Abstract

The scope of information flow control via static type systems is in principle much broader than information flow security, since this concept promises to cope with information flow correctness in full generality. Such a correctness policy can be expressed by extending the notion of a single stateless level-based interference relation applying throughout a program – addressed by the static security type systems described by Volpano, Smith, and Irvine, and formalized in Nipkow and Klein’s book on formal programming language semantics (in the version of February 2023) – to that of a stateful interference function mapping program states to (generally) intransitive interference relations.

This paper studies information flow control via stateful intransitive noninterference. First, the notion of termination-sensitive information flow security with respect to a level-based interference relation is generalized to that of termination-sensitive information flow correctness with respect to such a correctness policy. Then, a static type system is specified and is proven to be capable of enforcing such policies. Finally, the information flow correctness notion and the static type system introduced here are proven to degenerate to the counterparts formalized in Nipkow and Klein’s book in case of a stateless level-based information flow correctness policy. Although the operational semantics of the didactic programming language IMP employed in the book is used for this purpose, the introduced concepts apply to larger, real-world imperative programming languages as well.

Contents

1 Underlying concepts and formal definitions 2
1.1 Global context definitions 6
1.2 Local context definitions 7
In a passage of his book *Clean Architecture: A Craftsman’s Guide to Software Structure and Design* (Prentice Hall, 2017), Robert C. Martin defines a computer program as “a detailed description of the policy by which inputs are transformed into outputs”, remarking that “indeed, at its core, that’s all a computer program actually is”. Accordingly, the scope of information flow control via static type systems is in principle much broader than language-based information flow security, since this concept promises to cope with information flow correctness in full generality.

This is already shown by a basic program implementing the Euclidean algorithm, in Donald Knuth’s words “the granddaddy of all algorithms, because it is the oldest nontrivial algorithm that has survived to the present day” (from *The Art of Computer Programming, Volume 2: Seminumerical Algorithms*, third edition, Addison-Wesley, 1997). Here below is a sample such C program, where variables a and b initially contain two positive integers and a will finally contain the output, namely the greatest common divisor of those integers.
```plaintext
do  
{  
r = a % b;  
a = b;  
b = r;  
} while (b);
```

Even in a so basic program, information is not allowed to indistinctly flow from any variable to any other one, on pain of the program being incorrect. If an incautious programmer swapped `a` for `b` in the assignment at line 4, the greatest common divisor output for any two inputs `a` and `b` would invariably match `a`, whereas swapping the sides of the assignment at line 5 would give rise to an endless loop. Indeed, despite the marked differences in the resulting program behavior, both of these potential errors originate in information flowing between variables along paths other than the demanded ones. A sound implementation of the Euclidean algorithm does not provide for any information flow from `a` to `b`, or from `b` to `r`.

The static security type systems addressed in [11], [10], and [7] restrict the information flows occurring in a program based on a mapping of each of its variables to a `domain` along with an `interference relation` between such domains, including any pair of domains such that the former may interfere with the latter. Accordingly, if function `dom` stands for such a mapping, and infix notation `u ⇝ v` denotes the inclusion of any pair of domains `(u, v)` in such a relation (both notations are borrowed from [9]), the above errors would be detected at compile time by a static type system enforcing an interference relation such that:

- `dom a ⇝ dom r, dom b ⇝ dom r` (line 3),
- `dom b ⇝ dom a` (line 4),
- `dom r ⇝ dom b` (line 5),

and ruling out any other pair of distinct domains. Such an interference relation would also embrace the implicit information flow from `b` to the other two variables arising from the loop’s termination condition (line 6).

Remarkably, as `dom a ⇝ dom r` and `dom r ⇝ dom b` but ¬ `dom a ⇝ dom b`, this interference relation turns out to be intransitive. Therefore, unlike the security static type systems studied in [11] and [10], which deal with `level-based`, and then `transitive`, interference relations, a static type system aimed at enforcing information flow correctness in full generality must be capable of dealing with `intransitive` interference relations as well. This should come as no surprise, since [9] shows that this is the general
case already for interference relations expressing information flow security policies.

But the bar can be raised further. Considering the above program again, the information flows needed for its operation, as listed above, need not be allowed throughout the program. Indeed, information needs to flow from \( a \) and \( b \) to \( r \) at line 3, from \( b \) to \( a \) at line 4, from \( r \) to \( b \) at line 5, and then (implicitly) from \( b \) to the other two variables at line 6. Based on this observation, error detection at compile time can be made finer-grained by rewriting the program as follows, where \( i \) is a further integer variable introduced for this purpose.

```plaintext
1 do
2 {  
3     i = 0;
4     r = a \% b;
5     i = 1;
6     a = b;
7     i = 2;
8     b = r;
9     i = 3;
10    } while (b);
```

In this program, \( i \) serves as a state variable whose value in every execution step can be determined already at compile time. Since a distinct set of information flows is allowed for each of its values, a finer-grained information flow correctness policy for this program can be expressed by extending the concept of a single, *stateless* interference relation applying throughout the program to that of a *stateful interference function* mapping program states to interference relations (in this case, according to the value of \( i \)). As a result of this extension, for each program state, a distinct interference relation – that is, the one to which the applied interference function maps that state – can be enforced at compile time by a suitable static type system.

If mixfix notation \( s: u \rightsquigarrow v \) denotes the inclusion of any pair of domains \((u, v)\) in the interference relation associated with any state \( s \), a finer-grained information flow correctness policy for this program can then be expressed as an interference function such that:

- \( s: \text{dom } a \rightsquigarrow \text{dom } r \), \( s: \text{dom } b \rightsquigarrow \text{dom } r \) for any \( s \) where \( i = 0 \) (line 4),
- \( s: \text{dom } b \rightsquigarrow \text{dom } a \) for any \( s \) where \( i = 1 \) (line 6),
- \( s: \text{dom } r \rightsquigarrow \text{dom } b \) for any \( s \) where \( i = 2 \) (line 8),
- \( s: \text{dom } b \rightsquigarrow \text{dom } a, s: \text{dom } b \rightsquigarrow \text{dom } r, s: \text{dom } b \rightsquigarrow \text{dom } i \) for any \( s \) where \( i = 3 \) (line 10),
and ruling out any other pair of distinct domains in any state.

Notably, to enforce such an interference function, a static type system would not need to keep track of the full program state in every program execution step (which would be unfeasible, as the values of a, b, and r cannot be determined at compile time), but only of the values of some specified state variables (in this case, of i alone). Accordingly, term state variable will henceforth refer to any program variable whose value may affect that of the interference function expressing the information flow correctness policy in force, namely the interference relation to be applied.

Needless to say, there would be something artificial about the introduction of such a state variable into the above sample program, since it is indeed so basic as not to provide for a state machine on its own, so that i would be aimed exclusively at enabling the enforcement of such an information flow correctness policy. Yet, real-world imperative programs, for which error detection at compile time is truly meaningful, do typically provide for state machines such that only a subset of all the potential information flows is allowed in each state; and even for those which do not, the addition of some ad hoc state variable to enforce such a policy could likely be an acceptable trade-off.

Accordingly, the goal of this paper is to study information flow control via stateful intransitive noninterference. First, the notion of termination-sensitive information flow security with respect to a level-based interference relation, as defined in [7], section 9.2.6, is generalized to that of termination-sensitive information flow correctness with respect to a stateful interference function having (generally) intransitive interference relations as values. Then, a static type system is specified and is proven to be capable of enforcing such information flow correctness policies. Finally, the information flow correctness notion and the static type system introduced here are proven to degenerate to the counterparts addressed in [7], section 9.2.6, in case of a stateless level-based information flow correctness policy.

Although the operational semantics of the didactic imperative programming language IMP employed in [7] is used for this purpose, the introduced concepts are applicable to larger, real-world imperative programming languages as well, by just affording the additional type system complexity arising from richer language constructs. Accordingly, the informal explanations accompanying formal content in what follows will keep making use of sample C code snippets.

For further information about the formal definitions and proofs contained in this paper, see Isabelle documentation, particularly [8], [4], [2], [3], and [1].
1.1 Global context definitions

declare [[syntax-ambiguity-warning = false]]

datatype com-flow =
   Assign vname aexp (- ::= - [1000, 61] 70) |
   Observe vname set (\{\} [61] 70)

type-synonym flow = com-flow list

type-synonym config = state set \times vname set

type-synonym scope = config set \times bool

abbreviation eq-states :: state \Rightarrow state \Rightarrow vname set \Rightarrow bool
   ((\sim \equiv (\subseteq \sim)) [51, 51] 50) where
   s = t (\subseteq X) \equiv \forall x \in X. s x = t x

abbreviation univ-states :: state set \Rightarrow vname set \Rightarrow state set
   ((\Univ \equiv (\subseteq \equiv)) [51, 75] 75) where
   \Univ A (\subseteq X) \equiv \{s. \exists t \in A. s = t (\subseteq X)\}

abbreviation univ-vars-if :: state set \Rightarrow vname set \Rightarrow vname set
   ((\Univ?? \equiv -) [51, 75] 75) where
   \Univ?? A X \equiv if A = {} then UNIV else X

abbreviation tl2 xs \equiv tl (tl xs)

fun run-flow :: flow \Rightarrow state \Rightarrow state where
   run-flow (x ::= a \# cs) s = run-flow cs (s(x := aval a s)) |
   run-flow (- \# cs) s = run-flow cs s |
   run-flow - s = s

primrec no-upd :: flow \Rightarrow vname \Rightarrow bool where
   no-upd (c \# cs) x =
      ((case c of y ::= - \Rightarrow y \neq x | - \Rightarrow True) \wedge no-upd cs x) |
   no-upd [] - = True

primrec avars :: aexp \Rightarrow vname set where
   avars (N i) = \{} \} |
   avars (V x) = \{x\} |
   avars (Plus a1 a2) = avars a1 \cup avars a2

primrec bvars :: bexp \Rightarrow vname set where
   bvars (Bc v) = \{} \} |
   bvars (Not b) = bvars b |
   bvars (And b1 b2) = bvars b1 \cup bvars b2 |
   bvars (Less a1 a2) = avars a1 \cup avars a2
fun flow-aux :: com list ⇒ flow where
flow-aux ((x ::= a) # cs) = (x ::= a) # flow-aux cs |
flow-aux ((IF b THEN - ELSE -) # cs) = (bvars b) # flow-aux cs |
flow-aux ((c;; -) # cs) = flow-aux (c # cs) |
flow-aux [] = []
definition flow :: (com × state) list ⇒ flow where
flow cfs = flow-aux (map fst cfs)

function small-steps ::
com × state ⇒ (com × state) list ⇒ com × state ⇒ bool
((→ s { - }) [51, 51] 55)
where
cf → s{[[]]} cf' = (cf = cf') |
cf → s{cfs @ [cf']} cf'' = (cf → s{cfs} cf' ∧ cf' → cf'')
⟨proof⟩
termination ⟨proof⟩
lemmas small-steps-induct = small-steps.induct [split-format(complete)]

1.2 Local context definitions

In what follows, stateful intransitive noninterference will be formalized within
the local context defined by means of a locale [1], named noninterf. Later
on, this will enable to prove the degeneracy of the following definitions to
the stateless level-based counterparts addressed in [11], [10], and [7], and
formalized in [5] and [6], via a suitable locale interpretation.
Locale noninterf contains three parameters, as follows.

- A stateful interference function interf mapping program states to in-
  terference predicates of two domains, intended to be true just in case
  the former domain is allowed to interfere with the latter.

- A function dom mapping program variables to their respective do-
  mains.

- A set state collecting all state variables.

As the type of the domains is modeled using a type variable, it may be
assigned arbitrarily by any locale interpretation, which will enable to set it
to nat upon proving degeneracy. Moreover, the above mixfix notation s: u
⇝ v is adopted to express the fact that any two domains u, v satisfy the
interference predicate interf s associated with any state s, namely the fact
that u is allowed to interfere with v in state s.
Locale \textit{noninterf} also contains an assumption, named \textit{interf-state}, which serves the purpose of supplying parameter \textit{state} with its intended semantics, namely standing for the set of all state variables. The assumption is that function \textit{interf} maps any two program states agreeing on the values of all the variables in set \textit{state} to the same interference predicate. Correspondingly, any locale interpretation instantiating parameter \textit{state} as the empty set must instantiate parameter \textit{interf} as a function mapping any two program states, even if differing in the values of all variables, to the same interference predicate – namely, as a constant function. Hence, any such locale interpretation refers to a single, stateless interference predicate applying throughout the program. Unsurprisingly, this is the way how those parameters will be instantiated upon proving degeneracy.

The one just mentioned is the only locale assumption. Particularly, the following formalization does not rely upon the assumption that the interference predicates returned by function \textit{interf} be reflexive, although this will be the case for any meaningful real-world information flow correctness policy.

\texttt{locale noninterf =}
\begin{verbatim}
fixes
  interf :: state ⇒ 'd ⇒ 'd ⇒ bool
  ((::: ∼ ∼) [51, 51, 51] 50) and
  dom :: vname ⇒ 'd and
  state :: vname set
assumes
  interf-state: s = t (⊆ state) ⦨ interp s = interf t
\end{verbatim}

\texttt{context noninterf}
\begin{verbatim}
begin
\end{verbatim}

Locale parameters \textit{interf} and \textit{dom} are provided with their intended semantics by the definitions of functions \textit{sources} and \textit{correct}, which are formalized here below based on the following underlying ideas.

As long as a stateless transitive interference relation between domains is considered, the condition for the correctness of the value of a variable resulting from a full or partial program execution need not take into account the execution flow producing it, but rather the initial program state only. In fact, this is what happens with the stateless level-based correctness condition addressed in [11], [10], and [7]: the resulting value of a variable of level \( l \) is correct if the same value is produced for any initial state agreeing with the given one on the value of every variable of level not higher than \( l \).

Things are so simple because, for any variables \( x, y, \) and \( z \), if \( \text{dom } z \rightsquigarrow \text{dom } y \) and \( \text{dom } y \rightsquigarrow \text{dom } x \), transitivity entails \( \text{dom } z \rightsquigarrow \text{dom } x \), and these interference relationships hold statelessly. Therefore, \( z \) may be counted among
the variables whose initial values are allowed to affect \( x \) independently of whether some intermediate value of \( y \) may affect \( x \) within the actual execution flow.

Unfortunately, switching to stateful intransitive interference relations puts an end to that happy circumstance – indeed, even statefulness or intransitivity alone would suffice for this sad ending. In this context, deciding about the correctness of the resulting value of a variable \( x \) still demands the detection of the variables whose initial values are allowed to interfere with \( x \), but the execution flow leading from the initial program state to the resulting one needs to be considered to perform such detection.

This is precisely the task of function \( \text{sources} \), so named after its finite state machine counterpart defined in [9]. It takes as inputs an execution flow \( cs \), an initial program state \( s \), and a variable \( x \), and outputs the set of the variables whose values in \( s \) are allowed to affect the value of \( x \) in the state \( s' \) into which \( cs \) turns \( s \), according to \( cs \) as well as to the information flow correctness policy expressed by parameters \( \text{interf} \) and \( \text{dom} \).

In more detail, execution flows are modeled as lists comprising items of two possible kinds, namely an assignment of the value of an arithmetic expression \( a \) to a variable \( z \) or else an observation of the values of the variables in a set \( X \), denoted through notations \( z := a \) (same as with assignment commands) and \( \langle X \rangle \) and keeping track of explicit and implicit information flows, respectively. Particularly, item \( \langle X \rangle \) refers to the act of observing the values of the variables in \( X \) leaving the program state unaltered. During the execution of an IMP program, this happens upon any evaluation of a boolean expression containing all and only the variables in \( X \).

Function \( \text{sources} \) is defined along with an auxiliary function \( \text{sources-aux} \) by means of mutual recursion. Based on this definition, \( \text{sources} cs s x \) contains a variable \( y \) if there exist a descending sequence of left sublists \( cs_{n+1}, cs_n \atop [c_n], ..., cs_1 \atop [c_1] \) of \( cs \) and a sequence of variables \( y_{n+1}, ..., y_1 \), where \( n \geq 1 \), \( cs_{n+1} = cs, y_{n+1} = x \), and \( y_1 = y \), satisfying the following conditions.

- For each positive integer \( i \leq n \), \( c_i \) is an assignment \( y_{i+1} := a_i \) where:
  - \( y_i \in \text{avars } a_i \),
  - \( \text{run-flow } cs_i s: \text{dom } y_i \leadsto \text{dom } y_{i+1} \), and
  - the right sublist of \( cs_{i+1} \) complementary to \( cs_i \atop [c_i] \) does not comprise any assignment to variable \( y_{i+1} \) (as assignment \( c_i \) would otherwise be irrelevant),

or else an observation \( \langle X_i \rangle \) where:

- \( y_i \in X_i \) and
- \( \text{run-flow } cs_i s: \text{dom } y_i \leadsto \text{dom } y_{i+1} \).
• $cs_1$ does not comprise any assignment to variable $y$.

In addition, $sources~cs~s~x$ contains variable $x$ also if $cs$ does not comprise any assignment to variable $x$.

**function**

$sources :: flow \Rightarrow state \Rightarrow vname \Rightarrow vname\ set$ and

$sources-aux :: flow \Rightarrow state \Rightarrow vname \Rightarrow vname\ set$ where

$sources\ (cs@\ [c])\ s\ x = (case\ c\ of$

$z ::= a \Rightarrow if\ z = x$

then $sources-aux\ cs\ s\ x \cup \{sources\ cs\ s\ y\ |\ y\}$

$run-flow\ cs\ s:\ dom\ y \rightsquigarrow dom\ x \land y \in\ avars\ a\}$

else $sources\ cs\ s\ x$ |

$(X) \Rightarrow$

$sources\ cs\ s\ x \cup \{sources\ cs\ s\ y\ |\ y\}$

$run-flow\ cs\ s:\ dom\ y \rightsquigarrow dom\ x \land y \in\ X\}\}$

$sources\ []\ -\ x = \{x\}$ |

$sources-aux\ (cs@\ [c])\ s\ x = (case\ c\ of$

- ::= - \Rightarrow

$sources-aux\ cs\ s\ x$ |

$(X) \Rightarrow$

$sources-aux\ cs\ s\ x \cup \{sources\ cs\ s\ y\ |\ y\}$

$run-flow\ cs\ s:\ dom\ y \rightsquigarrow dom\ x \land y \in\ X\}\}$

$sources-aux\ []\ -\ - = \{\}$

⟨proof⟩

**termination** ⟨proof⟩

**lemmas** $sources\-induct = sources\-sources-aux\-induct$

Predicate correct takes as inputs a program $c$, a set of program states $A$, and a set of variables $X$. Its truth value equals that of the following termination-sensitive information flow correctness condition: for any state $s$ agreeing with a state in $A$ on the values of the state variables in $X$, if the small-step program semantics turns configuration $(c, s)$ into configuration $(c_1, s_1)$, and $(c_1, s_1)$ into configuration $(c_2, s_2)$, then for any state $t_1$ agreeing with $s_1$ on the values of the variables in $sources\ cs\ s_1\ x$, where $cs$ is the execution flow leading from $(c_1, s_1)$ to $(c_2, s_2)$, the small-step semantics turns $(c_1, t_1)$ into some configuration $(c_2', t_2)$ such that:

• $c_2' = SKIP$ (namely, $(c_2', t_2)$ is a final configuration) just in case $c_2 = SKIP$, and
• the value of variable \( x \) in state \( t_2 \) is the same as in state \( s_2 \).

Here below are some comments about this definition.

• As \( \textit{sources} cs \ s_1 \) \( x \) is the set of the variables whose values in \( s_1 \) are allowed to affect the value of \( x \) in \( s_2 \), this definition requires any state \( t_1 \) indistinguishable from \( s_1 \) in the values of those variables to produce a state where variable \( x \) has the same value as in \( s_2 \) in the continuation of program execution.

• Configuration \((c_2', t_2)\) must be the same one for \( \textit{any} \) variable \( x \) such that \( s_1 \) and \( t_1 \) agree on the values of any variable in \( \textit{sources} cs \ s_1 \) \( x \). Otherwise, even if states \( s_2 \) and \( t_2 \) agreed on the value of \( x \), they could be distinguished all the same based on a discrepancy between the respective values of some other variable. Likewise, if state \( t_2 \) alone had to be the same for any such \( x \), while command \( c_2' \) were allowed to vary, state \( t_1 \) could be distinguished from \( s_1 \) based on the continuation of program execution. This is the reason why the universal quantification over \( x \) is nested within the existential quantification over both \( c_2' \) and \( t_2 \).

• The state machine for a program typically provides for a set of initial states from which its execution is intended to start. In any such case, information flow correctness need not be assessed for arbitrary initial states, but just for those complying with the settled tuples of initial values for state variables. The values of any other variables do not matter, as they do not affect function \( \textit{interf}'s \) ones. This is the motivation for parameter \( A \), which then needs to contain just one state for each of such tuples, while parameter \( X \) enables to exclude the state variables, if any, whose initial values are not settled.

• If locale parameter \( \textit{state} \) matches the empty set, \( s \) will be any state agreeing with some state in \( A \) on the value of possibly even no variable at all, that is, a fully arbitrary state provided that \( A \) is nonempty. This makes \( s \) range over all possible states, as required for establishing the degeneracy of the present definition to the stateless level-based counterpart addressed in [7], section 9.2.6.

Why express information flow correctness in terms of the small-step program semantics, instead of resorting to the big-step one as happens with the stateless level-based correctness condition in [7], section 9.2.6? The answer is provided by the following sample C programs, where \( i \) is a state variable.

1
\begin{verbatim}
y = i;
\end{verbatim}

2
\begin{verbatim}
i = (i) ? 1 : 0;
\end{verbatim}

11
Let \( i \) be allowed to interfere with \( x \) just in case \( i \) matches 0 or 1, and \( y \) be never allowed to do so. If \( s_1 \) were constrained to be the initial state, for both programs \( i \) would be included among the variables on which \( t_1 \) needs to agree with \( s_1 \) in order to be indistinguishable from \( s_1 \) in the value of \( x \) resulting from the final assignment. Thus, both programs would fail to be labeled as wrong ones, although in both of them the information flow blatantly bypasses the sanitization of the initial value of \( i \), respectively due to an illegal explicit flow and an illegal implicit flow. On the contrary, the present information flow correctness definition detects any such illegal information flow by checking every partial program execution on its own.

**abbreviation** ok-flow :: com \( \Rightarrow \) com \( \Rightarrow \) state \( \Rightarrow \) state \( \Rightarrow \) flow \( \Rightarrow \) bool where

\[
\text{ok-flow } c_1 c_2 s_1 s_2 cfs \equiv \\
\forall t_1, \exists c_2', t_2, \forall x. \\
\quad s_1 = t_1 \left( \subseteq \text{sources } cs s_1 x \right) \rightarrow \\
\quad (c_1, t_1) \rightarrow (c_2', t_2) \land (c_2 = \text{SKIP}) = (c_2' = \text{SKIP}) \land s_2 x = t_2 x
\]

**definition** correct :: com \( \Rightarrow \) state set \( \Rightarrow \) vname set \( \Rightarrow \) bool where

\[
\text{correct } c A X \equiv \\
\forall s \in \text{Univ } A \left( \subseteq \text{state } \cap X \right). \forall c_1 c_2 s_1 s_2 cfs. \\
\quad (c, s) \rightarrow (c_1, s_1) \land (c_1, s_1) \rightarrow \{cfs\} (c_2, s_2) \rightarrow \\
\quad \text{ok-flow } c_1 c_2 s_1 s_2 \left( \text{flow } cfs \right)
\]

**abbreviation** interf-set :: state set \( \Rightarrow \) \('d set \Rightarrow \)'d set \( \Rightarrow \) bool \\
\((\cdot: \rightarrow \cdot) [51, 51, 51] 50) where

\( A: U \rightarrow W \equiv \forall s \in A. \forall u \in U. \forall w \in W. \ s: u \rightsquigarrow w \)

**abbreviation** ok-flow-aux :: 
config set \( \Rightarrow \) com \( \Rightarrow \) com \( \Rightarrow \) state \( \Rightarrow \) flow \( \Rightarrow \) bool where

\[
\text{ok-flow-aux } U c_1 c_2 s_1 s_2 cs \equiv \\
\forall t_1, \exists c_2', t_2, \forall x. \\
\quad (s_1 = t_1 \left( \subseteq \text{sources-aux } cs s_1 x \right) \rightarrow \\
\quad (c_1, t_1) \rightarrow (c_2', t_2) \land (c_2 = \text{SKIP}) = (c_2' = \text{SKIP}) \land \\
\quad s_2 x = t_2 x)
\]
\[ (\forall x. (\exists p \in U. \text{case } p \text{ of } (B, Y) \Rightarrow \\
\exists s \in B. \exists y \in Y. \neg s: \text{dom } y \not\leadsto \text{dom } x) \rightarrow \text{no-upd } cs \ x) \]

The next step is defining a static type system guaranteeing that well-typed programs satisfy this information flow correctness criterion. Whenever defining a function, and the pursued type system is obviously no exception, the primary question that one has to answer is: which inputs and outputs should it provide for? The type system formalized in [6] simply makes a pass/fail decision on an input program, based on an input security level, and outputs the verdict as a boolean value. Is this still enough in the present case? The answer can be found by considering again the above C program that computes the greatest common divisor of two positive integers \( a \), \( b \) using a state variable \( i \), along with its associated stateful interference function. For the reader’s convenience, the program is reported here below.

```c
1 do
2 { i = 0;
3   r = a % b;
4   i = 1;
5   a = b;
6   i = 2;
7   b = r;
8   i = 3;
9 } while (b);
```

As \( s: \text{dom } a \not\leadsto \text{dom } r \) only for a state \( s \) where \( i = 0 \), the type system cannot determine that the assignment \( r = a \% b \) at line 4 is well-typed without knowing that \( i = 0 \) whenever that step is executed. Consequently, upon checking the assignment \( i = 0 \) at line 3, the type system must output information indicating that \( i = 0 \) as a result of its execution. This information will then be input to the type system when it is recursively invoked to check line 4, so as to enable the well-typedness of the next assignment to be ascertained.

Therefore, in addition to the program under scrutiny, the type system needs to take a set of program states as input, and as long as the program is well-typed, the output must include a set of states covering any change to the values of the state variables possibly triggered by the input program. In other words, the type system has to simulate the execution of the input program at compile time as regards the values of its state variables. In the following formalization, this results in making the type system take an input of type \( \text{state set} \) and output a value of the same type. Yet, since state variables alone are relevant, a real-world implementation of the type system
would not need to work with full state values, but just with tuples of state variables’ values. Is the input/output of a set of program states sufficient to keep track of the possible values of the state variables at each execution step? Here below is a sample C program helping find an answer, which determines the minimum of two integers $a, b$ and assigns it to variable $a$ using a state variable $i$.

```c
1  i = (a > b) ? 1 : 0;
2  if (i > 0)
3  {
4    a = b;
5  }
```

Assuming that the initial value of $i$ is 0, the information flow correctness policy for this program will be such that:

- $s: \text{dom } a \rightsquigarrow \text{dom } i$, $s: \text{dom } b \rightsquigarrow \text{dom } i$ for any program state $s$ where $i = 0$ (line 1),
- $s: \text{dom } i \rightsquigarrow \text{dom } a$ for any $s$ where $i = 0$ or $i = 1$ (line 2, more on this later),
- $s: \text{dom } b \rightsquigarrow \text{dom } a$ for any $s$ where $i = 1$ (line 4),

ruling out any other pair of distinct domains in any state.

So far, everything has gone smoothly. However, what happens if the program is changed as follows?

```c
1  i = a - b;
2  if (i > 0)
3  {
4    a = b;
5  }
```

Upon simulating the execution of the former program, the type system can determine the set $\{0, 1\}$ of the possible values of variable $i$ arising from the conditional assignment $i = (a > b) ? 1 : 0$ at line 1. On the contrary, in the case of the latter program, the possible values of $i$ after the assignment $i = a - b$ at line 1 must be marked as being indeterminate, since they depend on the initial values of variables $a$ and $b$, which are unknown at compile time. Hence, the type system needs to provide for an additional input/output parameter of type $\text{vname set}$, whose input and output values shall collect
the variables whose possible values before and after the execution of the input program are *determinate*.

The correctness of the simulation of program execution by the type system can be expressed as the following condition. Suppose that the type system outputs a *state set* $A'$ and a *vname set* $X'$ when it is input a program $c$, a *state set* $A$, and a *vname set* $X$. Then, for any state $s$ agreeing with some state in $A$ on the value of every state variable in $X$, if $(c, s) \Rightarrow s'$, $s'$ must agree with some state in $A'$ on the value of every state variable in $X'$. This can be summarized by saying that the type system must *overapproximate* program semantics, since any algorithm simulating program execution cannot but be imprecise (see [7], incipit of chapter 13).

In turn, if the outputs for $c$, $A'$, $X'$ are $A''$, $X''$ and $(c, s) \Rightarrow s''$, $s''$ must agree with some state in $A''$ on the value of every state variable in $X''$. But if $c$ is a loop and $(c, s) \Rightarrow s'$, then $(c, s') \Rightarrow s''$ just in case $s' = s''$, so that the type system is guaranteed to overapproximate the semantics of $c$ only if states consistent with $A'$, $X'$ are also consistent with $A''$, $X''$ and vice versa. Thus, the type system needs to be *idempotent* if $c$ is a loop, that is, it must be such that $A' = A''$ and $X' = X''$ in this case. Since idempotence is not required for control structures other than loops, the main type system $ctyping^2$ formalized in what follows will delegate the simulation of the execution of loop bodies to an auxiliary, idempotent type system $ctyping^1$.

This type system keeps track of the program state updates possibly occurring in its input program using sets of lists of functions of type $\text{vname} \Rightarrow \text{val option option}$. Command $\text{SKIP}$ is mapped to a singleton made of the empty list, as no state update takes place. An assignment to a variable $x$ is mapped to a singleton made of a list comprising a single function, whose value is $\text{Some (Some } i \text{)}$ or $\text{Some None}$ for $x$ if it is a state variable and the right-hand side is a constant $N i$ or a non-constant expression, respectively, and $\text{None}$ otherwise. That is, $\text{None}$ stands for *unchanged/non-state variable* (remember, only state variable updates need to be tracked), whereas $\text{Some None}$ stands for *indeterminate variable*, since the value of a non-constant expression in a loop iteration (remember, $ctyping^1$ is meant for simulating the execution of loop bodies) is in general unknown at compile time.

At first glance, a conditional statement could simply be mapped to the union of the sets tracking the program state updates possibly occurring in its branches. However, things are not so simple, as shown by the sample C loop here below, which has a conditional statement as its body.

```c
for (i = 0; i < 2; i++)
{
    if (n % 2)
    {
```
If the initial value of the integer variable $n$ is even, the final values of variables $a$, $b$, and $c$ will be 1, 1, 2, whereas if the initial value of $n$ is odd, the final values of the aforesaid variables will be 2, 1, 2. Assuming that their initial value is 0, the potential final values tracked by considering each branch individually are 1, 1, 0 and 2, 0, 2 instead. These are exactly the values generated by a single loop iteration; if they are fed back into the loop body along with the increased value of $n$, the actual final values listed above are produced.

As a result, a mere union of the sets tracking the program state updates possibly occurring in each branch would not be enough for the type system to be idempotent. The solution is to rather construct every possible alternate concatenation without repetitions of the lists contained in each set, which is referred to as merging those sets in the following formalization. In fact, alternating the state updates performed by each branch in the previous example produces the actual final values listed above. Since the latest occurrence of a state update makes any previous occurrence irrelevant for the final state, repetitions need not be taken into account, which ensures the finiteness of the construction provided that the sets being merged are finite. In the special case where the boolean condition can be evaluated at compile time, considering the picked branch alone is of course enough.

Another case trickier than what one could expect at first glance is that of sequential composition. This is shown by the sample C loop here below, whose body consists of the sequential composition of some assignments with a conditional statement.
\begin{verbatim}
c = 2;
n++;
}
else
{
  b = 3;
  d = 3;
  n++;
}
\end{verbatim}

If the initial value of the integer variable \( n \) is even, the final values of variables \( a, b, c, \) and \( d \) will be 2, 1, 2, 3, whereas if the initial value of \( n \) is odd, the final values of the aforesaid variables will be 1, 3, 2, 3. Assuming that their initial value is 0, the potential final values tracked by considering the sequences of the state updates triggered by the starting assignments with the updates, simulated as described above, possibly triggered by the conditional statement rather are:

- 2, 1, 2, 0,
- 1, 3, 0, 3,
- 2, 3, 2, 3.

The first two tuples of values match the ones generated by a single loop iteration, and produce the actual final values listed above if they are fed back into the loop body along with the increased value of \( n \).

Hence, concatenating the lists tracking the state updates possibly triggered by the first command in the sequence (the starting assignment sequence in the previous example) with the lists tracking the updates possibly triggered by the second command in the sequence (the conditional statement in the previous example) would not suffice for the type system to be idempotent. The solution is to rather append the latter lists to those constructed by merging the sets tracking the state updates possibly performed by each command in the sequence. Again, provided that such sets are finite, this construction is finite, too. In the special case where the latter set is a singleton, the aforesaid merging is unnecessary, as it would merely insert a preceding occurrence of the single appended list into the resulting concatenated lists, and such repetitions are irrelevant as observed above.

Surprisingly enough, the case of loops is actually simpler than possible first-glance expectations. A loop defines two branches, namely its body and an implicit alternative branch doing nothing. Thus, it can simply be mapped to the union of the set tracking the state updates possibly occurring in its
body with a singleton made of the empty list. As happens with conditional statements, in the special case where the boolean condition can be evaluated at compile time, considering the selected branch alone is obviously enough. Type system `ctyping1` uses the set of lists resulting from this recursion over the input command to construct a set $F$ of functions of type $\text{vname} \Rightarrow \text{val option option}$, as follows: for each list $ys$ in the former set, $F$ contains the function mapping any variable $x$ to the rightmost occurrence, if any, of pattern $\text{Some v}$ to which $x$ is mapped by any function in $ys$ (that is, to the latest update, if any, of $x$ tracked in $ys$), or else to $\text{None}$. Then, if $A$, $X$ are the input state set and vname set, and $B$, $Y$ the output ones:

- $B$ is the set of the program states constructed by picking a function $f$ and a state $s$ from $F$ and $A$, respectively, and mapping any variable $x$ to $i$ if $f \ x = \text{Some (Some i)}$, or else to $s \ x$ if $f \ x = \text{None}$ (namely, to its value in the initial state $s$ if $f$ marks it as being unchanged).

- $Y$ is $\text{UNIV}$ if $A = \{\}$ (more on this later), or else the set of the variables not mapped to $\text{Some None}$ (that is, not marked as being indeterminate) by any function in $F$, and contained in $X$ (namely, being initially determinate) if mapped to $\text{None}$ (that is, if marked as being unchanged) by some function in $F$.

When can `ctyping1` evaluate the boolean condition of a conditional statement or a loop, so as to possibly detect and discard some “dead” branch? This question can be answered by examining the following sample C loop, where $n$ is a state variable, while integer $j$ is unknown at compile time.

```c
for (i = 0; i != j; i++)
{
  if (n == 1)
  {
    n = 2;
  }

  else if (n == 0)
  {
    n = 1;
  }
}
```

Assuming that the initial value of $n$ is 0, its final value will be 0, 1, or 2 according to whether $j$ matches 0, 1, or any other positive integer, respectively, whereas the loop will not even terminate if $j$ is negative. Consequently, the type system cannot avoid tracking the state updates possibly triggered in every branch, on pain of failing to be idempotent. As a result, evaluating
the boolean conditions in the conditional statement at compile time so as to discard some branch is not possible, even though they only depend on an initially determinate state variable. The conclusion is that \textit{ctyping1} may generally evaluate boolean conditions just in case they contain constants alone, namely only if they are trivial enough to be possibly eliminated by program optimization. This is exactly what \textit{ctyping1} does by passing any boolean condition found in the input program to the type system \textit{btyping1} for boolean expressions, defined here below as well.

\textbf{primrec} \textit{btyping1} :: \textit{bexp} \Rightarrow \textit{bool option} ((\vdash) [51] 55) \textbf{where}

\begin{align*}
\vdash Bc v &= \text{Some } v \\
\vdash \text{Not } b &= \begin{cases} 
\text{Some } (\neg v) & \text{if } v \vdash \text{True} \\
\text{None} & \text{otherwise}
\end{cases} \\
\vdash \text{And } b_1 b_2 &= \begin{cases} 
\text{Some } (v_1 \land v_2) & \text{if } v_1 \vdash \text{True} \land v_2 \vdash \text{True} \\
\text{None} & \text{otherwise}
\end{cases} \\
\vdash \text{Less } a_1 a_2 &= \begin{cases} 
\text{Some } (\lambda x. 0 < \lambda x. 0) & \text{if } a_1 \cup a_2 = \emptyset \\
\text{None} & \text{otherwise}
\end{cases}
\end{align*}

\textbf{type-synonym} \textit{state-upd} = \textit{vname} \Rightarrow \textit{val option option}

\textbf{inductive-set} \textit{ctyping1-merge-aux} :: \textit{state-upd list set} \Rightarrow \textit{state-upd list set} \Rightarrow (\textit{state-upd list \times bool} \textit{list set}) \textit{list set} (\text{infix} \ 55) \textbf{for } A \textit{ and } B \textbf{where}

\begin{align*}
x s \in A &\implies [(x s, \text{True})] \in A \biguplus B \\
y s \in B &\implies [(y s, \text{False})] \in A \biguplus B \\
\llbracket w s \in A \biguplus B; \neg \text{snd } (\text{last } w s); x s \in A; (x s, \text{True}) \notin \text{set } w s \rrbracket &\implies w s \at [(x s, \text{True})] \in A \biguplus B \\
\llbracket w s \in A \biguplus B; \text{snd } (\text{last } w s); y s \in B; (y s, \text{False}) \notin \text{set } w s \rrbracket &\implies w s \at [(y s, \text{False})] \in A \biguplus B
\end{align*}

\textbf{declare} \textit{ctyping1-merge-aux.intros} [intro]

\textbf{definition} \textit{ctyping1-append} :: \\
\textit{state-upd list set} \Rightarrow \textit{state-upd list set} \Rightarrow \textit{state-upd list set} (\text{infixl} @ 55) \textbf{where}

\begin{align*}
A \at B &\equiv \{ x s \at y s \mid x s y s, x s \in A \land y s \in B \}
\end{align*}

\textbf{definition} \textit{ctyping1-merge} :: \\
\textit{state-upd list set} \Rightarrow \textit{state-upd list set} \Rightarrow \textit{state-upd list set}

19
\[(\text{infixl} \sqcup 55) \text{ where}\]
\[A \sqcup B \equiv \{\text{concat (map fst ws)} \mid \text{ws. ws} \in A \sqcup B\}\]

\textbf{definition} \textit{ctyping1-merge-append} ::
\[\text{state-upd list set} \Rightarrow \text{state-upd list set} \Rightarrow \text{state-upd list set}\]
\[(\text{infixl} \sqcup_{\text{st}} 55) \text{ where}\]
\[A \sqcup_{\text{st}} B \equiv (\text{if card } B = \text{Suc } 0 \text{ then } A \text{ else } A \sqcup B) \circ B\]

\textbf{primrec} \textit{ctyping1-aux} :: \textit{com} \Rightarrow \text{state-upd list set}
\[(\text{(}\vdash \cdot \cdot \cdot)[51] 60) \text{ where}\]
\[\vdash \text{SKIP} = \{[]\} |\]
\[\vdash y ::= a = \{[\lambda x. \text{if } x = y \land y \in \text{state} \text{ then if avars } a = \{\} \text{ then Some (Some a (\lambda x. 0))) else Some None \text{ else None}\}\} |\]
\[\vdash c_1;; c_2 = \vdash c_1 \sqcup_{\text{st}} \vdash c_2 |\]
\[\vdash \text{IF } b \text{ THEN } c_1 \text{ ELSE } c_2 = (\text{let } f = \vdash b \text{ in}
\text{if } f \in \{\text{Some False, None}\} \text{ then } \vdash c_1 \text{ else } \{\}) \sqcup
\text{if } f \in \{\text{Some False, None}\} \text{ then } \{\} \text{ else } \{\}) |\]
\[\vdash \text{WHILE } b \text{ DO } c = (\text{let } f = \vdash b \text{ in}
\text{if } f \in \{\text{Some False, None}\} \text{ then } \{\} \text{ else } \{\}) \sqcup
\text{if } f \in \{\text{Some True, None}\} \text{ then } \vdash c \text{ else } \{\}) |\]

\textbf{definition} \textit{ctyping1-seq} :: \text{state-upd} \Rightarrow \text{state-upd} \Rightarrow \text{state-upd}
\[(\text{infixl} ;; 55) \text{ where}\]
\[S ;; T \equiv \lambda x. \text{case } T x \text{ of } \text{None } \Rightarrow S x \mid \text{Some v } \Rightarrow \text{Some v}\]

\textbf{definition} \textit{ctyping1} :: \text{com} \Rightarrow \text{state set} \Rightarrow \text{vname set} \Rightarrow \text{config}
\[(\text{(}\vdash \cdot \cdot \cdot [\subseteq - \cdot \cdot \cdot ])[51] 55) \text{ where}\]
\[\vdash c (\subseteq A, X) \equiv \text{let } \text{F} = \{\lambda x. \text{foldl (::)} (\lambda x. \text{None}) \text{ y s x} \mid \text{ys. ys} \in \vdash c\} \text{ in}
\{\lambda x. \text{case } f x \text{ of } \text{None } \Rightarrow s x \mid \text{Some None } \Rightarrow t x \mid \text{Some } (\text{Some } i) \Rightarrow i \mid
\text{f s t. f} \in F \land s \in A\},
\text{Univ?? } A \{x. \forall f \in F. f x \neq \text{Some None } \land (f x = \text{None } \Rightarrow x \in X)}\)

A further building block propaedeutic to the definition of the main type system \textit{ctyping2} is the definition of its own companion type system \textit{btyping2} for boolean expressions. The goal of \textit{btyping2} is splitting, whenever feasible at compile time, an input state set into two complementary subsets, respectively comprising the program states making the input boolean expression true or false. This enables \textit{ctyping2} to apply its information flow correctness checks to conditional branches by considering only the program states in which those branches are executed.
As opposed to $btyping_1$, $btyping_2$ may evaluate its input boolean expression even if it contains variables, provided that all of their values are known at compile time, namely that all of them are determinate state variables – hence $btyping_2$, like $ctyping_2$, needs to take a $vname$ set collecting determinate variables as an additional input. In fact, in the case of a loop body, the dirty work of covering any nested branch by skipping the evaluation of non-constant boolean conditions is already done by $ctyping_1$, so that any state set and $vname$ set input to $btyping_2$ already encompass every possible execution flow.

primrec $btyping_2$-aux :: $bexp$ ⇒ state set ⇒ $vname$ set ⇒ state set option

where

$\preceq \mathbf{Bc} \, v \, (\subseteq A, \cdot) = \mathbf{Some} \ (\text{if } v \text{ then } A \text{ else } \{\}) |$

$\preceq \mathbf{Not} \, b \, (\subseteq A, X) = \mathbf{case} \, (\preceq b, (\subseteq A, X)) \, \mathbf{of}$

$\mathbf{Some} \, B \Rightarrow \mathbf{Some} \, (A - B) | \cdot \Rightarrow \mathbf{None} |$

$\preceq \mathbf{And} \, b_1, b_2 \, (\subseteq A, X) = \mathbf{case} \, (\preceq b_1, (\subseteq A, X), \preceq b_2, (\subseteq A, X)) \, \mathbf{of}$

$(\mathbf{Some} \, B_1, \mathbf{Some} \, B_2) \Rightarrow \mathbf{Some} \, (B_1 \cap B_2) | \cdot \Rightarrow \mathbf{None} |$

$\preceq \mathbf{Less} \, a_1, a_2 \, (\subseteq A, X) = \mathbf{if} \, \text{avars} \, a_1 \cup \text{avars} \, a_2 \subseteq \text{state} \cap X \text{ then } \mathbf{Some} \ \{s. \ s \in A \land \text{aval} \ a_1 \, s < \text{aval} \ a_2 \, s\} \text{ else } \mathbf{None}$

definition $btyping_2$ :: $bexp$ ⇒ state set ⇒ $vname$ set ⇒ state set × state set

where

$\preceq b, (\subseteq A, X) \equiv \mathbf{case} \, \preceq b, (\subseteq A, X) \, \mathbf{of}$

$\mathbf{Some} \, A' \Rightarrow (A', A - A') | \cdot \Rightarrow (A, A)$

It is eventually time to define the main type system $ctyping_2$. Its output consists of the state set of the final program states and the $vname$ set of the finally determinate variables produced by simulating the execution of the input program, based on the state set of initial program states and the $vname$ set of initially determinate variables taken as inputs, if information flow correctness checks are passed; otherwise, the output is $\mathbf{None}$. An additional input is the counterpart of the level input to the security type systems formalized in [6], in that it specifies the scope in which information flow correctness is validated. It consists of a set of state set × $vname$ set pairs and a boolean flag. The set keeps track of the variables contained in the boolean conditions, if any, nesting the input program, in association with the program states in which they are evaluated. The flag is $\mathbf{False}$ if the input program is nested in a loop, in which case state variables set to non-constant expressions are marked as being indeterminate (as observed previously, the value of a non-constant expression in a loop iteration is in
general unknown at compile time).

In the recursive definition of \textit{ctyping2}, the equations dealing with conditional branches, namely those applying to conditional statements and loops, construct the output \textit{state set} and \textit{vname set} respectively as the \textit{union} and the \textit{intersection} of the sets computed for each branch. In fact, a possible final state is any one resulting from either branch, and a variable is finally determinate just in case it is such regardless of the branch being picked. Yet, a “dead” branch should have no impact on the determinateness of variables, as it only depends on the other branch. Accordingly, provided that information flow correctness checks are passed, the cases where the output is constructed non-recursively, namely those of \textit{SKIP}, assignments, and loops, return \textit{UNIV} as \textit{vname set} if the input \textit{state set} is empty. In the case of a loop, the \textit{state set} and the \textit{vname set} resulting from one or more iterations of its body are computed using the auxiliary type system \textit{ctyping1}. This explains why \textit{ctyping1} returns \textit{UNIV} as \textit{vname set} if the input \textit{state set} is empty, as mentioned previously.

As happens with the syntax-directed security type system formalized in [6], the cases performing non-recursive information flow correctness checks are those of assignments and loops. In the former case, \textit{ctyping2} verifies that the sets of variables contained in the scope, as well as any variable occurring in the expression on the right-hand side of the assignment, are allowed to interfere with the variable on the left-hand side, respectively in their associated sets of states and in the input \textit{state set}. In the latter case, \textit{ctyping2} verifies that the sets of variables contained in the scope, as well as any variable occurring in the boolean condition of the loop, are allowed to interfere with \textit{every} variable, respectively in their associated sets of states and in the states in which the boolean condition is evaluated. In both cases, if the applying interference relation is unknown as some state variable is indeterminate, each of those checks must be passed for \textit{any} possible state (unless the respective set of states is empty).

Why do the checks performed for loops test interference with \textit{every} variable? The answer is provided by the following sample C program, which sets variables \texttt{a} and \texttt{b} to the terms in the zero-based positions \texttt{j} and \texttt{j + 1} of the Fibonacci sequence.

```c
1  a = 0;
2  b = 1;
3 for (i = 0; i != j; i++)
4  {
5    c = b;
6    b += a;
7    a = c;
8  }
```
The loop in this program terminates for any nonnegative value of \( j \). For any variable \( x \), suppose that \( j \) is not allowed to interfere with \( x \) in such an initial state, say \( s \). According to the above information flow correctness definition, any initial state \( t \) differing from \( s \) in the value of \( j \) must make execution terminate all the same in order for the program to be correct. However, this is not the case, since execution does not terminate for any negative value of \( j \). Thus, the type system needs to verify that \( j \) may interfere with \( x \), on pain of returning a wrong pass verdict.

The cases that change the scope upon recursively calling the type system are those of conditional statements and loops. In the latter case, the boolean flag is set to \texttt{False}, and the set of state set \( \times \) vname set pairs is empty as the whole scope nesting the loop body, including any variable occurring in the boolean condition of the loop, must be allowed to interfere with every variable. In the former case, for both branches, the boolean flag is left unchanged, whereas the set of pairs is extended with the pair composed of the input state set (or of UNIV if some state variable is indeterminate, unless the input state set is empty) and of the set of the variables, if any, occurring in the boolean condition of the statement.

Why is the scope extended with the whole input state set for both branches, rather than just with the set of states in which each single branch is selected? Once more, the question can be answered by considering a sample C program, namely a previous one determining the minimum of two integers \( a \) and \( b \) using a state variable \( i \). For the reader’s convenience, the program is reported here below.

```
1  i = (a > b) ? 1 : 0;
2  if (i > 0)
3    {  
4      a = b;
5    }
```

Since the branch changing the value of variable \( a \) is executed just in case \( i = 1 \), suppose that in addition to \( b \), \( i \) also is not allowed to interfere with \( a \) for \( i = 0 \), and let \( s \) be any initial state where \( a \leq b \). Based on the above information flow correctness definition, any initial state \( t \) differing from \( s \) in the value of \( b \) (not bound by the interference of \( i \) with \( a \)) must produce the same final value of \( a \) in order for the program to be correct. However, this is not the case, as the final value of \( a \) will change for any state \( t \) where \( a > b \). Therefore, the type system needs to verify that \( i \) may interfere with \( a \) for \( i = 0 \), too, on pain of returning a wrong pass verdict. This is the reason why, as mentioned previously, an information flow correctness policy for this
program should be such that \( s: \text{dom} \, i \leadsto \text{dom} \, a \) even for any state \( s \) where \( i = 0 \).

An even simpler example explains why, in the case of an assignment or a loop, the information flow correctness checks described above need to be applied to the set of \( \text{state set} \times \text{vname set} \) pairs in the scope even if the input \( \text{state set} \) is empty, namely even if the assignment or the loop are nested in a “dead” branch. Here below is a sample C program showing this.

```c
if (i)
{
    a = 1;
}
```

Assuming that the initial value of \( i \) is 0, the assignment nested within the conditional statement is not executed, so that the final value of \( a \) matches the initial one, say 0. Suppose that \( i \) is not allowed to interfere with \( a \) in such an initial state, say \( s \). According to the above information flow correctness definition, any initial state \( t \) differing from \( s \) in the value of \( i \) must produce the same final value of \( a \) in order for the program to be correct. However, this is not the case, as the final value of \( a \) is 1 for any nonzero value of \( i \). Therefore, the type system needs to verify that \( i \) may interfere with \( a \) in state \( s \) even though the conditional branch is not executed in that state, on pain of returning a wrong pass verdict.

**abbreviation**  
\( \text{atyping} :: \text{bool} \Rightarrow \text{aexp} \Rightarrow \text{vname set} \Rightarrow \text{bool} \)

\( (\vdash - \subseteq \cdot) \left[ 51, 51 \right] \text{ where } 
\)

\( v \vdash a \subseteq X \equiv \text{avars} \, a = \{ \} \lor \text{avars} \, a \subseteq \text{state} \cap X \land v \)

**definition**  
\( \text{univ-states-if} :: \text{state set} \Rightarrow \text{vname set} \Rightarrow \text{state set} \)

\( ((\text{Univ}? \cdot \cdot) \left[ 51, 75 \right] 75) \text{ where } 
\)

\( \text{Univ}? \, A \, X \equiv \text{if state} \subseteq X \text{ then } A \text{ else } \text{Univ} \, A \, (\subseteq \{\}) \)

**fun**  
\( \text{ctyping2} :: \text{scope} \Rightarrow \text{com} \Rightarrow \text{state set} \Rightarrow \text{vname set} \Rightarrow \text{config option} \)

\( (\vdash - \cdot \cdot \cdot) \left[ 51, 51 \right] 55) \text{ where } 
\)

\( - \vdash \text{SKIP} \, (\subseteq A, X) = \text{Some} \, (A, \text{Univ}? \, A \, X) \mid 
\)

\( (U, v) \vdash x := a \subseteq A, X) = 
\)

\( (\text{if } (\forall (B, Y) \in \text{insert} \, (\text{Univ}? \, A \, X, \text{avars} \, a) \, U, B: \text{dom} \, Y \leadsto \{ \text{dom} \, x \}) 
\)

\( \text{then Some} \, (\text{if } x \in \text{state} \land A \not= \{\}) 
\)

\( \text{then if } v \vdash a \subseteq X 
\)

\( \text{then } ((\{s(x := \text{aval} \, a) \mid s, s \in A}, \text{insert} \, x \, X) \text{ else } (A, X \setminus \{x\}) 
\)

\( \text{else None}) \mid 
\)
\[(U, v) \vdash c_1 \sqcup c_2 (\subseteq A, X) =
\begin{cases}
\text{Some} (B, Y) \Rightarrow (U, v) \vdash c_2 (\subseteq B, Y) | - \Rightarrow \text{None} | \\
\end{cases}
\]

\[(U, v) \vdash \text{IF } b \text{ THEN } c_1 \text{ ELSE } c_2 (\subseteq A, X) =
\begin{cases}
\text{case } ((U', v) \vdash b (\subseteq A, X)) \text{ of } (U', B_1, B_2) \Rightarrow
\begin{cases}
\text{Some} (C_1, Y_1), \text{Some} (C_2, Y_2) \Rightarrow \text{Some} (C_1 \cup C_2, Y_1 \cap Y_2) | \\
- \Rightarrow \text{None} | \\
\end{cases}
\end{cases}
\]

\[(U, v) \vdash \text{WHILE } b \text{ DO } c (\subseteq A, X) =
\begin{cases}
\text{case } ((U', v) \vdash c (\subseteq B_1, X)) \text{ of } (U', C, Y) \Rightarrow
\begin{cases}
\text{if } \forall (B, W) \in \text{insert } (\text{Univ? } A X \cup \text{Univ? } C Y, \text{bvars } b) \ U.
\end{cases}
\end{cases}
\]

2 Idempotence of the auxiliary type system meant for loop bodies

theory Idempotence
  imports Definitions
begin

The purpose of this section is to prove that the auxiliary type system ctyping1 used to simulate the execution of loop bodies is idempotent, namely that if its output for a given input is the pair composed of state set B and vname set Y, then the same output is returned if B and Y are fed back into the type system (lemma ctyping1-idem).

2.1 Global context proofs

lemma remdups-filter-last:
  last [x←remdups xs. P x] = last [x←xs. P x]
  \langle proof \rangle

lemma remdups-append:
  set xs ⊆ set ys ⇒ remdups (xs @ ys) = remdups ys
  \langle proof \rangle
**lemma remdups-concat-1:**
\[\text{remdups} \left( \text{concat} \left( \text{remdups} \ [] \right) \right) = \text{remdups} \left( \text{concat} \ [] \right)\]
(proof)

**lemma remdups-concat-2:**
\[\text{remdups} \left( \text{concat} \left( \text{remdups} \ xs \right) \right) = \text{remdups} \left( \text{concat} \ xs \right) \Rightarrow \text{remdups} \left( \text{concat} \left( x \neq \text{xs} \right) \right) = \text{remdups} \left( \text{concat} \left( x \neq \text{xs} \right) \right)\]
(proof)

**lemma remdups-concat:**
\[\text{remdups} \left( \text{concat} \left( \text{remdups} \ xs \right) \right) = \text{remdups} \left( \text{concat} \ xs \right)\]
(proof)

**2.2 Local context proofs**

**context noninterf**

**begin**

**lemma ctyping1-seq-last:**
\[\text{foldl} \ (\_; \_ \ S \ xs) = (\lambda x \ . \ \text{let} \ \text{xs'} = [T \leftarrow \text{xs}. \ T x \neq \text{None}] \ \text{in} \ \text{if} \ \text{xs'} = [] \ \text{then} \ S \ x \ \text{else} \ \text{last} \ \text{xs'} \ x)\]
(proof)

**lemma ctyping1-seq-remdups:**
\[\text{foldl} \ (\_; \_ \ S \ (\text{remdups} \ xs)) = \text{foldl} \ (\_; \_ \ S \ xs)\]
(proof)

**lemma ctyping1-seq-remdups-concat:**
\[\text{foldl} \ (\_; \_ \ S \ (\text{concat} \ (\text{remdups} \ xs))) = \text{foldl} \ (\_; \_ \ S \ (\text{concat} \ xs))\]
(proof)

**lemma ctyping1-seq-eq:**
\[\text{assumes} \ A: \ \text{foldl} \ (\_; \_ \ S \ xs) = \text{foldl} \ (\_; \_ \ S \ ys)\]
\[\text{shows} \ \text{foldl} \ (\_; \_ \ S \ xs) = \text{foldl} \ (\_; \_ \ S \ ys)\]
(proof)

**lemma ctyping1-merge-aux-butlast:**
\[\text{ws} \in A \bigcup B; \ \text{butlast} \ \text{ws} \neq [] \Rightarrow \text{snd} \ (\text{last} \ \text{butlast} \ \text{ws})) = (\neg \text{snd} \ (\text{last} \ \text{ws}))\]
(proof)

**lemma ctyping1-merge-aux-distinct:**
\[\text{ws} \in A \bigcup B \Rightarrow \text{distinct} \ \text{ws}\]
(proof)

**lemma ctyping1-merge-aux-nonempty:**
\[\text{ws} \in A \bigcup B \Rightarrow \text{ws} \neq []\]
lemma ctyping1-merge-aux-item: 
\[ \{ ws \in A \uplus B; w \in set \ ws \} \implies \text{fst } w \in (\text{if snd } w \text{ then } A \text{ else } B) \]
(proof)

lemma ctyping1-merge-aux-take-1 [elim]:
\[ \{ \text{take } n \ ws \in A \uplus B; \neg \text{snd } (\text{last } ws); xs \in A; (xs, True) \notin set \ ws \} \implies \text{take } n \ ws @ \text{take } (n \ - \ \text{length } ws) [(xs, True)] \in A \uplus B \]
(proof)

lemma ctyping1-merge-aux-take-2 [elim]:
\[ \{ \text{take } n \ ws \in A \uplus B; \neg \text{snd } (\text{last } ws); ys \in B; (ys, False) \notin set \ ws \} \implies \text{take } n \ ws @ \text{take } (n \ - \ \text{length } ws) [(ys, False)] \in A \uplus B \]
(proof)

lemma ctyping1-merge-aux-take:
\[ \{ ws \in A \uplus B; 0 < n \} \implies \text{take } n \ ws \in A \uplus B \]
(proof)

lemma ctyping1-merge-aux-drop-1 [elim]:
assumes
A: xs \in A \ and
B: ys \in B
shows \text{drop } n [(xs, True)] @ [(ys, False)] \in A \uplus B
(proof)

lemma ctyping1-merge-aux-drop-2 [elim]:
assumes
A: xs \in A \ and
B: ys \in B
shows \text{drop } n [(ys, False)] @ [(xs, True)] \in A \uplus B
(proof)

lemma ctyping1-merge-aux-drop-3:
assumes
A: \forall xs. (xs, True) \notin set (\text{drop } n \ ws) \implies \text{xs } \in A \implies v \implies \text{drop } n \ ws @ [(xs, True)] \in A \uplus B \ and
B: xs \in A \ and
C: ys \in B \ and
D: (xs, True) \notin set \ ws \ and
E: (ys, False) \notin set (\text{drop } n \ ws)
shows \text{drop } n \ ws @ \text{drop } (n \ - \ \text{length } ws) [(xs, True)] @ [(ys, False)] \in A \uplus B
(proof)

lemma ctyping1-merge-aux-drop-4:
assumes
\begin{proof}
\end{proof}

\begin{lemma}
ctyping1-merge-aux-drop:
\begin{equation}
\begin{aligned}
\text{\textit{proof}} \quad \text{\textit{proof}} \quad \text{\textit{proof}} \quad \text{\textit{proof}}
\end{aligned}
\end{equation}
\end{lemma}

\begin{lemma}
ctyping1-merge-aux-closed-1:
\begin{equation}
\begin{aligned}
\text{\textit{proof}} \quad \text{\textit{proof}} \quad \text{\textit{proof}} \quad \text{\textit{proof}}
\end{aligned}
\end{equation}
\end{lemma}

\begin{lemma}
ctyping1-merge-aux-closed:
\begin{equation}
\begin{aligned}
\text{\textit{proof}} \quad \text{\textit{proof}} \quad \text{\textit{proof}} \quad \text{\textit{proof}}
\end{aligned}
\end{equation}
\end{lemma}

\begin{lemma}
ctyping1-merge-closed:
\begin{equation}
\begin{aligned}
\text{\textit{proof}} \quad \text{\textit{proof}} \quad \text{\textit{proof}} \quad \text{\textit{proof}}
\end{aligned}
\end{equation}
\end{lemma}
A: \( \forall xs \in A. \forall ys \in A. \exists zs \in A. \)
\[\text{foldl} (\_ \_ ) (\lambda x. \text{None}) zs = \text{foldl} (\_ \_ ) (\lambda x. \text{None}) (xs \oplus ys) \text{ and} \]

B: \( \forall xs \in B. \forall ys \in B. \exists zs \in B. \)
\[\text{foldl} (\_ \_ ) (\lambda x. \text{None}) zs = \text{foldl} (\_ \_ ) (\lambda x. \text{None}) (xs \oplus ys) \text{ and} \]

C: \( xs \in A \sqcup B \text{ and} \)

D: \( ys \in A \sqcup B \)

shows \( \exists zs \in A \sqcup B. \text{foldl} (\_ \_ ) (\lambda x. \text{None}) zs = \)
\[\text{foldl} (\_ \_ ) (\lambda x. \text{None}) (xs \oplus ys) \]

\langle proof \rangle

lemma ctyping1-merge-append-closed:

assumes
A: \( \forall xs \in A. \forall ys \in A. \exists zs \in A. \)
\[\text{foldl} (\_ \_ ) (\lambda x. \text{None}) zs = \text{foldl} (\_ \_ ) (\lambda x. \text{None}) (xs \oplus ys) \text{ and} \]

B: \( \forall xs \in B. \forall ys \in B. \exists zs \in B. \)
\[\text{foldl} (\_ \_ ) (\lambda x. \text{None}) zs = \text{foldl} (\_ \_ ) (\lambda x. \text{None}) (xs \oplus ys) \text{ and} \]

C: \( xs \in A \sqcup B \text{ and} \)

D: \( ys \in A \sqcup B \)

shows \( \exists zs \in A \sqcup B. \text{foldl} (\_ \_ ) (\lambda x. \text{None}) zs = \)
\[\text{foldl} (\_ \_ ) (\lambda x. \text{None}) (xs \oplus ys) \]

\langle proof \rangle

lemma ctyping1-aux-closed:

[ \( xs \in \vdash c ; ys \in \vdash c \) ] \implies \( \exists zs \in \vdash c. \text{foldl} (\_ \_ ) (\lambda x. \text{None}) zs = \)
\[\text{foldl} (\_ \_ ) (\lambda x. \text{None}) (xs \oplus ys) \]

\langle proof \rangle

lemma ctyping1-idem-1:

assumes
A: \( s \in A \text{ and} \)

B: \( xs \in \vdash c \text{ and} \)

C: \( ys \in \vdash c \)

shows \( \exists f r. \)
\[(\exists t. (\lambda x. \text{case foldl} (\_ \_ ) (\lambda x. \text{None}) ys x of \text{ None } \Rightarrow \text{case foldl} (\_ \_ ) (\lambda x. \text{None}) xs x of \text{ None } \Rightarrow s x | \text{Some None } \Rightarrow t’ x | \text{Some } (\text{Some i}) \Rightarrow i \text{ | } \text{Some None } \Rightarrow t’’ x | \text{Some } (\text{Some i}) \Rightarrow i = \text{ (\lambda x. case f x of } \text{ None } \Rightarrow \text{r x } | \text{Some None } \Rightarrow t x | \text{Some } (\text{Some i}) \Rightarrow i) \land \langle (\exists zs. f = \text{foldl} (\_ \_ ) (\lambda x. \text{None}) zs \land zs \in \vdash c) \land r \in A \rangle \text{ (proof) } \]

lemma ctyping1-idem-2:

assumes
A: \( s \in A \text{ and} \)

B: \( xs \in \vdash c \)
shows $\exists fr$. 

$\exists t. \ (\lambda x. case foldl (::) (\lambda x. None) xs x of
\hspace{1em} None \Rightarrow s x | Some None \Rightarrow t' x | Some (Some i) \Rightarrow i = \hspace{1em} (\lambda x. case f x of
\hspace{2em} None \Rightarrow r x | Some None \Rightarrow t x | Some (Some i) \Rightarrow i)) \land 
(\exists xs. f = foldl (::) (\lambda x. None) xs \land xs \in \vdash c) \land 
(\exists f s. \ (\exists t. r = (\lambda x. case f x of
\hspace{1em} None \Rightarrow s x | Some None \Rightarrow t x | Some (Some i) \Rightarrow i)) \land 
(\exists xs. f = foldl (::) (\lambda x. None) xs \land xs \in \vdash c) \land 
\hspace{1em} s \in A) \}$

⟨proof⟩

lemma ctyping1-idem:
$\vdash c (\subseteq A, X) = (B, Y) \implies \vdash c (\subseteq B, Y) = (B, Y)$

⟨proof⟩

end

end

3 Overapproximation of program semantics by the type system

theory Overapproximation
  imports Idempotence
begin

The purpose of this section is to prove that type system $ctyping2$ overapproximates program semantics, namely that if (a) $(c, s) \Rightarrow t$, (b) the type system outputs a state set $B$ and a vname set $Y$ when it is input program $c$, state set $A$, and vname set $X$, and (c) state $s$ agrees with a state in $A$ on the value of every state variable in $X$, then $t$ must agree with some state in $B$ on the value of every state variable in $Y$ (lemma $ctyping2$-approx).

This proof makes use of the lemma $ctyping1$-idem proven in the previous section.

3.1 Global context proofs

lemma avars-aval:
$s = t (\subseteq avars a) \implies aval a s = aval a t$

⟨proof⟩
3.2 Local context proofs

context noninterf
begin

lemma interf-set-mono:
\[ [ A' \subseteq A \land X \subseteq X' ; \forall (B', Y') \in U' . \exists (B, Y) \in U . B' \subseteq B \land Y' \subseteq Y ; \forall (B, Y) \in \text{insert (Univ A X, Z) U} . B; \text{dom} \ Y \rightsquigarrow W ] \implies \]
\[ \forall (B, Y) \in \text{insert (Univ A' X', Z) U'} . B; \text{dom} \ Y \rightsquigarrow W \]
\langle proof \rangle

lemma btyping1-btyping2-aux-1 [elim]:
assumes
\[ A: \text{avars } a_1 = \{ \} \text{ and } B: \text{avars } a_2 = \{ \} \text{ and } \]
\[ C: \text{aval } a_1 (\lambda x. 0) < \text{aval } a_2 (\lambda x. 0) \]
shows \text{aval } a_1 s < \text{aval } a_2 s
\langle proof \rangle

lemma btyping1-btyping2-aux-2 [elim]:
assumes
\[ A: \text{avars } a_1 = \{ \} \text{ and } B: \text{avars } a_2 = \{ \} \text{ and } \]
\[ C: \neg \text{aval } a_1 (\lambda x. 0) < \text{aval } a_2 (\lambda x. 0) \text{ and } \]
\[ D: \text{aval } a_1 s < \text{aval } a_2 s \]
shows False
\langle proof \rangle

lemma btyping1-btyping2-aux:
\[ \vdash b = \text{Some } v \implies \vdash b (\subseteq A, X) = \text{Some (if } v \text{ then } A \text{ else } \{ \}) \]
\langle proof \rangle

lemma btyping1-btyping2:
\[ \vdash b = \text{Some } v \implies \vdash b (\subseteq A, X) = (\text{if } v \text{ then } (A, \{ \}) \text{ else } (\{ \}, A)) \]
\langle proof \rangle

lemma btyping2-aux-subset:
\[ \vdash b (\subseteq A, X) = \text{Some } A' \implies A' = \{ s, s \in A \land \text{bval } b s \} \]
\langle proof \rangle

lemma btyping2-aux-diff:
\[ \llbracket \vdash b (\subseteq A, X) = \text{Some } B; \vdash b (\subseteq A', X') = \text{Some } B'; A' \subseteq A; B' \subseteq B \rrbracket \implies \]
\[ A' - B' \subseteq A - B \]
\langle proof \rangle

lemma btyping2-aux-mono:
\[ \llbracket \vdash b (\subseteq A, X) = \text{Some } B; A' \subseteq A; X \subseteq X' \rrbracket \implies \]
\[ \exists B'. \vdash b (\subseteq A', X') = \text{Some } B' \land B' \subseteq B \]
\[\begin{align*}
\text{lemma btyping2-mono:} & \quad \vdash b (\subseteq A, X) = (B_1, B_2); \quad \vdash b (\subseteq A', X') = (B_1', B_2'); \quad A' \subseteq A; \quad X \subseteq X' \quad \Rightarrow \\
& \quad B_1' \subseteq B_1 \land B_2' \subseteq B_2 \\
\text{lemma btyping2-un-eq:} & \quad \vdash b (\subseteq A, X) = (B_1, B_2) = \Rightarrow B_1 \cup B_2 = A \\
\text{lemma btyping2-fst-empty:} & \quad \vdash b (\subseteq \emptyset, X) = (\emptyset, \emptyset) \\
\text{lemma btyping2-aux-eq:} & \quad \vdash b (\subseteq A, X) = \text{Some } A' \land \quad s = t (\subseteq \text{state} \cap X) \quad \Rightarrow \\
& \quad \text{bval } b \ s = \text{bval } b \ t \\
\text{lemma ctyping1-merge-in:} & \quad xs \in A \cup B \Rightarrow xs \in A \cup B \\
\text{lemma ctyping1-merge-append-in:} & \quad [xs \in A; \quad ys \in B] \Rightarrow xs \circ ys \in A \cup @ B \\
\text{lemma ctyping1-aux-nonempty:} & \quad \vdash c \neq \emptyset \\
\text{lemma ctyping1-mono:} & \quad [(B, Y) \vdash c (\subseteq A, X); \quad (B', Y') = \vdash c (\subseteq A', X'); \quad A' \subseteq A; \quad X \subseteq X'] \Rightarrow \\
& \quad B' \subseteq B \land Y \subseteq Y' \\
\text{lemma ctyping1-fst-empty:} & \quad \text{Some } (B, Y) = (U, v) \vdash c (\subseteq \emptyset, X) \Rightarrow (B, Y) = (\emptyset, \text{UNIV}) \\
\text{lemma ctyping2-mono-assign [elim!]:} & \quad [(U, \text{False}) \models x ::= a (\subseteq A, X) = \text{Some } (C, Z); \quad A' \subseteq A; \quad X \subseteq X'; \\
& \quad \forall (B', Y') \in U'. \quad \exists (B, Y) \in U. \quad B' \subseteq B \land Y' \subseteq Y] \Rightarrow \\
& \quad \exists C' Z'. \quad (U', \text{False}) \models x ::= a (\subseteq A', X') = \text{Some } (C', Z') \land \\
& \quad C' \subseteq C \land Z \subseteq Z' \quad \Rightarrow \\
\end{align*}\]
lemma ctyping2-mono-seq:

assumes
A: \( \forall A' B' X' Y' U' \). 
\( (U, \text{False}) \models c_1 (\subseteq A, X) = \text{Some} (B, Y) \implies A' \subseteq A \implies X \subseteq X' \implies \forall (B', Y') \in U'. \exists (B, Y) \in U. B' \subseteq B \land Y' \subseteq Y \implies \exists B' Y'. (U', \text{False}) \models c_1 (\subseteq A', X') = \text{Some} (B', Y') \land B' \subseteq B \land Y \subseteq Y' \) and
\( B: \bigwedge p B Y B' C Y' Z U' . \)
\( (U, \text{False}) \models c_1 (\subseteq A, X) = \text{Some} p \implies (B, Y) = p \implies \forall (B', Y') \in U'. \exists (B, Y) \in U. B' \subseteq B \land Y' \subseteq Y \implies \exists C' Z'. (U', \text{False}) \models c_2 (\subseteq B', Y') = \text{Some} (C', Z') \land C' \subseteq C \land Z \subseteq Z' \) and
\( C: (U, \text{False}) \models c_1 ; c_2 (\subseteq A, X) = \text{Some} (C, Z) \) and
\( D: A' \subseteq A \) and
\( E: X \subseteq X' \) and
\( F: \forall (B', Y') \in U'. \exists (B, Y) \in U. B' \subseteq B \land Y' \subseteq Y \)
shows \( \exists C' Z'. (U', \text{False}) \models c_1 ; c_2 (\subseteq A', X') = \text{Some} (C', Z') \land C' \subseteq C \land Z \subseteq Z' \)
\( \) (proof)

lemma ctyping2-mono-if:

assumes
A: \( \forall W p B_1 B_2 B_1' C_1 X' Y_1 W'. (W, p) = \) 
\( (\text{insert} (\text{Univ?} A X, \text{bears} b) U, \models b (\subseteq A, X)) \implies (B_1, B_2) = p \implies \) 
\( (W, \text{False}) \models c_1 (\subseteq B_1, X) = \text{Some} (C_1, Y_1) \implies B_1' \subseteq B_1 \implies X \subseteq X' \implies \forall (B', Y') \in W'. \exists (B, Y) \in W. B' \subseteq B \land Y' \subseteq Y \implies \exists C_1 Y_1'. (W', \text{False}) \models c_1 (\subseteq B_1', X') = \text{Some} (C_1', Y_1') \land C_1' \subseteq C_1 \land Y_1 \subseteq Y_1' \) and
\( B: \bigwedge W p B_1 B_2 B_1' C_2 X' Y_2 W'. (W, p) = \)
\( (\text{insert} (\text{Univ?} A X, \text{bears} b) U, \models b (\subseteq A, X)) \implies (B_1, B_2) = p \implies \) 
\( (W, \text{False}) \models c_2 (\subseteq B_2, X) = \text{Some} (C_2, Y_2) \implies B_2' \subseteq B_2 \implies X \subseteq X' \implies \forall (B', Y') \in W'. \exists (B, Y) \in W. B' \subseteq B \land Y' \subseteq Y \implies \exists C_2 Y_2'. (W', \text{False}) \models c_2 (\subseteq B_2', X') = \text{Some} (C_2', Y_2') \land C_2' \subseteq C_2 \land Y_2 \subseteq Y_2' \) and
\( C: (U, \text{False}) \models \text{IF } b \text{ THEN } c_1 \text{ ELSE } c_2 (\subseteq A, X) = \text{Some} (C, Y) \) and
\( D: A' \subseteq A \) and
\( E: X \subseteq X' \) and
\( F: \forall (B', Y') \in U'. \exists (B, Y) \in U. B' \subseteq B \land Y' \subseteq Y \)
shows \( \exists C' Y'. (U', \text{False}) \models \text{IF } b \text{ THEN } c_1 \text{ ELSE } c_2 (\subseteq A', X') = \) 
\( \text{Some} (C', Y') \land C' \subseteq C \land Y \subseteq Y' \)
\( \) (proof)

lemma ctyping2-mono-while:

assumes
A: \( \forall B_1 B_2 C Y B_1' B_2' D_1 E X' V U'. (B_1, B_2) = \models b (\subseteq A, X) \implies (C, Y) = \models c (\subseteq B_1, X) \implies (B_1', B_2') = \models b (\subseteq C, Y) \implies \forall (B, W) \in \text{insert} (\text{Univ?} A X \cup \text{Univ?} C Y, \text{bears} b) U. \) 
\( B: \text{dom } W \sim \text{UNIV} \implies \)
⊢ c (∈ B₁, X) = Some (E, V) ⊢ D₁ ⊆ B₁ ⊢
X ⊆ X' ⊢ ∀ (B', Y') ∈ U'. ∃ (B, Y) ∈ {}. B' ⊆ B ∧ Y' ⊆ Y ⊢
∃ E' V'. (U', False) ⊢ c (∈ D₁, X') = Some (E', V') ∧
E' ⊆ E ∧ V ⊆ V' and
B: \wedge B₁ B₂ C Y B₁' B₂' D₁' F Y' W U' (B₁, B₂) = \vdash b (∈ A, X) ⊢
(C, Y) ⊢ c (∈ B₁, X) ⊢ (B₁', B₂') = \vdash b (∈ C, Y) ⊢
∀ (B, W) ∈ insert (Univ A X ∪ Univ C Y, bvars b) U.
B: dom ' W ~ UNIV ⊢
(\{\}, False) ⊢ c (∈ B₁', Y) = Some (F, W) ⊢ D₁' ⊆ B₁' ⊢
Y ⊆ Y' ⊢ ∀ (B', Y') ∈ U'. ∃ (B, Y) ∈ {}. B' ⊆ B ∧ Y' ⊆ Y ⊢
∃ F' W'. (U', False) ⊢ c (∈ D₁', Y') = Some (F', W') ∧
F' ⊆ F ∧ W ⊆ W' and
C: (U, False) ⊢ WHILE b DO c (∈ A, X) = Some (B, Z) and
D: A' ⊆ A and
E: X ⊆ X' and
F: ∀ (B', Y') ∈ U', ∃ (B, Y) ∈ U. B' ⊆ B ∧ Y' ⊆ Y shows
∃ B' Z'. (U', False) ⊢ WHILE b DO c (∈ A', X') = Some (B', Z') ∧
B' ⊆ B ∧ Z ⊆ Z' (proof)

lemma cttypeing2-mono:
[\{\}, False] \vdash c (∈ B₁, X) = Some (C, Z); A' ⊆ A; X ⊆ X'
\forall (B', Y') ∈ U'. ∃ (B, Y) ∈ U. B' ⊆ B ∧ Y' ⊆ Y \implies
∃ C' Z'. (U', False) \vdash c (∈ A', X') = Some (C', Z') ∧ C' ⊆ C ∧ Z ⊆ Z' (proof)

lemma cttypeing1-cttypeing2-fst-assign [elim!]:
assumes
A: (C, Z) = \vdash x ::= a (∈ A, X) and
B: Some (C', Z') = (U, False) \vdash x ::= a (∈ A, X)
shows C' ⊆ C (proof)

lemma cttypeing1-cttypeing2-fst-seq:
assumes
A: \wedge B B' Y Y', (B, Y) = \vdash c₁ (∈ A, X) \implies
Some (B', Y') = (U, False) \vdash c₁ (∈ A, X) \implies B' ⊆ B and
B: \wedge B Y C C' Z Z', (U, False) \vdash c₁ (∈ A, X) = Some p \implies
(B, Y) = p \implies (C, Z) = \vdash c₂ (∈ B, Y) \implies
Some (C', Z') = (U, False) \vdash c₂ (∈ B, Y) \implies C' ⊆ C and
C: (C, Z) = \vdash c₁:: c₂ (∈ A, X) and
D: Some (C', Z') = (U, False) \vdash c₁:: c₂ (∈ A, X)
shows C' ⊆ C (proof)

lemma cttypeing1-cttypeing2-fst-if:
assumes
A: \wedge U' p B₁ B₂ C₁ C₁' Y₁ Y₁'.

34
\[(U', p) = (\text{insert (Univ? }A \ X, \text{ bears } b) \ U, \vdash b (\subseteq A, X)) \Rightarrow\]
\[(B_1, B_2) = p \Rightarrow (C_1, Y_1) = \vdash c_1 (\subseteq B_1, X) \Rightarrow\]
Some \((C_1', Y_1') = (U', \text{ False}) \vdash c_1 (\subseteq B_1, X) \Rightarrow C_1' \subseteq C_1 \text{ and}\]
\[B: \bigwedge U' p B_1 B_2 C_2 C_2' Y_2 Y_2'.\]
\[(U', p) = (\text{insert (Univ? }A \ X, \text{ bears } b) \ U, \vdash b (\subseteq A, X)) \Rightarrow\]
\[(B_1, B_2) = p \Rightarrow (C_2, Y_2) = \vdash c_2 (\subseteq B_2, X) \Rightarrow\]
Some \((C_2', Y_2') = (U', \text{ False}) \vdash c_2 (\subseteq B_2, X) \Rightarrow C_2' \subseteq C_2 \text{ and}\]
\[C: (C, Y) = \vdash \text{ IF } b \text{ THEN } c_1 \text{ ELSE } c_2 (\subseteq A, X) \text{ and}\]
\[D: \text{ Some } (C', Y') = (U, \text{ False}) \vdash \text{ IF } b \text{ THEN } c_1 \text{ ELSE } c_2 (\subseteq A, X)\]
\textbf{shows}\ C' \subseteq C
\langle \text{proof} \rangle

\textbf{lemma ctyping1-ctyping2-fst-while:}
\textbf{assumes}\ A: (C, Y) = \vdash \text{ WHILE } b \text{ DO } c (\subseteq A, X) \text{ and}\]
\[B: \text{ Some } (C', Y') = (U, \text{ False}) \vdash \text{ WHILE } b \text{ DO } c (\subseteq A, X)\]
\textbf{shows}\ C' \subseteq C
\langle \text{proof} \rangle

\textbf{lemma ctyping1-ctyping2-fst:}
\[(C, Z) = \vdash x := a (\subseteq A, X); \text{ Some } (C', Z') = (U, \text{ False}) \vdash c (\subseteq A, X) \Rightarrow\]
\[C' \subseteq C\]
\langle \text{proof} \rangle

\textbf{lemma ctyping1-ctyping2-snd-assign [elim!]:}
\[(C, Z) = \vdash x := a (\subseteq A, X); \text{ Some } (C', Z') = (U, \text{ False}) \vdash x := a (\subseteq A, X) \Rightarrow Z \subseteq Z'\]
\langle \text{proof} \rangle

\textbf{lemma ctyping1-ctyping2-snd-seq:}
\textbf{assumes}\ A: \bigwedge B' \ Y' \ (B, Y) = \vdash c_1 (\subseteq A, X) \Rightarrow\]
\[\text{ Some } (B', Y') = (U, \text{ False}) \vdash c_1 (\subseteq A, X) \Rightarrow Y \subseteq Y' \text{ and}\]
\[B: \bigwedge B \ Y \ C \ C' \ Z \ Z'. \ (U, \text{ False}) \vdash c_1 (\subseteq A, X) \Rightarrow c_2 (\subseteq B, Y) \Rightarrow\]
\[B, Y) = p \Rightarrow (C, Z) = \vdash c_2 (\subseteq B, Y) \Rightarrow\]
\[\text{ Some } (C', Z') = (U, \text{ False}) \vdash c_2 (\subseteq B, Y) \Rightarrow Z \subseteq Z' \text{ and}\]
\[C: (C, Z) = \vdash c_1; c_2 (\subseteq A, X) \text{ and}\]
\[D: \text{ Some } (C', Z') = (U, \text{ False}) \vdash c_1; c_2 (\subseteq A, X)\]
\textbf{shows}\ Z \subseteq Z'
\langle \text{proof} \rangle

\textbf{lemma ctyping1-ctyping2-snd-if:}
\textbf{assumes}\ A: \bigwedge U' p B_1 B_2 C_1 C_1' Y_1 Y_1'.\]
\[(U', p) = (\text{insert (Univ? }A \ X, \text{ bears } b) \ U, \vdash b (\subseteq A, X)) \Rightarrow\]
\[(B_1, B_2) = p \Rightarrow (C_1, Y_1) = \vdash c_1 (\subseteq B_1, X) \Rightarrow\]
Some \((C_1', Y_1') = (U', \text{ False}) \vdash c_1 (\subseteq B_1, X) \Rightarrow Y_1 \subseteq Y_1' \text{ and}\]
\[B: \bigwedge U' p B_1 B_2 C_2 C_2' Y_2 Y_2'.\]
\((U', p) = (\text{insert} \ (\text{Unin}_? \ A \ X, \text{bvars} \ b) \ U, \models b (\subseteq A, X)) \implies (B_1, B_2) = p \implies (C_2, Y_2) = c_2 (\subseteq B_2, X) \implies \text{Some} \ (C_2', Y_2') = (U', \text{False}) \models c_2 (\subseteq B_2, X) \implies Y_2 \subseteq Y_2' \text{ and } C: (C, Y) = \models \text{IF} \ b \ \text{THEN} \ c_1 \ \text{ELSE} \ c_2 (\subseteq A, X) \text{ and } D: \text{Some} \ (C', Y') = (U, \text{False}) \models \text{IF} \ b \ \text{THEN} \ c_1 \ \text{ELSE} \ c_2 (\subseteq A, X) \text{ shows } Y \subseteq Y' \langle \text{proof} \rangle$

**lemma** \chyp{ctyping1-ctyping2-snd-while}:

**assumes**

\begin{align*}
A & : (C, Y) = \models \text{WHILE} \ b \ \text{DO} \ c (\subseteq A, X) \text{ and } \\
B & : \text{Some} \ (C', Y') = (U, \text{False}) \models \text{WHILE} \ b \ \text{DO} \ c (\subseteq A, X)
\end{align*}

**shows** \(Y \subseteq Y'\)

\langle \text{proof} \rangle

**lemma** \chyp{ctyping1-ctyping2-snd}:

\[
[(C, Z) = \models c (\subseteq A, X); \text{Some} \ (C', Z') = (U, \text{False}) \models c (\subseteq A, X)] \implies Z \subseteq Z'
\]

\langle \text{proof} \rangle

**lemma** \chyp{ctyping1-ctyping2}:

\[
[\models c (\subseteq A, X) = (C, Z); (U, \text{False}) \models c (\subseteq A, X) = \text{Some} \ (C', Z')] \implies C' \subseteq C \land Z \subseteq Z'
\]

\langle \text{proof} \rangle

**lemma** \chyp{btyping2-aux-approx-1} [elim]:

**assumes**

\begin{align*}
A & : \models b_1 (\subseteq A, X) = \text{Some} \ B_1 \text{ and } \\
B & : \models b_2 (\subseteq A, X) = \text{Some} \ B_2 \text{ and } \\
C & : \text{bval} \ b_1 \ s \ \text{and } \\
D & : \text{bval} \ b_2 \ s \ \text{and } \\
E & : r \in A \ \text{and } \\
F & : s = r (\subseteq \text{state} \cap X) \\
\text{shows} & \exists r' \in B_1 \cap B_2. \ r = r' (\subseteq \text{state} \cap X)
\end{align*}

\langle \text{proof} \rangle

**lemma** \chyp{btyping2-aux-approx-2} [elim]:

**assumes**

\begin{align*}
A & : \text{avars} \ a_1 \subseteq \text{state} \ \text{and } \\
B & : \text{avars} \ a_2 \subseteq \text{state} \ \text{and } \\
C & : \text{avars} \ a_1 \subseteq X \ \text{and } \\
D & : \text{avars} \ a_2 \subseteq X \ \text{and } \\
E & : \text{aval} \ a_1 \ s < \text{aval} \ a_2 \ s \ \text{and } \\
F & : r \in A \ \text{and } \\
G & : s = r (\subseteq \text{state} \cap X) \\
\text{shows} & \exists r'. \ r' \in A \land \text{aval} \ a_1 \ r' < \text{aval} \ a_2 \ r' \land r = r' (\subseteq \text{state} \cap X)
\end{align*}

\langle \text{proof} \rangle

36
lemma btyping2-aux-approx-3 [elim]:

assumes
  A: avars a1 ⊆ state and
  B: avars a2 ⊆ state and
  C: avars a1 ⊆ X and
  D: avars a2 ⊆ X and
  E: ¬ aval a1 s < aval a2 s and
  F: r ∈ A and
  G: s = r (⊆ state ∩ X)

shows ∃r′ ∈ A – {s ∈ A. aval a1 s < aval a2 s}. r = r′ (⊆ state ∩ X)
(proof)

lemma btyping2-aux-approx:

[|= b (⊆ A, X) = Some A′; s ∈ Univ A (⊆ state ∩ X)] ⊃
  s ∈ Univ (if bval b s then A′ else A – A′) (⊆ state ∩ X)
(proof)

lemma btyping2-approx:

[|= b (⊆ A, X) = (B1, B2); s ∈ Univ A (⊆ state ∩ X)] ⊃
  s ∈ Univ (if bval b s then B1 else B2) (⊆ state ∩ X)
(proof)

lemma ctyping2-approx-assign [elim!]:

[∀t′. aval a s = t′ x → (∀s. t′ = s(x := aval a s) → s /∈ A) ∨
  (∃y ∈ state ∩ X. y ≠ x ∧ t y ≠ t′ y);
  v |= a (⊆ X); t ∈ A; s = t (⊆ state ∩ X)] ⊃ False
(proof)

lemma ctyping2-approx-if-1:

[bval b s; |= b (⊆ A, X) = (B1, B2); r ∈ A; s = r (⊆ state ∩ X);
  (insert (Univ? A X, bvars b) U, v) |= c1 (⊆ B1, X) = Some (C1, Y1);
  \? A B X Y U v. (U, v) |= c1 (⊆ A, X) = Some (B, Y) ⊃
  \? r ∈ A. s = r (⊆ state ∩ X) ⊃ \? r′ ∈ B. t = r′ (⊆ state ∩ Y)] ⊃
  \? r′ ∈ C1 ∪ C2, t = r′ (⊆ state ∩ (Y1 ∩ Y2))
(proof)

lemma ctyping2-approx-if-2:

[¬ bval b s; |= b (⊆ A, X) = (B1, B2); r ∈ A; s = r (⊆ state ∩ X);
  (insert (Univ? A X, bvars b) U, v) |= c2 (⊆ B2, X) = Some (C2, Y2);
  \? A B X Y U v. (U, v) |= c2 (⊆ A, X) = Some (B, Y) ⊃
  \? r ∈ A. s = r (⊆ state ∩ X) ⊃ \? r′ ∈ B. t = r′ (⊆ state ∩ Y)] ⊃
  \? r′ ∈ C1 ∪ C2, t = r′ (⊆ state ∩ (Y1 ∩ Y2))
(proof)

lemma ctyping2-approx-while-1 [elim]:

[¬ bval b s; r ∈ A; s = r (⊆ state ∩ X); |= b (⊆ A, X) = (B, ∅)] ⊃
  \? t ∈ C. s = t (⊆ state ∩ Y)

37
lemma ctyping2-approx-while-2 [elim]:
\[ \forall t \in B_2 \cup B'_2. \exists x \in \text{state} \cap (X \cap Y), r \neq t x; \Rightarrow \text{bval} b s; \]
\[ r \in A; s = r (\subseteq \text{state} \cap X); \models b (\subseteq A, X) = (B_1, B_2) \implies \text{False} \]
(proof)

lemma ctyping2-approx-while-aux:
assumes
\[ A: \models b (\subseteq A, X) = (B_1, B_2) \quad \text{and} \quad B: \vdash c (\subseteq B_1, Y) = (C, Y) \quad \text{and} \quad C: \models b (\subseteq C, Y) = (B'_1, B'_2) \quad \text{and} \]
\[ D: (\{\}, \text{False}) \models c (\subseteq B_1, X) = \text{Some} (D, Z) \quad \text{and} \quad E: (\{\}, \text{False}) \models c (\subseteq B'_1, Y) = \text{Some} (D', Z') \quad \text{and} \]
\[ F: r_1 \in A \quad \text{and} \quad G: s_1 = r_1 (\subseteq \text{state} \cap X) \quad \text{and} \quad H: \text{bval} b s_1 \quad \text{and} \]
\[ I: \not\exists \ x \in C. \ s_2 = r (\subseteq \text{state} \cap Y) \Rightarrow \exists r \in B. \ s_3 = r (\subseteq \text{state} \cap W) \]
\[ (\text{is } \not\exists \ x \in C. \ s_2 = r (\subseteq \text{state} \cap Y) \Rightarrow \exists r \in B. \ s_3 = r (\subseteq \text{state} \cap W) \]
\[ J: \not\exists \ x \in C. \ s_1 = r (\subseteq \text{state} \cap X) \Rightarrow \exists r \in B. \ s_2 = r (\subseteq \text{state} \cap Y) \quad \text{and} \]
\[ K: \not\exists s \in \text{Univ}? C Y \cup \text{Univ}? C' Y'. \forall x \in \text{bvars} b. \text{All} (\text{interf s (dom x)}) \wedge \]
\[ \not\exists p \in U. \text{case p of } (B, W) \Rightarrow \forall s \in B. \forall x \in W. \text{All} (\text{interf s (dom x)}) \]
\[ \text{shows } \exists r \in B_2 \cup B'_2. \ s_3 = r (\subseteq \text{state} \cap \text{Univ}? B_2 X \cap Y) \]
(proof)

lemmas ctyping2-approx-while-3 =
ctyping2-approx-while-aux [where \( B_2 = \{\} \), simplified]

lemma ctyping2-approx-while-4:
\[ \models b (\subseteq A, X) = (B_1, B_2); \]
\[ \vdash c (\subseteq B_1, X) = (C, Y); \]
\[ \models b (\subseteq C, Y) = (B'_1, B'_2); \]
\[ (\{\}, \text{False}) \models c (\subseteq B_1, X) = \text{Some} (D, Z); \]
\[ (\{\}, \text{False}) \models c (\subseteq B'_1, Y) = \text{Some} (D', Z'); \]
\[ r_1 \in A; s_1 = r_1 (\subseteq \text{state} \cap X); \text{bval} b s_1; \]
\[ \not\exists \ x \in C. \ s_2 = r (\subseteq \text{state} \cap Y) \Rightarrow \exists r \in B. \ s_3 = r (\subseteq \text{state} \cap W) \]
\[ (\text{is } \not\exists \ x \in C. \ s_2 = r (\subseteq \text{state} \cap Y) \Rightarrow \exists r \in B. \ s_3 = r (\subseteq \text{state} \cap W) \]
\[ J: \not\exists \ x \in C. \ s_1 = r (\subseteq \text{state} \cap X) \Rightarrow \exists r \in B. \ s_2 = r (\subseteq \text{state} \cap Y) \quad \text{and} \]
\[ K: \not\exists s \in \text{Univ}? C Y \cup \text{Univ}? C' Y'. \forall x \in \text{bvars} b. \text{All} (\text{interf s (dom x)}) \wedge \]
\[ \not\exists p \in U. \text{case p of } (B, W) \Rightarrow \forall s \in B. \forall x \in W. \text{All} (\text{interf s (dom x)}) \]
\[ \text{shows } \exists r \in B_2 \cup B'_2. \ s_3 = r (\subseteq \text{state} \cap \text{Univ}? B_2 X \cap Y) \]

(proof)
if (∀ s ∈ Univ? C Y ∪ Univ? C' Y', ∀ x ∈ bvars b. All (interf s (dom x))) ∧
(∀ p ∈ U. case p of (B, W) ⇒ ∀ s ∈ B. ∀ x ∈ W. All (interf s (dom x)))
then case ({}), False) |= c (⊆ B_1', Y) of
  None ⇒ None | Some - ⇒ case ({}), False) |
  = c (⊆ B_1'', Y') of
  None ⇒ None | Some - ⇒ case ({}), False) |
else None = Some (B, W) =⇒
  ∃ r ∈ C. s_2 = r (⊆ state ∩ Y) ⇒ \exists r ∈ B. s_3 = r (⊆ state ∩ W)
  \bigwedge A B X Y U v. (U, v) |= c (⊆ A, X) = Some (B, Y) ⇒
  \exists r ∈ A. s_1 = r (⊆ state ∩ X) ⇒ \exists r ∈ B. s_2 = r (⊆ state ∩ Y)
  \forall s ∈ Univ? A X ∪ Univ? C Y. ∀ x ∈ bvars b. All (interf s (dom x))
  \forall p ∈ U. ∀ B W. p = (B, W) −→ (∀ s ∈ B. ∀ x ∈ W. All (interf s (dom x)))
  \forall r ∈ B_2 ∪ B_2'. \exists x ∈ state ∩ (X ∩ Y). s_3 x ≠ r x \] 
False

⟨proof⟩

lemma ctyping2-approx:
[(c, s) ⇒ t; (U, v) |= c (⊆ A, X) = Some (B, Y);
 s ∈ Univ A (⊆ state ∩ X)] ⇒ t ∈ Univ B (⊆ state ∩ Y)
⟨proof⟩

end

4 Sufficiency of well-typedness for information flow correctness

theory Correctness
  imports Overapproximation
begin

The purpose of this section is to prove that type system ctyping2 is correct in that it guarantees that well-typed programs satisfy the information flow correctness criterion expressed by predicate correct, namely that if the type system outputs a value other than None (that is, a pass verdict) when it is input program c, state set A, and vname set X, then correct c A X (theorem ctyping2-correct).

This proof makes use of the lemmas ctyping1-idem and ctyping2-approx proven in the previous sections.

4.1 Global context proofs

lemma flow-append-1:
  assumes A: \bigwedge \text{cfs} :: (com × state) list.
  c # map fst (cfs :: (com × state) list) = map fst cfs' \implies
  flow-aux (map fst cfs' @ map fst cfs'') =

39
flow-aux (map fst cfs') @ flow-aux (map fst cfs'')
shows flow-aux (c # map fst cfs @ map fst cfs'') = flow-aux (c # map fst cfs) @ flow-aux (map fst cfs'')

⟨proof⟩

lemma flow-append:
flow (cfs @ cfs') = flow cfs @ flow cfs'
⟨proof⟩

lemma flow-cons:
flow (cf # cfs) = flow-aux (fst cf # []) @ flow cfs
⟨proof⟩

lemma small-steps-append:
[(c, s) →*{cfs} (c', s'); (c', s') →*{cfs'} (c'', s'')] ⊃ (c, s) →*{cfs @ cfs'} (c'', s'')
⟨proof⟩

lemma small-steps-cons-1:
(c, s) →*{cf} (c'', s'') ⊃
   cf = (c, s) ∧
   (∃ c' s'. (c, s) → (c', s') ∧ (c', s') →*{cfs} (c'', s''))
⟨proof⟩

lemma small-steps-cons-2:
[(c, s) →*{cf # cfs} (c'', s'')] ⊃
   cf = (c, s) ∧
   (∃ c' s'. (c, s) → (c', s') ∧ (c', s') →*{cfs} (c'', s''))
   (c, s) →*{cf # cfs @ [[c''', s''']] (c''', s''))
⟨proof⟩

lemma small-steps-cons:
(c, s) →*{cf # cfs} (c'', s'') ⊃
   cf = (c, s) ∧
   (∃ c' s'. (c, s) → (c', s') ∧ (c', s') →*{cfs} (c'', s''))
⟨proof⟩

lemma small-steps-stepsl-1:
∃ cfs. (c, s) →*{cfs} (c, s)
⟨proof⟩

lemma small-steps-stepsl-2:
[(c, s) → (c', s'); (c', s') →*{cfs} (c'', s'')] ⊃
   ∃ cfs'. (c, s) →*{cfs'} (c'', s'')

40
(proof)

**Lemma** small-steps-stepsl:
\[(c, s) \to (c', s') \implies \exists cfs. (c, s) \to\{cfs\} (c', s')\]
(proof)

**Lemma** small-steps-steps:
\[(c, s) \to\{cfs\} (c', s') \implies (c, s) \to\{cfs\} (c', s')\]
(proof)

**Lemma** small-steps-skip:
\[(\text{SKIP}, s) \to\{cfs\} (c, t) \implies (c, t) = (\text{SKIP}, s) \land \text{flow cfs} = []\]
(proof)

**Lemma** small-steps-assign-1:
\[(x := a, s) \to\{[\text{a}]\} (c', s') \implies (c', s') = (x := a, s) \land \text{flow cfs} = [] \lor (c', s') = (\text{SKIP}, s(x := \text{val} a s)) \land \text{flow cfs} = [x := a]\]
(proof)

**Lemma** small-steps-assign-2:
\[(x := a, s) \to\{\text{cfs}\} (c', s') \implies (c', s') = (x := a, s) \land \text{flow cfs} = [] \lor (c', s') = (\text{SKIP}, s(x := \text{val} a s)) \land \text{flow cfs} = [x := a];\]
\[(x := a, s) \to\{\text{cfs} \otimes [(c', s')]\} (c'', s'') \implies (c'', s'') = (x := a, s) \land \text{flow (cfs} \otimes [(c', s')]) = [] \lor (c'', s'') = (\text{SKIP}, s(x := \text{val} a s)) \land \text{flow (cfs} \otimes [(c', s')]) = [x := a]\]
(proof)

**Lemma** small-steps-assign:
\[(x := a, s) \to\{\text{cfs}\} (c, t) \implies (c, t) = (x := a, s) \land \text{flow cfs} = [] \lor (c, t) = (\text{SKIP}, s(x := \text{val} a s)) \land \text{flow cfs} = [x := a]\]
(proof)

**Lemma** small-steps-seq-1:
\[(c_1 := c_2, s) \to\{[]\} (c', s') \implies \exists c'' \text{cfs}'. c' = c''; c_2 \land (c_1, s) \to\{\text{cfs}'\} (c'', s') \land \text{flow []} = \text{flow cfs'} \lor (\exists s'' \text{cfs' cfs'}. \text{length cfs''} < \text{length []} \land (c_1, s) \to\{\text{cfs}'\} (\text{SKIP}, s'') \land (c_2, s'') \to\{\text{cfs''}\} (c', s') \land \text{flow []} = \text{flow cfs'} \otimes \text{flow cfs'').\]
\langle proof \rangle

**lemma** small-steps-\textsl{seq}-2:

- **assumes**
  - A: \((c_1; c_2, s) \rightarrow \{cfs\} (c', s') \Rightarrow \)
  - \((\exists c'' cfs'. \; c' = c''; \; c_2 \land \)
  - \((c_1, s) \rightarrow \{cfs\} (c'', s') \land \)
  - \text{flow} cfs = \text{flow} cfs' \lor \)
  - \((\exists c'' cfs'. \; \text{length} cfs'' < \text{length} cfs \land \)
  - \((c_1, s) \rightarrow \{cfs\} (\text{SKIP}, s') \land \)
  - \((c_2, s'') \rightarrow \{cfs''\} (c', s') \land \)
  - \text{flow} cfs = \text{flow} cfs' @ \text{flow} cfs'' \)  

- **and**

- B: \((c_1; c_2, s) \rightarrow \{cfs @ [(c', s')]\} (c'', s'') \)

- **shows**

- \((\exists d \; cfs', \; c'' = d; \; c_2 \land \)
  - \((c_1, s) \rightarrow \{cfs\} (d, s'') \land \)
  - \text{flow} (cfs @ [(c', s')]) = \text{flow} cfs' \lor \)
  - \((\exists t \; cfs' cfs''. \; \text{length} cfs'' < \text{length} (cfs @ [(c', s')]) \land \)
  - \((c_1, s) \rightarrow \{cfs\} (\text{SKIP}, t) \land \)
  - \((c_2, t) \rightarrow \{cfs''\} (c', s'') \land \)
  - \text{flow} (cfs @ [(c', s')]) = \text{flow} cfs' @ \text{flow} cfs'' \)

- (is \(P \lor Q?))

\langle proof \rangle

**lemma** small-steps-\textsl{seq}:

- \((c_1; c_2, s) \rightarrow \{cfs\} (c, t) \Rightarrow \)
  - \((\exists c' cfs'. \; c = c'; \; c_2 \land \)
  - \((c_1, s) \rightarrow \{cfs\} (c', t) \land \)
  - \text{flow} cfs = \text{flow} cfs' \lor \)
  - \((\exists s' cfs' cfs'''. \; \text{length} cfs''' < \text{length} cfs \land \)
  - \((c_1, s) \rightarrow \{cfs\} (\text{SKIP}, s') \land \)
  - \((c_2, s') \rightarrow \{cfs''\} (c, t) \land \)
  - \text{flow} cfs = \text{flow} cfs' @ \text{flow} cfs'' \)

\langle proof \rangle

**lemma** small-steps-\textsl{-if}-1:

- \((\text{IF} \; b \; \text{THEN} \; c_1 \; \text{ELSE} \; c_2, s) \rightarrow \{[]\} (c', s') \Rightarrow \)
  - \((c', s') = (\text{IF} \; b \; \text{THEN} \; c_1 \; \text{ELSE} \; c_2, s) \land \)
  - \text{flow} [] = [] \lor \)
  - \text{bval} b s \land (c_1, s) \rightarrow \{tl []\} (c', s') \land \)
  - \text{flow} [] = (\text{bvars} b) \# \text{flow} (tl []) \lor \)
  - \neg \text{bval} b s \land (c_2, s) \rightarrow \{tl []\} (c', s') \land \)
  - \text{flow} [] = (\text{bvars} b) \# \text{flow} (tl [])

\langle proof \rangle

**lemma** small-steps-\textsl{-if}-2:

- **assumes**

- A: \((\text{IF} \; b \; \text{THEN} \; c_1 \; \text{ELSE} \; c_2, s) \rightarrow \{cfs\} (c', s') \Rightarrow \)
  - \((c', s') = (\text{IF} \; b \; \text{THEN} \; c_1 \; \text{ELSE} \; c_2, s) \land \)

42
\[\text{flow cfs} = \emptyset \lor \]
\[\text{bval b} \land (c_1, s) \rightarrow \{\text{tl cfs}\} \quad (c', s') \land \]
\[\text{flow cfs} = \langle \text{bvars b} \rangle \# \text{flow (tl cfs)} \lor \]
\[\neg \text{bval b} \land (c_2, s) \rightarrow \{\text{tl cfs}\} \quad (c', s') \land \]
\[\text{flow cfs} = \langle \text{bvars b} \rangle \# \text{flow (tl cfs)} \quad \text{and} \quad \]
\[B: \quad \langle \text{IF b THEN c_1 ELSE c_2, s} \rangle \rightarrow \{\text{cfs @ [(c', s')]}\} \quad (c'', s'')\]

\text{shows}
\((c'', s'') = (\text{IF b THEN c_1 ELSE c_2, s}) \land \]
\[\text{flow (cfs @ [(c', s')])} = \emptyset \lor \]
\[\text{bval b} \land (c_1, s) \rightarrow \{\text{tl cfs}\} \quad (c, t) \land \]
\[\text{flow cfs} = \langle \text{bvars b} \rangle \# \text{flow (tl cfs)} \lor \]
\[\neg \text{bval b} \land (c_2, s) \rightarrow \{\text{tl cfs}\} \quad (c, t) \land \]
\[\text{flow cfs} = \langle \text{bvars b} \rangle \# \text{flow (tl cfs)} \quad \text{and} \quad \]
\[\langle \text{is - } \lor \text{ ?P} \rangle\]

\text{proof}

\text{lemma small-stepsl-if:}
\[(\text{IF b THEN c_1 ELSE c_2, s}) \rightarrow \{\text{cfs}\} \quad (c, t) \implies \]
\[(c, t) = (\text{IF b THEN c_1 ELSE c_2, s}) \land \]
\[\text{flow cfs} = \emptyset \lor \]
\[\text{bval b} \land (c_1, s) \rightarrow \{\text{tl cfs}\} \quad (c, t) \land \]
\[\text{flow cfs} = \langle \text{bvars b} \rangle \# \text{flow (tl cfs)} \lor \]
\[\neg \text{bval b} \land (c_2, s) \rightarrow \{\text{tl cfs}\} \quad (c, t) \land \]
\[\text{flow cfs} = \langle \text{bvars b} \rangle \# \text{flow (tl cfs)} \quad \text{and} \quad \]
\[\langle \text{proof} \rangle\]

\text{lemma small-stepsl-while-1:}
\[(\text{WHILE b DO c, s}) \rightarrow \{\emptyset\} \quad (c', s') \implies \]
\[(c', s') = (\text{WHILE b DO c, s}) \land \text{flow} = \emptyset \lor \]
\[(\text{IF b THEN c; WHILE b DO c ELSE SKIP, s}) \rightarrow \{\text{tl []}\} \quad (c', s') \land \]
\[\text{flow []} = \text{flow (tl [])} \quad \text{and} \quad \]
\[\langle \text{proof} \rangle\]

\text{lemma small-stepsl-while-2:}
\text{assumes}
\[\langle \text{A: (WHILE b DO c, s)} \rightarrow \{\text{cfs}\} \quad (c', s') \implies \]
\[(c', s') = (\text{WHILE b DO c, s}) \land \]
\[\text{flow cfs} = \emptyset \lor \]
\[(\text{IF b THEN c; WHILE b DO c ELSE SKIP, s}) \rightarrow \{\text{tl cfs}\} \quad (c', s') \land \]
\[\text{flow cfs} = \text{flow (tl cfs)} \quad \text{and} \quad \]
\[\langle \text{B: (WHILE b DO c, s)} \rightarrow \{\text{cfs @ [(c', s')]}\} \quad (c'', s'')\]
\text{shows}
\[(c'', s'') = (\text{WHILE b DO c, s}) \land \]
\[\text{flow (cfs @ [(c', s')])} = \emptyset \lor \]
\[(\text{IF b THEN c; WHILE b DO c ELSE SKIP, s}) \rightarrow \{\text{tl (cfs @ [(c', s')])}\} \quad (c'', s'') \land \]
\[\text{flow (cfs @ [(c', s')])} = \text{flow (tl (cfs @ [(c', s')])}) \quad \text{and} \quad \]
\[\langle \text{is - } \lor \text{ ?P} \rangle\]

43
\[ \text{lemma } \text{small-stepsl-while:} \]
\[ (\text{WHILE } b \text{ DO } c, s) \rightarrow^* \{cfs\} \ (c', s') \implies \]
\[ (c', s') = (\text{WHILE } b \text{ DO } c, s) \land \]
\[ \text{flow } cfs = [] \lor \]
\[ (\text{IF } b \text{ THEN } c ;; \text{WHILE } b \text{ DO } c \text{ ELSE SKIP}, s) \rightarrow^* \{tlf cfs\} \ (c', s') \land \]
\[ \text{flow } cfs = \text{flow } (tlf cfs) \]
\[ \]
lemma no-upd-sources:
no-upd cs x ⇒ x ∈ sources cs s x
⟨proof⟩

lemma sources-aux-sources:
sources-aux cs s x ⊆ sources cs s x
⟨proof⟩

lemma sources-aux-append:
sources-aux cs s x ⊆ sources-aux (cs @ cs') s x
⟨proof⟩

lemma sources-aux-observe-hd-1:
∀ y ∈ X. s: dom y → dom x ⇒ X ⊆ sources-aux [(X)] s x
⟨proof⟩

lemma sources-aux-observe-hd-2:
∀ y ∈ X. dom y → dom x ⇒ X ⊆ sources-aux (⟨X⟩ # xs) s x
⟨proof⟩

lemma sources-observe-tl-1:
assumes
A: \( \forall z. c = (x := a :: \text{com-flow}) \Rightarrow z = x \Rightarrow \) sources-aux cs s x ⊆ sources-aux (⟨X⟩ # cs) s x and
B: \( \forall z. a y. c = (x := a :: \text{com-flow}) \Rightarrow z = x \Rightarrow \) sources cs s y ⊆ sources (⟨X⟩ # cs) s y and
C: \( \forall z. a. c = (z := a :: \text{com-flow}) \Rightarrow z ≠ x \Rightarrow \) sources cs s x ⊆ sources (⟨X⟩ # cs) s x and
D: \( \forall Y. y. c = (Y) \Rightarrow \) sources cs s y ⊆ sources (⟨X⟩ # cs) s y and
E: z ∈ (case c of
 \( z := a \Rightarrow \) if \( z = x \), then sources-aux cs s x ∪ \( \{ \text{sources cs s y | y, run-flow cs s: dom y → dom x ∧ y ∈ avars a} \} \)
 else sources cs s x |\( \langle X \rangle \Rightarrow \)
sources cs s x ∪ \( \{ \text{sources cs s y | y, run-flow cs s: dom y → dom x ∧ y ∈ X} \} \)
shows z ∈ sources (⟨X⟩ # cs @ [c]) s x
⟨proof⟩
lemma sources-observe-It-2:
assumes
A: \( \forall z \ a. \ c = (z := a :: \text{com-flow}) \Rightarrow \) sources-aux cs s x \( \subseteq \) sources-aux ((X) \# cs) s x and
B: \( \forall Y. \ c = \langle Y \rangle \Rightarrow \) sources-aux cs s x \( \subseteq \) sources-aux ((X) \# cs) s x and
C: \( \forall Y y. \ c = \langle Y \rangle \Rightarrow \) sources cs s y \( \subseteq \) sources ((X) \# cs) s y and
D: \( z \in (\text{case } c \ of} \) 
\( z := a \Rightarrow \) sources-aux cs s x | 
\( \langle X \rangle \Rightarrow \) sources-aux cs s x \( \cup \) \( \bigcup \) sources cs s y \( \cup \) \( \{ \text{sources cs s y | y. run-flow cs s: dom y \rightarrow dom x \land y \in X} \} \)
shows \( z \in \) sources-aux ((X) \# cs @ [c]) s x
(\text{proof})

lemma sources-observe-It:
sources cs s x \( \subseteq \) sources ((X) \# cs) s x
and sources-aux-observe-It:
sources-aux cs s x \( \subseteq \) sources-aux ((X) \# cs) s x
(\text{proof})

lemma sources-member-1:
assumes
A: \( \forall z \ a. \ c = (x := a :: \text{com-flow}) \Rightarrow z = x \Rightarrow \) y \( \in \) sources-aux cs' ((run-flow cs s) x \( \Rightarrow \) sources cs s y \( \subseteq \) sources-aux (cs @ cs') s x and
B: \( \forall z \ a w. \ c = (x := a :: \text{com-flow}) \Rightarrow z = x \Rightarrow \) y \( \in \) sources cs' ((run-flow cs s) w \( \Rightarrow \) sources cs s y \( \subseteq \) sources (cs @ cs') s w and
C: \( \forall z \ a. \ c = (z := a :: \text{com-flow}) \Rightarrow z \neq x \Rightarrow \) y \( \in \) sources cs' ((run-flow cs s) x \( \Rightarrow \) sources cs s y \( \subseteq \) sources (cs @ cs') s x and
D: \( \forall Y w. \ c = \langle Y \rangle \Rightarrow \) y \( \in \) sources cs' ((run-flow cs s) w \( \Rightarrow \) sources cs s y \( \subseteq \) sources (cs @ cs') s w and
E: \( y \in (\text{case } c \ of} \) 
\( z := a \Rightarrow \) if \( z = x \) 
then sources-aux cs' ((run-flow cs s) x \( \cup \) \( \bigcup \) sources cs' ((run-flow cs s) y \( \Rightarrow \) y. run-flow cs' ((run-flow cs s): dom y \rightarrow dom x \land y \in \text{avars a}) 
else sources cs' ((run-flow cs s) x \( \Rightarrow \) sources cs' ((run-flow cs s) x \( \cup \) \( \bigcup \) sources cs' ((run-flow cs s) y \( \Rightarrow \) y. run-flow cs' ((run-flow cs s): dom y \rightarrow dom x \land y \in X)) and
F: \( z \in \) sources cs s y
shows $z \in \text{sources } (cs \oplus cs' \oplus [c]) s x$

(\text{proof})

\textbf{lemma} sources-member-2:
assumes
A: $\forall z. a. c = (z::a :: \text{com-flow}) \implies$
\hspace{1cm} $y \in \text{sources-aux } cs' (\text{run-flow } cs s) x \implies$
\hspace{2cm} $\text{sources } cs s y \subseteq \text{sources-aux } (cs \oplus cs') s x$ \text{ and}
B: $\forall Y. c = (Y) \implies$
\hspace{1cm} $y \in \text{sources-aux } cs' (\text{run-flow } cs s) x \implies$
\hspace{2cm} $\text{sources } cs s y \subseteq \text{sources-aux } (cs \oplus cs') s x$ \text{ and}
C: $\forall w. c = (Y) \implies$
\hspace{1cm} $y \in \text{sources-aux cs} (\text{run-flow } cs s) w \implies$
\hspace{2cm} $\text{sources } cs s y \subseteq \text{sources-aux } cs' (cs \oplus cs') s x$ \text{ and}
D: $y \in \text{(case } c \text{ of}$
\hspace{1cm} $z ::= a \Rightarrow$
\hspace{2cm} $\text{sources-aux } cs' (\text{run-flow } cs s) x |$
\hspace{3cm} $(X) \Rightarrow$
\hspace{4cm} $\text{sources-aux } cs' (\text{run-flow } cs s) x \cup$
\hspace{5cm} $\bigcup \{ \text{sources } cs' (\text{run-flow } cs s) y \mid y.$
\hspace{6cm} $\text{run-flow } cs' (\text{run-flow } cs s); \text{dom } y \rightsquigarrow \text{dom } x \wedge y \in X)\} \text{ and}$
E: $z \in \text{sources } cs s y$
\text{shows } $z \in \text{sources-aux } (cs \oplus cs' \oplus [c]) s x$
(\text{proof})

\textbf{lemma} sources-member:
\hspace{1cm} $y \in \text{sources } cs' (\text{run-flow } cs s) x \implies$
\hspace{2cm} $\text{sources } cs s y \subseteq \text{sources-aux } (cs \oplus cs') s x$
\hspace{2cm} \text{and}
\textbf{sources-aux-member:}
\hspace{1cm} $y \in \text{sources-aux cs} (\text{run-flow } cs s) x \implies$
\hspace{2cm} $\text{sources } cs s y \subseteq \text{sources-aux } cs' (cs \oplus cs') s x$
(\text{proof})

\textbf{lemma} ctyping2-confine:
\hspace{1cm} $[(c, s) \Rightarrow s'; (U, v) \vdash c (\subseteq A, X) = \text{Some } (B, Y);$
\hspace{2cm} $\exists (C, Z) \in U. \neg C: \text{dom } ' Z \rightsquigarrow \{ \text{dom } x \}] \Rightarrow s' x = s x$
(\text{proof})

\textbf{lemma} ctyping2-term-if:
\hspace{1cm} $[\forall x'. y' z' s. x' = x \Rightarrow y' = y \Rightarrow z = z' \Rightarrow \exists s'. (c_1, s) \Rightarrow s';$
\hspace{2cm} $\forall x'. y' z' s. x' = x \Rightarrow y' = y \Rightarrow z' = z'' \Rightarrow \exists s'. (c_2, s) \Rightarrow s'] \Rightarrow$
\hspace{2cm} $\exists s'. (\text{IF } b \text{ THEN } c_1 \text{ ELSE } c_2, s) \Rightarrow s'$
(\text{proof})

\textbf{lemma} ctyping2-term:
\hspace{1cm} $[(U, v) \vdash c (\subseteq A, X) = \text{Some } (B, Y);$
\hspace{2cm} $\exists (C, Z) \in U. \neg C: \text{dom } ' Z \rightsquigarrow \text{UNIV} ] \Rightarrow \exists s'. (c, s) \Rightarrow s'$
(\text{proof})

47
\section*{lemmas}

**ctyping2-correct-aux-seq** [elim]:

\begin{align*}
& \text{[(}SKIP, s\text{) }\rightarrow\ast\{cfs_1\} (c_1, s_1); \ (c_1, s_1) \rightarrow\ast\{cfs_2\} (c_2, s_2)\text{]} \implies \\
& (\forall t_1, \exists c_1' t_2, \forall x. \\
& \quad (s_1 = t_1 (\subseteq \text{sources-aux (flow cfs_2)} \ s_1 \ x) \rightarrow \\
& \quad \quad (c_1, t_1) \rightarrow\ast (c_2', t_2) \land (c_2 = \text{SKIP}) = (c_2' = \text{SKIP}) \land \\
& \quad (s_1 = t_1 (\subseteq \text{sources (flow cfs_2)} \ s_1 \ x) \rightarrow s_2 x = t_2 x) \land \\
& \quad (\forall x. (\exists p \in U. \ case p \ of \ (B, W) \Rightarrow \\
& \quad \quad \exists s \in B. \exists y \in W. \neg s \Rightarrow \text{no-upd (flow cfs_2)} \ x) \\
& \quad \quad \{\text{proof}\})}
\end{align*}

**ctyping2-correct-aux-assign** [elim]:

\begin{align*}
& \text{assumes} \\
& A: (if (\forall s \in \text{Univ?} A X. \forall y \in \text{avars a. s: dom y }\rightarrow\text{ dom x}) \land \\
& \quad (\forall p \in U. \forall B Y. \ p = (B, Y)) \implies \\
& \quad (\forall s \in B. \forall y \in Y. \ s: \text{ dom y }\rightarrow\text{ dom x}) \\
& \quad \text{then Some (if } x \in \text{ state } \land A \neq \{\} \land \\
& \quad \quad \text{then } v \models a (\subseteq X) \land \\
& \quad \quad \text{then } (\{s(x := \text{ aval a s}) | s. s \in A\}, \text{ insert x X}) \land \\
& \quad \quad \text{else } (A, X - \{x\}) \land \\
& \quad \quad \text{else None) } = \text{ Some (B, Y)} \\
& \quad (\text{is (if } \text{IP then - else -) } = -) \text{ and} \\
& B: (x := a, s) \rightarrow\ast\{cfs_1\} (c_1, s_1) \text{ and} \\
& C: (c_1, s_1) \rightarrow\ast\{cfs_2\} (c_2, s_2) \text{ and} \\
& D: r \in A \text{ and} \\
& E: s = r (\subseteq \text{ state } \cap X) \\
& \text{shows} \\
& (\forall t_1, \exists c_1' t_2. \forall x. \\
& \quad (s_1 = t_1 (\subseteq \text{sources-aux (flow cfs_2)} \ s_1 \ x) \rightarrow \\
& \quad \quad (c_1, t_1) \rightarrow\ast (c_2', t_2) \land (c_2 = \text{SKIP}) = (c_2' = \text{SKIP}) \land \\
& \quad \quad (s_1 = t_1 (\subseteq \text{sources (flow cfs_2)} \ s_1 \ x) \rightarrow s_2 x = t_2 x) \land \\
& \quad \quad (\forall x. (\exists p \in U. \ case p \ of \ (B, Y) \Rightarrow \\
& \quad \quad \exists s \in B. \exists y \in Y. \neg s \Rightarrow \text{no-upd (flow cfs_2)} \ x) \\
& \quad \quad \{\text{proof}\})}
\end{align*}

**ctyping2-correct-aux-seq**:

\begin{align*}
& \text{assumes} \\
& A: \forall B' s' c''' s_1' s_2' cfs_1' cfs_2. \ B = B' \implies \\
& \quad \exists r \in A. \ s = r (\subseteq \text{ state } \cap X) \implies \\
& \quad (c_1, s) \rightarrow\ast\{cfs_1\} (c', s_1) \Rightarrow (c', s_1) \rightarrow\ast\{cfs_2\} (c'', s_2) \Rightarrow \\
& \quad (\forall t_1. \exists c_2' t_2. \forall x. \\
& \quad \quad (s_1 = t_1 (\subseteq \text{sources-aux (flow cfs_2)} \ s_1 \ x) \rightarrow \\
& \quad \quad \quad (c', t_1) \rightarrow\ast (c_2', t_2) \land (c'' = \text{SKIP}) = (c_2' = \text{SKIP}) \land \\
& \quad \quad \quad (s_1 = t_1 (\subseteq \text{sources (flow cfs_2)} \ s_1 \ x) \rightarrow s_2 x = t_2 x) \land \\
& \quad \quad \quad (\forall x. (\exists p \in U. \ case p \ of \ (B, W) \Rightarrow \\
& \quad \quad \quad \exists s \in B. \exists y \in W. \neg s \Rightarrow \text{no-upd (flow cfs_2)} \ x) \text{ and} \\
& \quad \quad \quad \{\text{proof}\})}
\end{align*}

48
lemma ctotyping2-correct-aux-if:

assumes

\( A: \bigwedge U' B C s c' c'' s_1 s_2 cfs_1 cfs_2.\)
\( U' = \text{insert} (\text{Univ}? A X, \text{bvars} b) U \implies B = B_1 \implies C_1 = C \implies\)
\( \exists r \in B_1. s = r (\subseteq \text{state} \cap X) \implies\)
\( (c_1, s) \to* (cfs_1) (c', s_1) \implies (c', s_1) \to* (cfs_2) (c'', s_2) \implies\)
\( (\forall t_1, \exists c_2', t_2. \forall x.\)
\( (s_1 = t_1 (\subseteq \text{sources-aux} (\text{flow} cfs_2) s_1 x) \implies\)
\( (c', t_1) \to* (c_2', t_2) \land (c'' = \text{SKIP}) = (c_2' = \text{SKIP})) \land\)
\( (s_1 = t_1 (\subseteq \text{sources} (\text{flow} cfs_2) s_1 x) \implies s_2 x = t_2 x) \land\)
\( (\forall x. (\exists p \in U. \text{case} p \text{ of } (B, W) \implies\)
\( \exists s \in B. \exists y \in W. \neg s: \text{dom} y \leadsto \text{dom} x) \implies \text{no-upd} (\text{flow} cfs_2) x)\) and

\( C: (U, v) = c_1 (\subseteq A, X) = \text{Some } (B, Y) \text{ and}\)
\( D: (U, v) = c_2 (\subseteq B, Y) = \text{Some } (C, Z) \text{ and}\)
\( E: (c_1; c_2, s) \to* (cfs_1) (c', s_1) \text{ and}\)
\( F: (c', s_1) \to* (cfs_2) (c'', s_2) \text{ and}\)
\( G: r \in A \text{ and}\)
\( H: s = r (\subseteq \text{state} \cap X)\) shows

\( (\forall t_1, \exists c_2' t_2. \forall x.\)
\( (s_1 = t_1 (\subseteq \text{sources-aux} (\text{flow} cfs_2) s_1 x) \implies\)
\( (c', t_1) \to* (c_2', t_2) \land (c'' = \text{SKIP}) = (c_2' = \text{SKIP})) \land\)
\( (s_1 = t_1 (\subseteq \text{sources} (\text{flow} cfs_2) s_1 x) \implies s_2 x = t_2 x) \land\)
\( (\forall x. (\exists p \in U. \text{case } p \text{ of } (B, W) \implies\)
\( \exists s \in B. \exists y \in W. \neg s: \text{dom } y \leadsto \text{dom } x) \implies \text{no-upd} (\text{flow} cfs_2) x)\) (proof)
\[(s_1 = t_1 (\subseteq sources (flow cfs_{s_2}) s_1 x) \rightarrow s_2 x = t_2 x)) \land
(\forall x, (\exists s \in Univ? A X. \exists y \in bvars b. \neg s. dom y \rightsquigarrow dom x) \rightarrow
\text{no-upd} (flow cfs_{s_2}) x) \land
((\exists p \in U. \text{case } p \text{ of } (B, W) \Rightarrow
\exists s \in B. \exists y \in W. \neg s. dom y \rightsquigarrow dom x) \rightarrow
\text{no-upd} (flow cfs_{s_2}) x)) \text{ and}
\]

\[C: \models b (\subseteq A, X) = (B_1, B_2) \text{ and}
D: (\text{insert} (Univ? A X, bvars b) U, v) \models c_1 (\subseteq B_1, X) =
\text{Some} (C_1, Y_1) \text{ and}
E: (\text{insert} (Univ? A X, bvars b) U, v) \models c_2 (\subseteq B_2, X) =
\text{Some} (C_2, Y_2) \text{ and}
F: (IF b \text{ THEN } c_1 \text{ ELSE } c_2, s) \rightarrow \{cfs_1\} (c', s_1) \text{ and}
G: (c', s_1) \rightarrow \{cfs_2\} (c'', s_2) \text{ and}
H: r \in A \text{ and}
I: s = r (\subseteq state \land X)
\]
\[\text{shows}
(\forall t_1. \exists c_2'. t_2. \forall x.
(s_1 = t_1 (\subseteq sources-aux (flow cfs_{s_2}) s_1 x) \rightarrow
(c', t_1) \rightarrow (c_2', t_2) \land (c'' = SKIP) = (c_2' = SKIP)) \land
(s_1 = t_1 (\subseteq sources (flow cfs_{s_2}) s_1 x) \rightarrow s_2 x = t_2 x)) \land
(\forall x. (\exists p \in U. \text{case } p \text{ of } (B, W) \Rightarrow
\exists s \in B. \exists y \in W. \neg s. dom y \rightsquigarrow dom x) \rightarrow
\text{no-upd} (flow cfs_{s_2}) x)
\]
\[\text{proof}
\]

\[\text{lemma ctyping2-correct-aux-while:}
\]
\[\text{assumes}
A: \forall B C' B' D' s c_1 c_2 s_1 s_2 cfs_1 cfs_2.
B = B_1 \implies C' = C \implies B' = B_1' \implies
(\forall s \in Univ? A X \cup Univ? C Y. \forall x \in bvars b. \text{All} (\text{interf } s (\text{dom } x))) \land
(\forall p \in U. \text{case } p \text{ of } (B, W) \Rightarrow \forall s \in B. \forall x \in W. \text{All} (\text{interf } s (\text{dom } x))) \implies
D = D' \implies \exists r \in B_1. s = r (\subseteq state \land X) \implies
(c, s) \rightarrow \{cfs_1\} (c_1, s_1) \implies (c_1, s_1) \rightarrow \{cfs_2\} (c_2, s_2) \implies
\forall t_1. \exists c_2'. t_2. \forall x.
(s_1 = t_1 (\subseteq sources-aux (flow cfs_{s_2}) s_1 x) \rightarrow
(c_1, t_1) \rightarrow (c_2', t_2) \land (c_2 = SKIP) = (c_2' = SKIP)) \land
(s_1 = t_1 (\subseteq sources (flow cfs_{s_2}) s_1 x) \rightarrow s_2 x = t_2 x)) \text{ and}
B: \forall B C' B' D'' s c_1 c_2 s_1 s_2 cfs_1 cfs_2.
B = B_1 \implies C' = C \implies B' = B_1' \implies
(\forall s \in Univ? A X \cup Univ? C Y. \forall x \in bvars b. \text{All} (\text{interf } s (\text{dom } x))) \land
(\forall p \in U. \text{case } p \text{ of } (B, W) \Rightarrow \forall s \in B. \forall x \in W. \text{All} (\text{interf } s (\text{dom } x))) \implies
D'' = D' \implies \exists r \in B_1'. s = r (\subseteq state \land Y) \implies
(c, s) \rightarrow \{cfs_1\} (c_1, s_1) \implies (c_1, s_1) \rightarrow \{cfs_2\} (c_2, s_2) \implies
\forall t_1. \exists c_2'. t_2. \forall x.
(s_1 = t_1 (\subseteq sources-aux (flow cfs_{s_2}) s_1 x) \rightarrow
(c_1, t_1) \rightarrow (c_2', t_2) \land (c_2 = SKIP) = (c_2' = SKIP)) \land
(s_1 = t_1 (\subseteq sources (flow cfs_{s_2}) s_1 x) \rightarrow s_2 x = t_2 x) \text{ and}
C: (if (\forall s \in Univ? A X \cup Univ? C Y. \forall x \in bvars b. \text{All} (\text{interf } s (\text{dom } x))) \land
(\forall p \in U. \forall B W. p = (B, W) \Rightarrow (\forall s \in B. \forall x \in W. \text{All} (\text{interf } s (\text{dom } x))))
\]
50
then Some \((B_2 \cup B_2', \text{Univ} \cap B_2 X \cap Y)\) else None) = Some \((B, W)\) and

\[ D: \models b \subseteq A, X = (B_1, B_2) \text{ and} \]
\[ E: \vdash c \subseteq B_1, X = (C, Y) \text{ and} \]
\[ F: \models b \subseteq C, Y = (B_1', B_2') \text{ and} \]
\[ G: (\emptyset, \text{False}) \models c \subseteq B_1, X = \text{Some} \((D, Z)\) \text{ and} \]
\[ H: (\emptyset, \text{False}) \models c \subseteq B_1', Y = \text{Some} \((D', Z')\) \text{ shows} \]
\[
\langle \text{proof} \rangle
\]

**lemma** ctyping2-correct-aux:

\[
\langle \text{proof} \rangle
\]

**theorem** ctyping2-correct:

\[
\langle \text{proof} \rangle
\]

\[
\text{5 Degeneracy to stateless level-based information flow control}
\]

**theory** Degeneracy

\[
\langle \text{proof} \rangle
\]

The goal of this concluding section is to prove the degeneracy of the information flow correctness notion and the static type system defined in this paper to the classical counterparts addressed in [7], section 9.2.6, and formalized in [5] and [6], in case of a stateless level-based information flow correctness policy.

First of all, locale noninterf is interpreted within the context of the class
sec defined in [5], as follows.

- Parameter dom is instantiated as function sec, which also sets the type variable standing for the type of the domains to nat.
- Parameter interf is instantiated as the predicate such that for any program state, the output is True just in case the former input level may interfere with, namely is not larger than, the latter one.
- Parameter state is instantiated as the empty set, consistently with the fact that the policy is represented by a single, stateless interference predicate.

Next, the information flow security notion implied by theorem noninterference in [6] is formalized as a predicate secure taking a program as input. This notion is then proven to be implied, in the degenerate interpretation described above, by the information flow correctness notion formalized as predicate correct (theorem correct-secure). Particularly:

- This theorem demands the additional assumption that the state set A input to correct is nonempty, since correct is vacuously true for A = {}.
- In order for this theorem to hold, predicate secure needs to slightly differ from the information flow security notion implied by theorem noninterference, in that it requires state t’ to exist if there also exists some variable with a level not larger than l, namely if condition \( s = t' (\leq l) \) is satisfied nontrivially – actually, no leakage may arise from two initial states disagreeing on the value of every variable. In fact, predicate correct requires a nontrivial configuration \((c_2', t_2)\) to exist in case condition \( s_1 = t_1 (\subseteq \text{sources es s}_1 x) \) is satisfied for some variable x.

Finally, the static type system ctyping\(^2\) is proven to be equivalent to the sec-type one defined in [6] in the above degenerate interpretation (theorems ctyping\(^2\)-sec-type and sec-type-ctyping\(^2\)). The former theorem, which proves that a pass verdict from ctyping\(^2\) implies the issuance of a pass verdict from sec-type as well, demands the additional assumptions that (a) the state set input to ctyping\(^2\) is nonempty, (b) the input program does not contain any loop with \( Bc \ True \) as boolean condition, and (c) the input program has undergone constant folding, as addressed in [7], section 3.1.3 for arithmetic expressions and in [7], section 3.2.1 for boolean expressions. Why? This need arises from the different ways in which the two type systems handle “dead” conditional branches. Type system sec-type does not try to detect “dead” branches; it simply applies its full range of information flow
security checks to any conditional branch contained in the input program, even if it actually is a “dead” one. On the contrary, type system \texttt{ctyping2} detects “dead” branches whenever boolean conditions can be evaluated at compile time, and applies only a subset of its information flow correctness checks to such branches.

As parameter \textit{state} is instantiated as the empty set, boolean conditions containing variables cannot be evaluated at compile time, yet they can if they only contain constants. Thus, assumption (a) prevents \texttt{ctyping2} from handling the entire input program as a “dead” branch, while assumptions (b) and (c) ensure that \texttt{ctyping2} will not detect any “dead” conditional branch within the program. On the whole, those assumptions guarantee that \texttt{ctyping2}, like \texttt{sec-type}, applies its full range of checks to any conditional branch contained in the input program, as required for theorem \texttt{ctyping2-sec-type} to hold.

5.1 Global context definitions and proofs

\begin{verbatim}
fun cgood :: com ⇒ bool where
cgood (c1;; c2) = (cgood c1 ∧ cgood c2) |
cgood (IF - THEN c1 ELSE c2) = (cgood c1 ∧ cgood c2) |
cgood (WHILE b DO c) = (b ≠ Bc True ∧ cgood c) |
cgood - = True

fun seq :: com ⇒ com ⇒ com where
seq SKIP c = c |
seq c SKIP = c |
seq c1 c2 = c1;; c2

fun ifc :: bexp ⇒ com ⇒ com ⇒ com ⇒ com where
ifc (Bc True) c - = c |
ifc (Bc False) c = c |
ifc b c1 c2 = (if c1 = c2 then IF b THEN c1 ELSE c2)

fun while :: bexp ⇒ com ⇒ com where
while (Bc False) - = SKIP |
while b c = WHILE b DO c

primrec csimp :: com ⇒ com where
csimp SKIP = SKIP |
csimp (x ::= a) = x ::= asimp a |
csimp (c1;; c2) = seq (csimp c1) (csimp c2) |
csimp (IF b THEN c1 ELSE c2) = ifc (bsimp b) (csimp c1) (csimp c2) |
csimp (WHILE b DO c) = while (bsimp b) (csimp c)

lemma not-size:
\end{verbatim}
\[\text{size (not } b) \leq \text{Suc (size } b)\]

\text{proof}

\text{lemma and-size:}
\[\text{size (and } b_1 \ b_2) \leq \text{Suc (size } b_1 + \text{size } b_2)\]

\text{proof}

\text{lemma less-size:}
\[\text{size (less } a_1 \ a_2) = 0\]

\text{proof}

\text{lemma bsimp-size:}
\[\text{size (bsimp } b) \leq \text{size } b\]

\text{proof}

\text{lemma seq-size:}
\[\text{size (seq } c_1 \ c_2) \leq \text{Suc (size } c_1 + \text{size } c_2)\]

\text{proof}

\text{lemma ifc-size:}
\[\text{size (ifc } b \ c_1 \ c_2) \leq \text{Suc (size } c_1 + \text{size } c_2)\]

\text{proof}

\text{lemma while-size:}
\[\text{size (while } b \ c) \leq \text{Suc (size } c)\]

\text{proof}

\text{lemma csimp-size:}
\[\text{size (csimp } c) \leq \text{size } c\]

\text{proof}

\text{lemma avars-asimp:}
\[\text{avars } a = \{\} \implies \exists \ i. \ \text{asimp } a = N i\]

\text{proof}

\text{lemma seq-match [dest!]:}
\[\text{seq (csimp } c_1) \ (\text{csimp } c_2) = c_1; \ c_2 \implies \text{csimp } c_1 = c_1 \land \text{csimp } c_2 = c_2\]

\text{proof}

\text{lemma ifc-match [dest!]:}
\[\text{ifc (bsimp } b) \ (\text{csimp } c_1) \ (\text{csimp } c_2) = \text{IF } b \ \text{THEN } c_1 \ \text{ELSE } c_2 \implies \text{bsimp } b = b \land (\forall \ v. \ b \neq Bc v) \land \text{csimp } c_1 = c_1 \land \text{csimp } c_2 = c_2\]

\text{proof}

\text{lemma while-match [dest!]:}
\[\text{while (bsimp } b) \ (\text{csimp } c) = \text{WHILE } b \ \text{DO } c \implies \text{bsimp } b = b \land b \neq Bc \text{ False} \land \text{csimp } c = c\]
5.2 Local context definitions and proofs

context sec
begin

interpretation noninterf λs. (≤) sec { }

notation interf-set ((-: -→ -) [51, 51, 51] 50)
notation univ-states-if (((Univ? -→) [51, 75] 75)
notation atyping ((| - | = -′ (⊆ -′)) [51, 51] 50)
notation btyping2-aux (|| = -′ (⊆ -′)) [51, 55]
notation btyping2 ((| - | = -′ (⊆ -′)) [51, 51] 55)
notation cttyping1 ((| - | = -′ (⊆ -′)) [51, 51] 55)
notation cttyping2 ((| - | = -′ (⊆ -′)) [51, 51] 55)

abbreviation eq-le-ext :: state ⇒ state ⇒ level ⇒ bool
((= - -′ (≤ -′)) [51, 51, 0] 50) where
s = t (≤ l) ≡ s = t (≤ l) ∧ (∃ x :: vname. sec x ≤ l)

definition secure :: com ⇒ bool where
secure c ≡ ∀ s s′ t l. (c, s) ⇒ s′ ∧ s = t (≤ l) —>
(∃ t′. (c, t) ⇒ t′ ∧ s′ = t′ (≤ l))

definition levels :: config set ⇒ level set where
levels U ≡ insert 0 (sec ∪ (snd' {(B, Y) ∈ U. B ≠ {}}))

lemma avars-finite:
finite (avars a)
⟨proof⟩

lemma avars-in:
n < sec a ⇒ sec a ∈ sec ′ avars a
⟨proof⟩

lemma avars-sec:
x ∈ avars a ⇒ sec x ≤ sec a
⟨proof⟩

lemma avars-ub:
sec a ≤ l = (∀ x ∈ avars a. sec x ≤ l)
⟨proof⟩
lemma bvars-finite:
finite (bvars b)
⟨proof⟩

lemma bvars-in:
n < sec b =⇒ sec b ∈ sec ' bvars b
⟨proof⟩

lemma bvars-sec:
x ∈ bvars b =⇒ sec x ≤ sec b
⟨proof⟩

lemma bvars-ub:
sec b ≤ l = (∀ x ∈ bvars b. sec x ≤ l)
⟨proof⟩

lemma levels-insert:
assumes
A: A ≠ {} and
B: finite (levels U)
shows finite (levels (insert (A, bvars b) U)) ∧
Max (levels (insert (A, bvars b) U)) = max (sec b) (Max (levels U))
(is finite (levels ?U') ∧ ?P)
⟨proof⟩

lemma sources-le:
y ∈ sources cs s x =⇒ y ≤ sec x
and sources-aux-le:
y ∈ sources-aux cs s x =⇒ y ≤ sec x
⟨proof⟩

lemma bsimp-btyping2-aux-not [intro]:
[bsimp b = b =⇒ ∀ v. b ≠ Be v =⇒ b (⊆ A, X) = None;
not (bsimp b) = Not b] =⇒ b (⊆ A, X) = None
⟨proof⟩

lemma bsimp-btyping2-aux-and [intro]:
assumes
A: [bsimp b₁ = b₁; ∀ v. b₁ ≠ Be v] =⇒ b₁ (⊆ A, X) = None and
B: and (bsimp b₁) (bsimp b₂) = And b₁ b₂
shows b₁ (⊆ A, X) = None
⟨proof⟩

lemma bsimp-btyping2-aux-less [elim]:
[less (asimp a₁) (asimp a₂) = Less a₁ a₂;
avars a₁ = {}; avars a₂ = {}] =⇒ False
⟨proof⟩

56
lemma bsimp-btyping2-aux:
\[ [ bsimp b = b; \forall v. b \neq Bc v ] \implies b (\subseteq A, X) = None \]
(proof)

lemma bsimp-btyping2:
\[ [ bsimp b = b; \forall v. b \neq Bc v ] \implies b (\subseteq A, X) = (A, A) \]
(proof)

lemma csimp-ctyping2-if:
\[ [ \forall \bar{U} B B'. \bar{U} = U \implies B = B_1 \implies \{ \} = B' \implies B_1 \neq \{ \} \implies False; s \in A; b (\subseteq A, X) = (B_1, B_2); bsimp b = b; \forall v. b \neq Bc v ] \implies False \]
(proof)

lemma csimp-ctyping2-while:
\[(if P then Some (B_2 \cup B_2', Y) else None) = Some (\{\}, Z); s \in A; b (\subseteq A, X) = (B_1, B_2); bsimp b = b; b \neq Bc True; b \neq Bc False \implies False \]
(proof)

lemma csimp-ctyping2:
\[ ((U, v) \models c (\subseteq A, X) = Some (B, Y); A \neq \{\}; cgood c; csimp c = c] \implies B \neq \{\} \]
(proof)

theorem correct-secure:
assumes
\[ A: correct c A X and \]
\[ B: A \neq \{\} \]
shows
secure c
(proof)

lemma ctyping2-sec-type-assign [elim]:
assumes
\[ A: (if (\exists s. s \in Univ? A X) \implies (\forall y \in avars a. \sec y \leq \sec x)) \land \]
\[ (\forall p \in U. \forall B Y. p = (B, Y) \implies B = \{\} \lor (\forall y \in Y. \sec y \leq \sec x)) \]
then Some (if x \in \{\} \land A \neq \{\}) \land then if v \models a (\subseteq X) \land \]
then (\{ s(x := aval a s) | s. s \in A \}, insert x X) else (A, X - \{x\}) \land \]
e else (A, Univ? A X)) \]
else None = Some (B, Y) \]
(is (if (- \longrightarrow \ ?P) \land \ ?Q then - else -) = -) and
\[ B: s \in A and \]
C: finite (levels U)
shows
Max (levels U) \vdash x ::= a

57
proof

lemma ctyping2-sec-type-seq:
  assumes A: \( \bigwedge B'\ s. B = B' \Rightarrow s \in A \Rightarrow \text{Max}(\text{levels}\ U) \vdash c_1 \) and  
  B: \( \bigwedge B'' C Z s'. B = B'' \Rightarrow B'' = B' \Rightarrow \) 
  \( (U, v) \vdash c_2 (\subseteq B', Y) = \text{Some}(C, Z) \Rightarrow \) 
  \( s' \in B' \Rightarrow \text{Max}(\text{levels}\ U) \vdash c_2 \) and  
  C: \( (U, v) \vdash c_1 (\subseteq A, X) = \text{Some}(B, Y) \) and  
  D: \( (U, v) \vdash c_2 (\subseteq B, Y) = \text{Some}(C, Z) \) and  
  E: s \in A and  
  F: cgood c_1 and  
  G: csimp c_1 = c_1  
  shows Max(levels U) \vdash c_1; c_2

(proof)

lemma ctyping2-sec-type-if:
  assumes A: \( \bigwedge U' B C s. U' = \text{insert}(\text{Univ} A X, \text{bvars}\ b) U \Rightarrow \) 
  B = B_1 \Rightarrow C_1 = C \Rightarrow s \in B_1 \Rightarrow  
  finite(levels(insert(\text{Univ} A X, \text{bvars}\ b) U)) \Rightarrow \) 
  Max(levels(insert(\text{Univ} A X, \text{bvars}\ b) U)) \vdash c_1  
  (is \( \wedge \ldots \Rightarrow \mathbf{?U'} \Rightarrow \ldots \Rightarrow \ldots \Rightarrow \mathbf{?} \)\)  
  assumes B: \( \bigwedge U' B C s. U' = \mathbf{?U'} \Rightarrow B = B_1 \Rightarrow C_2 = C \Rightarrow s \in B_2 \Rightarrow \) 
  finite(levels(\mathbf{?U'})) \Rightarrow Max(levels(\mathbf{?U'})) \vdash c_2 and  
  C: \( \equiv b (\subseteq A, X) = (B_1, B_2) \) and  
  D: s \in A and  
  E: bsimp b = b and  
  F: \( \forall v. b \neq B_2 v \) and  
  G: finite(levels U)  
  shows Max(levels U) \vdash IF b THEN c_1 ELSE c_2

(proof)

lemma ctyping2-sec-type-while:
  assumes A: \( \bigwedge B C' B' D' s. B = B_1 \Rightarrow C' = C \Rightarrow B' = B_1' \Rightarrow \) 
  \((\exists s. s \in \text{Univ}\ A X \lor s \in \text{Univ}\ C Y) \Rightarrow \) 
  \((\forall x \in \text{bvars}\ b. \text{All}(\subseteq (\text{sec}\ x))) \wedge \) 
  \((\forall x \in U. \text{case}\ p\ of\ (B, W) \Rightarrow (\exists s. s \in B) \Rightarrow \) 
  \((\forall x \in W. \text{All}(\subseteq (\text{sec}\ x))) \Rightarrow \) 
  D = D' \Rightarrow s \in B_1 \Rightarrow finite(levels\ \{\}) \Rightarrow Max(levels\ \{\}) \vdash c  
  (is \( \wedge \ldots \Rightarrow \ldots \Rightarrow \ldots \Rightarrow \ldots \Rightarrow \ldots \Rightarrow \ldots \Rightarrow \ldots \Rightarrow \ldots \Rightarrow \mathbf{?} \)\)  
  assumes B: \( \text{if}\ \mathbf{?P} \wedge (\forall p \in U. \forall B W. p = (B, W) \Rightarrow B = \{\} \lor \mathbf{?Q W}) \) \text{then Some}\ (B_2 \cup B_2', \text{Univ}\ B_2 X \cap Y) \text{else None} = \text{Some}\ (B, Z)  
  (is \( \text{if}\ \mathbf{?R then - else -} = - \)\) and  

58
C: \models b (\subseteq A, X) = (B_1, B_2) \textbf{ and}

D: s \in A \textbf{ and}

E: bsimp b = b \textbf{ and}

F: b \neq Bc \textbf{ False and}

G: b \neq Bc \textbf{ True and}

H: finite (\text{levels} U)

shows \text{Max} (\text{levels} U) \vdash \text{WHILE} b \text{ DO } c

(\text{proof})

\textbf{theorem} typing2-sec-type:

\[\[(U, v) \models c (\subseteq A, X) = \text{Some} (B, Y);\]
\[s \in A; \text{ecod } c; \text{csimp } c = c; \text{finite (levels} U)\]\[\implies \text{Max (levels} U) \vdash c\]

(\text{proof})

\textbf{lemma} sec-type-ctyping2-if:

\textbf{assumes}
\[A: \forall U' B_1 B_2. U' = \text{insert} (\text{Univ}? A X, \text{bvars} b) U \implies\]
\[(B_1, B_2) = \models b (\subseteq A, X) \implies\]
\[\text{Max (levels} (\text{insert} (\text{Univ}? A X, \text{bvars} b) U)) \vdash c_1 \implies\]
\[\text{finite (levels} (\text{insert} (\text{Univ}? A X, \text{bvars} b) U)) \implies\]
\[\exists C Y. (\text{insert} (\text{Univ}? A X, \text{bvars} b) U, v) \models c_1 (\subseteq B_1, X) = \text{Some} (C, Y)\]

(\text{is } \wedge - - - = ?U' \implies - \implies - \implies - \implies -)

\textbf{assumes}
\[B: \forall U' B_1 B_2. U' = ?U' \implies (B_1, B_2) = \models b (\subseteq A, X) \implies\]
\[\text{Max (levels} ?U') \vdash c_2 \implies \text{finite (levels} ?U') \implies\]
\[\exists C Y. (?U', v) \models c_2 (\subseteq B_2, X) = \text{Some} (C, Y) \textbf{ and}\]

\textbf{C: finite (levels} U) \textbf{ and}

D: max (sec b) (Max (levels U)) \vdash c_1 \textbf{ and}

E: max (sec b) (Max (levels U)) \vdash c_2

\textbf{shows} \exists C Y. (U, v) \models \text{IF} b \text{ THEN } c_1 \text{ ELSE } c_2 (\subseteq A, X) = \text{Some} (C, Y)

(\text{proof})

\textbf{lemma} sec-type-ctyping2-while:

\textbf{assumes}
\[A: \forall B_1 B_2 C Y B_1' B_2'. (B_1, B_2) = \models b (\subseteq A, X) \implies\]
\[(C, Y) = \models c (\subseteq B_1, X) \implies (B_1', B_2') = \models b (\subseteq C, Y) \implies\]
\[(\exists s. s \in \text{Univ}? A X \vee s \in \text{Univ}? C Y) \implies\]
\[(\forall x \in \text{bvars} b. \text{All} (((\leq) (\text{sec } x)))) \wedge\]
\[(\forall p \in U. \text{case } p \text{ of } (B, W) \implies (\exists s. s \in B) \implies\]
\[(\forall x \in W. \text{All} (((\leq) (\text{sec } x)))) \implies\]
\[\text{Max (levels} \{\}) \vdash c \implies \text{finite (levels} \{\}) \implies\]
\[\exists D Z. (\{\}, \text{False}) \models c (\subseteq B_1, X) = \text{Some} (D, Z)\]

(\text{is } \wedge - - C Y - - - \implies - \implies - \implies - \implies - \implies - \implies -)

\textbf{assumes}
\[B: \forall B_1 B_2 C Y B_1' B_2'. (B_1, B_2) = \models b (\subseteq A, X) \implies\]

59
\[(C, Y) = \vdash c (\subseteq B_1, X) \implies (B_1', B_2') = \models b (\subseteq C, Y) \implies
\]

\[
P C Y \implies \text{Max (levels \{\})} \vdash c \implies \text{finite (levels \{\})} \implies
\]

\[
\exists D Z. (\{\}, \text{False}) \models c (\subseteq B_1', Y) = \text{Some (D, Z) and}
\]

\[C: \text{finite (levels U) and}
\]

\[D: \text{Max (levels U) = 0 and}
\]

\[E: \text{sec } b = 0 \text{ and}
\]

\[F: 0 \vdash c \text{ and}
\]

\[\text{shows } \exists B Y. (U, v) \models \text{WHILE } b \text{ DO } c (\subseteq A, X) = \text{Some (B, Y)}
\]

\[\langle \text{proof} \rangle
\]

\[\text{proof}
\]

\[\text{end}
\]

\[\text{end}
\]

\section*{References}


