

Gödel's God in Isabelle/HOL

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| A1 | Either a property or its negation is positive, but not both: | $\forall\phi[P(\neg\phi) \leftrightarrow \neg P(\phi)]$ |
| A2 | A property necessarily implied by a positive property is positive: | $\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$ |
| T1 | Positive properties are possibly exemplified: | $\forall\phi[P(\phi) \rightarrow \Diamond\exists x\phi(x)]$ |
| D1 | A <i>God-like</i> being possesses all positive properties: | $G(x) \leftrightarrow \forall\phi[P(\phi) \rightarrow \phi(x)]$ |
| A3 | The property of being God-like is positive: | $P(G)$ |
| C | Possibly, God exists: | $\Diamond\exists xG(x)$ |
| A4 | Positive properties are necessarily positive: | $\forall\phi[P(\phi) \rightarrow \Box P(\phi)]$ |
| D2 | An <i>essence</i> of an individual is a property possessed by it and necessarily implying any of its properties: | $\phi \text{ ess. } x \leftrightarrow \phi(x) \wedge \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$ |
| T2 | Being God-like is an essence of any God-like being: | $\forall x[G(x) \rightarrow G \text{ ess. } x]$ |
| D3 | <i>Necessary existence</i> of an individual is the necessary exemplification of all its essences: | $NE(x) \leftrightarrow \forall\phi[\phi \text{ ess. } x \rightarrow \Box\exists y\phi(y)]$ |
| A5 | Necessary existence is a positive property: | $P(NE)$ |
| T3 | Necessarily, God exists: | $\Box\exists xG(x)$ |

Figure 1: Scott's version of Gödel's ontological argument [12].

1 Introduction

Dana Scott's version [12] (cf. Fig. 1) of Gödel's proof of God's existence [8] is formalized in quantified modal logic KB (QML KB) within the proof assistant Isabelle/HOL. QML KB is modeled as a fragment of classical higher-order logic (HOL); thus, the formalization is essentially a formalization in HOL. The employed embedding of QML KB in HOL is adapting the work of Benz Müller and Paulson [2, 1]. Note that the QML KB formalization employs quantification over individuals and quantification over sets of individuals (properties).

The gaps in Scott's proof have been automated with Sledgehammer [5], performing remote calls to the higher-order automated theorem prover LEO-II [3]. Sledgehammer suggests the Metis [9] calls, which result in proofs that are verified by Isabelle/HOL. For consistency checking, the model finder Nitpick [6] has been employed. The successful calls to Sledgehammer are deliberately kept as comments in the file for demonstration purposes (normally, they are automatically eliminated by Isabelle/HOL).

Isabelle is described in the textbook by Nipkow, Paulson, and Wenzel [10] and in tutorials available at: <http://isabelle.in.tum.de>.

1.1 Related Work

The formalization presented here is related to the THF [14] and Coq [4] formalizations at <https://github.com/FormalTheology/GoedelGod/tree/master/Formalizations/>.

An older ontological argument by Anselm was formalized in PVS by John Rushby [11].

2 An Embedding of QML KB in HOL

The types i for possible worlds and μ for individuals are introduced.

typeddecl i — the type for possible worlds
typeddecl μ — the type for individuals

Possible worlds are connected by an accessibility relation r .

consts $r :: i \Rightarrow i \Rightarrow \text{bool}$ (**infixr** $\langle r \rangle$ 70) — accessibility relation r

QML formulas are translated as HOL terms of type $i \Rightarrow \text{bool}$. This type is abbreviated as σ .

type-synonym $\sigma = (i \Rightarrow \text{bool})$

The classical connectives $\neg, \wedge, \rightarrow$, and \forall (over individuals and over sets of individuals) and \exists (over individuals) are lifted to type σ . The lifted connectives are $m\neg$, $m\wedge$, $m\rightarrow$, $m\forall$, and $m\exists$ (the latter two are modeled as constant symbols). Other connectives can be introduced analogously. We exemplarily do this for $m\vee$, $m\equiv$, and $mL=$ (Leibniz equality on individuals). Moreover, the modal operators \Box and \Diamond are introduced. Definitions could be used instead of abbreviations.

abbreviation $mnot :: \sigma \Rightarrow \sigma$ (**infixr** $\langle m\neg \rangle$) **where** $m\neg \varphi \equiv (\lambda w. \neg \varphi w)$
abbreviation $mand :: \sigma \Rightarrow \sigma \Rightarrow \sigma$ (**infixr** $\langle m\wedge \rangle$ 65) **where** $\varphi m\wedge \psi \equiv (\lambda w. \varphi w \wedge \psi w)$
abbreviation $mor :: \sigma \Rightarrow \sigma \Rightarrow \sigma$ (**infixr** $\langle m\vee \rangle$ 70) **where** $\varphi m\vee \psi \equiv (\lambda w. \varphi w \vee \psi w)$
abbreviation $mimplies :: \sigma \Rightarrow \sigma \Rightarrow \sigma$ (**infixr** $\langle m\rightarrow \rangle$ 74) **where** $\varphi m\rightarrow \psi \equiv (\lambda w. \varphi w \longrightarrow \psi w)$
abbreviation $mequiv :: \sigma \Rightarrow \sigma \Rightarrow \sigma$ (**infixr** $\langle m\equiv \rangle$ 76) **where** $\varphi m\equiv \psi \equiv (\lambda w. \varphi w \longleftrightarrow \psi w)$
abbreviation $mforall :: ('a \Rightarrow \sigma) \Rightarrow \sigma$ (**infixr** $\langle m\forall \rangle$) **where** $\forall \Phi \equiv (\lambda w. \forall x. \Phi x w)$
abbreviation $mexists :: ('a \Rightarrow \sigma) \Rightarrow \sigma$ (**infixr** $\langle m\exists \rangle$) **where** $\exists \Phi \equiv (\lambda w. \exists x. \Phi x w)$
abbreviation $mLeibeq :: \mu \Rightarrow \mu \Rightarrow \sigma$ (**infixr** $\langle mL= \rangle$ 90) **where** $x mL= y \equiv \forall (\lambda \varphi. (\varphi x m\rightarrow \varphi y))$
abbreviation $mbox :: \sigma \Rightarrow \sigma$ (**infixr** $\langle m\Box \rangle$) **where** $\Box \varphi \equiv (\lambda w. \forall v. w r v \longrightarrow \varphi v)$
abbreviation $mdia :: \sigma \Rightarrow \sigma$ (**infixr** $\langle m\Diamond \rangle$) **where** $\Diamond \varphi \equiv (\lambda w. \exists v. w r v \wedge \varphi v)$

For grounding lifted formulas, the meta-predicate *valid* is introduced.

abbreviation $valid :: \sigma \Rightarrow \text{bool}$ (**infixr** $\langle [-] \rangle$) **where** $[p] \equiv \forall w. p w$

3 Gödel's Ontological Argument

Constant symbol P (Gödel's 'Positive') is declared.

consts $P :: (\mu \Rightarrow \sigma) \Rightarrow \sigma$

The meaning of P is restricted by axioms $A1(a/b)$: $\forall\phi[P(\neg\phi) \leftrightarrow \neg P(\phi)]$ (Either a property or its negation is positive, but not both.) and $A2$: $\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$ (A property necessarily implied by a positive property is positive).

axiomatization where

$A1a$: $[\forall(\lambda\Phi. P(\lambda x. m \neg (\Phi x)) m \rightarrow m \neg (P \Phi))]$ **and**
 $A1b$: $[\forall(\lambda\Phi. m \neg (P \Phi) m \rightarrow P(\lambda x. m \neg (\Phi x)))]$ **and**
 $A2$: $[\forall(\lambda\Phi. \forall(\lambda\Psi. (P \Phi m \wedge \Box(\forall(\lambda x. \Phi x m \rightarrow \Psi x))) m \rightarrow P \Psi))]$

We prove theorem $T1$: $\forall\phi[P(\phi) \rightarrow \Diamond\exists x\phi(x)]$ (Positive properties are possibly exemplified). $T1$ is proved directly by Sledgehammer with command `sledgehammer [provers = remote-leo2]`. Sledgehammer suggests to call Metis with axioms $A1a$ and $A2$. Metis successfully generates a proof object that is verified in Isabelle/HOL's kernel.

theorem $T1$: $[\forall(\lambda\Phi. P \Phi m \rightarrow \Diamond(\exists \Phi))]$
 — sledgehammer [provers = remote_leo2]
 $\langle proof \rangle$

Next, the symbol G for ‘God-like’ is introduced and defined as $G(x) \leftrightarrow \forall\phi[P(\phi) \rightarrow \phi(x)]$ (A God-like being possesses all positive properties).

definition $G :: \mu \Rightarrow \sigma$ **where** $G = (\lambda x. \forall(\lambda\Phi. P \Phi m \rightarrow \Phi x))$

Axiom $A3$ is added: $P(G)$ (The property of being God-like is positive). Sledgehammer and Metis then prove corollary C : $\Diamond\exists xG(x)$ (Possibly, God exists).

axiomatization where $A3$: $[P G]$

corollary C : $[\Diamond(\exists G)]$
 — sledgehammer [provers = remote_leo2]
 $\langle proof \rangle$

Axiom $A4$ is added: $\forall\phi[P(\phi) \rightarrow \Box P(\phi)]$ (Positive properties are necessarily positive).

axiomatization where $A4$: $[\forall(\lambda\Phi. P \Phi m \rightarrow \Box(P \Phi))]$

Symbol ess for ‘Essence’ is introduced and defined as

$$\phi \text{ ess. } x \leftrightarrow \phi(x) \wedge \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$$

(An *essence* of an individual is a property possessed by it and necessarily implying any of its properties).

definition $ess :: (\mu \Rightarrow \sigma) \Rightarrow \mu \Rightarrow \sigma$ (**infixr** $\langle ess \rangle$ 85) **where**
 $\Phi \text{ ess } x = \Phi x m \wedge \forall(\lambda\Psi. \Psi x m \rightarrow \Box(\forall(\lambda y. \Phi y m \rightarrow \Psi y)))$

Next, Sledgehammer and Metis prove theorem $T2$: $\forall x[G(x) \rightarrow G \text{ ess. } x]$ (Being God-like is an essence of any God-like being).

theorem $T2$: $[\forall(\lambda x. G x m \rightarrow G \text{ ess } x)]$
 — sledgehammer [provers = remote_leo2]
 $\langle proof \rangle$

Symbol NE , for ‘Necessary Existence’, is introduced and defined as

$$NE(x) \leftrightarrow \forall\phi[\phi \text{ ess. } x \rightarrow \Box\exists y\phi(y)]$$

(Necessary existence of an individual is the necessary exemplification of all its essences).

definition $NE :: \mu \Rightarrow \sigma$ **where** $NE = (\lambda x. \forall (\lambda \Phi. \Phi \text{ ess } x \text{ m} \rightarrow \Box (\exists \Phi)))$

Moreover, axiom $A5$ is added: $P(NE)$ (Necessary existence is a positive property).

axiomatization where $A5$: $[P\ NE]$

The B axiom (symmetry) for relation r is stated. B is needed only for proving theorem $T3$ and for corollary $C2$.

axiomatization where sym : $x\ r\ y \longrightarrow y\ r\ x$

Finally, Sledgehammer and Metis prove the main theorem $T3$: $\Box \exists x G(x)$
(Necessarily, God exists).

theorem $T3$: $[\Box (\exists\ G)]$
— sledgehammer [provers = remote_leo2]
 $\langle proof \rangle$

Surprisingly, the following corollary can be derived even without the T axiom (reflexivity).

corollary $C2$: $[\exists\ G]$
— sledgehammer [provers = remote_leo2]
 $\langle proof \rangle$

The consistency of the entire theory is confirmed by Nitpick.

lemma *True nitpick* $[satisfy, \text{user-axioms}, \text{expect} = \text{genuine}] \langle proof \rangle$

4 Additional Results on Gödel's God.

Gödel's God is flawless: (s)he does not have non-positive properties.

theorem *Flawlessness*: $[\forall (\lambda \Phi. \forall (\lambda x. (G\ x\ \text{m} \rightarrow (m \neg (P\ \Phi)\ \text{m} \rightarrow m \neg (\Phi\ x)))))]$
— sledgehammer [provers = remote_leo2]
 $\langle proof \rangle$

There is only one God: any two God-like beings are equal.

theorem *Monotheism*: $[\forall (\lambda x. \forall (\lambda y. (G\ x\ \text{m} \rightarrow (G\ y\ \text{m} \rightarrow (x\ mL = y)))))]$
— sledgehammer [provers = remote_leo2]
 $\langle proof \rangle$

5 Modal Collapse

Gödel's axioms have been criticized for entailing the so-called modal collapse. The prover Satallax [7] confirms this. However, sledgehammer is not able to determine which axioms, definitions and previous theorems are used by Satallax; hence it suggests to call Metis using everything, but this (unsurprisingly) fails. Attempting to use 'Sledgehammer min' to minimize Sledgehammer's suggestion does not work. Calling Metis with $T2$, $T3$ and *ess-def* also does not work.

lemma MC : $[\forall (\lambda \Phi. (\Phi\ \text{m} \rightarrow (\Box\ \Phi)))]$
— sledgehammer [provers = remote_satallax]
— by (metis $T2\ T3\ \text{ess_def}$)
 $\langle proof \rangle$

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