



# The General Triangle Is Unique

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Some acute-angled triangles are special, e.g. right-angled or isosceles triangles. Some are not of this kind, but, without measuring angles, look as if they are. In that sense, there is exactly one general triangle. This well-known fact[1] is proven here formally.

```
theory GeneralTriangle
imports Complex-Main
begin
```

## 1 Type definitions

Since we are only considering acute-angled triangles, we define *angles* as numbers from the real interval  $[0 \dots 90]$ .

**abbreviation**  $angles \equiv \{ x::real . 0 \leq x \wedge x \leq 90 \}$

Triangles are represented as lists consisting of exactly three angles which add up to  $180^\circ$ . As we consider triangles up to similarity, we assume the angles to be given in ascending order.

Isabelle expects us to prove that the type is not empty, which we do by an example.

**definition**

```
triangle =
{ l . l ∈ lists angles ∧
  length l = 3 ∧
  sum-list l = 180 ∧
```

```

    sorted l
  }

```

```

typedef triangle = triangle
  unfolding triangle-def
  apply (rule-tac x = [45,45,90] in exI)
  apply auto
  done

```

For convenience, the following lemma gives us easy access to the three angles of a triangle and their properties.

**lemma** *unfold-triangle*:

```

obtains a b c
where Rep-triangle t = [a,b,c]
  and a ∈ angles
  and b ∈ angles
  and c ∈ angles
  and a + b + c = 180
  and a ≤ b
  and b ≤ c

```

**proof**–

```

obtain a b c where
  a = Rep-triangle t ! 0 and b = Rep-triangle t ! 1 and c = Rep-triangle t ! 2
  using Rep-triangle[of t]
  by (auto simp add:triangle-def)
hence Rep-triangle t = [a,b,c]
  using Rep-triangle[of t]
  apply (auto simp add:triangle-def)
  apply (cases Rep-triangle t, auto)
  apply (case-tac list, auto)
  apply (case-tac lista, auto)
  done
with that show thesis
  using Rep-triangle[of t]
  by (auto simp add:triangle-def)

```

**qed**

## 2 Property definitions

Two angles can be considered too similar if they differ by less than  $15^\circ$ . This number is obtained heuristically by a field experiment with an 11th grade class and was chosen that statistically, 99% will consider the angles as different.

```

definition similar-angle :: real ⇒ real ⇒ bool (infix ~ 50)
  where similar-angle x y = (abs (x - y) < 15)

```

The usual definitions of right-angled and isosceles, using the just introduced similarity for comparison of angles.

```

definition right-angled

```

**where** *right-angled*  $l = (\exists x \in \text{set } (\text{Rep-triangle } l). x \sim 90)$

**definition** *isosceles*

**where** *isosceles*  $l = ((\text{Rep-triangle } l) ! 0 \sim (\text{Rep-triangle } l) ! 1 \vee$   
 $(\text{Rep-triangle } l) ! 1 \sim (\text{Rep-triangle } l) ! (\text{Suc } 1))$

A triangle is special if it is isosceles or right-angled, and general if not. Equilateral triangle are isosceles and thus not mentioned on their own here.

**definition** *special*

**where** *special*  $t = (\text{isosceles } t \vee \text{right-angled } t)$

**definition** *general*

**where** *general*  $t = (\neg \text{special } t)$

### 3 The Theorem

**theorem**  $\exists! t. \text{general } t$

The proof proceeds in two steps: There is a general triangle, and it is unique. For the first step we give the triangle (angles  $45^\circ$ ,  $60^\circ$  and  $75^\circ$ ), show that it is a triangle and that it is general.

**proof**

**have** *is-t* [*simp*]:  $[45, 60, 75] \in \text{triangle}$  **by** (*auto simp add: triangle-def*)

**show** *general* (*Abs-triangle*  $[45, 60, 75]$ ) (**is general** ?*t*)

**by** (*auto simp add: general-def special-def isosceles-def right-angled-def*  
*Abs-triangle-inverse similar-angle-def*)

**next**

For the second step, we give names to the three angles and successively find upper bounds to them.

**fix** *t*

**obtain** *a b c* **where**

*abc*: *Rep-triangle*  $t = [a, b, c]$

**and**  $a \in \text{angles}$  **and**  $b \in \text{angles}$  **and**  $c \in \text{angles}$

**and**  $a \leq b$  **and**  $b \leq c$

**and**  $a + b + c = 180$

**by** (*rule unfold-triangle*)

**assume** *general* *t*

**hence** *ni*:  $\neg \text{isosceles } t$  **and** *nra*:  $\neg \text{right-angled } t$

**by** (*auto simp add: general-def special-def*)

**have**  $\neg c \sim 90$  **using** *nra abc*

**by** (*auto simp add: right-angled-def*)

**hence**  $c \leq 75$  **using**  $\langle c \in \text{angles} \rangle$

**by** (*auto simp add: similar-angle-def*)

**have**  $\neg b \sim c$  **using** *ni abc*

**by** (*auto simp add: isosceles-def*)

**hence**  $b \leq 60$  **using**  $\langle b \leq c \rangle$  **and**  $\langle c \leq 75 \rangle$

```

by (auto simp add:similar-angle-def)

have  $\neg a \sim b$  using ni abc
  by (auto simp add:isosceles-def)
hence  $a \leq 45$  using  $\langle a \leq b \rangle$  and  $\langle b \leq 60 \rangle$ 
  by (auto simp add:similar-angle-def)

The upper bound is actually the value, or we would not have a triangle
have  $\neg (c < 75 \vee b < 60 \vee a < 45)$ 
proof
  assume  $c < 75 \vee b < 60 \vee a < 45$ 
  hence  $a + b + c < 180$  using  $\langle c \leq 75 \rangle \langle b \leq 60 \rangle \langle a \leq 45 \rangle$ 
    and  $\langle a \in \text{angles} \rangle \langle b \in \text{angles} \rangle \langle c \in \text{angles} \rangle$ 
    by auto
  thus False using  $\langle a + b + c = 180 \rangle$  by auto
qed
hence  $c = 75$  and  $b = 60$  and  $a = 45$ 
  using  $\langle c \leq 75 \rangle \langle b \leq 60 \rangle \langle a \leq 45 \rangle$ 
  by auto

And this concludes the proof.

hence  $\text{Abs-triangle} (\text{Rep-triangle } t) = \text{Abs-triangle } [45, 60, 75]$ 
  using abc by simp
thus  $t = \text{Abs-triangle } [45, 60, 75]$  by (simp add: Rep-triangle-inverse)
qed

end

```

## References

- [1] B. Tergan. Das allgemeine Dreieck. *Praxis der Mathematik*, 23:48–51, 1981.