Irrationality Criteria for Series by Erdős and Straus

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Abstract

We formalise certain irrationality criteria for infinite series of the form:

$$\sum_n \frac{b_n}{\prod_{i \leq n} a_i}$$

where b_n, a_i are integers. The result is due to P. Erdős and E.G. Straus [1], and in particular we formalise Theorem 2.1, Corollary 2.10 and Theorem 3.1. The latter is an application of Theorem 2.1 involving the prime numbers.

Contents

theory *Irrational-Series-Erdos-Straus* **imports**
Prime-Number-Theorem.Prime-Number-Theorem
Prime-Distribution-Elementary.PNT-Consequences
begin

1 Miscellaneous

lemma *suminf-comparison*:
assumes *summable f* **and** *gf*: $\bigwedge n.$ *norm* $(g\ n) \leq f\ n$
shows *suminf g* $\leq$ *suminf f*
⟨*proof*⟩

lemma *tendsto-of-int-diff-0*:
assumes $(\lambda n.\ f\ n\ -\ \mathit{of\text{-}int}(g\ n)) \longrightarrow (0{::}\mathit{real})$ $\forall_F\ n\ \mathit{in\ sequentially}.\ f\ n > 0$
shows $\forall_F\ n\ \mathit{in\ sequentially}.\ 0 \leq g\ n$
⟨*proof*⟩

lemma *eventually-mono-sequentially*:
assumes *eventually P sequentially*
assumes $\bigwedge x.\ P\ (x{+}k) \Longrightarrow Q\ (x{+}k)$
shows *eventually Q sequentially*
⟨*proof*⟩

lemma *frequently-eventually-at-top*:
fixes $P\ Q{::}'a{::}\mathit{linorder} \Rightarrow \mathit{bool}$
assumes *frequently P at-top eventually Q at-top*
shows $\mathit{frequently}\ (\lambda x.\ P\ x \wedge (\forall y{\geq}x.\ Q\ y)\ )\ \mathit{at\text{-}top}$
⟨*proof*⟩

lemma *eventually-at-top-mono*:
fixes $P\ Q{::}'a{::}\mathit{linorder} \Rightarrow \mathit{bool}$
assumes *event-P*:*eventually P at-top*
assumes *PQ-imp*:$\bigwedge x.\ x{\geq}z \Longrightarrow \forall y{\geq}x.\ P\ y \Longrightarrow Q\ x$
shows *eventually Q at-top*
⟨*proof*⟩

lemma *frequently-at-top-elim*:
fixes $P\ Q{::}'a{::}\mathit{linorder} \Rightarrow \mathit{bool}$
assumes $\exists_F x\ \mathit{in\ at\text{-}top}.\ P\ x$
assumes $\bigwedge i.\ P\ i \Longrightarrow \exists j{>}i.\ Q\ j$
shows $\exists_F x\ \mathit{in\ at\text{-}top}.\ Q\ x$
⟨*proof*⟩

lemma *less-Liminf-iff*:
fixes $X\ {::}\ \text{-} \Rightarrow \text{-}\ {::}\ \mathit{complete\text{-}linorder}$
shows $\mathit{Liminf}\ F\ X < C \longleftrightarrow (\exists y{<}C.\ \mathit{frequently}\ (\lambda x.\ y \geq X\ x)\ F)$
⟨*proof*⟩

lemma *sequentially-even-odd-imp*:
assumes $\forall_F\ N\ \mathit{in\ sequentially}.\ P\ (2{*}N)$ $\forall_F\ N\ \mathit{in\ sequentially}.\ P\ (2{*}N{+}1)$
shows $\forall_F\ n\ \mathit{in\ sequentially}.\ P\ n$
⟨*proof*⟩

2 Theorem 2.1 and Corollary 2.10

context
fixes $a\ b\ {::}\mathit{nat}{\Rightarrow}\mathit{int}$
assumes *a-pos*: $\forall\ n.\ a\ n > 0$ **and** *a-large*: $\forall_F\ n\ \mathit{in\ sequentially}.\ a\ n > 1$
and *ab-tendsto*: $(\lambda n.\ |b\ n|\ /\ (a\ (n{-}1) * a\ n)) \longrightarrow 0$
begin

private lemma *aux-series-summable*: *summable* $(\lambda n.\ b\ n\ /\ (\prod k{\le}n.\ a\ k))$
$\langle proof \rangle$ **fun** *get-c*::$(nat \Rightarrow int) \Rightarrow (nat \Rightarrow int) \Rightarrow int \Rightarrow nat \Rightarrow (nat \Rightarrow int)$ **where**
  $get\text{-}c\ a'\ b'\ B\ N\ 0 = round\ (B * b'\ N\ /\ a'\ N)|$
  $get\text{-}c\ a'\ b'\ B\ N\ (Suc\ n) = get\text{-}c\ a'\ b'\ B\ N\ n * a'\ (n{+}N) - B * b'\ (n{+}N)$

lemma *ab-rationality-imp*:
  assumes *ab-rational*:$(\sum n.\ (b\ n\ /\ (\prod i \le n.\ a\ i))) \in \mathbb{Q}$
  shows $\exists\ (B{::}int){>}0.\ \exists\ c{::}nat{\Rightarrow}\ int.$
    $(\forall_F\ n\ in\ sequentially.\ B{*}b\ n = c\ n * a\ n - c(n{+}1) \wedge |c(n{+}1)|{<}a\ n/2)$
    $\wedge\ (\lambda n.\ c\ (Suc\ n)\ /\ a\ n) \longrightarrow 0$
$\langle proof \rangle$ **lemma** *imp-ab-rational*:
  assumes $\exists\ (B{::}int){>}0.\ \exists\ c{::}nat{\Rightarrow}\ int.$
    $(\forall_F\ n\ in\ sequentially.\ B{*}b\ n = c\ n * a\ n - c(n{+}1) \wedge |c(n{+}1)|{<}a\ n/2)$
  shows $(\sum n.\ (b\ n\ /\ (\prod i \le n.\ a\ i))) \in \mathbb{Q}$
$\langle proof \rangle$

theorem *theorem-2-1-Erdos-Straus* :
  $(\sum n.\ (b\ n\ /\ (\prod i \le n.\ a\ i))) \in \mathbb{Q} \longleftrightarrow (\exists\ (B{::}int){>}0.\ \exists\ c{::}nat{\Rightarrow}\ int.$
    $(\forall_F\ n\ in\ sequentially.\ B{*}b\ n = c\ n * a\ n - c(n{+}1) \wedge |c(n{+}1)|{<}a\ n/2))$
  $\langle proof \rangle$

The following is a Corollary to Theorem 2.1.

corollary *corollary-2-10-Erdos-Straus*:
  assumes *ab-event*:$\forall_F\ n\ in\ sequentially.\ b\ n > 0 \wedge a\ (n{+}1) \ge a\ n$
    and *ba-lim-leq*:$lim\ (\lambda n.\ (b(n{+}1) - b\ n\ )/a\ n) \le 0$
    and *ba-lim-exist*:$convergent\ (\lambda n.\ (b(n{+}1) - b\ n\ )/a\ n)$
    and $liminf\ (\lambda n.\ a\ n\ /\ b\ n) = 0$
  shows $(\sum n.\ (b\ n\ /\ (\prod i \le n.\ a\ i))) \notin \mathbb{Q}$
$\langle proof \rangle$

end

3 Some auxiliary results on the prime numbers.

lemma *nth-prime-nonzero*[*simp*]:$nth\text{-}prime\ n \neq 0$
  $\langle proof \rangle$

lemma *nth-prime-gt-zero*[*simp*]:$nth\text{-}prime\ n > 0$
  $\langle proof \rangle$

lemma *ratio-of-consecutive-primes*:
  $(\lambda n.\ nth\text{-}prime\ (n{+}1)/nth\text{-}prime\ n) \longrightarrow 1$
$\langle proof \rangle$

lemma *nth-prime-double-sqrt-less*:
  assumes $\varepsilon > 0$
  shows $\forall_F\ n\ in\ sequentially.\ (nth\text{-}prime\ (2{*}n) - nth\text{-}prime\ n)$
    $/\ sqrt\ (nth\text{-}prime\ n) < n\ powr\ (1/2{+}\varepsilon)$
$\langle proof \rangle$

4 Theorem 3.1

Theorem 3.1 is an application of Theorem 2.1 with the sequences considered involving the prime numbers.

theorem *theorem-3-10-Erdos-Straus*:
  fixes a::*nat* $\Rightarrow$ *int*
  assumes *a-pos*:$\forall$ $n.$ a n >0 **and** *mono a*
    and *nth-1*:($\lambda n.$ *nth-prime* n / (a n)^2) $\longrightarrow$ 0
    and *nth-2*:*liminf* ($\lambda n.$ a n / *nth-prime* n) = 0
  shows ($\sum n.$ (*nth-prime* n / ($\prod i \leq n.$ a i))) $\notin \mathbb{Q}$
$\langle$*proof*$\rangle$

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References

[1] P. Erdős and E. Straus. On the irrationality of certain series. *Pacific journal of mathematics*, 55(1):85–92, 1974.