An Isabelle/HOL Formalization of the Textbook Proof of Huffman’s Algorithm*

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Abstract
Huffman’s algorithm is a procedure for constructing a binary tree with minimum weighted path length. This report presents a formal proof of the correctness of Huffman’s algorithm written using Isabelle/HOL. Our proof closely follows the sketches found in standard algorithms textbooks, uncovering a few snags in the process. Another distinguishing feature of our formalization is the use of custom induction rules to help Isabelle’s automatic tactics, leading to very short proofs for most of the lemmas.

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1 Introduction

1.1 Binary Codes

Suppose we want to encode strings over a finite source alphabet to sequences of bits. The approach used by ASCII and most other charsets is to map each source symbol to a distinct k-bit code word, where k is fixed and is typically 8 or 16. To encode a string of symbols, we simply encode each symbol in turn. Decoding involves mapping each k-bit block back to the symbol it represents.

Fixed-length codes are simple and fast, but they generally waste space. If we know the frequency $w_a$ of each source symbol a, we can save space by using
shorter code words for the most frequent symbols. We say that a (variable-length) code is optimum if it minimizes the sum $\sum_a w_a \delta_a$, where $\delta_a$ is the length of the binary code word for $a$. Information theory tells us that a code is optimum if for each source symbol $c$ the code word representing $c$ has length

$$\delta_c = \log_2 \frac{1}{p_c},$$

where $p_c = \frac{w_c}{\sum_a w_a}$.

This number is generally not an integer, so we cannot use it directly. Nonetheless, the above criterion is a useful yardstick and paves the way for arithmetic coding [13], a generalization of the method presented here.

As an example, consider the source string ‘abacabad’. We have

$$p_a = \frac{1}{2}, \ p_b = \frac{1}{4}, \ p_c = \frac{1}{8}, \ p_d = \frac{1}{8}.$$  

The optimum lengths for the binary code words are all integers, namely

$$\delta_a = 1, \ \delta_b = 2, \ \delta_c = 3, \ \delta_d = 3,$$

and they are realized by the code

$$C_1 = \{a \mapsto 0, \ b \mapsto 10, \ c \mapsto 110, \ d \mapsto 111\}.$$  

Encoding ‘abacabad’ produces the 14-bit code word 01001100100111. The code $C_1$ is optimum: No code that unambiguously encodes source symbols one at a time could do better than $C_1$ on the input ‘abacabad’. In particular, with a fixed-length code such as

$$C_2 = \{a \mapsto 00, \ b \mapsto 01, \ c \mapsto 10, \ d \mapsto 11\}$$

we need at least 16 bits to encode ‘abacabad’.

### 1.2 Binary Trees

Inside a program, binary codes can be represented by binary trees. For example, the trees

```
0 1
a

0 1
b
0 1
c d
```

and

```
0 1
a
0 1
b
0 1
c d
```

correspond to $C_1$ and $C_2$. The code word for a given symbol can be obtained as follows: Start at the root and descend toward the leaf node associated with
the symbol one node at a time; generate a 0 whenever the left child of the current node is chosen and a 1 whenever the right child is chosen. The generated sequence of 0s and 1s is the code word.

To avoid ambiguities, we require that only leaf nodes are labeled with symbols. This ensures that no code word is a prefix of another, thereby eliminating the source of all ambiguities.\footnote{Strictly speaking, there is another potential source of ambiguity. If the alphabet consists of a single symbol \( a \), that symbol could be mapped to the empty code word, and then any string \( aa \ldots a \) would map to the empty bit sequence, giving the decoder no way to recover the original string’s length. This scenario can be ruled out by requiring that the alphabet has cardinality 2 or more.} Codes that have this property are called \textit{prefix codes}. As an example of a code that does not have this property, consider the code associated with the tree

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{tree_example}
\caption{Example of a code tree.}
\end{figure}

and observe that ‘bbb’, ‘bd’, and ‘db’ all map to the code word 111.

Each node in a code tree is assigned a \textit{weight}. For a leaf node, the weight is the frequency of its symbol; for an inner node, it is the sum of the weights of its subtrees. Code trees can be annotated with their weights:

\begin{figure}[h]
\centering
\includegraphics[width=0.6\textwidth]{weighted_tree}
\caption{Weighted code trees.}
\end{figure}

For our purposes, it is sufficient to consider only full binary trees (trees whose inner nodes all have two children). This is because any inner node with only one
child can advantageously be eliminated; for example,

1.3 Huffman’s Algorithm

David Huffman [7] discovered a simple algorithm for constructing an optimum code tree for specified symbol frequencies: Create a forest consisting of only leaf nodes, one for each symbol in the alphabet, taking the given symbol frequencies as initial weights for the nodes. Then pick the two trees with the lowest weights and replace them with the tree

Repeat this process until only one tree is left.

As an illustration, executing the algorithm for the frequencies

\[ f_d = 3, \ f_e = 11, \ f_t = 5, \ f_s = 7, \ f_z = 2 \]

gives rise to the following sequence of states:

\[
\begin{align*}
(1) & \quad \begin{array}{cccc}
  z & d & f & s & e \\
  2 & 3 & 5 & 7 & 11
\end{array} \\
(2) & \quad \begin{array}{cccc}
  5 & f & s & e \\
  & 3 & 7 & 11
\end{array}
\end{align*}
\]
Tree (5) is an optimum tree for the given frequencies.

1.4 The Textbook Proof

Why does the algorithm work? In his article, Huffman gave some motivation but no real proof. For a proof sketch, we turn to Donald Knuth [8, p. 403–404]:

It is not hard to prove that this method does in fact minimize the weighted path length (i.e., \( \sum_a w_a \delta_a \)), by induction on \( m \). Suppose we have \( w_1 \leq w_2 \leq w_3 \leq \cdots \leq w_m \), where \( m \geq 2 \), and suppose that we are given a tree that minimizes the weighted path length. (Such a tree certainly exists, since only finitely many binary trees with \( m \) terminal nodes are possible.) Let \( V \) be an internal node of maximum distance from the root. If \( w_1 \) and \( w_2 \) are not the weights already attached to the children of \( V \), we can interchange them with the values that are already there; such an interchange does not increase the weighted path length. Thus there is a tree that minimizes the weighted path length and contains the subtree

Now it is easy to prove that the weighted path length of such a tree is minimized if and only if the tree with

replaced by

has minimum path length for the weights \( w_1 + w_2, w_3, \ldots, w_m \).

There is, however, a small oddity in this proof: It is not clear why we must assert
the existence of an optimum tree that contains the subtree

Indeed, the formalization works without it.

Cormen et al. [4, p. 385–391] provide a very similar proof, articulated around the following propositions:

**Lemma 16.2**
Let $C$ be an alphabet in which each character $c \in C$ has frequency $f[c]$. Let $x$ and $y$ be two characters in $C$ having the lowest frequencies. Then there exists an optimal prefix code for $C$ in which the codewords for $x$ and $y$ have the same length and differ only in the last bit.

**Lemma 16.3**
Let $C$ be a given alphabet with frequency $f[c]$ defined for each character $c \in C$. Let $x$ and $y$ be two characters in $C$ with minimum frequency. Let $C'$ be the alphabet $C$ with characters $x$, $y$ removed and (new) character $z$ added, so that $C' = C - \{x, y\} \cup \{z\}$; define $f$ for $C'$ as for $C$, except that $f[z] = f[x] + f[y]$. Let $T'$ be any tree representing an optimal prefix code for the alphabet $C'$. Then the tree $T$, obtained from $T'$ by replacing the leaf node for $z$ with an internal node having $x$ and $y$ as children, represents an optimal prefix code for the alphabet $C$.

**Theorem 16.4**
Procedure HUFFMAN produces an optimal prefix code.

1.5 Overview of the Formalization
This document presents a formalization of the proof of Huffman’s algorithm written using Isabelle/HOL [12]. Our proof is based on the informal proofs given by Knuth and Cormen et al. The development was done independently of Laurent Théry’s Coq proof [14, 15], which through its “cover” concept represents a considerable departure from the textbook proof.

The development consists of a little under 100 lemmas and theorems. Most of them have very short proofs thanks to the extensive use of simplification rules and custom induction rules. The remaining proofs are written using the structured proof format Isar [16].

1.6 Head of the Theory File
The Isabelle theory starts in the standard way.
theory Huffman
imports Main
begin

We attach the simp attribute to some predefined lemmas to add them to the default set of simplification rules.

declare
Int_Un_distrib [simp]
Int_Un_distrib2 [simp]
max.absorb1 [simp]
max.absorb2 [simp]

2 Definition of Prefix Code Trees and Forests

2.1 Tree Type
A prefix code tree is a full binary tree in which leaf nodes are of the form Leaf w a, where a is a symbol and w is the frequency associated with a, and inner nodes are of the form Node w t1 t2, where t1 and t2 are the left and right subtrees and w caches the sum of the weights of t1 and t2. Prefix code trees are polymorphic on the symbol datatype α.

datatype α tree =
Leaf nat a
Node nat (α tree) (α tree)

2.2 Forest Type
The intermediate steps of Huffman’s algorithm involve a list of prefix code trees, or prefix code forest.

type_synonym α forest = α tree list

2.3 Alphabet
The alphabet of a code tree is the set of symbols appearing in the tree’s leaf nodes.

primrec alphabet :: α tree ⇒ α set where
alphabet (Leaf w a) = {a}
alphabet (Node w t1 t2) = alphabet t1 ∪ alphabet t2

For sets and predicates, Isabelle gives us the choice between inductive definitions (inductive_set and inductive) and recursive functions (primrec, fun, and function). In this development, we consistently favor recursion over induction, for two reasons:
Recursion gives rise to simplification rules that greatly help automatic proof tactics. In contrast, reasoning about inductively defined sets and predicates involves introduction and elimination rules, which are more clumsy than simplification rules.

Isabelle’s counterexample generator quickcheck [2], which we used extensively during the top-down development of the proof (together with sorry), has better support for recursive definitions.

The alphabet of a forest is defined as the union of the alphabets of the trees that compose it. Although Isabelle supports overloading for non-overlapping types, we avoid many type inference problems by attaching an ‘F’ subscript to the forest generalizations of functions defined on trees.

\[
\text{primrec } \text{alphabet}_F :: \alpha \text{ forest } \Rightarrow \alpha \text{ set where }
\]
\[
\text{alphabet}_F [\ ] = \emptyset \\
\text{alphabet}_F (t \cdot ts) = \text{alphabet } t \cup \text{alphabet}_F ts
\]

Alphabets are central to our proofs, and we need the following basic facts about them.

\[
\text{lemma } \text{finite_alphabet[simp]}:
\]
\[
\text{finite } (\text{alphabet } t) \\
\text{by } (\text{induct } t) \text{ auto}
\]

\[
\text{lemma } \text{exists_in_alphabet}:
\]
\[
\exists a. a \in \text{alphabet } t \\
\text{by } (\text{induct } t) \text{ auto}
\]

2.4 Consistency

A tree is consistent if for each inner node the alphabets of the two subtrees are disjoint. Intuitively, this means that every symbol in the alphabet occurs in exactly one leaf node. Consistency is a sufficient condition for \(\delta_a\) (the length of the unique code word for \(a\)) to be defined. Although this wellformedness property is not mentioned in algorithms textbooks [1, 4, 8], it is essential and appears as an assumption in many of our lemmas.

\[
\text{primrec } \text{consistent :: a tree } \Rightarrow \text{ bool where }
\]
\[
\text{consistent } (\text{Leaf } w a) = \text{True} \\
\text{consistent } (\text{Node } w t_1 t_2) = \\
(\text{consistent } t_1 \land \text{consistent } t_2 \land \text{alphabet } t_1 \cap \text{alphabet } t_2 = \emptyset)
\]

\[
\text{primrec } \text{consistent}_F :: \alpha \text{ forest } \Rightarrow \text{ bool where }
\]
\[
\text{consistent}_F [\ ] = \text{True} \\
\text{consistent}_F (t \cdot ts) = \\
(\text{consistent } t \land \text{consistent}_F ts \land \text{alphabet } t \cap \text{alphabet}_F ts = \emptyset)
\]
Several of our proofs are by structural induction on consistent trees \( t \) and involve one symbol \( a \). These proofs typically distinguish the following cases.

**BASE CASE:** \( t = \text{Leaf } w \ b \).

**INDUCTION STEP:** \( t = \text{Node } w \ t_1 \ t_2 \).

**SUBCASE 1:** \( a \) belongs to \( t_1 \) but not to \( t_2 \).

**SUBCASE 2:** \( a \) belongs to \( t_2 \) but not to \( t_1 \).

**SUBCASE 3:** \( a \) belongs to neither \( t_1 \) nor \( t_2 \).

Thanks to the consistency assumption, we can rule out the subcase where \( a \) belongs to both subtrees.

Instead of performing the above case distinction manually, we encode it in a custom induction rule. This saves us from writing repetitive proof scripts and helps Isabelle’s automatic proof tactics.

**lemma** tree_induct_consistent[consumes 1, case_names base step1 step2 step3]:

\[
\begin{align*}
& \text{consistent } t; \\
& \forall w_0 b a. P (\text{Leaf } w_0 b) \ a; \\
& \forall w \ t_1 \ t_2 a. \\
& \quad \text{consistent } t_1; \text{ consistent } t_2; \text{ alphabet } t_1 \cap \text{ alphabet } t_2 = \emptyset; \\
& \quad a \in \text{ alphabet } t_1; \ a \notin \text{ alphabet } t_2; \ P t_1 a; \ P t_2 a \implies \ P (\text{Node } w \ t_1 \ t_2) \ a; \\
& \forall w \ t_1 \ t_2 a. \\
& \quad \text{consistent } t_1; \text{ consistent } t_2; \text{ alphabet } t_1 \cap \text{ alphabet } t_2 = \emptyset; \\
& \quad a \notin \text{ alphabet } t_1; \ a \in \text{ alphabet } t_2; \ P t_1 a; \ P t_2 a \implies \ P (\text{Node } w \ t_1 \ t_2) \ a; \\
& \forall w \ t_1 \ t_2 a. \\
& \quad \text{consistent } t_1; \text{ consistent } t_2; \text{ alphabet } t_1 \cap \text{ alphabet } t_2 = \emptyset; \\
& \quad a \notin \text{ alphabet } t_1; \ a \notin \text{ alphabet } t_2; \ P t_1 a; \ P t_2 a \implies \ P (\text{Node } w \ t_1 \ t_2) \ a \implies \ P t a
\end{align*}
\]

The proof relies on the **induction_schema** and **lexicographic_order** tactics, which automate the most tedious aspects of deriving induction rules. The alternative would have been to perform a standard structural induction on \( t \) and proceed by cases, which is straightforward but long-winded.

**apply** rotate_tac

**apply** induction_schema

**apply** atomize_elim

**apply** (case_tac \( t \))

**apply** fastforce

**apply** fastforce

**by** lexicographic_order
The induction_schema tactic reduces the putative induction rule to simpler proof obligations. Internally, it reuses the machinery that constructs the default induction rules. The resulting proof obligations concern (a) case completeness, (b) invariant preservation (in our case, tree consistency), and (c) wellfoundedness. For tree_induct_consistent, the obligations (a) and (b) can be discharged using Isabelle’s simplifier and classical reasoner, whereas (c) requires a single invocation of lexicographic_order, a tactic that was originally designed to prove termination of recursive functions [3, 9, 10].

2.5 Symbol Depths

The depth of a symbol (which we denoted by δ_α in Section 1.1) is the length of the path from the root to the leaf node labeled with that symbol, or equivalently the length of the code word for the symbol. Symbols that do not occur in the tree or that occur at the root of a one-node tree have depth 0. If a symbol occurs in several leaf nodes (which may happen with inconsistent trees), the depth is arbitrarily defined in terms of the leftmost node labeled with that symbol.

\[
\text{primrec } \text{depth} :: \alpha \text{ tree } \Rightarrow \alpha \Rightarrow \text{nat where}
\]
\[
\text{depth} (\text{Leaf } w \ b) \ a = 0
\]
\[
\text{depth} (\text{Node } w \ t_1 \ t_2) \ a =
\]
\[
\begin{array}{l}
(\text{if } a \in \text{alphabet } t_1 \text{ then } \text{depth } t_1 \ a + 1 \\
(\text{else if } a \in \text{alphabet } t_2 \text{ then } \text{depth } t_2 \ a + 1 \\
(\text{else } 0))
\end{array}
\]

The definition may seem very inefficient from a functional programming point of view, but it does not matter, because unlike Huffman’s algorithm, the depth function is merely a reasoning tool and is never actually executed.

2.6 Height

The height of a tree is the length of the longest path from the root to a leaf node, or equivalently the length of the longest code word. This is readily generalized to forests by taking the maximum of the trees’ heights. Note that a tree has height 0 if and only if it is a leaf node, and that a forest has height 0 if and only if all its trees are leaf nodes.

\[
\text{primrec } \text{height} :: \alpha \text{ tree } \Rightarrow \text{nat where}
\]
\[
\text{height} (\text{Leaf } w \ a) = 0
\]
\[
\text{height} (\text{Node } w \ t_1 \ t_2) = \max (\text{height } t_1) (\text{height } t_2) + 1
\]

\[
\text{primrec } \text{height}_{F} :: \alpha \text{ forest } \Rightarrow \text{nat where}
\]
\[
\text{height}_{F} [] = 0
\]
\[
\text{height}_{F} (t \cdot ts) = \max (\text{height } t) (\text{height}_{F} ts)
\]

The depth of any symbol in the tree is bounded by the tree’s height, and there
exists a symbol with a depth equal to the height.

**Lemma** depth_le_height:

\[ \text{depth } t \ a \leq \text{height } t \]

by (induct t) auto

**Lemma** exists_at_height:

\[ \text{consistent } t \implies \exists a \in \text{alphabet } t. \text{depth } t \ a = \text{height } t \]

**Proof** (induct t)

  case Leaf thus case by simp

next

  case (Node w t_1 t_2)
  note hyps = Node
  let t = Node w t_1 t_2
  from hyps obtain b where b \in \text{alphabet } t_1 \text{ depth } t_1 \ b = \text{height } t_1 \ by \ auto
  from hyps obtain c where c \in \text{alphabet } t_2 \text{ depth } t_2 \ c = \text{height } t_2 \ by \ auto
  let a = if \text{height } t_1 \geq \text{height } t_2 \ then \ b \ else \ c
  from b c have \ a \in \text{alphabet } t \text{ depth } t \ a = \text{height } t \ using \ consistent \ b \ by \ auto
  thus \exists a \in \text{alphabet } t. \text{depth } t \ a = \text{height } t .

qed

The following elimination rules help Isabelle’s classical prover, notably the *auto* tactic. They are easy consequences of the inequation \( \text{depth } t \ a \leq \text{height } t \).

**Lemma** depth_max_heightE_left [elim!]:

\[ \text{depth } t_1 \ a = \max (\text{height } t_1) (\text{height } t_2); \]
\[ \text{depth } t_1 \ a = \text{height } t_1; \text{height } t_1 \geq \text{height } t_2 \implies P \]
\[ P \]

by (cut_tac t = t_1 and a = a in depth_le_height) simp

**Lemma** depth_max_heightE_right [elim!]:

\[ \text{depth } t_2 \ a = \max (\text{height } t_1) (\text{height } t_2); \]
\[ \text{depth } t_2 \ a = \text{height } t_2; \text{height } t_2 \geq \text{height } t_1 \implies P \]
\[ P \]

by (cut_tac t = t_2 and a = a in depth_le_height) simp

We also need the following lemma.

**Lemma** height_gt_0_alphabet_eq_imp_height_gt_0:

assumes \( \text{height } t > 0 \text{ consistent } t \text{ alphabet } t = \text{alphabet } u \)

shows \( \text{height } u > 0 \)

**Proof** (cases t)

  case Leaf thus thesis using assms by simp

next

  case (Node w t_1 t_2)
note \( t = \text{Node} \)

from \( \exists \text{ in alphabet} \) obtain \( b \) where \( b \in \text{alphabet } t \)

from \( \exists \text{ in alphabet} \) obtain \( c \) where \( c \in \text{alphabet } t \)

from \( b \ c \) have \( bc : b \neq c \) using \( t \) \( \text{consistent } t \) by fastforce

\[ \text{show thesis} \]

\[ \text{proof (cases } u \text{)} \]

\[ \text{case Leaf thus thesis using } b \ c \ t \ \text{assms by auto} \]

next

\[ \text{case Node thus thesis by simp} \]

qed

qed

2.7 Symbol Frequencies

The frequency of a symbol (which we denoted by \( w_a \) in Section 1.1) is the sum of the weights attached to the leaf nodes labeled with that symbol. If the tree is consistent, the sum comprises at most one nonzero term. The frequency is then the weight of the leaf node labeled with the symbol, or 0 if there is no such node. The generalization to forests is straightforward.

\[ \text{primrec } \text{freq} :: \alpha \text{ tree } \Rightarrow \alpha \Rightarrow \text{nat where} \]

\[ \text{freq } (\text{Leaf } w a) \ b = (\text{if } b = a \text{ then } w \text{ else } 0) \]

\[ \text{freq } (\text{Node } w \ t_1 \ t_2) \ b = \text{freq } t_1 \ b + \text{freq } t_2 \ b \]

\[ \text{primrec } \text{freq}_F :: \alpha \text{ forest } \Rightarrow \alpha \Rightarrow \text{nat where} \]

\[ \text{freq}_F [] \ b = 0 \]

\[ \text{freq}_F (t \cdot ts) \ b = \text{freq } t \ b + \text{freq}_F ts \ b \]

Alphabet and symbol frequencies are intimately related. Simplification rules ensure that sums of the form \( \text{freq } t_1 \ a + \text{freq } t_2 \ a \) collapse to a single term when we know which tree \( a \) belongs to.

\[ \text{lemma notin_alphabet_imp_freq_0[simp]:} \]

\[ a \notin \text{alphabet } t \Longrightarrow \text{freq } t \ a = 0 \]

by \text{ (induct } t \text{) simp}^+\]

\[ \text{lemma notin_alphabetF_imp_freqF_0[simp]:} \]

\[ a \notin \text{alphabet}_F ts \Longrightarrow \text{freq}_F ts \ a = 0 \]

by \text{ (induct } ts \text{) simp}^+\]

\[ \text{lemma freq_0_right[simp]:} \]

\[ [\text{alphabet } t_1 \cap \text{alphabet } t_2 = \emptyset ; a \in \text{alphabet } t_1] \Longrightarrow \text{freq } t_2 \ a = 0 \]

by \text{ (auto intro: notin_alphabet_imp_freq_0 simp: disjoint_iff_not_equal)}
Lemma freq_0_left[simp]:
\[ \text{alphabet } t_1 \cap \text{alphabet } t_2 = \emptyset; \ a \in \text{alphabet } t_2 \implies \text{freq } t_1 \ a = 0 \]
by (auto simp: disjoint_iff_not_equal)

Two trees are comparable if they have the same alphabet and symbol frequencies. This is an important concept, because it allows us to state not only that the tree constructed by Huffman’s algorithm is optimal but also that it has the expected alphabet and frequencies.

We close this section with a more technical lemma.

Lemma heightF_0_imp_Leaf_freqF_in_set:
\[ \text{consistent } F \ ts; \ heightF \ ts = 0; \ a \in \text{alphabet } F \ ts \implies \text{Leaf } (\text{freqF } ts \ a) \ a \in \text{set } ts \]
proof (induct ts)
  case Nil thus case by simp
next
  case (Cons t ts) show case using Cons by (cases t) auto
qed

2.8 Weight

The weight function returns the weight of a tree. In the Node case, we ignore the weight cached in the node and instead compute the tree’s weight recursively. This makes reasoning simpler because we can then avoid specifying cache correctness as an assumption in our lemmas.

Primrec weight :: \alpha \ tree \Rightarrow \mathbb{N} where
weight (Leaf w a) = w
weight (Node w t_1 t_2) = weight t_1 + weight t_2

The weight of a tree is the sum of the frequencies of its symbols.

Lemma weight_eq_Sum_freq:
consistent t \implies \text{weight } t = \sum_{a \in \text{alphabet } t} \text{freq } t \ a
by (induct t) (auto simp: sum.union_disjoint)

The assumption consistent t is not necessary, but it simplifies the proof by letting us invoke the lemma sum.union_disjoint:

\[ \text{finite } A; \ \text{finite } B; \ A \cap B = \emptyset \implies \sum_{x \in A} g \ x + \sum_{x \in B} g \ x = \sum_{x \in A \cup B} g \ x. \]
2.9 Cost

The cost of a consistent tree, sometimes called the weighted path length, is given by the sum \( \sum_{a \in \text{alphabet } t} \text{freq } t a \times \text{depth } t a \) (which we denoted by \( \sum_a w_a \delta_a \) in Section 1.1). It obeys a simple recursive law.

\[
\text{primrec cost :: } \alpha \text{ tree } \Rightarrow \text{nat where}
\begin{align*}
\text{cost } (\text{Leaf } w a) &= 0 \\
\text{cost } (\text{Node } w t_1 t_2) &= \text{weight } t_1 + \text{cost } t_1 + \text{weight } t_2 + \text{cost } t_2
\end{align*}
\]

One interpretation of this recursive law is that the cost of a tree is the sum of the weights of its inner nodes [8, p. 405]. (Recall that \( \text{weight } (\text{Node } w t_1 t_2) = \text{weight } t_1 + \text{weight } t_2 \).) Since the cost of a tree is such a fundamental concept, it seems necessary to prove that the above function definition is correct.

\[
\text{theorem cost_eq_Sum_freq_mult_depth:}
\begin{align*}
\text{consistent } t &\implies \text{cost } t = \sum_{a \in \text{alphabet } t} \text{freq } t a \times \text{depth } t a
\end{align*}
\]

The proof is by structural induction on \( t \). If \( t = \text{Leaf } w b \), both sides of the equation simplify to 0. This leaves the case \( t = \text{Node } w t_1 t_2 \). Let \( A, A_1, \) and \( A_2 \) stand for \( \text{alphabet } t, \text{alphabet } t_1, \) and \( \text{alphabet } t_2 \), respectively. We have

\[
\begin{align*}
\text{cost } t &= \text{weight } t_1 + \text{cost } t_1 + \text{weight } t_2 + \text{cost } t_2 \quad \text{(definition of cost)} \\
&= \text{weight } t_1 + \sum_{a \in A_1} \text{freq } t_1 a \times \text{depth } t_1 a + \text{weight } t_2 + \sum_{a \in A_2} \text{freq } t_2 a \times \text{depth } t_2 a \quad \text{(induction hypothesis)} \\
&= \text{weight } t_1 + \sum_{a \in A_1} \text{freq } t_1 a \times (\text{depth } t a - 1) + \text{weight } t_2 + \sum_{a \in A_2} \text{freq } t_2 a \times (\text{depth } t a - 1) \quad \text{(definition of depth, consistency)} \\
&= \text{weight } t_1 + \sum_{a \in A_1} \text{freq } t_1 a \times \text{depth } t a - \sum_{a \in A_1} \text{freq } t_1 a + \text{weight } t_2 + \sum_{a \in A_2} \text{freq } t_2 a \times \text{depth } t a - \sum_{a \in A_2} \text{freq } t_2 a \quad \text{(distributivity of } \times \text{ and } \sum \text{ over } -) \\
&= \sum_{a \in A_1} \text{freq } t_1 a \times \text{depth } t a + \sum_{a \in A_2} \text{freq } t_2 a \times \text{depth } t a \quad \text{(weight_eq_Sum_freq)} \\
&= \sum_{a \in A_1} \text{freq } t a \times \text{depth } t a + \sum_{a \in A_2} \text{freq } t a \times \text{depth } t a \quad \text{(definition of freq, consistency)} \\
&= \sum_{a \in A_1 \cup A_2} \text{freq } t a \times \text{depth } t a \quad \text{(sum.union_disjoint, consistency)} \\
&= \sum_{a \in A} \text{freq } t a \times \text{depth } t a \quad \text{(definition of alphabet)}
\end{align*}
\]

The structured proof closely follows this argument.
proof (induct t)
  case Leaf thus case by simp
next
case (Node w t1 t2)
  let t = Node w t1 t2
  let A = alphabet t and A1 = alphabet t1 and A2 = alphabet t2
  note c = (consistent t)
  note hyps = Node
  have d2: \forall a. \[\{A1 \cap A2 = \emptyset; \ a \in A2\} \implies \text{depth } t \ a = \text{depth } t2 \ a + 1\]
    by auto
  have cost t = weight t1 + cost t1 + weight t2 + cost t2 by simp
  also have \ldots = weight t1 + (\sum_{a \in A1} \text{freq } t1 \ a \times \text{depth } t1 \ a)
    + weight t2 + (\sum_{a \in A2} \text{freq } t2 \ a \times \text{depth } t2 \ a)
    using hyps by simp
  also have \ldots = weight t1 + (\sum_{a \in A1} \text{freq } t1 \ a \times (\text{depth } t \ a - 1))
    + weight t2 + (\sum_{a \in A2} \text{freq } t2 \ a \times (\text{depth } t \ a - 1))
    using c d2 by simp
  also have \ldots = weight t1 + (\sum_{a \in A1} \text{freq } t1 \ a \times \text{depth } t \ a)
    - (\sum_{a \in A1} \text{freq } t1 \ a)
    + weight t2 + (\sum_{a \in A2} \text{freq } t2 \ a \times \text{depth } t \ a)
    - (\sum_{a \in A2} \text{freq } t2 \ a)
    using c d2 by (simp add: sum.distrib)
  also have \ldots = (\sum_{a \in A1} \text{freq } t1 \ a \times \text{depth } t \ a)
    + (\sum_{a \in A2} \text{freq } t2 \ a \times \text{depth } t \ a)
    using c by (simp add: weight_eq_Sum_freq)
  also have \ldots = (\sum_{a \in A1} \text{freq } t \ a \times \text{depth } t \ a)
    + (\sum_{a \in A2} \text{freq } t \ a \times \text{depth } t \ a)
    using c by auto
  also have \ldots = (\sum_{a \in A1 \cup A2} \text{freq } t \ a \times \text{depth } t \ a)
    using c by (simp add: sum.union_disjoint)
  also have \ldots = (\sum_{a \in A} \text{freq } t \ a \times \text{depth } t \ a) by simp
  finally show case .
qed

Finally, it should come as no surprise that trees with height 0 have cost 0.

lemma height_0_imp_cost_0[simp]:
  height t = 0 \implies cost t = 0
by (case_tac t) simp

2.10 Optimality

A tree is optimum if and only if its cost is not greater than that of any comparable
tree. We can ignore inconsistent trees without loss of generality.
definition optimum :: α tree ⇒ bool where
optimum t =
(∀u. consistent u → alphabet t = alphabet u → freq t = freq u →
cost t ≤ cost u)

3 Functional Implementation of Huffman’s Algorithm

3.1 Cached Weight

The cached weight of a node is the weight stored directly in the node. Our arguments rely on the computed weight (embodied by the weight function) rather than the cached weight, but the implementation of Huffman’s algorithm uses the cached weight for performance reasons.

primrec cachedWeight :: α tree ⇒ nat where
cachedWeight (Leaf w a) = w
cachedWeight (Node w t₁ t₂) = w

The cached weight of a leaf node is identical to its computed weight.

lemma height_0_imp_cachedWeight_eq_weight[simp]:
height t = 0 ⇒ cachedWeight t = weight t
by (case_tac t) simp

3.2 Tree Union

The implementation of Huffman’s algorithm builds on two additional auxiliary functions. The first one, uniteTrees, takes two trees

and returns the tree

definition uniteTrees :: α tree ⇒ α tree ⇒ α tree where
uniteTrees t₁ t₂ = Node (cachedWeight t₁ + cachedWeight t₂) t₁ t₂
The alphabet, consistency, and symbol frequencies of a united tree are easy to connect to the homologous properties of the subtrees.

**Lemma** `alphabet_uniteTrees[simp]`:
\[
alphabet (\text{uniteTrees } t_1 t_2) = \alphabet t_1 \cup \alphabet t_2
\]
by `(simp add: uniteTrees_def)`

**Lemma** `consistent_uniteTrees[simp]`:
\[
[\text{consistent } t_1; \text{consistent } t_2; \alphabet t_1 \cap \alphabet t_2 = \emptyset] \implies \text{consistent } (\text{uniteTrees } t_1 t_2)
\]
by `(simp add: uniteTrees_def)`

**Lemma** `freq_uniteTrees[simp]`:
\[
freq (\text{uniteTrees } t_1 t_2) a = freq t_1 a + freq t_2 a
\]
by `(simp add: uniteTrees_def)`

### 3.3 Ordered Tree Insertion

The auxiliary function `insortTree` inserts a tree into a forest sorted by cached weight, preserving the sort order.

**Primrec** `insortTree :: a tree ⇒ a forest ⇒ a forest where`
\[
insortTree u [] = [u]
\]
\[
insortTree u (t \cdot ts) =
(\text{if } \text{cachedWeight } u \leq \text{cachedWeight } t \text{ then } u \cdot t \cdot ts \text{ else } t \cdot \text{insortTree } u ts)
\]

The resulting forest contains one more tree than the original forest. Clearly, it cannot be empty.

**Lemma** `length_insortTree[simp]`:
\[
\text{length } (\text{insortTree } t ts) = \text{length } ts + 1
\]
by `(induct ts) simp`

**Lemma** `insortTree_ne_Nil[simp]`:
\[
insortTree t ts \neq []
\]
by `(case_tac ts) simp`

The alphabet, consistency, symbol frequencies, and height of a forest after insertion are easy to relate to the homologous properties of the original forest and the inserted tree.

**Lemma** `alphabet_F_insortTree[simp]`:
\[
\alphabet_F (\text{insortTree } t ts) = \alphabet t \cup \alphabet_F ts
\]
by `(induct ts) auto`
lemma consistentF_insortTree[simp]:
consistentF (insortTree t ts) = consistentF (t · ts)
by (induct ts) auto

lemma freqF_insortTree[simp]:
freqF (insortTree t ts) = (λa. freq t a + freqF ts a)
by (induct ts) (simp add: ext) +

lemma heightF_insortTree[simp]:
heightF (insortTree t ts) = max (height t) (heightF ts)
by (induct ts) auto

3.4 The Main Algorithm

Huffman’s algorithm repeatedly unites the first two trees of the forest it receives
as argument until a single tree is left. It should initially be invoked with a list of
leaf nodes sorted by weight. Note that it is not defined for the empty list.

fun huffman :: α forest ⇒ α tree where
huffman [t] = t
huffman (t1 · t2 · ts) = huffman (insortTree (uniteTrees t1 t2) ts)

The time complexity of the algorithm is quadratic in the size of the forest. If
we eliminated the inner node’s cached weight component, and instead recomputed
the weight each time it is needed, the complexity would remain quadratic,
but with a larger constant. Using a binary search in insortTree, the correspond-
ing imperative algorithm is $O(n \log n)$ if we keep the weight cache and $O(n^2)$ if
we drop it. An $O(n)$ imperative implementation is possible by maintaining two
queues, one containing the unprocessed leaf nodes and the other containing the
combined trees [8, p. 404].

The tree returned by the algorithm preserves the alphabet, consistency, and
symbol frequencies of the original forest.

theorem alphabet_huffman[simp]:
ts ≠ [] ⇒ alphabet (huffman ts) = alphabetF ts
by (induct ts rule: huffman.induct) auto

theorem consistent_huffman[simp]:
[consistentF ts; ts ≠ []] ⇒ consistent (huffman ts)
by (induct ts rule: huffman.induct) simp +

theorem freq_huffman[simp]:
ts ≠ [] ⇒ freq (huffman ts) a = freqF ts a
by (induct ts rule: huffman.induct) auto
4 Definition of Auxiliary Functions Used in the Proof

4.1 Sibling of a Symbol

The sibling of a symbol $a$ in a tree $t$ is the label of the node that is the (left or right) sibling of the node labeled with $a$ in $t$. If the symbol $a$ is not in $t$’s alphabet or it occurs in a node with no sibling leaf, we define the sibling as being $a$ itself; this gives us the nice property that if $t$ is consistent, then $\text{sibling } t \ a \neq a$ if and only if $a$ has a sibling. As an illustration, we have $\text{sibling } t \ a = b$, $\text{sibling } t \ b = a$, and $\text{sibling } t \ c = c$ for the tree

![Tree Diagram]

fun sibling :: $\alpha$ tree $\Rightarrow$ $\alpha$ $\Rightarrow$ $\alpha$
where
sibling (Leaf $w$ $b$) $a$ = $a$

sibling (Node $w$ (Leaf $w$ $b$) (Leaf $w$ $c$)) $a$ =
   (if $a = b$ then $c$ else if $a = c$ then $b$ else $a$)

sibling (Node $w$ $t_1$ $t_2$) $a$ =
   (if $a \in$ alphabet $t_1$ then sibling $t_1$ $a$
    else if $a \in$ alphabet $t_2$ then sibling $t_2$ $a$
    else $a$)

Because $\text{sibling}$ is defined using sequential pattern matching [9, 10], reasoning about it can become tedious. Simplification rules therefore play an important role.

lemma notin_alphabet_imp_sibling_id[simp]:
$a \notin$ alphabet $t \implies$ sibling $t \ a = a$

by (cases rule: sibling.cases[where $x = (t, a)$]) simp+

lemma height_0_imp_sibling_id[simp]:
$height \ t = 0 \implies$ sibling $t \ a = a$

by (case_tac $t$) simp+

lemma height_gt_0_in_alphabet_imp_sibling_left[simp]:
$[height \ t_1 > 0; \ a \in$ alphabet $t_1] \implies$
$sibling \ (Node \ w \ t_1 \ t_2) \ a = sibling \ t_1 \ a$

by (case_tac $t_1$) simp+

lemma height_gt_0_in_alphabet_imp_sibling_right[simp]:
$[height \ t_2 > 0; \ a \in$ alphabet $t_1] \implies$
sibling (Node w t₁ t₂) a = sibling t₁ a
by (case_tac t₂) simp⁺

lemma height_gt_0_notin_alphabet_imp_sibling_left[simp]:
b \height t₁ > 0; a \notin alphabet t₁ \implies
sibling (Node w t₁ t₂) a = sibling t₂ a
by (case_tac t₁) simp⁺

lemma height_gt_0_notin_alphabet_imp_sibling_right[simp]:
b \height t₂ > 0; a \notin alphabet t₁ \implies
sibling (Node w t₁ t₂) a = sibling t₂ a
by (case_tac t₂) simp⁺

lemma either_height_gt_0_imp_sibling[simp]:
\height t₁ > 0 \lor \height t₂ > 0 \implies
sibling (Node w t₁ t₂) a =
(if a \in alphabet t₁ then sibling t₁ a else sibling t₂ a)
by auto

The following rules are also useful for reasoning about siblings and alphabets.

lemma in_alphabet_imp_sibling_in_alphabet:
a \in alphabet t \implies sibling t a \in alphabet t
by (induct t a rule: sibling.induct) auto

lemma sibling_ne_imp_sibling_in_alphabet:
sibling t a \neq a \implies sibling t a \in alphabet t
by (metis notin_alphabet_imp_sibling_id in_alphabet_imp_sibling_in_alphabet)

The default induction rule for sibling distinguishes four cases.

BASE CASE: \( t = \text{Leaf} w b \).

INDUCTION STEP 1: \( t = \text{Node} w (\text{Leaf} w b) (\text{Leaf} w c) \).

INDUCTION STEP 2: \( t = \text{Node} w (\text{Node} w₁ t₁₁ t₁₂) t₂ \).

INDUCTION STEP 3: \( t = \text{Node} w t₁ (\text{Node} w₂ t₂₁ t₂₂) \).

This rule leaves much to be desired. First, the last two cases overlap and can normally be handled the same way, so they should be combined. Second, the nested Node constructors in the last two cases reduce readability. Third, under the assumption that \( t \) is consistent, we would like to perform the same case distinction on \( a \) as we did for tree_induct_consistent, with the same benefits for automation. These observations lead us to develop a custom induction rule that distinguishes the following cases.

BASE CASE: \( t = \text{Leaf} w b \).
INDUCTION STEP 1: \( t = \text{Node } w (\text{Leaf } w_\mathit{b} b) (\text{Leaf } w_\mathit{c} c) \) with \( b \neq c \).

INDUCTION STEP 2: \( t = \text{Node } w t_1 t_2 \) and either \( t_1 \) or \( t_2 \) has nonzero height.

SUBCASE 1: \( a \) belongs to \( t_1 \) but not to \( t_2 \).

SUBCASE 2: \( a \) belongs to \( t_2 \) but not to \( t_1 \).

SUBCASE 3: \( a \) belongs to neither \( t_1 \) nor \( t_2 \).

The statement of the rule and its proof are similar to what we did for consistent trees, the main difference being that we now have two induction steps instead of one.

**lemma sibling_induct_consistent [consumes 1, case_names base step1 step2 step21 step22 step23]:**

\[
\forall w b a. P (\text{Leaf } w b) a;
\forall w w_\mathit{b} b w_\mathit{c} c a. b \neq c \implies P (\text{Node } w (\text{Leaf } w_\mathit{b} b) (\text{Leaf } w_\mathit{c} c)) a;
\forall w t_1 t_2 a.
\left[ \begin{align*}
\text{consistent } t_1; \\
\text{consistent } t_2; \\
\text{alphabet } t_1 \cap \text{alphabet } t_2 = \emptyset; \\
\text{height } t_1 > 0 \lor \text{height } t_2 > 0; \\
\text{sibling } t_1 a \in \text{alphabet } t_1; \\
\text{sibling } t_2 a \notin \text{alphabet } t_2; \\
P t_1 a & \implies P (\text{Node } w t_1 t_2) a;
\end{align*} \right]
\forall w t_1 t_2 a.
\left[ \begin{align*}
\text{consistent } t_1; \\
\text{consistent } t_2; \\
\text{alphabet } t_1 \cap \text{alphabet } t_2 = \emptyset; \\
\text{height } t_1 > 0 \lor \text{height } t_2 > 0; \\
\text{sibling } t_1 a \notin \text{alphabet } t_1; \\
\text{sibling } t_2 a \in \text{alphabet } t_2; \\
P t_2 a & \implies P (\text{Node } w t_1 t_2) a;
\end{align*} \right]
\forall w t_1 t_2 a.
\left[ \begin{align*}
\text{consistent } t_1; \\
\text{consistent } t_2; \\
\text{alphabet } t_1 \cap \text{alphabet } t_2 = \emptyset; \\
\text{height } t_1 > 0 \lor \text{height } t_2 > 0; \\
\text{sibling } t_2 a \notin \text{alphabet } t_2; \\
P t_2 a & \implies P (\text{Node } w t_1 t_2) a;
\end{align*} \right]
\]
apply fastforce
apply simp+
apply (auto intro: in_alphabet_imp_sibling_in_alphabet) [1]
by lexicographic_order

The custom induction rule allows us to prove new properties of \textit{Sibling} with little effort.

\textbf{lemma sibling_sibling_id[simp]:}
consistent t \implies sibling t (sibling t a) = a
\textbf{by (induct t a rule: sibling_induct_consistent) simp+}

\textbf{lemma sibling_reciprocal:}
[consistent t; sibling t a = b] \implies sibling t b = a
\textbf{by auto}

\textbf{lemma depth_height_imp_sibling_ne:}
[consistent t; depth t a = height t; height t > 0; a \in alphabet t] \implies sibling t a \neq a
\textbf{by (induct t a rule: sibling_induct_consistent) auto}

\textbf{lemma depth_sibling[simp]:}
consistent t \implies depth t (sibling t a) = depth t a
\textbf{by (induct t a rule: sibling_induct_consistent) simp+}

\subsection{Leaf Interchange}

The \textit{swapLeaves} function takes a tree \textit{t} together with two symbols \textit{a}, \textit{b} and their frequencies \textit{w}_a, \textit{w}_b, and returns the tree \textit{t} in which the leaf nodes labeled with \textit{a} and \textit{b} are exchanged. When invoking \textit{swapLeaves}, we normally pass \textit{freq t a} and \textit{freq t b} for \textit{w}_a and \textit{w}_b.

Note that we do not bother updating the cached weight of the ancestor nodes when performing the interchange. The cached weight is used only in the implementation of Huffman’s algorithm, which does not invoke \textit{swapLeaves}.

\textbf{primrec swapLeaves ::} \textit{\alpha} \textit{tree} \Rightarrow \mathbb{nat} \Rightarrow \alpha \Rightarrow \mathbb{nat} \Rightarrow \alpha \Rightarrow \alpha \textit{tree where}
\textit{swapLeaves (Leaf w c) w_a a w_b b} =
  (if \textit{c} = \textit{a} then \textit{Leaf w_b b} else if \textit{c} = \textit{b} then \textit{Leaf w_a a} else \textit{Leaf w_c c})
\textit{swapLeaves (Node w t_1 t_2) w_a a w_b b} =
  \textit{Node w (swapLeaves t_1 w_a a w_b b) (swapLeaves t_2 w_a a w_b b)}

Swapping a symbol \textit{a} with itself leaves the tree \textit{t} unchanged if \textit{a} does not belong to it or if the specified frequencies \textit{w}_a and \textit{w}_b equal \textit{freq t a}. 

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lemma swapLeaves_id_when_notin_alphabet[simp]:
a \not\in \text{alphabet } t \implies \text{swapLeaves } t \ a \ w' a = t
by (induct t) simp

lemma swapLeaves_id[simp]:
consistent t \implies \text{swapLeaves } t \ (\text{freq } t \ a) \ a \ (\text{freq } t \ a) \ a = t
by (induct t a rule: tree_induct_consistent) simp

The alphabet, consistency, symbol depths, height, and symbol frequencies of the tree \(\text{swapLeaves } t \ w_a \ a \ w_b \ b\) can be related to the homologous properties of \(t\).

lemma alphabet_swapLeaves:
\[\text{alphabet } (\text{swapLeaves } t \ w_a \ a \ w_b \ b) =\]
\[\begin{cases}
  (\text{if } a \in \text{alphabet } t \text{ then }) \\
  \quad (\text{if } b \in \text{alphabet } t \text{ then } \text{alphabet } t \setminus \{a\} \cup \{b\}) \\
  \quad \text{else } (\text{if } b \in \text{alphabet } t \text{ then } \text{alphabet } t \setminus \{b\} \cup \{a\} \text{ else alphabet } t) \\
\end{cases}\]
by (induct t) auto

lemma consistent_swapLeaves[simp]:
consistent t \implies \text{consistent } (\text{swapLeaves } t \ w_a \ a \ w_b \ b)
by (induct t) (auto simp: alphabet_swapLeaves)

lemma depth_swapLeaves_neither[simp]:
\[\text{consistent } t ; c \neq a ; c \neq b \implies \text{depth } (\text{swapLeaves } t \ w_a \ a \ w_b \ b) \ c = \text{depth } t \ c\]
by (induct t a rule: tree_induct_consistent) (auto simp: alphabet_swapLeaves)

lemma height_swapLeaves[simp]:
\[\text{height } (\text{swapLeaves } t \ w_a \ a \ w_b \ b) = \text{height } t\]
by (induct t) simp

lemma freq_swapLeaves[simp]:
\[\text{consistent } t ; a \neq b \implies \text{freq } (\text{swapLeaves } t \ w_a \ a \ w_b \ b) =\]
\[\begin{cases}
  (\lambda c. \text{if } c = a \text{ then if } b \in \text{alphabet } t \text{ then } w_a \text{ else } 0 \\
  \quad \text{else if } c = b \text{ then if } a \in \text{alphabet } t \text{ then } w_b \text{ else } 0 \\
  \quad \text{else freq } t \ c) \\
\end{cases}\]
apply (rule ext)
apply (induct t)
by auto

For the lemmas concerned with the resulting tree’s weight and cost, we avoid subtraction on natural numbers by rearranging terms. For example, we write
\[\text{weight } (\text{swapLeaves } t \ w_a \ a \ w_b \ b) + \text{freq } t \ a = \text{weight } t + w_b\]
rather than the more conventional

\[ \text{weight} \left( \text{swapLeaves} \ t \ w_a \ a \ w_b \ b \right) = \text{weight} \ t + w_b - \text{freq} \ t \ a. \]

In Isabelle/HOL, these two equations are not equivalent, because by definition

\[ m - n = 0 \text{ if } n > m. \]

We could use the second equation and additionally assert that \( \text{freq} \ t \ a \leq \text{weight} \ t \) (an easy consequence of \textit{weight\_eq\_Sum\_freq}), and then apply the \textit{arith} tactic, but it is much simpler to use the first equation and stay with \textit{simp} and \textit{auto}. Another option would be to use integers instead of natural numbers.

**Lemma weight\_swapLeaves:**

\[
\text{[consistent} \ t; \ a \neq b\text{]} \implies \begin{array}{l}
\text{if } a \in \text{alphabet} \ t \text{ then} \\
\quad \text{if } b \in \text{alphabet} \ t \text{ then} \\
\quad \quad \text{weight} \left( \text{swapLeaves} \ t \ w_a \ a \ w_b \ b \right) + \text{freq} \ t \ a + \text{freq} \ t \ b = \\
\quad \quad \text{weight} \ t + w_a + w_b \\
\quad \text{else} \\
\quad \quad \text{weight} \left( \text{swapLeaves} \ t \ w_a \ a \ w_b \ b \right) + \text{freq} \ t \ a = \text{weight} \ t + w_b \\
\quad \text{else} \\
\quad \quad \text{if } b \in \text{alphabet} \ t \text{ then} \\
\quad \quad \quad \text{weight} \left( \text{swapLeaves} \ t \ w_a \ a \ w_b \ b \right) + \text{freq} \ t \ b = \text{weight} \ t + w_a \\
\quad \quad \text{else} \\
\quad \quad \quad \text{weight} \left( \text{swapLeaves} \ t \ w_a \ a \ w_b \ b \right) = \text{weight} \ t
\end{array}
\]

**Proof**:

(induct \ t \ a \ rule: tree\_induct\_consistent)

— Base case: \( t = \text{Leaf} \ w \ b \)

\text{case base thus case by clarsimp}

next

— Induction step: \( t = \text{Node} \ w \ t_1 \ t_2 \)

— Subcase 1: \( a \) belongs to \( t_1 \) but not to \( t_2 \)

\text{case (step1 \ w \ t_1 \ t_2 \ a) show case}

**Proof** cases

\text{assume } b; \ b \in \text{alphabet} \ t_1

\text{hence } b \notin \text{alphabet} \ t_2 \text{ using step1 by auto}

\text{thus case using } b \text{ step1 by simp}

next

\text{assume } b \notin \text{alphabet} \ t_1 \text{ thus case using step1 by auto}

qed

next

— Subcase 2: \( a \) belongs to \( t_2 \) but not to \( t_1 \)

\text{case (step2 \ w \ t_1 \ t_2 \ a) show case}

**Proof** cases

\text{assume } b; \ b \in \text{alphabet} \ t_1
hence b \notin \text{alphabet } t_2 \text{ using step}_2 \text{ by auto}
thus \text{case using } b \text{ step}_2 \text{ by simp}

next
\text{assume } b \notin \text{alphabet } t_1 \text{ thus case using step}_2 \text{ by auto}
\text{qed}

next

— \text{SUBCASE 3: } a \text{ belongs to neither } t_1 \text{ nor } t_2
\text{case (step}_3 \text{ w } t_1 t_2 a) \text{ show case}

\text{proof cases}
\text{assume } b: b \in \text{alphabet } t_1
\text{hence } b \notin \text{alphabet } t_2 \text{ using step}_3 \text{ by auto}
\thus \text{case using } b \text{ step}_3 \text{ by simp}

next
\text{assume } b \notin \text{alphabet } t_1 \text{ thus case using step}_3 \text{ by auto}
\text{qed}
\text{qed}

\text{lemma } \text{cost swapLeaves:}
\langle \text{consistent } t; a \neq b \rangle \implies
\text{if } a \in \text{alphabet } t \text{ then}
\text{if } b \in \text{alphabet } t \text{ then}
\begin{align*}
\text{cost (swapLeaves } t w_a a w_b b) + \text{freq } t a \times \text{depth } t a + \text{freq } t b \times \text{depth } t b &= \\
\text{cost } t + w_a \times \text{depth } t b + w_b \times \text{depth } t a
\end{align*}
\text{else}
\begin{align*}
\text{cost (swapLeaves } t w_a a w_b b) + \text{freq } t a \times \text{depth } t a = \\
\text{cost } t + w_b \times \text{depth } t a
\end{align*}
\text{else}
\text{if } b \in \text{alphabet } t \text{ then}
\begin{align*}
\text{cost (swapLeaves } t w_a a w_b b) + \text{freq } t b \times \text{depth } t b &= \\
\text{cost } t + w_a \times \text{depth } t b
\end{align*}
\text{else}
\text{cost (swapLeaves } t w_a a w_b b) = \text{cost } t
\text{proof (induct } t)\text{ case Leaf show case by simp}

next
\text{case (Node } w t_1 t_2)\text{ note } c = \langle \text{consistent } (\text{Node } w t_1 t_2) \rangle\text{ note hyps = Node}
\text{have } w_1: \text{ if } a \in \text{alphabet } t_1 \text{ then}
\text{if } b \in \text{alphabet } t_1 \text{ then}
\begin{align*}
\text{weight (swapLeaves } t_1 w_a a w_b b) + \text{freq } t_1 a + \text{freq } t_1 b &= \\
\text{weight } t_1 + w_a + w_b
\end{align*}
\text{else
weight \( \text{swapLeaves} t \ w_a \ a \ w_b \ b \) + \text{freq} t \ a = \text{weight} t_1 + w_b

else

if \( b \in \text{alphabet} t_1 \) then

weight \( \text{swapLeaves} t \ w_a \ a \ w_b \ b \) + \text{freq} t \ b = \text{weight} t_1 + w_a

else

weight \( \text{swapLeaves} t \ w_a \ a \ w_b \ b \) = weight t_1 \textbf{using hyps}

by \( \text{simp add: weight_swapLeaves} \)

have \( w_2 \) if \( a \in \text{alphabet} t_2 \) then

if \( b \in \text{alphabet} t_2 \) then

weight \( \text{swapLeaves} t \ w_a \ a \ w_b \ b \) + \text{freq} t \ a + \text{freq} t \ b = \text{weight} t_2 + w_a + w_b

else

weight \( \text{swapLeaves} t \ w_a \ a \ w_b \ b \) + \text{freq} t \ b = weight t_2 + w_b

else

if \( b \in \text{alphabet} t_2 \) then

weight \( \text{swapLeaves} t \ w_a \ a \ w_b \ b \) + \text{freq} t \ b = weight t_2 + w_a

else

weight \( \text{swapLeaves} t \ w_a \ a \ w_b \ b \) = weight t_2 \textbf{using hyps}

by \( \text{simp add: weight_swapLeaves} \)

show case

proof cases

assume \( a_1 \) \( a \in \text{alphabet} t_1 \)

hence \( a_2 \) \( a \notin \text{alphabet} t_2 \textbf{ using c by auto} \)

show case

proof cases

assume \( b_1 \) \( b \in \text{alphabet} t_1 \)

hence \( b_2 \) \( b \notin \text{alphabet} t_2 \textbf{ using c by auto} \)

thus case \textbf{using} \( a_1 \ a_2 \ b_1 \ w_1 \ w_2 \textbf{ hyps by simp} \)

next

assume \( b_1 \) \( b \notin \text{alphabet} t_1 \textbf{ show case} \)

proof cases

assume \( b \in \text{alphabet} t_2 \textbf{ thus case using} \( a_1 \ a_2 \ b_1 \ w_1 \ w_2 \textbf{ hyps by simp} \)

next

assume \( b \notin \text{alphabet} t_2 \textbf{ thus case using} \( a_1 \ a_2 \ b_1 \ w_1 \ w_2 \textbf{ hyps by simp} \)

qed

qed

next

assume \( a_1 \) \( a \notin \text{alphabet} t_1 \textbf{ show case} \)

proof cases

assume \( a_2 \) \( a \in \text{alphabet} t_2 \textbf{ show case} \)

proof cases

assume \( b_1 \) \( b \in \text{alphabet} t_1 \)
hence $b \not\in \text{alphabet } t_2$ using $c$ by auto
thus case using $a_1$ $a_2$ $b_1$ $w_1$ $w_2$ hyps by simp
next
assume $b_1$: $b \not\in \text{alphabet } t_1$ show case
proof cases
  assume $b \in \text{alphabet } t_2$ thus case using $a_1$ $a_2$ $b_1$ $w_1$ $w_2$ hyps by simp
next
assume $b \not\in \text{alphabet } t_2$ thus case using $a_1$ $a_2$ $b_1$ $w_1$ $w_2$ hyps by simp
qed
qed

next
assume $a_2$: $a \not\in \text{alphabet } t_2$ show case
proof cases
  assume $b_1$: $b \in \text{alphabet } t_1$
hence $b \not\in \text{alphabet } t_2$ using $c$ by auto
thus case using $a_1$ $a_2$ $b_1$ $w_1$ $w_2$ hyps by simp
next
assume $b_1$: $b \not\in \text{alphabet } t_1$ show case
proof cases
  assume $b \in \text{alphabet } t_2$ thus case using $a_1$ $a_2$ $b_1$ $w_1$ $w_2$ hyps by simp
next
assume $b \not\in \text{alphabet } t_2$ thus case using $a_1$ $a_2$ $b_1$ $w_1$ $w_2$ hyps by simp
qed
qed
qed

dition of the following statement is valid: “If Astrid exchanges her house with Bernard’s neighbor, Bernard becomes Astrid’s new neighbor.” A similar property holds for binary trees.

\begin{description}
\item[Common sense] tells us that the following statement is valid: “If Astrid exchanges her house with Bernard’s neighbor, Bernard becomes Astrid’s new neighbor.” A similar property holds for binary trees.
\end{description}

\begin{enumerate}
\item \textbf{lemma} sibling\_swapLeaves\_sibling[simp]:
\item [consistent $t$; sibling $t$ $b \neq b$; $a \neq b$] $\implies$
\item sibling (swapLeaves $t$ $w$ $a$ $w_2$ (sibling $t$ $b$)) $a = b$
\item \textbf{proof} (induct $t$)
\item case Leaf \textbf{thus} case \textbf{by} simp
\item next
\item case (Node $w$ $t_1$ $t_2$)
\item note hyps = Node
\item show case
\item proof (cases height $t_1$ = 0)
\item case True
\item note $h_1 = True$
\end{enumerate}
show thesis
proof (cases \( t_1 \))
  case (Leaf \( w_c c \))
  note \( l_1 = \text{Leaf} \)
  show thesis
  proof (cases height \( t_2 = 0 \))
    case True
    note \( h_2 = \text{True} \)
    show thesis
    proof (cases \( t_2 \))
      case Leaf thus thesis using \( l_1 \) hyps by auto metis\(^+\)
    next
    case Node thus thesis using \( h_2 \) by simp
    qed
next
  case False
  note \( h_2 = \text{False} \)
  show thesis
  proof cases
    assume \( c = b \) thus thesis using \( l_1 h_2 \) hyps by simp
  next
    assume \( c \neq b \)
    have sibling \( t_2 b \in \text{alphabet} t_2 \) using \( c \neq b \) \( l_1 h_2 \) hyps
    by (simp add: sibling_ne_imp_sibling_in_alphabet)
    thus thesis using \( c \neq b \) \( l_1 h_2 \) hyps by auto
    qed
  qed
next
  case Node thus thesis using \( h_1 \) by simp
  qed
next
  case False
  note \( h_1 = \text{False} \)
  show thesis
  proof (cases height \( t_2 = 0 \))
    case True
    note \( h_2 = \text{True} \)
    show thesis
    proof (cases \( t_2 \))
      case (Leaf \( w_d d \))
      note \( l_2 = \text{Leaf} \)
      show thesis
      proof cases
assume \( d = b \) thus thesis using \( h_1 \) \( l_2 \) hyps by simp
next
assume \( d \neq b \) show thesis
proof (cases \( b \in \text{alphabet} \ t_1 \))
  case True
  hence sibling \( t_1 \) \( b \in \text{alphabet} \ t_1 \) using \( d \neq b \) \( h_1 \) \( l_2 \) hyps
  by (simp add: sibling_ne_imp_sibling_in_alphabet)
  thus thesis using \( d \neq b \) \( h_1 \) \( l_2 \) hyps
  by (simp add: alphabet_swapLeaves)
next
  case False thus thesis using \( d \neq b \) \( l_2 \) hyps by simp
qed
qed
next
case Node thus thesis using \( h_2 \) by simp
qed
next
case False
note \( h_2 = \text{False} \)
show thesis
proof (cases \( b \in \text{alphabet} \ t_1 \))
  case True thus thesis using \( h_1 \) \( h_2 \) hyps by auto
next
  case False
  note \( b_1 = \text{False} \)
  show thesis
  proof (cases \( b \in \text{alphabet} \ t_2 \))
    case True thus thesis using \( b_1 \) \( h_1 \) \( h_2 \) hyps
    by (auto simp: in_alphabet_imp_sibling_in_alphabet
        alphabet_swapLeaves)
next
  case False thus thesis using \( b_1 \) \( h_1 \) \( h_2 \) hyps by simp
qed
qed
qed
qed

4.3 Symbol Interchange

The \( \text{swapSyms} \) function provides a simpler interface to \( \text{swapLeaves} \), with \( \text{freq} \ t \ a \) and \( \text{freq} \ t \ b \) in place of \( w_a \) and \( w_b \). Most lemmas about \( \text{swapSyms} \) are directly adapted from the homologous results about \( \text{swapLeaves} \).
definition swapSyms :: α tree ⇒ α ⇒ α ⇒ α tree where
swapSyms t a b = swapLeaves t (freq t a) a (freq t b) b

lemma swapSyms_id[simp]:
consistent t ⇒ swapSyms t a a = t
by (simp add: swapSyms_def)

lemma alphabet_swapSyms[simp]:
[a ∈ alphabet t; b ∈ alphabet t] ⇒ alphabet (swapSyms t a b) = alphabet t
by (simp add: swapSyms_def alphabet_swapLeaves)

lemma consistent_swapSyms[simp]:
consistent t ⇒ consistent (swapSyms t a b)
by (simp add: swapSyms_def)

lemma depth_swapSyms_neither[simp]:
[consistent t; c ≠ a; c ≠ b] ⇒ depth (swapSyms t a b) c = depth t c
by (simp add: swapSyms_def)

lemma freq_swapSyms[simp]:
[consistent t; a ∈ alphabet t; b ∈ alphabet t] ⇒ freq (swapSyms t a b) = freq t
by (case_tac a = b) (simp add: swapSyms_def ext)

lemma cost_swapSyms:
assumes consistent t a ∈ alphabet t b ∈ alphabet t
shows cost (swapSyms t a b) + freq t a × depth t a + freq t b × depth t b =
cost t + freq t a × depth t b + freq t b × depth t a
proof cases
assume a = b thus thesis using assms by simp
next
assume a ≠ b
hence cost (swapLeaves t (freq t a) a (freq t b) b)
  + freq t a × depth t a + freq t b × depth t b =
cost t + freq t a × depth t b + freq t b × depth t a
using assms by (simp add: cost_swapLeaves)
thus thesis using assms by (simp add: swapSyms_def)
qed

If a’s frequency is lower than or equal to b’s, and a is higher up in the tree than b
or at the same level, then interchanging a and b does not increase the tree’s cost.

lemma le_le_imp_sum_mult_le_sum_mult:
assumes i ≤ j m ≤ (n::nat)
sshows i × n + j × m ≤ i × m + j × n
proof

\[
\begin{align*}
\text{have} & \ i \times m + i \times (n - m) + j \times m \leq i \times m + j \times m + j \times (n - m) \ \text{using} \ \text{assms} \\
\text{by} & \ \text{simp} \\
\text{thus} & \ \text{thesis using} \ \text{assms} \\
\text{by} & \ (\text{simp add: diff_mult_distrib})
\end{align*}
\]

qed

lemma cost_swapSyms_le:

assumes consistent t a \in alphabet t b \in alphabet t \ freq t a \leq freq t b \\
depth t a \leq depth t b

shows cost (swapSyms t a b) \leq cost t

proof

\[
\begin{align*}
\text{let} & \ aabb = \freq t a \times \depth t a + \freq t b \times \depth t b \\
\text{let} & \ abba = \freq t a \times \depth t b + \freq t b \times \depth t a \\
\text{have} & \ abba \leq aabb \ \text{using} \ \text{assms} (4-5) \\
\text{by} & \ (\text{rule le_le_imp_sum_mult_le_sum_mult}) \\
\text{have} & \ \text{cost} (\text{swapSyms} t a b) + aabb = \text{cost} t + abba \ \text{using} \ \text{assms} (1-3) \\
\text{by} & \ (\text{simp add: cost_swapSyms add.assoc[THEN sym]}) \\
\text{also have} & \ . . . \leq \text{cost} t + aabb \ \text{using} \ abba \leq aabb \ \text{by} \ \text{simp} \\
\text{finally show} & \ \text{thesis using} \ \text{assms} (4-5) \ \text{by} \ \text{simp}
\end{align*}
\]

qed

As stated earlier, “If Astrid exchanges her house with Bernard’s neighbor, Bernard becomes Astrid’s new neighbor.”

lemma sibling_swapSyms_sibling[simp]:

\[
[\text{consistent} t; \ \text{sibling} t b \neq b; a \neq b] \implies \text{sibling} (\text{swapSyms} t a (\text{sibling} t b)) a = b
\]

by (simp add: swapSyms_def)

“If Astrid exchanges her house with Bernard, Astrid becomes Bernard’s old neighbor’s new neighbor.”

lemma sibling_swapSyms_other_sibling[simp]:

\[
[\text{consistent} t; \ \text{sibling} t b \neq a; \ \text{sibling} t b \neq b; a \neq b] \implies \text{sibling} (\text{swapSyms} t a b) (\text{sibling} t b) = a
\]

by (metis consistent_swapSyms sibling_swapSyms_sibling sibling_reciprocal)

4.4 Four-Way Symbol Interchange

The \text{swapSyms} function exchanges two symbols \(a\) and \(b\). We use it to define the four-way symbol interchange function \text{swapFourSyms}, which takes four symbols \(a, b, c, d\) with \(a \neq b\) and \(c \neq d\), and exchanges them so that \(a\) and \(b\) occupy \(c\) and \(d\)’s positions.
A naive definition of this function would be

\[ \text{swapFourSyms } t \ a \ b \ c \ d = \text{swapSyms } (\text{swapSyms } t \ a \ c) \ b \ d. \]

This definition fails in the face of aliasing: If \( a = d \), but \( b \neq c \), then \( \text{swapFourSyms } a \ b \ c \ d \) would leave \( a \) in \( b \)'s position.\(^2\)

**definition** \( \text{swapFourSyms} :: \alpha \text{ tree } \Rightarrow \alpha \Rightarrow \alpha \Rightarrow \alpha \Rightarrow \alpha \Rightarrow \alpha \text{ tree} \) where

\[
\text{swapFourSyms } t \ a \ b \ c \ d = \\
(\text{if } a = d \text{ then } \text{swapSyms } t \ b \ c \\
\quad \text{else if } b = c \text{ then } \text{swapSyms } t \ a \ d \\
\quad \text{else } \text{swapSyms } (\text{swapSyms } t \ a \ c) \ b \ d)
\]

Lemmas about \( \text{swapFourSyms} \) are easy to prove by expanding its definition.

**lemma** \( \text{alphabet_swapFourSyms} [\text{simp}]: \)

\[
[a \in \text{alphabet } t; \ b \in \text{alphabet } t; \ c \in \text{alphabet } t; \ d \in \text{alphabet } t] \implies \text{alphabet } (\text{swapFourSyms } t \ a \ b \ c \ d) = \text{alphabet } t
\]

**by** (\( \text{simp add: swapFourSyms}_\text{def} \))

**lemma** \( \text{consistent_swapFourSyms} [\text{simp}]: \)

\[
\text{consistent } t \implies \text{consistent } (\text{swapFourSyms } t \ a \ b \ c \ d)
\]

**by** (\( \text{simp add: swapFourSyms}_\text{def} \))

**lemma** \( \text{freq_swapFourSyms} [\text{simp}]: \)

\[
[\text{consistent } t; \ a \in \text{alphabet } t; \ b \in \text{alphabet } t; \ c \in \text{alphabet } t; \ d \in \text{alphabet } t] \implies \text{freq } (\text{swapFourSyms } t \ a \ b \ c \ d) = \text{freq } t
\]

**by** (\( \text{auto simp: swapFourSyms}_\text{def} \))

“If Astrid and Bernard exchange their houses with Carmen and her neighbor, Astrid and Bernard will now be neighbors.”

**lemma** \( \text{sibling_swapFourSyms}_\text{when}_4\text{th}_\text{is}_\text{neighbor}: \)

**assumes** \n
\[
\text{consistent } t; \ a \in \text{alphabet } t; \ b \in \text{alphabet } t; \ c \in \text{alphabet } t; \ d \in \text{alphabet } t
\]

\[
a \neq b \quad \text{sibling } t \ c \neq c
\]

**shows** \( \text{sibling } (\text{swapFourSyms } t \ a \ b \ c \ (\text{sibling } t \ c)) \) \( a = b \)

**proof** (cases \( a \neq \text{sibling } t \ c \land b \neq c \))

**case** True 

**show** thesis

**proof** —

let \( d = \text{sibling } t \ c \)

let \( t_s = \text{swapFourSyms } t \ a \ b \ c \ d \)

have abba: \( (\text{sibling } t_s \ a = b) = (\text{sibling } t_s \ b = a) \) **using** \( \text{consistent } t; \)

**by** (\( \text{metis consistent_swapFourSyms}_\text{reciprocal} \))

have s: \( \text{sibling } t \ c = \text{sibling } (\text{swapSyms } t \ a \ c) \) **using** True **assms

---

\(^2\)Cormen et al. [4, p. 390] forgot to consider this case in their proof. Thomas Cormen indicated in a personal communication that this will be corrected in the next edition of the book.
by (metis sibling_reciprocal sibling_swapSyms_sibling)

have sibling \( t_5 \cdot b = \text{sibling} (\text{swapSyms} \; t \; a \; c) \) \( d \) using \( s \) True assms
  by (auto simp: swapFourSyms_def)

also have \( \ldots = a \) using True assms
  by (metis sibling_reciprocal sibling_swapSyms_other_sibling swapLeaves_id swapSyms_def)

finally have sibling \( t_5 \cdot b = a \).
  with \( \text{abba} \) show thesis ..
qed

next

case False thus thesis using assms
  by (auto intro: sibling_reciprocal simp: swapFourSyms_def)
qed

4.5 Sibling Merge

Given a symbol \( a \), the \text{mergeSibling} function transforms the tree

![Tree diagram]

into

The frequency of \( a \) in the result is the sum of the original frequencies of \( a \) and \( b \), so as not to alter the tree's weight.

fun mergeSibling :: \( \alpha \) tree \( \Rightarrow \) \( \alpha \) \Rightarrow \( \alpha \) tree where
mergeSibling (Leaf \( w_\cdot b \)) \( a \) = Leaf \( w_\cdot b \)
mergeSibling (Node \( w \) (Leaf \( w_\cdot b \)) (Leaf \( w_\cdot c \))) \( a \) =
  (if \( a = b \lor a = c \) then Leaf \( (w_\cdot b + w_\cdot c) \) \( a \)
    else Node \( w \) (Leaf \( w_\cdot b \)) (Leaf \( w_\cdot c \)))
mergeSibling (Node \( w \) \( t_1 \) \( t_2 \)) \( a \) =
  Node \( w \) (mergeSibling \( t_1 \) \( a \)) (mergeSibling \( t_2 \) \( a \))

The definition of \text{mergeSibling} has essentially the same structure as that of \text{sibling}. As a result, the custom induction rule that we derived for \text{sibling} works equally well for reasoning about \text{mergeSibling}.

lemmas mergeSibling_induct_consistent = sibling_induct_consistent
The properties of mergeSibling echo those of sibling. Like with sibling, simplification rules are crucial.

**lemma** notin_alphabet_imp_mergeSibling_id[simp]:
\[ a \notin \text{alphabet } t \implies \text{mergeSibling } t \ a = t \]
**by** (induct t a rule: mergeSibling.induct) simp^+

**lemma** height_gt_0_imp_mergeSibling_left[simp]:
\[ \text{height } t_1 > 0 \implies \text{mergeSibling } (\text{Node } w \ t_1 \ t_2) \ a = \]
\[ \text{Node } w \ (\text{mergeSibling } t_1 \ a \) (\text{mergeSibling } t_2 \ a) \]
**by** (case_tac t_1) simp^+

**lemma** height_gt_0_imp_mergeSibling_right[simp]:
\[ \text{height } t_2 > 0 \implies \text{mergeSibling } (\text{Node } w \ t_1 \ t_2) \ a = \]
\[ \text{Node } w \ (\text{mergeSibling } t_1 \ a \) (\text{mergeSibling } t_2 \ a) \]
**by** (case_tac t_2) simp^+

**lemma** either_height_gt_0_imp_mergeSibling[simp]:
\[ \text{height } t_1 > 0 \lor \text{height } t_2 > 0 \implies \]
\[ \text{mergeSibling } (\text{Node } w \ t_1 \ t_2) \ a = \]
\[ \text{Node } w \ (\text{mergeSibling } t_1 \ a \) (\text{mergeSibling } t_2 \ a) \]
**by** auto

**lemma** alphabet_mergeSibling[simp]:
\[ \text{consistent } t \ a \in \text{alphabet } t \implies \]
\[ \text{alphabet } (\text{mergeSibling } t \ a) = (\text{alphabet } t - \{ \text{sibling } t \ a \}) \cup \{ a \} \]
**by** (induct t a rule: mergeSibling_induct_consistent) auto

**lemma** consistent_mergeSibling[simp]:
\[ \text{consistent } t \implies \text{consistent } (\text{mergeSibling } t \ a) \]
**by** (induct t a rule: mergeSibling_induct_consistent) auto

**lemma** freq_mergeSibling:
\[ \text{consistent } t \ a \in \text{alphabet } t \ \text{sibling } t \ a \neq a \implies \]
\[ \text{freq } (\text{mergeSibling } t \ a) = \]
\[ (\lambda c. \text{if } c = a \text{ then } \text{freq } t \ a + \text{freq } t \ (\text{sibling } t \ a) \]
\[ \text{else if } c = \text{sibling } t \ a \text{ then } 0 \]
\[ \text{else } \text{freq } t \ c) \]
**by** (induct t a rule: mergeSibling_induct_consistent) (auto simp: fun_eq_iff)

**lemma** weight_mergeSibling[simp]:
\[ \text{weight } (\text{mergeSibling } t \ a) = \text{weight } t \]
**by** (induct t a rule: mergeSibling.induct) simp^+

If a has a sibling, merging a and its sibling reduces t’s cost by freq t a + freq t
(sibling $t \ a$).

**Lemma** cost_mergeSibling:

\[
\text{consistent } t \in \text{sibling } t \ a \neq a \implies \\
\text{cost} (\text{mergeSibling} t \ a) + \text{freq} t \ a + \text{freq} (\text{sibling } t \ a) = \text{cost} t
\]

by (induct $t \ a$ rule: mergeSibling_induct_consistent) auto

### 4.6 Leaf Split

The `splitLeaf` function undoes the merging performed by `mergeSibling`: Given two symbols $a, b$ and two frequencies $w_a, w_b$, it transforms

![Diagram of a tree with symbol $a$ at the root, split into two subtrees, and symbol $b$ at the root of one of the subtrees.]

In the resulting tree, $a$ has frequency $w_a$ and $b$ has frequency $w_b$. We normally invoke it with $w_a$ and $w_b$ such that freq $t \ a = w_a + w_b$.

**primrec** splitLeaf :: $\alpha$ tree $\Rightarrow$ nat $\Rightarrow$ $\alpha$ $\Rightarrow$ nat $\Rightarrow$ $\alpha$ $\Rightarrow$ $\alpha$ tree where

- splitLeaf (Leaf $w \ c$) $w_a$ $a$ $w_b$ $b$ =
  - (if $c = a$ then Node $w \ c$ (Leaf $w a$ $a$) (Leaf $w b$ $b$) else Leaf $w \ c$)

**primrec** splitLeaf $F$ :: $\alpha$ forest $\Rightarrow$ nat $\Rightarrow$ $\alpha$ $\Rightarrow$ nat $\Rightarrow$ $\alpha$ $\Rightarrow$ $\alpha$ forest where

- splitLeaf $F$ [] $w_a$ $a$ $w_b$ $b$ = []

Splitting leaf nodes affects the alphabet, consistency, symbol frequencies, weight, and cost in unsurprising ways.

**Lemma** notin_alphabet_imp_splitLeaf_id[simp]:

$a \notin \text{alphabet } t \implies \text{splitLeaf } t \ w_a \ a \ w_b \ b = t$

by (induct $t$) simp+

**Lemma** notin_alphabet$F$_imp_splitLeaf$F$_id[simp]:

$a \notin \text{alphabet } t \implies \text{splitLeaf } F \ts \ w_a \ a \ w_b \ b = \ts$

by (induct $ts$) simp+
lemma alphabet_splitLeaf [simp]:
alphabet (splitLeaf t w_1 a w_2 b) =
(if a ∈ alphabet t then alphabet t ∪ {b} else alphabet t)
by (induct t) simp+

lemma consistent_splitLeaf [simp]:
[consistent t; b ∉ alphabet t] ⇒ consistent (splitLeaf t w_1 a w_2 b)
by (induct t) auto

lemma freq_splitLeaf [simp]:
[frequent t; b /∈ alphabet t] ⇒ frequent (splitLeaf t w_1 a w_2 b)
by (induct t) auto

lemma weight_splitLeaf [simp]:
[frequent t; a ∈ alphabet t; frequent t a = w_1 + w_2] ⇒
weight (splitLeaf t w_1 a w_2 b) = weight t
by (induct t a rule: tree_induct_consistent) simp+

lemma cost_splitLeaf [simp]:
[frequent t; a ∈ alphabet t; frequent t a = w_1 + w_2] ⇒
cost (splitLeaf t w_1 a w_2 b) = cost t + w_1 + w_2
by (induct t a rule: tree_induct_consistent) simp+

4.7 Weight Sort Order

An invariant of Huffman’s algorithm is that the forest is sorted by weight. This
is expressed by the sortedByWeight function.

fun sortedByWeight :: a forest ⇒ bool where
sortedByWeight [] = True
sortedByWeight [t] = True
sortedByWeight (t₁ :: t₂ :: ts) =
(weight t₁ ≤ weight t₂ ∧ sortedByWeight (t₂ :: ts))

The function obeys the following fairly obvious laws.

lemma sortedByWeight_Cons_imp_sortedByWeight:
sortedByWeight (t :: ts) ⇒ sortedByWeight ts
by (case_tac ts) simp+

lemma sortedByWeight_Cons_imp_forall_weight_ge:
sortedByWeight (t :: ts) ⇒ ∀u ∈ set ts. weight u ≥ weight t
by (induct ts arbitrary: t) force+
**Lemma** `sortedByWeight_insortTree`:

\[
[\text{sortedByWeight ts; height } t = 0; \text{ heightF ts } = 0] \implies \text{sortedByWeight (insortTree t ts)}
\]

**by** (induct ts rule: `sortedByWeight.induct` auto)

---

### 4.8 Pair of Minimal Symbols

The `minima` predicate expresses that two symbols \( a, b \in \text{alphabet } t \) have the lowest frequencies in the tree \( t \). Minimal symbols need not be uniquely defined.

**Definition** `minima :: \( \alpha \) tree \Rightarrow \alpha \Rightarrow \alpha \Rightarrow \text{bool}`

\[\text{minima } t \ a \ b = \]
\[
(a \in \text{alphabet } t \land b \in \text{alphabet } t \land a \neq b \land (\forall c \in \text{alphabet } t. c \neq a \implies c \neq b \implies \text{freq } t \ c \geq \text{freq } t \ a \land \text{freq } t \ c \geq \text{freq } t \ b))
\]

---

### 5 Formalization of the Textbook Proof

#### 5.1 Four-Way Symbol Interchange Cost Lemma

If \( a \) and \( b \) are minima, and \( c \) and \( d \) are at the very bottom of the tree, then exchanging \( a \) and \( b \) with \( c \) and \( d \) does not increase the cost. Graphically, we have

\[
\text{cost} \leq \text{cost}
\]

This cost property is part of Knuth’s proof:

Let \( V \) be an internal node of maximum distance from the root. If \( w_1 \) and \( w_2 \) are not the weights already attached to the children of \( V \), we can interchange them with the values that are already there; such an interchange does not increase the weighted path length.

Lemma 16.2 in Cormen et al. [4, p. 389] expresses a similar property, which turns out to be a corollary of our cost property:
Let $C$ be an alphabet in which each character $c \in C$ has frequency $f[c]$. Let $x$ and $y$ be two characters in $C$ having the lowest frequencies. Then there exists an optimal prefix code for $C$ in which the codewords for $x$ and $y$ have the same length and differ only in the last bit.

**Lemma** `cost_swapFourSyms_le`:

**Assumes**
- \(\text{consistent } t\) \(\text{minima } t\ a\ b\ c \in \text{alphabet } t\ d \in \text{alphabet } t\)
- \(\text{depth } t\ c = \text{height } t\) \(\text{depth } t\ d = \text{height } t\) \(c \neq d\)

**Shows** \(\text{cost } (\text{swapFourSyms } t\ a\ b\ c\ d) \leq \text{cost } t\)

**Proof**

- **Note** `lens = swapFourSyms_def minima_def cost_swapSyms_le depth_le_height`
- **Show** `thesis`
- **Proof (cases \(a \neq d\) \(\land\) \(b \neq c\))**
  - **Case** `True` **show** `thesis`
    - **Proof** cases
      - **Assume** \(a = c\) **show** `thesis`
        - **Proof** cases
          - **Assume** \(b = d\) **thus** `thesis using` \(a = c\) `True assms`
            - **by** `(simp add: lems)`
          - **Next**
            - **Assume** \(b \neq d\) **thus** `thesis using` \(a = c\) `True assms`
              - **by** `(simp add: lems)`
            - **Qed**
          - **Next**
        - **Qed**
    - **Next**
      - **Assume** \(a \neq c\) **show** `thesis`
        - **Proof** cases
          - **Assume** \(b = d\) **thus** `thesis using` \(a \neq c\) `True assms`
            - **by** `(simp add: lems)`
          - **Next**
            - **Assume** \(b \neq d\)
              - **Have** `cost (swapFourSyms t a b c d) \leq cost (swapSyms t a c)`
                - **using** `(b \neq d) \(\land\) \(a \neq c\)` `True assms` **by** `(clarsimp simp: lems)`
              - **Also have** \(\ldots \leq \text{cost } t\) **using** `(b \neq d) \(\land\) \(a \neq c\)` `True assms`
                - **by** `(clarsimp simp: lems)`
            - **Finally show** `thesis`.
        - **Qed**
      - **Qed**
    - **Qed**
  - **Next**
    - **Case** `False` **thus** `thesis using` `assms` **by** `(clarsimp simp: lems)`
      - **Qed**
    - **Qed**

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5.2 Leaf Split Optimality Lemma

The tree \( \text{splitLeaf} t \ w_a \ a \ w_b \ b \) is optimum if \( t \) is optimum, under a few assumptions, notably that \( a \) and \( b \) are minima of the new tree and that \( \text{freq} \ t \ a = w_a + w_b \). Graphically:

![Diagram of leaf split](image)

This corresponds to the following fragment of Knuth’s proof:

Now it is easy to prove that the weighted path length of such a tree is minimized if and only if the tree with

\[
\begin{array}{c}
\text{optimum} \\
\end{array}
\begin{array}{c}
\text{optimum} \\
\end{array}
\begin{array}{c}
\text{replaced by} \\
w_1 + w_2
\end{array}
\]

has minimum path length for the weights \( w_1 + w_2, w_3, \ldots, w_m \).

We only need the “if” direction of Knuth’s equivalence. Lemma 16.3 in Cormen et al. [4, p. 391] expresses essentially the same property:

Let \( C \) be a given alphabet with frequency \( f[c] \) defined for each character \( c \in C \). Let \( x \) and \( y \) be two characters in \( C \) with minimum frequency. Let \( C' \) be the alphabet \( C \) with characters \( x, y \) removed and (new) character \( z \) added, so that \( C' = C \setminus \{x, y\} \cup \{z\} \); define \( f \) for \( C' \) as for \( C \), except that \( f[z] = f[x] + f[y] \). Let \( T' \) be any tree representing an optimal prefix code for the alphabet \( C' \). Then the tree \( T \), obtained from \( T' \) by replacing the leaf node for \( z \) with an internal node having \( x \) and \( y \) as children, represents an optimal prefix code for the alphabet \( C \).

The proof is as follows: We assume that \( t \) has a cost less than or equal to that of any other comparable tree \( v \) and show that \( \text{splitLeaf} t \ w_a \ a \ w_b \ b \) has a cost less than or equal to that of any other comparable tree \( u \). By \( \text{exists_at_height} \) and \( \text{depth_height_imp_sibling_ne} \), we know that some symbols \( c \) and \( d \) appear in
sibling nodes at the very bottom of $u$:

![Diagram of a tree with nodes labeled c, d, and question mark at the top.]

(The question mark is there to remind us that we know nothing specific about $u$’s structure.) From $u$ we construct a new tree $\text{swapFourSyms} \ u \ a \ b \ c \ d$ in which the minima $a$ and $b$ are siblings:

![Diagram of a tree with nodes labeled a, b, and question mark at the top.]

Merging $a$ and $b$ gives a tree comparable with $t$, which we can use to instantiate $v$ in the assumption:

![Diagram of a tree with node labeled a at the top.]

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With this instantiation, the proof is easy:

\[
\begin{align*}
\text{cost} \ (\text{splitLeaf } t \ a \ w_a \ b \ w_b) &= \text{(cost_splitLeaf)} \\
&= \text{cost} \ t + w_a + w_b \\
&\leq \underbrace{w}_v + \text{cost} \ (\text{mergeSibling} \ (\text{swapFourSyms } u \ a \ b \ c \ d) \ a) + w_a + w_b \\
&= \text{(cost_mergeSibling)} \\
&\leq \text{cost} \ (\text{swapFourSyms } u \ a \ b \ c \ d) \text{(cost_swapFourSyms_le)} \\
&\leq \text{cost} \ u.
\end{align*}
\]

In contrast, the proof in Cormen et al. is by contradiction: Essentially, they assume that there exists a tree \( u \) with a lower cost than \( \text{splitLeaf } t \ a \ w_a \ b \ w_b \) and show that there exists a tree \( v \) with a lower cost than \( t \), contradicting the hypothesis that \( t \) is optimum. In place of \( \text{cost_swapFourSyms_le} \), they invoke their lemma 16.2, which is questionable since \( u \) is not necessarily optimum.\(^3\)

Our proof relies on the following lemma, which asserts that \( a \) and \( b \) are minima of \( u \).

**lemma** \( \text{twice_freq_le_imp_minima} \):

\[
[\forall c \in \text{alphabet } t. \ w_a \leq \text{freq } t \ c \land w_b \leq \text{freq } t \ c; \\
\text{alphabet } u = \text{alphabet } t \cup \{ b \}; \ a \in \text{alphabet } u; \ a \neq b; \\
\text{freq } u = (\lambda c. \text{if } c = a \text{ then } w_a \text{ else if } c = b \text{ then } w_b \text{ else } \text{freq } t \ c)] \implies \text{minima } u \ a \ b
\]

**by** (simp add: minima_def)

Now comes the key lemma.

**lemma** \( \text{optimum_splitLeaf} \):

**assumes** consistent \( t \) optimum \( t \ a \in \text{alphabet } t \ b \not\in \text{alphabet } t \\
\text{freq } t \ a = w_a + \text{freq } t \ b \forall c \in \text{alphabet } t. \text{freq } t \ c \geq w_a \land \text{freq } t \ c \geq w_b
\]

**shows** optimum (splitLeaf \( t \ w_a \ a \ w_b \ b \))

**proof** (unfold optimum_def, clarify)

**fix** \( u \)

**let** \( t' = \text{splitLeaf } t \ w_a \ a \ w_b \)

**assume** \( c_u; \text{consistent } u \text{ and } a_u; \text{alphabet } t' = \text{alphabet } u \text{ and } \\
f_u; \text{freq } t' = \text{freq } u \)

**show** cost \( t' \leq \text{cost } u \)

**proof** (cases \( \text{height } t' = 0 \))

**case** True **thus** thesis by simp

**next**

**case** False

\(^3\)Thomas Cormen commented that this step will be clarified in the next edition of the book.
hence $h_u$: height $u > 0$ using $a_u$ assms 
  by (auto intro: height_gt_0_alphabet_eq_imp_height_gt_0)
have $a_u$: $a \in \text{alphabet } u$ using $a_u$ assms by fastforce
have $a_b$: $b \in \text{alphabet } u$ using $a_u$ assms by fastforce
have $ab$: $a \neq b$ using assms by blast
from exists_at_height[OF $c_u$]
obtain $c$ where $a_c$: $c \in \text{alphabet } u$ and $d_c$: depth $u c = \text{height } u$
let $d = \text{sibling } u c$

have $dc$: $d \neq c$ using $d_c c_u h_u a_c$ by (metis depth_height_imp_sibling_ne)
have $a_d$: $d \in \text{alphabet } u$ using $dc$
  by (rule sibling_ne_imp_sibling_in_alphabet)
have $ab$: $a \neq b$ using assms by blast
from exists_at_height[OF $c_u$]
obtain $c$ where $a_c$: $c \in \text{alphabet } u$ and $d_c$: depth $u c = \text{height } u$
let $d = \text{sibling } u c$

have $dc$: $d \neq c$ using $d_c c_u h_u a_c$ by (metis depth_height_imp_sibling_ne)
have $a_d$: $d \in \text{alphabet } u$ using $dc$
  by (rule sibling_ne_imp_sibling_in_alphabet)
have $ab$: $a \neq b$ using assms by blast

let $u' = \text{swapFourSyms } u a b c d$
have $c_u$: $\text{consistent } u'$ using $c_u$ by simp
have $a_u$: $\text{alphabet } u'$ = alphabet $u$ using $a_u a_b a_c a_d a_u$ by simp
have $f_u$: freq $u'$ = freq $u$ using $a_u a_b a_c a_d a_u f_u$ by simp
have $s_a$: sibling $u'a = b$ using $c_u a_u a_u a_b a_c b c d c_a$ using
  (rule sibling_swapFourSyms_when_4th_is_sibling)
have $c_v$: $\text{consistent } v$ using $c_v$ by simp
have $a_v$: $\text{alphabet } v$ = alphabet $t$ using $s_a c_u a_u a_u a_u$ assms by auto
have $f_v$: freq $v$ = freq $t$
  using $s_a c_u a_u f_u [\text{THEN sym}] ab a_u [\text{THEN sym}] assms$
  by (simp add: freq_mergeSibling ext)

have $cost t' = cost t + w_a + w_b$ using assms by simp
also have $\ldots \leq cost v + w_a + w_b$ using $c_v a_v f_v$ optimum $t$
  by (simp add: optimum_def)
also have $\ldots = cost u$
proof
  have $cost v + freq u'a + freq u'$ (sibling $u'a$) = cost $u'$
    using $c_u, s_a$ assms by (subst cost_mergeSibling) auto
  moreover have $w_a = freq u'a \ w_b = freq u'b$
    using $f_u f_u [\text{THEN sym}] assms by clarsimp$
  ultimately show thesis using $s_a$ by simp
qed
also have $\ldots \leq cost u$
proof
  have $\text{minima } u a b$ using $a_u f_u$ assms
    by (subst twice_freq_le_imp_minima) auto
  with $c_u$ show thesis using $a_u a_d d_c d_d$ by simp

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by \( \text{rule cost}\_\text{swapFourSyms}\_\text{le} \)

\[ \text{qed} \]

finally show thesis .

\[ \text{qed} \]

5.3 Leaf Split Commutativity Lemma

A key property of Huffman’s algorithm is that once it has combined two lowest-weight trees using \text{uniteTrees}, it does not visit these trees ever again. This suggests that splitting a leaf node before applying the algorithm should give the same result as applying the algorithm first and splitting the leaf node afterward. The diagram below illustrates the situation:

From the original forest (1), we can either run the algorithm (2a) and then split \( a \) (3a) or split \( a \) (2b) and then run the algorithm (3b). Our goal is to show that trees (3a) and (3b) are identical. Formally, we prove that

\[
\text{splitLeaf} (\text{huffman } ts) \ w_a \ a \ w_b = \text{huffman} (\text{splitLeaf}_F \ ts \ w_a \ a \ w_b)
\]

when \( ts \) is consistent, \( a \in \text{alphabet}_F \ ts \), \( b \notin \text{alphabet}_F \ ts \), and \( \text{freq}_F \ ts \ a = w_a + w_b \). But before we can prove this commutativity lemma, we need to introduce a few simple lemmas.
lemma cachedWeight_splitLeaf [simp]:
cachedWeight \( (\text{splitLeaf } t \ a \ w \ b) \) = cachedWeight \( t \)
by (case_tac \( t \)) simp +

lemma splitLeaf_F_insorTree_when_in_alphabet_left [simp]:
\( \begin{array}{l}
a \in \text{alphabet } t; \ \text{consistent } t; \ a ^{\not\in} \text{alphabet}_F ts; \ \text{freq}_F t a = w_a + w_b \end{array} \) \implies
splitLeaf_F (\text{insortTree } t ts) \ a \ a w \ b = \text{insortTree} (\text{splitLeaf } t w_a \ a \ w_b \ b) \ ts
by (induct ts) simp +

lemma splitLeaf_F_insorTree_when_in_alphabet_tail [simp]:
\( \begin{array}{l}
a \in \text{alphabet}_F ts; \ \text{consistent}_F ts; \ a ^{\not\in} \text{alphabet } t; \ \text{freq}_F ts a = w_a + w_b \end{array} \) \implies
\text{insortTree} t (\text{splitLeaf } F ts \ w_a \ a \ w_b \ b)
proof (induct ts)
  case Nil thus case by simp
next
case (Cons \( u \) \( us \)) show case
proof (cases \( a \in \text{alphabet } u \))
  case True
  hence \( a ^{\not\in} \text{alphabet}_F us \) using Cons by auto
  thus thesis using Cons by auto
next
case False thus thesis using Cons by simp
qed

We are now ready to prove the commutativity lemma.

lemma splitLeaf_huffman_commute:
\( \begin{array}{l}
\text{consistent}_F ts; ts \neq []; \ a \in \text{alphabet}_F ts; \ \text{freq}_F ts a = w_a + w_b \end{array} \) \implies
splitLeaf_F (huffman ts) \ a \ a w \ b = huffman (splitLeaf_F ts \ w_a \ a \ w_b \ b)
proof (induct \( ts \) rule: huffman.induct)
  case (1 \( t \)) thus case by simp
next
case (2 \( t_1 \) \( t_2 \) \( ts \))
note hyps = 2
show case
proof (cases \( a \in \text{alphabet } t_1 \))
  case True
  hence \( a ^{\not\in} \text{alphabet}_F t_2 \) \( a ^{\not\in} \text{alphabet}_F ts \) using hyps by auto
  thus thesis using hyps by (simp add: uniteTrees_def)
next
case False
note \( a_1 = False \)
show thesis
proof (cases $a \in \text{alphabet } t_2$)

- case True
  hence $a \notin \text{alphabet}_F \text{ ts using hyps by auto}$
  thus thesis using $a_1 \text{ hyps by } (\text{simp add: uniteTrees_def})$
  next
- case False
  thus thesis using $a_1 \text{ hyps by simp}$
  qed

next

- case 3 thus case by simp
  qed

An important consequence of the commutativity lemma is that applying Huffman’s algorithm on a forest of the form

![forest_diagram]

gives the same result as applying the algorithm on the “flat” forest

![flat_forest_diagram]

followed by splitting the leaf node $a$ into two nodes $a, b$ with frequencies $w_a, w_b$. The lemma effectively provides a way to flatten the forest at each step of the algorithm. Its invocation is implicit in the textbook proof.

5.4 Optimality Theorem

We are one lemma away from our main result.

lemma max_0_imp_0[simp]:
$(\operatorname{max} x y = (0::\text{nat})) = (x = 0 \land y = 0)$
by auto

theorem optimum_huffman:
$[\text{consistent}_F ts; \text{height}_F ts = 0; \text{sortedByWeight ts}; ts \neq [[]]] \implies \text{optimum} (\text{huffman ts})$

The input $ts$ is assumed to be a nonempty consistent list of leaf nodes sorted by weight. The proof is by induction on the length of the forest $ts$. Let $ts$ be

![ts_diagram]
with $w_a \leq w_b \leq w_c \leq w_d \leq \cdots \leq w_z$. If $ts$ consists of a single leaf node, the node has cost 0 and is therefore optimum. If $ts$ has length 2 or more, the first step of the algorithm leaves us with the term

$$\text{huffman} \begin{array}{c}
  c \\
  \frac{w_c}{w_a} \\
  a \\
  \frac{w_a}{w_b} \\
  b \\
  \frac{w_b}{w_z} \\
  d \\
  \frac{w_z}{w_d} \\
  \vdots \\
  z
\end{array}$$

In the diagram, we put the newly created tree at position 2 in the forest; in general, it could be anywhere. By $\text{splitLeaf}_\text{huffman}_\text{commute}$, the above tree equals

$$\text{splitLeaf} \left( \text{huffman} \begin{array}{c}
  c \\
  \frac{w_c}{w_a} \\
  a \\
  \frac{w_a}{w_b} \\
  b \\
  \frac{w_b}{w_z} \\
  d \\
  \frac{w_z}{w_d} \\
  \vdots \\
  z
\end{array} \right) w_a \cdot w_b \cdot w_d.$$ 

To prove that this tree is optimum, it suffices by $\text{optimum}_\text{splitLeaf}$ to show that

$$\text{huffman} \begin{array}{c}
  c \\
  \frac{w_c}{w_a} \\
  a \\
  \frac{w_a}{w_b} \\
  b \\
  \frac{w_b}{w_z} \\
  d \\
  \frac{w_z}{w_d} \\
  \vdots \\
  z
\end{array}$$

is optimum, which follows from the induction hypothesis.

**proof** (induct $ts$ rule: length_induct)

— COMPLETE INDUCTION STEP

**case** (1 $ts$)

**note** hyps = 1

**show** case

**proof** (cases $ts$)

  **case** Nil **thus** thesis **using** $ts \neq []$ **by** fast

**next**

  **case** (Cons $t_a \ ts'')$

  **note** $ts = \text{Cons}$

  **show** thesis

  **proof** (cases $ts''$)

    **case** Nil **thus** thesis **using** $ts$ hyps **by** (simp add: optimum_def)

**next**

  **case** (Cons $t_b \ ts'''$)

  **note** $ts'' = \text{Cons}$

  **show** thesis

  **proof** (cases $t_b$)

    **case** (Leaf $w_a \ a$)

    **note** $t_a = \text{Leaf}$

    **show** thesis

    **proof** (cases $t_b$)
So what have we achieved? Assuming that our definitions really mean what we intend them to mean, we established that our functional implementation of Huffman’s algorithm, when invoked properly, constructs a binary tree that represents an optimal prefix code for the specified alphabet and frequencies. Using Isabelle’s code generator [6], we can convert the Isabelle code into Standard ML, OCaml, or Haskell and use it in a real application.

As a side note, the optimum_huffman theorem assumes that the forest ts passed to huffman consists exclusively of leaf nodes. It is tempting to relax this restriction, by requiring instead that the forest ts has the lowest cost among forests of the
same size. We would define optimality of a forest as follows:

\[
optimum_F ts = (\forall us. \text{length } ts = \text{length } us \rightarrow \text{consistent}_F us \rightarrow \\
\text{alphabet}_F ts = \text{alphabet}_F us \rightarrow \text{freq}_F ts = \text{freq}_F us \rightarrow \\
\text{cost}_F ts \leq \text{cost}_F us)
\]

with \(\text{cost}_F [] = 0\) and \(\text{cost}_F (t \cdot ts) = \text{cost } t + \text{cost}_F ts\). However, the modified proposition does not hold. A counterexample is the optimum forest

\[
\begin{array}{c}
4 \\
5 \\
2 \quad 3 \quad 3
\end{array}
\]

for which the algorithm constructs the tree

\[
\begin{aligned}
14 & \quad 5 \\
5 & \quad 9 \\
2 \quad 3 \quad 4 & \quad 5 \\
2 \quad 3
\end{aligned}
\]

\[
\begin{aligned}
14 & \quad 8 \\
6 & \quad 3 \quad 4 & \quad 4 \\
3 \quad 3 \quad 4 & \quad 2 \quad 2
\end{aligned}
\]

of greater cost than

6 Related Work

Laurent Théry’s Coq formalization of Huffman’s algorithm [14, 15] is an obvious yardstick for our work. It has a somewhat wider scope, proving among others the isomorphism between prefix codes and full binary trees. With 291 theorems, it is also much larger.

Théry identified the following difficulties in formalizing the textbook proof:

1. The leaf interchange process that brings the two minimal symbols together is tedious to formalize.

2. The sibling merging process requires introducing a new symbol for the merged node, which complicates the formalization.

3. The algorithm constructs the tree in a bottom-up fashion. While top-down procedures can usually be proved by structural induction, bottom-up procedures often require more sophisticated induction principles and larger invariants.

4. The informal proof relies on the notion of depth of a node. Defining this notion formally is problematic, because the depth can only be seen as a function if the tree is composed of distinct nodes.
To circumvent these difficulties, Théry introduced the ingenious concept of cover. A forest $ts$ is a cover of a tree $t$ if $t$ can be built from $ts$ by adding inner nodes on top of the trees in $ts$. The term “cover” is easier to understand if the binary trees are drawn with the root at the bottom of the page, like natural trees. Huffman’s algorithm is a refinement of the cover concept. The main proof consists in showing that the cost of $huffman\ ts$ is less than or equal to that of any other tree for which $ts$ is a cover. It relies on a few auxiliary definitions, notably an “ordered cover” concept that facilitates structural induction and a four-argument depth predicate (confusingly called $height$). Permutations also play a central role.

Incidentally, our experience suggests that the potential problems identified by Théry can be overcome more directly without too much work, leading to a simpler proof:

1. Formalizing the leaf interchange did not prove overly tedious. Among our 95 lemmas and theorems, 24 concern $swapLeaves$, $swapSyms$, and $swapFourSyms$.

2. The generation of a new symbol for the resulting node when merging two sibling nodes in $mergeSibling$ was trivially solved by reusing one of the two merged symbols.

3. The bottom-up nature of the tree construction process was addressed by using the length of the forest as the induction measure and by merging the two minimal symbols, as in Knuth’s proof.

4. By restricting our attention to consistent trees, we were able to define the $depth$ function simply and meaningfully.

7 Conclusion

The goal of most formal proofs is to increase our confidence in a result. In the case of Huffman’s algorithm, however, the chances that a bug would have gone unnoticed for the 56 years since its publication, under the scrutiny of leading computer scientists, seem extremely low; and the existence of a Coq proof should be sufficient to remove any remaining doubts.

The main contribution of this document has been to demonstrate that the textbook proof of Huffman’s algorithm can be elegantly formalized using a state-of-the-art theorem prover such as Isabelle/HOL. In the process, we uncovered a few minor snags in the proof given in Cormen et al. [4].

We also found that custom induction rules, in combination with suitable simplification rules, greatly help the automatic proof tactics, sometimes reducing 30-line proof scripts to one-liners. We successfully applied this approach for handling both the ubiquitous “datatype + wellformedness predicate” combination ($a\ tree +\ consistent$) and functions defined by sequential pattern matching ($Sibling$,...)
and \texttt{mergeSibling}). Our experience suggests that such rules, which are uncommon in formalizations, are highly valuable and versatile. Moreover, Isabelle’s \texttt{induction\_schema} and \texttt{lexicographic\_order} tactics make these easy to prove.

Formalizing the proof of Huffman’s algorithm also led to a deeper understanding of this classic algorithm. Many of the lemmas, notably the leaf split commutativity lemma of Section 5.3, have not been found in the literature and express fundamental properties of the algorithm. Other discoveries did not find their way into the final proof. In particular, each step of the algorithm appears to preserve the invariant that the nodes in a forest are ordered by weight from left to right, bottom to top, as in the example below:

![Forest example](image)

It is not hard to prove formally that a tree exhibiting this property is optimum. On the other hand, proving that the algorithm preserves this invariant seems difficult—more difficult than formalizing the textbook proof—and remains a suggestion for future work.

A few other directions for future work suggest themselves. First, we could formalize some of our hypotheses, notably our restriction to full and consistent binary trees. The current formalization says nothing about the algorithm’s application for data compression, so the next step could be to extend the proof’s scope to cover \texttt{encode/decode} functions and show that full binary trees are isomorphic to prefix codes, as done in the Coq development. Independently, we could generalize the development to \textit{n}-ary trees.

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References


