Verification of the Deutsch-Schorr-Waite Graph Marking Algorithm using Data Refinement

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Abstract

The verification of the Deutsch-Schorr-Waite graph marking algorithm is used as a benchmark in many formalizations of pointer programs. The main purpose of this mechanization is to show how data refinement of invariant based programs can be used in verifying practical algorithms. The verification starts with an abstract algorithm working on a graph given by a relation *next* on nodes. Gradually the abstract program is refined into Deutsch-Schorr-Waite graph marking algorithm where only one bit per graph node of additional memory is used for marking.

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1 Introduction

The verification of the Deutsch-Schorr-Waite (DSW) [14, 10] graph marking algorithm is used as a benchmark in many formalizations of pointer programs [11, 1]. The main purpose of this mechanization is to show how data refinement [12] of invariant based programs [3, 4, 5, 6] can be used in verifying practical algorithms.

The DSW algorithm marks all nodes in a graph that are reachable from a *root* node. The marking is achieved using only one extra bit of memory for every node. The graph is given by two pointer functions, *left* and *right*, which for any given node return its left and right successors, respectively. While marking, the left and right functions are altered to represent a stack that describes the path from the root to the current node in the graph. On completion the original graph structure is restored. We construct the DSW algorithm by a sequence of three successive data refinement steps. One step in these refinements is a generalization of the DSW algorithm to an algorithm which marks a graph given by a family of pointer functions instead of left and right only.

Invariant based programming is an approach to construct correct programs where we start by identifying all basic situations (pre- and post-conditions, and loop invariants) that could arise during the execution of the algorithm. These situations are determined and described before any code is written. After that, we identify the transitions between the situations, which together determine the flow of control in the program. The transitions are verified at the same time as they are constructed. The correctness of the program is thus established as part of the construction process.

Data refinement [9, 2, 7, 8] is a technique of building correct programs working on concrete data structures as refinements of more abstract programs working on abstract data structures. The correctness of the final program follows from the correctness of the abstract program and from the correctness of the data refinement.

Both the semantics and the data refinement of invariant based programs were formalized in [13], and this verification is based on them.

We use a simple model of pointers where addresses (pointers, nodes) are the elements of a set and pointer fields are global pointer functions from addresses to addresses. Pointer updates ($x.left := y$) are done by modifying the global pointer function $left := left(x := y)$. Because of the nature of the marking algorithm where no allocation and disposal of memory are needed we do not treat these operations.

A number of Isabelle techniques are used here. The class mechanism is used for extending the complete lattice theories as well as for introducing well founded and transitive relations. The polimorphism is used for the state of the computation. In [13] the state of computation was introduced as a type variable, or even more generaly, state predicates were introduced as elements of a complete (boolean) lattice. Here the state of the computation is instantiated with various tuples ranging from the abstract data in the first algorithm to the concrete data in the final refinement. The locale mechanism of Isabelle is used to introduce the specification variables and their invariants. These specification variables are used for example to prove that the main variables are restored to their initial values when the algorithm terminates. The locale extension and partial instantiation mechanisms turn out to be also very useful in the data refinements of DSW. We start with a locale which fixes the abstract graph as a relation *next* on nodes. This locale is first partially interpreted into a locale which replaces *next* by a union of a family of pointer functions. In the final refinement step the locale of the pointer functions is interpreted into a locale with only two pointer functions, *left* and *right*.

2 Address Graph

theory *Graph*
imports *Main*
begin

This theory introduces the graph to be marked as a relation next on nodes (addresses). We assume that we have a special node nil (the null address). We have a node root from which we start marking the graph. We also assume that nil is not related by next to any node and any node is not related by next to nil.

locale *node* =

fixes *nil* :: *'node*
fixes *root* :: *'node*

locale *graph* = *node* +
fixes *next* :: (*'node* × *'node*) *set*
assumes *next-not-nil-left*: (!! *x* . (*nil*, *x*) ∉ *next*)
assumes *next-not-nil-right*: (!! *x* . (*x*, *nil*) ∉ *next*)
begin

On lists of nodes we introduce two operations similar to existing hd and tl for getting the head and the tail of a list. The new function head applied to a nonempty list returns the head of the list, and it reurns nil when applied to the empty list. The function tail returns the tail of the list when applied to a non-empty list, and it returns the empty list otherwise.

definition
head S ≡ (*if S* = [] *then nil else* (*hd S*))

definition
tail (*S*::*'a list*) ≡ (*if S* = [] *then* [] *else* (*tl S*))

lemma [*simp*]: ((*nil*, *x*) ∈ *next*) = *False*
by (*simp add*: *next-not-nil-left*)

lemma [*simp*]: ((*x*, *nil*) ∈ *next*) = *False*
by (*simp add*: *next-not-nil-right*)

theorem *head-not-nil* [*simp*]:
(*head S* ≠ *nil*) = (*head S* = *hd S* ∧ *tail S* = *tl S* ∧ *hd S* ≠ *nil* ∧ *S* ≠ [])
by (*simp add*: *head-def tail-def*)

theorem *nonempty-head* [*simp*]:
head (*x* # *S*) = *x*
by (*simp add*: *head-def*)

theorem *nonempty-tail* [*simp*]:
tail (*x* # *S*) = *S*
by (*simp add*: *tail-def*)

definition (**in** *graph*)
reach x ≡ {*y* . (*x*, *y*) ∈ $next^*$ ∧ *y* ≠ *nil*}

theorem (**in** *graph*) *reach-nil* [*simp*]: *reach nil* = {}
apply (*simp add*: *reach-def*, *safe*)
apply (*drule rtrancl-induct*)
by *auto*

theorem (**in** *graph*) *reach-next*: *b* ∈ *reach a* ⟹ (*b*, *c*) ∈ *next* ⟹ *c* ∈ *reach a*
apply (*simp add*: *reach-def*)

by *auto*

definition (**in** *graph*)
path S mrk $\equiv \{x\ .\ (\exists\ s\ .\ s \in S \wedge (s,\ x) \in next\ O\ (next \cap ((-mrk)\times(-mrk)))^*$
$)\}$
end

end

3 Marking Using a Set

theory *SetMark*
imports *Graph DataRefinementIBP.DataRefinement*
begin

We construct in this theory a diagram which computes all reachable nodes from a given root node in a graph. The graph is defined in the theory Graph and is given by a relation *next* on the nodes of the graph.

The diagram has only three ordered situation ($init > loop > final$). The termination variant is a pair of a situation and a natural number with the lexicographic ordering. The idea of this ordering is that we can go from a bigger situation to a smaller one, however if we stay in the same situation the second component of the variant must decrease.

The idea of the algorithm is that it starts with a set X containing the root element and the root is marked. As long as X is not empty, if $x \in X$ and y is an unmarked sucessor of x we add y to X. If $x \in X$ has no unmarked sucessors it is removed from X. The algorithm terminates when X is empty.

datatype $I = init \mid loop \mid final$

declare *I.split* [*split*]

instantiation I :: *well-founded-transitive*
begin

definition
less-I-def: $i < j \equiv\ (j = init \wedge (i = loop \vee i = final)) \vee (j = loop \wedge i = final)$

definition
less-eq-I-def: $(i{::}I) \leq (j{::}I) \equiv i = j \vee i < j$

instance
proof
fix $x\ y\ z :: I$
assume $x < y$ **and** $y < z$ **then** **show** $x < z$
apply (*simp add*: *less-I-def*)
by *auto*

next
  fix $x\ y :: I$
  show $x \le y \longleftrightarrow x = y \vee x < y$
    by (*simp add*: *less-eq-I-def*)
next
  fix P **fix** $a{::}I$
  show $P\ a$ **when** $\forall x.\ (\forall\ y.\ y < x \longrightarrow P\ y) \longrightarrow P\ x$
    apply (*insert that*)
    apply (*case-tac P final*)
    apply (*case-tac P loop*)
    apply (*simp-all add*: *less-I-def*)
    by *blast*
qed (*simp*)

end

The set *path S mrk* contains all reachable nodes from S along paths with unmarked nodes.

lemma *trascl-less*: $x \ne y \Longrightarrow (a,\ x) \in R^* \Longrightarrow$
    $((a,x) \in (R \cap (-\{y\}) \times (-\{y\}))^* \vee\ (y,x) \in R\ O\ (R \cap (-\{y\}) \times (-\{y\}))^*\ )$
  apply (*drule-tac*
    $b = x$ **and** $a = a$ **and** $r = R$ **and**
    $P = \lambda\ x.\ (x \ne y \longrightarrow ((a,x) \in (R \cap (-\{y\}) \times (-\{y\}))^* \vee (y,x) \in R\ O\ (R \cap (-\{y\}) \times (-\{y\}))^*\ ))$
    in *rtrancl-induct*)
  apply (*auto simp*: *Compl-insert*)
  apply (*case-tac* $ya = y$)
  apply *auto*
  apply (*rule-tac* $x = a$ **and** $y = ya$ **and** $z = z$ **and** $r = R \cap ((UNIV - \{y\}) \times (UNIV - \{y\}))$ **in** *rtrancl-trans*)
  apply *auto*
  apply (*case-tac* $za = y$)
  apply *auto*
  apply (*drule-tac* $x = ya$ **and** $y = za$ **and** $z = z$ **and** $r = (R \cap (UNIV - \{y\}) \times (UNIV - \{y\}))$ **in** *rtrancl-trans*)
  by *auto*

lemma (**in** *graph*) *add-set* [*simp*]: $x \ne y \Longrightarrow x \in path\ S\ mrk \Longrightarrow x \in path\ (insert\ y\ S)\ (insert\ y\ mrk)$
  apply (*simp add*: *path-def*)
  apply *clarify*
  apply (*drule-tac* $x = x$ **and** $y = y$ **and** $a = ya$ **and** $R = next \cap (-\ mrk) \times (-\ mrk)$ **in** *trascl-less*)
  apply *simp-all*
  apply (*case-tac* $(ya,\ x) \in (next \cap (-\ mrk) \times -\ mrk \cap (-\ \{y\}) \times -\ \{y\})^*$)
  apply (*rule-tac* $x = xa$ **in** *exI*)
  apply *simp-all*
  apply (*simp add*: *relcomp-unfold*)
  apply (*rule-tac* $x = ya$ **in** *exI*)

apply *simp*
apply (*case-tac* (*next* ∩ (− *mrk*) × − *mrk* ∩ (− {*y*}) × − {*y*}) = (*next* ∩ (− *insert y mrk*) × − *insert y mrk*))
apply *simp-all*
apply *safe*
apply *simp-all*
apply (*rule-tac x* = *y* **in** *exI*)
apply *simp*
apply (*simp add*: *relcomp-unfold*)
apply (*rule-tac x* = *yaa* **in** *exI*)
apply *simp*
apply (*case-tac* (*next* ∩ (− *mrk*) × − *mrk* ∩ (− {*y*}) × − {*y*}) = (*next* ∩ (− *insert y mrk*) × − *insert y mrk*))
apply *simp-all*
by *auto*

lemma (**in** *graph*) *add-set2*: *x* ∈ *path S mrk* ⟹ *x* ∉ *path* (*insert y S*) (*insert y mrk*) ⟹ *x* = *y*
apply (*case-tac x* ≠ *y*)
apply (*frule add-set*)
by *simp-all*

lemma (**in** *graph*) *del-stack* [*simp*]: (∀ *y* . (*t*, *y*) ∈ *next* ⟶ *y* ∈ *mrk*) ⟹ *x* ∉ *mrk* ⟹ *x* ∈ *path S mrk* ⟹ *x* ∈ *path* (*S* − {*t*}) *mrk*
apply (*simp add*: *path-def*)
apply *clarify*
apply (*rule-tac x* = *xa* **in** *exI*)
apply (*case-tac x* = *y*)
apply *auto*
apply (*drule-tac a* = *y* **and** *b* = *x* **and** *R* = (*next* ∩ (− *mrk*) × − *mrk*) **in** *rtranclD*)
apply *safe*
apply (*drule-tac x* = *y* **and** *y* = *x* **in** *tranclD*)
by *auto*

lemma (**in** *graph*) *init-set* [*simp*]: *x* ∈ *reach root* ⟹ *x* ≠ *root* ⟹ *x* ∈ *path* {*root*} {*root*}
apply (*simp add*: *reach-def path-def*)
apply (*case-tac root* ≠ *x*)
apply (*drule-tac a* = *root* **and** *x* = *x* **and** *y* = *root* **and** *R* = *next* **in** *trascl-less*)
apply (*simp-all add*: *Compl-insert*)
apply *safe*
apply (*drule-tac a* = *root* **and** *b* = *x* **and** *R* = (*next* ∩ (*UNIV* − {*root*}) × (*UNIV* − {*root*})) **in** *rtranclD*)
apply *safe*
apply (*drule-tac x* = *root* **and** *y* = *x* **in** *tranclD*)
by *auto*

lemma (**in** *graph*) *init-set2*: *x* ∈ *reach root* ⟹ *x* ∉ *path* {*root*} {*root*} ⟹ *x* =

root
  apply (*case-tac root* $\neq$ *x*)
  apply (*drule init-set*)
  by *simp-all*

3.1 Transitions

definition (**in** *graph*)
  $Q1\text{-}a \equiv [: X,\ mrk \leadsto X',\ mrk'.\ (root::'node) = nil \wedge X' = \{\} \wedge mrk' = mrk\ :]$

definition (**in** *graph*)
  $Q2\text{-}a \equiv [: X,\ mrk \leadsto X',\ mrk'\ .$
  $(root::'node) \neq nil \wedge X' = \{root::'node\} \wedge mrk' = \{root::'node\}\ :]$

definition (**in** *graph*)
  $Q3\text{-}a \equiv [: X,\ mrk \leadsto X',\ mrk'\ .$
  $(\exists\ x \in X\ .\ \exists\ y\ .\ (x,\ y) \in next \wedge y \notin mrk \wedge X' = X \cup \{y\} \wedge mrk' = mrk \cup \{y\}):]$

definition (**in** *graph*)
  $Q4\text{-}a \equiv [: X,\ mrk \leadsto X',\ mrk'\ .$
  $(\exists\ x \in X\ .\ (\forall\ y\ .\ (x,\ y) \in next \longrightarrow y \in mrk) \wedge X' = X - \{x\} \wedge mrk' = mrk):]$

definition (**in** *graph*)
  $Q5\text{-}a \equiv [: X,\ mrk \leadsto X',\ mrk'\ .\ X = \{\} \wedge mrk = mrk'\ :]$

3.2 Invariants

definition (**in** *graph*)
  $Loop \equiv \{\ (X,\ mrk)\ .$
  $finite\ (-mrk) \wedge finite\ X \wedge X \subseteq mrk \wedge$
  $mrk \subseteq reach\ root \wedge reach\ root \cap -mrk \subseteq path\ X\ mrk\}$

definition
  $trm \equiv \lambda\ (X,\ mrk)\ .\ 2 * card\ (-mrk) + card\ X$

definition
  $term\text{-}eq\ t\ w = \{s\ .\ t\ s = w\}$

definition
  $term\text{-}less\ t\ w = \{s\ .\ t\ s < w\}$

lemma *union-term-eq* [*simp*]: $(\bigcup\ w\ .\ term\text{-}eq\ t\ w) = UNIV$
  apply (*simp add*: *term-eq-def*)
  by *auto*

lemma *union-less-term-eq* [*simp*]: $(\bigcup v{\in}\{v.\ v < w\}.\ term\text{-}eq\ t\ v) = term\text{-}less\ t\ w$
  apply (*simp add*: *term-eq-def term-less-def*)
  by *auto*

definition (**in** *graph*)
Init ≡ { (*X*::(*'node set*), *mrk*::(*'node set*)) . *finite* (−*mrk*) ∧ *mrk* = {}}

definition (**in** *graph*)
Final ≡ { (*X*::(*'node set*), *mrk*::(*'node set*)) . *mrk* = *reach root*}

definition (**in** *graph*)
SetMarkInv i = (*case i of*
I.init ⇒ *Init* |
I.loop ⇒ *Loop* |
I.final ⇒ *Final*)

definition (**in** *graph*)
SetMarkInvFinal i = (*case i of*
I.final ⇒ *Final* |
- ⇒ {})

definition (**in** *graph*) [*simp*]:
SetMarkInvTerm w i = (*case i of*
I.init ⇒ *Init* |
I.loop ⇒ *Loop* ∩ {*s* . *trm s* = *w*} |
I.final ⇒ *Final*)

3.3 Diagram

definition (**in** *graph*)
SetMark ≡ λ (*i*, *j*) . (*case* (*i*, *j*) *of*
(*I.init*, *I.loop*) ⇒ *Q1-a* ⊓ *Q2-a* |
(*I.loop*, *I.loop*) ⇒ *Q3-a* ⊓ *Q4-a* |
(*I.loop*, *I.final*) ⇒ *Q5-a* |
- ⇒ *top*)

lemma (**in** *graph*) *SetMark-dmono* [*simp*]:
dmono SetMark
apply (*unfold dmono-def SetMark-def Q1-a-def Q2-a-def Q3-a-def Q4-a-def Q5-a-def*)
by *simp*

3.4 Correctness of the transitions

lemma (**in** *graph*) *init-loop-1-a*[*simp*]: ⊨ *Init* {| *Q1-a* |} *Loop*
apply (*unfold hoare-demonic Init-def Q1-a-def Loop-def*)
by *auto*

lemma (**in** *graph*) *init-loop-2-a*[*simp*]: ⊨ *Init* {| *Q2-a* |} *Loop*
apply (*simp add*: *hoare-demonic Init-def Q2-a-def Loop-def*)
apply *auto*
apply (*simp-all add*: *reach-def*)
apply (*rule init-set2*)
by (*simp-all add*: *reach-def*)

lemma (**in** *graph*) *loop-loop-1-a* [*simp*]: $\models$ (*Loop* $\cap$ {*s* . *trm s* = *w*}) {| *Q3-a* |} (*Loop* $\cap$ {*s*. *trm s* < *w*})
apply (*simp add*: *hoare-demonic Q3-a-def Loop-def trm-def*)
apply *safe*
apply (*simp-all*)
apply (*simp-all add*: *reach-def subset-eq*)
apply *safe*
apply (*simp-all add*: *Compl-insert*)
apply (*rule rtrancl-into-rtrancl*)
apply (*simp-all add*: *Int-def*)
apply (*rule add-set2*)
apply *simp-all*
apply (*case-tac card* (−*b*) > *0*)
by *auto*

lemma (**in** *graph*) *loop-loop-2-a*[*simp*]: $\models$ (*Loop* $\cap$ {*s* . *trm s* = *w*}) {| *Q4-a* |} (*Loop* $\cap$ {*s*. *trm s* < *w*})
apply (*simp add*: *hoare-demonic Q4-a-def Loop-def trm-def*)
apply *auto*
apply (*case-tac card a* > *0*)
by *auto*

lemma (**in** *graph*) *loop-final-a* [*simp*]: $\models$ (*Loop* $\cap$ {*s* . *trm s* = *w*}) {| *Q5-a* |} *Final*
apply (*simp add*: *hoare-demonic Q5-a-def Loop-def Final-def subset-eq Int-def path-def*)
by *auto*

lemma *union-term-w*[*simp*]: ($\bigcup$ *w*. {*s*. *t s* = *w*}) = *UNIV*
by *auto*

lemma *union-less-term-w*[*simp*]: ($\bigcup$ *v*∈{*v*. *v* < *w*}. {*s*. *t s* = *v*}) = {*s* . *t s* < *w*}
by *auto*

lemma *sup-union*[*simp*]: *Sup* (*range A*) *i* = ($\bigcup$ *w* . *A w i*)
by (*simp-all add*: *Sup-fun-def*)

lemma *forall-simp* [*simp*]: (∀ *a b*. ∀ *x*∈*A*. (*a* = (*t x*)) ⟶ (*h x*) ∨ *b* ≠ *u x*) = (∀ *x*∈ *A* . *h x*)
by *auto*

lemma *forall-simp2* [*simp*]: (∀ *a b*. ∀ *x*∈*A*. ∀ *y*. (*a* = *t x y*) ⟶ (*h x y*) ⟶ (*g x y*) ∨ *b* ≠ *u x y*) = (∀ *x*∈*A*. ∀ *y*. *h x y* ⟶ *g x y*)
by *auto*

3.5 Diagram correctness

The termination ordering for the *SetMark* diagram is the lexicographic ordering on pairs (i, n) where $i \in I$ and $n \in nat$.

interpretation *DiagramTermination* λ (*n*::*nat*) (*i* :: *I*) . (*i*, *n*)
done

theorem (**in** *graph*) *SetMark-correct*:
$\models$ *SetMarkInv* {|*pt SetMark*|} *SetMarkInvFinal*
proof (*rule-tac X* = *SetMarkInvTerm* **in** *hoare-diagram3*)
show *dmono SetMark* **by** *simp*
show $\forall$ *u i j*. $\models$ *SetMarkInvTerm u i*{| *SetMark* (*i*, *j*) |}
DiagramTermination.SUP-L-P (λ*n i*. (*i*, *n*)) *SetMarkInvTerm* (*i*, *u*) *j*
by (*auto simp add*: *SUP-L-P-def less-pair-def less-I-def hoare-choice SetMark-def*)
show *SetMarkInv* $\leq$ *Sup* (*range SetMarkInvTerm*)
apply (*simp add*: *le-fun-def*, *safe*)
apply (*simp-all add*: *SetMarkInv-def*)
apply (*case-tac x*)
apply *auto*
done
show *Sup* (*range SetMarkInvTerm*) $\sqcap$ $-$ *grd* (*step SetMark*) $\leq$ *SetMarkInvFinal*
apply (*simp add*: *le-fun-def inf-fun-def SetMarkInvFinal-def*)
apply *safe*
apply *simp-all*
apply (*drule-tac x*=*I.loop* **in** *spec*)
apply (*simp add*: *SetMark-def*)
apply (*simp add*: *Q1-a-def Q2-a-def*)
apply (*frule-tac x*=*I.loop* **in** *spec*)
apply (*drule-tac x*=*I.final* **in** *spec*)
apply (*simp add*: *SetMark-def*)
apply (*simp add*: *Q3-a-def Q4-a-def Q5-a-def*)
apply (*auto*)
done
qed

theorem (**in** *graph*) *SetMark-correct1* [*simp*]:
Hoare-dgr SetMarkInv SetMark (*SetMarkInv* $\sqcap$ ($-$ *grd* (*step SetMark*)))
apply (*simp add*: *Hoare-dgr-def*)
apply (*rule-tac x* = *SetMarkInvTerm* **in** *exI*)
apply (*subgoal-tac SetMarkInv* = $\bigsqcup$ *range SetMarkInvTerm*)
apply *simp*
apply *safe*
apply (*simp-all add*: *SetMark-def SUP-L-P-def*
less-pair-def less-I-def hoare-choice)
apply (*simp-all add*: *fun-eq-iff*)
apply *safe*
apply (*unfold SetMarkInv-def*)
by *auto*

theorem (**in** *graph*) *stack-not-nil* [*simp*]:
$(mrk, S) \in Loop \Longrightarrow x \in S \Longrightarrow x \neq nil$
apply (*simp add*: *Loop-def reach-def*)
by *auto*

end

4 Marking Using a Stack

theory *StackMark*
imports *SetMark DataRefinementIBP.DataRefinement*
begin

In this theory we refine the set marking diagram to a diagram in which the set is replaced by a list (stack). Iniatially the list contains the root element and as long as the list is nonempty and the top of the list has an unmarked successor y, then y is added to the top of the list. If the top does not have unmarked sucessors, it is removed from the list. The diagram terminates when the list is empty.

The data refinement relation of the two diagrams is true if the list has distinct elements and the elements of the list and the set are the same.

4.1 Transitions

definition (**in** *graph*)
$Q1'\text{-}a \equiv [:\lambda\ (stk::('node\ list),\ mrk::('node\ set))\ .\ \{(stk'::('node\ list),\ mrk')\ .\ root = nil \wedge stk' = [] \wedge mrk' = mrk\}:]$

definition (**in** *graph*)
$Q2'\text{-}a \equiv [:\lambda\ (stk::('node\ list),\ mrk::('node\ set))\ .\ \{(stk',\ mrk')\ .\ root \neq nil \wedge stk' = [root] \wedge mrk' = mrk \cup \{root\}\}:]$

definition (**in** *graph*)
$Q3'\text{-}a \equiv [:\lambda\ (stk,\ mrk)\ .\ \{(stk',\ mrk')\ .\ stk \neq [] \wedge (\exists\ y\ .\ (hd\ stk,\ y) \in next \wedge y \notin mrk \wedge stk' = y\ \#\ stk \wedge mrk' = mrk \cup \{y\})\}:]$

definition (**in** *graph*)
$Q4'\text{-}a \equiv [:\lambda\ (stk,\ mrk)\ .\ \{(stk',\ mrk')\ .\ stk \neq [] \wedge (\forall\ y\ .\ (hd\ stk,\ y) \in next \longrightarrow y \in mrk) \wedge stk' = tl\ stk \wedge mrk' = mrk\}:]$

definition
$Q5'\text{-}a \equiv [:\lambda\ (stk,\ mrk)\ .\ \{(stk',\ mrk')\ .\ stk = [] \wedge mrk' = mrk\}:]$

4.2 Invariants

definition
$Init' \equiv UNIV$

definition
Loop′ ≡ { (*stk*, *mrk*) . *distinct stk*}

definition
Final′ ≡ *UNIV*

definition [*simp*]:
StackMarkInv i = (*case i of*
I.init ⇒ *Init′* |
I.loop ⇒ *Loop′* |
I.final ⇒ *Final′*)

4.3 Data refinement relations

definition
R1-a ≡ {: *stk*, *mrk* ⇝ *X*, *mrk′* . *mrk′* = *mrk* :}

definition
R2-a ≡ {: *stk*, *mrk* ⇝ *X*, *mrk′* . *X* = *set stk* ∧ (*stk*, *mrk*) ∈ *Loop′* ∧ *mrk′* = *mrk* :}

lemma [*simp*]: *R1-a* ∈ *Apply.Disjunctive*
by (*simp add*: *R1-a-def*)

lemma [*simp*]: *R2-a* ∈ *Apply.Disjunctive* **by** (*simp add*: *R2-a-def*)

definition [*simp*]:
R-a i = (*case i of*
I.init ⇒ *R1-a* |
I.loop ⇒ *R2-a* |
I.final ⇒ *R1-a*)

lemma [*simp*]: *Disjunctive-fun R-a* **by** (*simp add*: *Disjunctive-fun-def*)

definition
angelic-fun r = (λ *i* . {:*r i*:})

definition (**in** *graph*)
StackMark-a = (λ (*i*, *j*) . (*case* (*i*, *j*) *of*
(*I.init*, *I.loop*) ⇒ *Q1′-a* ⊓ *Q2′-a* |
(*I.loop*, *I.loop*) ⇒ *Q3′-a* ⊓ *Q4′-a* |
(*I.loop*, *I.final*) ⇒ *Q5′-a* |
- ⇒ ⊤))

4.4 Data refinement of the transitions

theorem (**in** *graph*) *init-nil* [*simp*]:
DataRefinement ({.*Init*.} *o Q1-a*) *R1-a R2-a Q1′-a*
by (*simp add*: *data-refinement-hoare hoare-demonic Q1′-a-def Init-def*

Loop′-def R1-a-def R2-a-def Q1-a-def angelic-def subset-eq)

theorem (**in** *graph*) *init-root* [*simp*]:
DataRefinement ({.*Init*.} *o Q2-a*) *R1-a R2-a Q2′-a*
by (*simp add*: *data-refinement-hoare hoare-demonic Q2′-a-def Init-def*
Loop′-def R1-a-def R2-a-def Q2-a-def angelic-def subset-eq)

theorem (**in** *graph*) *step1* [*simp*]:
DataRefinement ({.*Loop*.} *o Q3-a*) *R2-a R2-a Q3′-a*
apply (*simp add*: *data-refinement-hoare hoare-demonic Loop-def*
Loop′-def R2-a-def Q3-a-def Q3′-a-def angelic-def subset-eq)
apply (*simp add*: *simp-eq-emptyset*)
by (*metis List.set-simps*(*2*) *hd-in-set distinct.simps*(*2*))

theorem (**in** *graph*) *step2* [*simp*]:
DataRefinement ({.*Loop*.} *o Q4-a*) *R2-a R2-a Q4′-a*
apply (*simp add*: *data-refinement-hoare hoare-demonic Loop-def*
Loop′-def R2-a-def Q4-a-def Q4′-a-def angelic-def subset-eq)
apply (*simp add*: *simp-eq-emptyset*)
apply *clarify*
apply (*case-tac a*)
by *auto*

theorem (**in** *graph*) *final* [*simp*]:
DataRefinement ({.*Loop*.} *o Q5-a*) *R2-a R1-a Q5′-a*
apply (*simp add*: *data-refinement-hoare hoare-demonic Loop-def*
Loop′-def R2-a-def R1-a-def Q5-a-def Q5′-a-def angelic-def subset-eq)
by (*simp add*: *simp-eq-emptyset*)

4.5 Diagram data refinement

lemma *assert-comp-choice*: {.*p*.} *o* ($S \sqcap T$) = ({.*p*.} *o S*) $\sqcap$ ({.*p*.} *o T*)
apply (*rule antisym*)
apply (*simp-all add*: *fun-eq-iff assert-def le-fun-def inf-fun-def inf-assoc*)
apply *safe*
apply (*rule-tac* $y = S\ x \sqcap T\ x$ **in** *order-trans*)
apply (*rule inf-le2*)
apply *simp*
apply (*rule-tac* $y = S\ x \sqcap T\ x$ **in** *order-trans*)
apply (*rule inf-le2*)
apply *simp*
apply (*rule-tac* $y = S\ x \sqcap (p \sqcap T\ x)$ **in** *order-trans*)
apply (*rule inf-le2*)
apply *simp*
apply (*rule-tac* $y = S\ x \sqcap (p \sqcap T\ x)$ **in** *order-trans*)
apply (*rule inf-le2*)
apply (*rule-tac* $y = p \sqcap T\ x$ **in** *order-trans*)
apply (*rule inf-le2*)
by *simp*

theorem (**in** *graph*) *StackMark-DataRefinement* [*simp*]:
DgrDataRefinement2 SetMarkInv SetMark R-a StackMark-a
by (*simp add*: *DgrDataRefinement2-def StackMark-a-def SetMark-def demonic-sup-inf*

SetMarkInv-def data-refinement-choice2 assert-comp-choice)

4.6 Diagram correctness

theorem (**in** *graph*) *StackMark-correct*:
Hoare-dgr (*R-a* .. *SetMarkInv*) *StackMark-a* ((*R-a* .. *SetMarkInv*) ⊓ (− *grd* (*step* (*StackMark-a*))))
apply (*rule-tac D* = *SetMark* **in** *Diagram-DataRefinement2*)
apply *auto*
by (*rule SetMark-correct1*)

end

5 Generalization of Deutsch-Schorr-Waite Algorithm

theory *LinkMark*
imports *StackMark*
begin

In the third step the stack diagram is refined to a diagram where no extra memory is used. The relation *next* is replaced by two new variables *link* and *label*. The variable *label* : *node* → *index* associates a label to every node and the variable *link* : *index* → *node* → *node* is a collection of pointer functions indexed by the set *index* of labels. For $x \in$ *node*, *link i x* is the successor node of x along the function *link i*. In this context a node x is reachable if there exists a path from the root to x along the links *link i* such that all nodes in this path are not *nil* and they are labeled by a special label *none* ∈ *index*.

The stack variable S is replaced by two new variables p and t ranging over nodes. Variable p stores the head of S, t stores the head of the tail of S, and the rest of S is stored by temporarily modifying the variables *link* and *label*.

This algorithm is a generalization of the Deutsch-Schorr-Waite graph marking algorithm because we have a collection of pointer functions instead of left and right only.

locale *pointer* = *node* +
fixes *none* :: *'index*
fixes *link0*::*'index* ⇒ *'node* ⇒ *'node*
fixes *label0* :: *'node* ⇒ *'index*

assumes $(nil::'node) = nil$
begin
definition $next = \{(a, b) . (\exists\ i . link0\ i\ a = b) \wedge a \neq nil \wedge b \neq nil \wedge label0\ a = none\}$
end

sublocale *pointer* $\subseteq$ *link?*: *graph nil root next*
apply *unfold-locales*
apply (*unfold next-def*)
by *auto*

The locale pointer fixes the initial values for the variables *link* and *label* and it defines the relation next as the union of all *link i* functions, excluding the mappings to *nil*, the mappings from *nil* as well as the mappings from elements which are not labeled by *none*.

The next two recursive functions, *label*_0, *link*_0 are used to compute the initial values of the variables *label* and *link* from their current values.

context *pointer*
begin
primrec
$label\text{-}0 :: ('node \Rightarrow 'index) \Rightarrow ('node\ list) \Rightarrow ('node \Rightarrow 'index)$ **where**
$label\text{-}0\ lbl\ [] = lbl\ |$
$label\text{-}0\ lbl\ (x \# l) = label\text{-}0\ (lbl(x := none))\ l$

lemma *label-cong* [*cong*]: $f = g \Longrightarrow xs = ys \Longrightarrow pointer.label\text{-}0\ n\ f\ xs = pointer.label\text{-}0\ n\ g\ ys$
by *simp*

primrec
$link\text{-}0 :: ('index \Rightarrow 'node \Rightarrow 'node) \Rightarrow ('node \Rightarrow 'index) \Rightarrow 'node \Rightarrow ('node\ list) \Rightarrow ('index \Rightarrow 'node \Rightarrow 'node)$ **where**
$link\text{-}0\ lnk\ lbl\ p\ [] = lnk\ |$
$link\text{-}0\ lnk\ lbl\ p\ (x \# l) = link\text{-}0\ (lnk((lbl\ x) := ((lnk\ (lbl\ x))(x := p))))\ lbl\ x\ l$

The function *stack* defined bellow is the main data refinement relation connecting the stack from the abstract algorithm to its concrete representation by temporarily modifying the variable *link* and *label*.

primrec
$stack :: ('index \Rightarrow 'node \Rightarrow 'node) \Rightarrow ('node \Rightarrow 'index) \Rightarrow 'node \Rightarrow ('node\ list) \Rightarrow bool$ **where**
$stack\ lnk\ lbl\ x\ [] = (x = nil)\ |$
$stack\ lnk\ lbl\ x\ (y \# l) =$
$(x \neq nil \wedge x = y \wedge \neg\ x \in set\ l \wedge stack\ lnk\ lbl\ (lnk\ (lbl\ x)\ x)\ l)$

lemma *label-out-range0* [*simp*]:
$\neg\ x \in set\ S \Longrightarrow label\text{-}0\ lbl\ S\ x = lbl\ x$

apply (*rule-tac P*=∀ *label* . ¬ *x* ∈ *set S* ⟶ *label-0 label S x* = *label x* **in** *mp*)
by (*simp*, *induct-tac S*, *auto*)

lemma *link-out-range0* [*simp*]:
¬ *x* ∈ *set S* ⟹ *link-0 link label p S i x* = *link i x*
apply (*rule-tac P*=∀ *link p* . ¬ *x* ∈ *set S* ⟶ *link-0 link label p S i x* = *link i x* **in** *mp*)
by (*simp*, *induct-tac S*, *auto*)

lemma *link-out-range* [*simp*]: ¬ *x* ∈ *set S* ⟹ *link-0 link* (*label*(*x* := *y*)) *p S* = *link-0 link label p S*
apply (*rule-tac P*=∀ *link p* . ¬ *x* ∈ *set S* ⟶ *link-0 link* (*label*(*x* := *y*)) *p S* = *link-0 link label p S* **in** *mp*)
by (*simp*, *induct-tac S*, *auto*)

lemma *empty-stack* [*simp*]: *stack link label nil S* = (*S* = [])
by (*case-tac S*, *simp-all*)

lemma *stack-out-link-range* [*simp*]: ¬ *p* ∈ *set S* ⟹ *stack* (*link*(*i* := (*link i*)(*p* := *q*))) *label x S* = *stack link label x S*
apply (*rule-tac P* = ∀ *link x* . ¬ *p* ∈ *set S* ⟶ *stack* (*link*(*i* := (*link i*)(*p* := *q*))) *label x S* = *stack link label x S* **in** *mp*)
by (*simp*, *induct-tac S*, *auto*)

lemma *stack-out-label-range* [*simp*]: ¬ *p* ∈ *set S* ⟹ *stack link* (*label*(*p* := *q*)) *x S* = *stack link label x S*
apply (*rule-tac P* = ∀ *link x* . ¬ *p* ∈ *set S* ⟶ *stack link* (*label*(*p* := *q*)) *x S* = *stack link label x S* **in** *mp*)
by (*simp*, *induct-tac S*, *auto*)

definition
g mrk lbl ptr x ≡ *ptr x* ≠ *nil* ∧ *ptr x* ∉ *mrk* ∧ *lbl x* = *none*

lemma *g-cong* [*cong*]: *mrk* = *mrk1* ⟹ *lbl* = *lbl1* ⟹ *ptr* = *ptr1* ⟹ *x* = *x1* ==>
pointer.g n m mrk lbl ptr x = *pointer.g n m mrk1 lbl1 ptr1 x1*
by *simp*

5.1 Transitions

definition
Q1''-a ≡ [: *p*, *t*, *lnk*, *lbl*, *mrk* ⇝ *p'*, *t'*, *lnk'*, *lbl'*, *mrk'* .
root = *nil* ∧ *p'* = *nil* ∧ *t'* = *nil* ∧ *lnk'* = *lnk* ∧ *lbl'* = *lbl* ∧ *mrk'* = *mrk*:]

definition
Q2''-a ≡ [: *p*, *t*, *lnk*, *lbl*, *mrk* ⇝ *p'*, *t'*, *lnk'*, *lbl'*, *mrk'* .
root ≠ *nil* ∧ *p'* = *root* ∧ *t'* = *nil* ∧ *lnk'* = *lnk* ∧ *lbl'* = *lbl* ∧ *mrk'* = *mrk* ∪ {*root*} :]

definition
$Q3''\text{-}a \equiv [: p, t, lnk, lbl, mrk \leadsto p', t', lnk', lbl', mrk'$.
$p \neq nil \wedge$
$(\exists\ i\ .\ g\ mrk\ lbl\ (lnk\ i)\ p \wedge$
$p' = lnk\ i\ p \wedge t' = p \wedge lnk' = lnk(i := (lnk\ i)(p := t)) \wedge lbl' = lbl(p := i) \wedge$
$mrk' = mrk \cup \{lnk\ i\ p\})$:]

definition
$Q4''\text{-}a \equiv [: p, t, lnk, lbl, mrk \leadsto p', t', lnk', lbl', mrk'$.
$p \neq nil \wedge$
$(\forall\ i\ .\ \neg\ g\ mrk\ lbl\ (lnk\ i)\ p) \wedge t \neq nil \wedge$
$p' = t \wedge t' = lnk\ (lbl\ t)\ t \wedge lnk' = lnk(lbl\ t := (lnk\ (lbl\ t))(t := p))$
$\wedge\ lbl' = lbl(t := none) \wedge mrk' = mrk$:]

definition
$Q5''\text{-}a \equiv [: p, t, lnk, lbl, mrk \leadsto p', t', lnk', lbl', mrk'$.
$p \neq nil \wedge$
$(\forall\ i\ .\ \neg\ g\ mrk\ lbl\ (lnk\ i)\ p) \wedge t = nil \wedge$
$p' = nil \wedge t' = t \wedge lnk' = lnk \wedge lbl' = lbl \wedge mrk' = mrk$:]

definition
$Q6''\text{-}a \equiv [: p, t, lnk, lbl, mrk \leadsto p', t', lnk', lbl', mrk'\ .\ p = nil \wedge$
$p' = p \wedge t' = t \wedge lnk' = lnk \wedge lbl' = lbl \wedge mrk' = mrk$:]

5.2 Invariants

definition
$Init'' \equiv \{\ (p, t, lnk, lbl, mrk)\ .\ lnk = link0 \wedge lbl = label0\}$

definition
$Loop'' \equiv UNIV$

definition
$Final'' \equiv Init''$

5.3 Data refinement relations

definition
$R1'\text{-}a \equiv \{: p, t, lnk, lbl, mrk \leadsto stk, mrk'\ .\ (p, t, lnk, lbl, mrk) \in Init'' \wedge mrk' = mrk$:}

definition
$R2'\text{-}a \equiv \{: p, t, lnk, lbl, mrk \leadsto stk, mrk'$.
$p = head\ stk \wedge$
$t = head\ (tail\ stk) \wedge$
$stack\ lnk\ lbl\ t\ (tail\ stk) \wedge$
$link0 = link\text{-}0\ lnk\ lbl\ p\ (tail\ stk) \wedge$
$label0 = label\text{-}0\ lbl\ (tail\ stk) \wedge$
$\neg\ nil \in set\ stk \wedge$

$mrk' = mrk$:}

lemma [*simp*]: $R1'\text{-}a \in Apply.Disjunctive$ **by** (*simp add*: $R1'$-*a-def*)

lemma [*simp*]: $R2'\text{-}a \in Apply.Disjunctive$ **by** (*simp add*: $R2'$-*a-def*)

definition [*simp*]:
R'-*a* i = (*case i of*
I.init ⇒ $R1'$-*a* |
I.loop ⇒ $R2'$-*a* |
I.final ⇒ $R1'$-*a*)

lemma [*simp*]: *Disjunctive-fun* R'-*a* **by** (*simp add*: *Disjunctive-fun-def*)

5.4 Diagram

definition
LinkMark = (λ (*i*, *j*) . (*case* (*i*, *j*) *of*
(*I.init*, *I.loop*) ⇒ $Q1''$-*a* ⊓ $Q2''$-*a* |
(*I.loop*, *I.loop*) ⇒ $Q3''$-*a* ⊓ ($Q4''$-*a* ⊓ $Q5''$-*a*) |
(*I.loop*, *I.final*) ⇒ $Q6''$-*a* |
- ⇒ ⊤))

definition [*simp*]:
LinkMarkInv i = (*case i of*
I.init ⇒ $Init''$ |
I.loop ⇒ $Loop''$ |
I.final ⇒ $Final''$)

5.5 Data refinement of the transitions

theorem *init1-a* [*simp*]:
DataRefinement ({.$Init'$.} *o* $Q1'$-*a*) $R1'$-*a* $R2'$-*a* $Q1''$-*a*
by (*simp add*: *data-refinement-hoare hoare-demonic* $Q1''$-*a-def* $Init'$-*def* $Init''$-*def*

$Loop''$-*def* $R1'$-*a-def* $R2'$-*a-def* $Q1'$-*a-def tail-def head-def angelic-def subset-eq*)

theorem *init2-a* [*simp*]:
DataRefinement ({.$Init'$.} *o* $Q2'$-*a*) $R1'$-*a* $R2'$-*a* $Q2''$-*a*
by (*simp add*: *data-refinement-hoare hoare-demonic* $Q2''$-*a-def* $Init'$-*def* $Init''$-*def*

$Loop''$-*def* $R1'$-*a-def* $R2'$-*a-def* $Q2'$-*a-def tail-def head-def angelic-def subset-eq*)

theorem *step1-a* [*simp*]:
DataRefinement ({.$Loop'$.} *o* $Q3'$-*a*) $R2'$-*a* $R2'$-*a* $Q3''$-*a*
apply (*simp add*: *data-refinement-hoare hoare-demonic* $Q3''$-*a-def* $Init'$-*def* $Init''$-*def*

$Loop'$-*def* $R1'$-*a-def* $Q3'$-*a-def tail-def head-def angelic-def subset-eq*)
apply (*unfold next-def*)
apply (*simp add*: $R2'$-*a-def*)

apply (*simp add*: *data-refinement-hoare*)
apply (*simp-all add*: *R2'-a-def angelic-def hoare-demonic simp-eq-emptyset*)
apply *auto*
apply (*rule-tac x* = *aa i* (*hd a*) # *a* **in** *exI*)
apply *safe*
apply *simp-all*
apply (*simp add*: *g-def neq-Nil-conv*)
apply *clarify*
apply (*simp add*: *g-def neq-Nil-conv*)
apply (*case-tac a*)
apply (*simp-all add*: *g-def neq-Nil-conv*)
apply (*case-tac a*)
apply *simp-all*
apply (*case-tac a*)
by *auto*

lemma *neqif* [*simp*]: $x \neq y \Longrightarrow$ (*if y* = *x then a else b*) = *b*
apply (*case-tac* $y \neq x$)
apply *simp-all*
done

theorem *step2-a* [*simp*]:
DataRefinement ({.*Loop'*.} *o Q4'-a*) *R2'-a R2'-a Q4''-a*
apply (*simp add*: *data-refinement-hoare hoare-demonic Q4''-a-def Init'-def Init''-def*

Loop'-def Q4'-a-def tail-def head-def angelic-def subset-eq)
apply (*unfold next-def*)
apply (*simp add*: *R2'-a-def*)
apply (*simp add*: *data-refinement-hoare*)
apply (*simp-all add*: *R2'-a-def angelic-def hoare-demonic simp-eq-emptyset*)
apply (*simp-all add*: *neq-Nil-conv*)
apply (*unfold g-def*)
apply (*simp add*: *head-def*)
apply *safe*
apply *auto* [*1*]
apply *auto* [*1*]
apply (*case-tac ysa*)
apply *simp-all*
apply *safe*
apply (*case-tac ab ya* = *i*)
by *auto*

lemma *setsimp*: *a* = *c* $\Longrightarrow$ ($x \in a$) = ($x \in c$)
apply *simp*
done

theorem *step3-a* [*simp*]:

DataRefinement ({.*Loop′*.} *o Q4′-a*) *R2′-a R2′-a Q5″-a*
apply (*simp add*: *data-refinement-hoare hoare-demonic Q5″-a-def Init′-def Init″-def*

Loop′-def Q4′-a-def angelic-def subset-eq)
apply (*unfold R2′-a-def*)
apply (*unfold next-def*)
apply (*simp add*: *data-refinement-hoare hoare-demonic angelic-def subset-eq simp-eq-emptyset g-def head-def tail-def*)
by *auto*

theorem *final-a* [*simp*]:
DataRefinement ({.*Loop′*.} *o Q5′-a*) *R2′-a R1′-a Q6″-a*
apply (*simp add*: *data-refinement-hoare hoare-demonic Q6″-a-def Init′-def Init″-def*

Loop′-def R2′-a-def R1′-a-def Q5′-a-def angelic-def subset-eq neq-Nil-conv tail-def head-def)
apply (*simp add*: *simp-eq-emptyset*)
apply *safe*
by *simp-all*

5.6 Diagram data refinement

lemma *apply-fun-index* [*simp*]: (*r* .. *P*) *i* = (*r i*) (*P i*) **by** (*simp add*: *apply-fun-def*)

lemma [*simp*]: *Disjunctive-fun* (*r*::(*′c* ⇒ *′a*::*complete-lattice* ⇒ *′b*::*complete-lattice*))

⟹ *mono-fun r*
by (*simp add*: *Disjunctive-fun-def mono-fun-def*)

theorem *LinkMark-DataRefinement-a* [*simp*]:
DgrDataRefinement2 (*R-a* .. *SetMarkInv*) *StackMark-a R′-a LinkMark*
apply (*rule-tac P* = *StackMarkInv* **in** *DgrDataRefinement-mono*)
apply (*simp add*: *le-fun-def SetMarkInv-def angelic-def R1-a-def R2-a-def Init′-def Final′-def*)
apply *safe*
apply *simp*
apply (*simp add*: *DgrDataRefinement2-def dgr-demonic-def LinkMark-def StackMark-a-def demonic-sup-inf data-refinement-choice2 assert-comp-choice*)
apply (*rule data-refinement-choice2*)
apply *simp-all*
apply (*rule data-refinement-choice1*)
by *simp-all*

lemma [*simp*]: *mono Q1′-a* **by** (*simp add*: *Q1′-a-def*)
lemma [*simp*]: *mono Q2′-a* **by** (*simp add*: *Q2′-a-def*)
lemma [*simp*]: *mono Q3′-a* **by** (*simp add*: *Q3′-a-def*)
lemma [*simp*]: *mono Q4′-a* **by** (*simp add*: *Q4′-a-def*)
lemma [*simp*]: *mono Q5′-a* **by** (*simp add*: *Q5′-a-def*)

lemma [*simp*]: *dmono StackMark-a*
  apply (*unfold dmono-def StackMark-a-def*)
  by *simp*

5.7 Diagram correctness

theorem *LinkMark-correct*:
  Hoare-dgr (R'-*a* .. (*R-a* .. *SetMarkInv*)) *LinkMark* ((R'-*a* .. (*R-a* .. *SetMarkInv*)) $\sqcap$ ($-$ *grd* (*step LinkMark*)))
  apply (*rule-tac D* = *StackMark-a* **in** *Diagram-DataRefinement2*)
  apply *simp-all*
  by (*rule StackMark-correct*)

end
end

6 Deutsch-Schorr-Waite Marking Algorithm

theory *DSWMark*
imports *LinkMark*
begin

Finally, we construct the Deutsch-Schorr-Waite marking algorithm by assuming that there are only two pointers (*left*, *right*) from every node. There is also a new variable, *atom* : *node* $\rightarrow$ *bool* which associates to every node a Boolean value. The data invariant of this refinement step requires that *index* has exactly two distinct elements *none* and *some*, *left* = *link none*, *right* = *link some*, and *atom x* is true if and only if *label x* = *some*.

We use a new locale which fixes the initial values of the variables *left*, *right*, and *atom* in *left0*, *right0*, and *atom0* respectively.

datatype *Index* = *none* | *some*

locale *classical* = *node* +
  fixes *left0* :: *'node* $\Rightarrow$ *'node*
  fixes *right0* :: *'node* $\Rightarrow$ *'node*
  fixes *atom0* :: *'node* $\Rightarrow$ *bool*
  assumes (*nil*::*'node*) = *nil*
  begin
    definition
      link0 i = (*if i* = (*none*::*Index*) *then left0 else right0*)

    definition
      label0 x = (*if atom0 x then* (*some*::*Index*) *else none*)
  end

sublocale *classical* $\subseteq$ *dsw?*: *pointer nil root none*::*Index link0 label0*
proof qed *auto*

context *classical*
begin

lemma [*simp*]:
$(label0 = (\lambda\ x\ .\ if\ atom\ x\ then\ some\ else\ none)) = (atom0 = atom)$
apply (*simp add*: *fun-eq-iff label0-def*)
by *auto*

definition
$gg\ mrk\ atom\ ptr\ x \equiv ptr\ x \neq nil \wedge ptr\ x \notin mrk \wedge \neg\ atom\ x$

6.1 Transitions

definition
$QQ1\text{-}a \equiv [:\ p,\ t,\ left,\ right,\ atom,\ mrk \rightsquigarrow p',\ t',\ left',\ right',\ atom',\ mrk'\ .$
$root = nil \wedge\ p' = nil \wedge t' = nil \wedge mrk' = mrk \wedge left' = left$
$\wedge\ right' = right \wedge atom' = atom\ :]$

definition
$QQ2\text{-}a \equiv [:\ p,\ t,\ left,\ right,\ atom,\ mrk \rightsquigarrow p',\ t',\ left',\ right',\ atom',\ mrk'\ .$
$root \neq nil \wedge\ p' = root \wedge t' = nil \wedge mrk' = mrk \cup \{root\}$
$\wedge\ left' = left \wedge right' = right \wedge atom' = atom\ :]$

definition
$QQ3\text{-}a \equiv [:\ p,\ t,\ left,\ right,\ atom,\ mrk \rightsquigarrow p',\ t',\ left',\ right',\ atom',\ mrk'\ .$
$p \neq nil \wedge gg\ mrk\ atom\ left\ p\ \wedge$
$p' = left\ p \wedge t' = p \wedge mrk' = mrk \cup \{left\ p\} \wedge$
$left' = left(p := t) \wedge right' = right \wedge atom' = atom:]$

definition
$QQ4\text{-}a \equiv [:\ p,\ t,\ left,\ right,\ atom,\ mrk \rightsquigarrow p',\ t',\ left',\ right',\ atom',\ mrk'\ .$
$p \neq nil \wedge gg\ mrk\ atom\ right\ p\ \wedge$
$p' = right\ p \wedge t' = p \wedge mrk' = mrk \cup \{right\ p\} \wedge$
$left' = left \wedge right' = right(p := t) \wedge atom' = atom(p := True)\ :]$

definition
$QQ5\text{-}a \equiv [:\ p,\ t,\ left,\ right,\ atom,\ mrk \rightsquigarrow p',\ t',\ left',\ right',\ atom',\ mrk'\ .$
$p \neq nil\ \wedge$ — not needed in the proof
$\neg\ gg\ mrk\ atom\ left\ p \wedge \neg\ gg\ mrk\ atom\ right\ p\ \wedge$
$t \neq nil \wedge \neg\ atom\ t\ \wedge$
$p' = t \wedge t' = left\ t \wedge mrk' = mrk\ \wedge$
$left' = left(t := p) \wedge right' = right \wedge atom' = atom\ :]$

definition
$QQ6\text{-}a \equiv [:\ p,\ t,\ left,\ right,\ atom,\ mrk \rightsquigarrow p',\ t',\ left',\ right',\ atom',\ mrk'\ .$
$p \neq nil\ \wedge$ — not needed in the proof
$\neg\ gg\ mrk\ atom\ left\ p \wedge \neg\ gg\ mrk\ atom\ right\ p\ \wedge$
$t \neq nil \wedge atom\ t\ \wedge$

$p' = t \wedge t' = right\ t \wedge mrk' = mrk \wedge$
$left' = left \wedge right' = right(t := p) \wedge atom' = atom(t := False)$:]

definition
QQ7-a ≡ [: $p, t, left, right, atom, mrk \leadsto p', t', left', right', atom', mrk'$.
$p \neq nil \wedge$
$\neg\ gg\ mrk\ atom\ left\ p \wedge \neg\ gg\ mrk\ atom\ right\ p \wedge$
$t = nil \wedge$
$p' = nil \wedge t' = t \wedge mrk' = mrk \wedge$
$left' = left \wedge right' = right \wedge atom' = atom$:]

definition
QQ8-a ≡ [: $p, t, left, right, atom, mrk \leadsto p', t', left', right', atom', mrk'$.
$p = nil \wedge p' = p \wedge t' = t \wedge mrk' = mrk \wedge left' = left \wedge right' = right \wedge atom' = atom$:]

7 Data refinement relation

definition
RR-a ≡ {: $p, t, left, right, atom, mrk \leadsto p', t', lnk, lbl, mrk'$.
$lnk\ none = left \wedge lnk\ some = right \wedge$
$lbl = (\lambda\ x\ .\ if\ atom\ x\ then\ some\ else\ none) \wedge$
$p' = p \wedge t' = t \wedge mrk' = mrk$:}

definition [*simp*]:
R''-a i = *RR-a*

definition
ClassicMark = (λ (*i*, *j*) . (*case* (*i*, *j*) *of*
(*I.init*, *I.loop*) ⇒ *QQ1-a* ⊓ *QQ2-a* |
(*I.loop*, *I.loop*) ⇒ (*QQ3-a* ⊓ *QQ4-a*) ⊓ ((*QQ5-a* ⊓ *QQ6-a*) ⊓ *QQ7-a*) |
(*I.loop*, *I.final*) ⇒ *QQ8-a* |
\- ⇒ ⊤))

7.1 Data refinement of the transitions

theorem *init1-a* [*simp*]:
DataRefinement ({*.Init''*.} *o Q1''-a*) *RR-a RR-a QQ1-a*
by (*simp add*: *data-refinement-hoare hoare-demonic angelic-def QQ1-a-def Q1''-a-def RR-a-def Init''-def subset-eq*)

theorem *init2-a* [*simp*]:
DataRefinement ({*.Init''*.} *o Q2''-a*) *RR-a RR-a QQ2-a*
by (*simp add*: *data-refinement-hoare hoare-demonic angelic-def QQ2-a-def Q2''-a-def RR-a-def Init''-def subset-eq*)

lemma *index-simp*:

$(u = v) = (u\ none = v\ none \wedge u\ some = v\ some)$
by (*safe*, *rule ext*, *case-tac x*, *auto*)

theorem *step1-a* [*simp*]:
DataRefinement ({.*Loop''*.} *o* *Q3''-a*) *RR-a* *RR-a* *QQ3-a*
apply (*simp add*: *data-refinement-hoare hoare-demonic angelic-def QQ3-a-def Q3''-a-def RR-a-def*
Loop''-def subset-eq g-def gg-def simp-eq-emptyset)
apply *safe*
apply (*rule-tac x*=λ *x* . *if x* = *some then ab some else* (*ab none*)(*a* := *aa*) **in** *exI*)
apply *simp*
apply (*rule-tac x*=*none* **in** *exI*)
apply (*simp add*: *index-simp*)
done

theorem *step2-a*[*simp*]:
DataRefinement ({.*Loop''*.} *o* *Q3''-a*) *RR-a* *RR-a* *QQ4-a*
apply (*simp add*: *data-refinement-hoare hoare-demonic angelic-def QQ4-a-def Q3''-a-def RR-a-def*
Loop''-def subset-eq g-def gg-def simp-eq-emptyset)
apply *safe*
apply (*rule-tac x*=λ *x* . *if x* = *none then ab none else* (*ab some*)(*a* := *aa*) **in** *exI*)
apply *simp*
apply (*rule-tac x*=*some* **in** *exI*)
apply (*simp add*: *index-simp*)
apply (*rule ext*)
apply *auto*
done

theorem *step3-a* [*simp*]:
DataRefinement ({.*Loop''*.} *o* *Q4''-a*) *RR-a* *RR-a* *QQ5-a*
apply (*simp add*: *data-refinement-hoare hoare-demonic angelic-def QQ5-a-def Q4''-a-def RR-a-def*
Loop''-def subset-eq g-def gg-def simp-eq-emptyset)
apply *clarify*
apply (*case-tac i*)
apply *auto*
done

lemma *if-set-elim*: $(x \in (\textit{if } b \textit{ then } A \textit{ else } B)) = ((b \wedge x \in A) \vee (\neg\ b \wedge x \in B))$
by *auto*

theorem *step4-a* [*simp*]:
DataRefinement ({.*Loop''*.} *o* *Q4''-a*) *RR-a* *RR-a* *QQ6-a*
apply (*simp add*: *data-refinement-hoare hoare-demonic angelic-def RR-a-def QQ6-a-def Q4''-a-def*

Loop''-def subset-eq simp-eq-emptyset g-def gg-def if-set-elim)
apply (*simp add*: *ext*)
apply *safe*
apply (*case-tac i*)
apply *simp-all*
apply (*case-tac i*)
apply *simp-all*
apply (*case-tac i*)
apply *simp-all*
apply (*case-tac i*)
by *simp-all*

theorem *step5-a* [*simp*]:
DataRefinement ({.*Loop''*.} *o Q5''-a*) *RR-a RR-a QQ7-a*
apply (*simp add*: *data-refinement-hoare hoare-demonic angelic-def Q5''-a-def QQ7-a-def*
Loop''-def subset-eq RR-a-def simp-eq-emptyset)
apply *safe*
apply (*simp-all add*: *g-def gg-def*)
apply (*case-tac i*)
by *auto*

theorem *final-step-a* [*simp*]:
DataRefinement ({.*Loop''*.} *o Q6''-a*) *RR-a RR-a QQ8-a*
by (*simp add*: *data-refinement-hoare hoare-demonic angelic-def Q6''-a-def QQ8-a-def*

Loop''-def subset-eq RR-a-def simp-eq-emptyset)

7.2 Diagram data refinement

lemma [*simp*]: *mono RR-a* **by** (*simp add*: *RR-a-def*)
lemma [*simp*]: *RR-a* ∈ *Apply.Disjunctive* **by** (*simp add*: *RR-a-def*)
lemma [*simp*]: *Disjunctive-fun R''-a* **by** (*simp add*: *Disjunctive-fun-def*)

lemma [*simp*]: *mono-fun R''-a* **by** *simp*

lemma [*simp*]: *mono Q1''-a* **by** (*simp add*: *Q1''-a-def*)
lemma [*simp*]: *mono Q2''-a* **by** (*simp add*: *Q2''-a-def*)
lemma [*simp*]: *mono Q3''-a* **by** (*simp add*: *Q3''-a-def*)
lemma [*simp*]: *mono Q4''-a* **by** (*simp add*: *Q4''-a-def*)
lemma [*simp*]: *mono Q5''-a* **by** (*simp add*: *Q5''-a-def*)
lemma [*simp*]: *mono Q6''-a* **by** (*simp add*: *Q6''-a-def*)

lemma [*simp*]: *dmono LinkMark*
apply (*unfold dmono-def LinkMark-def*)
by *simp*

theorem *ClassicMark-DataRefinement-a* [*simp*]:
DgrDataRefinement2 (*R'-a* .. (*R-a* .. *SetMarkInv*)) *LinkMark R''-a ClassicMark*

apply (*rule-tac P = LinkMarkInv* **in** *DgrDataRefinement-mono*)
apply (*simp add*: *le-fun-def SetMarkInv-def*
angelic-def R1′-a-def R2′-a-def Init′′-def Loop′′-def Final′′-def)
apply *auto*
apply (*simp add*: *DgrDataRefinement2-def dgr-demonic-def ClassicMark-def LinkMark-def*

demonic-sup-inf data-refinement-choice2 assert-comp-choice)
apply (*rule data-refinement-choice2*)
apply *simp*
apply (*rule data-refinement-choice1*)
apply *simp-all*
apply (*rule data-refinement-choice2*)
apply *simp-all*
apply (*rule data-refinement-choice1*)
by *simp-all*

7.3 Diagram corectness

theorem *ClassicMark-correct-a* [*simp*]:
Hoare-dgr (*R′′-a* .. (*R′-a* .. (*R-a* .. *SetMarkInv*))) *ClassicMark*
((*R′′-a* .. (*R′-a* ..(*R-a* .. *SetMarkInv*))) ⊓ (− *grd* (*step ClassicMark*)))
apply (*rule-tac D = LinkMark* **in** *Diagram-DataRefinement2*)
apply *auto*
by (*rule LinkMark-correct*)

We have proved the correctness of the final algorithm, but the pre and the post conditions involve the angelic choice operator and they depend on all data refinement steps we have used to prove the final diagram. We simplify these conditions and we show that we obtained indead the corectness of the marking algorithm.

The predicate *ClassicInit* which is true for the *init* situation states that initially the variables *left*, *right*, and *atom* are equal to their initial values and also that no node is marked.

The predicate *ClassicFinal* which is true for the $final$ situation states that at the end the values of the variables *left*, *right*, and *atom* are again equal to their initial values and the variable *mrk* records all reachable nodes. The reachable nodes are defined using our initial *next* relation, however if we unfold all locale interpretations and definitions we see easeily that a node x is reachable if there is a path from *root* to x along *left* and *right* functions, and all nodes in this path have the atom bit false.

definition
ClassicInit = {(*p*, *t*, *left*, *right*, *atom*, *mrk*) .
atom = *atom0* ∧ *left* = *left0* ∧ *right* = *right0* ∧
finite (− *mrk*) ∧ *mrk* = {}}

definition
ClassicFinal = {(*p*, *t*, *left*, *right*, *atom*, *mrk*) .

atom = *atom0* ∧ *left* = *left0* ∧ *right* = *right0* ∧
mrk = *reach root*}

theorem [*simp*]:
ClassicInit ⊆ (*RR-a* (*R1′-a* (*R1-a* (*SetMarkInv init*))))
apply (*simp add*: *SetMarkInv-def*)
apply (*simp add*: *ClassicInit-def angelic-def RR-a-def R1′-a-def R1-a-def Init-def Init″-def*)
apply *safe*
apply (*unfold simp-eq-emptyset*)
apply (*simp add*: *link0-def label0-def*)
apply (*simp add*: *fun-eq-iff*)
by (*simp add*: *label0-def*)

theorem [*simp*]:
(*RR-a* (*R1′-a* (*R1-a* (*SetMarkInv final*)))) ≤ *ClassicFinal*
apply (*simp add*: *SetMarkInv-def*)
apply (*simp add*: *ClassicFinal-def angelic-def RR-a-def R1′-a-def R1-a-def Final-def Final″-def Init″-def label0-def link0-def*)
apply (*simp add*: *simp-eq-emptyset inf-fun-def*)
apply *auto*
by (*simp-all add*: *link0-def*)

The indexed predicate *ClassicPre* is the precondition of the diagram, and since we are only interested in starting the marking diagram in the *init* situation we set *ClassicPre loop* = *ClassicPre final* = ∅.

definition [*simp*]:
ClassicPre i = (*case i of*
I.init ⇒ *ClassicInit* |
I.loop ⇒ {} |
I.final ⇒ {})

We are interested on the other hand that the marking diagram terminates only in the *final* situation. In order to achieve this we define the postcondition of the diagram as the indexed predicate *ClassicPost* which is empty on every situation except $final$.

definition [*simp*]:
ClassicPost i = (*case i of*
I.init ⇒ {} |
I.loop ⇒ {} |
I.final ⇒ *ClassicFinal*)

lemma *exists-or*:
(∃ *x* . *p x* ∨ *q x*) = ((∃ *x* . *p x*) ∨ (∃ *x* . *q x*))
by *auto*

lemma [*simp*]:
(− *grd* (*step ClassicMark*)) *init* = {}

apply (*simp add*: *grd-def step-def*)
apply *safe*
apply *simp*
apply (*drule-tac x* = *loop* **in** *spec*)
by (*simp add*: *ClassicMark-def QQ1-a-def QQ2-a-def demonic-def*)

lemma [*simp*]: *grd* $\top = \bot$
by (*simp add*: *grd-def top-fun-def*)

lemma [*simp*]:
(− *grd* (*step ClassicMark*)) *loop* = {}
apply *safe*
apply *simp*
apply (*frule-tac x* = *final* **in** *spec*)
apply (*drule-tac x* = *loop* **in** *spec*)
apply (*unfold ClassicMark-def QQ1-a-def QQ2-a-def QQ3-a-def QQ4-a-def QQ5-a-def QQ6-a-def QQ7-a-def QQ8-a-def*)
apply *simp*
apply (*case-tac a* ≠ *nil*)
by *auto*

The final theorem states the correctness of the marking diagram with respect to the precondition *ClassicPre* and the postcondition *ClassicPost*, that is, if the diagram starts in the initial situation, then it will terminate in the final situation, and it will mark all reachable nodes.

lemma [*simp*]: *mono QQ1-a* **by** (*simp add*: *QQ1-a-def*)
lemma [*simp*]: *mono QQ2-a* **by** (*simp add*: *QQ2-a-def*)
lemma [*simp*]: *mono QQ3-a* **by** (*simp add*: *QQ3-a-def*)
lemma [*simp*]: *mono QQ4-a* **by** (*simp add*: *QQ4-a-def*)
lemma [*simp*]: *mono QQ5-a* **by** (*simp add*: *QQ5-a-def*)
lemma [*simp*]: *mono QQ6-a* **by** (*simp add*: *QQ6-a-def*)
lemma [*simp*]: *mono QQ7-a* **by** (*simp add*: *QQ7-a-def*)
lemma [*simp*]: *mono QQ8-a* **by** (*simp add*: *QQ8-a-def*)

lemma [*simp*]: *dmono ClassicMark*
apply (*unfold dmono-def ClassicMark-def*)
by *simp*

theorem ⊨ *ClassicPre* {| *pt ClassicMark* |} *ClassicPost*
apply (*rule-tac P* = (*R''-a* .. (*R'-a* .. (*R-a* ..*SetMarkInv*))) **in** *hoare-pre*)
apply (*subst le-fun-def*)
apply *simp*
apply (*rule-tac Q* = ((*R''-a* .. (*R'-a* .. (*R-a* .. *SetMarkInv*))) ⊓ (− *grd* (*step* ((*ClassicMark*))))) **in** *hoare-mono*)
apply (*simp-all add*: *hoare-dgr-correctness*)
apply (*rule le-funI*)
apply (*case-tac x*)
apply (*simp-all add*: *inf-fun-def del*: *uminus-apply*)
apply (*rule-tac y* = *RR-a* (*R1'-a* (*R1-a* (*SetMarkInv final*))) **in** *order-trans*)

by *auto*

end

end

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