Irrational numbers from THE BOOK

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Abstract

An elementary proof is formalised: that $\exp r$ is irrational for every nonzero rational number r. The mathematical development comes from the well-known volume *Proofs from THE BOOK* [1, pp. 51–2], by Aigner and Ziegler, who credit the idea to Hermite. The development illustrates a number of basic Isabelle techniques: the manipulation of summations, the calculation of quite complicated derivatives and the estimation of integrals. We also see how to import another AFP entry (Stirling's formula) [2].

As for the theorem itself, note that a much stronger and more general result (the Hermite–Lindemann–Weierstraß transcendence theorem) is already available in the AFP [3].

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1 Some irrational numbers

From Aigner and Ziegler, *Proofs from THE BOOK* (Springer, 2018), Chapter 8, pp. 50–51.

theory *Irrationals-From-THEBOOK* **imports** *Stirling-Formula.Stirling-Formula*

begin

1.1 Basic definitions and their consequences

definition *hf* **where** $hf \equiv \lambda n.\ \lambda x{::}real.\ (x\widehat{\ }n * (1-x)\widehat{\ }n) \ /\ fact\ n$

definition *cf* **where** $cf \equiv \lambda n\ i.\ if\ i < n\ then\ 0\ else\ (n\ choose\ (i-n)) * (-1)\widehat{\ }(i-n)$

Mere knowledge that the coefficients are integers is not enough later on.

lemma *hf-int-poly*:
fixes $x{::}real$
shows $hf\ n = (\lambda x.\ (1\ /\ fact\ n) * (\sum i{=}0..2{*}n.\ real\text{-}of\text{-}int\ (cf\ n\ i) * x\widehat{\ }i))$
$\langle proof \rangle$

Lemma (ii) in the text has strict inequalities, but that's more work and is less useful.

lemma
assumes $0 \le x\ x \le 1$
shows *hf-nonneg*: $0 \le hf\ n\ x$ **and** *hf-le-inverse-fact*: $hf\ n\ x \le 1/fact\ n$
$\langle proof \rangle$

lemma *hf-differt* [*iff*]: *hf n differentiable at x*
$\langle proof \rangle$

lemma *deriv-sum-int*:
$deriv\ (\lambda x.\ \sum i{=}0..n.\ real\text{-}of\text{-}int\ (c\ i) * x\widehat{\ }i)\ x$
$= (if\ n{=}0\ then\ 0\ else\ (\sum i{=}0..n-1.\ of\text{-}int((i+1) * c(Suc\ i)) * x\widehat{\ }i))$
(**is** $deriv\ ?f\ x = (if\ n{=}0\ then\ 0\ else\ ?g)$)
$\langle proof \rangle$

We calculate the coefficients of the kth derivative precisely.

lemma *hf-deriv-int-poly*:
$(deriv\widehat{\ }\widehat{\ }k)\ (hf\ n) = (\lambda x.\ (1/fact\ n) * (\sum i{=}0..2{*}n-k.\ of\text{-}int\ (int(\prod \{i{<}..i+k\}) * cf\ n\ (i+k)) * x\widehat{\ }i))$
$\langle proof \rangle$

lemma *hf-deriv-0*: $(deriv\widehat{\ }\widehat{\ }k)\ (hf\ n)\ 0 \in \mathbb{Z}$
$\langle proof \rangle$

lemma *deriv-hf-minus*: $deriv\ (hf\ n) = (\lambda x.\ -\ deriv\ (hf\ n)\ (1-x))$
$\langle proof \rangle$

lemma *deriv-n-hf-diffr* [*iff*]: $(deriv\widehat{\ }\widehat{\ }k)\ (hf\ n)$ *field-differentiable at x*

$\langle proof \rangle$

lemma *deriv-n-hf-minus*: $(deriv \widehat{\ } \widehat{\ } k)$ $(hf\ n) = (\lambda x.\ (-1) \widehat{\ } k * (deriv \widehat{\ } \widehat{\ } k)\ (hf\ n)\ (1-x))$
$\langle proof \rangle$

1.2 Towards the main result

lemma *hf-deriv-1*: $(deriv \widehat{\ } \widehat{\ } k)$ $(hf\ n)$ $1 \in \mathbb{Z}$
$\langle proof \rangle$

lemma *hf-deriv-eq-0*: $k > 2*n \Longrightarrow (deriv \widehat{\ } \widehat{\ } k)\ (hf\ n) = (\lambda x.\ 0)$
$\langle proof \rangle$

The case for positive integers

lemma *exp-nat-irrational*:
assumes $s > 0$ **shows** $exp\ (real\text{-}of\text{-}int\ s) \notin \mathbb{Q}$
$\langle proof \rangle$

theorem *exp-irrational*:
fixes q::*real* **assumes** $q \in \mathbb{Q}$ $q \neq 0$ **shows** $exp\ q \notin \mathbb{Q}$
$\langle proof \rangle$

corollary *ln-irrational*:
fixes q::*real* **assumes** $q \in \mathbb{Q}$ $q > 0$ $q \neq 1$ **shows** $ln\ q \notin \mathbb{Q}$
$\langle proof \rangle$

end

References

[1] M. Aigner and G. M. Ziegler. *Proofs from THE BOOK*. Springer, 6th edition, 2018.

[2] M. Eberl. Stirling's formula. *Archive of Formal Proofs*, Sept. 2016. https://isa-afp.org/entries/Stirling_Formula.html, Formal proof development.

[3] M. Eberl. The Hermite–Lindemann–Weierstraß transcendence theorem. *Archive of Formal Proofs*, Mar. 2021. https://isa-afp.org/entries/Hermite_Lindemann.html, Formal proof development.