

The Hales–Jewett Theorem

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Abstract

This article is a formalisation of a proof of the Hales–Jewett theorem presented in the textbook *Ramsey Theory* by Graham et al. [1].

The Hales–Jewett theorem is a result in Ramsey Theory which states that, for any non-negative integers r and t , there exists a minimal dimension N , such that any r -coloured N' -dimensional cube over t elements (with $N' \geq N$) contains a monochromatic line. This theorem generalises Van der Waerden’s Theorem, which has already been formalised in another AFP entry [2].

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theory Hales-Jewett
  imports Main HOL-Library.Disjoint-Sets HOL-Library.FuncSet
begin

```

1 Preliminaries

The Hales–Jewett Theorem is at its core a statement about sets of tuples called the n -dimensional cube over t elements (denoted by C_t^n); i.e. the set $\{0, \dots, t-1\}^n$, where $\{0, \dots, t-1\}$ is called the base. We represent tuples by functions $f : \{0, \dots, n-1\} \rightarrow \{0, \dots, t-1\}$ because they’re easier to deal with. The set of tuples then becomes the function space $\{0, \dots, t-1\}^{\{0, \dots, n-1\}}$. Furthermore, r -colourings of the cube are represented by mappings from the function space to the set $\{0, \dots, r-1\}$.

1.1 The n -dimensional cube over t elements

Function spaces in Isabelle are supported by the library component FuncSet. In essence, $f \in A \rightarrow_E B$ means $a \in A \implies f a \in B$ and $a \notin A \implies f a = \text{undefined}$

The (canonical) n -dimensional cube over t elements is defined in the following using the variables:

```

n:  nat  dimension
t:  nat  number of elements

```

definition *cube* :: *nat* \Rightarrow *nat* \Rightarrow (*nat* \Rightarrow *nat*) *set*
where *cube* *n* *t* \equiv $\{..<n\} \rightarrow_E \{..<t\}$

For any function f whose image under a set A is a subset of another set B , there’s a unique function g in the function space B^A that equals f everywhere in A . The function g is usually written as $f|_A$ in the mathematical literature.

lemma *PiE-uniqueness*: $f \text{ ‘ } A \subseteq B \implies \exists! g \in A \rightarrow_E B. \forall a \in A. g a = f a$
<proof>

Any prefix of length j of an n -tuple (i.e. element of C_t^n) is a j -tuple (i.e. element of C_t^j).

lemma *cube-restrict*:
assumes $j < n$
and $y \in \text{cube } n \ t$
shows $(\lambda g \in \{..<j\}. y \ g) \in \text{cube } j \ t$ *<proof>*

Narrowing down the obvious fact $B^A \subseteq C^A$ if $B \subseteq C$ to a specific case for cubes.

lemma *cube-subset*: $\text{cube } n \ t \subseteq \text{cube } n \ (t + 1)$

<proof>

A simplifying definition for the 0-dimensional cube.

lemma *cube0-alt-def*: $\text{cube } 0 \ t = \{\lambda x. \text{undefined}\}$
<proof>

The cardinality of the n -dimensional over t elements is simply a consequence of the overarching definition of the cardinality of function spaces (over finite sets).

lemma *cube-card*: $\text{card} (\{..<n::\text{nat}\} \rightarrow_E \{..<t::\text{nat}\}) = t \wedge n$
<proof>

A simplifying definition for the n -dimensional cube over a single element, i.e. the single n -dimensional point $(0, \dots, 0)$.

lemma *cube1-alt-def*: $\text{cube } n \ 1 = \{\lambda x \in \{..<n\}. 0\}$ *<proof>*

1.2 Lines

The property of being a line in C_t^n is defined in the following using the variables:

L : $\text{nat} \Rightarrow \text{nat} \Rightarrow \text{nat}$ line
 n : nat dimension of cube
 t : nat the size of the cube's base

definition *is-line* :: $(\text{nat} \Rightarrow (\text{nat} \Rightarrow \text{nat})) \Rightarrow \text{nat} \Rightarrow \text{nat} \Rightarrow \text{bool}$

where *is-line* $L \ n \ t \equiv (L \in \{..<t\} \rightarrow_E \text{cube } n \ t \wedge$
 $((\forall j < n. (\forall x < t. \forall y < t. L \ x \ j = L \ y \ j) \vee (\forall s < t. L \ s \ j = s))$
 $\wedge (\exists j < n. (\forall s < t. L \ s \ j = s))))$

We introduce an elimination rule to relate lines with the more general definition of a subspace (see below).

lemma *is-line-elim-t-1*:

assumes *is-line* $L \ n \ t$ **and** $t = 1$

obtains $B_0 \ B_1$

where $B_0 \cup B_1 = \{..<n\} \wedge B_0 \cap B_1 = \{\}$ \wedge
 $B_0 \neq \{\} \wedge (\forall j \in B_1. (\forall x < t. \forall y < t. L \ x \ j = L \ y$
 $j)) \wedge (\forall j \in B_0. (\forall s < t. L \ s \ j = s))$

<proof>

The next two lemmas are used to simplify proofs by enabling us to use the resulting facts directly. This avoids having to unfold the definition of *is-line* each time.

lemma *line-points-in-cube*:

assumes *is-line* $L \ n \ t$

and $s < t$

shows $L \ s \in \text{cube } n \ t$

<proof>

lemma *line-points-in-cube-unfolded*:

assumes *is-line* L n t

and $s < t$

and $j < n$

shows L s $j \in \{..<t\}$

<proof>

The incrementation of all elements of a set is defined in the following using the variables:

n : *nat* increment size

S : *nat set* set

definition *set-incr* :: $nat \Rightarrow nat\ set \Rightarrow nat\ set$

where

$set-incr\ n\ S \equiv (\lambda a. a + n) ' S$

lemma *set-incr-disjnt*:

assumes *disjnt* A B

shows *disjnt* ($set-incr\ n\ A$) ($set-incr\ n\ B$)

<proof>

lemma *set-incr-disjoint-family*:

assumes *disjoint-family-on* B $\{..k\}$

shows *disjoint-family-on* $(\lambda i. set-incr\ n\ (B\ i))$ $\{..k\}$

<proof>

lemma *set-incr-altdef*: $set-incr\ n\ S = (+)\ n\ ' S$

<proof>

lemma *set-incr-image*:

assumes $(\bigcup i \in \{..k\}. B\ i) = \{..<n\}$

shows $(\bigcup i \in \{..k\}. set-incr\ m\ (B\ i)) = \{m..<m+n\}$

<proof>

Each tuple of dimension $k + 1$ can be split into a tuple of dimension 1 (the first entry) and a tuple of dimension k (the remaining entries).

lemma *split-cube*:

assumes $x \in cube\ (k+1)\ t$

shows $(\lambda y \in \{..<1\}. x\ y) \in cube\ 1\ t$

and $(\lambda y \in \{..<k\}. x\ (y + 1)) \in cube\ k\ t$

<proof>

1.3 Subspaces

The property of being a k -dimensional subspace of C_t^n is defined in the following using the variables:

S : $(nat \Rightarrow nat) \Rightarrow nat \Rightarrow nat$ the subspace
 k : nat the dimension of the subspace
 n : nat the dimension of the cube
 t : nat the size of the cube's base

definition *is-subspace*

where *is-subspace* $S k n t \equiv (\exists B f. disjoint-family-on B \{..k\} \wedge \bigcup (B \text{ ‘ } \{..k\}) = \{..<n\} \wedge (\{ \} \notin B \text{ ‘ } \{..<k\}) \wedge f \in (B k) \rightarrow_E \{..<t\} \wedge S \in (cube k t) \rightarrow_E (cube n t) \wedge (\forall y \in cube k t. (\forall i \in B k. S y i = f i) \wedge (\forall j < k. \forall i \in B j. (S y) i = y j)))$

A k -dimensional subspace of C_t^n can be thought of as an embedding of the C_t^k into C_t^n , akin to how a k -dimensional vector subspace of \mathbf{R}^n may be thought of as an embedding of \mathbf{R}^k into \mathbf{R}^n .

lemma *subspace-inj-on-cube*:

assumes *is-subspace* $S k n t$

shows *inj-on* $S (cube k t)$

<proof>

The following is required to handle base cases in the key lemmas.

lemma *dim0-subspace-ex*:

assumes $t > 0$

shows $\exists S. is-subspace S 0 n t$

<proof>

1.4 Equivalence classes

Defining the equivalence classes of *cube* $n (t + 1)$: $\{classes n t 0, \dots, classes n t n\}$

definition *classes*

where *classes* $n t \equiv (\lambda i. \{x . x \in (cube n (t + 1)) \wedge (\forall u \in \{(n-i)..<n\}. x u = t) \wedge t \notin x \text{ ‘ } \{..<(n - i)\}\})$

lemma *classes-subset-cube*: $classes n t i \subseteq cube n (t+1)$ *<proof>*

definition *layered-subspace*

where *layered-subspace* $S k n t r \chi \equiv (is-subspace S k n (t + 1) \wedge (\forall i \in \{..k\}. \exists c < r. \forall x \in classes k t i. \chi (S x) = c) \wedge \chi \in (cube n (t + 1) \rightarrow_E \{..<r\})$

lemma *layered-eq-classes*:

assumes *layered-subspace* $S k n t r \chi$

shows $\forall i \in \{..k\}. \forall x \in classes k t i. \forall y \in classes k t i.$

$\chi (S x) = \chi (S y)$

<proof>

lemma *dim0-layered-subspace-ex*:

assumes $\chi \in (cube n (t + 1)) \rightarrow_E \{..<r::nat\}$

shows $\exists S. \text{layered-subspace } S (0::\text{nat}) n t r \chi$
 $\langle \text{proof} \rangle$

lemma *disjoint-family-onI* [intro]:
assumes $\bigwedge m n. m \in S \implies n \in S \implies m \neq n$
 $\implies A m \cap A n = \{\}$
shows *disjoint-family-on* $A S$
 $\langle \text{proof} \rangle$

lemma *fun-ex*: $a \in A \implies b \in B \implies \exists f \in A$
 $\rightarrow_E B. f a = b$
 $\langle \text{proof} \rangle$

lemma *ex-bij-betw-nat-finite-2*:
assumes $\text{card } A = n$
and $n > 0$
shows $\exists f. \text{bij-betw } f A \{..<n\}$
 $\langle \text{proof} \rangle$

lemma *one-dim-cube-eq-nat-set*: $\text{bij-betw } (\lambda f. f 0) (\text{cube } 1 k) \{..<k\}$
 $\langle \text{proof} \rangle$

An alternative introduction rule for the $\exists!x$ quantifier, which means "there exists exactly one x ".

lemma *ex1I-alt*: $(\exists x. P x \wedge (\forall y. P y \longrightarrow x = y)) \implies (\exists!x. P x)$
 $\langle \text{proof} \rangle$

lemma *nat-set-eq-one-dim-cube*: $\text{bij-betw } (\lambda x. \lambda y \in \{..<1::\text{nat}\}. x) \{..<k::\text{nat}\} (\text{cube } 1 k)$
 $\langle \text{proof} \rangle$

A bijection f between domains A_1 and A_2 creates a correspondence between functions in $A_1 \rightarrow B$ and $A_2 \rightarrow B$.

lemma *bij-domain-PiE*:
assumes $\text{bij-betw } f A1 A2$
and $g \in A2 \rightarrow_E B$
shows $(\text{restrict } (g \circ f) A1) \in A1 \rightarrow_E B$
 $\langle \text{proof} \rangle$

The following three lemmas relate lines to 1-dimensional subspaces (in the natural way). This is a direct consequence of the elimination rule *is-line-elim* introduced above.

lemma *line-is-dim1-subspace-t-1*:
assumes $n > 0$
and $\text{is-line } L n 1$
shows $\text{is-subspace } (\text{restrict } (\lambda y. L (y 0)) (\text{cube } 1 1)) 1 n 1$
 $\langle \text{proof} \rangle$

lemma *line-is-dim1-subspace-t-ge-1*:

assumes $n > 0$
and $t > 1$
and *is-line* $L n t$
shows *is-subspace* (*restrict* ($\lambda y. L (y 0)$) (*cube* 1 t)) 1 $n t$
 ⟨*proof*⟩

lemma *line-is-dim1-subspace*:

assumes $n > 0$
and $t > 0$
and *is-line* $L n t$
shows *is-subspace* (*restrict* ($\lambda y. L (y 0)$) (*cube* 1 t)) 1 $n t$
 ⟨*proof*⟩

The key property of the existence of a minimal dimension N , such that for any r -colouring in $C_t^{N'}$ (for $N' \geq N$) there exists a monochromatic line is defined in the following using the variables:

r : *nat* the number of colours
 t : *nat* the size of of the base

definition *hj*

where $hj\ r\ t \equiv (\exists N > 0. \forall N' \geq N. \forall \chi. \chi \in (\textit{cube}\ N')$
 $t) \rightarrow_E \{..<r::nat\} \rightarrow (\exists L. \exists c < r. \textit{is-line}\ L\ N'\ t$
 $\wedge (\forall y \in L\ ' \{..<t\}. \chi\ y = c))$

The key property of the existence of a minimal dimension N , such that for any r -colouring in $C_t^{N'}$ (for $N' \geq N$) there exists a layered subspace of dimension k is defined in the following using the variables:

r : *nat* the number of colours
 t : *nat* the size of of the base
 k : *nat* the dimension of the subspace

definition *lhj*

where $lhj\ r\ t\ k \equiv (\exists N > 0. \forall N' \geq N. \forall \chi. \chi \in$
 $(\textit{cube}\ N'\ (t + 1)) \rightarrow_E \{..<r::nat\} \rightarrow (\exists S.$
 $\textit{layered-subspace}\ S\ k\ N'\ t\ r\ \chi))$

We state some useful facts about 1-dimensional subspaces.

lemma *dim1-subspace-elims*:

assumes *disjoint-family-on* $B\ \{..1::nat\}$ **and** $\bigcup (B\ ' \{..1::nat\}) = \{..<n\}$ **and**
 $(\{}$
 $\notin B\ ' \{..<1::nat\})$ **and** $f \in (B\ 1) \rightarrow_E \{..<t\}$ **and** $S \in (\textit{cube}\ 1$
 $t) \rightarrow_E (\textit{cube}\ n\ t)$ **and** $(\forall y \in \textit{cube}\ 1\ t. (\forall i \in B\ 1. S\ y\ i$
 $= f\ i) \wedge (\forall j < 1. \forall i \in B\ j. (S\ y)\ i = y\ j))$
shows $B\ 0 \cup B\ 1 = \{..<n\}$
and $B\ 0 \cap B\ 1 = \{}$
and $(\forall y \in \textit{cube}\ 1\ t. (\forall i \in B\ 1. S\ y\ i = f\ i) \wedge (\forall i \in B\ 0. (S\ y)\ i = y\ 0))$
and $B\ 0 \neq \{}$
 ⟨*proof*⟩

We state some properties of cubes.

lemma *cube-props*:

assumes $s < t$

shows $\exists p \in \text{cube } 1 \ t. \ p \ 0 = s$

and $(\text{SOME } p. \ p \in \text{cube } 1 \ t \wedge p \ 0 = s) \ 0 = s$

and $(\lambda s \in \{..<t\}. \ S \ (\text{SOME } p. \ p \in \text{cube } 1 \ t \wedge p \ 0 = s)) \ s =$

$(\lambda s \in \{..<t\}. \ S \ (\text{SOME } p. \ p \in \text{cube } 1 \ t \wedge p \ 0 = s)) \ ((\text{SOME } p. \ p \in \text{cube } 1 \ t$

$\wedge p \ 0 = s) \ 0)$

and $(\text{SOME } p. \ p \in \text{cube } 1 \ t \wedge p \ 0 = s) \in \text{cube } 1 \ t$

<proof>

The following lemma relates 1-dimensional subspaces to lines, thus establishing a bidirectional correspondence between the two together with *line-is-dim1-subspace*.

lemma *dim1-subspace-is-line*:

assumes $t > 0$

and *is-subspace* $S \ 1 \ n \ t$

shows *is-line* $(\lambda s \in \{..<t\}. \ S \ (\text{SOME } p. \ p \in \text{cube } 1 \ t \wedge p \ 0 = s)) \ n \ t$

<proof>

lemma *bij-unique-inv*:

assumes *bij-betw* $f \ A \ B$

and $x \in B$

shows $\exists! y \in A. \ (\text{the-inv-into } A \ f) \ x = y$

<proof>

lemma *inv-into-cube-props*:

assumes $s < t$

shows *the-inv-into* $(\text{cube } 1 \ t) \ (\lambda f. \ f \ 0) \ s \in \text{cube } 1 \ t$

and *the-inv-into* $(\text{cube } 1 \ t) \ (\lambda f. \ f \ 0) \ s \ 0 = s$

<proof>

lemma *some-inv-into*:

assumes $s < t$

shows $(\text{SOME } p. \ p \in \text{cube } 1 \ t \wedge p \ 0 = s) = (\text{the-inv-into } (\text{cube } 1 \ t) \ (\lambda f. \ f \ 0) \ s)$

<proof>

lemma *some-inv-into-2*:

assumes $s < t$

shows $(\text{SOME } p. \ p \in \text{cube } 1 \ (t+1) \wedge p \ 0 = s) = (\text{the-inv-into } (\text{cube } 1 \ t) \ (\lambda f. \ f \ 0) \ s)$

<proof>

lemma *dim1-layered-subspace-as-line*:

assumes $t > 0$

and *layered-subspace* $S \ 1 \ n \ t \ r \ \chi$

shows $\exists c1 \ c2. \ c1 < r \wedge c2 < r \wedge (\forall s < t. \ \chi \ (S \ (\text{SOME } p. \ p \in \text{cube } 1$

$(t+1) \wedge p \ 0 = s)) = c1) \wedge \chi \ (S \ (\text{SOME } p. \ p \in \text{cube } 1 \ (t+1) \wedge p \ 0 = t)) = c2$

<proof>

lemma *dim1-layered-subspace-mono-line*:

assumes $t > 0$
and *layered-subspace* $S\ 1\ n\ t\ r\ \chi$
shows $\forall s < t. \forall l < t. \chi (S (SOME\ p. p \in cube\ 1\ (t+1) \wedge p\ 0 = s)) =$
 $\chi (S (SOME\ p. p \in cube\ 1\ (t+1) \wedge p\ 0 = l)) \wedge \chi (S (SOME\ p. p \in cube\ 1$
 $(t+1) \wedge p\ 0 = s)) < r$
 $\langle proof \rangle$

definition $join :: (nat \Rightarrow 'a) \Rightarrow (nat \Rightarrow 'a) \Rightarrow nat$
 $\Rightarrow nat \Rightarrow (nat \Rightarrow 'a)$

where

$join\ f\ g\ n\ m \equiv (\lambda x. \text{if } x \in \{..<n\} \text{ then } f\ x \text{ else (if } x \in \{n..<n+m\} \text{ then } g$
 $(x - n) \text{ else undefined}))$

lemma *join-cubes*:

assumes $f \in cube\ n\ (t+1)$
and $g \in cube\ m\ (t+1)$
shows $join\ f\ g\ n\ m \in cube\ (n+m)\ (t+1)$
 $\langle proof \rangle$

lemma *subspace-elems-embed*:

assumes *is-subspace* $S\ k\ n\ t$
shows $S\ ' (cube\ k\ t) \subseteq cube\ n\ t$
 $\langle proof \rangle$

2 Core proofs

The numbering of the theorems has been borrowed from the textbook [1].

2.1 Theorem 4

2.1.1 Base case of Theorem 4

lemma *hj-imp-lhj-base*:

fixes $r\ t$
assumes $t > 0$
and $\bigwedge r'. hj\ r'\ t$
shows $lhj\ r\ t\ 1$
 $\langle proof \rangle$

2.1.2 Induction step of theorem 4

The proof has four parts:

1. We obtain two layered subspaces of dimension 1 and k (respectively), whose existence is guaranteed by the assumption *lhj* (i.e. the induction hypothesis). Additionally, we prove some useful facts about these.
2. We construct a $k+1$ -dimensional subspace with the goal of showing that it is layered.

3. We prove that our construction is a subspace in the first place.
4. We prove that it is a layered subspace.

lemma *hj-imp-lhj-step*:

fixes $r\ k$
assumes $t > 0$
and $k \geq 1$
and *True*
and $(\bigwedge r\ k'. k' \leq k \implies \text{lhj } r\ t\ k')$
and $r > 0$
shows $\text{lhj } r\ t\ (k+1)$
 $\langle \text{proof} \rangle$

Part 1: Obtaining the subspaces L and S

Recall that *lhj* claims the existence of a layered subspace for any colouring (of a fixed size, where the size of a colouring refers to the number of colours). Therefore, the colourings have to be defined first, before the layered subspaces can be obtained. The colouring χL here is χ^* in the book [1], an *s*-colouring; see the fact *s-coloured* a couple of lines below.

$\langle \text{proof} \rangle$

Part 2: Constructing the $(k + 1)$ -dimensional subspace T

Below, *Tset* is the set as defined in the book [1]. It represents the $(k + 1)$ -dimensional subspace. In this construction, subspaces (e.g. T) are functions whose image is a set. See the fact *im-T-eq-Tset* below.

Having obtained our subspaces S and L , we define the $(k + 1)$ -dimensional subspace very straightforwardly. Namely, $T = L \times S$. Since we represent tuples by function sets, we need an appropriate operator that mirrors the Cartesian product \times for these. We call this *join* and define it for elements of a function set.

$\langle \text{proof} \rangle$

Part 3: Proving that T is a subspace

To prove something is a subspace, we have to provide the B and f satisfying the subspace properties. We construct BT and fT from BS , fS and BL , fL , which correspond to the k -dimensional subspace S and the 1-dimensional subspace (i.e. line) L , respectively.

$\langle \text{proof} \rangle$

Part 4: Proving T is layered

The following redefinition of the classes makes proving the layered property easier.

<proof>

theorem *hj-imp-lhj*:

fixes k

assumes $\wedge r'. \text{hj } r' t$

shows $\text{lhj } r t k$

<proof>

2.2 Theorem 5

We provide a way to construct a monochromatic line in C_{t+1}^n from a k -dimensional k -coloured layered subspace S in C_{t+1}^n . The idea is to rely on the fact that there are $k + 1$ classes in S , but only k colours. It thus follows from the Pigeonhole Principle that two classes must share the same colour. The way classes are defined allows for a straightforward construction of a line with points only from those two classes. Thus we have our monochromatic line.

theorem *layered-subspace-to-mono-line*:

assumes *layered-subspace* $S k n t k \chi$

and $t > 0$

shows $(\exists L. \exists c < k. \text{is-line } L n (t+1) \wedge (\forall y \in L. \{..<t+1\}. \chi y = c))$

<proof>

2.3 Corollary 6

corollary *lhj-imp-hj*:

assumes $(\wedge r k. \text{lhj } r t k)$

and $t > 0$

shows $(\text{hj } r (t+1))$

<proof>

2.4 Main result

2.4.1 Edge cases and auxiliary lemmas

lemma *single-point-line*:

assumes $N > 0$

shows *is-line* $(\lambda s \in \{..<1\}. \lambda a \in \{..<N\}. 0) N 1$

<proof>

lemma *single-point-line-is-monochromatic*:

assumes $\chi \in \text{cube } N 1 \rightarrow_E \{..<r\} N > 0$

shows $(\exists c < r. \text{is-line } (\lambda s \in \{..<1\}. \lambda a \in \{..<N\}. 0) N 1 \wedge (\forall i \in$

$\{..<1\}. \lambda a \in \{..<N\}. 0) \{..<1\}. \chi i = c))$

<proof>

lemma *hj-r-nonzero-t-0*:

assumes $r > 0$

shows $hj\ r\ 0$
<proof>

Any cube over 1 element always has a single point, which also forms the only line in the cube. Since it's a single point line, it's trivially monochromatic. We show the result for dimension 1.

lemma *hj-t-1*: $hj\ r\ 1$
<proof>

2.4.2 Main theorem

We state the main result $hj\ r\ t$. The explanation for the choice of assumption is offered subsequently.

theorem *hales-jewett*:
assumes $\neg(r = 0 \wedge t = 0)$
shows $hj\ r\ t$
<proof>

We offer a justification for having excluded the special case $r = t = 0$ from the statement of the main theorem *hales-jewett*. The exclusion is a consequence of the fact that colourings are defined as members of the function set $cube\ n\ t \rightarrow_E \{..<r\}$, which for $r = t = 0$ means there's a dummy colouring $\lambda x. undefined$, even though $cube\ n\ 0 = \{\}$ for $n > 0$. Hence, in this case, no line exists at all (let alone one monochromatic under the aforementioned colouring). This means $hj\ 0\ 0 \implies False$ —but only because of the quirky behaviour of the $FuncSet\ cube\ n\ t \rightarrow_E \{..<r\}$. This could have been circumvented by letting colourings χ be arbitrary functions constraint only by $\chi\ `cube\ n\ t \subseteq \{..<r\}$. We avoided this in order to have consistency with the cube's definition, for which $FuncSets$ were crucial because the proof heavily relies on arguments about the cardinality of the cube. The constraint $x\ ` \{..<n\} \subseteq \{..<t\}$ for elements x of C_t^n would not have sufficed there, as there are infinitely many functions over the naturals satisfying it.

end

References

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