

# Gödel's God in Isabelle/HOL

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A1	Either a property or its negation is positive, but not both:	$\forall\phi[P(\neg\phi) \leftrightarrow \neg P(\phi)]$
A2	A property necessarily implied by a positive property is positive:	$\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$
T1	Positive properties are possibly exemplified:	$\forall\phi[P(\phi) \rightarrow \Diamond\exists x\phi(x)]$
D1	A <i>God-like</i> being possesses all positive properties:	$G(x) \leftrightarrow \forall\phi[P(\phi) \rightarrow \phi(x)]$
A3	The property of being God-like is positive:	$P(G)$
C	Possibly, God exists:	$\Diamond\exists xG(x)$
A4	Positive properties are necessarily positive:	$\forall\phi[P(\phi) \rightarrow \Box P(\phi)]$
D2	An <i>essence</i> of an individual is a property possessed by it and necessarily implying any of its properties:	$\phi \text{ ess. } x \leftrightarrow \phi(x) \wedge \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$
T2	Being God-like is an essence of any God-like being:	$\forall x[G(x) \rightarrow G \text{ ess. } x]$
D3	<i>Necessary existence</i> of an individual is the necessary exemplification of all its essences:	$NE(x) \leftrightarrow \forall\phi[\phi \text{ ess. } x \rightarrow \Box\exists y\phi(y)]$
A5	Necessary existence is a positive property:	$P(NE)$
T3	Necessarily, God exists:	$\Box\exists xG(x)$

Figure 1: Scott's version of Gödel's ontological argument [12].

## 1 Introduction

Dana Scott's version [12] (cf. Fig. 1) of Gödel's proof of God's existence [8] is formalized in quantified modal logic KB (QML KB) within the proof assistant Isabelle/HOL. QML KB is modeled as a fragment of classical higher-order logic (HOL); thus, the formalization is essentially a formalization in HOL. The employed embedding of QML KB in HOL is adapting the work of Benzmüller and Paulson [2, 1]. Note that the QML KB formalization employs quantification over individuals and quantification over sets of individuals (properties).

The gaps in Scott's proof have been automated with Sledgehammer [5], performing remote calls to the higher-order automated theorem prover LEO-II [3]. Sledgehammer suggests the Metis [9] calls, which result in proofs that are verified by Isabelle/HOL. For consistency checking, the model finder Nitpick [6] has been employed. The successful calls to Sledgehammer are deliberately kept as comments in the file for demonstration purposes (normally, they are automatically eliminated by Isabelle/HOL).

Isabelle is described in the textbook by Nipkow, Paulson, and Wenzel [10] and in tutorials available at: <http://isabelle.in.tum.de>.

## 1.1 Related Work

The formalization presented here is related to the THF [14] and Coq [4] formalizations at <https://github.com/FormalTheology/GoedelGod/tree/master/Formalizations/>.

An older ontological argument by Anselm was formalized in PVS by John Rushby [11].

## 2 An Embedding of QML KB in HOL

The types  $i$  for possible worlds and  $\mu$  for individuals are introduced.

**typedecl**  $i$  — the type for possible worlds  
**typedecl**  $\mu$  — the type for individuals

Possible worlds are connected by an accessibility relation  $r$ .

**consts**  $r :: i \Rightarrow i \Rightarrow bool$  (**infix**  $r$  70) — accessibility relation  $r$

QML formulas are translated as HOL terms of type  $i \Rightarrow bool$ . This type is abbreviated as  $\sigma$ .

**type-synonym**  $\sigma = (i \Rightarrow bool)$

The classical connectives  $\neg, \wedge, \rightarrow$ , and  $\forall$  (over individuals and over sets of individuals) and  $\exists$  (over individuals) are lifted to type  $\sigma$ . The lifted connectives are  $m\neg, m\wedge, m\rightarrow, \forall$ , and  $\exists$  (the latter two are modeled as constant symbols). Other connectives can be introduced analogously. We exemplarily do this for  $m\vee, m\equiv$ , and  $mL=$  (Leibniz equality on individuals). Moreover, the modal operators  $\Box$  and  $\Diamond$  are introduced. Definitions could be used instead of abbreviations.

**abbreviation**  $mnot :: \sigma \Rightarrow \sigma$  (**infix**  $m\neg$  65) **where**  $m\neg \varphi \equiv (\lambda w. \neg \varphi w)$   
**abbreviation**  $mand :: \sigma \Rightarrow \sigma \Rightarrow \sigma$  (**infix**  $m\wedge$  65) **where**  $m\wedge \varphi \psi \equiv (\lambda w. \varphi w \wedge \psi w)$   
**abbreviation**  $mor :: \sigma \Rightarrow \sigma \Rightarrow \sigma$  (**infix**  $m\vee$  70) **where**  $m\vee \varphi \psi \equiv (\lambda w. \varphi w \vee \psi w)$   
**abbreviation**  $mimplies :: \sigma \Rightarrow \sigma \Rightarrow \sigma$  (**infix**  $m\rightarrow$  74) **where**  $m\rightarrow \varphi \psi \equiv (\lambda w. \varphi w \longrightarrow \psi w)$   
**abbreviation**  $mequiv :: \sigma \Rightarrow \sigma \Rightarrow \sigma$  (**infix**  $m\equiv$  76) **where**  $m\equiv \varphi \psi \equiv (\lambda w. \varphi w \longleftrightarrow \psi w)$   
**abbreviation**  $mforall :: ('a \Rightarrow \sigma) \Rightarrow \sigma$  ( $\forall$ ) **where**  $\forall \Phi \equiv (\lambda w. \forall x. \Phi x w)$   
**abbreviation**  $mexists :: ('a \Rightarrow \sigma) \Rightarrow \sigma$  ( $\exists$ ) **where**  $\exists \Phi \equiv (\lambda w. \exists x. \Phi x w)$   
**abbreviation**  $mLeibeq :: \mu \Rightarrow \mu \Rightarrow \sigma$  (**infix**  $mL=$  90) **where**  $x mL= y \equiv \forall (\lambda \varphi. (\varphi x m\rightarrow \varphi y))$   
**abbreviation**  $mbox :: \sigma \Rightarrow \sigma$  ( $\Box$ ) **where**  $\Box \varphi \equiv (\lambda w. \forall v. w r v \longrightarrow \varphi v)$   
**abbreviation**  $mdia :: \sigma \Rightarrow \sigma$  ( $\Diamond$ ) **where**  $\Diamond \varphi \equiv (\lambda w. \exists v. w r v \wedge \varphi v)$

For grounding lifted formulas, the meta-predicate *valid* is introduced.

**abbreviation**  $valid :: \sigma \Rightarrow bool$  ( $[-]$ ) **where**  $[p] \equiv \forall w. p w$

## 3 Gödel's Ontological Argument

Constant symbol  $P$  (Gödel's 'Positive') is declared.

**consts**  $P :: (\mu \Rightarrow \sigma) \Rightarrow \sigma$

The meaning of  $P$  is restricted by axioms  $A1(a/b)$ :  $\forall\phi[P(\neg\phi) \leftrightarrow \neg P(\phi)]$  (Either a property or its negation is positive, but not both.) and  $A2$ :  $\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$  (A property necessarily implied by a positive property is positive).

**axiomatization where**

$A1a$ :  $[\forall(\lambda\Phi. P(\lambda x. m\neg(\Phi x)) m\rightarrow m\neg(P\Phi))]$  **and**  
 $A1b$ :  $[\forall(\lambda\Phi. m\neg(P\Phi) m\rightarrow P(\lambda x. m\neg(\Phi x)))]$  **and**  
 $A2$ :  $[\forall(\lambda\Phi. \forall(\lambda\Psi. (P\Phi m\wedge \Box(\forall(\lambda x. \Phi x m\rightarrow \Psi x))) m\rightarrow P\Psi)]]$

We prove theorem  $T1$ :  $\forall\phi[P(\phi) \rightarrow \Diamond\exists x\phi(x)]$  (Positive properties are possibly exemplified).  $T1$  is proved directly by Sledgehammer with command `sledgehammer [provers = remote-leo2]`. Sledgehammer suggests to call Metis with axioms  $A1a$  and  $A2$ . Metis successfully generates a proof object that is verified in Isabelle/HOL's kernel.

**theorem  $T1$** :  $[\forall(\lambda\Phi. P\Phi m\rightarrow \Diamond(\exists\Phi))]$   
— sledgehammer [provers = remote\_leo2]  
**by** (*metis A1a A2*)

Next, the symbol  $G$  for ‘God-like’ is introduced and defined as  $G(x) \leftrightarrow \forall\phi[P(\phi) \rightarrow \phi(x)]$  (A God-like being possesses all positive properties).

**definition  $G$**  ::  $\mu \Rightarrow \sigma$  **where**  $G = (\lambda x. \forall(\lambda\Phi. P\Phi m\rightarrow \Phi x))$

Axiom  $A3$  is added:  $P(G)$  (The property of being God-like is positive). Sledgehammer and Metis then prove corollary  $C$ :  $\Diamond\exists xG(x)$  (Possibly, God exists).

**axiomatization where  $A3$** :  $[P G]$

**corollary  $C$** :  $[\Diamond(\exists G)]$   
— sledgehammer [provers = remote\_leo2]  
**by** (*metis A3 T1*)

Axiom  $A4$  is added:  $\forall\phi[P(\phi) \rightarrow \Box P(\phi)]$  (Positive properties are necessarily positive).

**axiomatization where  $A4$** :  $[\forall(\lambda\Phi. P\Phi m\rightarrow \Box(P\Phi))]$

Symbol *ess* for ‘Essence’ is introduced and defined as

$$\phi \text{ ess. } x \leftrightarrow \phi(x) \wedge \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$$

(An *essence* of an individual is a property possessed by it and necessarily implying any of its properties).

**definition *ess*** ::  $(\mu \Rightarrow \sigma) \Rightarrow \mu \Rightarrow \sigma$  (**infixr *ess* 85**) **where**  
 $\Phi \text{ ess } x = \Phi x m\wedge \forall(\lambda\Psi. \Psi x m\rightarrow \Box(\forall(\lambda y. \Phi y m\rightarrow \Psi y)))$

Next, Sledgehammer and Metis prove theorem  $T2$ :  $\forall x[G(x) \rightarrow G \text{ ess. } x]$  (Being God-like is an essence of any God-like being).

**theorem  $T2$** :  $[\forall(\lambda x. G x m\rightarrow G \text{ ess } x)]$   
— sledgehammer [provers = remote\_leo2]  
**by** (*metis A1b A4 G-def ess-def*)

Symbol  $NE$ , for ‘Necessary Existence’, is introduced and defined as

$$NE(x) \leftrightarrow \forall\phi[\phi \text{ ess. } x \rightarrow \Box\exists y\phi(y)]$$

(Necessary existence of an individual is the necessary exemplification of all its essences).

**definition**  $NE :: \mu \Rightarrow \sigma$  **where**  $NE = (\lambda x. \forall (\lambda \Phi. \Phi \text{ ess } x \text{ m} \rightarrow \Box (\exists \Phi)))$

Moreover, axiom  $A5$  is added:  $P(NE)$  (Necessary existence is a positive property).

**axiomatization where**  $A5: [P\ NE]$

The  $B$  axiom (symmetry) for relation  $r$  is stated.  $B$  is needed only for proving theorem  $T3$  and for corollary  $C2$ .

**axiomatization where**  $sym: x\ r\ y \longrightarrow y\ r\ x$

Finally, Sledgehammer and Metis prove the main theorem  $T3: \Box \exists x G(x)$  (Necessarily, God exists).

**theorem**  $T3: [\Box (\exists G)]$   
 — sledgehammer [provers = remote\_leo2]  
**by** (*metis A5 C T2 sym G-def NE-def*)

Surprisingly, the following corollary can be derived even without the  $T$  axiom (reflexivity).

**corollary**  $C2: [\exists G]$   
 — sledgehammer [provers = remote\_leo2]  
**by** (*metis T1 T3 G-def sym*)

The consistency of the entire theory is confirmed by Nitpick.

**lemma** *True nitpick* [*satisfy, user-axioms, expect = genuine*] **oops**

## 4 Additional Results on Gödel’s God.

Gödel’s God is flawless: (s)he does not have non-positive properties.

**theorem** *Flawlessness*:  $[\forall (\lambda \Phi. \forall (\lambda x. (G\ x\ \text{m} \rightarrow (m \neg (P\ \Phi)\ \text{m} \rightarrow m \neg (\Phi\ x)))))]$   
 — sledgehammer [provers = remote\_leo2]  
**by** (*metis A1b G-def*)

There is only one God: any two God-like beings are equal.

**theorem** *Monotheism*:  $[\forall (\lambda x. \forall (\lambda y. (G\ x\ \text{m} \rightarrow (G\ y\ \text{m} \rightarrow (x\ \text{m}L= y)))))]$   
 — sledgehammer [provers = remote\_leo2]  
**by** (*metis Flawlessness G-def*)

## 5 Modal Collapse

Gödel’s axioms have been criticized for entailing the so-called modal collapse. The prover Satallax [7] confirms this. However, sledgehammer is not able to determine which axioms, definitions and previous theorems are used by Satallax; hence it suggests to call Metis using everything, but this (unsurprisingly) fails. Attempting to use ‘Sledgehammer min’ to minimize Sledgehammer’s suggestion does not work. Calling Metis with  $T2$ ,  $T3$  and *ess-def* also does not work.

**lemma** *MC*:  $[\forall (\lambda \Phi. (\Phi\ \text{m} \rightarrow (\Box \Phi)))]$   
 — sledgehammer [provers = remote\_satallax]  
 — by (*metis T2 T3 ess\_def*)  
**oops**

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