# An Isabelle/HOL formalisation of Green's Theorem

Mohammad Abdulaziz and Lawrence C. Paulson

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**Abstract**

We formalise a statement of Greens theorem—the first formalisation to our knowledge—in Isabelle/HOL. The theorem statement that we formalise is enough for most applications, especially in physics and engineering. Our formalisation is made possible by a novel proof that avoids the ubiquitous line integral cancellation argument. This eliminates the need to formalise orientations and region boundaries explicitly with respect to the outwards-pointing normal vector. Instead we appeal to a homological argument about equivalences between paths.

## 1 Acknowledgements

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**theory** *General-Utils*
  **imports** *HOL−Analysis.Analysis*
**begin**

**lemma** *lambda-skolem-gen*: $(\forall\, i.\ \exists f'::('a \mathbin{\hat{}} 'n) \Rightarrow 'a.\ P\ i\ f') \longleftrightarrow$
  $(\exists f'::('a \mathbin{\hat{}} 'n) \Rightarrow ('a \mathbin{\hat{}} 'n).\ \forall\, i.\ P\ i\ ((\lambda x.\ (f'\ x)\ \$\ i)))$ (**is** *?lhs* $\longleftrightarrow$ *?rhs*)

⟨*proof*⟩

**lemma** *lambda-skolem-euclidean*: $(\forall\, i{\in}Basis.\ \exists f'::('a::\{euclidean\text{-}space\}{\Rightarrow}real).\ P\ i\ f') \longleftrightarrow$
  $(\exists f'::('a::euclidean\text{-}space{\Rightarrow}'b::euclidean\text{-}space).\ \forall\, i{\in}Basis.\ P\ i\ ((\lambda x.\ (f'\ x) \cdot i)))$ (**is** *?lhs* $\longleftrightarrow$ *?rhs*)

⟨*proof*⟩

**lemma** *lambda-skolem-euclidean-explicit*: $(\forall\, i{\in}Basis.\ \exists f'::('a::\{euclidean\text{-}space\}{\Rightarrow}real).\ P\ i\ f') \longleftrightarrow$
  $(\exists f'::('a::\{euclidean\text{-}space\}{\Rightarrow}'a).\ \forall\, i{\in}Basis.\ P\ i\ ((\lambda x.\ (f'\ x) \cdot i)))$ (**is** *?lhs* $\longleftrightarrow$ *?rhs*)

$\langle proof \rangle$

**lemma** *indic-ident*:
$\bigwedge (f::'a \Rightarrow real)\ s.\ (\lambda x.\ (f\ x) * indicator\ s\ x) = (\lambda x.\ if\ x \in s\ then\ f\ x\ else\ 0)$
$\langle proof \rangle$

**lemma** *real-pair-basis*: $Basis = \{(1::real,0::real),\ (0::real,1::real)\}$
$\langle proof \rangle$

**lemma** *real-singleton-in-borel*:
**shows** $\{a::real\} \in sets\ borel$
$\langle proof \rangle$

**lemma** *real-singleton-in-lborel*:
**shows** $\{a::real\} \in sets\ lborel$
$\langle proof \rangle$

**lemma** *cbox-diff*:
**shows** $\{0::real..1\} - \{0,1\} = box\ 0\ 1$
$\langle proof \rangle$

**lemma** *sum-bij*:
**assumes** $bij\ F$
$\forall x \in s.\ f\ x = g\ (F\ x)$
**shows** $\bigwedge t.\ F\ `\ s = t \Longrightarrow sum\ f\ s = sum\ g\ t$
$\langle proof \rangle$

**abbreviation** *surj-on* **where**
$surj\text{-}on\ s\ f \equiv s \subseteq range\ f$

**lemma** *surj-on-image-vimage-eq*: $surj\text{-}on\ s\ f \Longrightarrow f\ `\ (f\ -`\ s) = s$
$\langle proof \rangle$

**end**
**theory** *Derivs*
**imports** *General-Utils*
**begin**

**lemma** *field-simp-has-vector-derivative* [*derivative-intros*]:
$(f\ has\text{-}field\text{-}derivative\ y)\ F \Longrightarrow (f\ has\text{-}vector\text{-}derivative\ y)\ F$
$\langle proof \rangle$

**lemma** *continuous-on-cases-empty* [*continuous-intros*]:
$[\![closed\ S;\ continuous\text{-}on\ S\ f;\ \bigwedge x.\ [\![x \in S;\ \neg\ P\ x]\!] \Longrightarrow f\ x = g\ x]\!] \Longrightarrow$
$continuous\text{-}on\ S\ (\lambda x.\ if\ P\ x\ then\ f\ x\ else\ g\ x)$
$\langle proof \rangle$

**lemma** *inj-on-cases*:
  **assumes** *inj-on f* (*Collect P* ∩ *S*) *inj-on g* (*Collect* (*Not* ∘ *P*) ∩ *S*)
    *f* ‘ (*Collect P* ∩ *S*) ∩ *g* ‘ (*Collect* (*Not* ∘ *P*) ∩ *S*) = {}
  **shows** *inj-on* (*λx. if P x then f x else g x*) *S*
  ⟨*proof*⟩

**lemma** *inj-on-arccos*: *S* ⊆ {−*1..1*} ⟹ *inj-on arccos S*
  ⟨*proof*⟩

**lemma** *has-vector-derivative-componentwise-within*:
  (*f has-vector-derivative f′*) (*at a within S*) ⟷
    (∀ *i* ∈ *Basis*. ((*λx. f x* • *i*) *has-vector-derivative* (*f′* • *i*)) (*at a within S*))
  ⟨*proof*⟩

**lemma** *has-vector-derivative-pair-within*:
  **fixes** *f* :: *real* ⇒ ′*a*::*euclidean-space* **and** *g* :: *real* ⇒ ′*b*::*euclidean-space*
  **assumes** ⋀*u*. *u* ∈ *Basis* ⟹ ((*λx. f x* • *u*) *has-vector-derivative f′* • *u*) (*at x within S*)
    ⋀*u*. *u* ∈ *Basis* ⟹ ((*λx. g x* • *u*) *has-vector-derivative g′* • *u*) (*at x within S*)
  **shows** ((*λx.* (*f x*, *g x*)) *has-vector-derivative* (*f′*,*g′*)) (*at x within S*)
  ⟨*proof*⟩

**lemma** *piecewise-C1-differentiable-const*:
  **shows** (*λx. c*) *piecewise-C1-differentiable-on s*
  ⟨*proof*⟩

**declare** *piecewise-C1-differentiable-const* [*simp*, *derivative-intros*]
**declare** *piecewise-C1-differentiable-neg* [*simp*, *derivative-intros*]
**declare** *piecewise-C1-differentiable-add* [*simp*, *derivative-intros*]
**declare** *piecewise-C1-differentiable-diff* [*simp*, *derivative-intros*]

**lemma** *piecewise-C1-differentiable-on-ident* [*simp*, *derivative-intros*]:
  **fixes** *f* :: *real* ⇒ ′*a*::*real-normed-vector*
  **shows** (*λx. x*) *piecewise-C1-differentiable-on S*
  ⟨*proof*⟩

**lemma** *piecewise-C1-differentiable-on-mult* [*simp*, *derivative-intros*]:
  **fixes** *f* :: *real* ⇒ ′*a*::*real-normed-algebra*
  **assumes** *f piecewise-C1-differentiable-on S g piecewise-C1-differentiable-on S*
  **shows** (*λx. f x* ∗ *g x*) *piecewise-C1-differentiable-on S*
  ⟨*proof*⟩

**lemma** *C1-differentiable-on-cdiv* [*simp*, *derivative-intros*]:
  **fixes** *f* :: *real* ⇒ ′*a* :: *real-normed-field*
  **shows** *f C1-differentiable-on S* ⟹ (*λx. f x* / *c*) *C1-differentiable-on S*
  ⟨*proof*⟩

**lemma** *piecewise-C1-differentiable-on-cdiv* [*simp*, *derivative-intros*]:
  **fixes** *f* :: *real* ⇒ *'a*::*real-normed-field*
  **assumes** *f piecewise-C1-differentiable-on S*
  **shows** (λ*x*. *f x* / *c*) *piecewise-C1-differentiable-on S*
  ⟨*proof*⟩

**lemma** *sqrt-C1-differentiable* [*simp*, *derivative-intros*]:
  **assumes** *f*: *f C1-differentiable-on S* **and** *fim*: *f* ‘ *S* ⊆ {*0*<..}
  **shows** (λ*x*. *sqrt* (*f x*)) *C1-differentiable-on S*
⟨*proof*⟩

**lemma** *sqrt-piecewise-C1-differentiable* [*simp*, *derivative-intros*]:
  **assumes** *f*: *f piecewise-C1-differentiable-on S* **and** *fim*: *f* ‘ *S* ⊆ {*0*<..}
  **shows** (λ*x*. *sqrt* (*f x*)) *piecewise-C1-differentiable-on S*
  ⟨*proof*⟩

**lemma**
  **fixes** *f* :: *real* ⇒ *'a*::{*banach*,*real-normed-field*}
  **assumes** *f*: *f C1-differentiable-on S*
 **shows** *sin-C1-differentiable* [*simp*, *derivative-intros*]: (λ*x*. *sin* (*f x*)) *C1-differentiable-on S*
 **and** *cos-C1-differentiable* [*simp*, *derivative-intros*]: (λ*x*. *cos* (*f x*)) *C1-differentiable-on S*
⟨*proof*⟩

**lemma** *has-derivative-abs*:
  **fixes** *a*::*real*
  **assumes** *a* ≠ *0*
  **shows** (*abs has-derivative* ((∗) (*sgn a*))) (*at a*)
⟨*proof*⟩

**lemma** *abs-C1-differentiable* [*simp*, *derivative-intros*]:
  **fixes** *f* :: *real* ⇒ *real*
  **assumes** *f*: *f C1-differentiable-on S* **and** *0* ∉ *f* ‘ *S*
  **shows** (λ*x*. *abs* (*f x*)) *C1-differentiable-on S*
⟨*proof*⟩

**lemma** *C1-differentiable-on-pair* [*simp*, *derivative-intros*]:
  **fixes** *f* :: *real* ⇒ *'a*::*euclidean-space* **and** *g* :: *real* ⇒ *'b*::*euclidean-space*
  **assumes** *f C1-differentiable-on S g C1-differentiable-on S*
  **shows** (λ*x*. (*f x*, *g x*)) *C1-differentiable-on S*
  ⟨*proof*⟩

**lemma** *piecewise-C1-differentiable-on-pair* [*simp*, *derivative-intros*]:
  **fixes** *f* :: *real* ⇒ *'a*::*euclidean-space* **and** *g* :: *real* ⇒ *'b*::*euclidean-space*
  **assumes** *f piecewise-C1-differentiable-on S g piecewise-C1-differentiable-on S*
  **shows** (λ*x*. (*f x*, *g x*)) *piecewise-C1-differentiable-on S*
  ⟨*proof*⟩

**lemma** *test2*:
  **assumes** *s*: $\bigwedge x.\ x \in \{0..1\} - s \Longrightarrow g$ *differentiable at* $x$
    **and** *fs*: *finite* $s$ **and** *uv*: $u \in \{0..1\}$ $v \in \{0..1\}$ $u \leq v$
    **and** $x \in \{0..1\}$ $x \notin (\lambda t.\ (v-u) *_R t + u) -` s$
  **shows** *vector-derivative* $(\lambda x.\ g\ ((v-u) * x + u))$ (*at* $x$ *within* $\{0..1\}$) $= (v-u)$ $*_R$ *vector-derivative* $g$ (*at* $((v-u) * x + u)$ *within*$\{0..1\}$)
⟨*proof*⟩

**lemma** *C1-differentiable-on-components*:
  **assumes** $\bigwedge i.\ i \in Basis \Longrightarrow (\lambda x.\ f\ x \cdot i)$ *C1-differentiable-on* $s$
  **shows** $f$ *C1-differentiable-on* $s$
⟨*proof*⟩

**lemma** *piecewise-C1-differentiable-on-components*:
  **assumes** *finite* $t$
    $\bigwedge i.\ i \in Basis \Longrightarrow (\lambda x.\ f\ x \cdot i)$ *C1-differentiable-on* $s - t$
    $\bigwedge i.\ i \in Basis \Longrightarrow$ *continuous-on* $s$ $(\lambda x.\ f\ x \cdot i)$
  **shows** $f$ *piecewise-C1-differentiable-on* $s$
  ⟨*proof*⟩

**lemma** *all-components-smooth-one-pw-smooth-is-pw-smooth*:
  **assumes** $\bigwedge i.\ i \in Basis - \{j\} \Longrightarrow (\lambda x.\ f\ x \cdot i)$ *C1-differentiable-on* $s$
  **assumes** $(\lambda x.\ f\ x \cdot j)$ *piecewise-C1-differentiable-on* $s$
  **shows** $f$ *piecewise-C1-differentiable-on* $s$
⟨*proof*⟩

**lemma** *derivative-component-fun-component*:
  **fixes** $i$::$'a$::*euclidean-space*
  **assumes** $f$ *differentiable* (*at* $x$)
  **shows** ((*vector-derivative* $f$ (*at* $x$)) $\cdot$ $i$) $=$ ((*vector-derivative* $(\lambda x.\ (f\ x) \cdot i)$ (*at* $x$)) )
⟨*proof*⟩

**lemma** *gamma-deriv-at-within*:
  **assumes** *a-leq-b*: $a < b$ **and**
    *x-within-bounds*: $x \in \{a..b\}$ **and**
    *gamma-differentiable*: $\forall x \in \{a\ ..\ b\}.\ \gamma$ *differentiable at* $x$
  **shows** *vector-derivative* $\gamma$ (*at* $x$ *within* $\{a..b\}$) $=$ *vector-derivative* $\gamma$ (*at* $x$)
  ⟨*proof*⟩

**lemma** *islimpt-diff-finite*:
  **assumes** *finite* ($t$::$'a$::*t1-space set*)
  **shows** $x$ *islimpt* $s - t = x$ *islimpt* $s$
⟨*proof*⟩

**lemma** *ivl-limpt-diff*:
  **assumes** *finite* $s$ $a < b$ $(x$::*real*$) \in \{a..b\} - s$
  **shows** $x$ *islimpt* $\{a..b\} - s$

$\langle proof \rangle$

**lemma** *ivl-closure-diff-del*:
  **assumes** *finite* $s$ $a < b$ $(x::real) \in \{a..b\} - s$
  **shows** $x \in$ *closure* $((\{a..b\} - s) - \{x\})$
  $\langle proof \rangle$

**lemma** *ivl-not-trivial-limit-within*:
  **assumes** *finite* $s$
    $a < b$
    $(x::real) \in \{a..b\} - s$
  **shows** *at* $x$ *within* $\{a..b\} - s \neq$ *bot*
  $\langle proof \rangle$

**lemma** *vector-derivative-at-within-non-trivial-limit*:
  *at* $x$ *within* $s \neq$ *bot* $\wedge$ ($f$ *has-vector-derivative* $f'$) (*at* $x$) $\Longrightarrow$
    *vector-derivative* $f$ (*at* $x$ *within* $s$) $= f'$
  $\langle proof \rangle$

**lemma** *vector-derivative-at-within-ivl-diff*:
  *finite* $s \wedge a < b \wedge (x::real) \in \{a..b\} - s \wedge$ ($f$ *has-vector-derivative* $f'$) (*at* $x$) $\Longrightarrow$
    *vector-derivative* $f$ (*at* $x$ *within* $\{a..b\} - s$) $= f'$
  $\langle proof \rangle$

**lemma** *gamma-deriv-at-within-diff*:
  **assumes** *a-leq-b*: $a < b$ **and**
    *x-within-bounds*: $x \in \{a..b\} - s$ **and**
    *gamma-differentiable*: $\forall x \in \{a .. b\} - s.$ $\gamma$ *differentiable at* $x$ **and**
    *s-subset*: $s \subseteq \{a..b\}$ **and**
    *finite-s*: *finite* $s$
  **shows** *vector-derivative* $\gamma$ (*at* $x$ *within* $\{a..b\} - s$)
              $=$ *vector-derivative* $\gamma$ (*at* $x$)
  $\langle proof \rangle$

**lemma** *gamma-deriv-at-within-gen*:
  **assumes** *a-leq-b*: $a < b$ **and**
    *x-within-bounds*: $x \in s$ **and**
    *s-subset*: $s \subseteq \{a..b\}$ **and**
    *gamma-differentiable*: $\forall x \in s.$ $\gamma$ *differentiable at* $x$
  **shows** *vector-derivative* $\gamma$ (*at* $x$ *within* $(\{a..b\})$) $=$ *vector-derivative* $\gamma$ (*at* $x$)
  $\langle proof \rangle$

**lemma** *derivative-component-fun-component-at-within-gen*:
  **assumes** *gamma-differentiable*: $\forall x \in s.$ $\gamma$ *differentiable at* $x$ **and** *s-subset*: $s \subseteq \{0..1\}$
  **shows** $\forall x \in s.$ *vector-derivative* $(\lambda x.\ \gamma\ x)$ (*at* $x$ *within* $\{0..1\}$) $\cdot$ ($i$::$'a$:: *euclidean-space*)
      $=$ *vector-derivative* $(\lambda x.\ \gamma\ x \cdot i)$ (*at* $x$ *within* $\{0..1\}$)
$\langle proof \rangle$

**lemma** *derivative-component-fun-component-at-within*:
**assumes** *gamma-differentiable*: $\forall x \in \{0 .. 1\}$. $\gamma$ *differentiable at x*
**shows** $\forall x \in \{0..1\}$. *vector-derivative* ($\lambda x$. $\gamma$ *x*) (*at x within* $\{0..1\}$) $\cdot$ (*i*::'*a*:: *euclidean-space*)
= *vector-derivative* ($\lambda x$. $\gamma$ *x* $\cdot$ *i*) (*at x within* $\{0..1\}$)
⟨*proof*⟩

**lemma** *straight-path-diffrentiable-x*:
**fixes** *b* :: *real* **and** *y1* ::*real*
**assumes** *gamma-def*: $\gamma = (\lambda x.\ (b,\ y2 + y1 * x))$
**shows** $\forall x$. $\gamma$ *differentiable at x*
⟨*proof*⟩

**lemma** *straight-path-diffrentiable-y*:
**fixes** *b* :: *real* **and**
*y1 y2* ::*real*
**assumes** *gamma-def*: $\gamma = (\lambda x.\ (y2 + y1 * x\ ,\ b))$
**shows** $\forall x$. $\gamma$ *differentiable at x*
⟨*proof*⟩

**lemma** *piecewise-C1-differentiable-on-imp-continuous-on*:
**assumes** *f piecewise-C1-differentiable-on s*
**shows** *continuous-on s f*
⟨*proof*⟩

**lemma** *boring-lemma1*:
**fixes** *f* :: *real*⇒*real*
**assumes** (*f has-vector-derivative D*) (*at x*)
**shows** (($\lambda x$. (*f x*, *0*)) *has-vector-derivative* ((*D*,*0*::*real*))) (*at x*)
⟨*proof*⟩

**lemma** *boring-lemma2*:
**fixes** *f* :: *real*⇒*real*
**assumes** (*f has-vector-derivative D*) (*at x*)
**shows** (($\lambda x$. (*0*, *f x*)) *has-vector-derivative* (*0*, *D*)) (*at x*)
⟨*proof*⟩

**lemma** *pair-prod-smooth-pw-smooth*:
**assumes** (*f*::*real*⇒*real*) *C1-differentiable-on s* (*g*::*real*⇒*real*) *piecewise-C1-differentiable-on s*
**shows** ($\lambda x$. (*f x*, *g x*)) *piecewise-C1-differentiable-on s*
⟨*proof*⟩

**lemma** *scale-shift-smooth*:
**shows** ($\lambda x$. $a + b * x$) *C1-differentiable-on s*
⟨*proof*⟩

**lemma** *open-diff*:

**assumes** *finite* $(t::'a::t1\text{-}space\ set)$
*open* $(s::'a\ set)$
**shows** *open* $(s - t)$
$\langle proof \rangle$

**lemma** *has-derivative-transform-within*:
**assumes** $0 < d$
**and** $x \in s$
**and** $\forall x' \in s.\ dist\ x'\ x < d \longrightarrow f\ x' = g\ x'$
**and** $(f\ has\text{-}derivative\ f')\ (at\ x\ within\ s)$
**shows** $(g\ has\text{-}derivative\ f')\ (at\ x\ within\ s)$
$\langle proof \rangle$

**lemma** *has-derivative-transform-within-ivl*:
**assumes** $(0::real) < b$
**and** $\forall x \in \{a..b\} - s.\ f\ x = g\ x$
**and** $x \in \{a..b\} - s$
**and** $(f\ has\text{-}derivative\ f')\ (at\ x\ within\ \{a..b\} - s)$
**shows** $(g\ has\text{-}derivative\ f')\ (at\ x\ within\ \{a..b\} - s)$
$\langle proof \rangle$

**lemma** *has-vector-derivative-transform-within-ivl*:
**assumes** $(0::real) < b$
**and** $\forall x \in \{a..b\} - s\ .\ f\ x = g\ x$
**and** $x \in \{a..b\} - s$
**and** $(f\ has\text{-}vector\text{-}derivative\ f')\ (at\ x\ within\ \{a..b\} - s)$
**shows** $(g\ has\text{-}vector\text{-}derivative\ f')\ (at\ x\ within\ \{a..b\} - s)$
$\langle proof \rangle$

**lemma** *has-derivative-transform-at*:
**assumes** $0 < d$
**and** $\forall x'.\ dist\ x'\ x < d \longrightarrow f\ x' = g\ x'$
**and** $(f\ has\text{-}derivative\ f')\ (at\ x)$
**shows** $(g\ has\text{-}derivative\ f')\ (at\ x)$
$\langle proof \rangle$

**lemma** *has-vector-derivative-transform-at*:
**assumes** $0 < d$
**and** $\forall x'.\ dist\ x'\ x < d \longrightarrow f\ x' = g\ x'$
**and** $(f\ has\text{-}vector\text{-}derivative\ f')\ (at\ x)$
**shows** $(g\ has\text{-}vector\text{-}derivative\ f')\ (at\ x)$
$\langle proof \rangle$

**lemma** *C1-diff-components-2*:
**assumes** $b \in Basis$
**assumes** $f\ C1\text{-}differentiable\text{-}on\ s$
**shows** $(\lambda x.\ f\ x \cdot b)\ C1\text{-}differentiable\text{-}on\ s$
$\langle proof \rangle$

**lemma** *eq-smooth*:
**assumes** $0 < d$
$\forall x\in s.\ \forall y.\ dist\ x\ y < d \longrightarrow f\ y = g\ y$
*f C1-differentiable-on s*
**shows** *g C1-differentiable-on s*
$\langle proof \rangle$

**lemma** *eq-pw-smooth*:
**assumes** $0 < d$
$\forall x\in s.\ \forall y.\ dist\ x\ y < d \longrightarrow f\ y = g\ y$
$\forall x\in s.\ f\ x = g\ x$
*f piecewise-C1-differentiable-on s*
**shows** *g piecewise-C1-differentiable-on s*
$\langle proof \rangle$

**lemma** *scale-piecewise-C1-differentiable-on*:
**assumes** *f piecewise-C1-differentiable-on s*
**shows** $(\lambda x.\ (c::real) * (f\ x))$ *piecewise-C1-differentiable-on s*
$\langle proof \rangle$

**lemma** *eq-smooth-gen*:
**assumes** *f C1-differentiable-on s*
$\forall x.\ f\ x = g\ x$
**shows** *g C1-differentiable-on s*
$\langle proof \rangle$

**lemma** *subpath-compose*:
**shows** $(subpath\ a\ b\ \gamma) = \gamma\ o\ (\lambda x.\ \ (b - a) * x + a)$
$\langle proof \rangle$

**lemma** *subpath-smooth*:
**assumes** $\gamma$ *C1-differentiable-on* $\{0..1\}$ $0 \le a$ $a < b$ $b \le 1$
**shows** $(subpath\ a\ b\ \gamma)$ *C1-differentiable-on* $\{0..1\}$
$\langle proof \rangle$

**lemma** *has-vector-derivative-divide*[*derivative-intros*]:
**fixes** $a :: 'a::real\text{-}normed\text{-}field$
**shows** $(f\ has\text{-}vector\text{-}derivative\ x)\ F \Longrightarrow ((\lambda x.\ f\ x\ /\ a)\ has\text{-}vector\text{-}derivative\ (x\ /\ a))\ F$
$\langle proof \rangle$

**end**
**theory** *Integrals*
**imports** *HOL−Analysis.Analysis General-Utils*
**begin**

**lemma** *gauge-integral-Fubini-universe-x*:
**fixes** $f :: ('a::euclidean\text{-}space * 'b::euclidean\text{-}space) \Rightarrow 'c::euclidean\text{-}space$
**assumes** *fun-lesbegue-integrable*: *integrable lborel f* **and**

*x-axis-integral-measurable*: $(\lambda x.\ integral\ UNIV\ (\lambda y.\ f(x,\ y))) \in borel\text{-}measurable$ *lborel*
**shows** $integral\ UNIV\ f = integral\ UNIV\ (\lambda x.\ integral\ UNIV\ (\lambda y.\ f(x,y)))$
$(\lambda x.\ integral\ UNIV\ (\lambda y.\ f(x,y)))\ integrable\text{-}on\ UNIV$
⟨*proof*⟩

**lemma** *gauge-integral-Fubini-universe-y*:
**fixes** $f :: ('a::euclidean\text{-}space * 'b::euclidean\text{-}space) \Rightarrow 'c::euclidean\text{-}space$
**assumes** *fun-lesbegue-integrable*: *integrable lborel f* **and**
*y-axis-integral-measurable*: $(\lambda x.\ integral\ UNIV\ (\lambda y.\ f(y,\ x))) \in borel\text{-}measurable$ *lborel*
**shows** $integral\ UNIV\ f = integral\ UNIV\ (\lambda x.\ integral\ UNIV\ (\lambda y.\ f(y,\ x)))$
$(\lambda x.\ integral\ UNIV\ (\lambda y.\ f(y,\ x)))\ integrable\text{-}on\ UNIV$
⟨*proof*⟩

**lemma** *gauge-integral-Fubini-curve-bounded-region-x*:
**fixes** $f\ g :: ('a::euclidean\text{-}space * 'b::euclidean\text{-}space) \Rightarrow 'c::euclidean\text{-}space$ **and**
$g1\ g2 :: 'a \Rightarrow 'b$ **and**
$s :: ('a * 'b)\ set$
**assumes** *fun-lesbegue-integrable*: *integrable lborel f* **and**
*x-axis-gauge-integrable*: $\bigwedge x.\ (\lambda y.\ f(x,\ y))\ integrable\text{-}on\ UNIV$ **and**

*x-axis-integral-measurable*: $(\lambda x.\ integral\ UNIV\ (\lambda y.\ f(x,\ y))) \in borel\text{-}measurable$ *lborel* **and**
*f-is-g-indicator*: $f = (\lambda x.\ if\ x \in s\ then\ g\ x\ else\ 0)$ **and**
*s-is-bounded-by-g1-and-g2*: $s = \{(x,y).\ (\forall i\in Basis.\ a \cdot i \le x \cdot i \wedge x \cdot i \le b \cdot i) \wedge$
$(\forall i\in Basis.\ (g1\ x) \cdot i \le y \cdot i \wedge y \cdot i \le (g2\ x) \cdot i)\}$
**shows** $integral\ s\ g = integral\ (cbox\ a\ b)\ (\lambda x.\ integral\ (cbox\ (g1\ x)\ (g2\ x))\ (\lambda y.\ g(x,y)))$
⟨*proof*⟩

**lemma** *gauge-integral-Fubini-curve-bounded-region-y*:
**fixes** $f\ g :: ('a::euclidean\text{-}space * 'b::euclidean\text{-}space) \Rightarrow 'c::euclidean\text{-}space$ **and**
$g1\ g2 :: 'b \Rightarrow 'a$ **and**
$s :: ('a * 'b)\ set$
**assumes** *fun-lesbegue-integrable*: *integrable lborel f* **and**
*y-axis-gauge-integrable*: $\bigwedge x.\ (\lambda y.\ f(y,\ x))\ integrable\text{-}on\ UNIV$ **and**

*y-axis-integral-measurable*: $(\lambda x.\ integral\ UNIV\ (\lambda y.\ f(y,\ x))) \in borel\text{-}measurable$ *lborel* **and**
*f-is-g-indicator*: $f = (\lambda x.\ if\ x \in s\ then\ g\ x\ else\ 0)$ **and**
*s-is-bounded-by-g1-and-g2*: $s = \{(y,\ x).\ (\forall i\in Basis.\ a \cdot i \le x \cdot i \wedge x \cdot i \le b \cdot i) \wedge$
$(\forall i\in Basis.\ (g1\ x) \cdot i \le y \cdot i \wedge y \cdot i \le (g2\ x) \cdot i)\}$
**shows** $integral\ s\ g = integral\ (cbox\ a\ b)\ (\lambda x.\ integral\ (cbox\ (g1\ x)\ (g2\ x))\ (\lambda y.\ g(y,\ x)))$
⟨*proof*⟩

**lemma** *gauge-integral-by-substitution*:
  **fixes** $f$::(*real* $\Rightarrow$ *real*) **and**
    $g$::(*real* $\Rightarrow$ *real*) **and**
    $g'$::*real* $\Rightarrow$ *real* **and**
    $a$::*real* **and**
    $b$::*real*
  **assumes** *a-le-b*: $a \leq b$ **and**
    *ga-le-gb*: $g\ a \leq g\ b$ **and**
    *g′-derivative*: $\forall\, x \in \{a..b\}$. ($g$ *has-vector-derivative* $(g'\ x)$) (*at* $x$ *within* $\{a..b\}$)
**and**
    *g′-continuous*: *continuous-on* $\{a..b\}$ $g'$ **and**
    *f-continuous*: *continuous-on* ($g$ ‘ $\{a..b\}$) $f$
  **shows** *integral* $\{g\ a..g\ b\}$ $(f)$ = *integral* $\{a..b\}$ $(\lambda x.\ f(g\ x) * (g'\ x))$
⟨*proof*⟩

**lemma** *frontier-ic*:
  **assumes** $a < (b$::*real*)
  **shows** *frontier* $\{a<..b\}$ $= \{a,b\}$
  ⟨*proof*⟩

**lemma** *frontier-ci*:
  **assumes** $a < (b$::*real*)
  **shows** *frontier* $\{a<..<b\}$ $= \{a,b\}$
  ⟨*proof*⟩

**lemma** *ic-not-closed*:
  **assumes** $a < (b$::*real*)
  **shows** $\neg$ *closed* $\{a<..b\}$
  ⟨*proof*⟩

**lemma** *closure-ic-union-ci*:
  **assumes** $a < (b$::*real*) $b < c$
  **shows** *closure* $(\{a..<b\} \cup \{b<..c\})$ $= \{a\ ..\ c\}$
  ⟨*proof*⟩

**lemma** *interior-ic-ci-union*:
  **assumes** $a < (b$::*real*) $b < c$
  **shows** $b \notin$ (*interior* $(\{a..<b\} \cup \{b<..c\})$)
⟨*proof*⟩

**lemma** *frontier-ic-union-ci*:
  **assumes** $a < (b$::*real*) $b < c$
  **shows** $b \in$ *frontier* $(\{a..<b\} \cup \{b<..c\})$
  ⟨*proof*⟩

**lemma** *ic-union-ci-not-closed*:
  **assumes** $a < (b$::*real*) $b < c$
  **shows** $\neg$ *closed* $(\{a..<b\} \cup \{b<..c\})$

⟨*proof*⟩

**lemma** *integrable-continuous-*:
  **fixes** $f :: 'b::euclidean\text{-}space \Rightarrow 'a::banach$
  **assumes** *continuous-on* $(cbox\ a\ b)\ f$
  **shows** $f$ *integrable-on cbox a b*
  ⟨*proof*⟩

**lemma** *removing-singletons-from-div*:
  **assumes** $\forall t \in S.\ \exists c\ d::real.\ c < d \wedge \{c..d\} = t$
    $\{x\} \cup \bigcup S = \{a..b\}$ $a < x$ $x < b$
    *finite* $S$
  **shows** $\exists t \in S.\ x \in t$
⟨*proof*⟩

**lemma** *remove-singleton-from-division-of*:
  **assumes** $A$ *division-of* $\{a::real..b\}$ $a < b$
  **assumes** $x \in \{a..b\}$
  **shows** $\exists c\ d.\ c < d \wedge \{c..d\} \in A \wedge x \in \{c..d\}$
⟨*proof*⟩

**lemma** *remove-singleton-from-tagged-division-of*:
  **assumes** $A$ *tagged-division-of* $\{a::real..b\}$ $a < b$
  **assumes** $x \in \{a..b\}$
  **shows** $\exists k\ c\ d.\ c < d \wedge (k, \{c..d\}) \in A \wedge x \in \{c..d\}$
  ⟨*proof*⟩

**lemma** *tagged-div-wo-singlestons*:
  **assumes** $p$ *tagged-division-of* $\{a::real..b\}$ $a < b$
  **shows** $(p - \{xk.\ \exists x\ y.\ xk = (x,\{y\})\})$ *tagged-division-of cbox a b*
  ⟨*proof*⟩

**lemma** *tagged-div-wo-empty*:
  **assumes** $p$ *tagged-division-of* $\{a::real..b\}$ $a < b$
  **shows** $(p - \{xk.\ \exists x.\ xk = (x,\{\})\})$ *tagged-division-of cbox a b*
  ⟨*proof*⟩

**lemma** *fine-diff*:
  **assumes** $\gamma$ *fine* $p$
  **shows** $\gamma$ *fine* $(p - s)$
  ⟨*proof*⟩

**lemma** *tagged-div-tage-notin-set*:
  **assumes** *finite* $(s::real\ set)$
    $p$ *tagged-division-of* $\{a..b\}$
    $\gamma$ *fine* $p$ $(\forall (x, K) \in p.\ \exists c\ d::real.\ c < d \wedge K = \{c..d\})$ *gauge* $\gamma$
  **shows** $\exists p'\ \gamma'.\ p'$ *tagged-division-of* $\{a..b\} \wedge$
      $\gamma'$ *fine* $p' \wedge (\forall (x, K) \in p'.\ x \notin s) \wedge$ *gauge* $\gamma'$
⟨*proof*⟩

**lemma** *has-integral-bound-spike-finite*:
**fixes** $f$ :: $'a$::*euclidean-space* ⇒ $'b$::*real-normed-vector*
**assumes** $0 \leq B$ **and** *finite* $S$
**and** $f$: ($f$ *has-integral* $i$) (*cbox* $a$ $b$)
**and** *leB*: $\bigwedge x.\ x \in$ *cbox* $a$ $b - S \Longrightarrow$ *norm* ($f$ $x$) $\leq B$
**shows** *norm* $i \leq B *$ *content* (*cbox* $a$ $b$)
⟨*proof*⟩

**lemma** *has-integral-bound-*:
**fixes** $f$ :: *real* ⇒ $'a$::*real-normed-vector*
**assumes** $a < b$
**and** $0 \leq B$
**and** $f$: ($f$ *has-integral* $i$) (*cbox* $a$ $b$)
**and** *finite* $s$
**and** $\forall x\in$(*cbox* $a$ $b$)$-s$. *norm* ($f$ $x$) $\leq B$
**shows** *norm* $i \leq B *$ *content* (*cbox* $a$ $b$)
⟨*proof*⟩

**corollary** *has-integral-bound-real'*:
**fixes** $f$ :: *real* ⇒ $'b$::*real-normed-vector*
**assumes** $0 \leq B$
**and** $f$: ($f$ *has-integral* $i$) (*cbox* $a$ $b$)
**and** *finite* $s$
**and** $\forall x\in$(*cbox* $a$ $b$)$-s$. *norm* ($f$ $x$) $\leq B$
**shows** *norm* $i \leq B *$ *content* $\{a..b\}$
⟨*proof*⟩

**lemma** *integral-has-vector-derivative-continuous-at'*:
**fixes** $f$ :: *real* ⇒ $'a$::*banach*
**assumes** *finite* $s$
**and** $f$: $f$ *integrable-on* $\{a..b\}$
**and** $x$: $x \in \{a..b\} - s$
**and** *fx*: *continuous* (*at* $x$ *within* ($\{a..b\} - s$)) $f$
**shows** (($\lambda u$. *integral* $\{a..u\}$ $f$) *has-vector-derivative* $f$ $x$) (*at* $x$ *within* ($\{a..b\} - s$))
⟨*proof*⟩

**lemma** *at-within-closed-interval-finite*:
**fixes** $x$::*real*
**assumes** $a < x$ $x < b$ $x \notin S$ *finite* $S$
**shows** (*at* $x$ *within* $\{a..b\} - S$) = *at* $x$
⟨*proof*⟩

**lemma** *fundamental-theorem-of-calculus-interior-stronger'*:
**fixes** $f$ :: *real* ⇒ $'a$::*banach*
**assumes** *finite* $S$
**and** $a \leq b$ $\bigwedge x.\ x \in \{a <..< b\} - S \Longrightarrow$ ($f$ *has-vector-derivative* $f'(x)$) (*at* $x$ *within* $\{a..b\} - S$)

**and** *continuous-on* $\{a \,..\, b\}$ *f*
**shows** $(f'$ *has-integral* $(f\ b - f\ a))$ $\{a \,..\, b\}$
⟨*proof*⟩

**lemma** *has-integral-substitution-general-*:
**fixes** *f* :: *real* ⇒ ′*a*::*euclidean-space* **and** *g* :: *real* ⇒ *real*
**assumes** *s*: *finite s* **and** *le*: $a \leq b$
**and** *subset*: *g* ‘ $\{a..b\} \subseteq \{c..d\}$
**and** *f*: *f integrable-on* $\{c..d\}$ *continuous-on* $(\{c..d\} - (g$ ‘ $s))$ *f*
**and** *g* : *continuous-on* $\{a..b\}$ *g inj-on g* $(\{a..b\} \cup s)$
**and** *deriv* [*derivative-intros*]:
$\bigwedge x.\ x \in \{a..b\} - s \Longrightarrow$ (*g has-field-derivative* $g'\ x$) (*at x within* $\{a..b\}$)
**shows** $((\lambda x.\ g'\ x *_R f\ (g\ x))$ *has-integral* (*integral* $\{g\ a..g\ b\}$ *f* − *integral* $\{g\ b..g\ a\}$ *f*)) $\{a..b\}$
⟨*proof*⟩

**lemma** *has-integral-substitution-general--*:
**fixes** *f* :: *real* ⇒ ′*a*::*euclidean-space* **and** *g* :: *real* ⇒ *real*
**assumes** *s*: *finite s* **and** *le*: $a \leq b$ **and** *s-subset*: $s \subseteq \{a..b\}$
**and** *subset*: *g* ‘ $\{a..b\} \subseteq \{c..d\}$
**and** *f*: *f integrable-on* $\{c..d\}$ *continuous-on* $(\{c..d\} - (g$ ‘ $s))$ *f*
**and** *g* : *continuous-on* $\{a..b\}$ *g inj-on g* $\{a..b\}$
**and** *deriv* [*derivative-intros*]:
$\bigwedge x.\ x \in \{a..b\} - s \Longrightarrow$ (*g has-field-derivative* $g'\ x$) (*at x within* $\{a..b\}$)
**shows** $((\lambda x.\ g'\ x *_R f\ (g\ x))$ *has-integral* (*integral* $\{g\ a..g\ b\}$ *f* − *integral* $\{g\ b..g\ a\}$ *f*)) $\{a..b\}$
⟨*proof*⟩

**lemma** *has-integral-substitution-general-*′:
**fixes** *f* :: *real* ⇒ ′*a*::*euclidean-space* **and** *g* :: *real* ⇒ *real*
**assumes** *s*: *finite s* **and** *le*: $a \leq b$ **and** $s'$: *finite* $s'$
**and** *subset*: *g* ‘ $\{a..b\} \subseteq \{c..d\}$
**and** *f*: *f integrable-on* $\{c..d\}$ *continuous-on* $(\{c..d\} - s')$ *f*
**and** *g* : *continuous-on* $\{a..b\}$ *g* $\forall x{\in}s'.$ *finite* $(g$ −‘ $\{x\})$ *surj-on* $s'$ *g inj-on g* $(\{a..b\} \cup ((s \cup g$ −‘ $s')))$
**and** *deriv* [*derivative-intros*]:
$\bigwedge x.\ x \in \{a..b\} - s \Longrightarrow$ (*g has-field-derivative* $g'\ x$) (*at x within* $\{a..b\}$)
**shows** $((\lambda x.\ g'\ x *_R f\ (g\ x))$ *has-integral* (*integral* $\{g\ a..g\ b\}$ *f* − *integral* $\{g\ b..g\ a\}$ *f*)) $\{a..b\}$
⟨*proof*⟩

**end**
**theory** *Paths*
**imports** *Derivs General-Utils Integrals*
**begin**

**lemma** *reverse-subpaths-join*:
**shows** *subpath 1 (1 / 2) p* +++ *subpath (1 / 2) 0 p* = *reversepath p*

⟨*proof*⟩

**definition** *line-integral*:: $('a::euclidean\text{-}space \Rightarrow 'a::euclidean\text{-}space) \Rightarrow (('a)\ set) \Rightarrow (real \Rightarrow 'a) \Rightarrow real$ **where**
*line-integral F basis g* $\equiv$ *integral* $\{0\ ..\ 1\}$ $(\lambda x.\ \sum b{\in}basis.\ (F(g\ x) \cdot b) * (vector\text{-}derivative\ g\ (at\ x\ within\ \{0..1\}) \cdot b))$

**definition** *line-integral-exists* **where**
*line-integral-exists F basis* $\gamma \equiv (\lambda x.\ \sum b{\in}basis.\ F(\gamma\ x) \cdot b * (vector\text{-}derivative\ \gamma\ (at\ x\ within\ \{0..1\}) \cdot b))$ *integrable-on* $\{0..1\}$

**lemma** *line-integral-on-pair-straight-path*:
**fixes** $F::('a::euclidean\text{-}space) \Rightarrow 'a$ **and** $g :: real \Rightarrow real$ **and** $\gamma$
**assumes** *gamma-const*: $\forall x.\ \gamma(x) \cdot i = a$
**and** *gamma-smooth*: $\forall x \in \{0\ ..\ 1\}.\ \gamma$ *differentiable at* $x$
**shows** (*line-integral* $F$ $\{i\}$ $\gamma$) = *0* (*line-integral-exists* $F$ $\{i\}$ $\gamma$)
⟨*proof*⟩

**lemma** *line-integral-on-pair-path-strong*:
**fixes** $F::('a::euclidean\text{-}space) \Rightarrow ('a)$ **and**
$g::real \Rightarrow 'a$ **and**
$\gamma::(real \Rightarrow 'a)$ **and**
$i::'a$
**assumes** *i-norm-1*: *norm* $i = 1$ **and**
*g-orthogonal-to-i*: $\forall x.\ g(x) \cdot i = 0$ **and**
*gamma-is-in-terms-of-i*: $\gamma = (\lambda x.\ f(x) *_R i + g(f(x)))$ **and**
*gamma-smooth*: $\gamma$ *piecewise-C1-differentiable-on* $\{0\ ..\ 1\}$ **and**
*g-continuous-on-f*: *continuous-on* $(f\ `\ \{0..1\})\ g$ **and**
*path-start-le-path-end*: $(pathstart\ \gamma) \cdot i \le (pathfinish\ \gamma) \cdot i$ **and**
*field-i-comp-cont*: *continuous-on* $(path\text{-}image\ \gamma)\ (\lambda x.\ F\ x \cdot i)$
**shows** *line-integral* $F$ $\{i\}$ $\gamma$
$= integral\ (cbox\ ((pathstart\ \gamma) \cdot i)\ ((pathfinish\ \gamma) \cdot i))\ (\lambda f\text{-}var.\ (F\ (f\text{-}var *_R i + g(f\text{-}var)) \cdot i))$
*line-integral-exists* $F$ $\{i\}$ $\gamma$
⟨*proof*⟩

**lemma** *line-integral-on-pair-path*:
**fixes** $F::('a::euclidean\text{-}space) \Rightarrow ('a)$ **and**
$g::real \Rightarrow 'a$ **and**
$\gamma::(real \Rightarrow 'a)$ **and**
$i::'a$
**assumes** *i-norm-1*: *norm* $i = 1$ **and**
*g-orthogonal-to-i*: $\forall x.\ g(x) \cdot i = 0$ **and**
*gamma-is-in-terms-of-i*: $\gamma = (\lambda x.\ f(x) *_R i + g(f(x)))$ **and**
*gamma-smooth*: $\gamma$ *C1-differentiable-on* $\{0\ ..\ 1\}$ **and**
*g-continuous-on-f*: *continuous-on* $(f\ `\ \{0..1\})\ g$ **and**
*path-start-le-path-end*: $(pathstart\ \gamma) \cdot i \le (pathfinish\ \gamma) \cdot i$ **and**

*field-i-comp-cont*: *continuous-on* (*path-image* $\gamma$) ($\lambda x$. *F x* $\cdot$ *i*)
**shows** (*line-integral F* {*i*} $\gamma$)
= *integral* (*cbox* ((*pathstart* $\gamma$) $\cdot$ *i*) ((*pathfinish* $\gamma$) $\cdot$ *i*)) ($\lambda$*f-var*. (*F* (*f-var* $*_R$ *i* + *g*(*f-var*)) $\cdot$ *i*))
$\langle$*proof*$\rangle$

**lemma** *content-box-cases*:
*content* (*box a b*) = (*if* $\forall$ *i*$\in$*Basis*. *a*$\cdot$*i* $\le$ *b*$\cdot$*i* *then prod* ($\lambda$*i*. *b*$\cdot$*i* $-$ *a*$\cdot$*i*) *Basis else* *0*)
$\langle$*proof*$\rangle$

**lemma** *content-box-cbox*:
**shows** *content* (*box a b*) = *content* (*cbox a b*)
$\langle$*proof*$\rangle$

**lemma** *content-eq-0*: *content* (*box a b*) = *0* $\longleftrightarrow$ ($\exists$ *i*$\in$*Basis*. *b*$\cdot$*i* $\le$ *a*$\cdot$*i*)
$\langle$*proof*$\rangle$

**lemma** *content-pos-lt-eq*: *0* < *content* (*cbox a* (*b*::'*a*::*euclidean-space*)) $\longleftrightarrow$ ($\forall$ *i*$\in$*Basis*. *a*$\cdot$*i* < *b*$\cdot$*i*)
$\langle$*proof*$\rangle$

**lemma** *content-lt-nz*: *0* < *content* (*box a b*) $\longleftrightarrow$ *content* (*box a b*) $\neq$ *0*
$\langle$*proof*$\rangle$

**lemma** *content-subset*: *cbox a b* $\subseteq$ *box c d* $\Longrightarrow$ *content* (*cbox a b*) $\le$ *content* (*box c d*)
$\langle$*proof*$\rangle$

**lemma** *sum-content-null*:
**assumes** *content* (*box a b*) = *0*
**and** *p tagged-division-of* (*box a b*)
**shows** *sum* ($\lambda$(*x*,*k*). *content k* $*_R$ *f x*) *p* = (*0*::'*a*::*real-normed-vector*)
$\langle$*proof*$\rangle$

**lemma** *has-integral-null* [*intro*]: *content*(*box a b*) = *0* $\Longrightarrow$ (*f has-integral 0*) (*box a b*)
$\langle$*proof*$\rangle$

**lemma** *line-integral-distrib*:
**assumes** *line-integral-exists f basis g1*
*line-integral-exists f basis g2*
*valid-path g1 valid-path g2*
**shows** *line-integral f basis* (*g1* +++ *g2*) = *line-integral f basis g1* + *line-integral f basis g2*
*line-integral-exists f basis* (*g1* +++ *g2*)
$\langle$*proof*$\rangle$

**lemma** *line-integral-exists-joinD1*:
**assumes** *line-integral-exists f basis* $(g1 +++ g2)$ *valid-path g1*
**shows** *line-integral-exists f basis g1*
⟨*proof*⟩

**lemma** *line-integral-exists-joinD2*:
**assumes** *line-integral-exists f basis* $(g1 +++ g2)$ *valid-path g2*
**shows** *line-integral-exists f basis g2*
⟨*proof*⟩

**lemma** *has-line-integral-on-reverse-path*:
**assumes** *g*: *valid-path g* **and** *int*:
$((\lambda x.\ \sum b{\in}basis.\ F\ (g\ x) \cdot b * (vector\text{-}derivative\ g\ (at\ x\ within\ \{0..1\}) \cdot b))$ *has-integral* $c)\{0..1\}$
**shows** $((\lambda x.\ \sum b{\in}basis.\ F\ ((reversepath\ g)\ x) \cdot b * (vector\text{-}derivative\ (reversepath\ g)\ (at\ x\ within\ \{0..1\}) \cdot b))$ *has-integral* $-c)\{0..1\}$
⟨*proof*⟩

**lemma** *line-integral-on-reverse-path*:
**assumes** *valid-path* $\gamma$ *line-integral-exists F basis* $\gamma$
**shows** *line-integral F basis* $\gamma = -$ (*line-integral F basis* (*reversepath* $\gamma$))
*line-integral-exists F basis* (*reversepath* $\gamma$)
⟨*proof*⟩

**lemma** *line-integral-exists-on-degenerate-path*:
**assumes** *finite basis*
**shows** *line-integral-exists F basis* $(\lambda x.\ c)$
⟨*proof*⟩

**lemma** *degenerate-path-is-valid-path*: *valid-path* $(\lambda x.\ c)$
⟨*proof*⟩

**lemma** *line-integral-degenerate-path*:
**assumes** *finite basis*
**shows** *line-integral F basis* $(\lambda x.\ c) = 0$
⟨*proof*⟩

**definition** *point-path* **where**
*point-path* $\gamma \equiv \exists\, c.\ \gamma = (\lambda x.\ c)$

**lemma** *line-integral-point-path*:
**assumes** *point-path* $\gamma$
**assumes** *finite basis*
**shows** *line-integral F basis* $\gamma = 0$
⟨*proof*⟩

**lemma** *line-integral-exists-point-path*:
**assumes** *finite basis point-path* $\gamma$
**shows** *line-integral-exists F basis* $\gamma$

⟨*proof*⟩

**lemma** *line-integral-exists-subpath*:
**assumes** *f*: *line-integral-exists f basis g* **and** *g*: *valid-path g*
**and** *uv*: $u \in \{0..1\}$ $v \in \{0..1\}$ $u \leq v$
**shows** (*line-integral-exists f basis* (*subpath u v g*))
⟨*proof*⟩

**type-synonym** *path* = *real* ⇒ (*real* ∗ *real*)
**type-synonym** *one-cube* = (*real* ⇒ (*real* ∗ *real*))
**type-synonym** *one-chain* = (*int* ∗ *path*) *set*
**type-synonym** *two-cube* = (*real* ∗ *real*) ⇒ (*real* ∗ *real*)
**type-synonym** *two-chain* = *two-cube set*

**definition** *one-chain-line-integral* :: ((*real* ∗ *real*) ⇒ (*real* ∗ *real*)) => ((*real*∗*real*) *set*) ⇒ *one-chain* ⇒ *real* **where**
*one-chain-line-integral F b C* ≡ ($\sum$ (*k*,*g*)∈*C*. *k* ∗ (*line-integral F b g*))

**definition** *boundary-chain* **where**
*boundary-chain s* ≡ (∀ (*k*, γ) ∈ *s*. *k* = *1* ∨ *k* = −*1*)

**fun** *coeff-cube-to-path*::(*int* ∗ *one-cube*) ⇒ *path*
**where** *coeff-cube-to-path* (*k*, γ) = (*if k* = *1 then* γ *else* (*reversepath* γ))

**fun** *rec-join* :: (*int*∗*path*) *list* ⇒ *path* **where**
*rec-join* [] = (λ*x*.*0*) |
*rec-join* [*oneC*] = *coeff-cube-to-path oneC* |
*rec-join* (*oneC* # *xs*) = *coeff-cube-to-path oneC* +++ (*rec-join xs*)

**fun** *valid-chain-list* **where**
*valid-chain-list* [] = *True* |
*valid-chain-list* [*oneC*] = *True* |
*valid-chain-list* (*oneC*#*l*) = (*pathfinish* (*coeff-cube-to-path* (*oneC*)) = *pathstart* (*rec-join l*) ∧ *valid-chain-list l*)

**lemma** *joined-is-valid*:
**assumes** *boundary-chain*: *boundary-chain* (*set l*) **and**
*valid-path*: ⋀*k* γ. (*k*, γ) ∈ *set l* ⟹ *valid-path* γ **and**
*valid-chain-list-ass*: *valid-chain-list l*
**shows** *valid-path* (*rec-join l*)
⟨*proof*⟩

**lemma** *pathstart-rec-join-1*:
*pathstart* (*rec-join* ((*1*, γ) # *l*)) = *pathstart* γ
⟨*proof*⟩

**lemma** *pathstart-rec-join-2*:
  *pathstart* (*rec-join* ((−*1*, $\gamma$) # *l*)) = *pathstart* (*reversepath* $\gamma$)
⟨*proof*⟩

**lemma** *pathstart-rec-join*:
  *pathstart* (*rec-join* ((*1*, $\gamma$) # *l*)) = *pathstart* $\gamma$
  *pathstart* (*rec-join* ((−*1*, $\gamma$) # *l*)) = *pathstart* (*reversepath* $\gamma$)
  ⟨*proof*⟩

**lemma** *line-integral-exists-on-rec-join*:
  **assumes** *boundary-chain*: *boundary-chain* (*set l*) **and**
    *valid-chain-list*: *valid-chain-list l* **and**
    *valid-path*: $\bigwedge k\ \gamma$. $(k, \gamma) \in$ *set l* $\Longrightarrow$ *valid-path* $\gamma$ **and**
    *line-integral-exists*: $\forall (k, \gamma) \in$ *set l*. *line-integral-exists F basis* $\gamma$
  **shows** *line-integral-exists F basis* (*rec-join l*)
  ⟨*proof*⟩

**lemma** *line-integral-exists-rec-join-cons*:
  **assumes** *line-integral-exists F basis* (*rec-join* ((*1*,$\gamma$) # *l*))
    ($\bigwedge k'\ \gamma'$. $(k', \gamma') \in$ *set* ((*1*,$\gamma$) # *l*) $\Longrightarrow$ *valid-path* $\gamma'$)
    *finite basis*
  **shows** *line-integral-exists F basis* ($\gamma$ +++ (*rec-join l*))
⟨*proof*⟩

**lemma** *line-integral-exists-rec-join-cons-2*:
  **assumes** *line-integral-exists F basis* (*rec-join* ((−*1*,$\gamma$) # *l*))
    ($\bigwedge k'\ \gamma'$. $(k', \gamma') \in$ *set* ((*1*,$\gamma$) # *l*) $\Longrightarrow$ *valid-path* $\gamma'$)
    *finite basis*
  **shows** *line-integral-exists F basis* ((*reversepath* $\gamma$) +++ (*rec-join l*))
⟨*proof*⟩

**lemma** *line-integral-exists-on-rec-join′*:
  **assumes** *boundary-chain*: *boundary-chain* (*set l*) **and**
    *valid-chain-list*: *valid-chain-list l* **and**
    *valid-path*: $\bigwedge k\ \gamma$. $(k, \gamma) \in$ *set l* $\Longrightarrow$ *valid-path* $\gamma$ **and**
    *line-integral-exists*: *line-integral-exists F basis* (*rec-join l*) **and**
    *finite-basis*: *finite basis*
  **shows** $\forall (k, \gamma) \in$ *set l*. *line-integral-exists F basis* $\gamma$
  ⟨*proof*⟩

**inductive** *chain-subdiv-path*
  **where** *I*: *chain-subdiv-path* $\gamma$ (*set l*) **if** *distinct l rec-join l* = $\gamma$ *valid-chain-list l*

**lemma** *valid-path-equiv-valid-chain-list*:
  **assumes** *path-eq-chain*: *chain-subdiv-path* $\gamma$ *one-chain*
    **and** *boundary-chain one-chain* $\forall (k, \gamma) \in$ *one-chain*. *valid-path* $\gamma$
  **shows** *valid-path* $\gamma$
⟨*proof*⟩

**lemma** *line-integral-rec-join-cons*:
  **assumes** *line-integral-exists F basis* $\gamma$
    *line-integral-exists F basis* (*rec-join* ((*l*)))
    $(\bigwedge k'\ \gamma'.\ (k', \gamma') \in$ *set* ((*1*,$\gamma$) # *l*) $\Longrightarrow$ *valid-path* $\gamma'$)
    *finite basis*
  **shows** *line-integral F basis* (*rec-join* ((*1*,$\gamma$) # *l*)) = *line-integral F basis* ($\gamma$ +++ (*rec-join l*))
⟨*proof*⟩

**lemma** *line-integral-rec-join-cons-2*:
  **assumes** *line-integral-exists F basis* $\gamma$
    *line-integral-exists F basis* (*rec-join* ((*l*)))
    $(\bigwedge k'\ \gamma'.\ (k', \gamma') \in$ *set* (($-1$,$\gamma$) # *l*) $\Longrightarrow$ *valid-path* $\gamma'$)
    *finite basis*
  **shows** *line-integral F basis* (*rec-join* (($-1$,$\gamma$) # *l*)) = *line-integral F basis* ((*reversepath* $\gamma$) +++ (*rec-join l*))
⟨*proof*⟩

**lemma** *one-chain-line-integral-rec-join*:
  **assumes** *l-props*: *set l* = *one-chain distinct l valid-chain-list l* **and**
    *boundary-chain*: *boundary-chain one-chain* **and**
    *line-integral-exists*: $\forall$ (*k*::*int*, $\gamma$) $\in$ *one-chain. line-integral-exists F basis* $\gamma$ **and**
    *valid-path*: $\forall$ (*k*::*int*, $\gamma$) $\in$ *one-chain. valid-path* $\gamma$ **and**
    *finite-basis*: *finite basis*
  **shows** *line-integral F basis* (*rec-join l*) = *one-chain-line-integral F basis one-chain*
⟨*proof*⟩

**lemma** *line-integral-on-path-eq-line-integral-on-equiv-chain*:
  **assumes** *path-eq-chain*: *chain-subdiv-path* $\gamma$ *one-chain* **and**
    *boundary-chain*: *boundary-chain one-chain* **and**
    *line-integral-exists*: $\forall$ (*k*::*int*, $\gamma$) $\in$ *one-chain. line-integral-exists F basis* $\gamma$ **and**
    *valid-path*: $\forall$ (*k*::*int*, $\gamma$) $\in$ *one-chain. valid-path* $\gamma$ **and**
    *finite-basis*: *finite basis*
  **shows** *one-chain-line-integral F basis one-chain* = *line-integral F basis* $\gamma$
    *line-integral-exists F basis* $\gamma$
    *valid-path* $\gamma$
⟨*proof*⟩

**lemma** *line-integral-on-path-eq-line-integral-on-equiv-chain′*:
  **assumes** *path-eq-chain*: *chain-subdiv-path* $\gamma$ *one-chain* **and**
    *boundary-chain*: *boundary-chain one-chain* **and**
    *line-integral-exists*: *line-integral-exists F basis* $\gamma$ **and**
    *valid-path*: $\forall$ (*k*, $\gamma$) $\in$ *one-chain. valid-path* $\gamma$ **and**
    *finite-basis*: *finite basis*
  **shows** *one-chain-line-integral F basis one-chain* = *line-integral F basis* $\gamma$
    $\forall$ (*k*, $\gamma$) $\in$ *one-chain. line-integral-exists F basis* $\gamma$
⟨*proof*⟩

**definition** *chain-subdiv-chain* **where**

*chain-subdiv-chain one-chain1 subdiv*
$\equiv \exists f.\ (\bigcup(f\ `\ \textit{one-chain1})) = \textit{subdiv} \wedge$
$(\forall\, c{\in}\textit{one-chain1}.\ \textit{chain-subdiv-path}\ (\textit{coeff-cube-to-path}\ c)\ (f\ c)) \wedge$
$\textit{pairwise}\ (\lambda\ p\ p'.\ f\ p \cap f\ p' = \{\})\ \textit{one-chain1} \wedge$
$(\forall\, x \in \textit{one-chain1}.\ \textit{finite}\ (f\ x))$

**lemma** *chain-subdiv-chain-character*:
**shows** $\textit{chain-subdiv-chain one-chain1 subdiv} \longleftrightarrow$
$(\exists f.\ \bigcup(f\ `\ \textit{one-chain1}) = \textit{subdiv} \wedge$
$(\forall\,(k,\gamma){\in}\textit{one-chain1}.$
$\textit{if}\ k = \mathit{1}$
$\textit{then chain-subdiv-path}\ \gamma\ (f\ (k,\gamma))$
$\textit{else chain-subdiv-path}\ (\textit{reversepath}\ \gamma)\ (f\ (k,\gamma))) \wedge$
$(\forall\, p{\in}\textit{one-chain1}.$
$\forall\, p'{\in}\textit{one-chain1}.\ p \neq p' \longrightarrow f\ p \cap f\ p' = \{\}) \wedge$
$(\forall\, x{\in}\textit{one-chain1}.\ \textit{finite}\ (f\ x)))$
⟨*proof*⟩

**lemma** *chain-subdiv-chain-imp-finite-subdiv*:
**assumes** *finite one-chain1*
*chain-subdiv-chain one-chain1 subdiv*
**shows** *finite subdiv*
⟨*proof*⟩

**lemma** *valid-subdiv-imp-valid-one-chain*:
**assumes** *chain1-eq-chain2*: *chain-subdiv-chain one-chain1 subdiv* **and**
*boundary-chain1*: *boundary-chain one-chain1* **and**
*boundary-chain2*: *boundary-chain subdiv* **and**
*valid-path*: $\forall\,(k,\gamma) \in \textit{subdiv}.\ \textit{valid-path}\ \gamma$
**shows** $\forall\,(k,\gamma) \in \textit{one-chain1}.\ \textit{valid-path}\ \gamma$
⟨*proof*⟩

**lemma** *one-chain-line-integral-eq-line-integral-on-sudivision*:
**assumes** *chain1-eq-chain2*: *chain-subdiv-chain one-chain1 subdiv* **and**
*boundary-chain1*: *boundary-chain one-chain1* **and**
*boundary-chain2*: *boundary-chain subdiv* **and**
*line-integral-exists-on-chain2*: $\forall\,(k,\gamma) \in \textit{subdiv}.\ \textit{line-integral-exists}\ F\ \textit{basis}\ \gamma$
**and**
*valid-path*: $\forall\,(k,\gamma) \in \textit{subdiv}.\ \textit{valid-path}\ \gamma$ **and**
*finite-chain1*: *finite one-chain1* **and**
*finite-basis*: *finite basis*
**shows** *one-chain-line-integral F basis one-chain1* = *one-chain-line-integral F basis subdiv*
$\forall\,(k,\gamma) \in \textit{one-chain1}.\ \textit{line-integral-exists}\ F\ \textit{basis}\ \gamma$
⟨*proof*⟩

**lemma** *one-chain-line-integral-eq-line-integral-on-sudivision′*:
**assumes** *chain1-eq-chain2*: *chain-subdiv-chain one-chain1 subdiv* **and**
*boundary-chain1*: *boundary-chain one-chain1* **and**

*boundary-chain2*: *boundary-chain subdiv* **and**
*line-integral-exists-on-chain1*: $\forall (k, \gamma) \in$ *one-chain1*. *line-integral-exists F basis* $\gamma$ **and**
*valid-path*: $\forall (k, \gamma) \in$ *subdiv*. *valid-path* $\gamma$ **and**
*finite-chain1*: *finite one-chain1* **and**
*finite-basis*: *finite basis*
**shows** *one-chain-line-integral F basis one-chain1* = *one-chain-line-integral F basis subdiv*
$\forall (k, \gamma) \in$ *subdiv*. *line-integral-exists F basis* $\gamma$
⟨*proof*⟩

**lemma** *line-integral-sum-gen*:
**assumes** *finite-basis*:
*finite basis* **and**
*line-integral-exists*:
*line-integral-exists F basis1* $\gamma$
*line-integral-exists F basis2* $\gamma$ **and**
*basis-partition*:
*basis1* $\cup$ *basis2* = *basis basis1* $\cap$ *basis2* = {}
**shows** *line-integral F basis* $\gamma$ = (*line-integral F basis1* $\gamma$) + (*line-integral F basis2* $\gamma$)
*line-integral-exists F basis* $\gamma$
⟨*proof*⟩

**definition** *common-boundary-sudivision-exists* **where**
*common-boundary-sudivision-exists one-chain1 one-chain2* $\equiv$
$\exists$ *subdiv*. *chain-subdiv-chain one-chain1 subdiv* $\wedge$
*chain-subdiv-chain one-chain2 subdiv* $\wedge$
($\forall (k, \gamma) \in$ *subdiv*. *valid-path* $\gamma$)$\wedge$
*boundary-chain subdiv*

**lemma** *common-boundary-sudivision-commutative*:
(*common-boundary-sudivision-exists one-chain1 one-chain2*) = (*common-boundary-sudivision-exists one-chain2 one-chain1*)
⟨*proof*⟩

**lemma** *common-subdivision-imp-eq-line-integral*:
**assumes** (*common-boundary-sudivision-exists one-chain1 one-chain2*)
*boundary-chain one-chain1*
*boundary-chain one-chain2*
$\forall (k, \gamma) \in$ *one-chain1*. *line-integral-exists F basis* $\gamma$
*finite one-chain1*
*finite one-chain2*
*finite basis*
**shows** *one-chain-line-integral F basis one-chain1* = *one-chain-line-integral F basis one-chain2*
$\forall (k, \gamma) \in$ *one-chain2*. *line-integral-exists F basis* $\gamma$
⟨*proof*⟩

**definition** *common-sudiv-exists* **where**
  *common-sudiv-exists one-chain1 one-chain2* ≡
    ∃ *subdiv ps1 ps2*. *chain-subdiv-chain* (*one-chain1* − *ps1*) *subdiv* ∧
        *chain-subdiv-chain* (*one-chain2* − *ps2*) *subdiv* ∧
        (∀ (*k*, *γ*) ∈ *subdiv*. *valid-path* *γ*) ∧
        (*boundary-chain subdiv*) ∧
        (∀ (*k*, *γ*) ∈ *ps1*. *point-path* *γ*) ∧
        (∀ (*k*, *γ*) ∈ *ps2*. *point-path* *γ*)

**lemma** *common-sudiv-exists-comm*:
  **shows** *common-sudiv-exists C1 C2* = *common-sudiv-exists C2 C1*
  ⟨*proof*⟩

**lemma** *line-integral-degenerate-chain*:
  **assumes** (∀ (*k*, *γ*) ∈ *chain*. *point-path* *γ*)
  **assumes** *finite basis*
  **shows** *one-chain-line-integral F basis chain* = *0*
⟨*proof*⟩

**lemma** *gen-common-subdiv-imp-common-subdiv*:
  **shows** (*common-sudiv-exists one-chain1 one-chain2*) = (∃ *ps1 ps2*. (*common-boundary-sudivision-exists* (*one-chain1* − *ps1*) (*one-chain2* − *ps2*)) ∧ (∀ (*k*, *γ*)∈*ps1*. *point-path* *γ*) ∧ (∀ (*k*, *γ*)∈*ps2*. *point-path* *γ*))
  ⟨*proof*⟩

**lemma** *common-subdiv-imp-gen-common-subdiv*:
  **assumes** (*common-boundary-sudivision-exists one-chain1 one-chain2*)
  **shows** (*common-sudiv-exists one-chain1 one-chain2*)
  ⟨*proof*⟩

**lemma** *one-chain-line-integral-point-paths*:
  **assumes** *finite one-chain*
  **assumes** *finite basis*
  **assumes** (∀ (*k*, *γ*)∈*ps*. *point-path* *γ*)
  **shows** *one-chain-line-integral F basis* (*one-chain* − *ps*) = *one-chain-line-integral F basis* (*one-chain*)
⟨*proof*⟩

**lemma** *boundary-chain-diff*:
  **assumes** *boundary-chain one-chain*
  **shows** *boundary-chain* (*one-chain* − *s*)
  ⟨*proof*⟩

**lemma** *gen-common-subdivision-imp-eq-line-integral*:
  **assumes** (*common-sudiv-exists one-chain1 one-chain2*)
    *boundary-chain one-chain1*
    *boundary-chain one-chain2*
    ∀ (*k*, *γ*)∈*one-chain1*. *line-integral-exists F basis* *γ*
    *finite one-chain1*

*finite one-chain2*
*finite basis*
**shows** *one-chain-line-integral F basis one-chain1* = *one-chain-line-integral F basis one-chain2*
$\forall (k, \gamma) \in$ *one-chain2*. *line-integral-exists F basis* $\gamma$
⟨*proof*⟩

**lemma** *common-sudiv-exists-refl*:
**assumes** *common-sudiv-exists C1 C2*
**shows** *common-sudiv-exists C2 C1*
⟨*proof*⟩

**lemma** *chain-subdiv-path-singleton*:
**shows** *chain-subdiv-path* $\gamma$ $\{(1,\gamma)\}$
⟨*proof*⟩

**lemma** *chain-subdiv-path-singleton-reverse*:
**shows** *chain-subdiv-path* (*reversepath* $\gamma$) $\{(-1,\gamma)\}$
⟨*proof*⟩

**lemma** *chain-subdiv-chain-refl*:
**assumes** *boundary-chain C*
**shows** *chain-subdiv-chain C C*
⟨*proof*⟩

**definition** *reparam-weak* **where**
*reparam-weak* $\gamma 1$ $\gamma 2 \equiv \exists \varphi. (\forall x \in \{0..1\}. \gamma 1\ x = (\gamma 2\ o\ \varphi)\ x) \wedge \varphi$ *piecewise-C1-differentiable-on* $\{0..1\} \wedge \varphi(0) = 0 \wedge \varphi(1) = 1 \wedge \varphi$ ‘ $\{0..1\} = \{0..1\}$

**definition** *reparam* **where**
*reparam* $\gamma 1$ $\gamma 2 \equiv \exists \varphi. (\forall x \in \{0..1\}. \gamma 1\ x = (\gamma 2\ o\ \varphi)\ x) \wedge \varphi$ *piecewise-C1-differentiable-on* $\{0..1\} \wedge \varphi(0) = 0 \wedge \varphi(1) = 1 \wedge$ *bij-betw* $\varphi$ $\{0..1\}$ $\{0..1\} \wedge \varphi -‘ \{0..1\} \subseteq \{0..1\}$ $\wedge (\forall x \in \{0..1\}.$ *finite* $(\varphi -‘ \{x\}))$

**lemma** *reparam-weak-eq-refl*:
**shows** *reparam-weak* $\gamma 1$ $\gamma 1$
⟨*proof*⟩

**lemma** *line-integral-exists-smooth-one-base*:
**assumes** $\gamma$ *C1-differentiable-on* $\{0..1\}$
*continuous-on* (*path-image* $\gamma$) $(\lambda x.\ F\ x \cdot b)$
**shows** *line-integral-exists F* $\{b\}$ $\gamma$
⟨*proof*⟩

**lemma** *contour-integral-primitive-lemma*:
**fixes** $f$ :: *complex* $\Rightarrow$ *complex* **and** $g$ :: *real* $\Rightarrow$ *complex*
**assumes** $a \leq b$

**and** $\bigwedge x.\ x \in s \Longrightarrow$ (*f has-field-derivative f′ x*) (*at x within s*)
**and** *g piecewise-differentiable-on* $\{a..b\}$ $\bigwedge x.\ x \in \{a..b\} \Longrightarrow g\ x \in s$
**shows** (($\lambda x.\ f'(g\ x) * $ *vector-derivative g* (*at x within* $\{a..b\}$))
*has-integral* $(f(g\ b) - f(g\ a)))\ \{a..b\}$
⟨*proof*⟩

**lemma** *line-integral-primitive-lemma*:
**fixes** *f* :: *′a*::{*euclidean-space*,*real-normed-field*} ⇒ *′a*::{*euclidean-space*,*real-normed-field*}
**and**
*g* :: *real* ⇒ *′a*
**assumes** $\bigwedge(a::'a).\ a \in s \Longrightarrow$ (*f has-field-derivative* (*f′ a*)) (*at a within s*)
**and** *g piecewise-differentiable-on* $\{0::real..1\}$ $\bigwedge x.\ x \in \{0..1\} \Longrightarrow g\ x \in s$
**and** *base-vec* ∈ *Basis*
**shows** (($\lambda x.$ (($f'(g\ x)$) ∗ (*vector-derivative g* (*at x within* $\{0..1\}$))) · *base-vec*)
*has-integral* ((($f(g\ 1)$) · *base-vec* − ($f(g\ 0)$) · *base-vec*))) $\{0..1\}$
⟨*proof*⟩

**lemma** *reparam-eq-line-integrals*:
**assumes** *reparam*: *reparam* $\gamma 1\ \gamma 2$ **and**
*pw-smooth*: $\gamma 2$ *piecewise-C1-differentiable-on* $\{0..1\}$ **and**
*cont*: *continuous-on* (*path-image* $\gamma 2$) ($\lambda x.\ F\ x \cdot b$) **and**
*line-integral-ex*: *line-integral-exists F* $\{b\}$ $\gamma 2$
**shows** *line-integral F* $\{b\}$ $\gamma 1$ = *line-integral F* $\{b\}$ $\gamma 2$
*line-integral-exists F* $\{b\}$ $\gamma 1$
⟨*proof*⟩

**lemma** *reparam-weak-eq-line-integrals*:
**assumes** *reparam-weak* $\gamma 1\ \gamma 2$
$\gamma 2$ *C1-differentiable-on* $\{0..1\}$
*continuous-on* (*path-image* $\gamma 2$) ($\lambda x.\ F\ x \cdot b$)
**shows** *line-integral F* $\{b\}$ $\gamma 1$ = *line-integral F* $\{b\}$ $\gamma 2$
*line-integral-exists F* $\{b\}$ $\gamma 1$
⟨*proof*⟩

**lemma** *line-integral-sum-basis*:
**assumes** *finite* (*basis*::(*′a*::*euclidean-space*) *set*) ∀ *b*∈*basis*. *line-integral-exists F*
$\{b\}$ $\gamma$
**shows** *line-integral F basis* $\gamma$ = ($\sum$ *b*∈*basis*. *line-integral F* $\{b\}$ $\gamma$)
*line-integral-exists F basis* $\gamma$
⟨*proof*⟩

**lemma** *reparam-weak-eq-line-integrals-basis*:
**assumes** *reparam-weak* $\gamma 1\ \gamma 2$
$\gamma 2$ *C1-differentiable-on* $\{0..1\}$
∀ *b*∈*basis*. *continuous-on* (*path-image* $\gamma 2$) ($\lambda x.\ F\ x \cdot b$)
*finite basis*
**shows** *line-integral F basis* $\gamma 1$ = *line-integral F basis* $\gamma 2$
*line-integral-exists F basis* $\gamma 1$
⟨*proof*⟩

**lemma** *reparam-eq-line-integrals-basis*:
**assumes** *reparam* $\gamma 1$ $\gamma 2$
$\gamma 2$ *piecewise-C1-differentiable-on* $\{0..1\}$
$\forall\, b \in basis.$ *continuous-on* (*path-image* $\gamma 2$) ($\lambda x.\ F\ x \cdot b$)
*finite basis*
$\forall\, b \in basis.$ *line-integral-exists* $F$ $\{b\}$ $\gamma 2$
**shows** *line-integral* $F$ *basis* $\gamma 1$ = *line-integral* $F$ *basis* $\gamma 2$
*line-integral-exists* $F$ *basis* $\gamma 1$
⟨*proof*⟩

**lemma** *line-integral-exists-smooth*:
**assumes** $\gamma$ *C1-differentiable-on* $\{0..1\}$
$\forall\, (b::'a::euclidean\text{-}space) \in basis.$ *continuous-on* (*path-image* $\gamma$) ($\lambda x.\ F\ x \cdot b$)
*finite basis*
**shows** *line-integral-exists* $F$ *basis* $\gamma$
⟨*proof*⟩

**lemma** *smooth-path-imp-reverse*:
**assumes** $g$ *C1-differentiable-on* $\{0..1\}$
**shows** (*reversepath* $g$) *C1-differentiable-on* $\{0..1\}$
⟨*proof*⟩

**lemma** *piecewise-smooth-path-imp-reverse*:
**assumes** $g$ *piecewise-C1-differentiable-on* $\{0..1\}$
**shows** (*reversepath* $g$) *piecewise-C1-differentiable-on* $\{0..1\}$
⟨*proof*⟩

**definition** *chain-reparam-weak-chain* **where**
*chain-reparam-weak-chain one-chain1 one-chain2* $\equiv$
$\exists f.$ *bij* $f$ $\wedge$ $f$ ‘ *one-chain1* = *one-chain2* $\wedge$ ($\forall\, (k,\gamma) \in$ *one-chain1*. *if* $k$ = *fst* ($f(k,\gamma)$) *then reparam-weak* $\gamma$ (*snd* ($f(k,\gamma)$)) *else reparam-weak* $\gamma$ (*reversepath* (*snd* ($f(k,\gamma)$))))

**lemma** *chain-reparam-weak-chain-line-integral*:
**assumes** *chain-reparam-weak-chain one-chain1 one-chain2*
$\forall\, (k2,\gamma 2) \in$ *one-chain2*. $\gamma 2$ *C1-differentiable-on* $\{0..1\}$
$\forall\, (k2,\gamma 2) \in$ *one-chain2*.$\forall\, b \in basis.$ *continuous-on* (*path-image* $\gamma 2$) ($\lambda x.\ F\ x \cdot b$)
*finite basis*
**and** *bound1*: *boundary-chain one-chain1*
**and** *bound2*: *boundary-chain one-chain2*
**shows** *one-chain-line-integral* $F$ *basis one-chain1* = *one-chain-line-integral* $F$ *basis one-chain2*
$\forall\, (k, \gamma) \in$ *one-chain1*. *line-integral-exists* $F$ *basis* $\gamma$
⟨*proof*⟩

**definition** *chain-reparam-chain* **where**
*chain-reparam-chain one-chain1 one-chain2* $\equiv$
$\exists f.$ *bij* $f$ $\wedge$ $f$ ‘ *one-chain1* = *one-chain2* $\wedge$ ($\forall\, (k,\gamma) \in$ *one-chain1*. *if* $k$ = *fst*

$(f(k,\gamma))$ *then reparam* $\gamma$ (*snd* $(f(k,\gamma)))$ *else reparam* $\gamma$ (*reversepath* (*snd* $(f(k,\gamma))))$)

**definition** *chain-reparam-weak-path*::((*real*) $\Rightarrow$ (*real* $*$ *real*)) $\Rightarrow$ ((*int* $*$ ((*real*) $\Rightarrow$ (*real* $*$ *real*))) *set*) $\Rightarrow$ *bool* **where**
  *chain-reparam-weak-path* $\gamma$ *one-chain*
    $\equiv \exists\, l.$ *set* $l$ = *one-chain* $\wedge$ *distinct* $l$ $\wedge$ *reparam* $\gamma$ (*rec-join* $l$) $\wedge$ *valid-chain-list* $l \wedge l \neq []$

**lemma** *chain-reparam-chain-line-integral*:
  **assumes** *chain-reparam-chain one-chain1 one-chain2*
    $\forall\,(k2,\gamma 2)\in$*one-chain2*. $\gamma 2$ *piecewise-C1-differentiable-on* $\{0..1\}$
    $\forall\,(k2,\gamma 2)\in$*one-chain2*.$\forall\, b\in$*basis*. *continuous-on* (*path-image* $\gamma 2$) ($\lambda x.\ F\ x \cdot b$)
    *finite basis*
  **and** *bound1*: *boundary-chain one-chain1*
  **and** *bound2*: *boundary-chain one-chain2*
  **and** *line*: $\forall\,(k2,\gamma 2)\in$*one-chain2*. ($\forall\, b\in$*basis*. *line-integral-exists* $F$ $\{b\}$ $\gamma 2$)
  **shows** *one-chain-line-integral* $F$ *basis one-chain1* = *one-chain-line-integral* $F$ *basis one-chain2*
    $\forall\,(k,\ \gamma)\in$*one-chain1*. *line-integral-exists* $F$ *basis* $\gamma$
$\langle$*proof*$\rangle$

**lemma** *path-image-rec-join*:
  **fixes** $\gamma$::*real* $\Rightarrow$ (*real* $\times$ *real*)
  **fixes** $k$::*int*
  **fixes** $l$
  **shows** $\bigwedge k\ \gamma.\ (k,\ \gamma) \in$ *set* $l \Longrightarrow$ *valid-chain-list* $l \Longrightarrow$ *path-image* $\gamma \subseteq$ *path-image* (*rec-join* $l$)
$\langle$*proof*$\rangle$

**lemma** *path-image-rec-join-2*:
  **fixes** $l$
  **shows** $l \neq [] \Longrightarrow$ *valid-chain-list* $l \Longrightarrow$ *path-image* (*rec-join* $l$) $\subseteq$ ($\bigcup(k,\ \gamma) \in$ *set* $l$. *path-image* $\gamma$)
$\langle$*proof*$\rangle$

**lemma** *continuous-on-closed-UN*:
  **assumes** *finite* $S$
  **shows** $((\bigwedge s.\ s \in S \Longrightarrow$ *closed* $s) \Longrightarrow (\bigwedge s.\ s \in S \Longrightarrow$ *continuous-on* $s\ f) \Longrightarrow$ *continuous-on* $(\bigcup S)\ f)$
  $\langle$*proof*$\rangle$

**lemma** *chain-reparam-weak-path-line-integral*:
  **assumes** *path-eq-chain*: *chain-reparam-weak-path* $\gamma$ *one-chain* **and**
    *boundary-chain*: *boundary-chain one-chain* **and**
    *line-integral-exists*: $\forall\, b\in$*basis*. $\forall\,(k$::*int*, $\gamma) \in$ *one-chain*. *line-integral-exists* $F$ $\{b\}$ $\gamma$ **and**
    *valid-path*: $\forall\,(k$::*int*, $\gamma) \in$ *one-chain*. *valid-path* $\gamma$ **and**
    *finite-basis*: *finite basis* **and**
    *cont*: $\forall\, b\in$*basis*. $\forall\,(k,\gamma 2)\in$*one-chain*. *continuous-on* (*path-image* $\gamma 2$) ($\lambda x.\ F\ x \cdot$

*b*) **and**
*finite-one-chain*: *finite one-chain*
**shows** *line-integral F basis* $\gamma$ = *one-chain-line-integral F basis one-chain*
*line-integral-exists F basis* $\gamma$

$\langle proof \rangle$

**definition** *chain-reparam-chain'* **where**
*chain-reparam-chain' one-chain1 subdiv*
$\equiv \exists f.\ ((\bigcup (f\ `\ one\text{-}chain1)) = subdiv) \wedge$
$(\forall\ cube \in one\text{-}chain1.\ chain\text{-}reparam\text{-}weak\text{-}path\ (rec\text{-}join\ [cube])\ (f\ cube))$
$\wedge$
$(\forall\ p \in one\text{-}chain1.\ \forall\ p' \in one\text{-}chain1.\ p \neq p' \longrightarrow f\ p \cap f\ p' = \{\}) \wedge$
$(\forall\ x \in one\text{-}chain1.\ finite\ (f\ x))$

**lemma** *chain-reparam-chain'-imp-finite-subdiv*:
**assumes** *finite one-chain1*
*chain-reparam-chain' one-chain1 subdiv*
**shows** *finite subdiv*
$\langle proof \rangle$

**lemma** *chain-reparam-chain'-line-integral*:
**assumes** *chain1-eq-chain2*: *chain-reparam-chain' one-chain1 subdiv* **and**
*boundary-chain1*: *boundary-chain one-chain1* **and**
*boundary-chain2*: *boundary-chain subdiv* **and**
*line-integral-exists-on-chain2*: $\forall\ b \in basis.\ \forall\ (k{::}int,\ \gamma) \in subdiv.$ *line-integral-exists*
*F* $\{b\}$ $\gamma$ **and**
*valid-path*: $\forall\ (k,\ \gamma) \in subdiv.$ *valid-path* $\gamma$ **and**
*valid-path-2*: $\forall\ (k,\ \gamma) \in one\text{-}chain1.$ *valid-path* $\gamma$ **and**
*finite-chain1*: *finite one-chain1* **and**
*finite-basis*: *finite basis* **and**
*cont-field*: $\forall\ b \in basis.\ \forall\ (k,\ \gamma 2) \in subdiv.\ continuous\text{-}on\ (path\text{-}image\ \gamma 2)\ (\lambda x.\ F$
$x \cdot b)$
**shows** *one-chain-line-integral F basis one-chain1* = *one-chain-line-integral F*
*basis subdiv*
$\forall\ (k,\ \gamma) \in one\text{-}chain1.$ *line-integral-exists F basis* $\gamma$
$\langle proof \rangle$

**lemma** *chain-reparam-chain'-line-integral-smooth-cubes*:
**assumes** *chain-reparam-chain' one-chain1 one-chain2*
$\forall\ (k2,\gamma 2) \in one\text{-}chain2.\ \gamma 2$ *C1-differentiable-on* $\{0..1\}$
$\forall\ b \in basis. \forall\ (k2,\gamma 2) \in one\text{-}chain2.\ continuous\text{-}on\ (path\text{-}image\ \gamma 2)\ (\lambda x.\ F\ x \cdot b)$
*finite basis*
*finite one-chain1*
*boundary-chain one-chain1*
*boundary-chain one-chain2*
$\forall\ (k,\gamma) \in one\text{-}chain1.$ *valid-path* $\gamma$
**shows** *one-chain-line-integral F basis one-chain1* = *one-chain-line-integral F*
*basis one-chain2*

$\forall (k, \gamma){\in}$*one-chain1*. *line-integral-exists F basis* $\gamma$
⟨*proof*⟩

**lemma** *chain-subdiv-path-pathimg-subset*:
**assumes** *chain-subdiv-path* $\gamma$ *subdiv*
**shows** $\forall (k',\gamma'){\in}$*subdiv*. (*path-image* $\gamma'$) $\subseteq$ *path-image* $\gamma$
⟨*proof*⟩

**lemma** *reparam-path-image*:
**assumes** *reparam* $\gamma 1$ $\gamma 2$
**shows** *path-image* $\gamma 1$ = *path-image* $\gamma 2$
⟨*proof*⟩

**lemma** *chain-reparam-weak-path-pathimg-subset*:
**assumes** *chain-reparam-weak-path* $\gamma$ *subdiv*
**shows** $\forall (k',\gamma'){\in}$*subdiv*. (*path-image* $\gamma'$) $\subseteq$ *path-image* $\gamma$
⟨*proof*⟩

**lemma** *chain-subdiv-chain-pathimg-subset'*:
**assumes** *chain-subdiv-chain one-chain subdiv*
**assumes** $(k,\gamma)\in$ *subdiv*
**shows** $\exists k'\ \gamma'.\ (k',\gamma')\in$ *one-chain* $\wedge$ *path-image* $\gamma \subseteq$ *path-image* $\gamma'$
⟨*proof*⟩

**lemma** *chain-subdiv-chain-pathimg-subset*:
**assumes** *chain-subdiv-chain one-chain subdiv*
**shows** $\bigcup$ (*path-image* ‘ $\{\gamma.\ \exists k.\ (k,\gamma)\in$ *subdiv*$\}$) $\subseteq \bigcup$ (*path-image* ‘ $\{\gamma.\ \exists k.\ (k,\gamma)\in$ *one-chain*$\}$)
⟨*proof*⟩

**lemma** *chain-reparam-chain'-pathimg-subset'*:
**assumes** *chain-reparam-chain' one-chain subdiv*
**assumes** $(k,\gamma)\in$ *subdiv*
**shows** $\exists k'\ \gamma'.\ (k',\gamma')\in$ *one-chain* $\wedge$ *path-image* $\gamma \subseteq$ *path-image* $\gamma'$
⟨*proof*⟩

**definition** *common-reparam-exists*:: (*int* $\times$ (*real* $\Rightarrow$ *real* $\times$ *real*)) *set* $\Rightarrow$ (*int* $\times$ (*real* $\Rightarrow$ *real* $\times$ *real*)) *set* $\Rightarrow$ *bool* **where**
*common-reparam-exists one-chain1 one-chain2* $\equiv$
($\exists$ *subdiv ps1 ps2*.
*chain-reparam-chain'* (*one-chain1* − *ps1*) *subdiv* $\wedge$
*chain-reparam-chain'* (*one-chain2* − *ps2*) *subdiv* $\wedge$
($\forall (k, \gamma){\in}$*subdiv*. $\gamma$ *C1-differentiable-on* $\{0..1\}$) $\wedge$
*boundary-chain subdiv* $\wedge$
($\forall (k, \gamma){\in}$*ps1*. *point-path* $\gamma$) $\wedge$
($\forall (k, \gamma){\in}$*ps2*. *point-path* $\gamma$))

**lemma** *common-reparam-exists-imp-eq-line-integral*:
**assumes** *finite-basis*: *finite basis* **and**

*finite one-chain1*
*finite one-chain2*
*boundary-chain* (*one-chain1*::(*int* × (*real* ⇒ *real* × *real*)) *set*)
*boundary-chain* (*one-chain2*::(*int* × (*real* ⇒ *real* × *real*)) *set*)
∀ (*k2*, *γ2*)∈*one-chain2*. ∀ *b*∈*basis*. *continuous-on* (*path-image γ2*) (*λx*. *F x* · *b*)
(*common-reparam-exists one-chain1 one-chain2*)
(∀ (*k*, *γ*)∈*one-chain1*. *valid-path γ*)
(∀ (*k*, *γ*)∈*one-chain2*. *valid-path γ*)
**shows** *one-chain-line-integral F basis one-chain1* = *one-chain-line-integral F basis one-chain2*
∀ (*k*, *γ*)∈*one-chain1*. *line-integral-exists F basis γ*
⟨*proof*⟩

**definition** *subcube* :: *real* ⇒ *real* ⇒ (*int* × (*real* ⇒ *real* × *real*)) ⇒ (*int* × (*real* ⇒ *real* × *real*)) **where**
*subcube a b cube* = (*fst cube*, *subpath a b* (*snd cube*))

**lemma** *subcube-valid-path*:
**assumes** *valid-path* (*snd cube*) *a* ∈ {*0..1*} *b* ∈ {*0..1*}
**shows** *valid-path* (*snd* (*subcube a b cube*))
⟨*proof*⟩

**end**
**theory** *Green*
**imports** *Paths Derivs Integrals General-Utils*
**begin**

**lemma** *frontier-Un-subset-Un-frontier*:
*frontier* (*s* ∪ *t*) ⊆ (*frontier s*) ∪ (*frontier t*)
⟨*proof*⟩

**definition** *has-partial-derivative*:: ((′*a*::*euclidean-space*) ⇒ ′*b*::*euclidean-space*) ⇒ ′*a* ⇒ (′*a* ⇒ ′*b*) ⇒ (′*a*) ⇒ *bool* **where**
*has-partial-derivative F base-vec F′ a*
≡ ((*λx*::′*a*::*euclidean-space*. *F*( (*a* − ((*a* · *base-vec*) $*_R$ *base-vec*)) + (*x* · *base-vec*) $*_R$ *base-vec* ))
*has-derivative F′*) (*at a*)

**definition** *has-partial-vector-derivative*:: ((′*a*::*euclidean-space*) ⇒ ′*b*::*euclidean-space*) ⇒ ′*a* ⇒ ( ′*b*) ⇒ (′*a*) ⇒ *bool* **where**
*has-partial-vector-derivative F base-vec F′ a*
≡ ((*λx*. *F*( (*a* − ((*a* · *base-vec*) $*_R$ *base-vec*)) + *x* $*_R$ *base-vec* ))
*has-vector-derivative F′*) (*at* (*a* · *base-vec*))

**definition** *partially-vector-differentiable* **where**
*partially-vector-differentiable F base-vec p* ≡ (∃ *F′*. *has-partial-vector-derivative F base-vec F′ p*)

**definition** *partial-vector-derivative*:: ((*'a*::*euclidean-space*) ⇒ *'b*::*euclidean-space*) ⇒ *'a* ⇒ *'a* ⇒ *'b* **where**
*partial-vector-derivative F base-vec a*
≡ (*vector-derivative* (λ*x*. *F*( (*a* − ((*a* · *base-vec*) $*_R$ *base-vec*)) + *x* $*_R$ *base-vec*)) (*at* (*a* · *base-vec*)))

**lemma** *partial-vector-derivative-works*:
**assumes** *partially-vector-differentiable F base-vec a*
**shows** *has-partial-vector-derivative F base-vec* (*partial-vector-derivative F base-vec a*) *a*
⟨*proof*⟩

**lemma** *fundamental-theorem-of-calculus-partial-vector*:
**fixes** *a b*:: *real* **and**
*F*:: (*'a*::*euclidean-space* ⇒ *'b*::*euclidean-space*) **and**
*i*:: *'a* **and**
*j*:: *'b* **and**
*F-j-i*:: (*'a*::*euclidean-space* ⇒ *real*)
**assumes** *a-leq-b*: *a* ≤ *b* **and**
*Base-vecs*: *i* ∈ *Basis j* ∈ *Basis* **and**
*no-i-component*: *c* · *i* = *0* **and**
*has-partial-deriv*: ∀ *p* ∈ *D*. *has-partial-vector-derivative* (λ*x*. (*F x*) · *j*) *i* (*F-j-i p*) *p* **and**
*domain-subset-of-D*: {*x* $*_R$ *i* + *c* |*x*. *a* ≤ *x* ∧ *x* ≤ *b*} ⊆ *D*
**shows** ((λ*x*. *F-j-i*( *x* $*_R$ *i* + *c*)) *has-integral*
*F*(*b* $*_R$ *i* + *c*) · *j* − *F*(*a* $*_R$ *i* + *c*) · *j*) (*cbox a b*)
⟨*proof*⟩

**lemma** *fundamental-theorem-of-calculus-partial-vector-gen*:
**fixes** *k1 k2*:: *real* **and**
*F*:: (*'a*::*euclidean-space* ⇒ *'b*::*euclidean-space*) **and**
*i*:: *'a* **and**
*F-i*:: (*'a*::*euclidean-space* ⇒ *'b*)
**assumes** *a-leq-b*: *k1* ≤ *k2* **and**
*unit-len*: *i* · *i* = *1* **and**
*no-i-component*: *c* · *i* = *0* **and**
*has-partial-deriv*: ∀ *p* ∈ *D*. *has-partial-vector-derivative F i* (*F-i p*) *p* **and**
*domain-subset-of-D*: {*v*. ∃ *x*. *k1* ≤ *x* ∧ *x* ≤ *k2* ∧ *v* = *x* $*_R$ *i* + *c*} ⊆ *D*
**shows** ( (λ*x*. *F-i*( *x* $*_R$ *i* + *c*)) *has-integral*
*F*(*k2* $*_R$ *i* + *c*) − *F*(*k1* $*_R$ *i* + *c*)) (*cbox k1 k2*)
⟨*proof*⟩

**lemma** *add-scale-img*:
**assumes** *a* < *b* **shows** (λ*x*::*real*. *a* + (*b* − *a*) ∗ *x*) ‘ {*0* .. *1*} = {*a* .. *b*}
⟨*proof*⟩

**lemma** *add-scale-img'*:
**assumes** *a* ≤ *b*
**shows** (λ*x*::*real*. *a* + (*b* − *a*) ∗ *x*) ‘ {*0* .. *1*} = {*a* .. *b*}

$\langle proof \rangle$

**definition** *analytically-valid*:: $'a$::*euclidean-space set* $\Rightarrow$ ($'a \Rightarrow$ $'b$::{*euclidean-space*,*times*,*zero-neq-one*}) $\Rightarrow$ $'a \Rightarrow$ *bool* **where**
*analytically-valid s F i* $\equiv$
($\forall$ *a* $\in$ *s*. *partially-vector-differentiable F i a*) $\wedge$
*continuous-on s F* $\wedge$ — TODO: should we replace this with saying that $F$ is partially diffrerentiable on $Dy$,
— i.e. there is a partial derivative on every dimension
*integrable lborel* ($\lambda p$. (*partial-vector-derivative F i*) *p* $*$ *indicator s p*) $\wedge$
($\lambda x$. *integral UNIV* ($\lambda y$. (*partial-vector-derivative F i* ($y *_R i + x *_R$ ($\sum b \in$(*Basis* $-$ $\{i\}$). *b*)))
$*$ (*indicator s* ($y *_R i + x *_R$ ($\sum b \in$ *Basis* $-$ $\{i\}$. *b*)) ))) $\in$ *borel-measurable lborel*

**lemma** *analytically-valid-imp-part-deriv-integrable-on*:
**assumes** *analytically-valid* (*s*::(*real*$*$*real*) *set*) (*f*::(*real*$*$*real*)$\Rightarrow$ *real*) *i*
**shows** (*partial-vector-derivative f i*) *integrable-on s*
$\langle proof \rangle$

**definition** *typeII-twoCube* :: ((*real* $*$ *real*) $\Rightarrow$ (*real* $*$ *real*)) $\Rightarrow$ *bool* **where**
*typeII-twoCube twoC*
$\equiv \exists$ *a b g1 g2*. $a < b \wedge (\forall x \in \{a..b\}.\ g2\ x \le g1\ x) \wedge$
*twoC* = ($\lambda$(*y*, *x*). ((*1* $-$ *y*) $*$ (*g2* ((*1*$-$*x*)$*$*a* + *x*$*$*b*)) + *y* $*$ (*g1* ((*1*$-$*x*)$*$*a* + *x*$*$*b*)),
(*1*$-$*x*)$*$*a* + *x*$*$*b*)) $\wedge$
*g1 piecewise-C1-differentiable-on* $\{a .. b\}$ $\wedge$
*g2 piecewise-C1-differentiable-on* $\{a .. b\}$

**abbreviation** *unit-cube* **where** *unit-cube* $\equiv$ *cbox* (*0*,*0*) (*1*::*real*,*1*::*real*)

**definition** *cubeImage*:: *two-cube* $\Rightarrow$ ((*real*$*$*real*) *set*) **where**
*cubeImage twoC* $\equiv$ (*twoC* ‘ *unit-cube*)

**lemma** *typeII-twoCubeImg*:
**assumes** *typeII-twoCube twoC*
**shows** $\exists$ *a b g1 g2*. $a < b \wedge (\forall x \in \{a .. b\}.\ g2\ x \le g1\ x) \wedge$
*cubeImage twoC* = {(*y*,*x*). $x \in \{a..b\} \wedge y \in \{g2\ x\ ..\ g1\ x\}$}
$\wedge$ *twoC* = ($\lambda$(*y*, *x*). ((*1* $-$ *y*) $*$ *g2* ((*1* $-$ *x*) $*$ *a* + *x* $*$ *b*) + *y* $*$ *g1* ((*1* $-$ *x*) $*$ *a* + *x* $*$ *b*), (*1* $-$ *x*) $*$ *a* + *x* $*$ *b*))
$\wedge$ *g1 piecewise-C1-differentiable-on* $\{a .. b\}$ $\wedge$ *g2 piecewise-C1-differentiable-on* $\{a .. b\}$
$\langle proof \rangle$

**definition** *horizontal-boundary* :: *two-cube* $\Rightarrow$ *one-chain* **where**

*horizontal-boundary twoC* ≡ $\{(1, (\lambda x.\ twoC(x,0))), (-1, (\lambda x.\ twoC(x,1)))\}$

**definition** *vertical-boundary* :: *two-cube* ⇒ *one-chain* **where**
*vertical-boundary twoC* ≡ $\{(-1, (\lambda y.\ twoC(0,y))), (1, (\lambda y.\ twoC(1,y)))\}$

**definition** *boundary* :: *two-cube* ⇒ *one-chain* **where**
*boundary twoC* ≡ *horizontal-boundary twoC* ∪ *vertical-boundary twoC*

**definition** *valid-two-cube* **where**
*valid-two-cube twoC* ≡ *card* (*boundary twoC*) = *4*

**definition** *two-chain-integral*:: *two-chain* ⇒ ((*real*∗*real*)⇒(*real*)) ⇒ *real* **where**
*two-chain-integral twoChain F* ≡ $\sum$ *C*∈*twoChain*. (*integral* (*cubeImage C*) *F*)

**definition** *valid-two-chain* **where**
*valid-two-chain twoChain* ≡ (∀ *twoCube* ∈ *twoChain*. *valid-two-cube twoCube*) ∧ *pairwise* (λ*c1 c2*. ((*boundary c1*) ∩ (*boundary c2*)) = {}) *twoChain* ∧ *inj-on cubeImage twoChain*

**definition** *two-chain-boundary*:: *two-chain* ⇒ *one-chain* **where**
*two-chain-boundary twoChain* == $\bigcup$(*boundary* ‘ *twoChain*)

**definition** *gen-division* **where**
*gen-division s S* ≡ (*finite S* ∧ ($\bigcup$*S* = *s*) ∧ *pairwise* (λ*X Y*. *negligible* (*X* ∩ *Y*)) *S*)

**definition** *two-chain-horizontal-boundary*:: *two-chain* ⇒ *one-chain* **where**
*two-chain-horizontal-boundary twoChain* ≡ $\bigcup$(*horizontal-boundary* ‘ *twoChain*)

**definition** *two-chain-vertical-boundary*:: *two-chain* ⇒ *one-chain* **where**
*two-chain-vertical-boundary twoChain* ≡ $\bigcup$(*vertical-boundary* ‘ *twoChain*)

**definition** *only-horizontal-division* **where**
*only-horizontal-division one-chain two-chain*
≡ ∃ $\mathcal{H}$ $\mathcal{V}$. *finite* $\mathcal{H}$ ∧ *finite* $\mathcal{V}$ ∧
(∀ (*k*,γ) ∈ $\mathcal{H}$.
(∃ (*k*′, γ′) ∈ *two-chain-horizontal-boundary two-chain*.
(∃ *a* ∈ {*0..1*}. ∃ *b* ∈ {*0..1*}. *a* ≤ *b* ∧ *subpath a b* γ′ = γ))) ∧
(*common-sudiv-exists* (*two-chain-vertical-boundary two-chain*) $\mathcal{V}$
∨ *common-reparam-exists* $\mathcal{V}$ (*two-chain-vertical-boundary two-chain*))
∧
*boundary-chain* $\mathcal{V}$ ∧
*one-chain* = $\mathcal{H}$ ∪ $\mathcal{V}$ ∧ (∀ (*k*,γ)∈$\mathcal{V}$. *valid-path* γ)

**lemma** *sum-zero-set*:
**assumes** ∀ *x* ∈ *s*. *f x* = *0 finite s finite t*
**shows** *sum f* (*s* ∪ *t*) = *sum f t*
⟨*proof*⟩

**abbreviation** *valid-typeII-division s twoChain* ≡ ((∀ *twoCube* ∈ *twoChain*. *typeII-twoCube twoCube*) ∧
(*gen-division s* (*cubeImage* ' *twoChain*)) ∧
(*valid-two-chain twoChain*))

**lemma** *two-chain-vertical-boundary-is-boundary-chain*:
**shows** *boundary-chain* (*two-chain-vertical-boundary twoChain*)
⟨*proof*⟩

**lemma** *two-chain-horizontal-boundary-is-boundary-chain*:
**shows** *boundary-chain* (*two-chain-horizontal-boundary twoChain*)
⟨*proof*⟩

**definition** *typeI-twoCube* :: *two-cube* ⇒ *bool* **where**
*typeI-twoCube* (*twoC*::*two-cube*)
≡ ∃ *a b g1 g2*. *a* < *b* ∧ (∀ *x* ∈ {*a*..*b*}. *g2 x* ≤ *g1 x*) ∧
*twoC* = (λ(*x*,*y*). ((*1*−*x*)∗*a* + *x*∗*b*,
(*1* − *y*) ∗ (*g2* ((*1*−*x*)∗*a* + *x*∗*b*)) + *y* ∗ (*g1* ((*1*−*x*)∗*a* + *x*∗*b*)))) ∧
*g1 piecewise-C1-differentiable-on* {*a*..*b*} ∧
*g2 piecewise-C1-differentiable-on* {*a*..*b*}

**lemma** *typeI-twoCubeImg*:
**assumes** *typeI-twoCube twoC*
**shows** ∃ *a b g1 g2*. *a* < *b* ∧ (∀ *x* ∈ {*a* .. *b*}. *g2 x* ≤ *g1 x*) ∧
*cubeImage twoC* = {(*x*,*y*). *x* ∈ {*a*..*b*} ∧ *y* ∈ {*g2 x* .. *g1 x*}} ∧
*twoC* = (λ(*x*, *y*). ((*1* − *x*) ∗ *a* + *x* ∗ *b*, (*1* − *y*) ∗ *g2* ((*1* − *x*) ∗ *a* + *x* ∗ *b*) + *y* ∗ *g1* ((*1* − *x*) ∗ *a* + *x* ∗ *b*))) ∧
*g1 piecewise-C1-differentiable-on* {*a* .. *b*} ∧ *g2 piecewise-C1-differentiable-on* {*a* .. *b*}
⟨*proof*⟩

**lemma** *typeI-cube-explicit-spec*:
**assumes** *typeI-twoCube twoC*
**shows** ∃ *a b g1 g2*. *a* < *b* ∧ (∀ *x* ∈ {*a* .. *b*}. *g2 x* ≤ *g1 x*) ∧
*cubeImage twoC* = {(*x*,*y*). *x* ∈ {*a*..*b*} ∧ *y* ∈ {*g2 x* .. *g1 x*}}
∧ *twoC* = (λ(*x*, *y*). ((*1* − *x*) ∗ *a* + *x* ∗ *b*, (*1* − *y*) ∗ *g2* ((*1* − *x*) ∗ *a* + *x* ∗ *b*) + *y* ∗ *g1* ((*1* − *x*) ∗ *a* + *x* ∗ *b*)))
∧ *g1 piecewise-C1-differentiable-on* {*a* .. *b*} ∧ *g2 piecewise-C1-differentiable-on* {*a* .. *b*}
∧ (λ*x*. *twoC*(*x*, *0*)) = (λ*x*. (*a* + (*b* − *a*) ∗ *x*, *g2* (*a* + (*b* − *a*) ∗ *x*)))
∧ (λ*y*. *twoC*(*1*, *y*)) = (λ*x*. (*b*, *g2 b* + *x* $*_R$ (*g1 b* − *g2 b*)))
∧ (λ*x*. *twoC*(*x*, *1*)) = (λ*x*. (*a* + (*b* − *a*) ∗ *x*, *g1* (*a* + (*b* − *a*) ∗ *x*)))
∧ (λ*y*. *twoC*(*0*, *y*)) = (λ*x*. (*a*, *g2 a* + *x* $*_R$ (*g1 a* − *g2 a*)))
⟨*proof*⟩

**lemma** *typeI-twoCube-smooth-edges*:
**assumes** *typeI-twoCube twoC*
$(k,\gamma) \in$ *boundary twoC*
**shows** $\gamma$ *piecewise-C1-differentiable-on* $\{0..1\}$
$\langle proof \rangle$

**lemma** *two-chain-integral-eq-integral-divisable*:
**assumes** *f-integrable*: $\forall$ *twoCube* $\in$ *twoChain*. *F integrable-on cubeImage twoCube*
**and**
*gen-division*: *gen-division s* (*cubeImage* ‘ *twoChain*) **and**
*valid-two-chain*: *valid-two-chain twoChain*
**shows** *integral s F* = *two-chain-integral twoChain F*
$\langle proof \rangle$

**definition** *only-vertical-division* **where**
*only-vertical-division one-chain two-chain* $\equiv$
$\exists \mathcal{V}\ \mathcal{H}$. *finite* $\mathcal{H} \wedge$ *finite* $\mathcal{V} \wedge$
$(\forall (k,\gamma) \in \mathcal{V}$.
$(\exists (k',\gamma') \in$ *two-chain-vertical-boundary two-chain*.
$(\exists a \in \{0..1\}. \exists b \in \{0..1\}. a \leq b \wedge$ *subpath a b* $\gamma' = \gamma))) \wedge$
(*common-sudiv-exists* (*two-chain-horizontal-boundary two-chain*) $\mathcal{H}$
$\vee$ *common-reparam-exists* $\mathcal{H}$ (*two-chain-horizontal-boundary two-chain*))
$\wedge$
*boundary-chain* $\mathcal{H} \wedge$ *one-chain* $= \mathcal{V} \cup \mathcal{H} \wedge$
$(\forall (k,\gamma) \in \mathcal{H}$. *valid-path* $\gamma)$

**abbreviation** *valid-typeI-division s twoChain*
$\equiv$ ($\forall$ *twoCube* $\in$ *twoChain*. *typeI-twoCube twoCube*) $\wedge$
*gen-division s* (*cubeImage* ‘ *twoChain*) $\wedge$ *valid-two-chain twoChain*

**lemma** *field-cont-on-typeI-region-cont-on-edges*:
**assumes** *typeI-twoC*: *typeI-twoCube twoC*
**and** *field-cont*: *continuous-on* (*cubeImage twoC*) *F*
**and** *member-of-boundary*: $(k,\gamma) \in$ *boundary twoC*
**shows** *continuous-on* ($\gamma$ ‘ $\{0\ ..\ 1\}$) *F*
$\langle proof \rangle$

**lemma** *typeII-cube-explicit-spec*:
**assumes** *typeII-twoCube twoC*
**shows** $\exists a\ b\ g1\ g2.\ a < b \wedge (\forall x \in \{a\ ..\ b\}.\ g2\ x \leq g1\ x) \wedge$
*cubeImage twoC* $= \{(y, x).\ x \in \{a..b\} \wedge y \in \{g2\ x\ ..\ g1\ x\}\}$
$\wedge$ *twoC* $= (\lambda(y, x).\ ((1 - y) * g2\ ((1 - x) * a + x * b) + y * g1$
$((1 - x) * a + x * b), (1 - x) * a + x * b))$
$\wedge$ *g1 piecewise-C1-differentiable-on* $\{a\ ..\ b\} \wedge$ *g2 piecewise-C1-differentiable-on*
$\{a\ ..\ b\}$
$\wedge (\lambda x.\ twoC(0, x)) = (\lambda x.\ (g2\ (a + (b - a) * x), a + (b - a) * x))$
$\wedge (\lambda y.\ twoC(y, 1)) = (\lambda x.\ (g2\ b + x *_R (g1\ b - g2\ b), b))$
$\wedge (\lambda x.\ twoC(1, x)) = (\lambda x.\ (g1\ (a + (b - a) * x), a + (b - a) * x))$

$\wedge$ (*λy. twoC*(*y*, *0*)) = (*λx.* (*g2 a* + *x* $*_R$ (*g1 a* − *g2 a*), *a*))
⟨*proof*⟩

**lemma** *typeII-twoCube-smooth-edges*:
**assumes** *typeII-twoCube twoC* (*k*,*γ*) ∈ *boundary twoC*
**shows** *γ piecewise-C1-differentiable-on* {*0*..*1*}
⟨*proof*⟩

**lemma** *field-cont-on-typeII-region-cont-on-edges*:
**assumes** *typeII-twoC*:
*typeII-twoCube twoC* **and**
*field-cont*:
*continuous-on* (*cubeImage twoC*) *F* **and**
*member-of-boundary*:
(*k*,*γ*) ∈ *boundary twoC*
**shows** *continuous-on* (*γ* ' {*0* .. *1*}) *F*
⟨*proof*⟩

**lemma** *two-cube-boundary-is-boundary*: *boundary-chain* (*boundary C*)
⟨*proof*⟩

**lemma** *common-boundary-subdiv-exists-refl*:
**assumes** ∀ (*k*,*γ*)∈*boundary twoC*. *valid-path γ*
**shows** *common-boundary-sudivision-exists* (*boundary twoC*) (*boundary twoC*)
⟨*proof*⟩

**lemma** *common-boundary-subdiv-exists-refl'*:
**assumes** ∀ (*k*,*γ*)∈*C*. *valid-path γ*
*boundary-chain* (*C*::(*int* × (*real* ⇒ *real* × *real*)) *set*)
**shows** *common-boundary-sudivision-exists* (*C*) (*C*)
⟨*proof*⟩

**lemma** *gen-common-boundary-subdiv-exists-refl-twochain-boundary*:
**assumes** ∀ (*k*,*γ*)∈*C*. *valid-path γ*
*boundary-chain* (*C*::(*int* × (*real* ⇒ *real* × *real*)) *set*)
**shows** *common-sudiv-exists* (*C*) (*C*)
⟨*proof*⟩

**lemma** *two-chain-boundary-is-boundary-chain*:
**shows** *boundary-chain* (*two-chain-boundary twoChain*)
⟨*proof*⟩

**lemma** *typeI-edges-are-valid-paths*:
**assumes** *typeI-twoCube twoC* (*k*,*γ*) ∈ *boundary twoC*
**shows** *valid-path γ*
⟨*proof*⟩

**lemma** *typeII-edges-are-valid-paths*:
**assumes** *typeII-twoCube twoC* (*k*,*γ*) ∈ *boundary twoC*

**shows** *valid-path* $\gamma$
⟨*proof*⟩

**lemma** *finite-two-chain-vertical-boundary*:
**assumes** *finite two-chain*
**shows** *finite* (*two-chain-vertical-boundary two-chain*)
⟨*proof*⟩

**lemma** *finite-two-chain-horizontal-boundary*:
**assumes** *finite two-chain*
**shows** *finite* (*two-chain-horizontal-boundary two-chain*)
⟨*proof*⟩

**locale** *R2* =
**fixes** *i j*
**assumes** *i-is-x-axis*: $i = (1::real, 0::real)$ **and**
*j-is-y-axis*: $j = (0::real,\ 1::real)$
**begin**

**lemma** *analytically-valid-y*:
**assumes** *analytically-valid s F i*
**shows** ($\lambda x$. *integral UNIV* ($\lambda y$. (*partial-vector-derivative F i*) $(y, x) *$ (*indicator s* $(y, x)$))) $\in$ *borel-measurable lborel*
⟨*proof*⟩

**lemma** *analytically-valid-x*:
**assumes** *analytically-valid s F j*
**shows** ($\lambda x$. *integral UNIV* ($\lambda y$. ((*partial-vector-derivative F j*) $(x, y)) *$ (*indicator s* $(x, y)$))) $\in$ *borel-measurable lborel*
⟨*proof*⟩

**lemma** *Greens-thm-type-I*:
**fixes** $F$:: $((real*real) \Rightarrow (real * real))$ **and**
*gamma1 gamma2 gamma3 gamma4* :: $(real \Rightarrow (real * real))$ **and**
$a$:: *real* **and** $b$:: *real* **and**
*g1*:: $(real \Rightarrow real)$ **and** *g2*:: $(real \Rightarrow real)$
**assumes** *Dy-def*: *Dy-pair* = $\{(x::real,y)\ .\ x \in cbox\ a\ b \wedge y \in cbox\ (g2\ x)\ (g1\ x)\}$ **and**
*gamma1-def*: $gamma1 = (\lambda x.\ (a + (b - a) * x,\ g2(a + (b - a) * x)))$ **and**
*gamma1-smooth*: *gamma1 piecewise-C1-differentiable-on* $\{0..1\}$ **and**
*gamma2-def*: $gamma2 = (\lambda x.\ (b,\ g2(b) + x *_R (g1(b) - g2(b))))$ **and**
*gamma3-def*: $gamma3 = (\lambda x.\ (a + (b - a) * x,\ g1(a + (b - a) * x)))$ **and**
*gamma3-smooth*: *gamma3 piecewise-C1-differentiable-on* $\{0..1\}$ **and**
*gamma4-def*: $gamma4 = (\lambda x.\ (a,\ g2(a) + x *_R (g1(a) - g2(a))))$ **and**
*F-i-analytically-valid*: *analytically-valid Dy-pair* $(\lambda p.\ F(p) \cdot i)$ $j$ **and**
*g2-leq-g1*: $\forall x \in cbox\ a\ b.\ (g2\ x) \leq (g1\ x)$ **and**
*a-lt-b*: $a < b$
**shows** (*line-integral F* $\{i\}$ *gamma1*) +
(*line-integral F* $\{i\}$ *gamma2*) −

(*line-integral* $F$ $\{i\}$ *gamma3*) $-$
(*line-integral* $F$ $\{i\}$ *gamma4*)
= (*integral Dy-pair* ($\lambda a.$ $-$ (*partial-vector-derivative* ($\lambda p.$ $F(p)$ $\cdot$ $i$) $j$ $a$)))
*line-integral-exists* $F$ $\{i\}$ *gamma4*
*line-integral-exists* $F$ $\{i\}$ *gamma3*
*line-integral-exists* $F$ $\{i\}$ *gamma2*
*line-integral-exists* $F$ $\{i\}$ *gamma1*
$\langle proof \rangle$

**theorem** *Greens-thm-type-II*:
**fixes** $F$:: ((*real*$*$*real*) $\Rightarrow$ (*real* $*$ *real*)) **and**
*gamma4 gamma3 gamma2 gamma1* :: (*real* $\Rightarrow$ (*real* $*$ *real*)) **and**
$a$:: *real* **and** $b$:: *real* **and**
*g1*:: (*real* $\Rightarrow$ *real*) **and** *g2*:: (*real* $\Rightarrow$ *real*)
**assumes** *Dx-def*: *Dx-pair* = $\{(x::real,y)$ . $y \in$ *cbox* $a$ $b$ $\wedge$ $x \in$ *cbox* (*g2* $y$) (*g1* $y$)$\}$ **and**
*gamma4-def*: *gamma4* = ($\lambda x.$ (*g2*($a$ + ($b$ $-$ $a$) $*$ $x$), $a$ + ($b$ $-$ $a$) $*$ $x$)) **and**
*gamma4-smooth*: *gamma4* *piecewise-C1-differentiable-on* $\{0..1\}$ **and**
*gamma3-def*: *gamma3* = ($\lambda x.$ (*g2*($b$) + $x$ $*_R$ (*g1*($b$) $-$ *g2*($b$)), $b$)) **and**
*gamma2-def*: *gamma2* = ($\lambda x.$ (*g1*($a$ + ($b$ $-$ $a$) $*$ $x$), $a$ + ($b$ $-$ $a$) $*$ $x$)) **and**
*gamma2-smooth*: *gamma2* *piecewise-C1-differentiable-on* $\{0..1\}$ **and**
*gamma1-def*: *gamma1* = ($\lambda x.$ (*g2*($a$) + $x$ $*_R$ (*g1*($a$) $-$ *g2*($a$)), $a$)) **and**
*F-j-analytically-valid*: *analytically-valid Dx-pair* ($\lambda p.$ $F(p)$ $\cdot$ $j$) $i$ **and**
*g2-leq-g1*: $\forall x \in$ *cbox* $a$ $b$. (*g2* $x$) $\leq$ (*g1* $x$) **and**
*a-lt-b*: $a < b$
**shows** $-$(*line-integral* $F$ $\{j\}$ *gamma4*) $-$
(*line-integral* $F$ $\{j\}$ *gamma3*) +
(*line-integral* $F$ $\{j\}$ *gamma2*) +
(*line-integral* $F$ $\{j\}$ *gamma1*)
= (*integral Dx-pair* ($\lambda a.$ (*partial-vector-derivative* ($\lambda a.$ ($F$ $a$) $\cdot$ $j$) $i$ $a$)))
*line-integral-exists* $F$ $\{j\}$ *gamma4*
*line-integral-exists* $F$ $\{j\}$ *gamma3*
*line-integral-exists* $F$ $\{j\}$ *gamma2*
*line-integral-exists* $F$ $\{j\}$ *gamma1*
$\langle proof \rangle$

**end**

**locale** *green-typeII-cube* = *R2* +
**fixes** *twoC F*
**assumes**
*two-cube*: *typeII-twoCube twoC* **and**
*valid-two-cube*: *valid-two-cube twoC* **and**
*f-analytically-valid*: *analytically-valid* (*cubeImage twoC*) ($\lambda x.$ ($F$ $x$) $\cdot$ $j$) $i$
**begin**

**lemma** *GreenThm-typeII-twoCube*:

**shows** *integral* (*cubeImage twoC*) ($\lambda a$. *partial-vector-derivative* ($\lambda x$. ($F$ $x$) $\cdot$ $j$) $i$ $a$) = *one-chain-line-integral* $F$ $\{j\}$ (*boundary twoC*)
$\forall$ ($k$,$\gamma$) $\in$ *boundary twoC*. *line-integral-exists* $F$ $\{j\}$ $\gamma$
⟨*proof*⟩

**lemma** *line-integral-exists-on-typeII-Cube-boundaries′*:
**assumes** ($k$,$\gamma$) $\in$ *boundary twoC*
**shows** *line-integral-exists* $F$ $\{j\}$ $\gamma$
⟨*proof*⟩

**end**

**locale** *green-typeII-chain* = *R2* +
**fixes** $F$ *two-chain* $s$
**assumes** *valid-typeII-div*: *valid-typeII-division* $s$ *two-chain* **and**
*F-anal-valid*: $\forall$ *twoC* $\in$ *two-chain*. *analytically-valid* (*cubeImage twoC*) ($\lambda x$. ($F$ $x$) $\cdot$ $j$) $i$
**begin**

**lemma** *two-chain-valid-valid-cubes*: $\forall$ *two-cube* $\in$ *two-chain*. *valid-two-cube two-cube*
⟨*proof*⟩

**lemma** *typeII-chain-line-integral-exists-boundary′*:
**shows** $\forall$ ($k$,$\gamma$) $\in$ *two-chain-vertical-boundary two-chain*. *line-integral-exists* $F$ $\{j\}$ $\gamma$
⟨*proof*⟩

**lemma** *typeII-chain-line-integral-exists-boundary″*:
$\forall$ ($k$,$\gamma$) $\in$ *two-chain-horizontal-boundary two-chain*. *line-integral-exists* $F$ $\{j\}$ $\gamma$
⟨*proof*⟩

**lemma** *typeII-cube-line-integral-exists-boundary*:
$\forall$ ($k$,$\gamma$) $\in$ *two-chain-boundary two-chain*. *line-integral-exists* $F$ $\{j\}$ $\gamma$
⟨*proof*⟩

**lemma** *type-II-chain-horiz-bound-valid*:
$\forall$ ($k$,$\gamma$) $\in$ *two-chain-horizontal-boundary two-chain*. *valid-path* $\gamma$
⟨*proof*⟩

**lemma** *type-II-chain-vert-bound-valid*:
$\forall$ ($k$,$\gamma$) $\in$ *two-chain-vertical-boundary two-chain*. *valid-path* $\gamma$
⟨*proof*⟩

**lemma** *members-of-only-horiz-div-line-integrable′*:
**assumes** *only-horizontal-division one-chain two-chain*
($k$::*int*, $\gamma$)$\in$*one-chain*
($k$::*int*, $\gamma$)$\in$*one-chain*
*finite two-chain*
$\forall$ *two-cube* $\in$ *two-chain*. *valid-two-cube two-cube*

**shows** *line-integral-exists* $F$ $\{j\}$ $\gamma$
⟨*proof*⟩

**lemma** *GreenThm-typeII-twoChain*:
**shows** *two-chain-integral two-chain* (*partial-vector-derivative* ($\lambda a$. ($F$ $a$) $\cdot$ $j$) $i$)
= *one-chain-line-integral* $F$ $\{j\}$ (*two-chain-boundary two-chain*)
⟨*proof*⟩

**lemma** *GreenThm-typeII-divisible*:
**assumes**
*gen-division*: *gen-division* $s$ (*cubeImage* ‘ *two-chain*)
**shows** *integral* $s$ (*partial-vector-derivative* ($\lambda x$. ($F$ $x$) $\cdot$ $j$) $i$) = *one-chain-line-integral*
$F$ $\{j\}$ (*two-chain-boundary two-chain*)
⟨*proof*⟩

**lemma** *GreenThm-typeII-divisible-region-boundary-gen*:
**assumes** *only-horizontal-division*: *only-horizontal-division* $\gamma$ *two-chain*
**shows** *integral* $s$ (*partial-vector-derivative* ($\lambda x$. ($F$ $x$) $\cdot$ $j$) $i$) = *one-chain-line-integral*
$F$ $\{j\}$ $\gamma$
⟨*proof*⟩

**lemma** *GreenThm-typeII-divisible-region-boundary*:
**assumes**
*two-cubes-trace-vertical-boundaries*:
*two-chain-vertical-boundary two-chain* $\subseteq$ $\gamma$ **and**
*boundary-of-region-is-subset-of-partition-boundary*:
$\gamma$ $\subseteq$ *two-chain-boundary two-chain*
**shows** *integral* $s$ (*partial-vector-derivative* ($\lambda x$. ($F$ $x$) $\cdot$ $j$) $i$) = *one-chain-line-integral*
$F$ $\{j\}$ $\gamma$
⟨*proof*⟩

**end**

**locale** *green-typeI-cube* = *R2* +
**fixes** *twoC* $F$
**assumes**
*two-cube*: *typeI-twoCube twoC* **and**
*valid-two-cube*: *valid-two-cube twoC* **and**
*f-analytically-valid*: *analytically-valid* (*cubeImage twoC*) ($\lambda x$. ($F$ $x$) $\cdot$ $i$) $j$
**begin**

**lemma** *GreenThm-typeI-twoCube*:
**shows** *integral* (*cubeImage twoC*) ($\lambda a$. − *partial-vector-derivative* ($\lambda p$. $F$ $p$ $\cdot$ $i$) $j$
$a$) = *one-chain-line-integral* $F$ $\{i\}$ (*boundary twoC*)
$\forall$ ($k$,$\gamma$) $\in$ *boundary twoC*. *line-integral-exists* $F$ $\{i\}$ $\gamma$
⟨*proof*⟩

**lemma** *line-integral-exists-on-typeI-Cube-boundaries'*:
**assumes** ($k$,$\gamma$) $\in$ *boundary twoC*

**shows** *line-integral-exists F* $\{i\}$ $\gamma$
⟨*proof*⟩

**end**

**locale** *green-typeI-chain* = *R2* +
**fixes** *F two-chain s*
**assumes** *valid-typeI-div*: *valid-typeI-division s two-chain* **and**
*F-anal-valid*: $\forall$ *twoC* $\in$ *two-chain. analytically-valid* (*cubeImage twoC*) ($\lambda x$. ($F$ $x$) $\cdot$ $i$) $j$
**begin**

**lemma** *two-chain-valid-valid-cubes*: $\forall$ *two-cube* $\in$ *two-chain. valid-two-cube two-cube*
⟨*proof*⟩

**lemma** *typeI-cube-line-integral-exists-boundary′*:
**assumes** $\forall$ *two-cube* $\in$ *two-chain. typeI-twoCube two-cube*
**assumes** $\forall$ *twoC* $\in$ *two-chain. analytically-valid* (*cubeImage twoC*) ($\lambda x$. ($F$ $x$) $\cdot$ $i$) $j$
**assumes** $\forall$ *two-cube* $\in$ *two-chain. valid-two-cube two-cube*
**shows** $\forall$ ($k$,$\gamma$) $\in$ *two-chain-vertical-boundary two-chain. line-integral-exists F* $\{i\}$ $\gamma$
⟨*proof*⟩

**lemma** *typeI-cube-line-integral-exists-boundary″*:
$\forall$ ($k$,$\gamma$) $\in$ *two-chain-horizontal-boundary two-chain. line-integral-exists F* $\{i\}$ $\gamma$
⟨*proof*⟩

**lemma** *typeI-cube-line-integral-exists-boundary*:
$\forall$ ($k$,$\gamma$) $\in$ *two-chain-boundary two-chain. line-integral-exists F* $\{i\}$ $\gamma$
⟨*proof*⟩

**lemma** *type-I-chain-horiz-bound-valid*:
$\forall$ ($k$,$\gamma$) $\in$ *two-chain-horizontal-boundary two-chain. valid-path* $\gamma$
⟨*proof*⟩

**lemma** *type-I-chain-vert-bound-valid*:
**assumes** $\forall$ *two-cube* $\in$ *two-chain. typeI-twoCube two-cube*
**shows** $\forall$ ($k$,$\gamma$) $\in$ *two-chain-vertical-boundary two-chain. valid-path* $\gamma$
⟨*proof*⟩

**lemma** *members-of-only-vertical-div-line-integrable′*:
**assumes** *only-vertical-division one-chain two-chain*
($k$::*int*, $\gamma$)$\in$*one-chain*
($k$::*int*, $\gamma$)$\in$*one-chain*
*finite two-chain*
**shows** *line-integral-exists F* $\{i\}$ $\gamma$
⟨*proof*⟩

**lemma** *GreenThm-typeI-two-chain*:
*two-chain-integral two-chain* ($\lambda a.$ $-$ *partial-vector-derivative* ($\lambda x.$ ($F$ $x$) $\cdot$ $i$) $j$ $a$)
= *one-chain-line-integral* $F$ $\{i\}$ (*two-chain-boundary two-chain*)
⟨*proof*⟩

**lemma** *GreenThm-typeI-divisible*:
**assumes** *gen-division*: *gen-division* $s$ (*cubeImage* ‘ *two-chain*)
**shows** *integral* $s$ ($\lambda x.$ $-$ *partial-vector-derivative* ($\lambda a.$ $F(a) \cdot i$) $j$ $x$) = *one-chain-line-integral* $F$ $\{i\}$ (*two-chain-boundary two-chain*)
⟨*proof*⟩

**lemma** *GreenThm-typeI-divisible-region-boundary*:
**assumes**
*gen-division*: *gen-division* $s$ (*cubeImage* ‘ *two-chain*) **and**
*two-cubes-trace-horizontal-boundaries*:
*two-chain-horizontal-boundary two-chain* $\subseteq \gamma$ **and**
*boundary-of-region-is-subset-of-partition-boundary*:
$\gamma \subseteq$ *two-chain-boundary two-chain*
**shows** *integral* $s$ ($\lambda x.$ $-$ *partial-vector-derivative* ($\lambda a.$ $F(a) \cdot i$) $j$ $x$) = *one-chain-line-integral* $F$ $\{i\}$ $\gamma$
⟨*proof*⟩

**lemma** *GreenThm-typeI-divisible-region-boundary-gen*:
**assumes** *valid-typeI-div*: *valid-typeI-division* $s$ *two-chain* **and**
*f-analytically-valid*: $\forall$ *twoC* $\in$ *two-chain*. *analytically-valid* (*cubeImage twoC*) ($\lambda a.$ $F(a) \cdot i$) $j$ **and**
*only-vertical-division*:
*only-vertical-division* $\gamma$ *two-chain*
**shows** *integral* $s$ ($\lambda x.$ $-$ *partial-vector-derivative* ($\lambda a.$ $F(a) \cdot i$) $j$ $x$) = *one-chain-line-integral* $F$ $\{i\}$ $\gamma$
⟨*proof*⟩

**end**

**locale** *green-typeI-typeII-chain* = *R2*: *R2* $i$ $j$ + *T1*: *green-typeI-chain* $i$ $j$ $F$ *two-chain-typeI* + *T2*: *green-typeII-chain* $i$ $j$ $F$ *two-chain-typeII* **for** $i$ $j$ $F$ *two-chain-typeI two-chain-typeII*
**begin**

**lemma** *GreenThm-typeI-typeII-divisible-region-boundary*:
**assumes**
*gen-divisions*: *gen-division* $s$ (*cubeImage* ‘ *two-chain-typeI*)
*gen-division* $s$ (*cubeImage* ‘ *two-chain-typeII*) **and**
*typeI-two-cubes-trace-horizontal-boundaries*:
*two-chain-horizontal-boundary two-chain-typeI* $\subseteq \gamma$ **and**
*typeII-two-cubes-trace-vertical-boundaries*:
*two-chain-vertical-boundary two-chain-typeII* $\subseteq \gamma$ **and**
*boundary-of-region-is-subset-of-partition-boundaries*:
$\gamma \subseteq$ *two-chain-boundary two-chain-typeI*
$\gamma \subseteq$ *two-chain-boundary two-chain-typeII*

**shows** *integral s* ($\lambda x$. *partial-vector-derivative* ($\lambda a$. $F$ $a$ $\cdot$ $j$) $i$ $x$ $-$ *partial-vector-derivative* ($\lambda a$. $F$ $a$ $\cdot$ $i$) $j$ $x$)
= *one-chain-line-integral* $F$ $\{i, j\}$ $\gamma$
⟨*proof*⟩

**lemma** *GreenThm-typeI-typeII-divisible-region′*:
**assumes**
*only-vertical-division*:
*only-vertical-division one-chain-typeI two-chain-typeI*
*boundary-chain one-chain-typeI* **and**
*only-horizontal-division*:
*only-horizontal-division one-chain-typeII two-chain-typeII*
*boundary-chain one-chain-typeII* **and**
*typeI-and-typII-one-chains-have-gen-common-subdiv*:
*common-sudiv-exists one-chain-typeI one-chain-typeII*
**shows** *integral s* ($\lambda x$. *partial-vector-derivative* ($\lambda x$. ($F$ $x$) $\cdot$ $j$) $i$ $x$ $-$ *partial-vector-derivative* ($\lambda x$. ($F$ $x$) $\cdot$ $i$) $j$ $x$) = *one-chain-line-integral* $F$ $\{i, j\}$ *one-chain-typeI*
*integral s* ($\lambda x$. *partial-vector-derivative* ($\lambda x$. ($F$ $x$) $\cdot$ $j$) $i$ $x$ $-$ *partial-vector-derivative* ($\lambda x$. ($F$ $x$) $\cdot$ $i$) $j$ $x$) = *one-chain-line-integral* $F$ $\{i, j\}$ *one-chain-typeII*
⟨*proof*⟩

**lemma** *GreenThm-typeI-typeII-divisible-region*:
**assumes** *only-vertical-division*:
*only-vertical-division one-chain-typeI two-chain-typeI*
*boundary-chain one-chain-typeI* **and**
*only-horizontal-division*:
*only-horizontal-division one-chain-typeII two-chain-typeII*
*boundary-chain one-chain-typeII* **and**
*typeI-and-typII-one-chains-have-common-subdiv*:
*common-boundary-sudivision-exists one-chain-typeI one-chain-typeII*
**shows** *integral s* ($\lambda x$. *partial-vector-derivative* ($\lambda x$. ($F$ $x$) $\cdot$ $j$) $i$ $x$ $-$ *partial-vector-derivative* ($\lambda x$. ($F$ $x$) $\cdot$ $i$) $j$ $x$) = *one-chain-line-integral* $F$ $\{i, j\}$ *one-chain-typeI*
*integral s* ($\lambda x$. *partial-vector-derivative* ($\lambda x$. ($F$ $x$) $\cdot$ $j$) $i$ $x$ $-$ *partial-vector-derivative* ($\lambda x$. ($F$ $x$) $\cdot$ $i$) $j$ $x$) = *one-chain-line-integral* $F$ $\{i, j\}$ *one-chain-typeII*
⟨*proof*⟩

**lemma** *GreenThm-typeI-typeII-divisible-region-finite-holes*:
**assumes** *valid-cube-boundary*: $\forall$ $(k,\gamma)\in$*boundary* $C$. *valid-path* $\gamma$ **and**
*only-vertical-division*:
*only-vertical-division* (*boundary* $C$) *two-chain-typeI* **and**
*only-horizontal-division*:
*only-horizontal-division* (*boundary* $C$) *two-chain-typeII* **and**
*s-is-oneCube*: $s$ = *cubeImage* $C$
**shows** *integral* (*cubeImage* $C$) ($\lambda x$. *partial-vector-derivative* ($\lambda x$. $F$ $x$ $\cdot$ $j$) $i$ $x$ $-$ *partial-vector-derivative* ($\lambda x$. $F$ $x$ $\cdot$ $i$) $j$ $x$) =
*one-chain-line-integral* $F$ $\{i, j\}$ (*boundary* $C$)
⟨*proof*⟩

**lemma** *GreenThm-typeI-typeII-divisible-region-equivallent-boundary*:

**assumes**
*gen-divisions*: *gen-division s* (*cubeImage ‘ two-chain-typeI*)
*gen-division s* (*cubeImage ‘ two-chain-typeII*) **and**
*typeI-two-cubes-trace-horizontal-boundaries*:
*two-chain-horizontal-boundary two-chain-typeI* $\subseteq$ *one-chain-typeI* **and**
*typeII-two-cubes-trace-vertical-boundaries*:
*two-chain-vertical-boundary two-chain-typeII* $\subseteq$ *one-chain-typeII* **and**
*boundary-of-region-is-subset-of-partition-boundaries*:
*one-chain-typeI* $\subseteq$ *two-chain-boundary two-chain-typeI*
*one-chain-typeII* $\subseteq$ *two-chain-boundary two-chain-typeII* **and**
*typeI-and-typII-one-chains-have-common-subdiv*:
*common-boundary-sudivision-exists one-chain-typeI one-chain-typeII*
**shows** *integral s* ($\lambda x$. *partial-vector-derivative* ($\lambda x$. ($F\ x$) $\cdot$ $j$) $i\ x$ $-$ *partial-vector-derivative* ($\lambda x$. ($F\ x$) $\cdot$ $i$) $j\ x$) $=$ *one-chain-line-integral* $F$ $\{i, j\}$ *one-chain-typeI*
*integral s* ($\lambda x$. *partial-vector-derivative* ($\lambda x$. ($F\ x$) $\cdot$ $j$) $i\ x$ $-$ *partial-vector-derivative* ($\lambda x$. ($F\ x$) $\cdot$ $i$) $j\ x$) $=$ *one-chain-line-integral* $F$ $\{i, j\}$ *one-chain-typeII*
⟨*proof*⟩

**end**
**end**
**theory** *SymmetricR2Shapes*
**imports** *Green*
**begin**

**context** *R2*
**begin**

**lemma** *valid-path-valid-swap*:
**assumes** *valid-path* ($\lambda x$::*real*. (($f\ x$)::*real*, ($g\ x$)::*real*))
**shows** *valid-path* (*prod.swap o* ($\lambda x$. ($f\ x$, $g\ x$)))
⟨*proof*⟩

**lemma** *pair-fun-components*: $C = (\lambda x.\ (C\ x \cdot i,\ C\ x \cdot j))$
⟨*proof*⟩

**lemma** *swap-pair-fun*: ($\lambda y$. *prod.swap* ($C$ ($y$, *0*))) $=$ ($\lambda x$. ($C$ ($x$, *0*) $\cdot$ $j$, $C$ ($x$, *0*) $\cdot$ $i$))
⟨*proof*⟩

**lemma** *swap-pair-fun′*: ($\lambda y$. *prod.swap* ($C$ ($y$, *1*))) $=$ ($\lambda x$. ($C$ ($x$, *1*) $\cdot$ $j$, $C$ ($x$, *1*) $\cdot$ $i$))
⟨*proof*⟩

**lemma** *swap-pair-fun″*: ($\lambda y$. *prod.swap* ($C$ (*0*, $y$))) $=$ ($\lambda x$. ($C$ (*0*,$x$) $\cdot$ $j$, $C$ (*0*,$x$) $\cdot$ $i$))
⟨*proof*⟩

**lemma** *swap-pair-fun‴*: ($\lambda y$. *prod.swap* ($C$ (*1*, $y$))) $=$ ($\lambda x$. ($C$ (*1*,$x$) $\cdot$ $j$, $C$ (*1*,$x$) $\cdot$ $i$))

⟨*proof*⟩

**lemma** *swap-valid-boundaries*:
**assumes** $\forall (k,\gamma)\in$*boundary C*. *valid-path* $\gamma$
**assumes** $(k,\gamma)\in$*boundary* (*prod.swap o C o prod.swap*)
**shows** *valid-path* $\gamma$
⟨*proof*⟩

**lemma** *prod-comp-eq*:
**assumes** *f* = *prod.swap o g*
**shows** *prod.swap o f* = *g*
⟨*proof*⟩

**lemma** *swap-typeI-is-typeII*:
**assumes** *typeI-twoCube C*
**shows** *typeII-twoCube* (*prod.swap o C o prod.swap*)
⟨*proof*⟩

**lemma** *valid-cube-valid-swap*:
**assumes** *valid-two-cube C*
**shows** *valid-two-cube* (*prod.swap o C o prod.swap*)
⟨*proof*⟩

**lemma** *twoChainVertDiv-of-itself*:
**assumes** *finite C*
$\forall (k, \gamma)\in$(*two-chain-boundary C*). *valid-path* $\gamma$
**shows** *only-vertical-division* (*two-chain-boundary C*) *C*
⟨*proof*⟩

**end**

**definition** *x-coord* **where** *x-coord* $\equiv$ ($\lambda t$::*real*. $t - 1/2$)

**lemma** *x-coord-smooth*: *x-coord C1-differentiable-on* $\{a..b\}$
⟨*proof*⟩

**lemma** *x-coord-bounds*:
**assumes** $(0::real) \leq x$ $x \leq 1$
**shows** $-1/2 \leq$ *x-coord* $x \wedge$ *x-coord* $x \leq 1/2$
⟨*proof*⟩

**lemma** *x-coord-img*: *x-coord* ‘ $\{(0::real)..1\} = \{-1/2 .. 1/2\}$
⟨*proof*⟩

**lemma** *x-coord-back-img*: *finite* ($\{0..1\} \cap$ *x-coord* −‘ $\{x::real\}$)
⟨*proof*⟩

**abbreviation** *rot-x t1 t2* ≡ (*if* (*t1* − *1*/*2*) ≤ *0 then* (*2* ∗ *t2* − *1*) ∗ *t1* + *1*/*2* ::*real else 2* ∗ *t2* − *2* ∗ *t1* ∗ *t2* +*t1* −*1*/*2*::*real*)

**lemma** *rot-x-ivl*:
  **assumes** *0* ≤ *x*
    *x* ≤ *1*
    *0* ≤ *y*
    *y* ≤ *1*
  **shows** *0* ≤ *rot-x x y* ∧ *rot-x x y* ≤ *1*
⟨*proof*⟩

**end**

# 2 The Circle Example

**theory** *CircExample*
  **imports** *Green SymmetricR2Shapes*

**begin**

**locale** *circle* = *R2* +
  **fixes** *d*::*real*
  **assumes** *d-gt-0*: *0* < *d*
**begin**

**definition** *circle-y* **where**
  *circle-y t* = *sqrt* (*1*/*4* − *t* ∗ *t*)

**definition** *circle-cube* **where**
  *circle-cube* = (λ(*x*,*y*). ((*x* − *1*/*2*) ∗ *d*, (*2* ∗ *y* − *1*) ∗ *d* ∗ *sqrt* (*1*/*4* − (*x* −*1*/*2*)∗(*x* −*1*/*2*))))

**lemma** *circle-cube-nice*:
  **shows** *circle-cube* = (λ(*x*,*y*). (*d* ∗ *x-coord x*, (*2* ∗ *y* − *1*) ∗ *d* ∗ *circle-y* (*x-coord x*)))
  ⟨*proof*⟩

**definition** *rot-circle-cube* **where**
  *rot-circle-cube* = *prod.swap* ∘ (*circle-cube*) ∘ *prod.swap*

**abbreviation** *rot-y t1 t2* ≡ ((*t1*−*1*/*2*)/(*2*∗ *circle-y* (*x-coord* (*rot-x t1 t2*))) +*1*/*2*::*real*)

**definition** *x-coord-inv* (*x*::*real*) = (*1*/*2*) + *x*

**lemma** *x-coord-inv-1*: *x-coord-inv* (*x-coord* (*x*::*real*)) = *x*
  ⟨*proof*⟩

**lemma** *x-coord-inv-2*: *x-coord* (*x-coord-inv* (*x*::*real*)) = *x*
  ⟨*proof*⟩

**definition** *circle-y-inv* = *circle-y*

**abbreviation** *rot-x''* (*x*::*real*) (*y*::*real*) ≡ (*x-coord-inv* ((*2* ∗ *y* − *1*) ∗ *circle-y* (*x-coord x*)))

**lemma** *circle-y-bounds*:
 **assumes** $-1/2 \leq$ (*x*::*real*) ∧ $x \leq 1/2$
 **shows** $0 \leq$ *circle-y x* *circle-y x* $\leq 1/2$
 ⟨*proof*⟩

**lemma** *circle-y-x-coord-bounds*:
 **assumes** $0 \leq$ (*x*::*real*) $x \leq 1$
 **shows** $0 \leq$ *circle-y* (*x-coord x*) ∧ *circle-y* (*x-coord x*) $\leq 1/2$
 ⟨*proof*⟩

**lemma** *rot-x-ivl*:
 **assumes** (*0*::*real*) ≤ *x* *x* ≤ *1*  *0* ≤ *y* *y* ≤ *1*
 **shows** *0* ≤ *rot-x''* *x y* ∧ *rot-x''* *x y* ≤ *1*
⟨*proof*⟩

**abbreviation** *rot-y''* (*x*::*real*) (*y*::*real*) ≡ (*x-coord x*)/(*2*∗ (*circle-y* (*x-coord* (*rot-x''* *x y*)))) + *1/2*

**lemma** *rot-y-ivl*:
 **assumes** (*0*::*real*) ≤ *x* *x* ≤ *1* *0* ≤ *y* *y* ≤ *1*
 **shows** *0* ≤ *rot-y''* *x y* ∧ *rot-y''* *x y* ≤ *1*
⟨*proof*⟩

**lemma** *circle-eq-rot-circle*:
 **assumes** *0* ≤ *x* *x* ≤ *1* *0* ≤ *y* *y* ≤ *1*
 **shows** (*circle-cube* (*x*, *y*)) = (*rot-circle-cube* (*rot-y''* *x y*, *rot-x''* *x y*))
⟨*proof*⟩

**lemma** *rot-circle-eq-circle*:
 **assumes** *0* ≤ *x* *x* ≤ *1* *0* ≤ *y* *y* ≤ *1*
 **shows** (*rot-circle-cube* (*x*, *y*)) = (*circle-cube* (*rot-x''* *y x*, *rot-y''* *y x*))
⟨*proof*⟩

**lemma** *rot-img-eq*:
 **assumes** *0* < *d*
 **shows** (*cubeImage* (*circle-cube* )) = (*cubeImage* (*rot-circle-cube*))
 ⟨*proof*⟩

**lemma** *rot-circle-div-circle*:
 **assumes** *0* < (*d*::*real*)
 **shows** *gen-division* (*cubeImage circle-cube*) (*cubeImage* ‘ {*rot-circle-cube*})
 ⟨*proof*⟩

**lemma** *circle-cube-boundary-valid*:
**assumes** $(k,\gamma) \in boundary\ circle\text{-}cube$
**shows** *valid-path* $\gamma$
$\langle proof \rangle$

**lemma** *rot-circle-cube-boundary-valid*:
**assumes** $(k,\gamma) \in boundary\ rot\text{-}circle\text{-}cube$
**shows** *valid-path* $\gamma$
$\langle proof \rangle$

**lemma** *diff-divide-cancel*:
**fixes** $z::real$ **shows** $z \neq 0 \implies (a * z - a * (b * z)) / z = (a - a * b)$
$\langle proof \rangle$

**lemma** *circle-cube-is-type-I*:
**assumes** $0 < d$
**shows** *typeI-twoCube circle-cube*
$\langle proof \rangle$

**lemma** *rot-circle-cube-is-type-II*:
**shows** *typeII-twoCube rot-circle-cube*
$\langle proof \rangle$

**definition** *circle-bot-edge* **where**
$circle\text{-}bot\text{-}edge = (1::int,\ \lambda t.\ (x\text{-}coord\ t * d,\ -\ d * circle\text{-}y\ (x\text{-}coord\ t)))$

**definition** *circle-top-edge* **where**
$circle\text{-}top\text{-}edge = (-\ 1::int,\ \lambda t.\ (x\text{-}coord\ t * d,\ d * circle\text{-}y\ (x\text{-}coord\ t)))$

**definition** *circle-right-edge* **where**
$circle\text{-}right\text{-}edge = (1::int,\ \lambda y.\ (d/2,\ 0))$

**definition** *circle-left-edge* **where**
$circle\text{-}left\text{-}edge = (-\ 1::int,\ \lambda y.\ (-\ (d/2),\ 0))$

**lemma** *circle-cube-boundary-explicit*:
$boundary\ circle\text{-}cube = \{circle\text{-}left\text{-}edge, circle\text{-}right\text{-}edge, circle\text{-}bot\text{-}edge, circle\text{-}top\text{-}edge\}$
$\langle proof \rangle$

**definition** *rot-circle-right-edge* **where**
$rot\text{-}circle\text{-}right\text{-}edge = (1::int,\ \lambda t.\ (d * circle\text{-}y\ (x\text{-}coord\ t),\ x\text{-}coord\ t * d))$

**definition** *rot-circle-left-edge* **where**
$rot\text{-}circle\text{-}left\text{-}edge = (-\ 1::int,\ \lambda t.\ (-\ d * circle\text{-}y\ (x\text{-}coord\ t),\ x\text{-}coord\ t * d))$

**definition** *rot-circle-top-edge* **where**
$rot\text{-}circle\text{-}top\text{-}edge = (-\ 1::int,\ \lambda y.\ (0,\ d/2))$

**definition** *rot-circle-bot-edge* **where**

*rot-circle-bot-edge* = (*1*::*int*, λ*y*. (*0*, − (*d*/*2*)))

**lemma** *rot-circle-cube-boundary-explicit*:
*boundary* (*rot-circle-cube*) =
{*rot-circle-top-edge*,*rot-circle-bot-edge*,*rot-circle-right-edge*,*rot-circle-left-edge*}
⟨*proof*⟩

**lemma** *rot-circle-cube-vertical-boundary-explicit*:
*vertical-boundary rot-circle-cube* = {*rot-circle-right-edge*,*rot-circle-left-edge*}
⟨*proof*⟩

**lemma** *circ-left-edge-neq-top*:
(− *1*::*int*, λ*y*::*real*. (− (*d*/*2*), *0*)) ≠ (− *1*, λ*x*. ((*x* − *1*/*2*) ∗ *d*, *d* ∗ *sqrt* (*1*/*4* − (*x* − *1*/*2*) ∗ (*x* − *1*/*2*))))
⟨*proof*⟩

**lemma** *circle-cube-valid-two-cube*: *valid-two-cube* (*circle-cube*)
⟨*proof*⟩

**lemma** *rot-circle-cube-valid-two-cube*:
**shows** *valid-two-cube rot-circle-cube*
⟨*proof*⟩

**definition** *circle-arc-0* **where** *circle-arc-0* = (*1*, λ*t*::*real*. (*0*,*0*))

**lemma** *circle-top-bot-edges-neq′* [*simp*]:
**shows** *circle-top-edge* ≠ *circle-bot-edge*
⟨*proof*⟩

**lemma** *rot-circle-top-left-edges-neq* [*simp*]: *rot-circle-top-edge* ≠ *rot-circle-left-edge*
⟨*proof*⟩

**lemma** *rot-circle-bot-left-edges-neq* [*simp*]: *rot-circle-bot-edge* ≠ *rot-circle-left-edge*
⟨*proof*⟩

**lemma** *rot-circle-top-right-edges-neq* [*simp*]: *rot-circle-top-edge* ≠ *rot-circle-right-edge*
⟨*proof*⟩

**lemma** *rot-circle-bot-right-edges-neq* [*simp*]: *rot-circle-bot-edge* ≠ *rot-circle-right-edge*
⟨*proof*⟩

**lemma** *rot-circle-right-top-edges-neq′* [*simp*]: *rot-circle-right-edge* ≠ *rot-circle-left-edge*
⟨*proof*⟩

**lemma** *rot-circle-left-bot-edges-neq* [*simp*]: *rot-circle-left-edge* ≠ *rot-circle-top-edge*
⟨*proof*⟩

**lemma** *circle-right-top-edges-neq* [*simp*]: *circle-right-edge* ≠ *circle-top-edge*
⟨*proof*⟩

**lemma** *circle-left-bot-edges-neq* [*simp*]: *circle-left-edge* $\neq$ *circle-bot-edge*
⟨*proof*⟩

**lemma** *circle-left-top-edges-neq* [*simp*]: *circle-left-edge* $\neq$ *circle-top-edge*
⟨*proof*⟩

**lemma** *circle-right-bot-edges-neq* [*simp*]: *circle-right-edge* $\neq$ *circle-bot-edge*
⟨*proof*⟩

**definition** *circle-polar* **where**
*circle-polar t* = ((*d*/*2*) ∗ *cos* (*2* ∗ *pi* ∗ *t*), (*d*/*2*) ∗ *sin* (*2* ∗ *pi* ∗ *t*))

**lemma** *circle-polar-smooth*: (*circle-polar*) *C1-differentiable-on* {*0*..*1*}
⟨*proof*⟩

**abbreviation** *custom-arccos* ≡ (λ*x*. (*if* −*1* ≤ *x* ∧ *x* ≤ *1* *then arccos x else* (*if x* < −*1* *then* −*x* + *pi else 1* −*x* )))

**lemma** *cont-custom-arccos*:
**assumes** *S* ⊆ {−*1*..*1*}
**shows** *continuous-on S custom-arccos*
⟨*proof*⟩

**lemma** *custom-arccos-has-deriv*:
**assumes** − *1* < *x x* < *1*
**shows** *DERIV custom-arccos x* :> *inverse* (− *sqrt* (*1* − $x^2$))
⟨*proof*⟩

**declare**
*custom-arccos-has-deriv*[*THEN DERIV-chain2*, *derivative-intros*]
*custom-arccos-has-deriv*[*THEN DERIV-chain2*, *unfolded has-field-derivative-def*, *derivative-intros*]

**lemma** *circle-boundary-reparams*:
**shows** *rot-circ-left-edge-reparam-polar-circ-split*:
*reparam* (*rec-join* [(*rot-circle-left-edge*)]) (*rec-join* [(*subcube* (*1*/*4*) (*1*/*2*) (*1*, *circle-polar*)), (*subcube* (*1*/*2*) (*3*/*4*) (*1*, *circle-polar*))])
(**is** *?P1*)
**and** *circ-top-edge-reparam-polar-circ-split*:
*reparam* (*rec-join* [(*circle-top-edge*)]) (*rec-join* [(*subcube 0* (*1*/*4*) (*1*, *circle-polar*)), (*subcube* (*1*/*4*) (*1*/*2*) (*1*, *circle-polar*))])
(**is** *?P2*)
**and** *circ-bot-edge-reparam-polar-circ-split*:
*reparam* (*rec-join* [(*circle-bot-edge*)]) (*rec-join* [(*subcube* (*1*/*2*) (*3*/*4*) (*1*, *circle-polar*)), (*subcube* (*3*/*4*) *1* (*1*, *circle-polar*))])
(**is** *?P3*)
**and** *rot-circ-right-edge-reparam-polar-circ-split*:
*reparam* (*rec-join* [(*rot-circle-right-edge*)]) (*rec-join* [(*subcube* (*3*/*4*) *1* (*1*, *cir-*

*cle-polar*)), (*subcube 0* (*1/4*) (*1*, *circle-polar*))])
(**is** *?P4*)
⟨*proof*⟩

**definition** *circle-cube-boundary-to-polarcircle* **where**
*circle-cube-boundary-to-polarcircle* $\gamma \equiv$
*if* ($\gamma$ = (*circle-top-edge*)) *then*
{*subcube 0* (*1/4*) (*1*, *circle-polar*), *subcube* (*1/4*) (*1/2*) (*1*, *circle-polar*)}
*else if* ($\gamma$ = (*circle-bot-edge*)) *then*
{*subcube* (*1/2*) (*3/4*) (*1*, *circle-polar*), *subcube* (*3/4*) *1* (*1*, *circle-polar*)}
*else* {}

**definition** *rot-circle-cube-boundary-to-polarcircle* **where**
*rot-circle-cube-boundary-to-polarcircle* $\gamma \equiv$
*if* ($\gamma$ = (*rot-circle-left-edge* )) *then*
{*subcube* (*1/4*) (*1/2*) (*1*, *circle-polar*), *subcube* (*1/2*) (*3/4*) (*1*, *circle-polar*)}
*else if* ($\gamma$ = (*rot-circle-right-edge*)) *then*
{*subcube* (*3/4*) *1* (*1*, *circle-polar*), *subcube 0* (*1/4*) (*1*, *circle-polar*)}
*else* {}

**lemma** *circle-arcs-neq*:
**assumes** $0 \le k$ $k \le 1$ $0 \le n$ $n \le 1$ $n < k$ $k + n < 1$
**shows** *subcube k m* (*1*, *circle-polar*) $\neq$ *subcube n q* (*1*, *circle-polar*)
⟨*proof*⟩

**lemma** *circle-arcs-neq-2*:
**assumes** $0 \le k$ $k \le 1$ $0 \le n$ $n \le 1$ $n < k$ $0 < n$ **and** *kn12*: $1/2 < k + n$ **and** $k + n < 3/2$
**shows** *subcube k m* (*1*, *circle-polar*) $\neq$ *subcube n q* (*1*, *circle-polar*)
⟨*proof*⟩

**lemma** *circle-cube-is-only-horizontal-div-of-rot*:
**shows** *only-horizontal-division* (*boundary* (*circle-cube*)) {*rot-circle-cube*}
⟨*proof*⟩

**lemma** *GreenThm-cirlce*:
**assumes** $\forall$ *twoC*$\in${*circle-cube*}. *analytically-valid* (*cubeImage twoC*) ($\lambda x.$ *F x* $\cdot$ *i*) *j*
$\forall$ *twoC*$\in${*rot-circle-cube*}. *analytically-valid* (*cubeImage twoC*) ($\lambda x.$ *F x* $\cdot$ *j*) *i*
**shows** *integral* (*cubeImage* (*circle-cube*)) ($\lambda x.$ *partial-vector-derivative* ($\lambda x.$ *F x* $\cdot$ *j*) *i x* $-$ *partial-vector-derivative* ($\lambda x.$ *F x* $\cdot$ *i*) *j x*) =
*one-chain-line-integral F* {*i*, *j*} (*boundary* (*circle-cube*))
⟨*proof*⟩
**end**
**end**

# 3 The Diamond Example

**theory** *DiamExample*
  **imports** *Green SymmetricR2Shapes*
**begin**

**lemma** *abs-if′*:
  **fixes** $a :: 'a :: \{abs\text{-}if, ordered\text{-}ab\text{-}group\text{-}add\}$
  **shows** $|a| = (if\ a \le 0\ then\ -\ a\ else\ a)$
  ⟨*proof*⟩

**locale** *diamond* = *R2* +
  **fixes** $d$::*real*
  **assumes** *d-gt-0*: $0 < d$
**begin**

**definition** *diamond-y-gen* :: *real* ⇒ *real* **where**
  *diamond-y-gen* $\equiv\lambda t.\ 1/2 - |t|$

**definition** *diamond-cube-gen*:: ((*real* ∗ *real*) ⇒ (*real* ∗ *real*)) **where**
  *diamond-cube-gen* ≡ ($\lambda$(*x*,*y*). (*d* ∗ *x-coord x*, (*2* ∗ *y* − *1*) ∗ (*d* ∗ *diamond-y-gen* (*x-coord x*))))

**lemma** *diamond-y-gen-valid*:
  **assumes** $a \le 0$ $0 \le b$
  **shows** *diamond-y-gen piecewise-C1-differentiable-on* $\{a..b\}$
  ⟨*proof*⟩

**lemma** *diamond-cube-gen-boundary-valid*:
  **assumes** ($k$,$\gamma$)∈*boundary* (*diamond-cube-gen*)
  **shows** *valid-path* $\gamma$
  ⟨*proof*⟩

**definition** *diamond-x* **where**
  *diamond-x* ≡ $\lambda t.\ (t - 1/2) * d$

**definition** *diamond-y* **where**
  *diamond-y* ≡ $\lambda t.\ d/2 - |t|$

**definition** *diamond-cube* **where**
  *diamond-cube* = ($\lambda$(*x*,*y*). (*diamond-x x*, (*2* ∗ *y* − *1*) ∗ (*diamond-y* (*diamond-x x*))))

**definition** *rot-diamond-cube* **where**
  *rot-diamond-cube* = *prod.swap o* (*diamond-cube*) *o prod.swap*

**lemma** *diamond-eq-characterisations*:
  **shows** *diamond-cube* (*x*,*y*)= *diamond-cube-gen* (*x*,*y*)
  ⟨*proof*⟩

**lemma** *diamond-eq-characterisations-fun*: *diamond-cube* = *diamond-cube-gen*
⟨*proof*⟩

**lemma** *diamond-y-valid*:
**shows** *diamond-y piecewise-C1-differentiable-on* $\{-d/2..d/2\}$ (**is** *?P*)
($\lambda x.$ *diamond-y x*) *piecewise-C1-differentiable-on* $\{-d/2..d/2\}$ (**is** *?Q*)
⟨*proof*⟩

**lemma** *diamond-cube-boundary-valid*:
**assumes** $(k,\gamma) \in$ *boundary* (*diamond-cube*)
**shows** *valid-path* $\gamma$
⟨*proof*⟩

**lemma** *diamond-cube-is-type-I*:
**shows** *typeI-twoCube* (*diamond-cube*)
⟨*proof*⟩

**lemma** *diamond-cube-valid-two-cube*:
**shows** *valid-two-cube* (*diamond-cube*)
⟨*proof*⟩

**lemma** *rot-diamond-cube-boundary-valid*:
**assumes** $(k,\gamma)\in$*boundary* (*rot-diamond-cube*)
**shows** *valid-path* $\gamma$
⟨*proof*⟩

**lemma** *rot-diamond-cube-is-type-II*:
**shows** *typeII-twoCube* (*rot-diamond-cube*)
⟨*proof*⟩

**lemma** *rot-diamond-cube-valid-two-cube*: *valid-two-cube* (*rot-diamond-cube*)
⟨*proof*⟩

**definition** *diamond-top-edges* **where**
*diamond-top-edges* = (− *1*::*int*, $\lambda x.$ (*diamond-x x*, *diamond-y* (*diamond-x x*)))

**definition** *diamond-bot-edges* **where**
*diamond-bot-edges* = (*1*::*int*, $\lambda x.$ (*diamond-x x*, − *diamond-y* (*diamond-x x*)))

**lemma** *diamond-cube-boundary-explicit*:
*boundary* (*diamond-cube* ) =
{*diamond-top-edges*,
*diamond-bot-edges*,
(− *1*::*int*, $\lambda y.$ (*diamond-x 0*, (*2* ∗ *y* − *1*) ∗ *diamond-y* (*diamond-x 0*))),
(*1*::*int*, $\lambda y.$ (*diamond-x 1*, (*2* ∗ *y* − *1*) ∗ *diamond-y* (*diamond-x 1*)))}
⟨*proof*⟩

**definition** *diamond-top-left-edge* **where**

*diamond-top-left-edge* = (− *1*::*int*, (λ*x*. (*diamond-x* (*1*/*2* ∗ *x*), (*diamond-x* (*1*/*2* ∗ *x*)) + *d*/*2*)))

**definition** *diamond-top-right-edge* **where**
*diamond-top-right-edge* = (− *1*::*int*, (λ*x*. (*diamond-x* (*1*/*2* ∗ *x* + *1*/*2*), (−(*diamond-x* (*1*/*2* ∗ *x* + *1*/*2*)) + *d*/*2*))))

**definition** *diamond-bot-left-edge* **where**
*diamond-bot-left-edge* = (*1*::*int*, (λ*x*. (*diamond-x* (*1*/*2* ∗ *x*), − (*diamond-x* (*1*/*2* ∗ *x*) + *d*/*2*))))

**definition** *diamond-bot-right-edge* **where**
*diamond-bot-right-edge* = (*1*::*int*, (λ*x*. (*diamond-x* (*1*/*2* ∗ *x* + *1*/*2*), − (−(*diamond-x* (*1*/*2* ∗ *x* + *1*/*2*)) + *d*/*2*))))

**lemma** *diamond-edges-are-valid*:
*valid-path* (*snd* (*diamond-top-left-edge*))
*valid-path* (*snd* (*diamond-top-right-edge*))
*valid-path* (*snd* (*diamond-bot-left-edge*))
*valid-path* (*snd* (*diamond-bot-right-edge*))
⟨*proof*⟩

**definition** *diamond-cube-boundary-to-subdiv* **where**
*diamond-cube-boundary-to-subdiv* (*gamma*::(*int* × (*real* ⇒ *real* × *real*))) ≡
*if* (*gamma* = *diamond-top-edges*) *then*
{*diamond-top-left-edge*, *diamond-top-right-edge*}
*else if* (*gamma* = *diamond-bot-edges*) *then*
{*diamond-bot-left-edge*, *diamond-bot-right-edge*}
*else* {}

**lemma** *rot-diam-edge-1*:
(*1*::*int*, λ*x*::*real*. ((*x*::*real*) ∗ (*2* ∗ *diamond-y* (*diamond-x* *0*)) − *1* ∗ *diamond-y* (*diamond-x* *0*), *diamond-x* *0*)) =
(*1*, λ*x*. (*x* ∗ (*2* ∗ *diamond-y* (*diamond-x* *0*)) − (*diamond-y* (*diamond-x* *0*)), *diamond-x* *0*))
⟨*proof*⟩

**definition** *diamond-left-edges* **where**
*diamond-left-edges* = (− *1*, λ*y*. (− *diamond-y* (*diamond-x* *y*), *diamond-x* *y*))

**definition** *diamond-right-edges* **where**
*diamond-right-edges* = (*1*, λ*y*. ( *diamond-y* (*diamond-x* *y*), *diamond-x* *y*))

**lemma** *rot-diamond-cube-boundary-explicit*:
*boundary* (*rot-diamond-cube*) = {(*1*::*int*, λ*x*::*real*. ((*2* ∗ *x* − *1*) ∗ *diamond-y* (*diamond-x* *0*), *diamond-x* *0*)),
(− *1*, λ*x*. ((*2* ∗ *x* − *1*) ∗ *diamond-y* (*diamond-x* *1*), *diamond-x* *1*)),
*diamond-left-edges*, *diamond-right-edges*}

⟨*proof*⟩

**lemma** *rot-diamond-cube-vertical-boundary-explicit*:
*vertical-boundary* (*rot-diamond-cube*) = {*diamond-left-edges*, *diamond-right-edges*}
⟨*proof*⟩

**definition** *rot-diamond-cube-boundary-to-subdiv* **where**
*rot-diamond-cube-boundary-to-subdiv* (*gamma*::(*int* × (*real* ⇒ *real* × *real*))) ≡
*if* (*gamma* = *diamond-left-edges* ) *then* {*diamond-bot-left-edge* , *diamond-top-left-edge*}
*else if* (*gamma* = *diamond-right-edges*) *then* {*diamond-bot-right-edge*, *diamond-top-right-edge*}
*else* {}

**definition** *diamond-boundaries-reparam-map* **where**
*diamond-boundaries-reparam-map* ≡ *id*

**lemma** *diamond-boundaries-reparam-map-bij*:
*bij* (*diamond-boundaries-reparam-map*)
⟨*proof*⟩

**lemma** *diamond-bot-edges-neq-diamond-top-edges*:
*diamond-bot-edges* ≠ *diamond-top-edges*
⟨*proof*⟩

**lemma** *diamond-top-left-edge-neq-diamond-top-right-edge*:
*diamond-top-left-edge* ≠ *diamond-top-right-edge*
⟨*proof*⟩

**lemma** *neqs1*:
**shows** ($\lambda x$. (*diamond-x x*, *diamond-y* (*diamond-x x*))) ≠ ($\lambda x$. (*diamond-x x*, − *diamond-y* (*diamond-x x*)))
**and** ($\lambda y$. (− *diamond-y* (*diamond-x y*), *diamond-x y*)) ≠ ($\lambda y$. (*diamond-y* (*diamond-x y*), *diamond-x y*))
**and** ($\lambda x$. (*diamond-x*($x/2$ + $1/2$), *diamond-x*($x/2$ + $1/2$) − $d/2$)) ≠ ($\lambda x$. (*diamond-x*($x/2$), − *diamond-x*($x/2$) − $d/2$))
**and** ($\lambda x$. (*diamond-x*($x/2$ + $1/2$), $d/2$ − *diamond-x*($x/2$ + $1/2$))) ≠ ($\lambda x$. (*diamond-x*($x/2$), *diamond-x*($x/2$) + $d/2$))
**and** ($\lambda x$. (*diamond-x*($x/2$), − *diamond-x*($x/2$) − $d/2$)) ≠ ($\lambda x$. (*diamond-x*($x/2$ + $1/2$), *diamond-x*($x/2$ + $1/2$) − $d/2$))
**and** ($\lambda x$. (*diamond-x*($x/2$), *diamond-x*($x/2$) + $d/2$)) ≠ ($\lambda x$. (*diamond-x*($x/2$ + $1/2$), $d/2$ − *diamond-x*($x/2$ + $1/2$)))
⟨*proof*⟩

**lemma** *neqs2*:
**shows** ($\lambda x$. (*diamond-x x*, *diamond-y* (*diamond-x x*))) ≠ ($\lambda x$. (($2 * x - 1$) * *diamond-y* (*diamond-x 1*), *diamond-x 1*))
**and** ($\lambda x$. (*diamond-x x*, − *diamond-y* (*diamond-x x*))) ≠ ($\lambda x$. (($2 * x - 1$) * *diamond-y* (*diamond-x 0*), *diamond-x 0*))
⟨*proof*⟩

**lemma** *diamond-cube-is-only-horizontal-div-of-rot*:
**shows** *only-horizontal-division* (*boundary* (*diamond-cube*)) {*rot-diamond-cube*}
⟨*proof*⟩

**abbreviation** *rot-y t1 t2* $\equiv$ (*t1* $-$ *1/2*) / (*2*$*$ *diamond-y-gen* (*x-coord* (*rot-x t1 t2*))) + *1/2*

**lemma** *rot-y-ivl*:
**assumes** (*0*::*real*) $\leq$ *x x* $\leq$ *1 0* $\leq$ *y y* $\leq$ *1*
**shows** *0* $\leq$ *rot-y x y* $\wedge$ *rot-y x y* $\leq$ *1*
⟨*proof*⟩

**lemma** *diamond-gen-eq-rot-diamond*:
**assumes***0* $\leq$ *x x* $\leq$ *1 0* $\leq$ *y y* $\leq$ *1*
**shows** (*diamond-cube-gen* (*x*, *y*)) = (*rot-diamond-cube* (*rot-y x y*, *rot-x x y*))
⟨*proof*⟩

**lemma** *rot-diamond-eq-diamond-gen*:
**assumes** *0* $\leq$ *x x* $\leq$ *1 0* $\leq$ *y y* $\leq$ *1*
**shows** *rot-diamond-cube* (*x*, *y*) = *diamond-cube-gen* (*rot-x y x*, *rot-y y x*)
⟨*proof*⟩

**lemma** *rot-img-eq*: *cubeImage* (*diamond-cube-gen*) = *cubeImage* (*rot-diamond-cube*)
⟨*proof*⟩

**lemma** *rot-diamond-gen-div-diamond-gen*:
**shows** *gen-division* (*cubeImage* (*diamond-cube-gen*)) (*cubeImage* ‘ {*rot-diamond-cube*})
⟨*proof*⟩

**lemma** *rot-diamond-gen-div-diamond*:
**shows** *gen-division* (*cubeImage* (*diamond-cube*)) (*cubeImage* ‘ {*rot-diamond-cube*})
⟨*proof*⟩

**lemma** *GreenThm-diamond*:
**assumes** *analytically-valid* (*cubeImage* (*diamond-cube*)) ($\lambda$*x*. *F x* $\cdot$ *i*) *j*
*analytically-valid* (*cubeImage* (*diamond-cube*)) ($\lambda$*x*. *F x* $\cdot$ *j*) *i*
**shows** *integral* (*cubeImage* (*diamond-cube*)) ($\lambda$*x*. *partial-vector-derivative* ($\lambda$*x*. *F x* $\cdot$ *j*) *i x* $-$ *partial-vector-derivative* ($\lambda$*x*. *F x* $\cdot$ *i*) *j x*) =
*one-chain-line-integral F* {*i*, *j*} (*boundary* (*diamond-cube*))
⟨*proof*⟩
**end**
**end**