Abstract

This entry contains formalisations of the answers to three of the six problem of the International Mathematical Olympiad 2019, namely Q1, Q4, and Q5. The reason why these problems were chosen is that they are particularly amenable to formalisation: they can be solved with minimal use of libraries. The remaining three concern geometry and graph theory, which, in the author’s opinion, are more difficult to formalise resp. require a more complex library.

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1 Q1

theory IMO2019-Q1
  imports Main
begin

Consider a function \( f : \mathbb{Z} \to \mathbb{Z} \) that fulfils the functional equation \( f(2a) + 2f(b) = f(f(a+b)) \) for all \( a, b \in \mathbb{Z} \).

Then \( f \) is either identically 0 or of the form \( f(x) = 2x + c \) for some constant \( c \in \mathbb{Z} \).

context
fixes \( f :: \text{int} \Rightarrow \text{int} \) and \( m :: \text{int} \)
assumes \( f\text{-eq} : f (2 * a) + 2 * f b = f (f (a + b)) \)
defines \( m \equiv (f 0 - f (-2)) \div 2 \)
begin

We first show that \( f \) is affine with slope \( (f(0) - f(-2)) / 2 \). This follows from plugging in \( (0, b) \) and \( (-1, b + 1) \) into the functional equation.

lemma \( f\text{-eq}' : f x = m * x + f 0 \)
⟨proof⟩

This version is better for the simplifier because it prevents it from looping.

lemma \( f\text{-eq}'\text{-aux} [\text{simp}] : \text{NO-MATCH} 0 x \implies f x = m * x + f 0 \)
⟨proof⟩

Plugging in \( (0, 0) \) and \( (0, 1) \).

lemma \( f\text{-classification} : (\forall x. f x = 0) \lor (\forall x. f x = 2 * x + f 0) \)
⟨proof⟩

end

It is now easy to derive the full characterisation of the functions we considered:

theorem
fixes \( f :: \text{int} \Rightarrow \text{int} \)
shows \( (\forall a b. f (2 * a) + 2 * f b = f (f (a + b))) \iff (\forall x. f x = 0) \lor (\forall x. f x = 2 * x + f 0) \) (is \(?\text{lhs} \leftrightarrow ?\text{rhs}\))
⟨proof⟩

end

2 Q4

theory IMO2019-Q4
  imports Prime-Distribution-Elementary.More-Dirichlet-Misc
begin

Find all pairs \((k, n)\) of positive integers such that \( k! = \prod_{i=0}^{n-1} (2^n - 2^i) \).
2.1 Auxiliary facts

**lemma** Sigma-insert: \( \Sigma (\text{insert } x A) f = (\lambda y. (x, y)) \circ (f \cup \Sigma A) f \)

(proof)

**lemma** atLeastAtMost-nat-numeral:

\( \{ (m::\text{nat})..\text{numeral } k \} = \)

\( \begin{cases} \text{if } m \leq \text{numeral } k \text{ then insert } (\text{numeral } k) \{ m..\text{pred-numeral } k \} \text{ else } \{ \} \end{cases} \)

(proof)

**lemma** greaterThanAtMost-nat-numeral:

\( \{ (m::\text{nat})<..\text{numeral } k \} = \)

\( \begin{cases} \text{if } m < \text{numeral } k \text{ then insert } (\text{numeral } k) \{ m<..\text{pred-numeral } k \} \text{ else } \{ \} \end{cases} \)

(proof)

**lemma** fact-ge-power:

fixes \( c :: \text{nat} \)

assumes \( \text{fact } n0 \geq c \cdot n0 \) \( c \leq n0 + 1 \)

assumes \( n \geq n0 \)

shows \( \text{fact } n \geq c \cdot n \)

(proof)

**lemma** prime-multiplicity-prime:

fixes \( p q :: \alpha :: \text{factorial-semiring} \)

assumes \( \text{prime } p \text{ prime } q \)

shows \( \text{multiplicity } p q = (\text{if } p = q \text{ then } 1 \text{ else } 0) \)

(proof)

We use Legendre’s identity from the library. One could easily prove the property in question without the library, but it probably still saves a few lines.

**lemma** legendre-aux (related to Legendre’s identity) is the multiplicity of a given prime in the prime factorisation of \( n! \).

**lemma** multiplicity-prime-fact:

fixes \( p :: \text{nat} \)

assumes \( \text{prime } p \)

shows \( \text{multiplicity } p (\text{fact } n) = \text{legendre-aux } n p \)

(proof)

The following are simple and trivial lower and upper bounds for **legendre-aux**:

**lemma** legendre-aux-ge:

assumes \( \text{prime } p \ k \geq 1 \)

shows \( \text{legendre-aux } k p \geq \text{nat } \lfloor k / p \rfloor \)

(proof)

**lemma** legendre-aux-less:

assumes \( \text{prime } p \ k \geq 1 \)

shows \( \text{legendre-aux } k p < k / (p - 1) \)

(proof)
2.2 Main result

Now we move on to the main result: We fix two numbers \( n \) and \( k \) with the property in question and derive facts from that. The triangle number \( T = n(n + 1)/2 \) is of particular importance here, so we introduce an abbreviation for it.

**context**

\[\text{fixes } k, n :: \text{nat} \quad \text{and} \quad \text{rhs } T :: \text{nat}\]

\[\text{defines } \text{rhs} \equiv (\prod_{i<n} 2^i \cdot n - 2^i)\]

\[\text{defines } T \equiv (n \cdot (n - 1)) \div 2\]

\[\text{assumes } \text{pos}: k > 0 \quad n > 0\]

\[\text{assumes } k-n: \text{fact } k = \text{rhs}\]

**begin**

We can rewrite the right-hand side into a more convenient form:

**lemma** rhs-altdef: \( \text{rhs} = 2^T \cdot (\prod_{i=1..n} 2^i - 1) \)

**⟨proof⟩**

The multiplicity of 2 in the prime factorisation of the right-hand side is precisely \( T \).

**lemma** multiplicity-2-rhs [simp]: \( \text{multiplicity } 2 \text{ rhs} = T \)

**⟨proof⟩**

From Legendre’s identities and the associated bounds, it can easily be seen that \( \lfloor k/2 \rfloor \leq T < k \):

**lemma** k-gt-T: \( k > T \)

**⟨proof⟩**

**lemma** T-ge-half-k: \( T \geq k \div 2 \)

**⟨proof⟩**

It can also be seen fairly easily that the right-hand side is strictly smaller than \( 2n^2 \):

**lemma** rhs-less: \( \text{rhs} < 2^n \cdot n^2 \)

**⟨proof⟩**

It is clear that \( 2n^2 \leq 8^T \) and that \( 8^T < T! \) if \( T \) is sufficiently big. In this case, ‘sufficiently big’ means \( T \geq 20 \) and thereby \( n \geq 7 \). We can therefore conclude that \( n \) must be less than 7.

**lemma** n-less-7: \( n < 7 \)

**⟨proof⟩**

We now only have 6 values for \( n \) to check. Together with the bounds that we obtained on \( k \), this only leaves a few combinations of \( n \) and \( k \) to check, and we do precisely that and find that \( n = k = 1 \) and \( n = 2, k = 3 \) are the only possible combinations.
lemma n-k-in-set: $(n, k) \in \{(1, 1), (2, 3)\}$
(proof)
end

Using this, deriving the final result is now trivial:

theorem \{(n, k). n > 0 \land k > 0 \land \text{fact } k = (\prod_{i<n} 2 \cdot n - 2 \cdot i :: \text{nat})\} = \{(1, 1), (2, 3)\}
(is ?lhs = ?rhs)
(proof)
end

3 Q5

theory IMO2019-Q5
  imports Complex-Main
begin

Given a sequence $(c_1, \ldots, c_n)$ of coins, each of which can be heads $(H)$ or tails $(T)$. Harry performs the following process: Let $k$ be the number of coins that show $H$. If $k > 0$, flip the $k$-th coin and repeat the process. Otherwise, stop.

What is the average number of steps that this process takes, averaged over all $2^n$ coin sequences of length $n$?

3.1 Definition

We represent coins as Booleans, where $\text{True}$ indicates $H$ and $\text{False}$ indicates $T$. Coin sequences are then simply lists of Booleans.

The following function flips the $i$-th coin in the sequence (in Isabelle, the convention is that the first list element is indexed with 0).

definition flip :: bool list $\Rightarrow$ nat $\Rightarrow$ bool list where
flip xs i = xs[i := $\neg$xs ! i]

lemma flip-Cons-pos [simp]: $n > 0 \implies \text{flip } (x \# xs) n = x \# \text{flip } xs (n - 1)$
(proof)

lemma flip-Cons-0 [simp]: flip (x # xs) 0 = (¬x) # xs
(proof)

lemma flip-append1 [simp]: $n < \text{length } xs \implies \text{flip } (xs @ ys) n = \text{flip } xs n @ ys$
and flip-append2 [simp]: $n \geq \text{length } xs \implies n < \text{length } xs + \text{length } ys \implies$

  flip (xs @ ys) n = xs @ flip ys (n - \text{length } xs)
(proof)
lemma length-flip [simp]: length (flip xs i) = length xs
  ⟨proof⟩

The following function computes the number of $H$ in a coin sequence.
definition heads :: bool list ⇒ nat where heads xs = length (filter id xs)

lemma heads-True [simp]: heads (True # xs) = 1 + heads xs
  and heads-False [simp]: heads (False # xs) = heads xs
  and heads-append [simp]: heads (xs @ ys) = heads xs + heads ys
  and heads-Nil [simp]: heads [] = 0
  ⟨proof⟩

lemma heads-Cons: heads (x # xs) = (if x then heads xs + 1 else heads xs)
  ⟨proof⟩

lemma heads-pos: True ∈ set xs ⇒ heads xs > 0
  ⟨proof⟩

lemma heads-eq-0 [simp]: True /∈ set xs ⇒ heads xs = 0
  ⟨proof⟩

lemma heads-eq-0-iff [simp]: heads xs = 0 ←→ True /∈ set xs
  ⟨proof⟩

lemma heads-pos-iff [simp]: heads xs > 0 ←→ True ∈ set xs
  ⟨proof⟩

lemma heads-le-length: heads xs ≤ length xs
  ⟨proof⟩

The following function performs a single step of Harry’s process.
definition harry-step :: bool list ⇒ bool list where
  harry-step xs = flip xs (heads xs − 1)

lemma length-harry-step [simp]: length (harry-step xs) = length xs
  ⟨proof⟩

The following is the measure function for Harry’s process, i.e. how many steps the process takes to terminate starting from the given sequence. We define it like this now and prove the correctness later.
function harry-meas where
  harry-meas xs = 
    (if xs = [] then 0
      else if hd xs then 1 + harry-meas (tl xs)
      else if ¬last xs then harry-meas (butlast xs)
      else let n = length xs in harry-meas (take (n − 2) (tl xs)) + 2 * n − 1)
  ⟨proof⟩

termination ⟨proof⟩
We now prove some simple properties of \emph{harry-meas} and \emph{harry-step}.

We prove a more convenient case distinction rule for lists that allows us to distinguish between lists starting with True, ending with False, and starting with False and ending with True.

\begin{lemma}
\textbf{head-last-cases} [\textbf{case-names} Nil True False False-True]:
\begin{align*}
\text{assumes } & \text{xs} = [] \implies P \\
\text{assumes } & \forall ys. \text{xs} = \text{True} \# \text{ys} \implies P \land \forall ys. \text{xs} = \text{ys} \@ [\text{False}] \implies P \\
\text{shows } & P
\end{align*}
\end{lemma}

\begin{proof}
\end{proof}

\begin{lemma}
\textbf{harry-meas-Nil} [\textbf{simp}]: \text{harry-meas} [] = 0
\end{lemma}

\begin{proof}
\end{proof}

\begin{lemma}
\textbf{harry-meas-True-start} [\textbf{simp}]: \text{harry-meas} (\text{True} \# \text{xs}) = 1 + \text{harry-meas} \text{xs}
\end{lemma}

\begin{proof}
\end{proof}

\begin{lemma}
\textbf{harry-meas-False-end} [\textbf{simp}]: \text{harry-meas} (\text{xs} \@ [\text{False}]) = \text{harry-meas} \text{xs}
\end{lemma}

\begin{proof}
\end{proof}

\begin{lemma}
\textbf{harry-meas-False-True}: \text{harry-meas} (\text{False} \# \text{xs} \@ [\text{True}]) = \text{harry-meas} \text{xs} + 2 \ast \text{length} \text{xs} + 3
\end{lemma}

\begin{proof}
\end{proof}

\begin{lemma}
\textbf{harry-meas-eq-0} [\textbf{simp}]:
\begin{align*}
\text{assumes } & \text{True} \notin \text{set} \text{xs} \\
\text{shows } & \text{harry-meas} \text{xs} = 0
\end{align*}
\end{lemma}

\begin{proof}
\end{proof}

If the sequence starts with \emph{H}, the process runs on the remaining sequence until it terminates and then flips this \emph{H} in another single step.

\begin{lemma}
\textbf{harry-step-True-start} [\textbf{simp}]:
\begin{align*}
\text{harry-step} (\text{True} \# \text{xs}) = (\text{if } \text{True} \in \text{set} \text{xs} \text{ then } \text{True} \# \text{harry-step} \text{xs} \text{ else } \text{False} \# \text{xs})
\end{align*}
\end{lemma}

\begin{proof}
\end{proof}

If the sequence ends in \emph{T}, the process simply runs on the remaining sequence as if it were not present.

\begin{lemma}
\textbf{harry-step-False-end} [\textbf{simp}]:
\begin{align*}
\text{assumes } & \text{True} \in \text{set} \text{xs} \\
\text{shows } & \text{harry-step} (\text{xs} \@ [\text{False}]) = \text{harry-step} \text{xs} \@ [\text{False}]
\end{align*}
\end{lemma}

\begin{proof}
\end{proof}

If the sequence starts with \emph{T} and ends with \emph{H}, the process runs on the
remaining sequence inbetween as if these two were not present, eventually
leaving a sequence that consists entirely if $T$ except for a single final $H$.

**Lemma** harry-step-False-True:

**Assumes** $\text{True} \in \text{set } xs$

**Shows** $\text{harry-step } (\text{False } \# xs @ [\text{True}]) = \text{False } \# \text{harry-step xs } @ [\text{True}]

⟨proof⟩

That sequence consisting only of $T$ except for a single final $H$ is then turned into an all-$T$ sequence in $2n+1$ steps.

**Lemma** harry-meas-Falses-True [simp]: $\text{harry-meas } (\text{replicate } n \text{ False } @ [\text{True}]) = 2 \times n + 1$

⟨proof⟩

**Lemma** harry-step-Falses-True [simp]:

$n > 0 \implies \text{harry-step } (\text{replicate } n \text{ False } @ [\text{True}]) = \text{True } \# \text{replicate } (n - 1) \text{ False } @ [\text{True}]

⟨proof⟩

### 3.2 Correctness of the measure

We will now show that $\text{harry-meas}$ indeed counts the length of the process. As a first step, we will show that if there is a $H$ in a sequence, applying a single step decreases the measure by one.

**Lemma** harry-meas-step-aux:

**Assumes** $\text{True} \in \text{set } xs$

**Shows** $\text{harry-meas xs} = \text{Suc } (\text{harry-meas } (\text{harry-step xs}))$

⟨proof⟩

**Lemma** harry-meas-step: $\text{True} \in \text{set } xs \implies \text{harry-meas } (\text{harry-step xs}) = \text{harry-meas xs } - 1$

⟨proof⟩

Next, we show that the measure is zero if and only if there is no $H$ left in the sequence.

**Lemma** harry-meas-eq-0-iff [simp]: $\text{harry-meas xs} = 0 \iff \text{True} \notin \text{set } xs$

⟨proof⟩

It follows by induction that if the measure of a sequence is $n$, then iterating the step less than $n$ times yields a sequence with at least one $H$ in it, but iterating it exactly $n$ times yields a sequence that contains no more $H$.

**Lemma** True-in-funpow-harry-step:

**Assumes** $n < \text{harry-meas xs}$

**Shows** $\text{True} \in \text{set } ((\text{harry-step } \cdot\cdot\cdot n) \text{ xs})$

⟨proof⟩

**Lemma** True-notin-funpow-harry-step: $\text{True} \notin \text{set } ((\text{harry-step } \cdot\cdot\cdot \text{harry-meas xs}) \text{ xs})$
This shows that the measure is indeed the correct one: It is the smallest number such that iterating Harry’s step that often yields a sequence with no heads in it.

**Theorem** harry-meas \( xs \) = (LEAST \( n \). True \notin set ((harry-step ^ n \) \( xs \))

**Proof**

### 3.3 Average-case analysis

The set of all coin sequences of a given length.

**Definition** `seqs` where `seqs n = {xs :: bool list . length xs = n}`

**Lemma** `length-seqs` [dest]: `xs ∈ seqs n → length xs = n`

**Lemma** `seqs-0` [simp]: `seqs 0 = {[[]]}`

The coin sequences of length \( n + 1 \) are simply what is obtained by appending either \( H \) or \( T \) to each coin sequence of length \( n \).

**Lemma** `seqs-Suc`: `seqs (Suc n) = (λxs. True # xs) ' seqs n ∪ (λxs. False # xs) ' seqs n`

**Lemma** `seqs-rev` [simp]: `rev ' seqs n = seqs n`

Hence we get a similar decomposition theorem that appends at the end.

**Lemma** `seqs-Suc'`: `seqs (Suc n) = (λxs. xs @ [True]) ' seqs n ∪ (λxs. xs @ [False]) ' seqs n`

The set of coin sequences of length \( n \) is invariant under reversal.

**Lemma** `seqs-rev` [simp]: `rev ' seqs n = seqs n`

**Lemma** `finite-seqs` [intro]: `finite (seqs n)`

**Lemma** `card-seqs` [simp]: `card (seqs n) = 2 ^ n`

**Lemma** `seqs-code` [code] = `seqs-0 seqs-Suc`

The sum of the measures over all possible coin sequences of a given length (defined as a recurrence relation; correctness proven later).

**Function** `harry-sum :: nat ⇒ nat where`

\[
\begin{align*}
\text{harry-sum} \ 0 &= 0 \\
\text{harry-sum} \ (\text{Suc} \ 0) &= 1
\end{align*}
\]
| harry-sum (Suc (Suc n)) = 2 * harry-sum (Suc n) + (2 * n + 4) * 2 ^ n

**lemma** Suc-Suc-induct: P 0 \implies P (Suc 0) \implies (\forall n. P n \implies P (Suc n)) \implies P (Suc (Suc n)) \implies P n

(\langle proof \rangle)

The recurrence relation really does describe the sum over all measures:

**lemma** harry-sum-correct: harry-sum n = sum harry-meas (seqs n)

(\langle proof \rangle)

**lemma** harry-sum-closed-form-aux: 4 * harry-sum n = n * (n + 1) * 2 ^ n

(\langle proof \rangle)

Solving the recurrence gives us the following solution:

**theorem** harry-sum-closed-form: harry-sum n = n * (n + 1) * 2 ^ n div 4

(\langle proof \rangle)

The average is now a simple consequence:

**definition** harry-avg where harry-avg n = harry-sum n / card (seqs n)

**corollary** harry-avg n = n * (n + 1) / 4

(\langle proof \rangle)

**end**

**References**