

# An Isabelle/HOL Formalization of the Textbook Proof of Huffman’s Algorithm\*

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## Abstract

Huffman’s algorithm is a procedure for constructing a binary tree with minimum weighted path length. This report presents a formal proof of the correctness of Huffman’s algorithm written using Isabelle/HOL. Our proof closely follows the sketches found in standard algorithms textbooks, uncovering a few snags in the process. Another distinguishing feature of our formalization is the use of custom induction rules to help Isabelle’s automatic tactics, leading to very short proofs for most of the lemmas.

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# 1 Introduction

## 1.1 Binary Codes

Suppose we want to encode strings over a finite source alphabet to sequences of bits. The approach used by ASCII and most other charsets is to map each source symbol to a distinct  $k$ -bit code word, where  $k$  is fixed and is typically 8 or 16. To encode a string of symbols, we simply encode each symbol in turn. Decoding involves mapping each  $k$ -bit block back to the symbol it represents.

Fixed-length codes are simple and fast, but they generally waste space. If we know the frequency  $w_a$  of each source symbol  $a$ , we can save space by using shorter code words for the most frequent symbols. We say that a (variable-length) code is *optimum* if it minimizes the sum  $\sum_a w_a \delta_a$ , where  $\delta_a$  is the length of the binary code word for  $a$ . Information theory tells us that a code is optimum if for each source symbol  $c$  the code word representing  $c$  has length

$$\delta_c = \log_2 \frac{1}{p_c}, \quad \text{where } p_c = \frac{w_c}{\sum_a w_a}.$$

This number is generally not an integer, so we cannot use it directly. Nonetheless, the above criterion is a useful yardstick and paves the way for arithmetic coding [13], a generalization of the method presented here.

As an example, consider the source string 'abacabad'. We have

$$p_a = \frac{1}{2}, \quad p_b = \frac{1}{4}, \quad p_c = \frac{1}{8}, \quad p_d = \frac{1}{8}.$$

The optimum lengths for the binary code words are all integers, namely

$$\delta_a = 1, \quad \delta_b = 2, \quad \delta_c = 3, \quad \delta_d = 3,$$

and they are realized by the code

$$C_1 = \{a \mapsto 0, b \mapsto 10, c \mapsto 110, d \mapsto 111\}.$$

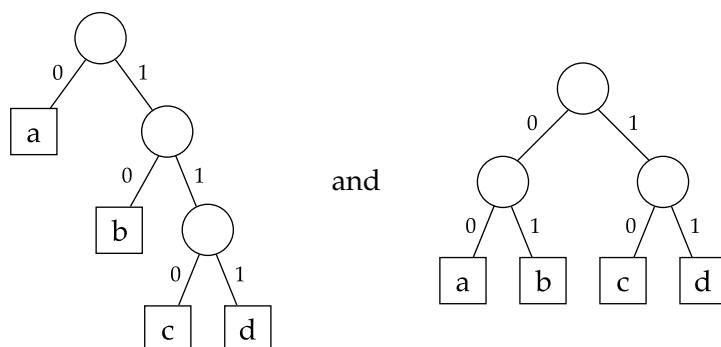
Encoding 'abacabad' produces the 14-bit code word 01001100100111. The code  $C_1$  is optimum: No code that unambiguously encodes source symbols one at a time could do better than  $C_1$  on the input 'abacabad'. In particular, with a fixed-length code such as

$$C_2 = \{a \mapsto 00, b \mapsto 01, c \mapsto 10, d \mapsto 11\}$$

we need at least 16 bits to encode 'abacabad'.

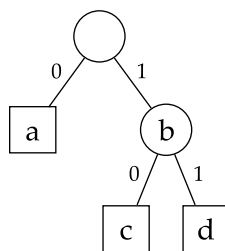
## 1.2 Binary Trees

Inside a program, binary codes can be represented by binary trees. For example, the trees



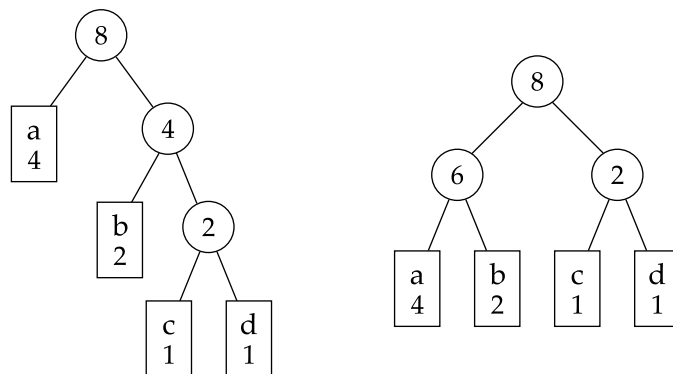
correspond to  $C_1$  and  $C_2$ . The code word for a given symbol can be obtained as follows: Start at the root and descend toward the leaf node associated with the symbol one node at a time; generate a 0 whenever the left child of the current node is chosen and a 1 whenever the right child is chosen. The generated sequence of 0s and 1s is the code word.

To avoid ambiguities, we require that only leaf nodes are labeled with symbols. This ensures that no code word is a prefix of another, thereby eliminating the source of all ambiguities.<sup>1</sup> Codes that have this property are called *prefix codes*. As an example of a code that does not have this property, consider the code associated with the tree



and observe that 'bbb', 'bd', and 'db' all map to the code word 111.

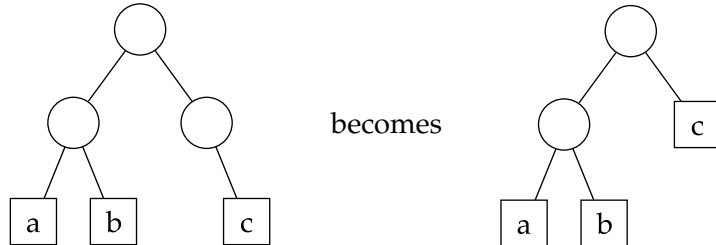
Each node in a code tree is assigned a *weight*. For a leaf node, the weight is the frequency of its symbol; for an inner node, it is the sum of the weights of its subtrees. Code trees can be annotated with their weights:



For our purposes, it is sufficient to consider only full binary trees (trees whose inner nodes all have two children). This is because any inner node with only one

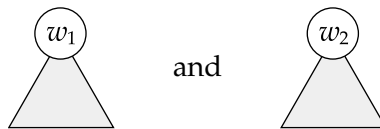
<sup>1</sup>Strictly speaking, there is another potential source of ambiguity. If the alphabet consists of a single symbol  $a$ , that symbol could be mapped to the empty code word, and then any string  $aa \dots a$  would map to the empty bit sequence, giving the decoder no way to recover the original string's length. This scenario can be ruled out by requiring that the alphabet has cardinality 2 or more.

child can advantageously be eliminated; for example,

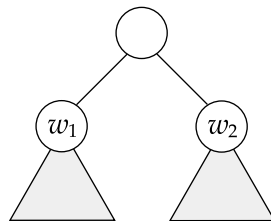


### 1.3 Huffman's Algorithm

David Huffman [7] discovered a simple algorithm for constructing an optimum code tree for specified symbol frequencies: Create a forest consisting of only leaf nodes, one for each symbol in the alphabet, taking the given symbol frequencies as initial weights for the nodes. Then pick the two trees



with the lowest weights and replace them with the tree

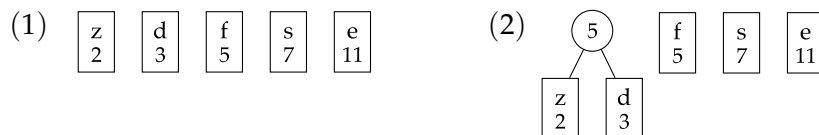


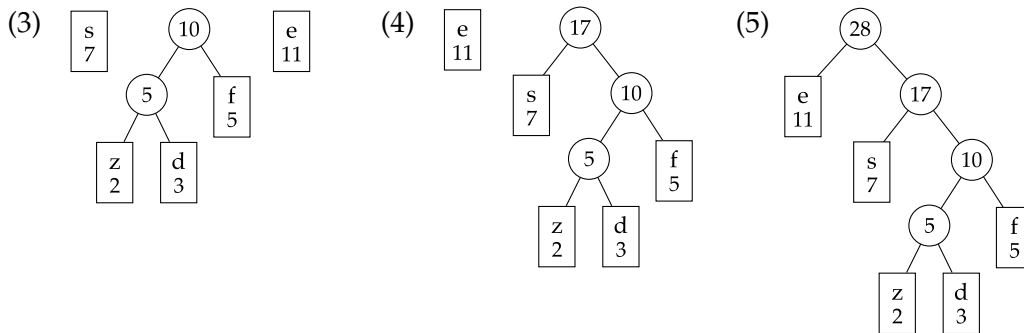
Repeat this process until only one tree is left.

As an illustration, executing the algorithm for the frequencies

$$f_d = 3, f_e = 11, f_f = 5, f_s = 7, f_z = 2$$

gives rise to the following sequence of states:



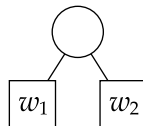


Tree (5) is an optimum tree for the given frequencies.

### 1.4 The Textbook Proof

Why does the algorithm work? In his article, Huffman gave some motivation but no real proof. For a proof sketch, we turn to Donald Knuth [8, p. 403–404]:

It is not hard to prove that this method does in fact minimize the weighted path length [i.e.,  $\sum_a w_a \delta_a$ ], by induction on  $m$ . Suppose we have  $w_1 \leq w_2 \leq w_3 \leq \dots \leq w_m$ , where  $m \geq 2$ , and suppose that we are given a tree that minimizes the weighted path length. (Such a tree certainly exists, since only finitely many binary trees with  $m$  terminal nodes are possible.) Let  $V$  be an internal node of maximum distance from the root. If  $w_1$  and  $w_2$  are not the weights already attached to the children of  $V$ , we can interchange them with the values that are already there; such an interchange does not increase the weighted path length. Thus there is a tree that minimizes the weighted path length and contains the subtree



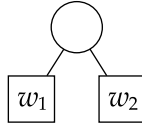
Now it is easy to prove that the weighted path length of such a tree is minimized if and only if the tree with



has minimum path length for the weights  $w_1 + w_2, w_3, \dots, w_m$ .

There is, however, a small oddity in this proof: It is not clear why we must assert

the existence of an optimum tree that contains the subtree



Indeed, the formalization works without it.

Cormen et al. [4, p. 385–391] provide a very similar proof, articulated around the following propositions:

**Lemma 16.2**

Let  $C$  be an alphabet in which each character  $c \in C$  has frequency  $f[c]$ . Let  $x$  and  $y$  be two characters in  $C$  having the lowest frequencies. Then there exists an optimal prefix code for  $C$  in which the codewords for  $x$  and  $y$  have the same length and differ only in the last bit.

**Lemma 16.3**

Let  $C$  be a given alphabet with frequency  $f[c]$  defined for each character  $c \in C$ . Let  $x$  and  $y$  be two characters in  $C$  with minimum frequency. Let  $C'$  be the alphabet  $C$  with characters  $x, y$  removed and (new) character  $z$  added, so that  $C' = C - \{x, y\} \cup \{z\}$ ; define  $f$  for  $C'$  as for  $C$ , except that  $f[z] = f[x] + f[y]$ . Let  $T'$  be any tree representing an optimal prefix code for the alphabet  $C'$ . Then the tree  $T$ , obtained from  $T'$  by replacing the leaf node for  $z$  with an internal node having  $x$  and  $y$  as children, represents an optimal prefix code for the alphabet  $C$ .

**Theorem 16.4**

Procedure HUFFMAN produces an optimal prefix code.

## 1.5 Overview of the Formalization

This document presents a formalization of the proof of Huffman’s algorithm written using Isabelle/HOL [12]. Our proof is based on the informal proofs given by Knuth and Cormen et al. The development was done independently of Laurent Théry’s Coq proof [14, 15], which through its “cover” concept represents a considerable departure from the textbook proof.

The development consists of 90 lemmas and 5 theorems. Most of them have very short proofs thanks to the extensive use of simplification rules and custom induction rules. The remaining proofs are written using the structured proof format Isar [16] and are accompanied by informal arguments and diagrams.

The document is organized as follows. Section 2 defines the datatypes for binary code trees and forests and develops a small library of related functions. (Incidentally, there is nothing special about binary codes and binary trees. Huffman’s algorithm and its proof can be generalized to  $n$ -ary trees [8, p. 405 and

595].) Section 3 presents a functional implementation of the algorithm. Section 4 defines several tree manipulation functions needed for the proof. Section 5 presents three key lemmas and concludes with the optimality theorem. Section 6 compares our work with Théry’s Coq proof. Finally, Section 7 concludes the document.

## 1.6 Overview of Isabelle’s HOL Logic

This section presents a brief overview of the Isabelle/HOL logic, so that readers not familiar with the system can at least understand the lemmas and theorems, if not the proofs. Readers who already know Isabelle are encouraged to skip this section.

Isabelle is a generic theorem prover whose built-in metalogic is an intuitionistic fragment of higher-order logic [5, 12]. The metalogical operators are material implication, written  $[[\varphi_1; \dots; \varphi_n]] \implies \psi$  (“if  $\varphi_1$  and  $\dots$  and  $\varphi_n$ , then  $\psi$ ”), universal quantification, written  $\bigwedge x_1 \dots x_n. \psi$  (“for all  $x_1, \dots, x_n$  we have  $\psi$ ”), and equality, written  $t \equiv u$ .

The incarnation of Isabelle that we use in this development, Isabelle/HOL, provides a more elaborate version of higher-order logic, complete with the familiar connectives and quantifiers ( $\neg$ ,  $\wedge$ ,  $\vee$ ,  $\longrightarrow$ ,  $\forall$ , and  $\exists$ ) on terms of type *bool*. In addition,  $=$  expresses equivalence. The formulas  $\bigwedge x_1 \dots x_m. [[\varphi_1; \dots; \varphi_n]] \implies \psi$  and  $\forall x_1. \dots \forall x_m. \varphi_1 \wedge \dots \wedge \varphi_n \longrightarrow \psi$  are logically equivalent, but they interact differently with Isabelle’s proof tactics.

The term language consists of simply typed  $\lambda$ -terms written in an ML-like syntax [11]. Function application expects no parentheses around the argument list and no commas between the arguments, as in  $f x y$ . Syntactic sugar provides an infix syntax for common operators, such as  $x = y$  and  $x + y$ . Types are inferred automatically in most cases, but they can always be supplied using an annotation  $t::\tau$ , where  $t$  is a term and  $\tau$  is its type. The type of total functions from  $\alpha$  to  $\beta$  is written  $\alpha \Rightarrow \beta$ . Variables may range over functions.

The type of natural numbers is called *nat*. The type of lists over type  $\alpha$ , written  $\alpha$  *list*, features the empty list  $[]$ , the infix constructor  $x \cdot xs$  (where  $x$  is an element of type  $\alpha$  and  $xs$  is a list over  $\alpha$ ), and the conversion function *set* from lists to sets. The type of sets over  $\alpha$  is written  $\alpha$  *set*. Operations on sets are written using traditional mathematical notation.

## 1.7 Head of the Theory File

The Isabelle theory starts in the standard way.

```
theory Huffman
imports Main
begin
```

We attach the *simp* attribute to some predefined lemmas to add them to the de-



fault set of simplification rules.

```
declare Int_Un_distrib [simp]  
         Int_Un_distrib2 [simp]  
         max.absorb1 [simp]  
         max.absorb2 [simp]
```

## 2 Definition of Prefix Code Trees and Forests

### 2.1 Tree Datatype

A *prefix code tree* is a full binary tree in which leaf nodes are of the form *Leaf w a*, where *a* is a symbol and *w* is the frequency associated with *a*, and inner nodes are of the form *InnerNode w t<sub>1</sub> t<sub>2</sub>*, where *t<sub>1</sub>* and *t<sub>2</sub>* are the left and right subtrees and *w* caches the sum of the weights of *t<sub>1</sub>* and *t<sub>2</sub>*. Prefix code trees are polymorphic on the symbol datatype  $\alpha$ .

```
datatype  $\alpha$  tree =  
  Leaf nat  $\alpha$   
  InnerNode nat ( $\alpha$  tree) ( $\alpha$  tree)
```

### 2.2 Forest Datatype

The intermediate steps of Huffman's algorithm involve a list of prefix code trees, or *prefix code forest*.

```
type_synonym  $\alpha$  forest =  $\alpha$  tree list
```

### 2.3 Alphabet

The *alphabet* of a code tree is the set of symbols appearing in the tree's leaf nodes.

```
primrec alphabet ::  $\alpha$  tree  $\Rightarrow$   $\alpha$  set where  
alphabet (Leaf w a) = {a}  
alphabet (InnerNode w t1 t2) = alphabet t1  $\cup$  alphabet t2
```

For sets and predicates, Isabelle gives us the choice between inductive definitions (**inductive\_set** and **inductive**) and recursive functions (**primrec**, **fun**, and **function**). In this development, we consistently favor recursion over induction, for two reasons:

- Recursion gives rise to simplification rules that greatly help automatic proof tactics. In contrast, reasoning about inductively defined sets and predicates involves introduction and elimination rules, which are more clumsy than simplification rules.

- Isabelle’s counterexample generator **quickcheck** [2], which we used extensively during the top-down development of the proof (together with **sorry**), has better support for recursive definitions.

The alphabet of a forest is defined as the union of the alphabets of the trees that compose it. Although Isabelle supports overloading for non-overlapping types, we avoid many type inference problems by attaching an ‘*F*’ subscript to the forest generalizations of functions defined on trees.

**primrec**  $alphabet_F :: \alpha \text{ forest} \Rightarrow \alpha \text{ set}$  **where**  
 $alphabet_F [] = \emptyset$   
 $alphabet_F (t \cdot ts) = alphabet\ t \cup alphabet_F\ ts$

Alphabets are central to our proofs, and we need the following basic facts about them.

**lemma** *finite\_alphabet* [simp]:  
 $finite\ (alphabet\ t)$   
**by** (induct *t*) auto

**lemma** *exists\_in\_alphabet*:  
 $\exists a. a \in alphabet\ t$   
**by** (induct *t*) auto

## 2.4 Consistency

A tree is *consistent* if for each inner node the alphabets of the two subtrees are disjoint. Intuitively, this means that every symbol in the alphabet occurs in exactly one leaf node. Consistency is a sufficient condition for  $\delta_a$  (the length of the *unique* code word for *a*) to be defined. Although this wellformedness property is not mentioned in algorithms textbooks [1, 4, 8], it is essential and appears as an assumption in many of our lemmas.

**primrec**  $consistent :: \alpha \text{ tree} \Rightarrow bool$  **where**  
 $consistent\ (Leaf\ w\ a) = True$   
 $consistent\ (InnerNode\ w\ t_1\ t_2) =$   
 $(consistent\ t_1 \wedge consistent\ t_2 \wedge alphabet\ t_1 \cap alphabet\ t_2 = \emptyset)$

**primrec**  $consistent_F :: \alpha \text{ forest} \Rightarrow bool$  **where**  
 $consistent_F\ [] = True$   
 $consistent_F\ (t \cdot ts) =$   
 $(consistent\ t \wedge consistent_F\ ts \wedge alphabet\ t \cap alphabet_F\ ts = \emptyset)$

Several of our proofs are by structural induction on consistent trees *t* and involve one symbol *a*. These proofs typically distinguish the following cases.

BASE CASE:  $t = Leaf\ w\ b$ .

INDUCTION STEP:  $t = \text{InnerNode } w \ t_1 \ t_2$ .

SUBCASE 1:  $a$  belongs to  $t_1$  but not to  $t_2$ .

SUBCASE 2:  $a$  belongs to  $t_2$  but not to  $t_1$ .

SUBCASE 3:  $a$  belongs to neither  $t_1$  nor  $t_2$ .

Thanks to the consistency assumption, we can rule out the subcase where  $a$  belongs to both subtrees.

Instead of performing the above case distinction manually, we encode it in a custom induction rule. This saves us from writing repetitive proof scripts and helps Isabelle's automatic proof tactics.

**lemma** *tree\_induct\_consistent* [*consumes* 1, *case\_names* *base* *step1* *step2* *step3*]:

```

[[consistent t;
   $\wedge w_b \ b \ a. \ P \ (\text{Leaf } w_b \ b) \ a;$ 
   $\wedge w \ t_1 \ t_2 \ a.$ 
    [[consistent t1; consistent t2; alphabet t1  $\cap$  alphabet t2 =  $\emptyset$ ;
       $a \in \text{alphabet } t_1; a \notin \text{alphabet } t_2; P \ t_1 \ a; P \ t_2 \ a]] \implies$ 
       $P \ (\text{InnerNode } w \ t_1 \ t_2) \ a;$ 
     $\wedge w \ t_1 \ t_2 \ a.$ 
      [[consistent t1; consistent t2; alphabet t1  $\cap$  alphabet t2 =  $\emptyset$ ;
         $a \notin \text{alphabet } t_1; a \in \text{alphabet } t_2; P \ t_1 \ a; P \ t_2 \ a]] \implies$ 
         $P \ (\text{InnerNode } w \ t_1 \ t_2) \ a;$ 
       $\wedge w \ t_1 \ t_2 \ a.$ 
        [[consistent t1; consistent t2; alphabet t1  $\cap$  alphabet t2 =  $\emptyset$ ;
           $a \notin \text{alphabet } t_1; a \notin \text{alphabet } t_2; P \ t_1 \ a; P \ t_2 \ a]] \implies$ 
           $P \ (\text{InnerNode } w \ t_1 \ t_2) \ a]] \implies$ 
   $P \ t \ a$ 

```

The proof relies on the *induction\_schema* and *lexicographic\_order* tactics, which automate the most tedious aspects of deriving induction rules. The alternative would have been to perform a standard structural induction on  $t$  and proceed by cases, which is straightforward but long-winded.

```

apply rotate_tac
apply induction_schema
  apply atomize_elim
  apply (case_tac t)
  apply fastforce
  apply fastforce
by lexicographic_order

```

The *induction\_schema* tactic reduces the putative induction rule to simpler proof obligations. Internally, it reuses the machinery that constructs the default induction rules. The resulting proof obligations concern (a) case completeness,

(b) invariant preservation (in our case, tree consistency), and (c) wellfoundedness. For *tree\_induct\_consistent*, the obligations (a) and (b) can be discharged using Isabelle’s simplifier and classical reasoner, whereas (c) requires a single invocation of *lexicographic\_order*, a tactic that was originally designed to prove termination of recursive functions [3, 9, 10].

## 2.5 Symbol Depths

The *depth* of a symbol (which we denoted by  $\delta_a$  in Section 1.1) is the length of the path from the root to the leaf node labeled with that symbol, or equivalently the length of the code word for the symbol. Symbols that do not occur in the tree or that occur at the root of a one-node tree have depth 0. If a symbol occurs in several leaf nodes (which may happen with inconsistent trees), the depth is arbitrarily defined in terms of the leftmost node labeled with that symbol.

```
primrec depth ::  $\alpha$  tree  $\Rightarrow$   $\alpha$   $\Rightarrow$  nat where
depth (Leaf w b) a = 0
depth (InnerNode w t1 t2) a =
  (if a  $\in$  alphabet t1 then depth t1 a + 1
   else if a  $\in$  alphabet t2 then depth t2 a + 1
   else 0)
```

The definition may seem very inefficient from a functional programming point of view, but it does not matter, because unlike Huffman’s algorithm, the *depth* function is merely a reasoning tool and is never actually executed.

## 2.6 Height

The *height* of a tree is the length of the longest path from the root to a leaf node, or equivalently the length of the longest code word. This is readily generalized to forests by taking the maximum of the trees’ heights. Note that a tree has height 0 if and only if it is a leaf node, and that a forest has height 0 if and only if all its trees are leaf nodes.

```
primrec height ::  $\alpha$  tree  $\Rightarrow$  nat where
height (Leaf w a) = 0
height (InnerNode w t1 t2) = max (height t1) (height t2) + 1
```

```
primrec heightF ::  $\alpha$  forest  $\Rightarrow$  nat where
heightF [] = 0
heightF (t · ts) = max (height t) (heightF ts)
```

The depth of any symbol in the tree is bounded by the tree’s height, and there exists a symbol with a depth equal to the height.

**lemma** *depth\_le\_height*:

*depth t a ≤ height t*

**by** (*induct t*) *auto*

**lemma** *exists\_at\_height*:

*consistent t ⇒ ∃a ∈ alphabet t. depth t a = height t*

**proof** (*induct t*)

**case** *Leaf* **thus** **case** **by** *simp*

**next**

**case** (*InnerNode w t<sub>1</sub> t<sub>2</sub>*)

**note** *hyps = InnerNode*

**let** *t = InnerNode w t<sub>1</sub> t<sub>2</sub>*

**from** *hyps* **obtain** *b* **where** *b: b ∈ alphabet t<sub>1</sub> depth t<sub>1</sub> b = height t<sub>1</sub>* **by** *auto*

**from** *hyps* **obtain** *c* **where** *c: c ∈ alphabet t<sub>2</sub> depth t<sub>2</sub> c = height t<sub>2</sub>* **by** *auto*

**let** *a = if height t<sub>1</sub> ≥ height t<sub>2</sub> then b else c*

**from** *b c* **have** *a ∈ alphabet t depth t a = height t*

**using** (*consistent t*) **by** *auto*

**thus** *∃a ∈ alphabet t. depth t a = height t ..*

**qed**

The following elimination rules help Isabelle's classical prover, notably the *auto* tactic. They are easy consequences of the inequation *depth t a ≤ height t*.

**lemma** *depth\_max\_heightE\_left* [*elim!*]:

$\llbracket \text{depth } t_1 \ a = \max (\text{height } t_1) (\text{height } t_2);$

$\llbracket \text{depth } t_1 \ a = \text{height } t_1; \text{height } t_1 \geq \text{height } t_2 \rrbracket \implies P \rrbracket \implies P$

**by** (*cut\_tac t = t<sub>1</sub> and a = a in depth\_le\_height*) *simp*

**lemma** *depth\_max\_heightE\_right* [*elim!*]:

$\llbracket \text{depth } t_2 \ a = \max (\text{height } t_1) (\text{height } t_2);$

$\llbracket \text{depth } t_2 \ a = \text{height } t_2; \text{height } t_2 \geq \text{height } t_1 \rrbracket \implies P \rrbracket \implies P$

**by** (*cut\_tac t = t<sub>2</sub> and a = a in depth\_le\_height*) *simp*

We also need the following lemma.

**lemma** *height\_gt\_0\_alphabet\_eq\_imp\_height\_gt\_0*:

**assumes** *height t > 0 consistent t alphabet t = alphabet u*

**shows** *height u > 0*

**proof** (*cases t*)

**case** *Leaf* **thus** **thesis** **using** *assms* **by** *simp*

**next**

**case** (*InnerNode w t<sub>1</sub> t<sub>2</sub>*)

**note** *t = InnerNode*

```

from exists_in_alphabet obtain b where b: b ∈ alphabet t1 ..
from exists_in_alphabet obtain c where c: c ∈ alphabet t2 ..
from b c have bc: b ≠ c using t (consistent t) by fastforce
show thesis
proof (cases u)
  case Leaf thus thesis using b c bc t assms by auto
next
  case InnerNode thus thesis by simp
qed
qed

```

## 2.7 Symbol Frequencies

The *frequency* of a symbol (which we denoted by  $w_a$  in Section 1.1) is the sum of the weights attached to the leaf nodes labeled with that symbol. If the tree is consistent, the sum comprises at most one nonzero term. The frequency is then the weight of the leaf node labeled with the symbol, or 0 if there is no such node. The generalization to forests is straightforward.

```

primrec freq ::  $\alpha$  tree  $\Rightarrow$   $\alpha$   $\Rightarrow$  nat where
freq (Leaf w a) = ( $\lambda b$ . if b = a then w else 0)
freq (InnerNode w t1 t2) = ( $\lambda b$ . freq t1 b + freq t2 b)

```

```

primrec freqF ::  $\alpha$  forest  $\Rightarrow$   $\alpha$   $\Rightarrow$  nat where
freqF [] = ( $\lambda b$ . 0)
freqF (t · ts) = ( $\lambda b$ . freq t b + freqF ts b)

```

Alphabet and symbol frequencies are intimately related. Simplification rules ensure that sums of the form  $\text{freq } t_1 a + \text{freq } t_2 a$  collapse to a single term when we know which tree  $a$  belongs to.

```

lemma notin_alphabet_imp_freq_0 [simp]:
a ∉ alphabet t  $\implies$  freq t a = 0
by (induct t) simp+

```

```

lemma notin_alphabetF_imp_freqF_0 [simp]:
a ∉ alphabetF ts  $\implies$  freqF ts a = 0
by (induct ts) simp+

```

```

lemma freq_0_right [simp]:
[[alphabet t1 ∩ alphabet t2 = ∅; a ∈ alphabet t1]]  $\implies$  freq t2 a = 0
by (auto intro: notin_alphabet_imp_freq_0 simp: disjoint_iff_not_equal)

```

```

lemma freq_0_left [simp]:
[[alphabet t1 ∩ alphabet t2 = ∅; a ∈ alphabet t2]]  $\implies$  freq t1 a = 0
by (auto simp: disjoint_iff_not_equal)

```

Two trees are *comparable* if they have the same alphabet and symbol frequencies. This is an important concept, because it allows us to state not only that the tree constructed by Huffman's algorithm is optimal but also that it has the expected alphabet and frequencies.

We close this section with a more technical lemma.

**lemma** *height<sub>F</sub>\_0\_imp\_Leaf\_freq<sub>F</sub>\_in\_set*:  
 $\llbracket \text{consistent}_F \text{ ts}; \text{height}_F \text{ ts} = 0; a \in \text{alphabet}_F \text{ ts} \rrbracket \implies$   
*Leaf* (*freq<sub>F</sub> ts a*)  $a \in \text{set ts}$   
**proof** (*induct ts*)  
  **case Nil thus case by simp**  
**next**  
  **case** (*Cons t ts*) **show case using Cons**  
  **proof** (*cases t*)  
    **case Leaf thus thesis using Cons by clarsimp**  
  **next**  
    **case InnerNode thus thesis using Cons by clarsimp**  
  **qed**  
**qed**

## 2.8 Weight

The *weight* function returns the weight of a tree. In the *InnerNode* case, we ignore the weight cached in the node and instead compute the tree's weight recursively. This makes reasoning simpler because we can then avoid specifying cache correctness as an assumption in our lemmas.

**primrec** *weight* ::  $\alpha \text{ tree} \Rightarrow \text{nat}$  **where**  
*weight* (*Leaf w a*) = *w*  
*weight* (*InnerNode w t<sub>1</sub> t<sub>2</sub>*) = *weight t<sub>1</sub>* + *weight t<sub>2</sub>*

The weight of a tree is the sum of the frequencies of its symbols.

**lemma** *weight\_eq\_Sum\_freq*:  
 $\text{consistent } t \implies \text{weight } t = \sum_{a \in \text{alphabet } t} \text{freq } t a$   
**by** (*induct t*) (*auto simp: sum.union\_disjoint*)

The assumption *consistent t* is not necessary, but it simplifies the proof by letting us invoke the lemma *sum.union\_disjoint*:

$$\llbracket \text{finite } A; \text{finite } B; A \cap B = \emptyset \rrbracket \implies \sum_{x \in A} g x + \sum_{x \in B} g x = \sum_{x \in A \cup B} g x.$$

## 2.9 Cost

The cost of a consistent tree, sometimes called the *weighted path length*, is given by the sum  $\sum_{a \in \text{alphabet } t} \text{freq } t a \times \text{depth } t a$  (which we denoted by  $\sum_a w_a \delta_a$  in Section 1.1). It obeys a simple recursive law.

**primrec**  $\text{cost} :: \alpha \text{ tree} \Rightarrow \text{nat}$  **where**

$\text{cost} (\text{Leaf } w a) = 0$

$\text{cost} (\text{InnerNode } w t_1 t_2) = \text{weight } t_1 + \text{cost } t_1 + \text{weight } t_2 + \text{cost } t_2$

One interpretation of this recursive law is that the cost of a tree is the sum of the weights of its inner nodes [8, p. 405]. (Recall that  $\text{weight} (\text{InnerNode } w t_1 t_2) = \text{weight } t_1 + \text{weight } t_2$ .) Since the cost of a tree is such a fundamental concept, it seems necessary to prove that the above function definition is correct.

**theorem**  $\text{cost\_eq\_Sum\_freq\_mult\_depth}$ :

$\text{consistent } t \implies \text{cost } t = \sum_{a \in \text{alphabet } t} \text{freq } t a \times \text{depth } t a$

The proof is by structural induction on  $t$ . If  $t = \text{Leaf } w b$ , both sides of the equation simplify to 0. This leaves the case  $t = \text{InnerNode } w t_1 t_2$ . Let  $A$ ,  $A_1$ , and  $A_2$  stand for  $\text{alphabet } t$ ,  $\text{alphabet } t_1$ , and  $\text{alphabet } t_2$ , respectively. We have

$$\begin{aligned}
& \text{cost } t && \text{(definition of cost)} \\
= & \text{weight } t_1 + \text{cost } t_1 + \text{weight } t_2 + \text{cost } t_2 && \text{(induction hypothesis)} \\
= & \text{weight } t_1 + \sum_{a \in A_1} \text{freq } t_1 a \times \text{depth } t_1 a + && \\
& \text{weight } t_2 + \sum_{a \in A_2} \text{freq } t_2 a \times \text{depth } t_2 a && \text{(definition of depth, consistency)} \\
= & \text{weight } t_1 + \sum_{a \in A_1} \text{freq } t_1 a \times (\text{depth } t a - 1) + && \\
& \text{weight } t_2 + \sum_{a \in A_2} \text{freq } t_2 a \times (\text{depth } t a - 1) && \text{(distributivity of } \times \text{ and } \sum \text{ over } -) \\
= & \text{weight } t_1 + \sum_{a \in A_1} \text{freq } t_1 a \times \text{depth } t a - \sum_{a \in A_1} \text{freq } t_1 a + && \\
& \text{weight } t_2 + \sum_{a \in A_2} \text{freq } t_2 a \times \text{depth } t a - \sum_{a \in A_2} \text{freq } t_2 a && \text{(weight\_eq\_Sum\_freq)} \\
= & \sum_{a \in A_1} \text{freq } t_1 a \times \text{depth } t a + \sum_{a \in A_2} \text{freq } t_2 a \times \text{depth } t a && \text{(definition of freq, consistency)} \\
= & \sum_{a \in A_1} \text{freq } t a \times \text{depth } t a + \sum_{a \in A_2} \text{freq } t a \times \text{depth } t a && \text{(sum.union\_disjoint, consistency)} \\
= & \sum_{a \in A_1 \cup A_2} \text{freq } t a \times \text{depth } t a && \text{(definition of alphabet)} \\
= & \sum_{a \in A} \text{freq } t a \times \text{depth } t a.
\end{aligned}$$

The structured proof closely follows this argument.



**proof** (*induct t*)  
**case** *Leaf* **thus** *case by simp*  
**next**  
**case** (*InnerNode w t<sub>1</sub> t<sub>2</sub>*)  
**let**  $t = \text{InnerNode } w \ t_1 \ t_2$   
**let**  $A = \text{alphabet } t$  **and**  $A_1 = \text{alphabet } t_1$  **and**  $A_2 = \text{alphabet } t_2$   
**note**  $c = \langle \text{consistent } t \rangle$   
**note**  $\text{hyps} = \text{InnerNode}$   
**have**  $d_2: \bigwedge a. [[A_1 \cap A_2 = \emptyset; a \in A_2]] \implies \text{depth } t \ a = \text{depth } t_2 \ a + 1$   
**by** *auto*  
**have**  $\text{cost } t = \text{weight } t_1 + \text{cost } t_1 + \text{weight } t_2 + \text{cost } t_2$  **by** *simp*  
**also have**  $\dots = \text{weight } t_1 + (\sum_{a \in A_1} \text{freq } t_1 \ a \times \text{depth } t_1 \ a) +$   
 $\text{weight } t_2 + (\sum_{a \in A_2} \text{freq } t_2 \ a \times \text{depth } t_2 \ a)$   
**using**  $\text{hyps}$  **by** *simp*  
**also have**  $\dots = \text{weight } t_1 + (\sum_{a \in A_1} \text{freq } t_1 \ a \times (\text{depth } t \ a - 1)) +$   
 $\text{weight } t_2 + (\sum_{a \in A_2} \text{freq } t_2 \ a \times (\text{depth } t \ a - 1))$   
**using**  $c \ d_2$  **by** *simp*  
**also have**  $\dots = \text{weight } t_1 + (\sum_{a \in A_1} \text{freq } t_1 \ a \times \text{depth } t \ a)$   
 $- (\sum_{a \in A_1} \text{freq } t_1 \ a) +$   
 $\text{weight } t_2 + (\sum_{a \in A_2} \text{freq } t_2 \ a \times \text{depth } t \ a)$   
 $- (\sum_{a \in A_2} \text{freq } t_2 \ a)$   
**using**  $c \ d_2$  **by** (*simp add: sum.distrib*)  
**also have**  $\dots = (\sum_{a \in A_1} \text{freq } t_1 \ a \times \text{depth } t \ a) +$   
 $(\sum_{a \in A_2} \text{freq } t_2 \ a \times \text{depth } t \ a)$   
**using**  $c$  **by** (*simp add: weight\_eq\_Sum\_freq*)  
**also have**  $\dots = (\sum_{a \in A_1} \text{freq } t \ a \times \text{depth } t \ a) +$   
 $(\sum_{a \in A_2} \text{freq } t \ a \times \text{depth } t \ a)$   
**using**  $c$  **by** *auto*  
**also have**  $\dots = (\sum_{a \in A_1 \cup A_2} \text{freq } t \ a \times \text{depth } t \ a)$   
**using**  $c$  **by** (*simp add: sum.union\_disjoint*)  
**also have**  $\dots = (\sum_{a \in A} \text{freq } t \ a \times \text{depth } t \ a)$  **by** *simp*  
**finally show** *case* .  
**qed**

Finally, it should come as no surprise that trees with height 0 have cost 0.

**lemma** *height\_0\_imp\_cost\_0* [*simp*]:

*height t = 0*  $\implies$  *cost t = 0*

**by** (*case\_tac t*) *simp*<sup>+</sup>

## 2.10 Optimality

A tree is optimum if and only if its cost is not greater than that of any comparable tree. We can ignore inconsistent trees without loss of generality.

**definition**  $optimum :: \alpha \text{ tree} \Rightarrow \text{bool}$  **where**

$optimum\ t \equiv$

$\forall u. \text{consistent } u \longrightarrow \text{alphabet } t = \text{alphabet } u \longrightarrow \text{freq } t = \text{freq } u \longrightarrow$   
 $\text{cost } t \leq \text{cost } u$

### 3 Functional Implementation of Huffman's Algorithm

#### 3.1 Cached Weight

The *cached weight* of a node is the weight stored directly in the node. Our arguments rely on the computed weight (embodied by the *weight* function) rather than the cached weight, but the implementation of Huffman's algorithm uses the cached weight for performance reasons.

**primrec**  $cachedWeight :: \alpha \text{ tree} \Rightarrow \text{nat}$  **where**

$cachedWeight\ (Leaf\ w\ a) = w$

$cachedWeight\ (InnerNode\ w\ t_1\ t_2) = w$

The cached weight of a leaf node is identical to its computed weight.

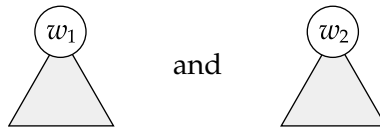
**lemma**  $height\_0\_imp\_cachedWeight\_eq\_weight$  [simp]:

$height\ t = 0 \implies cachedWeight\ t = weight\ t$

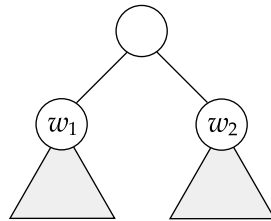
**by** (case\_tac t) simp<sup>+</sup>

#### 3.2 Tree Union

The implementation of Huffman's algorithm builds on two additional auxiliary functions. The first one, *uniteTrees*, takes two trees



and returns the tree



**definition**  $uniteTrees :: \alpha \text{ tree} \Rightarrow \alpha \text{ tree} \Rightarrow \alpha \text{ tree}$  **where**

$uniteTrees\ t_1\ t_2 \equiv InnerNode\ (cachedWeight\ t_1 + cachedWeight\ t_2)\ t_1\ t_2$

The alphabet, consistency, and symbol frequencies of a united tree are easy to connect to the homologous properties of the subtrees.

**lemma** *alphabet\_uniteTrees* [simp]:  
 $alphabet (uniteTrees t_1 t_2) = alphabet t_1 \cup alphabet t_2$   
**by** (simp add: uniteTrees\_def)

**lemma** *consistent\_uniteTrees* [simp]:  
 $\llbracket consistent t_1; consistent t_2; alphabet t_1 \cap alphabet t_2 = \emptyset \rrbracket \implies$   
 $consistent (uniteTrees t_1 t_2)$   
**by** (simp add: uniteTrees\_def)

**lemma** *freq\_uniteTrees* [simp]:  
 $freq (uniteTrees t_1 t_2) = (\lambda a. freq t_1 a + freq t_2 a)$   
**by** (simp add: uniteTrees\_def)

### 3.3 Ordered Tree Insertion

The auxiliary function *insortTree* inserts a tree into a forest sorted by cached weight, preserving the sort order.

**primrec** *insortTree* ::  $\alpha$  tree  $\Rightarrow$   $\alpha$  forest  $\Rightarrow$   $\alpha$  forest **where**  
 $insortTree u [] = [u]$   
 $insortTree u (t \cdot ts) =$   
     (if  $cachedWeight u \leq cachedWeight t$  then  $u \cdot t \cdot ts$   
     else  $t \cdot insortTree u ts$ )

The resulting forest contains one more tree than the original forest. Clearly, it cannot be empty.

**lemma** *length\_insortTree* [simp]:  
 $length (insortTree t ts) = length ts + 1$   
**by** (induct ts) simp<sup>+</sup>

**lemma** *insortTree\_ne\_Nil* [simp]:  
 $insortTree t ts \neq []$   
**by** (case\_tac ts) simp<sup>+</sup>

The alphabet, consistency, symbol frequencies, and height of a forest after insertion are easy to relate to the homologous properties of the original forest and the inserted tree.

**lemma** *alphabet\_F\_insortTree* [simp]:  
 $alphabet_F (insortTree t ts) = alphabet t \cup alphabet_F ts$   
**by** (induct ts) auto

**lemma** *consistent<sub>F</sub>\_insertTree* [simp]:  
 $consistent_F (insertTree\ t\ ts) = consistent_F (t \cdot ts)$   
**by** (induct ts) auto

**lemma** *freq<sub>F</sub>\_insertTree* [simp]:  
 $freq_F (insertTree\ t\ ts) = (\lambda a. freq\ t\ a + freq_F\ ts\ a)$   
**by** (induct ts) (simp add: ext)<sup>+</sup>

**lemma** *height<sub>F</sub>\_insertTree* [simp]:  
 $height_F (insertTree\ t\ ts) = \max (height\ t) (height_F\ ts)$   
**by** (induct ts) auto

### 3.4 The Main Algorithm

Huffman's algorithm repeatedly unites the first two trees of the forest it receives as argument until a single tree is left. It should initially be invoked with a list of leaf nodes sorted by weight. Note that it is not defined for the empty list.

**fun** *huffman* ::  $\alpha$  forest  $\Rightarrow$   $\alpha$  tree **where**  
*huffman* [] = t  
*huffman* (t<sub>1</sub> · t<sub>2</sub> · ts) = *huffman* (insertTree (uniteTrees t<sub>1</sub> t<sub>2</sub>) ts)

The time complexity of the algorithm is quadratic in the size of the forest. If we eliminated the inner node's cached weight component, and instead recomputed the weight each time it is needed, the complexity would remain quadratic, but with a larger constant. Using a binary search in *insertTree*, the corresponding imperative algorithm is  $O(n \log n)$  if we keep the weight cache and  $O(n^2)$  if we drop it. An  $O(n)$  imperative implementation is possible by maintaining two queues, one containing the unprocessed leaf nodes and the other containing the combined trees [8, p. 404].

The tree returned by the algorithm preserves the alphabet, consistency, and symbol frequencies of the original forest.

**theorem** *alphabet\_huffman* [simp]:  
 $ts \neq [] \implies alphabet (huffman\ ts) = alphabet_F\ ts$   
**by** (induct ts rule: *huffman.induct*) auto

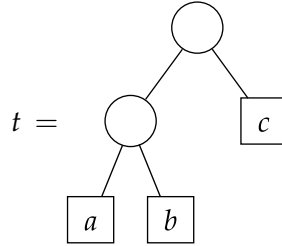
**theorem** *consistent\_huffman* [simp]:  
 $\llbracket consistent_F\ ts; ts \neq [] \rrbracket \implies consistent (huffman\ ts)$   
**by** (induct ts rule: *huffman.induct*) simp<sup>+</sup>

**theorem** *freq\_huffman* [simp]:  
 $ts \neq [] \implies freq (huffman\ ts) = freq_F\ ts$   
**by** (induct ts rule: *huffman.induct*) (auto simp: ext)

## 4 Definition of Auxiliary Functions Used in the Proof

### 4.1 Sibling of a Symbol

The *sibling* of a symbol  $a$  in a tree  $t$  is the label of the node that is the (left or right) sibling of the node labeled with  $a$  in  $t$ . If the symbol  $a$  is not in  $t$ 's alphabet or it occurs in a node with no sibling leaf, we define the sibling as being  $a$  itself; this gives us the nice property that if  $t$  is consistent, then  $\text{sibling } t a \neq a$  if and only if  $a$  has a sibling. As an illustration, we have  $\text{sibling } t a = b$ ,  $\text{sibling } t b = a$ , and  $\text{sibling } t c = c$  for the tree



```

fun sibling ::  $\alpha$  tree  $\Rightarrow$   $\alpha \Rightarrow \alpha$  where
sibling (Leaf  $w_b$   $b$ )  $a = a$ 
sibling (InnerNode  $w$  (Leaf  $w_b$   $b$ ) (Leaf  $w_c$   $c$ ))  $a =$ 
  (if  $a = b$  then  $c$  else if  $a = c$  then  $b$  else  $a$ )
sibling (InnerNode  $w$   $t_1$   $t_2$ )  $a =$ 
  (if  $a \in \text{alphabet } t_1$  then  $\text{sibling } t_1 a$ 
  else if  $a \in \text{alphabet } t_2$  then  $\text{sibling } t_2 a$ 
  else  $a$ )
  
```

Because *sibling* is defined using sequential pattern matching [9, 10], reasoning about it can become tedious. Simplification rules therefore play an important role.

```

lemmanotin_alphabet_imp_sibling_id [simp]:
 $a \notin \text{alphabet } t \implies \text{sibling } t a = a$ 
by (cases rule: sibling.cases [where  $x = (t, a)$ ]) simp+
  
```

```

lemma height_0_imp_sibling_id [simp]:
 $\text{height } t = 0 \implies \text{sibling } t a = a$ 
by (case_tac  $t$ ) simp+
  
```

```

lemma height_gt_0_in_alphabet_imp_sibling_left [simp]:
 $\llbracket \text{height } t_1 > 0; a \in \text{alphabet } t_1 \rrbracket \implies$ 
 $\text{sibling (InnerNode } w t_1 t_2) a = \text{sibling } t_1 a$ 
by (case_tac  $t_1$ ) simp+
  
```

```

lemma height_gt_0_in_alphabet_imp_sibling_right [simp]:
 $\llbracket \text{height } t_2 > 0; a \in \text{alphabet } t_1 \rrbracket \implies$ 
  
```

$sibling (InnerNode\ w\ t_1\ t_2)\ a = sibling\ t_1\ a$   
**by**  $(case\_tac\ t_2)\ simp^+$

**lemma**  $height\_gt\_0\_notin\_alphabet\_imp\_sibling\_left$   $[simp]$ :  
 $\llbracket height\ t_1 > 0; a \notin alphabet\ t_1 \rrbracket \implies$   
 $sibling (InnerNode\ w\ t_1\ t_2)\ a = sibling\ t_2\ a$   
**by**  $(case\_tac\ t_1)\ simp^+$

**lemma**  $height\_gt\_0\_notin\_alphabet\_imp\_sibling\_right$   $[simp]$ :  
 $\llbracket height\ t_2 > 0; a \notin alphabet\ t_1 \rrbracket \implies$   
 $sibling (InnerNode\ w\ t_1\ t_2)\ a = sibling\ t_2\ a$   
**by**  $(case\_tac\ t_2)\ simp^+$

**lemma**  $either\_height\_gt\_0\_imp\_sibling$   $[simp]$ :  
 $height\ t_1 > 0 \vee height\ t_2 > 0 \implies$   
 $sibling (InnerNode\ w\ t_1\ t_2)\ a =$   
 $(if\ a \in alphabet\ t_1\ then\ sibling\ t_1\ a\ else\ sibling\ t_2\ a)$   
**by** *auto*

The following rules are also useful for reasoning about siblings and alphabets.

**lemma**  $in\_alphabet\_imp\_sibling\_in\_alphabet$ :  
 $a \in alphabet\ t \implies sibling\ t\ a \in alphabet\ t$   
**by**  $(induct\ t\ a\ rule: sibling.induct)\ auto$

**lemma**  $sibling\_ne\_imp\_sibling\_in\_alphabet$ :  
 $sibling\ t\ a \neq a \implies sibling\ t\ a \in alphabet\ t$   
**by**  $(metis\ notin\_alphabet\_imp\_sibling\_id\ in\_alphabet\_imp\_sibling\_in\_alphabet)$

The default induction rule for *sibling* distinguishes four cases.

BASE CASE:  $t = Leaf\ w\ b$ .

INDUCTION STEP 1:  $t = InnerNode\ w\ (Leaf\ w_b\ b)\ (Leaf\ w_c\ c)$ .

INDUCTION STEP 2:  $t = InnerNode\ w\ (InnerNode\ w_1\ t_{11}\ t_{12})\ t_2$ .

INDUCTION STEP 3:  $t = InnerNode\ w\ t_1\ (InnerNode\ w_2\ t_{21}\ t_{22})$ .

This rule leaves much to be desired. First, the last two cases overlap and can normally be handled the same way, so they should be combined. Second, the nested *InnerNode* constructors in the last two cases reduce readability. Third, under the assumption that *t* is consistent, we would like to perform the same case distinction on *a* as we did for *tree\_induct\_consistent*, with the same benefits for automation. These observations lead us to develop a custom induction rule that distinguishes the following cases.

BASE CASE:  $t = Leaf\ w\ b$ .

INDUCTION STEP 1:  $t = \text{InnerNode } w (\text{Leaf } w_b b) (\text{Leaf } w_c c)$  with  $b \neq c$ .

INDUCTION STEP 2:  $t = \text{InnerNode } w t_1 t_2$  and either  $t_1$  or  $t_2$  has nonzero height.

SUBCASE 1:  $a$  belongs to  $t_1$  but not to  $t_2$ .

SUBCASE 2:  $a$  belongs to  $t_2$  but not to  $t_1$ .

SUBCASE 3:  $a$  belongs to neither  $t_1$  nor  $t_2$ .

The statement of the rule and its proof are similar to what we did for consistent trees, the main difference being that we now have two induction steps instead of one.

**lemma** *sibling\_induct\_consistent*

[consumes 1, case\_names base step<sub>1</sub> step<sub>21</sub> step<sub>22</sub> step<sub>23</sub>]:

[[consistent  $t$ ;

$\wedge w b a. P (\text{Leaf } w b) a$ ;

$\wedge w w_b b w_c c a. b \neq c \implies P (\text{InnerNode } w (\text{Leaf } w_b b) (\text{Leaf } w_c c)) a$ ;

$\wedge w t_1 t_2 a.$

[[consistent  $t_1$ ; consistent  $t_2$ ; alphabet  $t_1 \cap \text{alphabet } t_2 = \emptyset$ ;

height  $t_1 > 0 \vee \text{height } t_2 > 0$ ;  $a \in \text{alphabet } t_1$ ;

sibling  $t_1 a \in \text{alphabet } t_1$ ;  $a \notin \text{alphabet } t_2$ ;

sibling  $t_1 a \notin \text{alphabet } t_2$ ;  $P t_1 a$ ]]  $\implies$

$P (\text{InnerNode } w t_1 t_2) a$ ;

$\wedge w t_1 t_2 a.$

[[consistent  $t_1$ ; consistent  $t_2$ ; alphabet  $t_1 \cap \text{alphabet } t_2 = \emptyset$ ;

height  $t_1 > 0 \vee \text{height } t_2 > 0$ ;  $a \notin \text{alphabet } t_1$ ;

sibling  $t_2 a \notin \text{alphabet } t_1$ ;  $a \in \text{alphabet } t_2$ ;

sibling  $t_2 a \in \text{alphabet } t_2$ ;  $P t_2 a$ ]]  $\implies$

$P (\text{InnerNode } w t_1 t_2) a$ ;

$\wedge w t_1 t_2 a.$

[[consistent  $t_1$ ; consistent  $t_2$ ; alphabet  $t_1 \cap \text{alphabet } t_2 = \emptyset$ ;

height  $t_1 > 0 \vee \text{height } t_2 > 0$ ;  $a \notin \text{alphabet } t_1$ ;  $a \notin \text{alphabet } t_2$ ]]  $\implies$

$P (\text{InnerNode } w t_1 t_2) a$ ]]  $\implies$

$P t a$

**apply** *rotate\_tac*

**apply** *induction\_schema*

**apply** *atomize\_elim*

**apply** (*case\_tac*  $t$ , *simp*)

**apply** *clarsimp*

**apply** (*rename\_tac*  $a t_1 t_2$ )

**apply** (*case\_tac* height  $t_1 = 0 \wedge \text{height } t_2 = 0$ )

**apply** *simp*

**apply** (*case\_tac*  $t_1$ )

```

apply (case_tac t2)
apply fastforce
apply simp+
apply (auto intro: in_alphabet_imp_sibling_in_alphabet)[1]
by lexicographic_order

```

The custom induction rule allows us to prove new properties of *sibling* with little effort.

```

lemma sibling_sibling_id [simp]:
consistent t  $\implies$  sibling t (sibling t a) = a
by (induct t a rule: sibling_induct_consistent) simp+

```

```

lemma sibling_reciprocal:
 $\llbracket$ consistent t; sibling t a = b $\rrbracket \implies$  sibling t b = a
by auto

```

```

lemma depth_height_imp_sibling_ne:
 $\llbracket$ consistent t; depth t a = height t; height t > 0; a  $\in$  alphabet t $\rrbracket \implies$ 
sibling t a  $\neq$  a
by (induct t a rule: sibling_induct_consistent) auto

```

```

lemma depth_sibling [simp]:
consistent t  $\implies$  depth t (sibling t a) = depth t a
by (induct t a rule: sibling_induct_consistent) simp+

```

## 4.2 Leaf Interchange

The *swapLeaves* function takes a tree *t* together with two symbols *a*, *b* and their frequencies  $w_a$ ,  $w_b$ , and returns the tree *t* in which the leaf nodes labeled with *a* and *b* are exchanged. When invoking *swapLeaves*, we normally pass *freq t a* and *freq t b* for  $w_a$  and  $w_b$ .

Note that we do not bother updating the cached weight of the ancestor nodes when performing the interchange. The cached weight is used only in the implementation of Huffman's algorithm, which does not invoke *swapLeaves*.

```

primrec swapLeaves ::  $\alpha$  tree  $\Rightarrow$  nat  $\Rightarrow$   $\alpha$   $\Rightarrow$  nat  $\Rightarrow$   $\alpha$   $\Rightarrow$   $\alpha$  tree where
swapLeaves (Leaf wc c) wa a wb b =
  (if c = a then Leaf wb b else if c = b then Leaf wa a else Leaf wc c)
swapLeaves (InnerNode w t1 t2) wa a wb b =
  InnerNode w (swapLeaves t1 wa a wb b) (swapLeaves t2 wa a wb b)

```

Swapping a symbol *a* with itself leaves the tree *t* unchanged if *a* does not belong to it or if the specified frequencies  $w_a$  and  $w_b$  equal *freq t a*.



**lemma** *swapLeaves\_id\_when\_notin\_alphabet* [simp]:  
 $a \notin \text{alphabet } t \implies \text{swapLeaves } t \ w \ a \ w' \ a = t$   
**by** (induct t) simp<sup>+</sup>

**lemma** *swapLeaves\_id* [simp]:  
 $\text{consistent } t \implies \text{swapLeaves } t \ (\text{freq } t \ a) \ a \ (\text{freq } t \ a) \ a = t$   
**by** (induct t a rule: tree\_induct\_consistent) simp<sup>+</sup>

The alphabet, consistency, symbol depths, height, and symbol frequencies of the tree  $\text{swapLeaves } t \ w_a \ a \ w_b \ b$  can be related to the homologous properties of  $t$ .

**lemma** *alphabet\_swapLeaves*:  
 $\text{alphabet } (\text{swapLeaves } t \ w_a \ a \ w_b \ b) =$   
 (if  $a \in \text{alphabet } t$  then  
   if  $b \in \text{alphabet } t$  then  $\text{alphabet } t$  else  $(\text{alphabet } t - \{a\}) \cup \{b\}$   
 else  
   if  $b \in \text{alphabet } t$  then  $(\text{alphabet } t - \{b\}) \cup \{a\}$  else  $\text{alphabet } t$ )  
**by** (induct t) auto

**lemma** *consistent\_swapLeaves* [simp]:  
 $\text{consistent } t \implies \text{consistent } (\text{swapLeaves } t \ w_a \ a \ w_b \ b)$   
**by** (induct t) (auto simp: alphabet\_swapLeaves)

**lemma** *depth\_swapLeaves\_neither* [simp]:  
 $\llbracket \text{consistent } t; c \neq a; c \neq b \rrbracket \implies \text{depth } (\text{swapLeaves } t \ w_a \ a \ w_b \ b) \ c = \text{depth } t \ c$   
**by** (induct t a rule: tree\_induct\_consistent) (auto simp: alphabet\_swapLeaves)

**lemma** *height\_swapLeaves* [simp]:  
 $\text{height } (\text{swapLeaves } t \ w_a \ a \ w_b \ b) = \text{height } t$   
**by** (induct t) simp<sup>+</sup>

**lemma** *freq\_swapLeaves* [simp]:  
 $\llbracket \text{consistent } t; a \neq b \rrbracket \implies$   
 $\text{freq } (\text{swapLeaves } t \ w_a \ a \ w_b \ b) =$   
 ( $\lambda c.$  if  $c = a$  then if  $b \in \text{alphabet } t$  then  $w_a$  else 0  
   else if  $c = b$  then if  $a \in \text{alphabet } t$  then  $w_b$  else 0  
   else  $\text{freq } t \ c$ )  
**apply** (rule ext)  
**apply** (induct t)  
**by** auto

For the lemmas concerned with the resulting tree's weight and cost, we avoid subtraction on natural numbers by rearranging terms. For example, we write

$$\text{weight } (\text{swapLeaves } t \ w_a \ a \ w_b \ b) + \text{freq } t \ a = \text{weight } t + w_b$$

rather than the more conventional

$$\text{weight} (\text{swapLeaves } t \ w_a \ a \ w_b \ b) = \text{weight } t + w_b - \text{freq } t \ a.$$

In Isabelle/HOL, these two equations are not equivalent, because by definition  $m - n = 0$  if  $n > m$ . We could use the second equation and additionally assert that  $\text{freq } t \ a \leq \text{weight } t$  (an easy consequence of *weight\_eq\_Sum\_freq*), and then apply the *arith* tactic, but it is much simpler to use the first equation and stay with *simp* and *auto*. Another option would be to use integers instead of natural numbers.

**lemma** *weight\_swapLeaves*:

$\llbracket \text{consistent } t; a \neq b \rrbracket \implies$

if  $a \in \text{alphabet } t$  then

  if  $b \in \text{alphabet } t$  then

$$\text{weight} (\text{swapLeaves } t \ w_a \ a \ w_b \ b) + \text{freq } t \ a + \text{freq } t \ b = \\ \text{weight } t + w_a + w_b$$

  else

$$\text{weight} (\text{swapLeaves } t \ w_a \ a \ w_b \ b) + \text{freq } t \ a = \text{weight } t + w_b$$

else

  if  $b \in \text{alphabet } t$  then

$$\text{weight} (\text{swapLeaves } t \ w_a \ a \ w_b \ b) + \text{freq } t \ b = \text{weight } t + w_a$$

  else

$$\text{weight} (\text{swapLeaves } t \ w_a \ a \ w_b \ b) = \text{weight } t$$

**proof** (*induct t a rule: tree\_induct\_consistent*)

— BASE CASE:  $t = \text{Leaf } w \ b$

**case base thus case by** *clarsimp*

**next**

— INDUCTION STEP:  $t = \text{InnerNode } w \ t_1 \ t_2$

— SUBCASE 1:  $a$  belongs to  $t_1$  but not to  $t_2$

**case** (*step<sub>1</sub> w t<sub>1</sub> t<sub>2</sub> a*) **show case**

**proof cases**

**assume**  $b \in \text{alphabet } t_1$

**moreover hence**  $b \notin \text{alphabet } t_2$  **using** *step<sub>1</sub>* **by** *auto*

**ultimately show case using** *step<sub>1</sub>* **by** *simp*

**next**

**assume**  $b \notin \text{alphabet } t_1$  **thus case using** *step<sub>1</sub>* **by** *auto*

**qed**

**next**

— SUBCASE 2:  $a$  belongs to  $t_2$  but not to  $t_1$

**case** (*step<sub>2</sub> w t<sub>1</sub> t<sub>2</sub> a*) **show case**

**proof cases**

**assume**  $b \in \text{alphabet } t_1$

**moreover hence**  $b \notin \text{alphabet } t_2$  **using**  $\text{step}_2$  **by** *auto*  
**ultimately show case using**  $\text{step}_2$  **by** *simp*  
**next**  
**assume**  $b \notin \text{alphabet } t_1$  **thus case using**  $\text{step}_2$  **by** *auto*  
**qed**  
**next**  
— SUBCASE 3:  $a$  belongs to neither  $t_1$  nor  $t_2$   
**case**  $(\text{step}_3 \ w \ t_1 \ t_2 \ a)$  **show case**  
**proof cases**  
**assume**  $b \in \text{alphabet } t_1$   
**moreover hence**  $b \notin \text{alphabet } t_2$  **using**  $\text{step}_3$  **by** *auto*  
**ultimately show case using**  $\text{step}_3$  **by** *simp*  
**next**  
**assume**  $b \notin \text{alphabet } t_1$  **thus case using**  $\text{step}_3$  **by** *auto*  
**qed**  
**qed**

**lemma**  $\text{cost\_swapLeaves}$ :  
 $\llbracket \text{consistent } t; a \neq b \rrbracket \implies$   
**if**  $a \in \text{alphabet } t$  **then**  
**if**  $b \in \text{alphabet } t$  **then**  
 $\text{cost } (\text{swapLeaves } t \ w_a \ a \ w_b \ b) + \text{freq } t \ a \times \text{depth } t \ a$   
 $+ \text{freq } t \ b \times \text{depth } t \ b =$   
 $\text{cost } t + w_a \times \text{depth } t \ b + w_b \times \text{depth } t \ a$   
**else**  
 $\text{cost } (\text{swapLeaves } t \ w_a \ a \ w_b \ b) + \text{freq } t \ a \times \text{depth } t \ a =$   
 $\text{cost } t + w_b \times \text{depth } t \ a$   
**else**  
**if**  $b \in \text{alphabet } t$  **then**  
 $\text{cost } (\text{swapLeaves } t \ w_a \ a \ w_b \ b) + \text{freq } t \ b \times \text{depth } t \ b =$   
 $\text{cost } t + w_a \times \text{depth } t \ b$   
**else**  
 $\text{cost } (\text{swapLeaves } t \ w_a \ a \ w_b \ b) = \text{cost } t$

**proof** (*induct t*)  
**case** *Leaf* **show case by** *simp*  
**next**  
**case** (*InnerNode*  $w \ t_1 \ t_2$ )  
**note**  $c = \langle \text{consistent } (\text{InnerNode } w \ t_1 \ t_2) \rangle$   
**note**  $\text{hyps} = \text{InnerNode}$   
**have**  $w_1$ : **if**  $a \in \text{alphabet } t_1$  **then**  
**if**  $b \in \text{alphabet } t_1$  **then**  
 $\text{weight } (\text{swapLeaves } t_1 \ w_a \ a \ w_b \ b) + \text{freq } t_1 \ a + \text{freq } t_1 \ b =$   
 $\text{weight } t_1 + w_a + w_b$   
**else**

$weight (swapLeaves t_1 w_a a w_b b) + freq t_1 a = weight t_1 + w_b$   
else  
if  $b \in alphabet t_1$  then  
 $weight (swapLeaves t_1 w_a a w_b b) + freq t_1 b = weight t_1 + w_a$   
else  
 $weight (swapLeaves t_1 w_a a w_b b) = weight t_1$  **using** *hyps*  
**by** (*simp add: weight\_swapLeaves*)  
**have**  $w_2$ : if  $a \in alphabet t_2$  then  
if  $b \in alphabet t_2$  then  
 $weight (swapLeaves t_2 w_a a w_b b) + freq t_2 a + freq t_2 b =$   
 $weight t_2 + w_a + w_b$   
else  
 $weight (swapLeaves t_2 w_a a w_b b) + freq t_2 a = weight t_2 + w_b$   
else  
if  $b \in alphabet t_2$  then  
 $weight (swapLeaves t_2 w_a a w_b b) + freq t_2 b = weight t_2 + w_a$   
else  
 $weight (swapLeaves t_2 w_a a w_b b) = weight t_2$  **using** *hyps*  
**by** (*simp add: weight\_swapLeaves*)  
**show** *case*  
**proof** *cases*  
**assume**  $a_1: a \in alphabet t_1$   
**hence**  $a_2: a \notin alphabet t_2$  **using** *c* **by** *auto*  
**show** *case*  
**proof** *cases*  
**assume**  $b_1: b \in alphabet t_1$   
**hence**  $b \notin alphabet t_2$  **using** *c* **by** *auto*  
**thus** *case* **using**  $a_1 a_2 b_1 w_1 w_2$  *hyps* **by** *simp*  
**next**  
**assume**  $b_1: b \notin alphabet t_1$  **show** *case*  
**proof** *cases*  
**assume**  $b \in alphabet t_2$  **thus** *case* **using**  $a_1 a_2 b_1 w_1 w_2$  *hyps* **by** *simp*  
**next**  
**assume**  $b \notin alphabet t_2$  **thus** *case* **using**  $a_1 a_2 b_1 w_1 w_2$  *hyps* **by** *simp*  
**qed**  
**qed**  
**next**  
**assume**  $a_1: a \notin alphabet t_1$  **show** *case*  
**proof** *cases*  
**assume**  $a_2: a \in alphabet t_2$  **show** *case*  
**proof** *cases*  
**assume**  $b_1: b \in alphabet t_1$

```

    hence  $b \notin \text{alphabet } t_2$  using  $c$  by auto
    thus case using  $a_1 a_2 b_1 w_1 w_2$  hyps by simp
  next
    assume  $b_1: b \notin \text{alphabet } t_1$  show case
    proof cases
      assume  $b \in \text{alphabet } t_2$  thus case using  $a_1 a_2 b_1 w_1 w_2$  hyps by simp
    next
      assume  $b \notin \text{alphabet } t_2$  thus case using  $a_1 a_2 b_1 w_1 w_2$  hyps by simp
    qed
  qed
next
  assume  $a_2: a \notin \text{alphabet } t_2$  show case
  proof cases
    assume  $b_1: b \in \text{alphabet } t_1$ 
    hence  $b \notin \text{alphabet } t_2$  using  $c$  by auto
    thus case using  $a_1 a_2 b_1 w_1 w_2$  hyps by simp
  next
    assume  $b_1: b \notin \text{alphabet } t_1$  show case
    proof cases
      assume  $b \in \text{alphabet } t_2$  thus case using  $a_1 a_2 b_1 w_1 w_2$  hyps by simp
    next
      assume  $b \notin \text{alphabet } t_2$  thus case using  $a_1 a_2 b_1 w_1 w_2$  hyps by simp
    qed
  qed
qed
qed
qed

```

Common sense tells us that the following statement is valid: “If Astrid exchanges her house with Bernard’s neighbor, Bernard becomes Astrid’s new neighbor.” A similar property holds for binary trees.

```

lemma sibling_swapLeaves_sibling [simp]:
   $\llbracket \text{consistent } t; \text{sibling } t \ b \neq b; a \neq b \rrbracket \implies$ 
  sibling (swapLeaves  $t \ w_a \ a \ w_s$  (sibling  $t \ b$ ))  $a = b$ 
proof (induct  $t$ )
  case Leaf thus case by simp
next
  case (InnerNode  $w \ t_1 \ t_2$ )
  note hyps = InnerNode
  show case
  proof (cases height  $t_1 = 0$ )
  case True
  note  $h_1 = \text{True}$ 

```

```

show thesis
proof (cases  $t_1$ )
  case (Leaf  $w_c$   $c$ )
    note  $l_1 = \text{Leaf}$ 
    show thesis
    proof (cases  $\text{height } t_2 = 0$ )
      case True
        note  $h_2 = \text{True}$ 
        show thesis
        proof (cases  $t_2$ )
          case Leaf thus thesis using  $l_1$  hyps by auto metis+
        next
          case InnerNode thus thesis using  $h_2$  by simp
        qed
      next
        case False
          note  $h_2 = \text{False}$ 
          show thesis
          proof cases
            assume  $c = b$  thus thesis using  $l_1$   $h_2$  hyps by simp
          next
            assume  $c \neq b$ 
            have sibling  $t_2$   $b \in \text{alphabet } t_2$  using  $\langle c \neq b \rangle$   $l_1$   $h_2$  hyps
              by (simp add: sibling_ne_imp_sibling_in_alphabet)
            thus thesis using  $\langle c \neq b \rangle$   $l_1$   $h_2$  hyps by auto
          qed
        qed
      next
        case InnerNode thus thesis using  $h_1$  by simp
        qed
    next
      case False
        note  $h_1 = \text{False}$ 
        show thesis
        proof (cases  $\text{height } t_2 = 0$ )
          case True
            note  $h_2 = \text{True}$ 
            show thesis
            proof (cases  $t_2$ )
              case (Leaf  $w_d$   $d$ )
                note  $l_2 = \text{Leaf}$ 
                show thesis
                proof cases

```

```

    assume  $d = b$  thus thesis using  $h_1$   $l_2$  hyps by simp
  next
    assume  $d \neq b$  show thesis
    proof (cases  $b \in \text{alphabet } t_1$ )
      case True
        hence sibling  $t_1$   $b \in \text{alphabet } t_1$  using  $\langle d \neq b \rangle$   $h_1$   $l_2$  hyps
          by (simp add: sibling_ne_imp_sibling_in_alphabet)
        thus thesis using True  $\langle d \neq b \rangle$   $h_1$   $l_2$  hyps
          by (simp add: alphabet_swapLeaves)
      next
        case False thus thesis using  $\langle d \neq b \rangle$   $l_2$  hyps by simp
    qed
  qed
next
  case InnerNode thus thesis using  $h_2$  by simp
qed
next
  case False
  note  $h_2 = \text{False}$ 
  show thesis
  proof (cases  $b \in \text{alphabet } t_1$ )
    case True thus thesis using  $h_1$   $h_2$  hyps by auto
  next
    case False
    note  $b_1 = \text{False}$ 
    show thesis
    proof (cases  $b \in \text{alphabet } t_2$ )
      case True thus thesis using  $b_1$   $h_1$   $h_2$  hyps
        by (auto simp: in_alphabet_imp_sibling_in_alphabet
            alphabet_swapLeaves)
    next
      case False thus thesis using  $b_1$   $h_1$   $h_2$  hyps by simp
    qed
  qed
qed
qed
qed

```

### 4.3 Symbol Interchange

The *swapSyms* function provides a simpler interface to *swapLeaves*, with *freq t a* and *freq t b* in place of  $w_a$  and  $w_b$ . Most lemmas about *swapSyms* are directly adapted from the homologous results about *swapLeaves*.

**definition**  $\text{swapSyms} :: \alpha \text{ tree} \Rightarrow \alpha \Rightarrow \alpha \Rightarrow \alpha \text{ tree}$  **where**  
 $\text{swapSyms } t \ a \ b \equiv \text{swapLeaves } t \ (\text{freq } t \ a) \ a \ (\text{freq } t \ b) \ b$

**lemma**  $\text{swapSyms\_id}$  [simp]:  
 $\text{consistent } t \Longrightarrow \text{swapSyms } t \ a \ a = t$   
**by** (simp add: swapSyms\_def)

**lemma**  $\text{alphabet\_swapSyms}$  [simp]:  
 $\llbracket a \in \text{alphabet } t; b \in \text{alphabet } t \rrbracket \Longrightarrow \text{alphabet } (\text{swapSyms } t \ a \ b) = \text{alphabet } t$   
**by** (simp add: swapSyms\_def alphabet\_swapLeaves)

**lemma**  $\text{consistent\_swapSyms}$  [simp]:  
 $\text{consistent } t \Longrightarrow \text{consistent } (\text{swapSyms } t \ a \ b)$   
**by** (simp add: swapSyms\_def)

**lemma**  $\text{depth\_swapSyms\_neither}$  [simp]:  
 $\llbracket \text{consistent } t; c \neq a; c \neq b \rrbracket \Longrightarrow$   
 $\text{depth } (\text{swapSyms } t \ a \ b) \ c = \text{depth } t \ c$   
**by** (simp add: swapSyms\_def)

**lemma**  $\text{freq\_swapSyms}$  [simp]:  
 $\llbracket \text{consistent } t; a \in \text{alphabet } t; b \in \text{alphabet } t \rrbracket \Longrightarrow$   
 $\text{freq } (\text{swapSyms } t \ a \ b) = \text{freq } t$   
**by** (case\_tac  $a = b$ ) (simp add: swapSyms\_def ext)<sup>+</sup>

**lemma**  $\text{cost\_swapSyms}$ :  
**assumes**  $\text{consistent } t \ a \in \text{alphabet } t \ b \in \text{alphabet } t$   
**shows**  $\text{cost } (\text{swapSyms } t \ a \ b) + \text{freq } t \ a \times \text{depth } t \ a + \text{freq } t \ b \times \text{depth } t \ b =$   
 $\text{cost } t + \text{freq } t \ a \times \text{depth } t \ b + \text{freq } t \ b \times \text{depth } t \ a$

**proof** cases

**assume**  $a = b$  **thus** **thesis** **using** **assms** **by** **simp**

**next**

**assume**  $a \neq b$

**moreover** **hence**  $\text{cost } (\text{swapLeaves } t \ (\text{freq } t \ a) \ a \ (\text{freq } t \ b) \ b)$   
 $+ \text{freq } t \ a \times \text{depth } t \ a + \text{freq } t \ b \times \text{depth } t \ b =$   
 $\text{cost } t + \text{freq } t \ a \times \text{depth } t \ b + \text{freq } t \ b \times \text{depth } t \ a$

**using** **assms** **by** (simp add: cost\_swapLeaves)

**ultimately show** **thesis** **using** **assms** **by** (simp add: swapSyms\_def)

**qed**

If  $a$ 's frequency is lower than or equal to  $b$ 's, and  $a$  is higher up in the tree than  $b$  or at the same level, then interchanging  $a$  and  $b$  does not increase the tree's cost.

**lemma**  $\text{le\_le\_imp\_sum\_mult\_le\_sum\_mult}$ :  
 $\llbracket i \leq j; m \leq (n::\text{nat}) \rrbracket \Longrightarrow i \times n + j \times m \leq i \times m + j \times n$   
**apply** (subgoal\_tac  $i \times m + i \times (n - m) + j \times m \leq i \times m + j \times m + j \times (n - m)$ )



**apply** (*simp add: diff\_mult\_distrib2*)  
**by** *simp*

**lemma** *cost\_swapSyms\_le*:

**assumes** *consistent t a ∈ alphabet t b ∈ alphabet t freq t a ≤ freq t b*  
*depth t a ≤ depth t b*

**shows** *cost (swapSyms t a b) ≤ cost t*

**proof** –

**let** *aabb = freq t a × depth t a + freq t b × depth t b*

**let** *abba = freq t a × depth t b + freq t b × depth t a*

**have** *abba ≤ aabb* **using** *assms(4–5)*

**by** (*rule le\_le\_imp\_sum\_mult\_le\_sum\_mult*)

**have** *cost (swapSyms t a b) + aabb = cost t + abba* **using** *assms(1–3)*

**by** (*simp add: cost\_swapSyms add.assoc [THEN sym]*)

**also have** *... ≤ cost t + aabb* **using** (*abba ≤ aabb*) **by** *simp*

**finally show** *thesis* **using** *assms(4–5)* **by** *simp*

**qed**

As stated earlier, “If Astrid exchanges her house with Bernard’s neighbor, Bernard becomes Astrid’s new neighbor.”

**lemma** *sibling\_swapSyms\_sibling [simp]*:

$\llbracket \text{consistent } t; \text{ sibling } t b \neq b; a \neq b \rrbracket \implies$

*sibling (swapSyms t a (sibling t b)) a = b*

**by** (*simp add: swapSyms\_def*)

“If Astrid exchanges her house with Bernard, Astrid becomes Bernard’s old neighbor’s new neighbor.”

**lemma** *sibling\_swapSyms\_other\_sibling [simp]*:

$\llbracket \text{consistent } t; \text{ sibling } t b \neq a; \text{ sibling } t b \neq b; a \neq b \rrbracket \implies$

*sibling (swapSyms t a b) (sibling t b) = a*

**by** (*metis consistent\_swapSyms sibling\_swapSyms\_sibling sibling\_reciprocal*)

#### 4.4 Four-Way Symbol Interchange

The *swapSyms* function exchanges two symbols *a* and *b*. We use it to define the four-way symbol interchange function *swapFourSyms*, which takes four symbols *a, b, c, d* with *a ≠ b* and *c ≠ d*, and exchanges them so that *a* and *b* occupy *c* and *d*’s positions.

A naive definition of this function would be

$$\text{swapFourSyms } t a b c d \equiv \text{swapSyms } (\text{swapSyms } t a c) b d.$$

This definition fails in the face of aliasing: If *a = d*, but *b ≠ c*, then *swapFourSyms*

$a b c d$  would leave  $a$  in  $b$ 's position.<sup>2</sup>

**definition**  $swapFourSyms :: \alpha tree \Rightarrow \alpha \Rightarrow \alpha \Rightarrow \alpha \Rightarrow \alpha \Rightarrow \alpha tree$  **where**  
 $swapFourSyms t a b c d \equiv$   
 if  $a = d$  then  $swapSyms t b c$   
 else if  $b = c$  then  $swapSyms t a d$   
 else  $swapSyms (swapSyms t a c) b d$

Lemmas about  $swapFourSyms$  are easy to prove by expanding its definition.

**lemma**  $alphabet\_swapFourSyms [simp]:$   
 $\llbracket a \in alphabet\ t; b \in alphabet\ t; c \in alphabet\ t; d \in alphabet\ t \rrbracket \implies$   
 $alphabet\ (swapFourSyms\ t\ a\ b\ c\ d) = alphabet\ t$   
**by** ( $simp\ add: swapFourSyms\_def$ )

**lemma**  $consistent\_swapFourSyms [simp]:$   
 $consistent\ t \implies consistent\ (swapFourSyms\ t\ a\ b\ c\ d)$   
**by** ( $simp\ add: swapFourSyms\_def$ )

**lemma**  $freq\_swapFourSyms [simp]:$   
 $\llbracket consistent\ t; a \in alphabet\ t; b \in alphabet\ t; c \in alphabet\ t;$   
 $d \in alphabet\ t \rrbracket \implies$   
 $freq\ (swapFourSyms\ t\ a\ b\ c\ d) = freq\ t$   
**by** ( $auto\ simp: swapFourSyms\_def$ )

More Astrid and Bernard insanity: "If Astrid and Bernard exchange their houses with Carmen and her neighbor, Astrid and Bernard will now be neighbors."

**lemma**  $sibling\_swapFourSyms\_when\_4th\_is\_sibling:$   
**assumes**  $consistent\ t\ a \in alphabet\ t\ b \in alphabet\ t\ c \in alphabet\ t$   
 $a \neq b\ sibling\ t\ c \neq c$   
**shows**  $sibling\ (swapFourSyms\ t\ a\ b\ c\ (sibling\ t\ c))\ a = b$   
**proof** ( $cases\ a \neq sibling\ t\ c \wedge b \neq c$ )  
**case**  $True$  **show**  $thesis$   
**proof** –  
 let  $d = sibling\ t\ c$   
 let  $t_s = swapFourSyms\ t\ a\ b\ c\ d$   
**have**  $abba: (sibling\ t_s\ a = b) = (sibling\ t_s\ b = a)$  **using** ( $consistent\ t$ )  
**by** ( $metis\ consistent\_swapFourSyms\ sibling\_reciprocal$ )  
**have**  $s: sibling\ t\ c = sibling\ (swapSyms\ t\ a\ c)\ a$  **using**  $True\ assms$   
**by** ( $metis\ sibling\_reciprocal\ sibling\_swapSyms\_sibling$ )  
**have**  $sibling\ t_s\ b = sibling\ (swapSyms\ t\ a\ c)\ d$  **using**  $s\ True\ assms$   
**by** ( $auto\ simp: swapFourSyms\_def$ )  
**also have**  $\dots = a$  **using**  $True\ assms$

---

<sup>2</sup>Cormen et al. [4, p. 390] forgot to consider this case in their proof. Thomas Cormen indicated in a personal communication that this will be corrected in the next edition of the book.

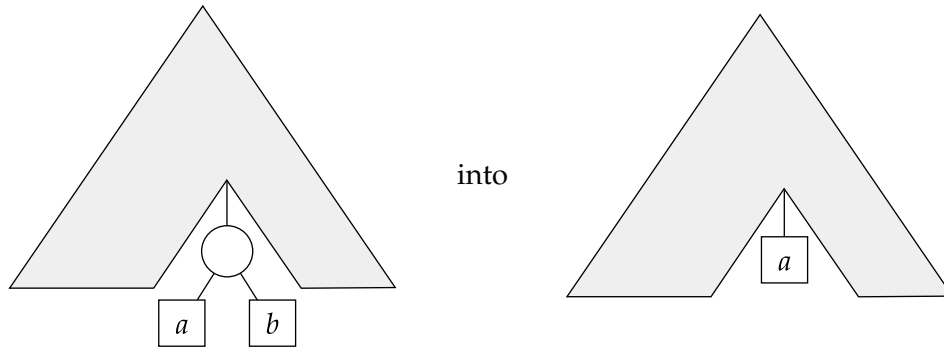
```

    by (metis sibling_reciprocal sibling_swapSyms_other_sibling
        swapLeaves_id swapSyms_def)
  finally have sibling  $t_s b = a$  .
  with abba show thesis ..
qed
next
case False thus thesis using assms
  by (auto intro: sibling_reciprocal simp: swapFourSyms_def)
qed

```

## 4.5 Sibling Merge

Given a symbol  $a$ , the *mergeSibling* function transforms the tree



The frequency of  $a$  in the result is the sum of the original frequencies of  $a$  and  $b$ , so as not to alter the tree's weight.

```

fun mergeSibling ::  $\alpha$  tree  $\Rightarrow$   $\alpha$   $\Rightarrow$   $\alpha$  tree where
mergeSibling (Leaf  $w_b$   $b$ )  $a$  = Leaf  $w_b$   $b$ 
mergeSibling (InnerNode  $w$  (Leaf  $w_b$   $b$ ) (Leaf  $w_c$   $c$ ))  $a$  =
  (if  $a = b \vee a = c$  then Leaf  $(w_b + w_c)$   $a$ 
   else InnerNode  $w$  (Leaf  $w_b$   $b$ ) (Leaf  $w_c$   $c$ ))
mergeSibling (InnerNode  $w$   $t_1$   $t_2$ )  $a$  =
  InnerNode  $w$  (mergeSibling  $t_1$   $a$ ) (mergeSibling  $t_2$   $a$ )

```

The definition of *mergeSibling* has essentially the same structure as that of *sibling*. As a result, the custom induction rule that we derived for *sibling* works equally well for reasoning about *mergeSibling*.

**lemmas** *mergeSibling\_induct\_consistent* = *sibling\_induct\_consistent*

The properties of *mergeSibling* echo those of *sibling*. Like with *sibling*, simplification rules are crucial.

**lemma** *notin\_alphabet\_imp\_mergeSibling\_id* [simp]:

$a \notin \text{alphabet } t \implies \text{mergeSibling } t a = t$

**by** (induct  $t$  a rule: *mergeSibling.induct*) *simp*<sup>+</sup>

**lemma** *height\_gt\_0\_imp\_mergeSibling\_left* [simp]:

$\text{height } t_1 > 0 \implies$

$\text{mergeSibling } (\text{InnerNode } w \ t_1 \ t_2) \ a =$   
 $\text{InnerNode } w \ (\text{mergeSibling } t_1 \ a) \ (\text{mergeSibling } t_2 \ a)$

**by** (*case\_tac*  $t_1$ ) *simp*<sup>+</sup>

**lemma** *height\_gt\_0\_imp\_mergeSibling\_right* [simp]:

$\text{height } t_2 > 0 \implies$

$\text{mergeSibling } (\text{InnerNode } w \ t_1 \ t_2) \ a =$   
 $\text{InnerNode } w \ (\text{mergeSibling } t_1 \ a) \ (\text{mergeSibling } t_2 \ a)$

**by** (*case\_tac*  $t_2$ ) *simp*<sup>+</sup>

**lemma** *either\_height\_gt\_0\_imp\_mergeSibling* [simp]:

$\text{height } t_1 > 0 \vee \text{height } t_2 > 0 \implies$

$\text{mergeSibling } (\text{InnerNode } w \ t_1 \ t_2) \ a =$   
 $\text{InnerNode } w \ (\text{mergeSibling } t_1 \ a) \ (\text{mergeSibling } t_2 \ a)$

**by** *auto*

**lemma** *alphabet\_mergeSibling* [simp]:

$\llbracket \text{consistent } t; a \in \text{alphabet } t \rrbracket \implies$

$\text{alphabet } (\text{mergeSibling } t \ a) = (\text{alphabet } t - \{\text{sibling } t \ a\}) \cup \{a\}$

**by** (induct  $t$  a rule: *mergeSibling\_induct\_consistent*) *auto*

**lemma** *consistent\_mergeSibling* [simp]:

$\text{consistent } t \implies \text{consistent } (\text{mergeSibling } t \ a)$

**by** (induct  $t$  a rule: *mergeSibling\_induct\_consistent*) *auto*

**lemma** *freq\_mergeSibling*:

$\llbracket \text{consistent } t; a \in \text{alphabet } t; \text{sibling } t \ a \neq a \rrbracket \implies$

$\text{freq } (\text{mergeSibling } t \ a) =$   
 $(\lambda c. \text{if } c = a \text{ then } \text{freq } t \ a + \text{freq } t \ (\text{sibling } t \ a)$   
 $\text{else if } c = \text{sibling } t \ a \text{ then } 0$   
 $\text{else } \text{freq } t \ c)$

**by** (induct  $t$  a rule: *mergeSibling\_induct\_consistent*)  
(*auto simp: fun\_eq\_iff*)

**lemma** *weight\_mergeSibling* [simp]:

$\text{weight } (\text{mergeSibling } t \ a) = \text{weight } t$

**by** (induct  $t$  a rule: *mergeSibling.induct*) *simp*<sup>+</sup>

If  $a$  has a sibling, merging  $a$  and its sibling reduces  $t$ 's cost by  $\text{freq } t \ a + \text{freq } t \ (\text{sibling } t \ a)$ .

**lemma** *cost\_mergeSibling*:

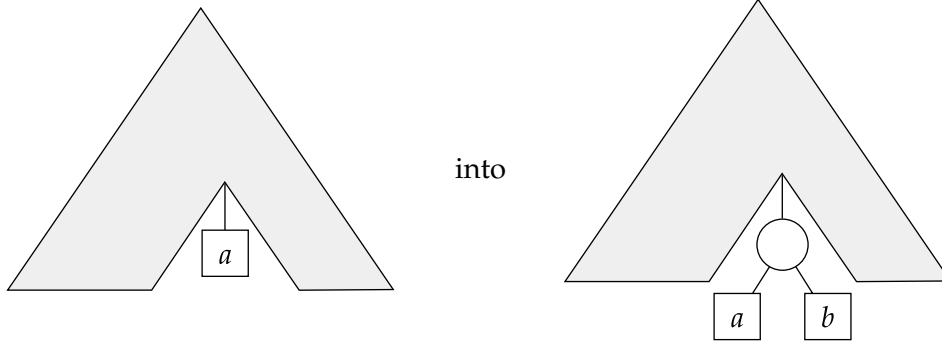
$\llbracket \text{consistent } t; \text{ sibling } t \ a \neq a \rrbracket \implies$

$\text{cost } (\text{mergeSibling } t \ a) + \text{freq } t \ a + \text{freq } t \ (\text{sibling } t \ a) = \text{cost } t$

**by** (*induct* *t* a rule: *mergeSibling\_induct\_consistent*) *auto*

## 4.6 Leaf Split

The *splitLeaf* function undoes the merging performed by *mergeSibling*: Given two symbols *a*, *b* and two frequencies  $w_a, w_b$ , it transforms



In the resulting tree, *a* has frequency  $w_a$  and *b* has frequency  $w_b$ . We normally invoke it with  $w_a$  and  $w_b$  such that  $\text{freq } t \ a = w_a + w_b$ .

**primrec** *splitLeaf*  $:: \alpha \text{ tree} \Rightarrow \text{nat} \Rightarrow \alpha \Rightarrow \text{nat} \Rightarrow \alpha \Rightarrow \alpha \text{ tree}$  **where**

*splitLeaf* (*Leaf*  $w_c \ c$ )  $w_a \ w_b \ b =$

(if  $c = a$  then *InnerNode*  $w_c$  (*Leaf*  $w_a \ a$ ) (*Leaf*  $w_b \ b$ ) else *Leaf*  $w_c \ c$ )

*splitLeaf* (*InnerNode*  $w \ t_1 \ t_2$ )  $w_a \ w_b \ b =$

*InnerNode*  $w$  (*splitLeaf*  $t_1 \ w_a \ w_b \ b$ ) (*splitLeaf*  $t_2 \ w_a \ w_b \ b$ )

**primrec** *splitLeaf<sub>F</sub>*  $:: \alpha \text{ forest} \Rightarrow \text{nat} \Rightarrow \alpha \Rightarrow \text{nat} \Rightarrow \alpha \Rightarrow \alpha \text{ forest}$  **where**

*splitLeaf<sub>F</sub>* []  $w_a \ w_b \ b = []$

*splitLeaf<sub>F</sub>* ( $t \cdot ts$ )  $w_a \ w_b \ b =$

*splitLeaf*  $t \ w_a \ w_b \ b \cdot \text{splitLeaf}_F \ ts \ w_a \ w_b \ b$

Splitting leaf nodes affects the alphabet, consistency, symbol frequencies, weight, and cost in unsurprising ways.

**lemma** *notin\_alphabet\_imp\_splitLeaf\_id* [*simp*]:

$a \notin \text{alphabet } t \implies \text{splitLeaf } t \ w_a \ w_b \ b = t$

**by** (*induct* *t*) *simp*<sup>+</sup>

**lemma** *notin\_alphabet<sub>F</sub>\_imp\_splitLeaf<sub>F</sub>\_id* [*simp*]:

$a \notin \text{alphabet}_F \ ts \implies \text{splitLeaf}_F \ ts \ w_a \ w_b \ b = ts$

**by** (*induct* *ts*) *simp*<sup>+</sup>

**lemma** *alphabet\_splitLeaf* [simp]:  
*alphabet (splitLeaf t w<sub>a</sub> a w<sub>b</sub> b) =*  
     *(if a ∈ alphabet t then alphabet t ∪ {b} else alphabet t)*  
**by** (*induct t*) *simp*<sup>+</sup>

**lemma** *consistent\_splitLeaf* [simp]:  
 $\llbracket \text{consistent } t; b \notin \text{alphabet } t \rrbracket \implies \text{consistent } (\text{splitLeaf } t \ w_a \ a \ w_b \ b)$   
**by** (*induct t*) *auto*

**lemma** *freq\_splitLeaf* [simp]:  
 $\llbracket \text{consistent } t; b \notin \text{alphabet } t \rrbracket \implies$   
*freq (splitLeaf t w<sub>a</sub> a w<sub>b</sub> b) =*  
     *(if a ∈ alphabet t then*  
         *(λc. if c = a then w<sub>a</sub> else if c = b then w<sub>b</sub> else freq t c)*  
     *else*  
         *freq t)*  
**by** (*induct t b rule: tree\_induct\_consistent*) (*rule ext, auto*)<sup>+</sup>

**lemma** *weight\_splitLeaf* [simp]:  
 $\llbracket \text{consistent } t; a \in \text{alphabet } t; \text{freq } t \ a = w_a + w_b \rrbracket \implies$   
*weight (splitLeaf t w<sub>a</sub> a w<sub>b</sub> b) = weight t*  
**by** (*induct t a rule: tree\_induct\_consistent*) *simp*<sup>+</sup>

**lemma** *cost\_splitLeaf* [simp]:  
 $\llbracket \text{consistent } t; a \in \text{alphabet } t; \text{freq } t \ a = w_a + w_b \rrbracket \implies$   
*cost (splitLeaf t w<sub>a</sub> a w<sub>b</sub> b) = cost t + w<sub>a</sub> + w<sub>b</sub>*  
**by** (*induct t a rule: tree\_induct\_consistent*) *simp*<sup>+</sup>

## 4.7 Weight Sort Order

An invariant of Huffman's algorithm is that the forest is sorted by weight. This is expressed by the *sortedByWeight* function.

**fun** *sortedByWeight* ::  $\alpha$  *forest*  $\Rightarrow$  *bool* **where**  
*sortedByWeight [] = True*  
*sortedByWeight [t] = True*  
*sortedByWeight (t<sub>1</sub> · t<sub>2</sub> · ts) =*  
     *(weight t<sub>1</sub> ≤ weight t<sub>2</sub> ∧ sortedByWeight (t<sub>2</sub> · ts))*

The function obeys the following fairly obvious laws.

**lemma** *sortedByWeight\_Cons\_imp\_sortedByWeight*:  
 $\text{sortedByWeight } (t \cdot ts) \implies \text{sortedByWeight } ts$   
**by** (*case\_tac ts*) *simp*<sup>+</sup>

**lemma** *sortedByWeight\_Cons\_imp\_forall\_weight\_ge*:  
 $\text{sortedByWeight } (t \cdot ts) \implies \forall u \in \text{set } ts. \text{weight } u \geq \text{weight } t$   
**proof** (*induct ts arbitrary: t*)  
 case Nil **thus** case **by** *simp*  
**next**  
 case (Cons u us) **thus** case **by** *simp (metis le\_trans)*  
**qed**

**lemma** *sortedByWeight\_insortTree*:  
 $\llbracket \text{sortedByWeight } ts; \text{height } t = 0; \text{height}_F ts = 0 \rrbracket \implies$   
 $\text{sortedByWeight } (\text{insortTree } t ts)$   
**by** (*induct ts rule: sortedByWeight.induct*) *auto*

## 4.8 Pair of Minimal Symbols

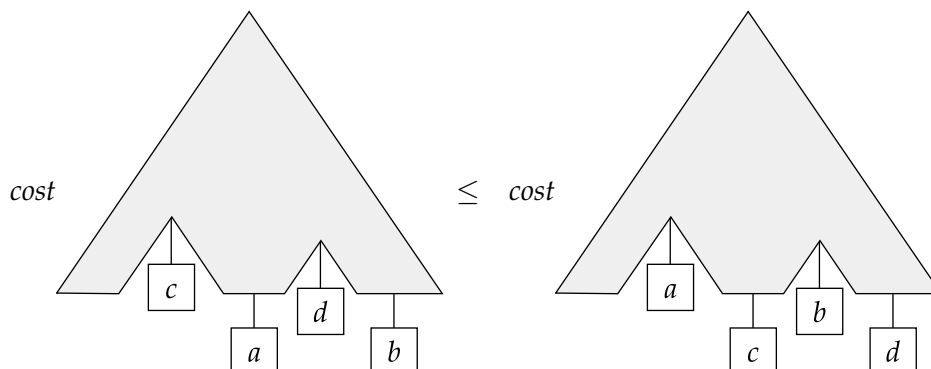
The *minima* predicate expresses that two symbols  $a, b \in \text{alphabet } t$  have the lowest frequencies in the tree  $t$  and that  $\text{freq } t a \leq \text{freq } t b$ . Minimal symbols need not be uniquely defined.

**definition** *minima* ::  $\alpha \text{ tree} \Rightarrow \alpha \Rightarrow \alpha \Rightarrow \text{bool}$  **where**  
 $\text{minima } t a b \equiv$   
 $a \in \text{alphabet } t \wedge b \in \text{alphabet } t \wedge a \neq b \wedge \text{freq } t a \leq \text{freq } t b$   
 $\wedge (\forall c \in \text{alphabet } t. c \neq a \longrightarrow c \neq b \longrightarrow$   
 $\text{freq } t c \geq \text{freq } t a \wedge \text{freq } t c \geq \text{freq } t b)$

## 5 Formalization of the Textbook Proof

### 5.1 Four-Way Symbol Interchange Cost Lemma

If  $a$  and  $b$  are minima, and  $c$  and  $d$  are at the very bottom of the tree, then exchanging  $a$  and  $b$  with  $c$  and  $d$  does not increase the cost. Graphically, we have



This cost property is part of Knuth's proof:

Let  $V$  be an internal node of maximum distance from the root. If  $w_1$  and  $w_2$  are not the weights already attached to the children of  $V$ , we can interchange them with the values that are already there; such an interchange does not increase the weighted path length.

Lemma 16.2 in Cormen et al. [4, p. 389] expresses a similar property, which turns out to be a corollary of our cost property:

Let  $C$  be an alphabet in which each character  $c \in C$  has frequency  $f[c]$ . Let  $x$  and  $y$  be two characters in  $C$  having the lowest frequencies. Then there exists an optimal prefix code for  $C$  in which the codewords for  $x$  and  $y$  have the same length and differ only in the last bit.

```

lemma cost_swapFourSyms_le:
assumes consistent t minima t a b c ∈ alphabet t d ∈ alphabet t
           depth t c = height t depth t d = height t c ≠ d
shows cost (swapFourSyms t a b c d) ≤ cost t
proof –
  note lems = swapFourSyms_def minima_def cost_swapSyms_le depth_le_height
  show thesis
  proof (cases a ≠ d ∧ b ≠ c)
    case True show thesis
    proof cases
      assume a = c show thesis
      proof cases
        assume b = d thus thesis using ⟨a = c⟩ True assms
          by (simp add: lems)
        next
          assume b ≠ d thus thesis using ⟨a = c⟩ True assms
            by (simp add: lems)
      qed
    next
      assume a ≠ c show thesis
      proof cases
        assume b = d thus thesis using ⟨a ≠ c⟩ True assms
          by (simp add: lems)
        next
          assume b ≠ d
          have cost (swapFourSyms t a b c d) ≤ cost (swapSyms t a c)
            using ⟨b ≠ d⟩ ⟨a ≠ c⟩ True assms by (clarsimp simp: lems)
          also have ... ≤ cost t using ⟨b ≠ d⟩ ⟨a ≠ c⟩ True assms
            by (clarsimp simp: lems)
          finally show thesis .
      qed

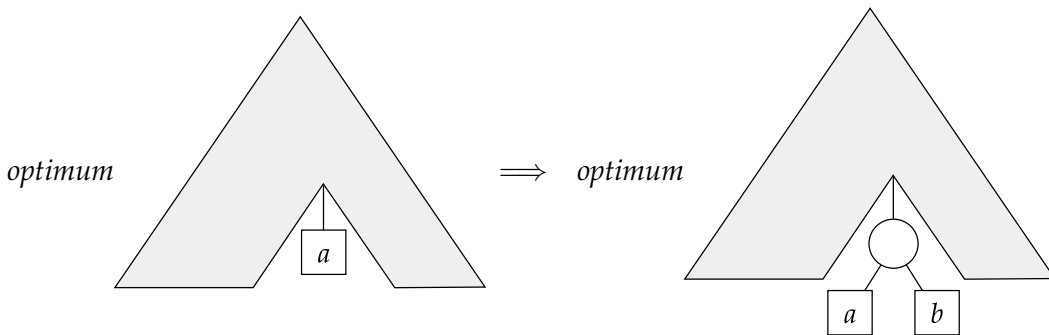
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qed  
 next  
 case False thus thesis using assms by (clarsimp simp: lems)  
 qed  
 qed

## 5.2 Leaf Split Optimality Lemma

The tree  $\text{splitLeaf } t \ w_a \ a \ w_b \ b$  is optimum if  $t$  is optimum, under a few assumptions, notably that  $a$  and  $b$  are minima of the new tree and that  $\text{freq } t \ a = w_a + w_b$ . Graphically:



This corresponds to the following fragment of Knuth's proof:

Now it is easy to prove that the weighted path length of such a tree is minimized if and only if the tree with

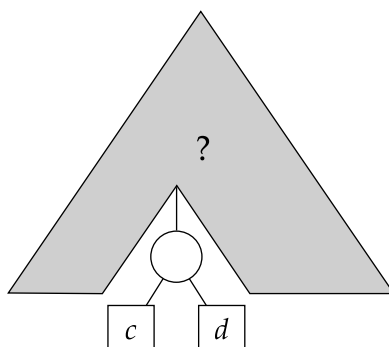


has minimum path length for the weights  $w_1 + w_2, w_3, \dots, w_m$ .

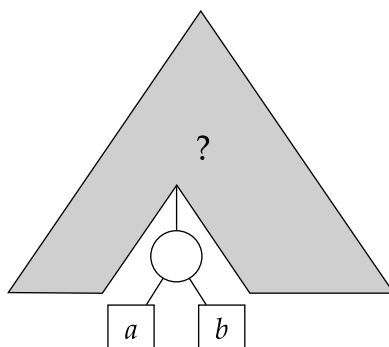
We only need the "if" direction of Knuth's equivalence. Lemma 16.3 in Cormen et al. [4, p. 391] expresses essentially the same property:

Let  $C$  be a given alphabet with frequency  $f[c]$  defined for each character  $c \in C$ . Let  $x$  and  $y$  be two characters in  $C$  with minimum frequency. Let  $C'$  be the alphabet  $C$  with characters  $x, y$  removed and (new) character  $z$  added, so that  $C' = C - \{x, y\} \cup \{z\}$ ; define  $f$  for  $C'$  as for  $C$ , except that  $f[z] = f[x] + f[y]$ . Let  $T'$  be any tree representing an optimal prefix code for the alphabet  $C'$ . Then the tree  $T$ , obtained from  $T'$  by replacing the leaf node for  $z$  with an internal node having  $x$  and  $y$  as children, represents an optimal prefix code for the alphabet  $C$ .

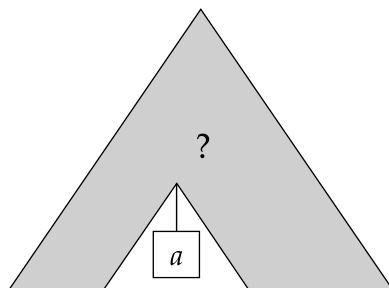
The proof is as follows: We assume that  $t$  has a cost less than or equal to that of any other comparable tree  $v$  and show that  $\text{splitLeaf } t \ w_a \ a \ w_b \ b$  has a cost less than or equal to that of any other comparable tree  $u$ . By  $\text{exists\_at\_height}$  and  $\text{depth\_height\_imp\_sibling\_ne}$ , we know that some symbols  $c$  and  $d$  appear in sibling nodes at the very bottom of  $u$ :



(The question mark is there to remind us that we know nothing specific about  $u$ 's structure.) From  $u$  we construct a new tree  $\text{swapFourSyms } u \ a \ b \ c \ d$  in which the minima  $a$  and  $b$  are siblings:



Merging  $a$  and  $b$  gives a tree comparable with  $t$ , which we can use to instantiate  $v$  in the assumption:



With this instantiation, the proof is easy:

$$\begin{aligned}
& \text{cost } (\text{splitLeaf } t \ a \ w_a \ b \ w_b) \\
= & \hspace{20em} (\text{cost\_splitLeaf}) \\
& \text{cost } t + w_a + w_b \\
\leq & \hspace{20em} (\text{assumption}) \\
& \text{cost } (\overbrace{\text{mergeSibling } (\text{swapFourSyms } u \ a \ b \ c \ d) \ a}^v) + w_a + w_b \\
= & \hspace{20em} (\text{cost\_mergeSibling}) \\
& \text{cost } (\text{swapFourSyms } u \ a \ b \ c \ d) \\
\leq & \hspace{20em} (\text{cost\_swapFourSyms\_le}) \\
& \text{cost } u.
\end{aligned}$$

In contrast, the proof in Cormen et al. is by contradiction: Essentially, they assume that there exists a tree  $u$  with a lower cost than  $\text{splitLeaf } t \ a \ w_a \ b \ w_b$  and show that there exists a tree  $v$  with a lower cost than  $t$ , contradicting the hypothesis that  $t$  is optimum. In place of  $\text{cost\_swapFourSyms\_le}$ , they invoke their lemma 16.2, which is questionable since  $u$  is not necessarily optimum.<sup>3</sup>

Our proof relies on the following lemma, which asserts that  $a$  and  $b$  are minima of  $u$ .

**lemma** *twice\_freq\_le\_imp\_minima*:

$\llbracket \forall c \in \text{alphabet } t. w_a \leq \text{freq } t \ c \wedge w_b \leq \text{freq } t \ c;$   
 $\text{alphabet } u = \text{alphabet } t \cup \{b\}; a \in \text{alphabet } u; a \neq b;$   
 $\text{freq } u = (\lambda c. \text{if } c = a \text{ then } w_a \text{ else if } c = b \text{ then } w_b \text{ else } \text{freq } t \ c);$   
 $w_a \leq w_b \rrbracket \implies$   
*minima*  $u \ a \ b$

**by** (*simp add: minima\_def*)

Now comes the key lemma.

**lemma** *optimum\_splitLeaf*:

**assumes** *consistent*  $t$  *optimum*  $t$   $a \in \text{alphabet } t$   $b \notin \text{alphabet } t$   
 $\text{freq } t \ a = w_a + w_b \ \forall c \in \text{alphabet } t. \text{freq } t \ c \geq w_a \wedge \text{freq } t \ c \geq w_b$   
 $w_a \leq w_b$

**shows** *optimum* ( $\text{splitLeaf } t \ w_a \ a \ w_b \ b$ )

**proof** (*unfold optimum\_def, clarify*)

**fix**  $u$

**let**  $t' = \text{splitLeaf } t \ w_a \ a \ w_b \ b$

**assume**  $c_u$ : *consistent*  $u$

**and**  $a_u$ :  $\text{alphabet } t' = \text{alphabet } u$

**and**  $f_u$ :  $\text{freq } t' = \text{freq } u$

**show**  $\text{cost } t' \leq \text{cost } u$

**proof** (*cases height*  $t' = 0$ )

---

<sup>3</sup>Thomas Cormen commented that this step will be clarified in the next edition of the book.

```

case True thus thesis by simp
next
case False
hence  $h_u$ : height  $u > 0$  using  $a_u$  assms
  by (auto intro: height_gt_0_alphabet_eq_imp_height_gt_0)
have  $a_a$ :  $a \in \text{alphabet } u$  using  $a_u$  assms by fastforce
have  $a_b$ :  $b \in \text{alphabet } u$  using  $a_u$  assms by fastforce
have  $ab$ :  $a \neq b$  using assms by blast
from exists_at_height [OF  $c_u$ ]
obtain  $c$  where  $a_c$ :  $c \in \text{alphabet } u$  and  $d_c$ : depth  $u \ c = \text{height } u ..$ 
let  $d = \text{sibling } u \ c$ 
have  $d_c$ :  $d \neq c$  using  $d_c \ c_u \ h_u \ a_c$  by (metis depth_height_imp_sibling_ne)
have  $a_d$ :  $d \in \text{alphabet } u$  using  $d_c$ 
  by (rule sibling_ne_imp_sibling_in_alphabet)
have  $d_d$ : depth  $u \ d = \text{height } u$  using  $d_c \ c_u$  by simp

let  $u' = \text{swapFourSyms } u \ a \ b \ c \ d$ 
have  $c_{u'}$ : consistent  $u'$  using  $c_u$  by simp
have  $a_{u'}$ : alphabet  $u' = \text{alphabet } u$  using  $a_a \ a_b \ a_c \ a_d \ a_u$  by simp
have  $f_{u'}$ : freq  $u' = \text{freq } u$  using  $a_a \ a_b \ a_c \ a_d \ c_u \ f_u$  by simp
have  $s_a$ : sibling  $u' \ a = b$  using  $c_u \ a_a \ a_b \ a_c \ ab \ dc$ 
  by (rule sibling_swapFourSyms_when_4th_is_sibling)

let  $v = \text{mergeSibling } u' \ a$ 
have  $c_v$ : consistent  $v$  using  $c_{u'}$  by simp
have  $a_v$ : alphabet  $v = \text{alphabet } t$  using  $s_a \ c_{u'} \ a_{u'} \ a_a \ a_u$  assms by auto
have  $f_v$ : freq  $v = \text{freq } t$ 
  using  $s_a \ c_{u'} \ a_{u'} \ f_{u'} \ f_u$  [THEN sym]  $ab \ a_u$  [THEN sym] assms
  by (simp add: freq_mergeSibling ext)

have cost  $t' = \text{cost } t + w_a + w_b$  using assms by simp
also have  $\dots \leq \text{cost } v + w_a + w_b$  using  $c_v \ a_v \ f_v$  (optimum  $t$ )
  by (simp add: optimum_def)
also have  $\dots = \text{cost } u'$ 
  proof –
    have cost  $v + \text{freq } u' \ a + \text{freq } u' \ (\text{sibling } u' \ a) = \text{cost } u'$ 
      using  $c_{u'} \ s_a$  assms by (subst cost_mergeSibling) auto
    moreover have  $w_a = \text{freq } u' \ a \ w_b = \text{freq } u' \ b$ 
      using  $f_{u'} \ f_u$  [THEN sym] assms by clarsimp+
    ultimately show thesis using  $s_a$  by simp
  qed
also have  $\dots \leq \text{cost } u$ 
  proof –
    have minima  $u \ a \ b$  using  $a_u \ f_u$  assms

```

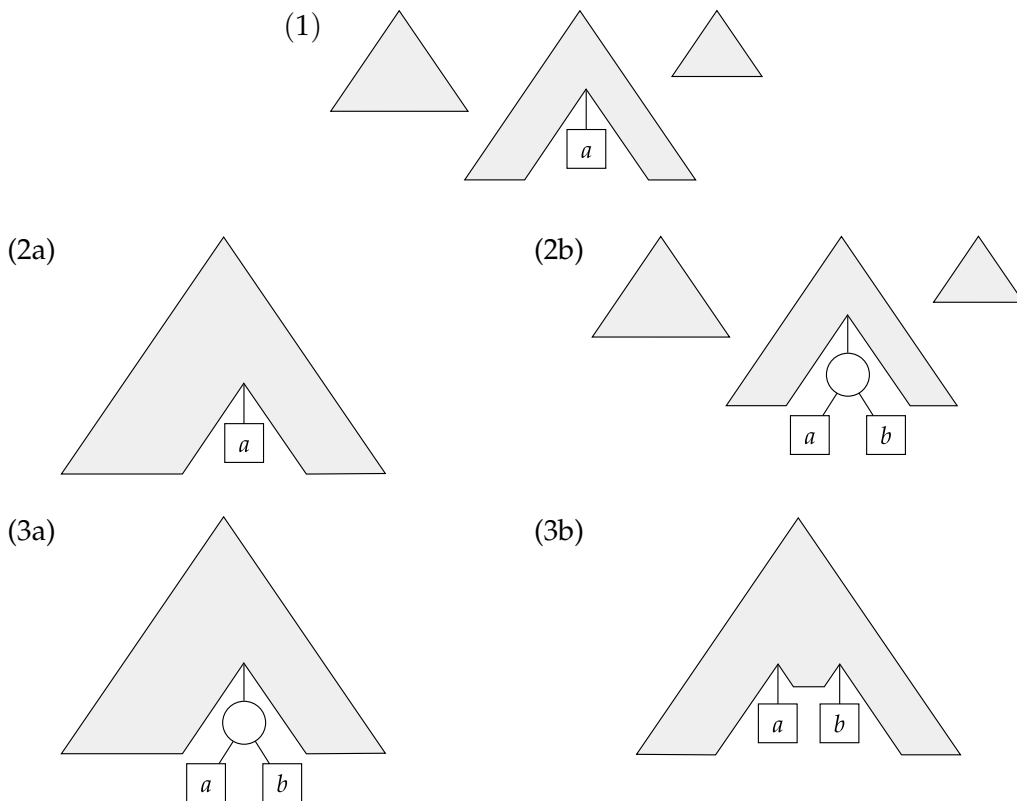
```

    by (subst twice_freq_le_imp_minima) auto
  with  $c_u$  show thesis using  $a_c a_d d_c d_d dc$  [THEN not_sym]
  by (rule cost_swapFourSyms_le)
qed
finally show thesis .
qed
qed

```

### 5.3 Leaf Split Commutativity Lemma

A key property of Huffman's algorithm is that once it has combined two lowest-weight trees using *uniteTrees*, it does not visit these trees ever again. This suggests that splitting a leaf node before applying the algorithm should give the same result as applying the algorithm first and splitting the leaf node afterward. The diagram below illustrates the situation:



From the original forest (1), we can either run the algorithm (2a) and then split  $a$  (3a) or split  $a$  (2b) and then run the algorithm (3b). Our goal is to show that trees (3a) and (3b) are identical. Formally, we prove that

$$\text{splitLeaf} (\text{huffman } ts) w_a a w_b b = \text{huffman} (\text{splitLeaf}_F ts w_a a w_b b)$$

when  $ts$  is consistent,  $a \in \text{alphabet}_F ts$ ,  $b \notin \text{alphabet}_F ts$ , and  $\text{freq}_F ts a = w_a + w_b$ . But before we can prove this commutativity lemma, we need to introduce a few simple lemmas.

**lemma** *cachedWeight\_splitLeaf [simp]:*  
 $\text{cachedWeight} (\text{splitLeaf } t \ w_a \ a \ w_b \ b) = \text{cachedWeight } t$   
**by** (*case\_tac*  $t$ ) *simp*<sup>+</sup>

**lemma** *splitLeaf\_F\_insortTree\_when\_in\_alphabet\_left [simp]:*  
 $\llbracket a \in \text{alphabet } t; \text{consistent } t; a \notin \text{alphabet}_F ts; \text{freq } t \ a = w_a + w_b \rrbracket \implies$   
 $\text{splitLeaf}_F (\text{insortTree } t \ ts) \ w_a \ a \ w_b \ b = \text{insortTree} (\text{splitLeaf } t \ w_a \ a \ w_b \ b) \ ts$   
**by** (*induct*  $ts$ ) *simp*<sup>+</sup>

**lemma** *splitLeaf\_F\_insortTree\_when\_in\_alphabet\_F\_tail [simp]:*  
 $\llbracket a \in \text{alphabet}_F ts; \text{consistent}_F ts; a \notin \text{alphabet } t; \text{freq}_F ts \ a = w_a + w_b \rrbracket \implies$   
 $\text{splitLeaf}_F (\text{insortTree } t \ ts) \ w_a \ a \ w_b \ b =$   
 $\text{insortTree } t (\text{splitLeaf}_F \ ts \ w_a \ a \ w_b \ b)$

**proof** (*induct*  $ts$ )  
**case** *Nil* **thus** *case by simp*  
**next**  
**case** (*Cons*  $u \ us$ ) **show** *case*  
**proof** (*cases*  $a \in \text{alphabet } u$ )  
**case** *True*  
**moreover** **hence**  $a \notin \text{alphabet}_F us$  **using** *Cons* **by** *auto*  
**ultimately** **show** *thesis using Cons by auto*  
**next**  
**case** *False* **thus** *thesis using Cons by simp*  
**qed**  
**qed**

We are now ready to prove the commutativity lemma.

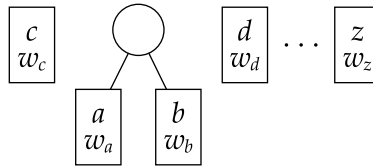
**lemma** *splitLeaf\_huffman\_commute:*  
 $\llbracket \text{consistent}_F ts; ts \neq []; a \in \text{alphabet}_F ts; \text{freq}_F ts \ a = w_a + w_b \rrbracket \implies$   
 $\text{splitLeaf} (\text{huffman } ts) \ w_a \ a \ w_b \ b = \text{huffman} (\text{splitLeaf}_F \ ts \ w_a \ a \ w_b \ b)$   
**proof** (*induct*  $ts$  *rule: huffman.induct*)  
— **BASE CASE 1:**  $ts = []$   
**case** 3 **thus** *case by simp*  
**next**  
— **BASE CASE 2:**  $ts = [t]$   
**case** (1  $t$ ) **thus** *case by simp*  
**next**  
— **INDUCTION STEP:**  $ts = t_1 \cdot t_2 \cdot ts$   
**case** (2  $t_1 \ t_2 \ ts$ )  
**note** *hyps = 2*

```

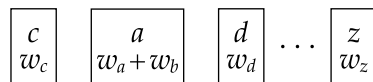
show case
proof (cases a ∈ alphabet t1)
  case True
    moreover hence a ∉ alphabet t2 a ∉ alphabetF ts using hyps by auto
    ultimately show thesis using hyps by (simp add: uniteTrees_def)
  next
    case False
      note a1 = False
      show thesis
      proof (cases a ∈ alphabet t2)
        case True
          moreover hence a ∉ alphabetF ts using hyps by auto
          ultimately show thesis using a1 hyps by (simp add: uniteTrees_def)
        next
          case False
            thus thesis using a1 hyps by simp
      qed
    qed
  qed

```

An important consequence of the commutativity lemma is that applying Huffman's algorithm on a forest of the form



gives the same result as applying the algorithm on the “flat” forest



followed by splitting the leaf node  $a$  into two nodes  $a, b$  with frequencies  $w_a, w_b$ . The lemma effectively provides a way to flatten the forest at each step of the algorithm. Its invocation is implicit in the textbook proof.

## 5.4 Optimality Theorem

We are one lemma away from our main result.

```

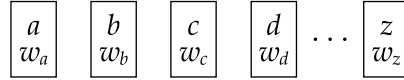
lemma max_0_imp_0 [simp]:
  (max x y = (0::nat)) = (x = 0 ∧ y = 0)
by auto

```

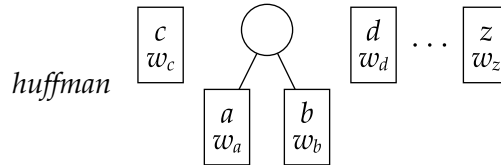
**theorem** *optimum\_huffman*:

$\llbracket \text{consistent}_F ts; \text{height}_F ts = 0; \text{sortedByWeight } ts; ts \neq [] \rrbracket \implies$   
*optimum (huffman ts)*

The input *ts* is assumed to be a nonempty consistent list of leaf nodes sorted by weight. The proof is by induction on the length of the forest *ts*. Let *ts* be



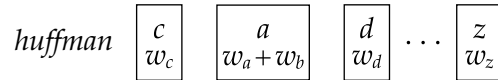
with  $w_a \leq w_b \leq w_c \leq w_d \leq \dots \leq w_z$ . If *ts* consists of a single leaf node, the node has cost 0 and is therefore optimum. If *ts* has length 2 or more, the first step of the algorithm leaves us with the term



In the diagram, we put the newly created tree at position 2 in the forest; in general, it could be anywhere. By *splitLeaf\_huffman\_commute*, the above tree equals

$$\text{splitLeaf} \left( \text{huffman} \begin{array}{c} c \\ w_c \end{array} \begin{array}{c} a \\ w_a + w_b \end{array} \begin{array}{c} d \\ w_d \end{array} \dots \begin{array}{c} z \\ w_z \end{array} \right) w_a a w_b b.$$

To prove that this tree is optimum, it suffices by *optimum\_splitLeaf* to show that



is optimum, which follows from the induction hypothesis.

**proof** (*induct ts rule: length\_induct*)

— COMPLETE INDUCTION STEP

**case** (1 *ts*)

**note** *hyps* = 1

**show** *case*

**proof** (*cases ts*)

**case** *Nil* **thus** *thesis* **using** (*ts*  $\neq$  []) **by** *fast*

**next**

**case** (*Cons t<sub>a</sub> ts'*)

**note** *ts* = *Cons*

**show** *thesis*

**proof** (*cases ts'*)



```

    case Nil thus thesis using ts hyps by (simp add: optimum_def)
next
  case (Cons tb ts'')
  note ts' = Cons
  show thesis
  proof (cases ta)
    case (Leaf wa a)
    note la = Leaf
    show thesis
    proof (cases tb)
      case (Leaf wb b)
      note lb = Leaf
      show thesis
      proof -
        let us = insertTree (uniteTrees ta tb) ts''
        let us' = insertTree (Leaf (wa + wb) a) ts''
        let ts = splitLeaf (huffman us') wa a wb b
        have e1: huffman ts = huffman us using ts' ts by simp
        have e2: huffman us = ts using la lb ts' ts hyps
          by (auto simp: splitLeaf_huffman_commute uniteTrees_def)

        have optimum (huffman us') using la ts' ts hyps
          by (drule_tac x = us' in spec)
            (auto dest: sortedByWeight_Cons_imp_sortedByWeight
              simp: sortedByWeight_insertTree)
        hence optimum ts using la lb ts' ts hyps
          apply simp
          apply (rule optimum_splitLeaf)
          by (auto dest!: height_F_0_imp_Leaf_freq_F_in_set
            sortedByWeight_Cons_imp_forall_weight_ge)
        thus optimum (huffman ts) using e1 e2 by simp
      qed
    next
      case InnerNode thus thesis using ts' ts hyps by simp
    qed
  next
    case InnerNode thus thesis using ts' ts hyps by simp
  qed
qed
qed
qed
qed
end

```

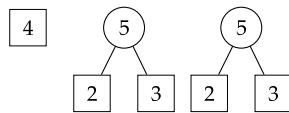
So what have we achieved? Assuming that our definitions really mean what

we intend them to mean, we established that our functional implementation of Huffman’s algorithm, when invoked properly, constructs a binary tree that represents an optimal prefix code for the specified alphabet and frequencies. Using Isabelle’s code generator [6], we can convert the Isabelle code into Standard ML, OCaml, or Haskell and use it in a real application.

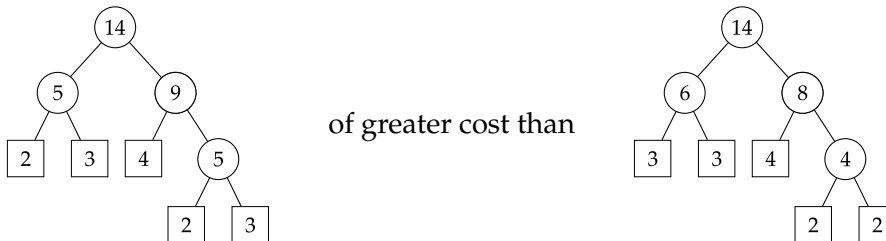
As a side note, the *optimum\_huffman* theorem assumes that the forest *ts* passed to *huffman* consists exclusively of leaf nodes. It is tempting to relax this restriction, by requiring instead that the forest *ts* has the lowest cost among forests of the same size. We would define optimality of a forest as follows:

$$\begin{aligned} \text{optimum}_F ts \equiv & (\forall us. \text{length } ts = \text{length } us \longrightarrow \text{consistent}_F us \longrightarrow \\ & \text{alphabet}_F ts = \text{alphabet}_F us \longrightarrow \text{freq}_F ts = \text{freq}_F us \longrightarrow \\ & \text{cost}_F ts \leq \text{cost}_F us) \end{aligned}$$

with  $\text{cost}_F [] = 0$  and  $\text{cost}_F (t \cdot ts) = \text{cost } t + \text{cost}_F ts$ . However, the modified proposition does not hold. A counterexample is the optimum forest



for which the algorithm constructs the tree



## 6 Related Work

Laurent Théry’s Coq formalization of Huffman’s algorithm [14, 15] is an obvious yardstick for our work. It has a somewhat wider scope, proving among others the isomorphism between prefix codes and full binary trees. With 291 theorems, it is also much larger.

Théry identified the following difficulties in formalizing the textbook proof:

1. The leaf interchange process that brings the two minimal symbols together is tedious to formalize.
2. The sibling merging process requires introducing a new symbol for the merged node, which complicates the formalization.

3. The algorithm constructs the tree in a bottom-up fashion. While top-down procedures can usually be proved by structural induction, bottom-up procedures often require more sophisticated induction principles and larger invariants.
4. The informal proof relies on the notion of depth of a node. Defining this notion formally is problematic, because the depth can only be seen as a function if the tree is composed of distinct nodes.

To circumvent these difficulties, Théry introduced the ingenious concept of cover. A forest  $ts$  is a *cover* of a tree  $t$  if  $t$  can be built from  $ts$  by adding inner nodes on top of the trees in  $ts$ . The term “cover” is easier to understand if the binary trees are drawn with the root at the bottom of the page, like natural trees. Huffman’s algorithm is a refinement of the cover concept. The main proof consists in showing that the cost of *huffman*  $ts$  is less than or equal to that of any other tree for which  $ts$  is a cover. It relies on a few auxiliary definitions, notably an “ordered cover” concept that facilitates structural induction and a four-argument depth predicate (confusingly called *height*). Permutations also play a central role.

Incidentally, our experience suggests that the potential problems identified by Théry can be overcome more directly without too much work, leading to a simpler proof:

1. Formalizing the leaf interchange did not prove overly tedious. Among our 95 lemmas and theorems, 24 concern *swapLeaves*, *swapSyms*, and *swapFourSyms*.
2. The generation of a new symbol for the resulting node when merging two sibling nodes in *mergeSibling* was trivially solved by reusing one of the two merged symbols.
3. The bottom-up nature of the tree construction process was addressed by using the length of the forest as the induction measure and by merging the two minimal symbols, as in Knuth’s proof.
4. By restricting our attention to consistent trees, we were able to define the *depth* function simply and meaningfully.

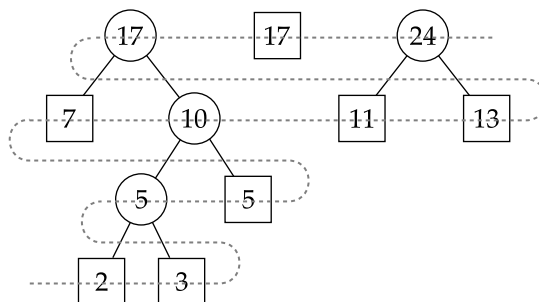
## 7 Conclusion

The goal of most formal proofs is to increase our confidence in a result. In the case of Huffman’s algorithm, however, the chances that a bug would have gone unnoticed for the 56 years since its publication, under the scrutiny of leading computer scientists, seem extremely low; and the existence of a Coq proof should be sufficient to remove any remaining doubts.

The main contribution of this document has been to demonstrate that the textbook proof of Huffman’s algorithm can be elegantly formalized using a state-of-the-art theorem prover such as Isabelle/HOL. In the process, we uncovered a few minor snags in the proof given in Cormen et al. [4].

We also found that custom induction rules, in combination with suitable simplification rules, greatly help the automatic proof tactics, sometimes reducing 30-line proof scripts to one-liners. We successfully applied this approach for handling both the ubiquitous “datatype + wellformedness predicate” combination ( $\alpha$  tree + consistent) and functions defined by sequential pattern matching (*sibling* and *mergeSibling*). Our experience suggests that such rules, which are uncommon in formalizations, are highly valuable and versatile. Moreover, Isabelle’s *induction\_schema* and *lexicographic\_order* tactics make these easy to prove.

Formalizing the proof of Huffman’s algorithm also led to a deeper understanding of this classic algorithm. Many of the lemmas, notably the leaf split commutativity lemma of Section 5.3, have not been found in the literature and express fundamental properties of the algorithm. Other discoveries did not find their way into the final proof. In particular, each step of the algorithm appears to preserve the invariant that the nodes in a forest are ordered by weight from left to right, bottom to top, as in the example below:



It is not hard to prove formally that a tree exhibiting this property is optimum. On the other hand, proving that the algorithm preserves this invariant seems difficult—more difficult than formalizing the textbook proof—and remains a suggestion for future work.

A few other directions for future work suggest themselves. First, we could formalize some of our hypotheses, notably our restriction to full and consistent binary trees. The current formalization says nothing about the algorithm’s application for data compression, so the next step could be to extend the proof’s scope to cover *encode/decode* functions and show that full binary trees are isomorphic to prefix codes, as done in the Coq development. Independently, we could generalize the development to  $n$ -ary trees.

## Acknowledgments

I am grateful to several people for their help in producing this report. Tobias Nipkow suggested that I cut my teeth on Huffman coding and discussed several (sometimes flawed) drafts of the proof. He also provided many insights into Isabelle, which led to considerable simplifications. Alexander Krauss answered all my Isabelle questions and helped me with the trickier proofs. Thomas Cormen and Donald Knuth were both gracious enough to discuss their proofs with me, and Donald Knuth also suggested a terminology change. Finally, Mark Summerfield and the anonymous reviewers of the corresponding journal paper proposed many textual improvements.

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