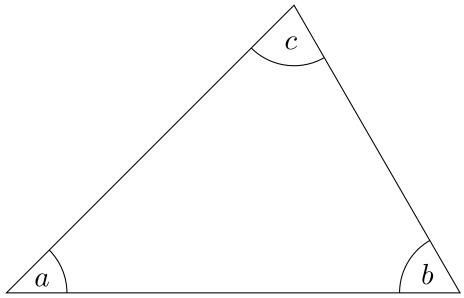

The General Triangle Is Unique

Joachim Breitner

April 1, 2011

Some acute-angled triangles are special, e.g. right-angled or isosceles triangles. Some are not of this kind, but, without measuring angles, look as if they are. In that sense, there is exactly one general triangle. This well-known fact[1] is proven here formally.

theory *GeneralTriangle*
imports *Complex-Main*
begin

1 Type definitions

Since we are only considering acute-angled triangles, we define *angles* as numbers from the real interval $[0 \ldots 90]$.

abbreviation *angles* ≡ { *x*::*real* . *0* ≤ *x* ∧ *x* ≤ *90* }

Triangles are represented as lists consisting of exactly three angles which add up to 180°. As we consider triangles up to similarity, we assume the angles to be given in ascending order.

Isabelle expects us to prove that the type is not empty, which we do by an example.

definition
  triangle =
    { *l* . *l* ∈ *lists angles* ∧
          length l = *3* ∧
          sum-list l = *180* ∧

sorted l
}

typedef *triangle* = *triangle*
⟨*proof*⟩

For convenience, the following lemma gives us easy access to the three angles of a triangle and their properties.

lemma *unfold-triangle*:
obtains *a b c*
where *Rep-triangle t* = [*a*,*b*,*c*]
and $a \in$ *angles*
and $b \in$ *angles*
and $c \in$ *angles*
and $a + b + c = 180$
and $a \leq b$
and $b \leq c$
⟨*proof*⟩

2 Property definitions

Two angles can be considered too similar if they differ by less than 15°. This number is obtained heuristically by a field experiment with an 11th grade class and was chosen that statistically, 99% will consider the angles as different.

definition *similar-angle* :: *real* ⇒ *real* ⇒ *bool* (**infix** ‹∼› *50*)
where *similar-angle x y* = (*abs* (*x* − *y*) < *15*)

The usual definitions of right-angled and isosceles, using the just introduced similarity for comparison of angles.

definition *right-angled*
where *right-angled l* = (∃ *x* ∈ *set* (*Rep-triangle l*). *x* ∼ *90*)

definition *isosceles*
where *isosceles l* = ((*Rep-triangle l*) ! *0* ∼ (*Rep-triangle l*) ! *1* ∨
(*Rep-triangle l*) ! *1* ∼ (*Rep-triangle l*) ! (*Suc 1*))

A triangle is special if it is isosceles or right-angled, and general if not. Equilateral triangle are isosceles and thus not mentioned on their own here.

definition *special*
where *special t* = (*isosceles t* ∨ *right-angled t*)

definition *general*
where *general t* = (¬ *special t*)

3 The Theorem

theorem ∃! *t*. *general t*

The proof proceeds in two steps: There is a general triangle, and it is unique. For the first step we give the triangle (angles 45°, 60°and 75°), show that it is a triangle and that it is general.

⟨*proof*⟩

end

References

[1] B. Tergan. Das allgemeine Dreieck. *Praxis der Mathematik*, 23:48–51, 1981.