THEBOOK* **imports** *Stirling-Formula.Stirling-Formula*

begin

1.1 Library additions

context *comm-monoid-set*
begin

lemma *atLeast-atMost-pred-shift*:
  $F\ (g \circ (\lambda n.\ n - Suc\ 0))\ \{Suc\ m..Suc\ n\} = F\ g\ \{m..n\}$
  ⟨*proof*⟩

end

lemma *field-differentiable-diff-const* [*simp*,*derivative-intros*]:
  $(-)c$ *field-differentiable* F
  ⟨*proof*⟩

1.2 Basic definitions and their consequences

definition *hf* **where** $hf \equiv \lambda n.\ \lambda x{::}real.\ (x\hat{\ }n * (1-x)\hat{\ }n)\ /\ fact\ n$

definition *cf* **where** $cf \equiv \lambda n\ i.\ if\ i < n\ then\ 0\ else\ (n\ choose\ (i-n)) * (-1)\hat{\ }(i-n)$

Mere knowledge that the coefficients are integers is not enough later on.

lemma *hf-int-poly*:
  fixes $x{::}real$
  shows $hf\ n = (\lambda x.\ (1\ /\ fact\ n) * (\sum i{=}0..2{*}n.\ real\text{-}of\text{-}int\ (cf\ n\ i) * x\hat{\ }i))$
⟨*proof*⟩

Lemma (ii) in the text has strict inequalities, but it takes more work and is less useful.

lemma
  assumes $0 \le x\ x \le 1$
  shows *hf-nonneg*: $0 \le hf\ n\ x$ **and** *hf-le-inverse-fact*: $hf\ n\ x \le 1/fact\ n$
  ⟨*proof*⟩

lemma *hf-differt* [*iff*]: *hf n differentiable at x*
  ⟨*proof*⟩

lemma *deriv-sum-int*:
  $deriv\ (\lambda x.\ \sum i{=}0..n.\ real\text{-}of\text{-}int\ (c\ i) * x\hat{\ }i)\ x$
    $= (if\ n{=}0\ then\ 0\ else\ (\sum i{=}0..n - Suc\ 0.\ real\text{-}of\text{-}int\ ((int\ i + 1) * c\ (Suc\ i)) * x\hat{\ }i))$

(**is** *deriv ?f x* = (*if n=0 then 0 else ?g*))
⟨*proof*⟩

We calculate the coefficients of the kth derivative precisely.

lemma *hf-deriv-int-poly*:
(*deriv*^^*k*) (*hf n*) = (λx. (*1* / *fact n*) $*$ ($\sum$ *i=0..2$*$n$-$k*. *real-of-int* (*int*($\prod$ {*i<..i+k*}) $*$ *cf n* (*i+k*)) $*$ *x*^*i*))
⟨*proof*⟩

lemma *hf-deriv-0*: (*deriv*^^*k*) (*hf n*) *0* $\in \mathbb{Z}$
⟨*proof*⟩

lemma *deriv-hf-minus*: *deriv* (*hf n*) = (λx. $-$ *deriv* (*hf n*) (*1*$-$*x*))
⟨*proof*⟩

lemma *deriv-n-hf-diffr* [*iff*]: (*deriv*^^*k*) (*hf n*) *field-differentiable at x*
⟨*proof*⟩

lemma *deriv-n-hf-minus*: (*deriv*^^*k*) (*hf n*) = (λx. ($-$*1*)^*k* $*$ (*deriv*^^*k*) (*hf n*) (*1*$-$*x*))
⟨*proof*⟩

1.3 Towards the main result

lemma *hf-deriv-1*: (*deriv*^^*k*) (*hf n*) *1* $\in \mathbb{Z}$
⟨*proof*⟩

lemma *hf-deriv-eq-0*: *k* > *2$*$n* $\Longrightarrow$ (*deriv*^^*k*) (*hf n*) = (λx. *0*)
⟨*proof*⟩

The case for positive integers

lemma *exp-nat-irrational*:
assumes *s* > *0* **shows** *exp* (*real-of-int s*) $\notin \mathbb{Q}$
⟨*proof*⟩

theorem *exp-irrational*:
fixes *q*::*real* **assumes** *q* $\in \mathbb{Q}$ *q* $\neq$ *0* **shows** *exp q* $\notin \mathbb{Q}$
⟨*proof*⟩

end

References

[1] M. Aigner and G. M. Ziegler. *Proofs from THE BOOK*. Springer, 6th edition, 2018.

[2] M. Eberl. Stirling's formula. *Archive of Formal Proofs*, Sept. 2016. https://isa-afp.org/entries/Stirling_Formula.html, Formal proof development.

[3] M. Eberl. The Hermite–Lindemann–Weierstraß transcendence theorem. *Archive of Formal Proofs*, Mar. 2021. https://isa-afp.org/entries/Hermite_Lindemann.html, Formal proof development.