The Hahn and Jordan Decomposition Theorems

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March 17, 2025

Contents

1 Introduction

Signed measures are a generalization of measures that can map measurable sets to negative values. In this work we formalize the Hahn decomposition theorem for signed measures, namely that if $(\Omega, \mathcal{A}, \mu)$ is a measure space for a signed measure μ, then Ω can be decomposed as $\Omega^+ \cup \Omega^-$, where every measurable subset of Ω^+ has a positive measure, and every measurable subset of Ω^- has a negative measure. We then prove that this decomposition is essentially unique, meaning that if $X^+ \cup X^-$ is another such decomposition, then any measurable subset in $(\Omega^+ \triangle X^+) \cup (\Omega^- \triangle X^-)$ has a zero measure.

We also formalize the Jordan decomposition theorem as a corollary, which states that the signed measure μ admits a unique decomposition into a difference $\mu = \mu^+ - \mu^-$ of two positive measures, at least one of which is finite, and such that for any Hahn decomposition $\Omega^+ \cup \Omega^-$ and measurable set A, if $A \subseteq \Omega^-$ then $\mu^+(A) = 0$ and if $A \subseteq \Omega^+$ then $\mu^-(A) = 0$. The formalization is mostly based on [1], Section 16 of Chapter 4.

2 Signed measures

In this section we define signed measures. These are generalizations of measures that can also take negative values but cannot contain both ∞ and $-\infty$ in their range.

2.1 Basic definitions

theory *Hahn-Jordan-Decomposition* **imports**
HOL−Probability.Probability
Hahn-Jordan-Prelims
begin

definition $\mathit{signed\text{-}measure}{::}'a\ \mathit{measure} \Rightarrow ('a\ \mathit{set} \Rightarrow \mathit{ereal}) \Rightarrow \mathit{bool}$ **where**
$\mathit{signed\text{-}measure}\ M\ \mu \longleftrightarrow \mu\ \{\} = 0 \wedge (-\infty \notin \mathit{range}\ \mu \vee \infty \notin \mathit{range}\ \mu) \wedge$
$(\forall A.\ \mathit{range}\ A \subseteq \mathit{sets}\ M \longrightarrow \mathit{disjoint\text{-}family}\ A \longrightarrow \bigcup\ (\mathit{range}\ A) \in \mathit{sets}\ M \longrightarrow$
$(\lambda i.\ \mu\ (A\ i))\ \mathit{sums}\ \mu\ (\bigcup\ (\mathit{range}\ A))) \wedge$
$(\forall A.\ \mathit{range}\ A \subseteq \mathit{sets}\ M \longrightarrow \mathit{disjoint\text{-}family}\ A \longrightarrow \bigcup\ (\mathit{range}\ A) \in \mathit{sets}\ M \longrightarrow$
$|\mu\ (\bigcup\ (\mathit{range}\ A))| < \infty \longrightarrow \mathit{summable}\ (\lambda i.\ \mathit{real\text{-}of\text{-}ereal}\ |\mu\ (A\ i)|))$

lemma *signed-measure-empty*:
assumes *signed-measure* $M\ \mu$
shows $\mu\ \{\} = 0$ ⟨*proof*⟩

lemma *signed-measure-sums*:
assumes *signed-measure* $M\ \mu$
and $\mathit{range}\ A \subseteq M$
and *disjoint-family* A
and $\bigcup\ (\mathit{range}\ A) \in \mathit{sets}\ M$
shows $(\lambda i.\ \mu\ (A\ i))\ \mathit{sums}\ \mu\ (\bigcup\ (\mathit{range}\ A))$
⟨*proof*⟩

lemma *signed-measure-summable*:
assumes *signed-measure* $M\ \mu$
and $\mathit{range}\ A \subseteq M$
and *disjoint-family* A
and $\bigcup\ (\mathit{range}\ A) \in \mathit{sets}\ M$
and $|\mu\ (\bigcup\ (\mathit{range}\ A))| < \infty$
shows $\mathit{summable}\ (\lambda i.\ \mathit{real\text{-}of\text{-}ereal}\ |\mu\ (A\ i)|)$
⟨*proof*⟩

lemma *signed-measure-inf-sum*:
assumes *signed-measure* $M\ \mu$
and $\mathit{range}\ A \subseteq M$
and *disjoint-family* A
and $\bigcup\ (\mathit{range}\ A) \in \mathit{sets}\ M$
shows $(\sum i.\ \mu\ (A\ i)) = \mu\ (\bigcup\ (\mathit{range}\ A))$ ⟨*proof*⟩

lemma *signed-measure-abs-convergent*:

assumes *signed-measure M μ*
and *range A* ⊆ *sets M*
and *disjoint-family A*
and $\bigcup$ (*range A*) ∈ *sets M*
and |*μ* ($\bigcup$ (*range A*))| < ∞
shows *summable* (λ*i. real-of-ereal* |*μ* (*A i*)|) ⟨*proof*⟩

lemma *signed-measure-additive*:
assumes *signed-measure M μ*
shows *additive M μ*
⟨*proof*⟩

lemma *signed-measure-add*:
assumes *signed-measure M μ*
and *a*∈ *sets M*
and *b*∈ *sets M*
and *a*∩ *b* = {}
shows *μ* (*a*∪ *b*) = *μ a* + *μ b* ⟨*proof*⟩

lemma *signed-measure-disj-sum*:
shows *finite I* ⟹ *signed-measure M μ* ⟹ *disjoint-family-on A I* ⟹
($\bigwedge$*i. i* ∈ *I* ⟹ *A i* ∈ *sets M*) ⟹ *μ* ($\bigcup$ *i*∈ *I. A i*) = ($\sum$ *i*∈ *I. μ* (*A i*))
⟨*proof*⟩

lemma *pos-signed-measure-count-additive*:
assumes *signed-measure M μ*
and ∀ *E* ∈ *sets M. 0* ≤ *μ E*
shows *countably-additive* (*sets M*) (λ*A. e2ennreal* (*μ A*))
⟨*proof*⟩

lemma *signed-measure-minus*:
assumes *signed-measure M μ*
shows *signed-measure M* (λ*A.* − *μ A*) ⟨*proof*⟩

locale *near-finite-function* =
fixes *μ*:: *'b set* ⇒ *ereal*
assumes *inf-range*: − ∞ ∉ *range μ* ∨ ∞ ∉ *range μ*

lemma (**in** *near-finite-function*) *finite-subset*:
assumes |*μ E*| < ∞
and *A*⊆ *E*
and *μ E* = *μ A* + *μ* (*E* − *A*)
shows |*μ A*| < ∞
⟨*proof*⟩

locale *signed-measure-space*=
fixes *M*::*'a measure* **and** *μ*
assumes *sgn-meas*: *signed-measure M μ*

sublocale *signed-measure-space* $\subseteq$ *near-finite-function*
$\langle proof \rangle$

context *signed-measure-space*
begin
lemma *signed-measure-finite-subset*:
  assumes $E \in sets\ M$
    and $|\mu\ E| < \infty$
    and $A \in sets\ M$
    and $A \subseteq E$
  shows $|\mu\ A| < \infty$
$\langle proof \rangle$

lemma *measure-space-e2ennreal* :
  assumes *measure-space* $(space\ M)$ $(sets\ M)$ $m \wedge (\forall E \in sets\ M.\ m\ E < \infty) \wedge$
    $(\forall E \in sets\ M.\ m\ E \geq 0)$
  shows $\forall E \in sets\ M.\ e2ennreal\ (m\ E) < \infty$
$\langle proof \rangle$

2.2 Positive and negative subsets

The Hahn decomposition theorem is based on the notions of positive and negative measurable sets. A measurable set is positive (resp. negative) if all its measurable subsets have a positive (resp. negative) measure by μ. The decomposition theorem states that any measure space for a signed measure can be decomposed into a positive and a negative measurable set.

definition *pos-meas-set* **where**
  $pos\text{-}meas\text{-}set\ E \longleftrightarrow E \in sets\ M \wedge (\forall A \in sets\ M.\ A \subseteq E \longrightarrow 0 \leq \mu\ A)$

definition *neg-meas-set* **where**
  $neg\text{-}meas\text{-}set\ E \longleftrightarrow E \in sets\ M \wedge (\forall A \in sets\ M.\ A \subseteq E \longrightarrow \mu\ A \leq 0)$

lemma *pos-meas-setI*:
  assumes $E \in sets\ M$
    and $\bigwedge A.\ A \in sets\ M \Longrightarrow A \subseteq E \Longrightarrow 0 \leq \mu\ A$
  shows *pos-meas-set* E $\langle proof \rangle$

lemma *pos-meas-setD1* :
  assumes *pos-meas-set* E
  shows $E \in sets\ M$
  $\langle proof \rangle$

lemma *neg-meas-setD1* :
  assumes *neg-meas-set* E
  shows $E \in sets\ M$ $\langle proof \rangle$

lemma *neg-meas-setI*:
  assumes $E \in sets\ M$

and $\bigwedge A.\ A \in sets\ M \Longrightarrow A \subseteq E \Longrightarrow \mu\ A \leq 0$
shows *neg-meas-set E* $\langle proof \rangle$

lemma *pos-meas-self*:
assumes *pos-meas-set E*
shows $0 \leq \mu\ E$ $\langle proof \rangle$

lemma *empty-pos-meas-set*:
shows *pos-meas-set* $\{\}$
$\langle proof \rangle$

lemma *empty-neg-meas-set*:
shows *neg-meas-set* $\{\}$
$\langle proof \rangle$

lemma *pos-measure-meas*:
assumes *pos-meas-set E*
and $A \subseteq E$
and $A \in sets\ M$
shows $0 \leq \mu\ A$ $\langle proof \rangle$

lemma *pos-meas-subset*:
assumes *pos-meas-set A*
and $B \subseteq A$
and $B \in sets\ M$
shows *pos-meas-set B* $\langle proof \rangle$

lemma *neg-meas-subset*:
assumes *neg-meas-set A*
and $B \subseteq A$
and $B \in sets\ M$
shows *neg-meas-set B* $\langle proof \rangle$

lemma *pos-meas-set-Union*:
assumes $\bigwedge(i::nat).\ pos\text{-}meas\text{-}set\ (A\ i)$
and $\bigwedge i.\ A\ i \in sets\ M$
and $|\mu\ (\bigcup\ i.\ A\ i)| < \infty$
shows *pos-meas-set* $(\bigcup\ i.\ A\ i)$
$\langle proof \rangle$

lemma *pos-meas-set-pos-lim*:
assumes $\bigwedge(i::nat).\ pos\text{-}meas\text{-}set\ (A\ i)$
and $\bigwedge i.\ A\ i \in sets\ M$
shows $0 \leq \mu\ (\bigcup\ i.\ A\ i)$
$\langle proof \rangle$

lemma *pos-meas-disj-union*:
assumes *pos-meas-set A*
and *pos-meas-set B*

and $A\cap B = \{\}$
shows *pos-meas-set* $(A \cup B)$ ⟨*proof*⟩

lemma *pos-meas-set-union*:
assumes *pos-meas-set A*
and *pos-meas-set B*
shows *pos-meas-set* $(A \cup B)$
⟨*proof*⟩

lemma *neg-meas-disj-union*:
assumes *neg-meas-set A*
and *neg-meas-set B*
and $A\cap B = \{\}$
shows *neg-meas-set* $(A \cup B)$ ⟨*proof*⟩

lemma *neg-meas-set-union*:
assumes *neg-meas-set A*
and *neg-meas-set B*
shows *neg-meas-set* $(A \cup B)$
⟨*proof*⟩

lemma *neg-meas-self* :
assumes *neg-meas-set E*
shows $\mu\ E \leq 0$ ⟨*proof*⟩

lemma *pos-meas-set-opp*:
assumes *signed-measure-space.pos-meas-set* $M\ (\lambda\ A.\ -\ \mu\ A)\ A$
shows *neg-meas-set A*
⟨*proof*⟩

lemma *neg-meas-set-opp*:
assumes *signed-measure-space.neg-meas-set* $M\ (\lambda\ A.\ -\ \mu\ A)\ A$
shows *pos-meas-set A*
⟨*proof*⟩
end

lemma *signed-measure-inter*:
assumes *signed-measure* $M\ \mu$
and $A \in$ *sets M*
shows *signed-measure* $M\ (\lambda E.\ \mu\ (E \cap A))$ ⟨*proof*⟩

context *signed-measure-space*
begin
lemma *pos-signed-to-meas-space* :
assumes *pos-meas-set M1*
and $m1 = (\lambda A.\ \mu\ (A \cap M1))$
shows *measure-space* (*space M*) (*sets M*) *m1* ⟨*proof*⟩

lemma *neg-signed-to-meas-space* :

assumes *neg-meas-set M2*
and *m2* = (λ*A*. −μ (*A* ∩ *M2*))
shows *measure-space* (*space M*) (*sets M*) *m2* ⟨*proof*⟩

lemma *pos-part-meas-nul-neg-set* :
assumes *pos-meas-set M1*
and *neg-meas-set M2*
and *m1* = (λ*A*. μ (*A* ∩ *M1*))
and *E* ∈ *sets M*
and *E* ⊆ *M2*
shows *m1 E* = *0*
⟨*proof*⟩

lemma *neg-part-meas-nul-pos-set* :
assumes *pos-meas-set M1*
and *neg-meas-set M2*
and *m2* = (λ*A*. −μ (*A* ∩ *M2*))
and *E* ∈ *sets M*
and *E* ⊆ *M1*
shows *m2 E* = *0*
⟨*proof*⟩

definition *pos-sets* **where**
pos-sets = {*A*. *A* ∈ *sets M* ∧ *pos-meas-set A*}

definition *pos-img* **where**
pos-img = {μ *A*|*A*. *A*∈ *pos-sets*}

2.3 Essential uniqueness

In this part, under the assumption that a measure space for a signed measure admits a decomposition into a positive and a negative set, we prove that this decomposition is essentially unique; in other words, that if two such decompositions (P, N) and (X, Y) exist, then any measurable subset of $(P \triangle X) \cup (N \triangle Y)$ has a null measure.

definition *hahn-space-decomp* **where**
hahn-space-decomp M1 M2 ≡ (*pos-meas-set M1*) ∧ (*neg-meas-set M2*) ∧ (*space M* = *M1* ∪ *M2*) ∧ (*M1* ∩ *M2* = {})

lemma *pos-neg-null-set*:
assumes *pos-meas-set A*
and *neg-meas-set A*
shows μ *A* = *0* ⟨*proof*⟩

lemma *pos-diff-neg-meas-set*:
assumes (*pos-meas-set M1*)
and (*neg-meas-set N2*)
and (*space M* = *N1* ∪ *N2*)

and *N1* ∈ *sets M*
shows *neg-meas-set* ((*M1* − *N1*) ∩ *space M*) ⟨*proof*⟩

lemma *neg-diff-pos-meas-set*:
assumes (*neg-meas-set M2*)
and (*pos-meas-set N1*)
and (*space M* = *N1* ∪ *N2*)
and *N2* ∈ *sets M*
shows *pos-meas-set* ((*M2* − *N2*) ∩ *space M*)
⟨*proof*⟩

lemma *pos-sym-diff-neg-meas-set*:
assumes *hahn-space-decomp M1 M2*
and *hahn-space-decomp N1 N2*
shows *neg-meas-set* ((*sym-diff M1 N1*) ∩ *space M*) ⟨*proof*⟩

lemma *neg-sym-diff-pos-meas-set*:
assumes *hahn-space-decomp M1 M2*
and *hahn-space-decomp N1 N2*
shows *pos-meas-set* ((*sym-diff M2 N2*) ∩ *space M*) ⟨*proof*⟩

lemma *pos-meas-set-diff*:
assumes *pos-meas-set A*
and *B*∈ *sets M*
shows *pos-meas-set* ((*A* − *B*) ∩ (*space M*)) ⟨*proof*⟩

lemma *pos-meas-set-sym-diff*:
assumes *pos-meas-set A*
and *pos-meas-set B*
shows *pos-meas-set* ((*sym-diff A B*) ∩ *space M*) ⟨*proof*⟩

lemma *neg-meas-set-diff*:
assumes *neg-meas-set A*
and *B*∈ *sets M*
shows *neg-meas-set* ((*A* − *B*) ∩ (*space M*)) ⟨*proof*⟩

lemma *neg-meas-set-sym-diff*:
assumes *neg-meas-set A*
and *neg-meas-set B*
shows *neg-meas-set* ((*sym-diff A B*) ∩ *space M*) ⟨*proof*⟩

lemma *hahn-decomp-space-diff*:
assumes *hahn-space-decomp M1 M2*
and *hahn-space-decomp N1 N2*
shows *pos-meas-set* ((*sym-diff M1 N1* ∪ *sym-diff M2 N2*) ∩ *space M*)
neg-meas-set ((*sym-diff M1 N1* ∪ *sym-diff M2 N2*) ∩ *space M*)
⟨*proof*⟩

lemma *hahn-decomp-ess-unique*:

assumes *hahn-space-decomp M1 M2*
and *hahn-space-decomp N1 N2*
and $C \subseteq$ *sym-diff M1 N1* $\cup$ *sym-diff M2 N2*
and $C \in$ *sets M*
shows μ $C = 0$
⟨*proof*⟩

3 Existence of a positive subset

The goal of this part is to prove that any measurable set of finite and positive measure must contain a positive subset with a strictly positive measure.

3.1 A sequence of negative subsets

definition *inf-neg* **where**
inf-neg A = (*if* ($A \notin$ *sets M* $\vee$ *pos-meas-set A*) *then* (*0*::*nat*)
else Inf {*n*|*n*. (*1*::*nat*) $\leq n \wedge (\exists B \in$ *sets M*. $B \subseteq A \wedge \mu$ $B <$ *ereal*(−*1*/*n*))})

lemma *inf-neg-ne*:
assumes $A \in$ *sets M*
and $\neg$ *pos-meas-set A*
shows {*n*::*nat*|*n*. (*1*::*nat*) $\leq n \wedge$
($\exists B \in$ *sets M*. $B \subseteq A \wedge \mu$ $B <$ *ereal* (−*1*/*n*))} $\neq$ {}
⟨*proof*⟩

lemma *inf-neg-ge-1*:
assumes $A \in$ *sets M*
and $\neg$ *pos-meas-set A*
shows (*1*::*nat*) $\leq$ *inf-neg A*
⟨*proof*⟩

lemma *inf-neg-pos*:
assumes $A \in$ *sets M*
and $\neg$ *pos-meas-set A*
shows $\exists$ $B \in$ *sets M*. $B \subseteq A \wedge \mu$ $B <$ −*1*/(*inf-neg A*)
⟨*proof*⟩

definition *rep-neg* **where**
rep-neg A = (*if* ($A \notin$ *sets M* $\vee$ *pos-meas-set A*) *then* {} *else*
SOME B. $B \in$ *sets M* $\wedge B \subseteq A \wedge \mu$ $B \leq$ *ereal* (−*1* / (*inf-neg A*)))

lemma *g-rep-neg*:
assumes $A \in$ *sets M*
and $\neg$ *pos-meas-set A*
shows *rep-neg A* $\in$ *sets M* *rep-neg A* $\subseteq A$
μ (*rep-neg A*) $\leq$ *ereal* (−*1* / (*inf-neg A*))
⟨*proof*⟩

lemma *rep-neg-sets*:
shows *rep-neg A* ∈ *sets M*
⟨*proof*⟩

lemma *rep-neg-subset*:
shows *rep-neg A* ⊆ *A*
⟨*proof*⟩

lemma *rep-neg-less*:
assumes *A*∈ *sets M*
and ¬ *pos-meas-set A*
shows μ (*rep-neg A*) ≤ *ereal* (−*1* / (*inf-neg A*)) ⟨*proof*⟩

lemma *rep-neg-leq*:
shows μ (*rep-neg A*) ≤ *0*
⟨*proof*⟩

3.2 Construction of the positive subset

fun *pos-wtn*
where
pos-wtn-base: *pos-wtn E 0* = *E*|
pos-wtn-step: *pos-wtn E* (*Suc n*) = *pos-wtn E n* − *rep-neg* (*pos-wtn E n*)

lemma *pos-wtn-subset*:
shows *pos-wtn E n* ⊆ *E*
⟨*proof*⟩

lemma *pos-wtn-sets*:
assumes *E*∈ *sets M*
shows *pos-wtn E n* ∈ *sets M*
⟨*proof*⟩

definition *neg-wtn* **where**
neg-wtn E (*n*::*nat*) = *rep-neg* (*pos-wtn E n*)

lemma *neg-wtn-neg-meas*:
shows μ (*neg-wtn E n*) ≤ *0* ⟨*proof*⟩

lemma *neg-wtn-sets*:
shows *neg-wtn E n* ∈ *sets M* ⟨*proof*⟩

lemma *neg-wtn-subset*:
shows *neg-wtn E n* ⊆ *E* ⟨*proof*⟩

lemma *neg-wtn-union-subset*:
shows (⋃ *i* ≤ *n*. *neg-wtn E i*) ⊆ *E* ⟨*proof*⟩

lemma *pos-wtn-Suc*:

shows *pos-wtn E (Suc n)* = *E* − ($\bigcup$ *i* ≤ *n*. *neg-wtn E i*) ⟨*proof*⟩

definition *pos-sub* **where**
pos-sub E = ($\bigcap$ *n*. *pos-wtn E n*)

lemma *pos-sub-sets*:
assumes *E*∈ *sets M*
shows *pos-sub E* ∈ *sets M* ⟨*proof*⟩

lemma *pos-sub-subset*:
shows *pos-sub E* ⊆ *E* ⟨*proof*⟩

lemma *pos-sub-infty*:
assumes *E* ∈ *sets M*
and |*μ E*| < ∞
shows |*μ* (*pos-sub E*)| < ∞ ⟨*proof*⟩

lemma *neg-wtn-djn*:
shows *disjoint-family* (*λn*. *neg-wtn E n*) ⟨*proof*⟩
end

lemma *disjoint-family-imp-on*:
assumes *disjoint-family A*
shows *disjoint-family-on A S*
⟨*proof*⟩

context *signed-measure-space*
begin
lemma *neg-wtn-union-neg-meas*:
shows *μ* ($\bigcup$ *i* ≤ *n*. *neg-wtn E i*) ≤ *0*
⟨*proof*⟩

lemma *pos-wtn-meas-gt*:
assumes *0* < *μ E*
and *E*∈ *sets M*
shows *0* < *μ* (*pos-wtn E n*)
⟨*proof*⟩

definition *union-wit* **where**
union-wit E = ($\bigcup$ *n*. *neg-wtn E n*)

lemma *union-wit-sets*:
shows *union-wit E* ∈ *sets M* ⟨*proof*⟩

lemma *union-wit-subset*:
shows *union-wit E* ⊆ *E*
⟨*proof*⟩

lemma *pos-sub-diff*:

shows *pos-sub E* = *E* − *union-wit E*
⟨*proof*⟩

definition *num-wtn* **where**
num-wtn E n = *inf-neg* (*pos-wtn E n*)

lemma *num-wtn-geq*:
shows μ (*neg-wtn E n*) ≤ *ereal* (−*1*/(*num-wtn E n*))
⟨*proof*⟩

lemma *neg-wtn-infty*:
assumes $E \in$ *sets M*
and $|\mu\ E| < \infty$
shows $|\mu$ (*neg-wtn E i*)$| < \infty$
⟨*proof*⟩

lemma *union-wit-infty*:
assumes $E \in$ *sets M*
and $|\mu\ E| < \infty$
shows $|\mu$ (*union-wit E*)$| < \infty$ ⟨*proof*⟩

lemma *neg-wtn-summable*:
assumes $E \in$ *sets M*
and $|\mu\ E| < \infty$
shows *summable* ($\lambda i.$ − *real-of-ereal* (μ (*neg-wtn E i*)))
⟨*proof*⟩

lemma *inv-num-wtn-summable*:
assumes $E \in$ *sets M*
and $|\mu\ E| < \infty$
shows *summable* ($\lambda n.$ *1*/(*num-wtn E n*))
⟨*proof*⟩

lemma *inv-num-wtn-shift-summable*:
assumes $E \in$ *sets M*
and $|\mu\ E| < \infty$
shows *summable* ($\lambda n.$ *1*/(*num-wtn E n* − *1*))
⟨*proof*⟩

lemma *neg-wtn-meas-sums*:
assumes $E \in$ *sets M*
and $|\mu\ E| < \infty$
shows ($\lambda i.$ − (μ (*neg-wtn E i*))) *sums*
suminf ($\lambda i.$ − *real-of-ereal* (μ (*neg-wtn E i*)))
⟨*proof*⟩

lemma *neg-wtn-meas-suminf-le*:
assumes $E \in$ *sets M*
and $|\mu\ E| < \infty$

shows *suminf* (λ*i*. μ (*neg-wtn E i*)) ≤ − *suminf* (λ*n*. *1*/(*num-wtn E n*))
⟨*proof*⟩

lemma *union-wit-meas-le*:
assumes *E* ∈ *sets M*
and |μ *E*| < ∞
shows μ (*union-wit E*) ≤ − *suminf* (λ*n*. *1* / *real* (*num-wtn E n*))
⟨*proof*⟩

lemma *pos-sub-pos-meas*:
assumes *E* ∈ *sets M*
and |μ *E*| < ∞
and *0* < μ *E*
and ¬ *pos-meas-set E*
shows *0* < μ (*pos-sub E*)
⟨*proof*⟩

lemma *num-wtn-conv*:
assumes *E* ∈ *sets M*
and |μ *E*| < ∞
shows (λ*n*. *1*/(*num-wtn E n*)) ⟶ *0*
⟨*proof*⟩

lemma *num-wtn-shift-conv*:
assumes *E* ∈ *sets M*
and |μ *E*| < ∞
shows (λ*n*. *1*/(*num-wtn E n* − *1*)) ⟶ *0*
⟨*proof*⟩

lemma *inf-neg-E-set*:
assumes *0* < *inf-neg E*
shows *E* ∈ *sets M* ⟨*proof*⟩

lemma *inf-neg-pos-meas*:
assumes *0* < *inf-neg E*
shows ¬ *pos-meas-set E* ⟨*proof*⟩

lemma *inf-neg-mem*:
assumes *0* < *inf-neg E*
shows *inf-neg E* ∈ {*n*::*nat*|*n*. (*1*::*nat*) ≤ *n* ∧
(∃ *B* ∈ *sets M*. *B* ⊆ *E* ∧ μ *B* < *ereal* (−*1*/*n*))}
⟨*proof*⟩

lemma *prec-inf-neg-pos*:
assumes *0* < *inf-neg E* − *1*
and *B* ∈ *sets M*
and *B*⊆ *E*
shows −*1*/(*inf-neg E* − *1*) ≤ μ *B*
⟨*proof*⟩

lemma *pos-wtn-meas-ge*:
  assumes $E \in sets\ M$
    and $|\mu\ E| < \infty$
    and $C \in sets\ M$
    and $\bigwedge n.\ C \subseteq pos\text{-}wtn\ E\ n$
    and $\bigwedge n.\ 0 < num\text{-}wtn\ E\ n$
  shows $\exists N.\ \forall n \geq N.\ -\ 1/\ (num\text{-}wtn\ E\ n\ -\ 1) \leq \mu\ C$
⟨*proof*⟩

lemma *pos-sub-pos-meas-subset*:
  assumes $E \in sets\ M$
    and $|\mu\ E| < \infty$
    and $C \in sets\ M$
    and $C \subseteq (pos\text{-}sub\ E)$
    and $\bigwedge n.\ 0 < num\text{-}wtn\ E\ n$
  shows $0 \leq \mu\ C$
⟨*proof*⟩

lemma *pos-sub-pos-meas′*:
  assumes $E \in sets\ M$
    and $|\mu\ E| < \infty$
    and $0 < \mu\ E$
    and $\forall n.\ 0 < num\text{-}wtn\ E\ n$
  shows $0 < \mu\ (pos\text{-}sub\ E)$
⟨*proof*⟩

We obtain the main result of this part on the existence of a positive subset.

lemma *exists-pos-meas-subset*:
  assumes $E \in sets\ M$
    and $|\mu\ E| < \infty$
    and $0 < \mu\ E$
  shows $\exists A.\ A \subseteq E \wedge pos\text{-}meas\text{-}set\ A \wedge 0 < \mu\ A$
⟨*proof*⟩

4 The Hahn decomposition theorem

definition *seq-meas* **where**
  $seq\text{-}meas = (SOME\ f.\ incseq\ f \wedge range\ f \subseteq pos\text{-}img\ \wedge \bigsqcup\ pos\text{-}img = \bigsqcup\ range\ f)$

lemma *seq-meas-props*:
  shows $incseq\ seq\text{-}meas \wedge range\ seq\text{-}meas \subseteq pos\text{-}img\ \wedge$
    $\bigsqcup\ pos\text{-}img = \bigsqcup\ range\ seq\text{-}meas$
⟨*proof*⟩

definition *seq-meas-rep* **where**
  $seq\text{-}meas\text{-}rep\ n = (SOME\ A.\ A \in pos\text{-}sets \wedge seq\text{-}meas\ n = \mu\ A)$

lemma *seq-meas-rep-ex*:
  shows $seq\text{-}meas\text{-}rep\ n \in pos\text{-}sets \wedge \mu\ (seq\text{-}meas\text{-}rep\ n) = seq\text{-}meas\ n$
⟨*proof*⟩

lemma *seq-meas-rep-pos*:
  assumes $\forall E \in sets\ M.\ \mu\ E < \infty$
  shows $pos\text{-}meas\text{-}set\ (\bigcup\ i.\ seq\text{-}meas\text{-}rep\ i)$
⟨*proof*⟩

lemma *sup-seq-meas-rep*:
  assumes $\forall E \in sets\ M.\ \mu\ E < \infty$
    and $S = (\bigsqcup\ pos\text{-}img)$
    and $A = (\bigcup\ i.\ seq\text{-}meas\text{-}rep\ i)$
  shows $\mu\ A = S$
⟨*proof*⟩

lemma *seq-meas-rep-compl*:
  assumes $\forall E \in sets\ M.\ \mu\ E < \infty$
    and $A = (\bigcup\ i.\ seq\text{-}meas\text{-}rep\ i)$
  shows $neg\text{-}meas\text{-}set\ ((space\ M) - A)$ ⟨*proof*⟩

lemma *hahn-decomp-finite*:
  assumes $\forall E \in sets\ M.\ \mu\ E < \infty$
  shows $\exists\ M1\ M2.\ hahn\text{-}space\text{-}decomp\ M1\ M2$ ⟨*proof*⟩

theorem *hahn-decomposition*:
  shows $\exists\ M1\ M2.\ hahn\text{-}space\text{-}decomp\ M1\ M2$
⟨*proof*⟩

5 The Jordan decomposition theorem

definition *jordan-decomp* **where**
  $jordan\text{-}decomp\ m1\ m2 \longleftrightarrow ((measure\text{-}space\ (space\ M)\ (sets\ M)\ m1) \wedge$
    $(measure\text{-}space\ (space\ M)\ (sets\ M)\ m2) \wedge$
    $(\forall A \in sets\ M.\ 0 \leq m1\ A) \wedge$
    $(\forall\ A \in sets\ M.\ 0 \leq m2\ A) \wedge$
    $(\forall A \in sets\ M.\ \mu\ A = (m1\ A) - (m2\ A)) \wedge$
    $(\forall\ P\ N\ A.\ hahn\text{-}space\text{-}decomp\ P\ N \longrightarrow$
      $(A \in sets\ M \longrightarrow A \subseteq P \longrightarrow (m2\ A) = 0) \wedge$
      $(A \in sets\ M \longrightarrow A \subseteq N \longrightarrow (m1\ A) = 0)) \wedge$
    $((\forall A \in sets\ M.\ m1\ A < \infty) \vee (\forall A \in sets\ M.\ m2\ A < \infty)))$

lemma *jordan-decomp-pos-meas*:
  assumes $jordan\text{-}decomp\ m1\ m2$
    and $hahn\text{-}space\text{-}decomp\ P\ N$
    and $A \in sets\ M$
  shows $m1\ A = \mu\ (A \cap P)$
⟨*proof*⟩

lemma *jordan-decomp-neg-meas*:
assumes *jordan-decomp m1 m2*
and *hahn-space-decomp P N*
and $A \in sets\ M$
shows $m2\ A = -\mu\ (A \cap N)$
⟨*proof*⟩

lemma *pos-inter-neg-0*:
assumes *hahn-space-decomp M1 M2*
and *hahn-space-decomp P N*
and $A \in sets\ M$
and $A \subseteq N$
shows $\mu\ (A \cap M1) = 0$
⟨*proof*⟩

lemma *neg-inter-pos-0*:
assumes *hahn-space-decomp M1 M2*
and *hahn-space-decomp P N*
and $A \in sets\ M$
and $A \subseteq P$
shows $\mu\ (A \cap M2) = 0$
⟨*proof*⟩

lemma *jordan-decomposition* :
shows $\exists\ m1\ m2.\ jordan\text{-}decomp\ m1\ m2$
⟨*proof*⟩

lemma *jordan-decomposition-unique* :
assumes *jordan-decomp m1 m2*
and *jordan-decomp n1 n2*
and $A \in sets\ M$
shows $m1\ A = n1\ A\ m2\ A = n2\ A$
⟨*proof*⟩

end

end

References

[1] E. DiBenedetto. *Real Analysis*. Birkhäuser Advanced Texts. Birkhäuser.