Gauss-Jordan Elimination for Matrices Represented as Functions

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Abstract

This theory provides a compact formulation of Gauss-Jordan elimination for matrices represented as functions. Its distinctive feature is succinctness. It is not meant for large computations.

1 Gauss-Jordan elimination algorithm

theory *Gauss-Jordan-Elim-Fun*
imports
HOL−Combinatorics.Transposition
begin

Matrices are functions:

type-synonym *'a matrix* = *nat* ⇒ *nat* ⇒ *'a*

In order to restrict to finite matrices, a matrix is usually combined with one or two natural numbers indicating the maximal row and column of the matrix.

Gauss-Jordan elimination is parameterized with a natural number n. It indicates that the matrix A has n rows and columns. In fact, A is the augmented matrix with $n+1$ columns. Column n is the "right-hand side", i.e. the constant vector b. The result is the unit matrix augmented with the solution in column n; see the correctness theorem below.

```
fun gauss-jordan :: ('a::field)matrix ⇒ nat ⇒ ('a)matrix option where
gauss-jordan A 0 = Some(A) |
gauss-jordan A (Suc m) =
 (case dropWhile (λi. A i m = 0) [0..<Suc m] of
   [] ⇒ None |
   p # - ⇒
    (let Ap' = (λj. A p j / A p m);
         A' = (λi. if i=p then Ap' else (λj. A i j − A i m ∗ Ap' j))
     in gauss-jordan (Fun.swap p m A') m))
```

Some auxiliary functions:

definition *solution* :: ('*a*::*field*)*matrix* ⇒ *nat* ⇒ (*nat* ⇒ '*a*) ⇒ *bool* **where**
solution A n x = (∀ *i*<*n*. (∑ *j*=*0*..<*n*. *A i j* ∗ *x j*) = *A i n*)

definition *unit* :: ('*a*::*field*)*matrix* ⇒ *nat* ⇒ *nat* ⇒ *bool* **where**
unit A m n =
(∀ *i j*::*nat*. *m*≤*j* ⟶ *j*<*n* ⟶ *A i j* = (*if i*=*j then 1 else 0*))

lemma *solution-swap*:
assumes *p1* < *n p2* < *n*
shows *solution* (*Fun.swap p1 p2 A*) *n x* = *solution A n x* (**is** *?L* = *?R*)
⟨*proof*⟩

lemma *solution-upd1*:
c ≠ *0* ⟹ *solution* (*A*(*p*:=(λ*j*. *A p j* / *c*))) *n x* = *solution A n x*
⟨*proof*⟩

lemma *solution-upd-but1*: ⟦ *ap* = *A p*; ∀ *i j*. *i*≠*p* ⟶ *a i j* = *A i j*; *p*<*n* ⟧ ⟹
solution (λ*i*. *if i*=*p then ap else* (λ*j*. *a i j* − *c i* ∗ *ap j*)) *n x* =
solution A n x
⟨*proof*⟩

1.1 Correctness

The correctness proof:

lemma *gauss-jordan-lemma*: *m*≤*n* ⟹ *unit A m n* ⟹ *gauss-jordan A m* = *Some B* ⟹
unit B 0 n ∧ *solution A n* (λ*j*. *B j n*)
⟨*proof*⟩

theorem *gauss-jordan-correct*:
gauss-jordan A n = *Some B* ⟹ *solution A n* (λ*j*. *B j n*)
⟨*proof*⟩

definition *solution2* :: ('*a*::*field*)*matrix* ⇒ *nat* ⇒ *nat* ⇒ (*nat* ⇒ '*a*) ⇒ *bool*
where *solution2 A m n x* = (∀ *i*<*m*. (∑ *j*=*0*..<*m*. *A i j* ∗ *x j*) = *A i n*)

definition *usolution A m n x* ⟷
solution2 A m n x ∧ (∀ *y*. *solution2 A m n y* ⟶ (∀ *j*<*m*. *y j* = *x j*))

lemma *non-null-if-pivot*:
assumes *usolution A m n x* **and** *q* < *m* **shows** ∃ *p*<*m*. *A p q* ≠ *0*
⟨*proof*⟩

lemma *lem1*:
fixes *f* :: '*a* ⇒ '*b*::*field*
shows (∑ *x*∈*A*. *f x* ∗ (*a* ∗ *g x*)) = *a* ∗ (∑ *x*∈*A*. *f x* ∗ *g x*)
⟨*proof*⟩

lemma *lem2*:
fixes $f :: 'a \Rightarrow 'b$::*field*
shows $(\sum x{\in}A.\ f\ x * (g\ x * a)) = a * (\sum x{\in}A.\ f\ x * g\ x)$
⟨*proof*⟩

1.2 Complete

lemma *gauss-jordan-complete*:
$m \leq n \Longrightarrow$ *usolution A m n x* $\Longrightarrow \exists B.$ *gauss-jordan A m* $=$ *Some B*
⟨*proof*⟩

Future work: extend the proof to matrix inversion.

hide-const (**open**) *unit*

end