# The Hales–Jewett Theorem

Ujkan Sulejmani, Manuel Eberl, Katharina Kreuzer

March 17, 2025

**Abstract**

This article is a formalisation of a proof of the Hales–Jewett theorem presented in the textbook *Ramsey Theory* by Graham et al. [1].

The Hales–Jewett theorem is a result in Ramsey Theory which states that, for any non-negative integers $r$ and $t$, there exists a minimal dimension $N$, such that any $r$-coloured $N'$-dimensional cube over $t$ elements (with $N' \geq N$) contains a monochromatic line. This theorem generalises Van der Waerden's Theorem, which has already been formalised in another AFP entry [2].

# Contents

**theory** *Hales-Jewett*
  **imports** *Main HOL−Library.Disjoint-Sets HOL−Library.FuncSet*
**begin**

# 1 Preliminaries

The Hales–Jewett Theorem is at its core a statement about sets of tuples called the $n$-dimensional cube over $t$ elements (denoted by $C_t^n$); i.e. the set $\{0, \ldots, t-1\}^n$, where $\{0, \ldots, t-1\}$ is called the base. We represent tuples by functions $f : \{0, \ldots, n-1\} \to \{0, \ldots, t-1\}$ because they're easier to deal with. The set of tuples then becomes the function space $\{0, \ldots, t-1\}^{\{0,\ldots,n-1\}}$. Furthermore, $r$-colourings of the cube are represented by mappings from the function space to the set $\{0, \ldots, r-1\}$.

## 1.1 The $n$-dimensional cube over $t$ elements

Function spaces in Isabelle are supported by the library component FuncSet. In essence, $f \in A \to_E B$ means $a \in A \Longrightarrow f\ a \in B$ and $a \notin A \Longrightarrow f\ a =$ *undefined*

The (canonical) $n$-dimensional cube over $t$ elements is defined in the following using the variables:

| | | |
|---|---|---|
| $n$: | *nat* | dimension |
| $t$: | *nat* | number of elements |

**definition** *cube* :: *nat* ⇒ *nat* ⇒ (*nat* ⇒ *nat*) *set*
  **where** *cube n t* ≡ {..<*n*} $\to_E$ {..<*t*}

For any function $f$ whose image under a set $A$ is a subset of another set $B$, there's a unique function $g$ in the function space $B^A$ that equals $f$ everywhere in $A$. The function $g$ is usually written as $f|_A$ in the mathematical literature.

**lemma** *PiE-uniqueness*: *f* ‘ *A* ⊆ *B* ⟹ ∃!*g* ∈ *A*
$\to_E$ *B*. ∀ *a*∈*A*. *g a* = *f a*
  ⟨*proof*⟩

Any prefix of length $j$ of an $n$-tuple (i.e. element of $C_t^n$) is a $j$-tuple (i.e. element of $C_t^j$).

**lemma** *cube-restrict*:
  **assumes** *j* < *n*
    **and** *y* ∈ *cube n t*
  **shows** (λ*g* ∈ {..<*j*}. *y g*) ∈ *cube j t* ⟨*proof*⟩

Narrowing down the obvious fact $B^A \subseteq C^A$ if $B \subseteq C$ to a specific case for cubes.

**lemma** *cube-subset*: *cube n t* ⊆ *cube n* (*t* + *1*)

⟨*proof*⟩

A simplifying definition for the 0-dimensional cube.

**lemma** *cube0-alt-def*: $cube\ 0\ t = \{\lambda x.\ undefined\}$
⟨*proof*⟩

The cardinality of the $n$-dimensional over $t$ elements is simply a consequence of the overarching definition of the cardinality of function spaces (over finite sets).

**lemma** *cube-card*: $card\ (\{..<n::nat\} \rightarrow_E \{..<t::nat\}) = t\ \hat{}\ n$
⟨*proof*⟩

A simplifying definition for the $n$-dimensional cube over a single element, i.e. the single $n$-dimensional point $(0, \ldots, 0)$.

**lemma** *cube1-alt-def*: $cube\ n\ 1 = \{\lambda x \in \{..<n\}.\ 0\}$ ⟨*proof*⟩

## 1.2 Lines

The property of being a line in $C_t^n$ is defined in the following using the variables:

| | | |
|---|---|---|
| $L$: | $nat \Rightarrow nat \Rightarrow nat$ | line |
| $n$: | $nat$ | dimension of cube |
| $t$: | $nat$ | the size of the cube's base |

**definition** *is-line* :: $(nat \Rightarrow (nat \Rightarrow nat)) \Rightarrow nat \Rightarrow$
$nat \Rightarrow bool$
**where** $is\text{-}line\ L\ n\ t \equiv (L \in \{..<t\} \rightarrow_E cube\ n\ t\ \wedge$
$((\forall j<n.\ (\forall x<t.\ \forall y<t.\ L\ x\ j = L\ y\ j) \vee (\forall s<t.\ L\ s\ j = s))$
$\wedge\ (\exists j < n.\ (\forall s < t.\ L\ s\ j = s))))$

We introduce an elimination rule to relate lines with the more general definition of a subspace (see below).

**lemma** *is-line-elim-t-1*:
**assumes** $is\text{-}line\ L\ n\ t$ **and** $t = 1$
**obtains** $B_0\ B_1$
**where** $B_0 \cup B_1 = \{..<n\} \wedge B_0 \cap B_1 = \{\} \wedge$
$B_0 \neq \{\} \wedge (\forall j \in B_1.\ (\forall x<t.\ \forall y<t.\ L\ x\ j = L\ y$
$j)) \wedge (\forall j \in B_0.\ (\forall s<t.\ L\ s\ j = s))$
⟨*proof*⟩

The next two lemmas are used to simplify proofs by enabling us to use the resulting facts directly. This avoids having to unfold the definition of *is-line* each time.

**lemma** *line-points-in-cube*:
**assumes** $is\text{-}line\ L\ n\ t$
**and** $s < t$
**shows** $L\ s \in cube\ n\ t$

⟨*proof*⟩

**lemma** *line-points-in-cube-unfolded*:
  **assumes** *is-line L n t*
    **and** $s < t$
    **and** $j < n$
  **shows** $L\ s\ j \in \{..<t\}$
  ⟨*proof*⟩

The incrementation of all elements of a set is defined in the following using the variables:

| | | |
|---|---|---|
| $n$: | *nat* | increment size |
| $S$: | *nat set* | set |

**definition** *set-incr* :: *nat* ⇒ *nat set* ⇒ *nat set*
  **where**
   *set-incr n S* ≡ $(\lambda a.\ a + n)$ ‘ $S$

**lemma** *set-incr-disjnt*:
  **assumes** *disjnt A B*
  **shows** *disjnt* (*set-incr n A*) (*set-incr n B*)
  ⟨*proof*⟩

**lemma** *set-incr-disjoint-family*:
  **assumes** *disjoint-family-on B* $\{..k\}$
  **shows** *disjoint-family-on* ($\lambda i.$ *set-incr n* $(B\ i)$) $\{..k\}$
  ⟨*proof*⟩

**lemma** *set-incr-altdef*: *set-incr n S* = (+) *n* ‘ *S*
  ⟨*proof*⟩

**lemma** *set-incr-image*:
  **assumes** $(\bigcup i{\in}\{..k\}.\ B\ i) = \{..<n\}$
  **shows** $(\bigcup i{\in}\{..k\}.$ *set-incr m* $(B\ i)) = \{m..<m{+}n\}$
  ⟨*proof*⟩

Each tuple of dimension $k + 1$ can be split into a tuple of dimension 1 (the first entry) and a tuple of dimension $k$ (the remaining entries).

**lemma** *split-cube*:
  **assumes** $x \in$ *cube* $(k{+}1)\ t$
  **shows** $(\lambda y \in \{..<1\}.\ x\ y) \in$ *cube* $1\ t$
    **and** $(\lambda y \in \{..<k\}.\ x\ (y + 1)) \in$ *cube* $k\ t$
  ⟨*proof*⟩

## 1.3 Subspaces

The property of being a $k$-dimensional subspace of $C_t^n$ is defined in the following using the variables:

| | | |
|---|---|---|
| $S$: | $(nat \Rightarrow nat) \Rightarrow nat \Rightarrow nat$ | the subspace |
| $k$: | $nat$ | the dimension of the subspace |
| $n$: | $nat$ | the dimension of the cube |
| $t$: | $nat$ | the size of the cube's base |

**definition** *is-subspace*
**where** $is\text{-}subspace\ S\ k\ n\ t \equiv (\exists B\, f.\ disjoint\text{-}family\text{-}on\ B\ \{..k\} \wedge \bigcup(B\ `\ \{..k\}) = \{..<n\} \wedge (\{\} \notin B\ `\ \{..<k\}) \wedge f \in (B\ k) \rightarrow_E \{..<t\}$
$\wedge\ S \in (cube\ k\ t) \rightarrow_E (cube\ n\ t) \wedge (\forall y \in cube\ k\ t.$
$(\forall i \in B\ k.\ S\ y\ i = f\ i) \wedge (\forall j{<}k.\ \forall i \in B\ j.\ (S\ y)\ i = y\ j)))$

A $k$-dimensional subspace of $C_t^n$ can be thought of as an embedding of the $C_t^k$ into $C_t^n$, akin to how a $k$-dimensional vector subspace of $\mathbf{R}^n$ may be thought of as an embedding of $\mathbf{R}^k$ into $\mathbf{R}^n$.

**lemma** *subspace-inj-on-cube*:
**assumes** *is-subspace S k n t*
**shows** *inj-on S (cube k t)*
⟨*proof*⟩

The following is required to handle base cases in the key lemmas.

**lemma** *dim0-subspace-ex*:
**assumes** $t > 0$
**shows** $\exists S.\ is\text{-}subspace\ S\ 0\ n\ t$
⟨*proof*⟩

## 1.4 Equivalence classes

Defining the equivalence classes of *cube n (t + 1)*: $\{classes\ n\ t\ 0, \ldots, classes\ n\ t\ n\}$

**definition** *classes*
**where** $classes\ n\ t \equiv (\lambda i.\ \{x\ .\ x \in (cube\ n\ (t + 1)) \wedge (\forall u \in \{(n-i)..<n\}.\ x\ u = t) \wedge t \notin x\ `\ \{..<(n - i)\}\})$

**lemma** *classes-subset-cube*: $classes\ n\ t\ i \subseteq cube\ n\ (t{+}1)$ ⟨*proof*⟩

**definition** *layered-subspace*
**where** $layered\text{-}subspace\ S\ k\ n\ t\ r\ \chi \equiv (is\text{-}subspace\ S\ k\ n\ (t + 1) \wedge (\forall i \in \{..k\}.\ \exists c{<}r.\ \forall x \in classes\ k\ t\ i.\ \chi\ (S\ x) = c)) \wedge \chi \in cube\ n\ (t + 1) \rightarrow_E \{..<r\}$

**lemma** *layered-eq-classes*:
**assumes** *layered-subspace S k n t r* $\chi$
**shows** $\forall i \in \{..k\}.\ \forall x \in classes\ k\ t\ i.\ \forall y \in classes\ k\ t\ i.$
$\chi\ (S\ x) = \chi\ (S\ y)$
⟨*proof*⟩

**lemma** *dim0-layered-subspace-ex*:
**assumes** $\chi \in (cube\ n\ (t + 1)) \rightarrow_E \{..<r{::}nat\}$

**shows** $\exists S.$ *layered-subspace S (0::nat) n t r* $\chi$
⟨*proof*⟩

**lemma** *disjoint-family-onI* [*intro*]:
**assumes** $\bigwedge m\ n.\ m \in S \Longrightarrow n \in S \Longrightarrow m \neq n \Longrightarrow A\ m \cap A\ n = \{\}$
**shows** *disjoint-family-on A S*
⟨*proof*⟩

**lemma** *fun-ex*: $a \in A \Longrightarrow b \in B \Longrightarrow \exists f \in A \rightarrow_E B.\ f\ a = b$
⟨*proof*⟩

**lemma** *ex-bij-betw-nat-finite-2*:
**assumes** *card* $A = n$
**and** $n > 0$
**shows** $\exists f.$ *bij-betw f A* $\{..<n\}$
⟨*proof*⟩

**lemma** *one-dim-cube-eq-nat-set*: *bij-betw* $(\lambda f.\ f\ 0)$ (*cube 1 k*) $\{..<k\}$
⟨*proof*⟩

An alternative introduction rule for the $\exists!x$ quantifier, which means "there exists exactly one $x$".

**lemma** *ex1I-alt*: $(\exists x.\ P\ x \wedge (\forall y.\ P\ y \longrightarrow x = y)) \Longrightarrow (\exists!x.\ P\ x)$
⟨*proof*⟩
**lemma** *nat-set-eq-one-dim-cube*: *bij-betw* $(\lambda x.\ \lambda y{\in}\{..<1::nat\}.\ x)$ $\{..<k::nat\}$ (*cube 1 k*)
⟨*proof*⟩

A bijection $f$ between domains $A_1$ and $A_2$ creates a correspondence between functions in $A_1 \rightarrow B$ and $A_2 \rightarrow B$.

**lemma** *bij-domain-PiE*:
**assumes** *bij-betw f A1 A2*
**and** $g \in A2 \rightarrow_E B$
**shows** (*restrict* $(g \circ f)$ *A1*) $\in A1 \rightarrow_E B$
⟨*proof*⟩

The following three lemmas relate lines to 1-dimensional subspaces (in the natural way). This is a direct consequence of the elimination rule *is-line-elim* introduced above.

**lemma** *line-is-dim1-subspace-t-1*:
**assumes** $n > 0$
**and** *is-line L n 1*
**shows** *is-subspace* (*restrict* $(\lambda y.\ L\ (y\ 0))$ (*cube 1 1*)) *1 n 1*
⟨*proof*⟩

**lemma** *line-is-dim1-subspace-t-ge-1*:

**assumes** $n > 0$
**and** $t > 1$
**and** *is-line* $L$ $n$ $t$
**shows** *is-subspace* (*restrict* ($\lambda y.$ $L$ ($y$ $0$)) (*cube* $1$ $t$)) $1$ $n$ $t$
⟨*proof*⟩

**lemma** *line-is-dim1-subspace*:
**assumes** $n > 0$
**and** $t > 0$
**and** *is-line* $L$ $n$ $t$
**shows** *is-subspace* (*restrict* ($\lambda y.$ $L$ ($y$ $0$)) (*cube* $1$ $t$)) $1$ $n$ $t$
⟨*proof*⟩

The key property of the existence of a minimal dimension $N$, such that for any $r$-colouring in $C_t^{N'}$ (for $N' \geq N$) there exists a monochromatic line is defined in the following using the variables:

| | | |
|---|---|---|
| $r$: | *nat* | the number of colours |
| $t$: | *nat* | the size of of the base |

**definition** *hj*
**where** *hj* $r$ $t$ $\equiv$ ($\exists N{>}0.\ \forall N' \geq N.\ \forall \chi.\ \chi \in$ (*cube* $N'$ $t$) $\rightarrow_E$ {..<$r$::*nat*} $\longrightarrow$ ($\exists L.\ \exists c{<}r.$ *is-line* $L$ $N'$ $t$ $\wedge$ ($\forall y \in L$ ‘ {..<$t$}. $\chi$ $y$ = $c$)))

The key property of the existence of a minimal dimension $N$, such that for any $r$-colouring in $C_t^{N'}$ (for $N' \geq N$) there exists a layered subspace of dimension $k$ is defined in the following using the variables:

| | | |
|---|---|---|
| $r$: | *nat* | the number of colours |
| $t$: | *nat* | the size of of the base |
| $k$: | *nat* | the dimension of the subspace |

**definition** *lhj*
**where** *lhj* $r$ $t$ $k$ $\equiv$ ($\exists N > 0.\ \forall N' \geq N.\ \forall \chi.\ \chi \in$ (*cube* $N'$ ($t$ + $1$)) $\rightarrow_E$ {..<$r$::*nat*} $\longrightarrow$ ($\exists S.$ *layered-subspace* $S$ $k$ $N'$ $t$ $r$ $\chi$))

We state some useful facts about 1-dimensional subspaces.

**lemma** *dim1-subspace-elims*:
**assumes** *disjoint-family-on* $B$ {..*1*::*nat*} **and** $\bigcup$($B$ ‘ {..*1*::*nat*}) = {..<$n$} **and** ({}
$\notin$ $B$ ‘ {..<*1*::*nat*}) **and** $f \in$ ($B$ $1$) $\rightarrow_E$ {..<$t$} **and** $S \in$ (*cube* $1$ $t$) $\rightarrow_E$ (*cube* $n$ $t$) **and** ($\forall y \in$ *cube* $1$ $t$. ($\forall i \in B$ $1$. $S$ $y$ $i$ = $f$ $i$) $\wedge$ ($\forall j{<}1.\ \forall i \in B$ $j$. ($S$ $y$) $i$ = $y$ $j$))
**shows** $B$ $0$ $\cup$ $B$ $1$ = {..<$n$}
**and** $B$ $0$ $\cap$ $B$ $1$ = {}
**and** ($\forall y \in$ *cube* $1$ $t$. ($\forall i \in B$ $1$. $S$ $y$ $i$ = $f$ $i$) $\wedge$ ($\forall i \in B$ $0$. ($S$ $y$) $i$ = $y$ $0$))
**and** $B$ $0 \neq$ {}
⟨*proof*⟩

We state some properties of cubes.

**lemma** *cube-props*:
  **assumes** $s < t$
  **shows** $\exists p \in cube\ 1\ t.\ p\ 0 = s$
    **and** $(SOME\ p.\ p \in cube\ 1\ t \wedge p\ 0 = s)\ 0 = s$
    **and** $(\lambda s{\in}\{..{<}t\}.\ S\ (SOME\ p.\ p{\in}cube\ 1\ t \wedge p\ 0 = s))\ s =$
    $(\lambda s{\in}\{..{<}t\}.\ S\ (SOME\ p.\ p{\in}cube\ 1\ t \wedge p\ 0 = s))\ ((SOME\ p.\ p \in cube\ 1\ t$
    $\wedge\ p\ 0 = s)\ 0)$
    **and** $(SOME\ p.\ p \in cube\ 1\ t \wedge p\ 0 = s) \in cube\ 1\ t$
⟨*proof*⟩

The following lemma relates 1-dimensional subspaces to lines, thus establishing a bidirectional correspondence between the two together with *line-is-dim1-subspace*.

**lemma** *dim1-subspace-is-line*:
  **assumes** $t > 0$
    **and** *is-subspace* $S\ 1\ n\ t$
  **shows** *is-line* $(\lambda s{\in}\{..{<}t\}.\ S\ (SOME\ p.\ p{\in}cube\ 1\ t \wedge p\ 0 = s))\ n\ t$
⟨*proof*⟩

**lemma** *bij-unique-inv*:
  **assumes** *bij-betw* $f\ A\ B$
    **and** $x \in B$
  **shows** $\exists!y \in A.\ (the\text{-}inv\text{-}into\ A\ f)\ x = y$
  ⟨*proof*⟩

**lemma** *inv-into-cube-props*:
  **assumes** $s < t$
  **shows** *the-inv-into* $(cube\ 1\ t)\ (\lambda f.\ f\ 0)\ s \in cube\ 1\ t$
    **and** *the-inv-into* $(cube\ 1\ t)\ (\lambda f.\ f\ 0)\ s\ 0 = s$
  ⟨*proof*⟩

**lemma** *some-inv-into*:
  **assumes** $s < t$
  **shows** $(SOME\ p.\ p{\in}cube\ 1\ t \wedge p\ 0 = s) = (the\text{-}inv\text{-}into\ (cube\ 1\ t)\ (\lambda f.\ f\ 0)\ s)$
  ⟨*proof*⟩

**lemma** *some-inv-into-2*:
  **assumes** $s < t$
  **shows** $(SOME\ p.\ p{\in}cube\ 1\ (t{+}1) \wedge p\ 0 = s) = (the\text{-}inv\text{-}into\ (cube\ 1\ t)\ (\lambda f.\ f\ 0)$
$s)$
⟨*proof*⟩

**lemma** *dim1-layered-subspace-as-line*:
  **assumes** $t > 0$
    **and** *layered-subspace* $S\ 1\ n\ t\ r\ \chi$
  **shows** $\exists\, c1\ c2.\ c1{<}r \wedge c2{<}r \wedge (\forall\, s{<}t.\ \chi\ (S\ (SOME\ p.\ p{\in}cube\ 1$
  $(t{+}1) \wedge p\ 0 = s)) = c1) \wedge \chi\ (S\ (SOME\ p.\ p{\in}cube\ 1\ (t{+}1) \wedge p\ 0 = t)) = c2$
⟨*proof*⟩

**lemma** *dim1-layered-subspace-mono-line*:

**assumes** *t > 0*
**and** *layered-subspace S 1 n t r χ*
**shows** ∀ *s<t.* ∀ *l<t. χ (S (SOME p. p∈cube 1 (t+1) ∧ p 0 = s)) = χ (S (SOME p. p∈cube 1 (t+1) ∧ p 0 = l)) ∧ χ (S (SOME p. p∈cube 1 (t+1) ∧ p 0 = s)) < r*
⟨*proof*⟩

**definition** *join* :: *(nat ⇒ 'a) ⇒ (nat ⇒ 'a) ⇒ nat ⇒ nat ⇒ (nat ⇒ 'a)*
**where**
*join f g n m ≡ (λx. if x ∈ {..<n} then f x else (if x ∈ {n..<n+m} then g (x − n) else undefined))*

**lemma** *join-cubes*:
**assumes** *f ∈ cube n (t+1)*
**and** *g ∈ cube m (t+1)*
**shows** *join f g n m ∈ cube (n+m) (t+1)*
⟨*proof*⟩

**lemma** *subspace-elems-embed*:
**assumes** *is-subspace S k n t*
**shows** *S ' (cube k t) ⊆ cube n t*
⟨*proof*⟩

# 2 Core proofs

The numbering of the theorems has been borrowed from the textbook [1].

## 2.1 Theorem 4

### 2.1.1 Base case of Theorem 4

**lemma** *hj-imp-lhj-base*:
**fixes** *r t*
**assumes** *t > 0*
**and** ⋀*r'. hj r' t*
**shows** *lhj r t 1*
⟨*proof*⟩

### 2.1.2 Induction step of theorem 4

The proof has four parts:

1. We obtain two layered subspaces of dimension 1 and k (respectively), whose existence is guaranteed by the assumption *lhj* (i.e. the induction hypothesis). Additionally, we prove some useful facts about these.

2. We construct a *k+1*-dimensional subspace with the goal of showing that it is layered.

3. We prove that our construction is a subspace in the first place.

4. We prove that it is a layered subspace.

**lemma** *hj-imp-lhj-step*:
**fixes** $r\ k$
**assumes** $t > 0$
**and** $k \geq 1$
**and** *True*
**and** $(\bigwedge r\ k'.\ k' \leq k \Longrightarrow lhj\ r\ t\ k')$
**and** $r > 0$
**shows** $lhj\ r\ t\ (k+1)$
⟨*proof*⟩

**Part 1: Obtaining the subspaces $L$ and $S$**

Recall that *lhj* claims the existence of a layered subspace for any colouring (of a fixed size, where the size of a colouring refers to the number of colours). Therefore, the colourings have to be defined first, before the layered subspaces can be obtained. The colouring $\chi L$ here is $\chi^*$ in the book [1], an *s*-colouring; see the fact *s-coloured* a couple of lines below.

⟨*proof*⟩

**Part 2: Constructing the $(k+1)$-dimensional subspace $T$**

Below, *Tset* is the set as defined in the book [1]. It represents the $(k+1)$-dimensional subspace. In this construction, subspaces (e.g. $T$) are functions whose image is a set. See the fact *im-T-eq-Tset* below.

Having obtained our subspaces $S$ and $L$, we define the $(k+1)$-dimensional subspace very straightforwardly Namely, T = L ×S. Since we represent tuples by function sets, we need an appropriate operator that mirrors the Cartesian product $\times$ for these. We call this *join* and define it for elements of a function set.

⟨*proof*⟩

**Part 3: Proving that $T$ is a subspace**

To prove something is a subspace, we have to provide the $B$ and $f$ satisfying the subspace properties. We construct $BT$ and $fT$ from $BS$, $fS$ and $BL$, $fL$, which correspond to the $k$-dimensional subspace $S$ and the 1-dimensional subspace (i.e. line) $L$, respectively.

⟨*proof*⟩

**Part 4: Proving $T$ is layered**

The following redefinition of the classes makes proving the layered property easier.

⟨*proof*⟩

**theorem** *hj-imp-lhj*:
  **fixes** *k*
  **assumes** $\bigwedge r'.\ hj\ r'\ t$
  **shows** *lhj r t k*
⟨*proof*⟩

## 2.2 Theorem 5

We provide a way to construct a monochromatic line in $C^n_{t+1}$ from a $k$-dimensional $k$-coloured layered subspace $S$ in $C^n_{t+1}$. The idea is to rely on the fact that there are $k+1$ classes in $S$, but only $k$ colours. It thus follows from the Pigeonhole Principle that two classes must share the same colour. The way classes are defined allows for a straightforward construction of a line with points only from those two classes. Thus we have our monochromatic line.

**theorem** *layered-subspace-to-mono-line*:
  **assumes** *layered-subspace S k n t k* $\chi$
    **and** $t > 0$
  **shows** $(\exists L.\ \exists c{<}k.\ is\text{-}line\ L\ n\ (t{+}1) \land (\forall y \in L\ \text{‘}\ \{..{<}t{+}1\}.\ \chi\ y = c))$
⟨*proof*⟩

## 2.3 Corollary 6

**corollary** *lhj-imp-hj*:
  **assumes** $(\bigwedge r\ k.\ lhj\ r\ t\ k)$
    **and** $t{>}0$
  **shows** $(hj\ r\ (t{+}1))$
  ⟨*proof*⟩

## 2.4 Main result

### 2.4.1 Edge cases and auxiliary lemmas

**lemma** *single-point-line*:
  **assumes** $N > 0$
  **shows** $is\text{-}line\ (\lambda s{\in}\{..{<}1\}.\ \lambda a{\in}\{..{<}N\}.\ 0)\ N\ 1$
  ⟨*proof*⟩

**lemma** *single-point-line-is-monochromatic*:
  **assumes** $\chi \in cube\ N\ 1 \rightarrow_E \{..{<}r\}\ N > 0$
  **shows** $(\exists c < r.\ is\text{-}line\ (\lambda s{\in}\{..{<}1\}.\ \lambda a{\in}\{..{<}N\}.\ 0)\ N\ 1 \land (\forall i \in (\lambda s{\in}\{..{<}1\}.\ \lambda a{\in}\{..{<}N\}.\ 0)\ \text{‘}\ \{..{<}1\}.\ \chi\ i = c))$
⟨*proof*⟩

**lemma** *hj-r-nonzero-t-0*:
  **assumes** $r > 0$

**shows** *hj r 0*
⟨*proof*⟩

Any cube over 1 element always has a single point, which also forms the only line in the cube. Since it's a single point line, it's trivially monochromatic. We show the result for dimension 1.

**lemma** *hj-t-1*: *hj r 1*
⟨*proof*⟩

### 2.4.2 Main theorem

We state the main result *hj r t*. The explanation for the choice of assumption is offered subsequently.

**theorem** *hales-jewett*:
**assumes** $\neg(r = 0 \wedge t = 0)$
**shows** *hj r t*
⟨*proof*⟩

We offer a justification for having excluded the special case $r = t = 0$ from the statement of the main theorem *hales-jewett*. The exclusion is a consequence of the fact that colourings are defined as members of the function set $\mathit{cube}\ n\ t \rightarrow_E \{..<r\}$, which for $r = t = 0$ means there's a dummy colouring $\lambda x.\ \mathit{undefined}$, even though $\mathit{cube}\ n\ 0 = \{\}$ for $n > 0$. Hence, in this case, no line exists at all (let alone one monochromatic under the aforementioned colouring). This means $\mathit{hj}\ 0\ 0 \Longrightarrow \mathit{False}$—but only because of the quirky behaviour of the FuncSet $\mathit{cube}\ n\ t \rightarrow_E \{..<r\}$. This could have been circumvented by letting colourings $\chi$ be arbitrary functions constraint only by $\chi\ \text{`}\ \mathit{cube}\ n\ t \subseteq \{..<r\}$. We avoided this in order to have consistency with the cube's definition, for which FuncSets were crucial because the proof heavily relies on arguments about the cardinality of the cube. he constraint $x\ \text{`}\ \{..<n\} \subseteq \{..<t\}$ for elements $x$ of $C_t^n$ would not have sufficed there, as there are infinitely many functions over the naturals satisfying it.

**end**

# References


[1] R. L. Graham, B. L. Rothschild, and J. H. Spencer. *Ramsey Theory, 2nd Edition*. Wiley-Interscience, March 1990.

[2] K. Kreuzer and M. Eberl. Van der Waerden's Theorem. *Archive of Formal Proofs*, June 2021. https://isa-afp.org/entries/Van_der_Waerden.html, Formal proof development.