

# Proving the Impossibility of Trisecting an Angle and Doubling the Cube

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## Abstract

Squaring the circle, doubling the cube and trisecting an angle, using a compass and straightedge alone, are classic unsolved problems first posed by the ancient Greeks. All three problems were proved to be impossible in the 19th century. The following document presents the proof of the impossibility of solving the latter two problems using Isabelle/HOL, following a proof by Carrega [Car81]. The proof uses elementary methods: no Galois theory or field extensions. The set of points constructible using a compass and straightedge is defined inductively. Radical expressions, which involve only square roots and arithmetic of rational numbers, are defined, and we find that all constructive points have radical coordinates. Finally, doubling the cube and trisecting certain angles requires solving certain cubic equations that can be proved to have no rational roots. The Isabelle proofs require a great many detailed calculations.

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# 1 Proving the impossibility of trisecting an angle and doubling the cube

```
theory Impossible-Geometry
imports Complex-Main
begin
```

## 2 Formal Proof

### 2.1 Definition of the set of Points

```
datatype point = Point real real
```

```
definition points-def:
```

```
points = {M.  $\exists x \in \mathbb{R}. \exists y \in \mathbb{R}. (M = \text{Point } x \ y)$ }
```

```
primrec abscissa :: point => real
```

```
where abscissa: abscissa (Point x y) = x
```

```
primrec ordinate :: point => real
```

```
where ordinate: ordinate (Point x y) = y
```

```
lemma point-surj [simp]:
```

```
Point (abscissa M) (ordinate M) = M
```

```
by (induct M) simp
```

```
lemma point-eqI [intro?]:
```

```
 $\llbracket \text{abscissa } M = \text{abscissa } N; \text{ordinate } M = \text{ordinate } N \rrbracket \implies M = N$ 
```

```
by (induct M, induct N) simp
```

```
lemma point-eq-iff:
```

```
 $M = N \iff \text{abscissa } M = \text{abscissa } N \wedge \text{ordinate } M = \text{ordinate } N$ 
```

```
by (induct M, induct N) simp
```

### 2.2 Subtraction

Datatype point has a structure of abelian group

```
instantiation point :: ab-group-add
```

```
begin
```

```
definition point-zero-def:
```

$0 = \text{Point } 0 \ 0$

**definition** *point-one-def*:

$\text{point-one} = \text{Point } 1 \ 0$

**definition** *point-add-def*:

$A + B = \text{Point } (\text{abscissa } A + \text{abscissa } B) \ (\text{ordinate } A + \text{ordinate } B)$

**definition** *point-minus-def*:

$- A = \text{Point } (- \text{abscissa } A) \ (- \text{ordinate } A)$

**definition** *point-diff-def*:

$A - (B::\text{point}) = A + - B$

**lemma** *Point-eq-0* [*simp*]:

$\text{Point } xA \ yA = 0 \longleftrightarrow (xA = 0 \wedge yA = 0)$

**by** (*simp add: point-zero-def*)

**lemma** *point-abscissa-zero* [*simp*]:

$\text{abscissa } 0 = 0$

**by** (*simp add: point-zero-def*)

**lemma** *point-ordinate-zero* [*simp*]:

$\text{ordinate } 0 = 0$

**by** (*simp add: point-zero-def*)

**lemma** *point-add* [*simp*]:

$\text{Point } xA \ yA + \text{Point } xB \ yB = \text{Point } (xA + xB) \ (yA + yB)$

**by** (*simp add: point-add-def*)

**lemma** *point-abscissa-add* [*simp*]:

$\text{abscissa } (A + B) = \text{abscissa } A + \text{abscissa } B$

**by** (*simp add: point-add-def*)

**lemma** *point-ordinate-add* [*simp*]:

$\text{ordinate } (A + B) = \text{ordinate } A + \text{ordinate } B$

**by** (*simp add: point-add-def*)

**lemma** *point-minus* [*simp*]:

$-(\text{Point } xA \ yA) = \text{Point } (- xA) \ (- yA)$

**by** (*simp add: point-minus-def*)

**lemma** *point-abscissa-minus* [*simp*]:

$\text{abscissa } (- A) = - \text{abscissa } (A)$

**by** (*simp add: point-minus-def*)

**lemma** *point-ordinate-minus* [*simp*]:

$\text{ordinate } (- A) = - \text{ordinate } (A)$

**by** (*simp add: point-minus-def*)

**lemma** *point-diff* [simp]:  
 $Point\ xA\ yA - Point\ xB\ yB = Point\ (xA - xB)\ (yA - yB)$   
**by** (*simp add: point-diff-def*)

**lemma** *point-abscissa-diff* [simp]:  
 $abscissa\ (A - B) = abscissa\ (A) - abscissa\ (B)$   
**by** (*simp add: point-diff-def*)

**lemma** *point-ordinate-diff* [simp]:  
 $ordinate\ (A - B) = ordinate\ (A) - ordinate\ (B)$   
**by** (*simp add: point-diff-def*)

**instance**  
**by** *intro-classes (simp-all add: point-add-def point-diff-def)*

**end**

## 2.3 Metric Space

We can also define a distance, hence point is also a metric space

**instantiation** *point* :: *metric-space*  
**begin**

**definition** *point-dist-def*:  
 $dist\ A\ B = sqrt\ ((abscissa\ (A - B))^2 + (ordinate\ (A - B))^2)$

**definition**  
*(uniformity* :: *(point × point) filter*) = (*INF*  $e:\{0 < ..\}$ . *principal*  $\{(x, y). dist\ x\ y < e\}$ )

**definition**  
 $open\ (S :: point\ set) = (\forall x \in S. \forall_F (x', y)\ in\ uniformity. x' = x \longrightarrow y \in S)$

**lemma** *point-dist* [simp]:  
 $dist\ (Point\ xA\ yA)\ (Point\ xB\ yB) = sqrt\ ((xA - xB)^2 + (yA - yB)^2)$   
**unfolding** *point-dist-def*  
**by** *simp*

**lemma** *real-sqrt-diff-squares-triangle-ineq*:  
**fixes**  $a\ b\ c\ d :: real$   
**shows**  $sqrt\ ((a - c)^2 + (b - d)^2) \leq sqrt\ (a^2 + b^2) + sqrt\ (c^2 + d^2)$   
**proof** –  
**have**  $sqrt\ ((a - c)^2 + (b - d)^2) \leq sqrt\ (a^2 + b^2) + sqrt\ ((-c)^2 + (-d)^2)$   
**by** (*metis diff-conv-add-uminus real-sqrt-sum-squares-triangle-ineq*)  
**also have**  $... = sqrt\ (a^2 + b^2) + sqrt\ (c^2 + d^2)$   
**by** *simp*  
**finally show** *?thesis* .

**qed**

**instance**

**proof**

**fix**  $A B C :: \text{point}$  **and**  $S :: \text{point set}$

**show**  $(\text{dist } A B = 0) = (A = B)$

**by**  $(\text{induct } A, \text{induct } B) (\text{simp add: point-dist-def})$

**show**  $(\text{dist } A B) \leq (\text{dist } A C) + (\text{dist } B C)$

**proof**  $-$

**have**  $\text{sqrt } ((\text{abscissa } (A - B))^2 + (\text{ordinate } (A - B))^2) \leq$

$\text{sqrt } ((\text{abscissa } (A - C))^2 + (\text{ordinate } (A - C))^2) +$

$\text{sqrt } ((\text{abscissa } (B - C))^2 + (\text{ordinate } (B - C))^2)$

**using**  $\text{real-sqrt-diff-squares-triangle-ineq}$

$[\text{of } \text{abscissa } (A) - \text{abscissa } (C) \text{ abscissa } (B) - \text{abscissa } (C)$

$\text{ordinate } (A) - \text{ordinate } (C) \text{ ordinate } (B) - \text{ordinate } (C)]$

**by**  $(\text{simp only: point-diff-def}) (\text{simp add: algebra-simps})$

**thus**  $?thesis$

**by**  $(\text{simp add: point-dist-def})$

**qed**

**qed**  $(\text{rule uniformity-point-def open-point-def})+$

**end**

## 2.4 Geometric Definitions

These geometric definitions will later be used to define constructible points

The distance between two points is defined with the distance of the metric space point

**definition**  $\text{distance-def}$ :

$\text{distance } A B = \text{dist } A B$

$\text{parallel } A B C D$  is true if the lines  $AB$  and  $CD$  are parallel. If not it is false.

**definition**  $\text{parallel-def}$ :

$\text{parallel } A B C D = ((\text{abscissa } A - \text{abscissa } B) * (\text{ordinate } C - \text{ordinate } D) =$   
 $(\text{ordinate } A - \text{ordinate } B) * (\text{abscissa } C - \text{abscissa } D))$

Three points  $A B C$  are collinear if and only if the lines  $AB$  and  $AC$  are parallel

**definition**  $\text{collinear-def}$ :

$\text{collinear } A B C = \text{parallel } A B A C$

The point  $M$  is the intersection of two lines  $AB$  and  $CD$  if and only if the points  $A, M$  and  $B$  are collinear and the points  $C, M$  and  $D$  are also collinear

**definition**  $\text{is-intersection-def}$ :

$\text{is-intersection } M A B C D = (\text{collinear } A M B \wedge \text{collinear } C M D)$

## 2.5 Reals definable with square roots

The inductive set *radical-sqrt* defines the reals that can be defined with square roots. If  $x$  is in the following set, then it depends only upon rational expressions and square roots. For example, suppose  $x$  is of the form :  $x = (\sqrt{a + \sqrt{b}} + \sqrt{c + \sqrt{d * e + f}}) / (\sqrt{a} + \sqrt{b}) + (a + \sqrt{b}) / \sqrt{g}$ , where  $a, b, c, d, e, f$  and  $g$  are rationals. Then  $x$  is in *radical-sqrt* because it is only defined with rationals and square roots of radicals.

**inductive-set** *radical-sqrt* :: *real set*

**where**

$x \in \mathbb{Q} \implies x \in \text{radical-sqrt}$

$x \in \text{radical-sqrt} \implies -x \in \text{radical-sqrt}$

$x \in \text{radical-sqrt} \implies x \neq 0 \implies 1/x \in \text{radical-sqrt}$

$x \in \text{radical-sqrt} \implies y \in \text{radical-sqrt} \implies x+y \in \text{radical-sqrt}$

$x \in \text{radical-sqrt} \implies y \in \text{radical-sqrt} \implies x*y \in \text{radical-sqrt}$

$x \in \text{radical-sqrt} \implies x \geq 0 \implies \text{sqrt } x \in \text{radical-sqrt}$

Here, we list some rules that will be used to prove that a given real is in *radical-sqrt*.

Given two reals in *radical-sqrt*  $x$  and  $y$ , the subtraction  $x - y$  is also in *radical-sqrt*.

**lemma** *radical-sqrt-rule-subtraction*:

$x \in \text{radical-sqrt} \implies y \in \text{radical-sqrt} \implies x-y \in \text{radical-sqrt}$

**by** (*metis diff-conv-add-uminus radical-sqrt.intros(2) radical-sqrt.intros(4)*)

Given two reals in *radical-sqrt*  $x$  and  $y$ , and  $y \neq 0$ , the division  $x/y$  is also in *radical-sqrt*.

**lemma** *radical-sqrt-rule-division*:

$x \in \text{radical-sqrt} \implies y \in \text{radical-sqrt} \implies y \neq 0 \implies x/y \in \text{radical-sqrt}$

**by** (*metis divide-real-def radical-sqrt.intros(3) radical-sqrt.intros(5) real-scaleR-def real-vector.scale-one*)

Given a positive real  $x$  in *radical-sqrt*, its square  $x^2$  is also in *radical-sqrt*.

**lemma** *radical-sqrt-rule-power2*:

$x \in \text{radical-sqrt} \implies x \geq 0 \implies x^2 \in \text{radical-sqrt}$

**by** (*metis power2-eq-square radical-sqrt.intros(5)*)

Given a positive real  $x$  in *radical-sqrt*, its cube  $x^3$  is also in *radical-sqrt*.

**lemma** *radical-sqrt-rule-power3*:

$x \in \text{radical-sqrt} \implies x \geq 0 \implies x^3 \in \text{radical-sqrt}$

**by** (*metis power3-eq-cube radical-sqrt.intros(5)*)

## 2.6 Introduction of the datatype *expr* which represents radical expressions

An expression *expr* is either a rational constant: *Const* or the negation of an expression or the inverse of an expression or the addition of two expressions

or the multiplication of two expressions or the square root of an expression.

**datatype** *expr* = *Const rat* | *Negation expr* | *Inverse expr* | *Addition expr expr* | *Multiplication expr expr* | *Sqrt expr*

The function *translation* translates a given expression into its equivalent real.

```
fun translation :: expr => real ((2{|-|}))
  where
    translation (Const x) = of-rat x |
    translation (Negation e) = - translation e |
    translation (Inverse e) = (1::real) / translation e |
    translation (Addition e1 e2) = translation e1 + translation e2 |
    translation (Multiplication e1 e2) = translation e1 * translation e2 |
    translation (Sqrt e) = (if translation e < 0 then 0 else sqrt (translation e))
```

Define the set of all the radicals of a given expression. For example, suppose *expr* is of the form : *expr* = *Addition (Sqrt (Addition (Const a) Sqrt (Const b))) (Sqrt (Addition (Const c) (Sqrt (Sqrt (Const d))))))*, where *a*, *b*, *c* and *d* are rationals. This can be translated as follows:  $\{expr\} = \sqrt{a + \sqrt{b}} + \sqrt{c + \sqrt{\sqrt{d}}}$ . Moreover, the set *radicals* of this expression is :  $\{Addition (Const a) (Sqrt (Const b)), Const b, Addition (Const c) (Sqrt (Sqrt (Const d))), Sqrt (Const d), Const d\}$ .

```
fun radicals :: expr => expr set
  where
    radicals (Const x) = {} |
    radicals (Negation e) = (radicals e) |
    radicals (Inverse e) = (radicals e) |
    radicals (Addition e1 e2) = ((radicals e1) ∪ (radicals e2)) |
    radicals (Multiplication e1 e2) = ((radicals e1) ∪ (radicals e2)) |
    radicals (Sqrt e) = (if {e} < 0 then radicals e else {e} ∪ (radicals e))
```

If *r* is in *radicals* of *e* then the set *radical-sqrt* of *r* is a subset (strictly speaking) of the set *radicals* of *e*.

**lemma** *radicals-expr-subset*:  $r \in \text{radicals } e \implies \text{radicals } r \subset \text{radicals } e$   
**by** (*induct e*, *auto simp add: if-split-asm*)

If *x* is in *radical-sqrt* then there exists a radical expression *e* which translation is *x* (it is important to notice that this expression is not necessarily unique).

```
lemma radical-sqrt-correct-expr:
   $x \in \text{radical-sqrt} \implies (\exists e. \{e\} = x)$ 
  apply (rule radical-sqrt.induct)
  apply auto
  apply (erule Rats-induct)
  apply (metis translation.simps(1))
```

```

apply (metis translation.simps(2))
apply (metis translation.simps(3))
apply (metis translation.simps(4))
apply (metis translation.simps(5))
apply (metis linorder-not-less translation.simps(6))
done

```

The order of an expression is the maximum number of radicals one over another occurring in a given expression. Using the example above, suppose  $expr$  is of the form :  $expr = \text{Addition} (\text{Sqrt} (\text{Addition} (\text{Const } a) \text{Sqrt} (\text{Const } b))) (\text{Sqrt} (\text{Addition} (\text{Const } c) (\text{Sqrt} (\text{Sqrt} (\text{Const } d))))))$ , where  $a$ ,  $b$ ,  $c$  and  $d$  are rationals and which can be translated as follows:  $\{\{expr\}\} = \sqrt{a + \sqrt{b} + \sqrt{c + \sqrt{\sqrt{d}}}}$ . The order of  $expr$  is  $\max(2, 3) = 3$ .

```

fun order :: expr => nat
  where
    order (Const x) = 0 |
    order (Negation e) = order e |
    order (Inverse e) = order e |
    order (Addition e1 e2) = max (order e1) (order e2) |
    order (Multiplication e1 e2) = max (order e1) (order e2) |
    order (Sqrt e) = 1 + order e

```

If an expression  $s$  is one of the radicals (or in *radicals*) of the expression  $r$ , then its order is smaller (strictly speaking) then the order of  $r$ .

```

lemma in-radicals-smaller-order:
  s ∈ radicals r ⇒ (order s) < (order r)
apply (induct r, auto)
apply (metis insert-iff insert-is-Un less-Suc-eq)
done

```

The following theorem is the converse of the previous lemma.

```

lemma in-radicals-smaller-order-contrap:
  (order s) ≥ (order r) ⇒ ¬ (s ∈ radicals r)
by (metis in-radicals-smaller-order leD)

```

An expression  $r$  cannot be one of its own radicals.

```

lemma not-in-own-radicals:
  ¬ (r ∈ radicals r)
by (metis in-radicals-smaller-order order-less-irrefl)

```

If an expression  $e$  is a radical expression and it has no radicals then its translation is a rational.

```

lemma radicals-empty-rational: radicals e = {} ⇒ {e} ∈ ℚ
by (induct e, auto)

```

A finite non-empty set of natural numbers has necessarily a maximum.

**lemma** *finite-set-has-max*:  
 $finite (s :: nat set) \implies s \neq \{\} \implies \exists k \in s. \forall p \in s. p \leq k$   
**by** (*metis Max-ge Max-in*)

There is a finite number of radicals in an expression.

**lemma** *finite-radicals*:  $finite (radicals e)$   
**by** (*induct e, auto*)

We define here a new set corresponding to the orders of each element in the set *radicals* of an expression *expr*. Using the example above, suppose *expr* is of the form :  $expr = Addition (Sqrt (Addition (Const a) Sqrt (Const b))) (Sqrt (Addition (Const c) (Sqrt (Sqrt (Const d)))))$ , where *a*, *b*, *c* and *d* are rationals and which can be translated as follows:  $\{expr\} = \sqrt{a + \sqrt{b}} + \sqrt{c + \sqrt{\sqrt{d}}}$ . The set *radicals* of *expr* is  $\{Addition (Const a) Sqrt (Const b), Const b, Addition (Const c) (Sqrt (Sqrt (Const d))), Sqrt (Const d), Const d\}$ ; therefore, the set *order-radicals* of this set is  $\{1, 0, 2, 1, 0\}$ .

**fun** *order-radicals*::  $expr set \Rightarrow nat set$   
**where** *order-radicals*  $s = \{y. \exists x \in s. y = order x\}$

If the set of radicals of an expression *e* is not empty and is finite then the set *order-radicals* of the set of radicals of *e* is not empty and is also finite.

**lemma** *finite-order-radicals*:  
 $radicals e \neq \{\} \implies finite (radicals e) \implies$   
 $order-radicals (radicals e) \neq \{\} \wedge finite (order-radicals (radicals e))$   
**by** *simp (metis equalsOI)*

The following lemma states that given an expression *e*, if the set *order-radicals* of the set *radicals e* is not empty and is finite, then there exists a radical *r* of *e* which is of highest order among the radicals of *e*.

**lemma** *finite-order-radicals-has-max*:  
 $order-radicals (radicals e) \neq \{\} \implies$   
 $finite (order-radicals (radicals e)) \implies$   
 $\exists r. (r \in radicals e) \wedge (\forall s \in (radicals e). (order r \geq order s))$   
**using** *finite-set-has-max [of order-radicals (radicals e)]*  
**by** *auto*

This important lemma states that in an expression that has at least one radical, we can find an upmost radical *r* which is not radical of any other term of the expression *e*. It is also important to notice that this upmost radical is not necessarily unique and is not the term of highest order of the expression *e*. Using the example above, suppose *e* is of the form :  $e = Addition (Sqrt (Addition (Const a) Sqrt (Const b))) (Sqrt (Addition (Const c) (Sqrt (Sqrt (Const d)))))$ , where *a*, *b*, *c* and *d* are rationals and which can be translated as follows:  $\{e\} = \sqrt{a + \sqrt{b}} + \sqrt{c + \sqrt{\sqrt{d}}}$ . The possible

upmost radicals in this expression are Addition (Const  $a$ ) (Sqrt (Const  $b$ )) or Addition (Const  $c$ ) (Sqrt (Sqrt (Const  $d$ ))).

**lemma** *upmost-radical-sqrt2*:

*radicals e*  $\neq \{\}$   $\implies$

$\exists r \in \text{radicals } e. \forall s \in \text{radicals } e. r \notin \text{radicals } s$

**using** *in-radicals-smaller-order-contrap* [of  $r$   $s$ ] *finite-radicals* [of  $e$ ]

**by** (*metis finite-order-radicals finite-order-radicals-has-max in-radicals-smaller-order-contrap*)

The following 7 lemmas are used to prove the main lemma *radical-sqrt-normal-form* which states that if an expression  $e$  has at least one radical then it can be written in a normal form. This means that there exist three radical expressions  $a$ ,  $b$  and  $r$  such that  $\{e\} = \{a\} + \{b\} * \sqrt{\{r\}}$  and the radicals of  $a$  are radicals of  $e$  but are not  $r$ , and the same goes for the radicals of  $b$  and  $r$ . It is important to notice that  $a$ ,  $b$  and  $r$  are not unique and *Sqrt*  $r$  is not necessarily the term of highest order.

**lemma** *radical-sqrt-normal-form-sublemma*:

$((a::\text{real}) - b) * (a + b) = a * a - b * b$

**by** (*simp add: field-simps*)

**lemma** *eq-sqrt-squared*:

$(x::\text{real}) \geq 0 \implies (\text{sqrt } x) * (\text{sqrt } x) = x$

**by** (*metis abs-of-nonneg real-sqrt-abs2 real-sqrt-mult*)

**lemma** *radical-sqrt-normal-form-lemma4*:

**assumes**  $z \geq 0$   $x \neq y * \text{sqrt } z$

**shows**

$1 / (x + y * \text{sqrt } z) =$

$x / (x * x - y * y * z) - (y * \text{sqrt } z) / (x * x - y * y * z)$

**proof** –

**have**  $1 / (x + y * \text{sqrt } z) = ((x - y * \text{sqrt } z) / (x + y * \text{sqrt } z)) / (x - y * \text{sqrt } z)$

**by** (*auto simp add: eq-divide-imp assms*)

**also have**  $\dots = x / (x * x - y * y * z) - (y * \text{sqrt } z) / (x * x - y * y * z)$

**by** (*auto simp add: algebra-simps eq-sqrt-squared diff-divide-distrib assms*)

**finally show** *?thesis* .

**qed**

**lemma** *radical-sqrt-normal-form-lemma*:

**fixes**  $e::\text{expr}$

**assumes** *radicals e*  $\neq \{\}$

**and**  $\forall s \in \text{radicals } e. r \notin \text{radicals } s$

**and**  $r : \text{radicals } e$

**shows**  $\exists a b. 0 \leq \{r\} \ \& \ \{e\} = \{a\} + \{b\} * \text{sqrt } \{r\} \ \&$

$\text{radicals } a \cup \text{radicals } b \cup \text{radicals } r \subseteq \text{radicals } e \ \&$

$r \notin \text{radicals } a \cup \text{radicals } b$

(**is**  $\exists a b. ?\text{concl } e \ a \ b$ )

**using** *assms*

**proof** (*induct e*)

```

    case (Const rat) thus ?case
      by auto
next
case (Negation e)
  obtain a b
    where a2: ?concl e a b
      by (metis Negation radicals.simps(2))
  hence {Negation e} = {Negation a} + {Negation b} * sqrt {r}
    by simp
  thus ?case using a2
    by (metis radicals.simps(2))
next
case (Inverse e)
  obtain a b
    where ?concl e a b
      by (metis Inverse radicals.simps(3))
  thus ?case
    apply (case-tac {b} * sqrt {r} = {a})
    apply simp
    apply (case-tac {a} = 0)
    apply (metis add-0-right div-by-0 mult-zero-right)
    apply (rule-tac x = Multiplication (Const 1) (Inverse (Multiplication (Const
2) a)) in exI)
    apply (rule-tac x = Const 0 in exI, simp)
    apply (rule-tac x = Multiplication a (Inverse (Addition (Multiplication a a)
(Negation (Multiplication (Multiplication b b) r)))) in exI)
    apply (rule-tac x = Negation (Multiplication b (Inverse (Addition (Multiplication
a a) (Negation (Multiplication (Multiplication b b) r)))) in exI)
    apply (simp add: algebra-simps not-in-own-radicals eq-diff-eq' radical-sqrt-normal-form-lemma4)
    done
next
case (Addition e1 e2)
  hence d1:  $\forall s \in \text{radicals } e1 \cup \text{radicals } e2. r \notin \text{radicals } s$ 
    by (metis radicals.simps(4))
  show ?case
  proof (cases r: radicals e1 & r : radicals e2)
    case True
      obtain a1 b1 a2 b2
        where ab: ?concl e1 a1 b1
          and bb: ?concl e2 a2 b2
          using Addition.hyps
          by (simp add: d1) (metis True empty-iff)
      thus ?thesis
        apply simp
        apply (rule-tac x = Addition a1 a2 in exI)
        apply (rule-tac x = Addition b1 b2 in exI)
        apply (auto simp add: comm-semiring-class.distrib)
        done
  next

```

```

case False
thus ?thesis
proof (cases r: radicals e1)
  case True
  obtain a1 b1
  where  $0 \leq \{r\}$  ?concl e1 a1 b1
  using Addition.hyps
  by (auto simp: d1) (metis True empty-iff)
  thus ?thesis
  apply (rule-tac x = Addition a1 e2 in exI)
  apply (rule-tac x = b1 in exI) using False True
  apply auto
  done
next
  case False
  obtain a2 b2
  where  $0 \leq \{r\}$  ?concl e2 a2 b2
  using Addition d1
  by (metis False Un-iff empty-iff radicals.simps(4))
  thus ?thesis
  apply (rule-tac x = Addition a2 e1 in exI)
  apply (rule-tac x = b2 in exI) using False
  apply auto
  done
qed
qed
next
case (Multiplication e1 e2)
show ?case
proof (cases r: radicals e1 & r : radicals e2)
  case True
  then obtain a1 b1 a2 b2
  where ?concl e1 a1 b1 ?concl e2 a2 b2
  using Multiplication
  by simp (metis True empty-iff)
  thus ?thesis
  apply (rule-tac x = Addition (Multiplication a1 a2) (Multiplication r (Multiplication
b1 b2)) in exI)
  apply (rule-tac x = Addition (Multiplication a1 b2) (Multiplication a2 b1)
in exI)
  apply (auto simp add: not-in-own-radicals algebra-simps eq-sqrt-squared)
  done
next
  case False
  thus ?thesis
proof (cases r: radicals e1)
  case True
  then obtain a1 b1
  where ?concl e1 a1 b1

```

```

    using Multiplication.hyps Multiplication(4)
    by auto (metis True empty-iff)
  thus ?thesis
    apply (rule-tac x = Multiplication a1 e2 in exI)
    apply (rule-tac x = Multiplication b1 e2 in exI)
    apply (simp add: algebra-simps)
    by (metis False True le-supI1 radicals.simps(5))
next
case False
then obtain a2 b2
  where ?concl e2 a2 b2
  using Multiplication.hyps Multiplication(4) Multiplication(5)
  by auto blast
thus ?thesis
  apply (rule-tac x = Multiplication a2 e1 in exI)
  apply (rule-tac x = Multiplication b2 e1 in exI)
  apply (simp add: algebra-simps)
  by (metis False le-supI2)
qed
qed
next
case (Sqrt e)
show ?case
proof (cases {e} < 0)
case True thus ?thesis
  using Sqrt
  apply (rule-tac x = Const 0 in exI)
  apply (rule-tac x = Const 0 in exI)
  apply auto
  done
next
case False thus ?thesis
  apply (rule-tac x = Const 0 in exI)
  apply (rule-tac x = Const 1 in exI) using Sqrt
  apply (auto simp add: linorder-not-less)
  done
qed
qed

```

This main lemma is essential for the remaining part of the proof.

**theorem** *radical-sqrt-normal-form*:

$radicals\ e \neq \{\} \implies$

$\exists r \in radicals\ e.$

$\exists a\ b. \{e\} = \{Addition\ a\ (Multiplication\ b\ (Sqrt\ r))\} \wedge \{r\} \geq 0 \wedge$   
 $radicals\ a \cup radicals\ b \cup radicals\ r \subseteq radicals\ e \ \&$   
 $r \notin radicals\ a \cup radicals\ b \cup radicals\ r$

using *upmost-radical-sqrt2* [of *e*] *radical-sqrt-normal-form-lemma*

by *auto* (metis *all-not-in-conv leD*)

## 2.7 Important properties of the roots of a cubic equation

The following 7 lemmas are used to prove a main result about the properties of the roots of a cubic equation (*cubic-root-radical-sqrt-rational*) which states that assuming that  $a$ ,  $b$  and  $c$  are rationals and that  $x$  is a radical satisfying  $x^3 + ax^2 + bx + c = 0$  then there exists a rational root. This lemma will be used in the proof of the impossibility of trisection an angle and of duplicating a cube.

**lemma** *cubic-root-radical-sqrt-steplemma*:

**fixes**  $P :: \text{real set}$   
**assumes**  $\text{Nats } [THEN \text{ set-mp, intro}]: \text{Nats} \subseteq P$   
**and**  $\text{Neg}: \forall x \in P. -x \in P$   
**and**  $\text{Inv}: \forall x \in P. x \neq 0 \longrightarrow 1/x \in P$   
**and**  $\text{Add}: \forall x \in P. \forall y \in P. x+y \in P$   
**and**  $\text{Mult}: \forall x \in P. \forall y \in P. x*y \in P$   
**and**  $a: (a \in P)$  **and**  $b: (b \in P)$  **and**  $c: (c \in P)$   
**and**  $\text{eq0}: z^3 + a * z^2 + b * z + c = 0$   
**and**  $u: (u \in P)$   
**and**  $v: (v \in P)$   
**and**  $s: ((s * s) \in P)$   
**and**  $z: (z = u + v * s)$   
**shows**  $\exists w \in P. w^3 + a * w^2 + b * w + c = 0$   
**proof** (*cases v \* s = 0*)  
**case** *True*  
**thus** *?thesis*  
**by** (*metis eq0 u z add-0-iff*)  
**next**  
**case** *False*  
**hence**  $sl0: v \neq 0$   
**by** (*metis mult-eq-0-iff*)  
**from**  $\text{Add Neg have Minus}: \forall x \in P. \forall y \in P. x - y \in P$  **by** (*simp only: diff-conv-add-uminus*) *blast*  
**have**  $l2: (u^3 + 3 * u * v^2 * s^2 + a * u^2 + a * v^2 * s^2 + b * u + c) + (3 * u^2 * v + v^3 * s^2 + 2 * a * u * v + b * v) * s = 0$   
**using**  $\text{eq0 } z$   
**by** *algebra*  
**show** *?thesis*  
**proof** (*cases 3 \* u^2 \* v + v^3 \* s^2 + 2 \* a \* u \* v + b \* v \neq 0*)  
**case** *True*  
**hence**  $s * ((3 * u^2 * v + v^3 * s^2 + 2 * a * u * v + b * v) * (1 / (3 * u^2 * v + v^3 * s^2 + 2 * a * u * v + b * v))) = - (u^3 + 3 * u * v^2 * s^2 + a * u^2 + a * v^2 * s^2 + b * u + c) * (1 / (3 * u^2 * v + v^3 * s^2 + 2 * a * u * v + b * v))$   
**using**  $l2$   
**by** *algebra*  
**hence**  $s * ((3 * u^2 * v + v^3 * s^2 + 2 * a * u * v + b * v) / (3 * u^2 * v + v^3 * s^2 + 2 * a * u * v + b * v)) = - (u^3 + 3 * u * v^2 * s^2 + a * u^2 + a * v^2 * s^2 + b * u +$

c) \*  
      $(1 / (\mathfrak{I} * u^2 * v + v^3 * s^2 + 2 * a * u * v + b * v))$   
     by *auto*  
     hence  $s = - (u^3 + \mathfrak{I} * u * v^2 * s^2 + a * u^2 + a * v^2 * s^2 + b * u + c) *$   
      $(1 / (\mathfrak{I} * u^2 * v + v^3 * s^2 + 2 * a * u * v + b * v))$   
     by (*metis mult-1-right True divide-self-if*)  
     hence *l10*:  $s = - (u * u * u + \mathfrak{I} * u * v * v * (s * s) + a * u * u + a * v * v * (s * s) + b * u + c) *$   
      $(1 / (\mathfrak{I} * u * u * v + v * v * v * (s * s) + 2 * a * u * v + b * v))$   
     by (*simp add: algebra-simps power2-eq-square power3-eq-cube*)  
     have  $(\mathfrak{I} * u * u * v + v * v * v * (s * s) + 2 * a * u * v + b * v) \in P$   
     using *a b u v s Nats Mult Add*  
     by *auto*  
     hence *l103*:  $1 / (\mathfrak{I} * u * u * v + v * v * v * (s * s) + 2 * a * u * v + b * v) \in P$   
*P*  
     using *Inv True*  
     by *auto*  
     have  $-(u * u * u + \mathfrak{I} * u * v * v * (s * s) + a * u * u + a * v * v * (s * s) + b * u + c) \in P$   
     using *a b c u v s Mult Add Neg Minus Nats*  
     by *simp*  
     hence  $-(u * u * u + \mathfrak{I} * u * v * v * (s * s) + a * u * u + a * v * v * (s * s) + b * u + c) * (1 / (\mathfrak{I} * u * u * v + v * v * v * (s * s) + 2 * a * u * v + b * v)) \in P$   
     using *l103 Mult*  
     by *metis*  
     hence  $s \in P$   
     using *l10*  
     by *auto*  
     hence  $z \in P$   
     using *z u v Mult Add*  
     by *auto*  
     thus *?thesis*  
     using *eq0*  
     by *auto*  
 next  
   case *False*  
   have  $(-a - 2 * u)^3 + a * (-a - 2 * u)^2 + b * (-a - 2 * u) + c =$   
      $(-a - 2 * u)^3 + a * (-a - 2 * u)^2 + (-(\mathfrak{I} * u^2 + v^2 * s^2 + 2 * a * u)) *$   
      $(-a - 2 * u) + (- (u^3) - \mathfrak{I} * u * v^2 * s^2 - a * u^2 - a * v^2 * s^2 + \mathfrak{I} * u^3 + v^2 * s^2 * u + 2 * a * u^2)$   
     using *l2 False sl0*  
     by *algebra*  
   also have  $\dots = 0$   
   by (*simp add: algebra-simps power-def*)  
   finally show *?thesis*  
   by (*metis a u Add Neg diff-conv-add-uminus mult-2*)  
 qed

qed

**lemma** *cubic-root-radical-sqrt-steplemma-sqrt*:

**assumes** *Nats* [*THEN set-mp, intro*]:  $Nats \subseteq P$

**and** *Neg*:  $\forall x \in P. -x \in P$

**and** *Inv*:  $\forall x \in P. x \neq 0 \longrightarrow 1/x \in P$

**and** *Add*:  $\forall x \in P. \forall y \in P. x+y \in P$

**and** *Mult*:  $\forall x \in P. \forall y \in P. x*y \in P$

**and** *a*: ( $a \in P$ ) **and** *b*: ( $b \in P$ ) **and** *c*: ( $c \in P$ )

**and** *eq0*:  $z^3 + a * z^2 + b * z + c = 0$

**and** *u*: ( $u \in P$ )

**and** *v*: ( $v \in P$ )

**and** *s*: ( $s \in P$ )

**and** *sPositive*:  $s \geq 0$

**and** *z*:  $z = u + v * \text{sqrt } s$

**shows**  $\exists w \in P. w^3 + a * w^2 + b * w + c = 0$

**proof** –

**have**  $(\text{sqrt } s) * (\text{sqrt } s) \in P$

**by** (*metis eq-sqrt-squared s sPositive*)

**thus** *?thesis*

**using** *cubic-root-radical-sqrt-steplemma* [*of P a b c z u v sqrt s*]

*Neg Add Mult Inv a b c u v s eq0 z*

**by** *auto*

qed

**lemma** *cubic-root-radical-sqrt-lemma*:

**fixes** *e*::*expr*

**assumes** *a*:  $a \in \mathbb{Q}$  **and** *b*:  $b \in \mathbb{Q}$  **and** *c*:  $c \in \mathbb{Q}$

**and** *notEmpty*: *radicals e*  $\neq \{\}$

**and** *eq0*:  $\{\!|e|\!\}^3 + a * \{\!|e|\!\}^2 + b * \{\!|e|\!\} + c = 0$

**shows**  $\exists e1. \text{radicals } e1 \subseteq \text{radicals } e \ \& \ (\{\!|e1|\!\}^3 + a * \{\!|e1|\!\}^2 + b * \{\!|e1|\!\} + c = 0)$

**proof** –

**obtain** *r u v*

**where** *hypsruv*:  $\{\!|r|\!\} \geq 0 \ r \in \text{radicals } e$

$\{\!|e|\!\} = \{\!| \text{Addition } u \ (\text{Multiplication } v \ (\text{Sqrt } r)) |\!\}$

$\text{radicals } u \cup \text{radicals } v \cup \text{radicals } r \subseteq \text{radicals } e$

$r \notin \text{radicals } u \cup \text{radicals } v \ r \notin \text{radicals } r$

**using** *notEmpty radical-sqrt-normal-form* [*of e*]

**by** *blast*

**let**  $?E = \{x. \exists ex. (\{\!|ex|\!\} = x) \ \& \ ((\text{radicals } ex) \subseteq (\text{radicals } e)) \ \& \ (r \notin (\text{radicals } ex))\}$

**have** *NatsE*:  $Nats \subseteq ?E$

**by** (*force elim: Nats-cases intro: exI[of - Const (rat-of-nat n) for n]*)

**have** *negE*:  $\forall x \in ?E. -x \in ?E$

**using** *hypsruv* **by** (*force intro: exI[of - Negation ex for ex]*)

**have** *invE*:  $\forall x \in ?E. x \neq 0 \longrightarrow 1/x \in ?E$

**using** *hypsruv* **by** (*force intro: exI[of - Inverse ex for ex]*)

**have** *addE*:  $\forall x \in ?E. \forall y \in ?E. x+y \in ?E$

```

  using hypsrw by (force intro: exI[of - Addition ex1 ex2 for ex1 ex2])
  have multE:  $\forall x \in ?E. \forall y \in ?E. x*y \in ?E$ 
  using hypsrw by (force intro: exI[of - Multiplication ex1 ex2 for ex1 ex2])
  obtain ra rb rc
  where hypsra:  $a = \text{of-rat } ra$ 
    and hypsrb:  $b = \text{of-rat } rb$ 
    and hypsrc:  $c = \text{of-rat } rc$ 
  unfolding Rats-def
  by (metis Rats-cases a b c)
  have  $a \in ?E \ \& \ b \in ?E \ \& \ c \in ?E \ \& \ \{u\} \in ?E \ \& \ \{v\} \in ?E \ \& \ \{r\} \in ?E \ \& \ \{r\} \geq 0 \ \& \ \{e\} = \{u\} + \{v\} * \text{sqrt } \{r\}$ 
  using a b c notEmpty hypsrw hypsra hypsrb hypsrc
  by (auto intro: exI[of - Const x for x])
  with eq0 hypsrw NatsE negE invE addE multE
  cubic-root-radical-sqrt-steplemma-sqrt [of ?E a b c {e} {u} {v} {r}]
  obtain w where  $w \in ?E \ \& \ (w^3 + a * w^2 + b * w + c = 0)$ 
  by auto
  then obtain e2
  where  $\{e2\} = w \ \text{radicals } e2 \subseteq \text{radicals } e \ r \notin \text{radicals } e2$ 
     $\{e2\}^3 + a * \{e2\}^2 + b * \{e2\} + c = 0$ 
  by auto
  with hypsrw show ?thesis
  by (metis subset-iff-psubset-eq)
qed

```

**lemma** *cubic-root-radical-sqrt*:

```

  assumes abc:  $a \in \mathbb{Q} \ b \in \mathbb{Q} \ c \in \mathbb{Q}$ 
  shows  $\text{card } (\text{radicals } e) = n \implies \{e\}^3 + a * \{e\}^2 + b * \{e\} + c = 0 \implies \exists x \in \mathbb{Q}. x^3 + a * x^2 + b * x + c = 0$ 
proof (induct n arbitrary: e rule: less-induct)
  case (less n)
  thus ?case
  proof cases
  assume n:  $n = 0$ 
  thus ?thesis
  using less.prem1 radicals-empty-rational [of e] finite-radicals [of e]
  by (auto simp add: card-eq-0-iff n)
  next
  assume  $n \neq 0$ 
  hence  $\text{card } (\text{radicals } e) \neq 0$ 
  using less.prem1 by auto
  hence  $\text{radicals } e \neq \{\}$ 
  by (metis card.empty)
  hence  $\exists e1. \text{radicals } e1 \subset \text{radicals } e \ \& \ (\{e1\}^3 + a * \{e1\}^2 + b * \{e1\} + c = 0)$ 
  using abc less.prem1 cubic-root-radical-sqrt-lemma [of a b c e]
  by auto
  then obtain e1
  where hypse1:  $\text{radicals } e1 \subset \text{radicals } e \ \& \ (\{e1\}^3 + a * \{e1\}^2 + b * \{e1\} + c = 0)$ 

```

```

+ c = 0)
  by auto
  hence card (radicals e1) < card (radicals e)
  by (metis finite-radicals psubset-card-mono)
  hence card (radicals e1) < n & a : Rats & b : Rats & c : Rats & {e1}^3 +
a * {e1}^2 + b * {e1} + c = 0
  using hypse1 less.prem1 abc
  by auto
  thus ?thesis using less.hyps [of - e1]
  by auto
qed

```

Now we can prove the final result about the properties of the roots of a cubic equation.

**theorem** *cubic-root-radical-sqrt-rational:*

**assumes**  $a: a \in \mathbb{Q}$  **and**  $b: b \in \mathbb{Q}$  **and**  $c: c \in \mathbb{Q}$

**and**  $x: x \in \text{radical-sqrt}$

**and**  $x\text{-eqn}: x^3 + a * x^2 + b * x + c = 0$

**shows**  $c: \exists x \in \mathbb{Q}. x^3 + a * x^2 + b * x + c = 0$

**proof**—

**obtain**  $e\ n$

**where**  $\{e\} = x$  &  $(\{e\}^3 + a * \{e\}^2 + b * \{e\} + c = 0)$   $n = \text{card}(\text{radicals } e)$

**using**  $x\ x\text{-eqn}\ \text{radical-sqrt-correct-expr}\ [of\ x]$

**by** *auto*

**thus** ?thesis

**using** *cubic-root-radical-sqrt* [OF a b c]

**by** *auto*

qed

## 2.8 Important properties of radicals

**lemma** *sqrt-roots:*

$y^2 = x \implies x \geq 0$  &  $(\text{sqrt}(x) = y \mid \text{sqrt}(x) = -y)$

**apply** (*simp add: power-def*)

**by** (*metis abs-of-nonneg abs-of-nonpos real-sqrt-abs2 zero-le-mult-iff zero-le-square*)

**lemma** *radical-sqrt-linear-equation:*

**assumes**  $a: a \in \text{radical-sqrt}$

**and**  $b: b \in \text{radical-sqrt}$

**and**  $ab\text{NotNull}: \neg (a = 0 \ \& \ b = 0)$

**and**  $eq0: a * x + b = 0$

**shows**  $x \in \text{radical-sqrt}$

**proof** (*cases a=0*)

**case** *True*

**thus** ?thesis

**using**  $ab\text{NotNull}\ eq0$

**by** *auto*

**next**  
**case** *False*  
**hence**  $l0: a \neq 0$   
**by** *simp*  
**hence**  $x = -b/a$   
**using** *eq0* **by** (*simp add: field-simps*)  
**also have**  $\dots \in \text{radical-sqrt}$   
**using** *a b radical-sqrt.simps l0*  
**by** (*metis radical-sqrt.intros(2) radical-sqrt-rule-division*)  
**finally show** *?thesis* .  
**qed**

**lemma** *radical-sqrt-simultaneous-linear-equation:*

**assumes**  $a: a \in \text{radical-sqrt}$   
**and**  $b: b \in \text{radical-sqrt}$   
**and**  $c: c \in \text{radical-sqrt}$   
**and**  $d: d \in \text{radical-sqrt}$   
**and**  $e: e \in \text{radical-sqrt}$   
**and**  $f: f \in \text{radical-sqrt}$   
**and** *NotNull*:  $\neg (a*e - b*d = 0 \ \& \ a*f - c*d = 0 \ \& \ e*c = b*f)$   
**and** *eq0*:  $a*x + b*y = c$   
**and** *eq1*:  $d*x + e*y = f$   
**shows**  $x \in \text{radical-sqrt} \ \& \ y \in \text{radical-sqrt}$   
**proof** (*cases a\*e - b\*d = 0*)  
**case** *False*  
**hence**  $(a*e - b*d) * x = (e*c - b*f)$  **using** *eq0 eq1*  
**by** *algebra*  
**hence**  $x: x = (e*c - b*f) / (a*e - b*d)$   
**using** *False* **by** (*simp add: field-simps*)  
**hence**  $(a*e - b*d) * y = (a*f - d*c)$  **using** *eq0 eq1*  
**by** *algebra*  
**hence**  $y: y = (a*f - d*c) / (a*e - b*d)$   
**using** *False* **by** (*simp add: field-simps*)  
**have** *ae-rad*:  $(a*e - b*d) \in \text{radical-sqrt}$   
**using** *a e b d radical-sqrt.simps*  
**by** (*metis radical-sqrt.intros(5) radical-sqrt-rule-subtraction*)  
**hence**  $((e*c - b*f) / (a*e - b*d)) \in \text{radical-sqrt} \ ((a*f - d*c) / (a*e - b*d)) \in \text{radical-sqrt}$   
**by** (*metis False a b c d e f radical-sqrt.intros(5) radical-sqrt-rule-division radical-sqrt-rule-subtraction*)  
**thus** *?thesis*  
**by** (*simp add: x y*)  
**next**  
**case** *True*  
**hence**  $(a*e - b*d) * x = (e*c - b*f) \ (a*e - b*d) * y = (a*f - d*c)$  **using** *eq0 eq1*  
**by** *algebra+*  
**thus** *?thesis* **using** *NotNull True*

by simp  
qed

**lemma** *radical-sqrt-quadratic-equation*:

**assumes**  $a: a \in \text{radical-sqrt}$   
**and**  $b: b \in \text{radical-sqrt}$   
**and**  $c: c \in \text{radical-sqrt}$   
**and**  $eq0: a*x^2+b*x+c=0$   
**and**  $NotNull: \neg (a = 0 \ \& \ b = 0 \ \& \ c = 0)$   
**shows**  $x \in \text{radical-sqrt}$   
**proof** (*cases a=0*)  
**case** *True*  
**have**  $\neg (b = 0 \ \& \ c = 0)$   
**by** (*metis True NotNull*)  
**thus** *?thesis*  
**using**  $b \ c \ \text{radical-sqrt-linear-equation} \ [of \ b \ c \ x]$   
**by** (*metis True add-0 eq0 mult-zero-left*)  
**next**  
**case** *False*  
**hence**  $(2*a*x+b)^2 = 4*a*(-c)+b^2$  **using**  $eq0$   
**by** *algebra*  
**hence**  $(b^2 - 4*a*c) \geq 0 \ \& \ (\text{sqrt} ((b^2 - 4*a*c)) = (2*a*x+b) \ | \ \text{sqrt} ((b^2 - 4*a*c)) = -(2*a*x+b))$   
**using**  $\text{sqrt-roots} \ [of \ 2*a*x+b \ b^2 - 4*a*c]$   
**by** *auto*  
**hence**  $l12: b^2 - 4*a*c \geq 0 \ \& \ ((-b + \text{sqrt} (b^2 - 4*a*c)) / (2*a) = x \ | \ (-b - \text{sqrt} (b^2 - 4*a*c)) / (2*a) = x)$   
**using** *False*  
**by** *auto*  
**have**  $4*a*c \in \text{radical-sqrt}$   
**using**  $a \ c \ \text{radical-sqrt.simps}$   
**by** (*metis Rats-number-of radical-sqrt.intros(1) radical-sqrt.intros(5)*)  
**hence**  $b^2 - 4*a*c \in \text{radical-sqrt}$  **using**  $a \ b \ c$   
**by** (*metis power2-eq-square radical-sqrt.intros(5) radical-sqrt-rule-subtraction*)  
**hence**  $l22: \text{sqrt} (b^2 - 4*a*c) \in \text{radical-sqrt}$   
**using**  $l12$   
**by** (*metis radical-sqrt.intros(6)*)  
**hence**  $l23: (-b + \text{sqrt} (b^2 - 4*a*c)) / (2*a) \in \text{radical-sqrt}$   
**using**  $b \ a \ \text{False}$   
**apply** (*simp add: algebra-simps*)  
**apply** (*metis radical-sqrt-rule-division radical-sqrt-rule-subtraction double-zero-sym mult-2-right mult-2-right radical-sqrt.intros(4)*)  
**done**  
**have**  $(-b - \text{sqrt} (b^2 - 4*a*c)) / (2*a) \in \text{radical-sqrt}$   
**using**  $a \ b \ \text{False} \ l22$   
**by** (*metis div-by-0 mult-2 radical-sqrt.intros(2) radical-sqrt.intros(4) radical-sqrt-rule-division radical-sqrt-rule-subtraction*)  
**thus** *?thesis*

by (*metis l12 l23*)  
qed

**lemma** *radical-sqrt-simultaneous-linear-quadratic*:

**assumes** *a*:  $a \in \text{radical-sqrt}$   
**and** *b*:  $b \in \text{radical-sqrt}$   
**and** *c*:  $c \in \text{radical-sqrt}$   
**and** *d*:  $d \in \text{radical-sqrt}$   
**and** *e*:  $e \in \text{radical-sqrt}$   
**and** *f*:  $f \in \text{radical-sqrt}$   
**and** *NotNull*:  $\neg(d=0 \ \& \ e=0 \ \& \ f=0)$   
**and** *eq0*:  $(x-a)^2 + (y-b)^2 = c$   
**and** *eq1*:  $d*x + e*y = f$   
**shows**  $x \in \text{radical-sqrt} \ \& \ y \in \text{radical-sqrt}$   
**proof** (*cases d=0 & e=0*)  
**case** *True*  
**thus** *?thesis*  
by (*metis add-0 eq1 mult-zero-left NotNull*)  
**next**  
**case** *False*  
**hence** *l10*:  $(e^2 + d^2) * x^2 + (2*e*b*d - 2*a*e^2 - 2*d*f)*x + (a^2 * e^2 + f^2 - 2* e *b* f + b^2 * e^2 - e^2 *c) = 0$   
**using** *eq0 eq1*  
**by** *algebra*  
**have** *l12*:  $\neg(e^2 + d^2 = 0 \ \& \ 2*e*b*d - 2*a*e^2 - 2*d*f = 0 \ \& \ a^2 * e^2 + f^2 - 2* e *b* f + b^2 * e^2 - e^2 *c = 0)$   
**using** *False power-def*  
**by** *auto*  
**have** *l13*:  $(e^2 + d^2) \in \text{radical-sqrt}$   
**using** *e d*  
**by** (*metis power2-eq-square radical-sqrt.intros(4) radical-sqrt.intros(5)*)  
**have** *2*:  $2 \in \text{radical-sqrt}$   
**by** (*auto intro: radical-sqrt.intros*)  
**have** *sl1*:  $(2*e*b*d) \in \text{radical-sqrt}$   
**using** *e b d*  
**by** (*metis (lifting) mult-2 radical-sqrt.intros(4) radical-sqrt.intros(5)*)  
**hence** *sl2*:  $(- 2*a*e^2) \in \text{radical-sqrt}$  **using** *radical-sqrt.intros*  
**by** (*simp add: field-simps*) (*metis mult-2 power2-eq-square a e*)  
**have**  $(- 2*d*f) \in \text{radical-sqrt}$  **using** *radical-sqrt.intros*  
**using** *d f* **by** (*auto intro: radical-sqrt.intros simp add: power2-eq-square*)  
**hence** *sl4*:  $((2*e*b*d) + (- 2*a*e^2) + (- 2*d*f)) \in \text{radical-sqrt}$   
**using** *sl1 sl2*  
**by** (*metis radical-sqrt.intros(4)*)  
**have** *sl5*:  $2*e*b*d - 2*a*e^2 - 2*d*f = (2*e*b*d) + (- 2*a*e^2) + (- 2*d*f)$   
**by** *auto*  
**hence** *l14*:  $(2*e*b*d - 2*a*e^2 - 2*d*f) \in \text{radical-sqrt}$   
**using** *sl4*

```

    by metis
  have sl6: (a^2 * e^2) ∈ radical-sqrt
    using a e by (auto intro: radical-sqrt.intros simp add: power2-eq-square)
  have sl7: (f^2) ∈ radical-sqrt
    using f by (auto intro: radical-sqrt.intros simp add: power2-eq-square)
  have sl8: (- 2 * e * b * f) ∈ radical-sqrt
    using e b f 2 by (auto intro: radical-sqrt.intros simp add: power2-eq-square)
  have sl9: (b^2 * e^2) ∈ radical-sqrt
    using b e by (auto intro: radical-sqrt.intros simp add: power2-eq-square)
  have sl10: (- c * e^2) ∈ radical-sqrt
    using c e by (auto intro: radical-sqrt.intros simp add: power2-eq-square)
  have sl6: (a^2 * e^2 + f^2 + (- 2 * e * b * f) + b^2 * e^2 + (- c * e^2)) ∈
radical-sqrt
    using a e f b c sl6 sl7 sl8 sl9 sl10
    by (metis (hide-lams, no-types) power2-eq-square radical-sqrt.intros(4))
  have a^2 * e^2 + f^2 - 2 * e * b * f + b^2 * e^2 - e^2 * c = a^2 * e^2 + f^2
+ (- 2 * e * b * f) + b^2 * e^2 + (- c * e^2)
    by auto
  hence (a^2 * e^2 + f^2 - 2 * e * b * f + b^2 * e^2 - e^2 * c) ∈ radical-sqrt
    using sl6
    by metis
  hence x: x ∈ radical-sqrt
    using radical-sqrt-quadratic-equation [of e^2 + d^2 2*e*b*d - 2*a*e^2 -
2*d*f a^2 * e^2 + f^2 - 2 * e * b * f + b^2 * e^2 - e^2 * c] l13 l14 l12 l10
    by auto
  have l18: e*y + (d*x - f) = 0
    using eq1
    by auto
  hence y: y ∈ radical-sqrt
    using e d f x False
  proof (cases e = 0)
  case True
  hence l22: 1 * y^2 + (- 2 * b) * y + (b^2 + (x - a)^2 - c) = 0
    using eq0
    by algebra
  have l24: 1 ∈ radical-sqrt
    by (metis Rats-1 radical-sqrt.intros(1))
  have l25: (- 2 * b) ∈ radical-sqrt
    using b
    by (metis minus-mult-commute mult-2 radical-sqrt.intros(2) radical-sqrt.intros(4))
  have l26: (b^2 + (x - a)^2 - c) ∈ radical-sqrt
    using a b c x
    by (auto intro: radical-sqrt.intros radical-sqrt-rule-subtraction simp add:
power2-eq-square)
  thus ?thesis
    using radical-sqrt-quadratic-equation [of 1::real - 2 * b b^2 + (x - a)^2 -
c y] l22 l24 l25 l26
    by auto
  next

```

```

case False
hence l29:  $\neg (e=0 \ \& \ d*x-f = 0)$ 
  by simp
have  $(d*x - f) \in \text{radical-sqrt}$ 
  using d f x
  by (metis radical-sqrt.intros(5) radical-sqrt-rule-subtraction)
thus ?thesis
  using radical-sqrt-linear-equation [of e d*x - f y] e d f l18 l29
  by auto
qed
show ?thesis
  by (metis x y)
qed

```

**lemma** *radical-sqrt-simultaneous-quadratic-quadratic:*

```

assumes a:  $a \in \text{radical-sqrt}$ 
  and b:  $b \in \text{radical-sqrt}$ 
  and c:  $c \in \text{radical-sqrt}$ 
  and d:  $d \in \text{radical-sqrt}$ 
  and e:  $e \in \text{radical-sqrt}$ 
  and f:  $f \in \text{radical-sqrt}$ 
  and NotEqual:  $\neg (a = d \ \& \ b = e \ \& \ c = f)$ 
  and eq0:  $(x - a)^2 + (y - b)^2 = c$ 
  and eq1:  $(x - d)^2 + (y - e)^2 = f$ 
shows  $x \in \text{radical-sqrt} \ \& \ y \in \text{radical-sqrt}$ 
proof -
  have  $(x^2 - 2*a*x + a^2 + y^2 - 2*y*b + b^2) - (x^2 - 2*d*x + d^2 + y^2 - 2*y*e + e^2) = (c - f)$ 
  using eq0 eq1
  by (simp add: algebra-simps power-def)
  hence l4:  $(2*d - 2*a)*x + (2*e - 2*b)*y + (b^2 - e^2 + a^2 - d^2 + f - c) = 0$ 
  by algebra
  hence l6:  $\neg ((2*d - 2*a) = 0 \ \& \ (2*e - 2*b) = 0 \ \& \ (b^2 - e^2) + (a^2 - d^2) + (f - c) = 0)$ 
  using NotEqual
  by algebra
  have l7:  $(2*d - 2*a) \in \text{radical-sqrt}$ 
  by (metis a d mult-2 radical-sqrt.intros(4) radical-sqrt-rule-subtraction)
  have l8:  $(2*e - 2*b) \in \text{radical-sqrt}$ 
  by (metis b e mult-2 radical-sqrt.intros(4) radical-sqrt-rule-subtraction)
  have be-rad:  $(b^2 - e^2) \in \text{radical-sqrt}$ 
  by (metis b e power2-eq-square radical-sqrt.intros(5) radical-sqrt-rule-subtraction)
  have ad-rad:  $(a^2 - d^2) \in \text{radical-sqrt}$ 
  by (metis a d power2-eq-square radical-sqrt.intros(5) radical-sqrt-rule-subtraction)
  have  $(f - c) \in \text{radical-sqrt}$ 
  using f c
  by (metis radical-sqrt-rule-subtraction)
  hence  $-((b^2 - e^2) + (a^2 - d^2) + (f - c)) \in \text{radical-sqrt}$ 

```

```

    using radical-sqrt.intros
    by (metis be-rad ad-rad)
  thus ?thesis
    using radical-sqrt-simultaneous-linear-quadratic [of a b c (2*d - 2*a) (2*e
- 2*b) - ((b^2 - e^2) + (a^2 - d^2) + (f - c)) x y] l7 l8 l6 l4 a b c d e f
  NotEqual eq0 eq1
    by simp
qed

```

## 2.9 Important properties of geometrical points which coordinates are radicals

**lemma** *radical-sqrt-line-line-intersection*:

```

  assumes absA: (abscissa A) ∈ radical-sqrt
    and ordA: (ordinate A) ∈ radical-sqrt
    and absB: (abscissa B) ∈ radical-sqrt
    and ordB: (ordinate B) ∈ radical-sqrt
    and absC: (abscissa C) ∈ radical-sqrt
    and ordC: (ordinate C) ∈ radical-sqrt
    and absD: (abscissa D) ∈ radical-sqrt
    and ordD: (ordinate D) ∈ radical-sqrt
    and notParallel: ¬ (parallel A B C D)
    and isIntersec: is-intersection X A B C D
  shows (abscissa X) ∈ radical-sqrt & (ordinate X) ∈ radical-sqrt
proof –
  have l2: (abscissa A - abscissa X) * (ordinate A - ordinate B) = (ordinate A -
ordinate X) * (abscissa A - abscissa B) & (abscissa C - abscissa X) * (ordinate
C - ordinate D) = (ordinate C - ordinate X) * (abscissa C - abscissa D)
    using isIntersec is-intersection-def collinear-def parallel-def
    by auto
  hence l4: (- (ordinate A - ordinate B)) * abscissa X + (abscissa A - abscissa
B) * ordinate X = (- abscissa A * (ordinate A - ordinate B) + ordinate A *
(abscissa A - abscissa B))
    by (simp add: algebra-simps)
  have l6: (- (ordinate C - ordinate D)) * abscissa X + (abscissa C - abscissa
D) * ordinate X = (- abscissa C * (ordinate C - ordinate D) + ordinate C *
(abscissa C - abscissa D))
    using l2
    by (simp add: algebra-simps)
  have sl1: (- (ordinate A - ordinate B)) ∈ radical-sqrt
    by (metis ordA ordB minus-diff-eq radical-sqrt-rule-subtraction)
  have sl2: (abscissa A - abscissa B) ∈ radical-sqrt
    by (metis absA absB radical-sqrt-rule-subtraction)
  have sl3: (- abscissa A * (ordinate A - ordinate B) + ordinate A * (abscissa
A - abscissa B)) ∈ radical-sqrt
    using absA ordA ordB absB
    by (metis diff-conv-add-uminus radical-sqrt.intros(2) radical-sqrt.intros(4) radical-sqrt.intros(5))
  have sl4: (- (ordinate C - ordinate D)) ∈ radical-sqrt
    by (metis ordC ordD minus-diff-eq radical-sqrt-rule-subtraction)

```

**have** *sl5*: (*abscissa C* - *abscissa D*) ∈ *radical-sqrt*  
**by** (*metis absC absD radical-sqrt-rule-subtraction*)  
**have** *sl6*: (- *abscissa C* \* (*ordinate C* - *ordinate D*) + *ordinate C* \* (*abscissa C* - *abscissa D*)) ∈ *radical-sqrt*  
**using** *absC ordC absD ordD*  
**by** (*metis diff-conv-add-uminus radical-sqrt.intros(2) radical-sqrt.intros(4) radical-sqrt.intros(5)*)  
**have** (- (*ordinate A* - *ordinate B*)) \* (*abscissa C* - *abscissa D*) ≠ (*abscissa A* - *abscissa B*) \* (- (*ordinate C* - *ordinate D*))  
**using** *notParallel parallel-def*  
**by** (*simp add: algebra-simps*)  
**thus** ?*thesis*  
**using** *radical-sqrt-simultaneous-linear-equation* [*of* - (*ordinate A* - *ordinate B*) (*abscissa A* - *abscissa B*) - *abscissa A* \* (*ordinate A* - *ordinate B*) + *ordinate A* \* (*abscissa A* - *abscissa B*) - (*ordinate C* - *ordinate D*) *abscissa C* - *abscissa D* - *abscissa C* \* (*ordinate C* - *ordinate D*) + *ordinate C* \* (*abscissa C* - *abscissa D*) *abscissa X* *ordinate X*] *absA ordA absB ordB absC ordC absD ordD l4 sl1 sl2 sl3 sl4 sl5 sl6 l6*  
**by** *simp*  
**qed**

**lemma** *radical-sqrt-line-circle-intersection*:

**assumes** *absA*: (*abscissa A*) ∈ *radical-sqrt* **and** *ordA*: (*ordinate A*) ∈ *radical-sqrt*  
**and** *absB*: (*abscissa B*) ∈ *radical-sqrt* **and** *ordB*: (*ordinate B*) ∈ *radical-sqrt*  
**and** *absC*: (*abscissa C*) ∈ *radical-sqrt* **and** *ordC*: (*ordinate C*) ∈ *radical-sqrt*  
**and** *absD*: (*abscissa D*) ∈ *radical-sqrt* **and** *ordD*: (*ordinate D*) ∈ *radical-sqrt*  
**and** *absE*: (*abscissa E*) ∈ *radical-sqrt* **and** *ordE*: (*ordinate E*) ∈ *radical-sqrt*  
**and** *notEqual*: *A* ≠ *B*  
**and** *colin*: *collinear A X B*  
**and** *eqDist*: (*distance C X* = *distance D E*)  
**shows** (*abscissa X*) ∈ *radical-sqrt* & (*ordinate X*) ∈ *radical-sqrt*  
**proof** -  
**have** *l3*: (- (*ordinate A* - *ordinate B*)) \* *abscissa X* + (*abscissa A* - *abscissa B*) \* *ordinate X* = (- *abscissa A* \* (*ordinate A* - *ordinate B*) + *ordinate A* \* (*abscissa A* - *abscissa B*))  
**using** *colin unfolding collinear-def parallel-def*  
**by** *algebra*  
**have** *sqrt* ((*abscissa X* - *abscissa C*)<sup>2</sup> + (*ordinate X* - *ordinate C*)<sup>2</sup>) = *sqrt* ((*abscissa D* - *abscissa E*)<sup>2</sup> + (*ordinate D* - *ordinate E*)<sup>2</sup>)  
**using** *eqDist distance-def*  
**by** (*metis (no-types) minus-diff-eq point-abscissa-diff point-dist-def point-ordinate-diff power2-minus*)  
**hence** *l6*: (*abscissa X* - *abscissa C*)<sup>2</sup> + (*ordinate X* - *ordinate C*)<sup>2</sup> = (*abscissa D* - *abscissa E*)<sup>2</sup> + (*ordinate D* - *ordinate E*)<sup>2</sup>  
**by** *auto*  
**have** *l8*: ¬ (- (*ordinate A* - *ordinate B*) = 0 & (*abscissa A* - *abscissa B*) = 0 & (- *abscissa A* \* (*ordinate A* - *ordinate B*) + *ordinate A* \* (*abscissa A* - *abscissa B*)) = 0)  
**using** *notEqual unfolding point-eq-iff*

**by auto**  
**have**  $sl1: (- (ordinate A - ordinate B)) \in radical\text{-}sqrt$   
**by** (*metis ordA ordB minus-diff-eq radical-sqrt-rule-subtraction*)  
**have**  $sl2: (abscissa A - abscissa B) \in radical\text{-}sqrt$   
**by** (*metis absA absB radical-sqrt-rule-subtraction*)  
**have**  $sl3: (- abscissa A * (ordinate A - ordinate B) + ordinate A * (abscissa A - abscissa B)) \in radical\text{-}sqrt$   
**by** (*metis absA ordA absB ordB diff-conv-add-uminus radical-sqrt.intros(2) radical-sqrt.intros(4) radical-sqrt.intros(5)*)  
**have**  $(abscissa D - abscissa E)^2 + (ordinate D - ordinate E)^2 \in radical\text{-}sqrt$   
**by** (*metis power2-eq-square absD absE ordD ordE radical-sqrt-rule-subtraction radical-sqrt.intros(5) radical-sqrt.intros(4)*)  
**thus** *?thesis*  
**using** *radical-sqrt-simultaneous-linear-quadratic*  
[*of abscissa C ordinate C*  
 $(abscissa D - abscissa E)^2 + (ordinate D - ordinate E)^2$   
 $- (ordinate A - ordinate B) abscissa A - abscissa B$   
 $- abscissa A * (ordinate A - ordinate B) + ordinate A * (abscissa A - abscissa B)$   
 $abscissa X ordinate X$ ]  
*l3 absC ordC sl1 sl2 sl3 l6 l8*  
**by simp**  
**qed**

**lemma** *radical-sqrt-circle-circle-intersection:*

**assumes**  $absA: (abscissa A) \in radical\text{-}sqrt$  **and**  $ordA: (ordinate A) \in radical\text{-}sqrt$   
**and**  $absB: (abscissa B) \in radical\text{-}sqrt$  **and**  $ordB: (ordinate B) \in radical\text{-}sqrt$   
**and**  $absC: (abscissa C) \in radical\text{-}sqrt$  **and**  $ordC: (ordinate C) \in radical\text{-}sqrt$   
**and**  $absD: (abscissa D) \in radical\text{-}sqrt$  **and**  $ordD: (ordinate D) \in radical\text{-}sqrt$   
**and**  $absE: (abscissa E) \in radical\text{-}sqrt$  **and**  $ordE: (ordinate E) \in radical\text{-}sqrt$   
**and**  $absF: (abscissa F) \in radical\text{-}sqrt$  **and**  $ordF: (ordinate F) \in radical\text{-}sqrt$   
**and**  $eqDist0: distance A X = distance B C$   
**and**  $eqDist1: distance D X = distance E F$   
**and**  $notEqual: \neg (A = D \ \& \ distance B C = distance E F)$   
**shows**  $(abscissa X) \in radical\text{-}sqrt \ \& \ (ordinate X) \in radical\text{-}sqrt$   
**proof**–  
**have**  $sqrt ((abscissa X - abscissa A)^2 + (ordinate X - ordinate A)^2) = sqrt ((abscissa B - abscissa C)^2 + (ordinate B - ordinate C)^2)$   
**by** (*metis (no-types) eqDist0 distance-def minus-diff-eq point-abscissa-diff point-dist-def point-ordinate-diff power2-minus*)  
**hence**  $(sqrt ((abscissa X - abscissa A)^2 + (ordinate X - ordinate A)^2))^2 = (sqrt ((abscissa B - abscissa C)^2 + (ordinate B - ordinate C)^2))^2$   
**by** (*auto simp add: power-def*)  
**hence**  $l3: (abscissa X - abscissa A)^2 + (ordinate X - ordinate A)^2 = (abscissa B - abscissa C)^2 + (ordinate B - ordinate C)^2$   
**by auto**  
**have**  $sqrt ((abscissa X - abscissa D)^2 + (ordinate X - ordinate D)^2) =$

$\text{sqrt } ((\text{abscissa } E - \text{abscissa } F)^2 + (\text{ordinate } E - \text{ordinate } F)^2)$   
**by** (metis (no-types) eqDist1 distance-def minus-diff-eq point-abscissa-diff point-dist-def point-ordinate-diff power2-minus)  
**hence** l3bis:  $(\text{abscissa } X - \text{abscissa } D)^2 + (\text{ordinate } X - \text{ordinate } D)^2 = (\text{abscissa } E - \text{abscissa } F)^2 + (\text{ordinate } E - \text{ordinate } F)^2$   
**by** auto  
**have** l4:  $\neg (\text{abscissa } A = \text{abscissa } D \ \& \ \text{ordinate } A = \text{ordinate } D)$   
**by** (metis point-eq-iff notEqual eqDist0 eqDist1)  
**have**  $(\text{abscissa } B - \text{abscissa } C) \in \text{radical-sqrt}$   
**by** (metis absB absC radical-sqrt-rule-subtraction)  
**hence** sl1:  $((\text{abscissa } B - \text{abscissa } C)^2) \in \text{radical-sqrt}$   
**by** (auto intro: radical-sqrt.intros simp add: power2-eq-square)  
**have**  $(\text{ordinate } B - \text{ordinate } C) \in \text{radical-sqrt}$   
**by** (metis ordB ordC radical-sqrt-rule-subtraction)  
**hence**  $(\text{ordinate } B - \text{ordinate } C)^2 \in \text{radical-sqrt}$   
**by** (auto intro: radical-sqrt.intros simp add: power2-eq-square)  
**hence** sl3:  $((\text{abscissa } B - \text{abscissa } C)^2 + (\text{ordinate } B - \text{ordinate } C)^2) \in \text{radical-sqrt}$   
**by** (metis radical-sqrt.intros(4) sl1)  
**have**  $(\text{abscissa } E - \text{abscissa } F) \in \text{radical-sqrt}$   
**by** (metis absE absF radical-sqrt-rule-subtraction)  
**hence** sl4:  $((\text{abscissa } E - \text{abscissa } F)^2) \in \text{radical-sqrt}$   
**by** (auto intro: radical-sqrt.intros simp add: power2-eq-square)  
**have**  $(\text{ordinate } E - \text{ordinate } F) \in \text{radical-sqrt}$   
**by** (metis ordE ordF radical-sqrt-rule-subtraction)  
**hence**  $(\text{ordinate } E - \text{ordinate } F)^2 \in \text{radical-sqrt}$   
**by** (auto intro: radical-sqrt.intros simp add: power2-eq-square)  
**hence**  $((\text{abscissa } E - \text{abscissa } F)^2 + (\text{ordinate } E - \text{ordinate } F)^2) \in \text{radical-sqrt}$   
**by** (metis radical-sqrt.intros(4) sl4)  
**thus** ?thesis  
**using** radical-sqrt-simultaneous-quadratic-quadratic  
 $[\text{of abscissa } A \ \text{ordinate } A \ (\text{abscissa } B - \text{abscissa } C)^2 + (\text{ordinate } B - \text{ordinate } C)^2$   
 $\text{abscissa } D \ \text{ordinate } D \ (\text{abscissa } E - \text{abscissa } F)^2 + (\text{ordinate } E - \text{ordinate } F)^2$   
 $\text{abscissa } X \ \text{ordinate } X]$   
absA ordA absD ordD l3 l3bis l4 sl3  
**by** auto  
**qed**

## 2.10 Definition of the set of constructible points

**inductive-set** *constructible* :: *point set*

**where**

$(M \in \text{points} \ \& \ (\text{abscissa } M) \in \mathbb{Q} \ \& \ (\text{ordinate } M) \in \mathbb{Q}) \implies M \in \text{constructible} |$   
 $(A \in \text{constructible} \ \& \ B \in \text{constructible} \ \& \ C \in \text{constructible} \ \& \ D \in \text{constructible} |$   
 $\ \& \ \neg \text{parallel } A \ B \ C \ D \ \& \ \text{is-intersection } M \ A \ B \ C \ D) \implies M \in \text{constructible} |$   
 $(A \in \text{constructible} \ \& \ B \in \text{constructible} \ \& \ C \in \text{constructible} \ \& \ D \in \text{constructible} |$   
 $\ \& \ E \in \text{constructible} \ \& \ \neg A = B \ \& \ \text{collinear } A \ M \ B \ \& \ \text{distance } C \ M = \text{distance } D$

$E) \implies M \in \text{constructible}$   
 $(A \in \text{constructible} \wedge B \in \text{constructible} \wedge C \in \text{constructible} \wedge D \in \text{constructible}$   
 $\wedge E \in \text{constructible} \wedge F \in \text{constructible} \wedge \neg (A = D \wedge \text{distance } B C = \text{distance}$   
 $E F) \wedge \text{distance } A M = \text{distance } B C \wedge \text{distance } D M = \text{distance } E F) \implies M \in$   
 $\text{constructible}$

## 2.11 An important property about constructible points: their coordinates are radicals

**lemma** *constructible-radical-sqrt*:  
**assumes**  $h: M \in \text{constructible}$   
**shows**  $(\text{abscissa } M) \in \text{radical-sqrt} \ \& \ (\text{ordinate } M) \in \text{radical-sqrt}$   
**apply**  $(\text{rule } \text{constructible.induct})$   
**apply**  $(\text{metis } \text{assms})$   
**apply**  $(\text{metis } \text{radical-sqrt.intros}(1))$   
**apply**  $(\text{metis } \text{radical-sqrt-line-line-intersection})$   
**apply**  $(\text{metis } \text{radical-sqrt-line-circle-intersection})$   
**apply**  $(\text{metis } \text{radical-sqrt-circle-circle-intersection})$   
**done**

## 2.12 Proving the impossibility of duplicating the cube

**lemma** *impossibility-of-doubling-the-cube-lemma*:  
**assumes**  $x: x \in \text{radical-sqrt}$   
**and**  $x\text{-eqn}: x^3 = 2$   
**shows** *False*  
**proof** –  
**have**  $\exists x \in \text{Rats}. x^3 + 0 * x^2 + 0 * x + (- 2) = (0::\text{real})$   
**using**  $x \text{ x-eqn } \text{cubic-root-radical-sqrt-rational} \ [\text{of } 0 \ 0 \ - \ 2]$   
**by** *auto*  
**then obtain**  $y::\text{real}$  **where**  $\text{hypsy}: y: \text{Rats} \ \& \ y^3 = 2$   
**by**  $(\text{simp } \text{only}: \text{left-minus } \text{mult-zero-left } \text{add-0-right } \text{real-add-minus-iff}) \ \text{auto}$   
**then obtain**  $r$  **where**  $\text{hypsr}: y = \text{of-rat } r$   
**unfolding** *Rats-def*  
**by**  $(\text{metis } \text{Rats-cases } \text{hypsy})$   
**hence**  $\exists! p. r = \text{Fract } (\text{fst } p) (\text{snd } p) \ \& \ \text{snd } p > 0 \ \& \ \text{coprime } (\text{fst } p) (\text{snd } p)$   
**by**  $(\text{metis } \text{quotient-of-unique})$   
**then obtain**  $p \ q$  **where**  $\text{hypsp}: r = \text{Fract } p \ q \ q > 0 \ \text{coprime } p \ q$   
**by** *auto*  
**have**  $l6: r^3 = 2$   
**by**  $(\text{metis } (\text{lifting}) \ \text{hypsy } \text{hypsr } \text{of-rat-eq-iff } \text{of-rat-numeral-eq } \text{of-rat-power})$   
**have**  $l7: r^3 = \text{Fract } (p^3) (q^3)$   
**by**  $(\text{metis } (\text{no-types}) \ \text{hypsp } \text{mult-rat } \text{power3-eq-cube})$   
**have**  $l8: q^3 > 0 \ \& \ \text{coprime } (p^3) (q^3)$   
**by**  $(\text{metis } \text{hypsp } \text{gcd-exp } \text{power-one } \text{zero-less-power})$   
**have**  $\text{Fract } (p^3) (q^3) = 2$   
**using**  $l6 \ l7$   
**by** *auto*  
**hence**  $\text{Fract } (p^3) (q^3) = \text{Fract } 2 \ 1$

```

    by (metis rat-number-expand(3))
  hence l12:  $p^3 = q^3 * 2$  using hypsp
    by (simp add: eq-rat)
  hence even (p ^ 3)
    by (auto intro: dvdI)
  then have even p
    by auto
  then have 8 dvd p ^ 3
    by (auto simp add: dvd-def power-def)
  then have 8 dvd q ^ 3 * 2
    using l12 by auto
  then have even (q ^ 3)
    by (auto simp add: dvd-def)
  then have even q
    by auto
  with ⟨even p⟩ have 2 dvd gcd p q
    by (rule gcd-greatest)
  with ⟨coprime p q⟩ show False by simp
qed

```

**theorem** *impossibility-of-doubling-the-cube:*

$x^3 = 2 \implies (\text{Point } x \ 0) \notin \text{constructible}$

by (metis abscissa.simps constructible-radical-sqrt impossibility-of-doubling-the-cube-lemma)

## 2.13 Proving the impossibility of trisecting an angle

**lemma** *impossibility-of-trisecting-pi-over-3-lemma:*

assumes  $x \in \text{radical-sqrt}$

and  $x\text{-eqn}: x^3 - 3 * x - 1 = 0$

shows False

**proof**—

have  $\exists x \in \text{Rats}. x^3 + (-3) * x = (1::\text{real})$

using  $x\text{-eqn}$  cubic-root-radical-sqrt-rational [of 0 - 3 - 1] x

by force

then obtain  $y :: \text{real}$  where  $\text{hypsy}: y \in \text{Rats} \wedge y^3 - 3 * y = 1$  by auto

then obtain  $r$  where  $\text{hypsr}: y = \text{of-rat } r$

by (metis Rats-cases)

then obtain  $p$  where  $\text{hypsp}: r = \text{Fract } (\text{fst } p) (\text{snd } p) \ \& \ \text{snd } p > 0 \ \& \ \text{coprime } (\text{fst } p) (\text{snd } p)$

using quotient-of-unique hypsy

by blast

have  $r3eq: r^3 - 3 * r = 1$

using hypsy hypsr [[hypsubst-thin = true]]

by auto (metis (hide-lams, no-types) of-rat-1 of-rat-diff of-rat-eq-iff of-rat-mult of-rat-numeral-eq of-rat-power)

have  $l7: (\text{snd } p)^3 > 0 \ \& \ \text{coprime } ((\text{fst } p)^3) ((\text{snd } p)^3)$

by (metis hypsp gcd-exp power-one zero-less-power)

have  $r^3 = \text{Fract } ((\text{fst } p)^3) ((\text{snd } p)^3)$

by (metis (no-types) mult-rat power3-eq-cube hypsp)  
 then have  $\text{Fract } ((fst\ p)^3) ((snd\ p)^3) - (\text{Fract } (3 * (fst\ p)) (snd\ p)) = 1$   
 using r3eq hypsp  
 by (simp add: Fract-of-int-quotient)  
 then have l10:  $\text{Fract } ((fst\ p)^3) ((snd\ p)^3) - \text{Fract } (3 * (fst\ p) * (snd\ p)^2)$   
 $((snd\ p)^3) = 1$   
 using hypsp  
 by (simp add: power-def algebra-simps Fract-of-int-quotient)  
 have  $\text{Fract } ((fst\ p)^3 - (3 * (fst\ p) * (snd\ p)^2)) ((snd\ p)^3) =$   
 $\text{Fract } (((fst\ p)^3 - (3 * (fst\ p) * (snd\ p)^2)) * (snd\ p)^3) (((snd\ p)^3) * (snd\ p)^3)$   
 using l7  
 mult-rat-cancel [of (snd p)^3 ((fst p)^3 - (3 \* (fst p) \* (snd p)^2)) (snd p)^3]  
 by (auto simp add: algebra-simps)  
 also have ... =  $\text{Fract } 1\ 1$   
 by (metis l7 l10 one-rat diff-rat mult-neg-pos not-square-less-zero int-distrib(3))  
 finally have  $(fst\ p)^3 - 3 * (fst\ p) * (snd\ p)^2 = (snd\ p)^3$  using hypsp  
 by (simp add: eq-rat)  
 hence  $(fst\ p) * ((fst\ p)^2 - 3 * (snd\ p)^2) = (snd\ p)^3$   
 $(snd\ p) * ((snd\ p)^2 + 3 * (fst\ p) * (snd\ p)) = (fst\ p)^3$   
 by (auto simp add: power-def algebra-simps)  
 hence (fst p) dvd ((snd p)^3) (snd p) dvd ((fst p)^3)  
 apply (auto simp add: dvd-def)  
 apply (rule-tac x = (fst p)^2 - 3 \* (snd p)^2 in exI)  
 apply (rule-tac [2] x = (snd p)^2 + 3 \* (fst p) \* (snd p) in exI)  
 apply auto  
 done  
 moreover have coprime (fst p) ((snd p)^3) coprime ((fst p)^3) (snd p)  
 using hypsp  
 by (auto simp add: coprime-exp gcd commute)  
 ultimately have  $(fst\ p) = 1 \mid (fst\ p) = -1$  (snd p) = 1  
 using hypsp  
 by auto  
 hence  $r = 1 \mid r = -1$   
 by (metis hypsp minus-rat one-rat)  
 with r3eq show False  
 by (auto simp add: power-def algebra-simps)  
 qed

**theorem impossibility-of-trisecting-angle-pi-over-3:**

*Point (cos (pi / 9)) 0 ∉ constructible*

**proof –**

have  $\cos (3 * (pi/9)) = 4 * (\cos (pi/9))^3 - 3 * \cos (pi/9)$

using cos-treble-cos [of pi / 9]

by auto

hence  $1/2 = 4 * (\cos (pi/9))^3 - 3 * \cos (pi/9)$

by (simp add: cos-60)

**hence**  $8 * (\cos (\pi / 9))^3 - 6 * \cos (\pi / 9) - 1 = 0$   
**by** (*simp add: algebra-simps*)  
**hence**  $(2 * \cos (\pi / 9))^3 - 3 * (2 * \cos (\pi / 9)) - 1 = 0$   
**by** (*simp add: algebra-simps power-def*)  
**hence**  $\neg (2 * \cos (\pi / 9)) \in \text{radical-sqrt}$   
**by** (*metis impossibility-of-trisecting-pi-over-3-lemma*)  
**hence**  $\neg (\cos (\pi / 9)) \in \text{radical-sqrt}$   
**by** (*metis divide-self-if mult-zero-right one-add-one radical-sqrt.intros(4) radical-sqrt.intros(5)*  
*radical-sqrt-rule-division*)  
**thus** *?thesis*  
**by** (*metis abscissa.simps constructible-radical-sqrt*)  
**qed**  
**end**

## References

- [Car81] J. C. Carrega. *Théorie des corps : la règle et le compas*. Hermann, 1981.