From Abstract to Concrete Gödel's Incompleteness Theorems—Part I

Andrei Popescu Dmitriy Traytel

September 18, 2020

Abstract

We validate an abstract formulation of Gödel's First and Second Incompleteness Theorems from a separate AFP entry by instantiating them to the case of *finite sound extensions of the Hereditarily Finite (HF) Set theory*, i.e., FOL theories extending the HF Set theory with a finite set of axioms that are sound in the standard model. The concrete results had been previously formalised in an AFP entry by Larry Paulson; our instantiation reuses the infrastructure developed in that entry.

Contents

1 The Instantiation

definition *Fvars t* = {*a* :: *name*. ¬ *atom a* ♯ *t*}

lemma *Fvars_tm_simps*[*simp*]:
  Fvars Zero = {}
  Fvars (*Var a*) = {*a*}
  Fvars (*Eats x y*) = *Fvars x* ∪ *Fvars y*
  ⟨*proof*⟩

lemma *finite_Fvars_tm*[*simp*]:
  fixes *t* :: *tm*
  shows *finite* (*Fvars t*)
  ⟨*proof*⟩

lemma *Fvars_fm_simps*[*simp*]:
  Fvars (*x IN y*) = *Fvars x* ∪ *Fvars y*
  Fvars (*x EQ y*) = *Fvars x* ∪ *Fvars y*
  Fvars (*A OR B*) = *Fvars A* ∪ *Fvars B*
  Fvars (*A AND B*) = *Fvars A* ∪ *Fvars B*
  Fvars (*A IMP B*) = *Fvars A* ∪ *Fvars B*
  Fvars Fls = {}
  Fvars (*Neg A*) = *Fvars A*
  Fvars (*Ex a A*) = *Fvars A* − {*a*}
  Fvars (*All a A*) = *Fvars A* − {*a*}
  ⟨*proof*⟩

lemma *finite_Fvars_fm*[*simp*]:
  fixes *A* :: *fm*
  shows *finite* (*Fvars A*)

⟨*proof*⟩

lemma *subst_tm_subst_tm*[*simp*]:
$x \neq y \Longrightarrow atom\ x \sharp\ u \Longrightarrow subst\ y\ u\ (subst\ x\ t\ v) = subst\ x\ (subst\ y\ u\ t)\ (subst\ y\ u\ v)$
⟨*proof*⟩

lemma *subst_fm_subst_fm*[*simp*]:
$x \neq y \Longrightarrow atom\ x \sharp\ u \Longrightarrow (A(x{::=}t))(y{::=}u) = (A(y{::=}u))(x{::=}subst\ y\ u\ t)$
⟨*proof*⟩

lemma *Fvars_ground_aux*: $Fvars\ t \subseteq B \Longrightarrow ground\_aux\ t\ (atom\ `\ B)$
⟨*proof*⟩

lemma *ground_Fvars*: $ground\ t \longleftrightarrow Fvars\ t = \{\}$
⟨*proof*⟩

lemma *Fvars_ground_fm_aux*: $Fvars\ A \subseteq B \Longrightarrow ground\_fm\_aux\ A\ (atom\ `\ B)$
⟨*proof*⟩

lemma *ground_fm_Fvars*: $ground\_fm\ A \longleftrightarrow Fvars\ A = \{\}$
⟨*proof*⟩

interpretation *Generic_Syntax* **where**
$var = UNIV :: name\ set$
and $trm = UNIV :: tm\ set$
and $fmla = UNIV :: fm\ set$
and $Var = Var$
and $FvarsT = Fvars$
and $substT = \lambda t\ u\ x.\ subst\ x\ u\ t$
and $Fvars = Fvars$
and $subst = \lambda A\ u\ x.\ subst\_fm\ A\ x\ u$
⟨*proof*⟩

lemma *coding_tm_Fvars_empty*[*simp*]: $coding\_tm\ t \Longrightarrow Fvars\ t = \{\}$
⟨*proof*⟩

lemma *Fvars_empty_ground*[*simp*]: $Fvars\ t = \{\} \Longrightarrow ground\ t$
⟨*proof*⟩

interpretation *Syntax_with_Numerals* **where**
$var = UNIV :: name\ set$
and $trm = UNIV :: tm\ set$
and $fmla = UNIV :: fm\ set$
and $num = \{t.\ ground\ t\}$
and $Var = Var$
and $FvarsT = Fvars$
and $substT = \lambda t\ u\ x.\ subst\ x\ u\ t$
and $Fvars = Fvars$
and $subst = \lambda A\ u\ x.\ subst\_fm\ A\ x\ u$
⟨*proof*⟩

declare *FvarsT_num*[*simp del*]

interpretation *Deduct2_with_False* **where**
$var = UNIV :: name\ set$
and $trm = UNIV :: tm\ set$
and $fmla = UNIV :: fm\ set$
and $num = \{t.\ ground\ t\}$

and *Var* = *Var*
and *FvarsT* = *Fvars*
and *substT* = λ*t u x*. *subst x u t*
and *Fvars* = *Fvars*
and *subst* = λ*A u x*. *subst_fm A x u*
and *eql* = (*EQ*)
and *cnj* = (*AND*)
and *imp* = (*IMP*)
and *all* = *All*
and *exi* = *Ex*
and *fls* = *Fls*
and *prv* = (⊢) {}
and *bprv* = (⊢) {}
⟨*proof*⟩

interpretation *HBL1* **where**
var = *UNIV* :: *name set*
and *trm* = *UNIV* :: *tm set*
and *fmla* = *UNIV* :: *fm set*
and *num* = {*t*. *ground t*}
and *Var* = *Var*
and *FvarsT* = *Fvars*
and *substT* = λ*t u x*. *subst x u t*
and *Fvars* = *Fvars*
and *subst* = λ*A u x*. *subst_fm A x u*
and *eql* = (*EQ*)
and *cnj* = (*AND*)
and *imp* = (*IMP*)
and *all* = *All*
and *exi* = *Ex*
and *prv* = (⊢) {}
and *bprv* = (⊢) {}
and *enc* = *quot*
and *P* = *PfP* (*Var xx*)
⟨*proof*⟩

interpretation *Goedel_Form* **where**
var = *UNIV* :: *name set*
and *trm* = *UNIV* :: *tm set*
and *fmla* = *UNIV* :: *fm set*
and *num* = {*t*. *ground t*}
and *Var* = *Var*
and *FvarsT* = *Fvars*
and *substT* = λ*t u x*. *subst x u t*
and *Fvars* = *Fvars*
and *subst* = λ*A u x*. *subst_fm A x u*
and *eql* = (*EQ*)
and *cnj* = (*AND*)
and *imp* = (*IMP*)
and *all* = *All*
and *exi* = *Ex*
and *fls* = *Fls*
and *prv* = (⊢) {}
and *bprv* = (⊢) {}
and *enc* = *quot*
and *S* = *KRP* (*quot* (*Var xx*)) (*Var xx*) (*Var yy*)
and *P* = *PfP* (*Var xx*)
⟨*proof*⟩

interpretation *g2*: *Goedel_Second_Assumptions* **where**
var = *UNIV* :: *name set*
and *trm* = *UNIV* :: *tm set*
and *fmla* = *UNIV* :: *fm set*
and *num* = {*t*. *ground t*}
and *Var* = *Var*
and *FvarsT* = *Fvars*
and *substT* = λ*t u x*. *subst x u t*
and *Fvars* = *Fvars*
and *subst* = λ*A u x*. *subst_fm A x u*
and *eql* = (*EQ*)
and *cnj* = (*AND*)
and *imp* = (*IMP*)
and *all* = *All*
and *exi* = *Ex*
and *fls* = *Fls*
and *prv* = (⊢) {}
and *bprv* = (⊢) {}
and *enc* = *quot*
and *S* = *KRP* (*quot* (*Var xx*)) (*Var xx*) (*Var yy*)
and *P* = *PfP* (*Var xx*)
⟨*proof*⟩

theorem *Goedel_II*: ¬ {} ⊢ *Fls* ⟹ ¬ {} ⊢ *neg* (*PfP* ≪*Fls*≫)
⟨*proof*⟩

lemma *ground_fm_PrfP*[*simp*]:
ground_fm (*PrfP s k t*) ⟷ *ground s* ∧ *ground k* ∧ *ground t*
⟨*proof*⟩

lemma *Fvars_HPair*[*simp*]: *Fvars* (*HPair t u*) = *Fvars t* ∪ *Fvars u*
⟨*proof*⟩

lemma *ground_HPair*[*simp*]: *ground* (*HPair t u*) ⟷ *ground t* ∧ *ground u*
⟨*proof*⟩

interpretation *dwfd*: *Deduct2_with_False_Disj* **where**
var = *UNIV* :: *name set*
and *trm* = *UNIV* :: *tm set*
and *fmla* = *UNIV* :: *fm set*
and *num* = {*t*. *ground t*}
and *Var* = *Var*
and *FvarsT* = *Fvars*
and *substT* = λ*t u x*. *subst x u t*
and *Fvars* = *Fvars*
and *subst* = λ*A u x*. *subst_fm A x u*
and *eql* = (*EQ*)
and *cnj* = (*AND*)
and *dsj* = (*OR*)
and *imp* = (*IMP*)
and *all* = *All*
and *exi* = *Ex*
and *fls* = *Fls*
and *prv* = (⊢) {}
and *bprv* = (⊢) {}
⟨*proof*⟩

interpretation *Minimal_Truth_Soundness* **where**
$var = UNIV :: name\ set$
and $trm = UNIV :: tm\ set$
and $fmla = UNIV :: fm\ set$
and $num = \{t.\ ground\ t\}$
and $Var = Var$
and $FvarsT = Fvars$
and $substT = \lambda t\ u\ x.\ subst\ x\ u\ t$
and $Fvars = Fvars$
and $subst = \lambda A\ u\ x.\ subst\_fm\ A\ x\ u$
and $eql = (EQ)$
and $cnj = (AND)$
and $dsj = (OR)$
and $imp = (IMP)$
and $all = All$
and $exi = Ex$
and $fls = Fls$
and $prv = (\vdash)\ \{\}$
and $isTrue = eval\_fm\ e0$
$\langle proof\rangle$

lemma *neg_Neg*:
$\{\} \vdash neg\ \varphi\ IFF\ Neg\ \varphi$
$\langle proof\rangle$

lemma *ground_aux_mono*: $A \subseteq B \Longrightarrow ground\_aux\ t\ A \Longrightarrow ground\_aux\ t\ B$
$\langle proof\rangle$

interpretation *g1*: *Goedel_Form_Minimal_Truth_Soundness_HBL1iff_prv_Compl_Pf_Classical* **where**
$var = UNIV :: name\ set$
and $trm = UNIV :: tm\ set$
and $fmla = UNIV :: fm\ set$
and $num = \{t.\ ground\ t\}$
and $Var = Var$
and $FvarsT = Fvars$
and $substT = \lambda t\ u\ x.\ subst\ x\ u\ t$
and $Fvars = Fvars$
and $subst = \lambda A\ u\ x.\ subst\_fm\ A\ x\ u$
and $eql = (EQ)$
and $cnj = (AND)$
and $dsj = (OR)$
and $imp = (IMP)$
and $all = All$
and $exi = Ex$
and $fls = Fls$
and $prv = (\vdash)\ \{\}$
and $bprv = (\vdash)\ \{\}$
and $enc = quot$
and $S = KRP\ (quot\ (Var\ xx))\ (Var\ xx)\ (Var\ yy)$
and $P = PfP\ (Var\ xx)$
and $isTrue = eval\_fm\ e0$
and $Pf = Ex\ xx'\ (Ex\ yy'\ (Var\ yy\ EQ\ HPair\ (Var\ xx')\ (Var\ yy')\ AND\ PrfP\ (Var\ xx')\ (Var\ yy')\ (Var\ xx)))$
$\langle proof\rangle$

theorem *Goedel_I*: $\exists\varphi.\ \neg\ \{\} \vdash \varphi \wedge \neg\ \{\} \vdash Neg\ \varphi \wedge eval\_fm\ e0\ \varphi$

⟨*proof*⟩

The following interpretation is redundant, because *Goedel_Form_Minimal_Truth_Soundness_HBL1iff_prv_Compl_Pf_Cl*
(interpreted above) is a sublocale of *Goedel_Form_Classical_HBL1_rev_prv_Minimal_Truth_Soundness_TIP*.
However, the latter requires less infrastructure (no Pf formula).

The definition of isTrue prevents Isabelle from noticing that the locale has already been interpreted via the above g1 interpretation of *Goedel_Form_Minimal_Truth_Soundness_HBL1iff_prv_Compl_Pf_Classical*.

definition *isTrue* **where**
isTrue = eval_fm e0

interpretation *g1′*: *Goedel_Form_Classical_HBL1_rev_prv_Minimal_Truth_Soundness_TIP* **where**
var = UNIV :: name set
and *trm = UNIV :: tm set*
and *fmla = UNIV :: fm set*
and *num = {t. ground t}*
and *Var = Var*
and *FvarsT = Fvars*
and *substT = λt u x. subst x u t*
and *Fvars = Fvars*
and *subst = λA u x. subst_fm A x u*
and *eql = (EQ)*
and *cnj = (AND)*
and *dsj = (OR)*
and *imp = (IMP)*
and *all = All*
and *exi = Ex*
and *fls = Fls*
and *prv = (⊢) {}*
and *bprv = (⊢) {}*
and *enc = quot*
and *S = KRP (quot (Var xx)) (Var xx) (Var yy)*
and *P = PfP (Var xx)*
and *isTrue = isTrue*
⟨*proof*⟩

theorem *Goedel_I′*: $\exists\varphi.\ \neg\ \{\} \vdash \varphi \wedge \neg\ \{\} \vdash Neg\ \varphi \wedge isTrue\ \varphi$
⟨*proof*⟩