

# Verified Efficient Implementation of Gabow's Strongly Connected Components Algorithm

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## Abstract

We present an Isabelle/HOL formalization of Gabow's algorithm for finding the strongly connected components of a directed graph. Using data refinement techniques, we extract efficient code that performs comparable to a reference implementation in Java. Our style of formalization allows for re-using large parts of the proofs when defining variants of the algorithm. We demonstrate this by verifying an algorithm for the emptiness check of generalized Büchi automata, re-using most of the existing proofs.

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# 1 Introduction

A strongly connected component (SCC) of a directed graph is a maximal subset of mutually reachable nodes. Finding the SCCs is a standard problem from graph theory with applications in many fields ([19, Chap. 4.2]).

This formalization accompanies our conference paper [11], where we describe the used formalization techniques.

There are several algorithms to partition the nodes of a graph into SCCs, the main ones being the Kosaraju-Sharir algorithm[20], Tarjan’s algorithm[21], and the class of path-based algorithms[17, 15, 3, 1, 5].

In this formalization, we present the verification of Gabow’s path-based SCC-algorithm[5] within the theorem prover Isabelle/HOL[16]. Using refinement techniques and a collection of efficient verified data structures, we extract Standard ML (SML)[14] code from the formalization. Our verified algorithm has a performance comparable to a reference implementation in Java, taken from Sedgewick and Wayne’s textbook on algorithms[19, Chap. 4.2].

Our main interest in SCC-algorithms stems from the fact that they can be used for the emptiness check of generalized Büchi automata (GBA), a problem that arises in LTL model checking[22, 6, 2]. Towards this end, we extend the algorithm to check the emptiness of generalized Büchi automata, re-using many of the proofs from the original verification.

**Contributions and Related Work** Up to our knowledge, we present the first mechanically verified SCC-algorithm, as well as the first mechanically verified SCC-based emptiness check for GBA. Path-based algorithms have already been regarded for the emptiness check of GBAs[18]. However, we are the first to use the data structure proposed by Gabow[5].<sup>1</sup> Finally, our development is a case study for using the Isabelle/HOL Monadic Refinement and Collection Frameworks[12, 13, 9, 10] to engineer a verified, efficient implementation of a quite complex algorithm, while keeping proofs modular and re-usable.

This development is part of the CAVA project[4] to produce a fully verified LTL model checker.

**Outline** The rest of this formalization is organized as follows: In Section 1.1, we define a skeleton algorithm and show preservation of some general-purpose invariants. In Section 1.5, we define and prove correct an algorithm that takes a directed graph and computes a list of SCCs in topological order. In Section 1.12 we provide a simple safety property mod-

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<sup>1</sup>Although called Gabow-based algorithm in [18], a union-find data structure is used to implement collapsing of nodes, while Gabow proposes a different data structure[5, pg. 109]

elchecker, which tries to find a path to a node violating a given property in a graph. This is used in Section 1.14, where we define an algorithm that checks the language of a given generalized Büchi graph<sup>2</sup> (GBG) for emptiness, and returns a counterexample in case of non-emptiness. In the next three sections (1.23, 1.27, 1.31,1.36) we use the Autoref Tool[10] to refine the above algorithms to efficient data structures, and extract SML code using Isabelle/HOL’s code generator[7, 8].

## 1.1 Skeleton for Gabow’s SCC Algorithm

```
theory Gabow-Skeleton
imports CAVA-Automata.Digraph
begin
```

```
locale fr-graph =
  graph G
  for G :: ('v, 'more) graph-rec-scheme
  +
  assumes finite-reachableE-V0[simp, intro!]: finite ( $E^* \text{ “ } V0$ )
```

In this theory, we formalize a skeleton of Gabow’s SCC algorithm. The skeleton serves as a starting point to develop concrete algorithms, like enumerating the SCCs or checking emptiness of a generalized Büchi automaton.

## 1.2 Statistics Setup

We define some dummy-constants that are included into the generated code, and may be mapped to side-effecting ML-code that records statistics and debug information about the execution. In the skeleton algorithm, we count the number of visited nodes, and include a timing for the whole algorithm.

```
definition stat-newnode :: unit => unit — Invoked if new node is visited
where [code]: stat-newnode ≡ λ-. ()
```

```
definition stat-start :: unit => unit — Invoked once if algorithm starts
where [code]: stat-start ≡ λ-. ()
```

```
definition stat-stop :: unit => unit — Invoked once if algorithm stops
where [code]: stat-stop ≡ λ-. ()
```

```
lemma [autoref-rules]:
  (stat-newnode,stat-newnode) ∈ unit-rel → unit-rel
  (stat-start,stat-start) ∈ unit-rel → unit-rel
  (stat-stop,stat-stop) ∈ unit-rel → unit-rel
by auto
```

---

<sup>2</sup>GBGs are generalized Büchi automata without labels.

**abbreviation**  $stat\_newnode\_nres \equiv RETURN (stat\_newnode ())$

**abbreviation**  $stat\_start\_nres \equiv RETURN (stat\_start ())$

**abbreviation**  $stat\_stop\_nres \equiv RETURN (stat\_stop ())$

**lemma**  $discard\_stat\_refine[refine]$ :

$m1 \leq m2 \implies stat\_newnode\_nres \gg m1 \leq m2$

$m1 \leq m2 \implies stat\_start\_nres \gg m1 \leq m2$

$m1 \leq m2 \implies stat\_stop\_nres \gg m1 \leq m2$

**by**  $simp\_all$

### 1.3 Abstract Algorithm

In this section, we formalize an abstract version of a path-based SCC algorithm. Later, this algorithm will be refined to use Gabow's data structure.

#### 1.3.1 Preliminaries

**definition**  $path\_seg :: 'a \text{ set list} \Rightarrow nat \Rightarrow nat \Rightarrow 'a \text{ set}$

— Set of nodes in a segment of the path

**where**  $path\_seg \ p \ i \ j \equiv \bigcup \{p!k \mid k. i \leq k \wedge k < j\}$

**lemma**  $path\_seg\_simps[simp]$ :

$j \leq i \implies path\_seg \ p \ i \ j = \{\}$

$path\_seg \ p \ i \ (Suc \ i) = p!i$

**unfolding**  $path\_seg\_def$

**apply**  $auto \ []$

**apply**  $(auto \ simp: \ le\_less\_Suc\_eq) \ []$

**done**

**lemma**  $path\_seg\_drop$ :

$\bigcup (set \ (drop \ i \ p)) = path\_seg \ p \ i \ (length \ p)$

**unfolding**  $path\_seg\_def$

**by**  $(fastforce \ simp: \ in\_set\_drop\_conv\_nth \ Bex\_def)$

**lemma**  $path\_seg\_butlast$ :

$p \neq [] \implies path\_seg \ p \ 0 \ (length \ p - Suc \ 0) = \bigcup (set \ (butlast \ p))$

**apply**  $(cases \ p \ rule: \ rev\_cases, \ simp)$

**apply**  $(fastforce \ simp: \ path\_seg\_def \ nth\_append \ in\_set\_conv\_nth)$

**done**

**definition**  $idx\_of :: 'a \text{ set list} \Rightarrow 'a \Rightarrow nat$

— Index of path segment that contains a node

**where**  $idx\_of \ p \ v \equiv THE \ i. \ i < length \ p \wedge v \in p!i$

**lemma**  $idx\_of\_props$ :

**assumes**

$p\_disjoint\_sym: \forall \ i \ j \ v. \ i < length \ p \wedge j < length \ p \wedge v \in p!i \wedge v \in p!j \longrightarrow i = j$

**assumes**  $ON\_STACK: v \in \bigcup (set \ p)$

**shows**  
 $idx\text{-of } p \ v < length \ p$  **and**  
 $v \in p \ ! \ idx\text{-of } p \ v$   
**proof** –  
**from** *ON-STACK* **obtain**  $i$  **where**  $i < length \ p \quad v \in p \ ! \ i$   
**by** (*auto simp add: in-set-conv-nth*)  
**moreover hence**  $\forall j < length \ p. v \in p \ ! \ j \longrightarrow i=j$   
**using** *p-disjoint-sym* **by** *auto*  
**ultimately show**  $idx\text{-of } p \ v < length \ p$   
**and**  $v \in p \ ! \ idx\text{-of } p \ v$  **unfolding** *idx-of-def*  
**by** (*metis (lifting) theI'*)  
**qed**

**lemma** *idx-of-uniq*:  
**assumes**  
 $p\text{-disjoint-sym}: \forall i \ j \ v. i < length \ p \wedge j < length \ p \wedge v \in p \ ! \ i \wedge v \in p \ ! \ j \longrightarrow i=j$   
**assumes**  $A: i < length \ p \quad v \in p \ ! \ i$   
**shows**  $idx\text{-of } p \ v = i$   
**proof** –  
**from**  $A$  *p-disjoint-sym* **have**  $\forall j < length \ p. v \in p \ ! \ j \longrightarrow i=j$  **by** *auto*  
**with**  $A$  **show** *?thesis*  
**unfolding** *idx-of-def*  
**by** (*metis (lifting) the-equality*)  
**qed**

### 1.3.2 Invariants

The state of the inner loop consists of the path  $p$  of collapsed nodes, the set  $D$  of finished (done) nodes, and the set  $pE$  of pending edges.

**type-synonym**  $'v \text{ abs-state} = 'v \text{ set list} \times 'v \text{ set} \times ('v \times 'v) \text{ set}$

**context** *fr-graph*

**begin**

**definition**  $touched :: 'v \text{ set list} \Rightarrow 'v \text{ set} \Rightarrow 'v \text{ set}$

– Touched: Nodes that are done or on path

**where**  $touched \ p \ D \equiv D \cup \bigcup (set \ p)$

**definition**  $vE :: 'v \text{ set list} \Rightarrow 'v \text{ set} \Rightarrow ('v \times 'v) \text{ set} \Rightarrow ('v \times 'v) \text{ set}$

– Visited edges: No longer pending edges from touched nodes

**where**  $vE \ p \ D \ pE \equiv (E \cap (touched \ p \ D \times UNIV)) - pE$

**lemma**  $vE\text{-ss-}E: vE \ p \ D \ pE \subseteq E$  – Visited edges are edges

**unfolding** *vE-def* **by** *auto*

**end**

**locale** *outer-invar-loc* – Invariant of the outer loop

= *fr-graph*  $G$  **for**  $G :: ('v, 'more) \text{ graph-rec-scheme} +$

**fixes**  $it :: 'v \text{ set}$  – Remaining nodes to iterate over

**fixes**  $D :: 'v \text{ set}$  — Finished nodes

**assumes**  $it\text{-initial}: it \subseteq V0$  — Only start nodes to iterate over

**assumes**  $it\text{-done}: V0 - it \subseteq D$  — Nodes already iterated over are visited  
**assumes**  $D\text{-reachable}: D \subseteq E^* \text{ `` } V0$  — Done nodes are reachable  
**assumes**  $D\text{-closed}: E \text{ `` } D \subseteq D$  — Done is closed under transitions

**begin**

**lemma**  $locale\text{-this}: outer\text{-invar}\text{-loc } G \text{ it } D \text{ by } unfold\text{-locales}$

**definition** (in  $fr\text{-graph}$ )  $outer\text{-invar} \equiv \lambda it D. outer\text{-invar}\text{-loc } G \text{ it } D$

**lemma**  $outer\text{-invar}\text{-this}[simp, intro!]: outer\text{-invar } it D$   
**unfolding**  $outer\text{-invar}\text{-def}$  **apply**  $simp$  **by**  $unfold\text{-locales}$

**end**

**locale**  $invar\text{-loc}$  — Invariant of the inner loop  
 $= fr\text{-graph } G$   
**for**  $G :: ('v, 'more) graph\text{-rec}\text{-scheme} +$   
**fixes**  $v0 :: 'v$   
**fixes**  $D0 :: 'v \text{ set}$   
**fixes**  $p :: 'v \text{ set list}$   
**fixes**  $D :: 'v \text{ set}$   
**fixes**  $pE :: ('v \times 'v) \text{ set}$

**assumes**  $v0\text{-initial}[simp, intro!]: v0 \in V0$   
**assumes**  $D\text{-incr}: D0 \subseteq D$

**assumes**  $pE\text{-E}\text{-from}\text{-p}: pE \subseteq E \cap (\bigcup (set p)) \times UNIV$   
— Pending edges are edges from path  
**assumes**  $E\text{-from}\text{-p}\text{-touched}: E \cap (\bigcup (set p)) \times UNIV \subseteq pE \cup UNIV \times touched$   
 $p D$   
— Edges from path are pending or touched  
**assumes**  $D\text{-reachable}: D \subseteq E^* \text{ `` } V0$  — Done nodes are reachable  
**assumes**  $p\text{-connected}: Suc i < length p \implies p!i \times p!Suc i \cap (E - pE) \neq \{\}$   
— CNodes on path are connected by non-pending edges

**assumes**  $p\text{-disjoint}: \llbracket i < j; j < length p \rrbracket \implies p!i \cap p!j = \{\}$   
— CNodes on path are disjoint  
**assumes**  $p\text{-sc}: U \in set p \implies U \times U \subseteq (vE p D pE \cap U \times U)^*$   
— Nodes in CNodes are mutually reachable by visited edges

**assumes**  $root\text{-}v0: p \neq [] \implies v0 \in hd p$  — Root CNode contains start node  
**assumes**  $p\text{-empty}\text{-}v0: p = [] \implies v0 \in D$  — Start node is done if path empty

**assumes**  $D\text{-closed}: E \text{ `` } D \subseteq D$  — Done is closed under transitions

**assumes**  $vE$ -no-back:  $\llbracket i < j; j < \text{length } p \rrbracket \implies vE \ p \ D \ pE \cap \ p!j \times \ p!i = \{\}$   
 — Visited edges do not go back on path  
**assumes**  $p$ -not- $D$ :  $\bigcup (\text{set } p) \cap D = \{\}$  — Path does not contain done nodes

**begin**

**abbreviation**  $ltouched$  **where**  $ltouched \equiv touched \ p \ D$

**abbreviation**  $lvE$  **where**  $lvE \equiv vE \ p \ D \ pE$

**lemma**  $locale$ -this:  $\text{invar-loc } G \ v0 \ D0 \ p \ D \ pE$  **by**  $\text{unfold-locales}$

**definition** (in  $\text{fr-graph}$ )

$\text{invar} \equiv \lambda v0 \ D0 \ (p, D, pE). \ \text{invar-loc } G \ v0 \ D0 \ p \ D \ pE$

**lemma**  $\text{invar-this}$ [ $\text{simp}$ ,  $\text{intro!$ ]:  $\text{invar } v0 \ D0 \ (p, D, pE)$

**unfolding**  $\text{invar-def}$  **apply**  $\text{simp}$  **by**  $\text{unfold-locales}$

**lemma**  $\text{finite-reachableE-v0}$ [ $\text{simp}$ ,  $\text{intro!$ ]:  $\text{finite } (E^* \ \{\{v0\}\})$

**apply** ( $\text{rule } \text{finite-subset}[OF \ - \ \text{finite-reachableE-V0}]$ )

**using**  $v0$ -initial **by**  $\text{auto}$

**lemma**  $D$ -vis:  $E \cap D \times UNIV \subseteq lvE$  — All edges from done nodes are visited

**unfolding**  $vE$ -def  $touched$ -def **using**  $pE$ - $E$ -from- $p$   $p$ -not- $D$  **by**  $\text{blast}$

**lemma**  $vE$ -touched:  $lvE \subseteq ltouched \times ltouched$

— Visited edges only between touched nodes

**using**  $E$ -from- $p$ -touched  $D$ -closed **unfolding**  $vE$ -def  $touched$ -def **by**  $\text{blast}$

**lemma**  $lvE$ -ss- $E$ :  $lvE \subseteq E$  — Visited edges are edges

**unfolding**  $vE$ -def **by**  $\text{auto}$

**lemma**  $\text{path-touched}$ :  $\bigcup (\text{set } p) \subseteq ltouched$  **by** ( $\text{auto } \text{simp}$ :  $\text{touched-def}$ )

**lemma**  $D$ -touched:  $D \subseteq ltouched$  **by** ( $\text{auto } \text{simp}$ :  $\text{touched-def}$ )

**lemma**  $pE$ -by- $vE$ :  $pE = (E \cap \bigcup (\text{set } p) \times UNIV) - lvE$

— Pending edges are edges from path not yet visited

**unfolding**  $vE$ -def  $touched$ -def

**using**  $pE$ - $E$ -from- $p$

**by**  $\text{auto}$

**lemma**  $\text{pick-pending}$ :  $p \neq [] \implies pE \cap \text{last } p \times UNIV = (E - lvE) \cap \text{last } p \times UNIV$

— Pending edges from end of path are non-visited edges from end of path

**apply** ( $\text{subst } pE$ -by- $vE$ )

**by**  $\text{auto}$

**lemma**  $p$ -connected':

**assumes**  $A$ :  $\text{Suc } i < \text{length } p$

**shows**  $p!i \times p!\text{Suc } i \cap lvE \neq \{\}$

**proof** —

**from**  $A$   $p$ -not- $D$  **have**  $p!i \in \text{set } p \quad p!\text{Suc } i \in \text{set } p$  **by**  $\text{auto}$

**with** *p-connected*[*OF A*] **show** *?thesis unfolding vE-def touched-def*  
**by** *blast*  
**qed**

**end**

**Termination context** *fr-graph*  
**begin**

The termination argument is based on unprocessed edges: Reachable edges from untouched nodes and pending edges.

**definition** *unproc-edges v0 p D pE*  $\equiv (E \cap (E^* \{v0\} - (D \cup \bigcup (set\ p)))) \times UNIV) \cup pE$

In each iteration of the loop, either the number of unprocessed edges decreases, or the path length decreases.

**definition** *abs-wf-rel v0*  $\equiv inv\ image\ (finite\ psubset\ <*lex*>\ measure\ length)$   
 $(\lambda(p,D,pE). (unproc\ edges\ v0\ p\ D\ pE, p))$

**lemma** *abs-wf-rel-wf[simp, intro!]*: *wf (abs-wf-rel v0)*

**unfolding** *abs-wf-rel-def*

**by** *auto*

**end**

### 1.3.3 Abstract Skeleton Algorithm

**context** *fr-graph*  
**begin**

**definition** (**in** *fr-graph*) *initial*  $:: 'v \Rightarrow 'v\ set \Rightarrow 'v\ abs\ state$   
**where** *initial v0 D*  $\equiv ([\{v0\}], D, (E \cap \{v0\} \times UNIV))$

**definition** (**in**  $-$ ) *collapse-aux*  $:: 'a\ set\ list \Rightarrow nat \Rightarrow 'a\ set\ list$   
**where** *collapse-aux p i*  $\equiv take\ i\ p\ @\ [\bigcup (set\ (drop\ i\ p))]$

**definition** (**in**  $-$ ) *collapse*  $:: 'a \Rightarrow 'a\ abs\ state \Rightarrow 'a\ abs\ state$

**where** *collapse v PDPE*  $\equiv$

*let*

$(p,D,pE) = PDPE;$

$i = idx\ of\ p\ v;$

$p = collapse\ aux\ p\ i$

*in*  $(p,D,pE)$

**definition** (**in**  $-$ )

*select-edge*  $:: 'a\ abs\ state \Rightarrow ('a\ option \times 'a\ abs\ state)\ nres$

**where**

*select-edge PDPE*  $\equiv do\ \{$

*let*  $(p,D,pE) = PDPE;$

$e \leftarrow SELECT\ (\lambda e. e \in pE \cap last\ p \times UNIV);$

```

    case e of
      None  $\Rightarrow$  RETURN (None,(p,D,pE))
    | Some (u,v)  $\Rightarrow$  RETURN (Some v, (p,D,pE - {(u,v)}))
  }

```

**definition** (in *fr-graph*) *push* :: 'v  $\Rightarrow$  'v abs-state  $\Rightarrow$  'v abs-state  
**where** *push v PDPE*  $\equiv$

```

let
  (p,D,pE) = PDPE;
  p = p@[{v}];
  pE = pE  $\cup$  (E $\cap$ {v}  $\times$  UNIV)
in
  (p,D,pE)

```

**definition** (in  $-$ ) *pop* :: 'v abs-state  $\Rightarrow$  'v abs-state

```

where pop PDPE  $\equiv$  let
  (p,D,pE) = PDPE;
  (p, V) = (butlast p, last p);
  D = V  $\cup$  D
in
  (p,D,pE)

```

The following lemmas match the definitions presented in the paper:

```

lemma select-edge (p,D,pE)  $\equiv$  do {
  e  $\leftarrow$  SELECT ( $\lambda e. e \in pE \cap$  last p  $\times$  UNIV);
  case e of
    None  $\Rightarrow$  RETURN (None,(p,D,pE))
  | Some (u,v)  $\Rightarrow$  RETURN (Some v, (p,D,pE - {(u,v)}))
}
unfolding select-edge-def by simp

```

```

lemma collapse v (p,D,pE)
 $\equiv$  let i=idx-of p v in (take i p @ [ $\bigcup$ (set (drop i p))],D,pE)
unfolding collapse-def collapse-aux-def by simp

```

```

lemma push v (p, D, pE)  $\equiv$  (p @ [{v}], D, pE  $\cup$  E  $\cap$  {v}  $\times$  UNIV)
unfolding push-def by simp

```

```

lemma pop (p, D, pE)  $\equiv$  (butlast p, last p  $\cup$  D, pE)
unfolding pop-def by auto

```

**thm** *pop-def*[*unfolded Let-def, no-vars*]

**thm** *select-edge-def*[*unfolded Let-def*]

**definition** *skeleton* :: 'v set nres  
— Abstract Skeleton Algorithm  
**where**

```

skeleton  $\equiv$  do {
  let D = {};
  r  $\leftarrow$  FOREACHi outer-invar V0 ( $\lambda v0 D0$ . do {
    if  $v0 \notin D0$  then do {
      let s = initial v0 D0;

      (p,D,pE)  $\leftarrow$  WHILEIT (invar v0 D0)
        ( $\lambda(p,D,pE)$ . p  $\neq$  []) ( $\lambda(p,D,pE)$ ).
      do {
        — Select edge from end of path
        (vo,(p,D,pE))  $\leftarrow$  select-edge (p,D,pE);

        ASSERT (p $\neq$ []);
        case vo of
          Some v  $\Rightarrow$  do { — Found outgoing edge to node v
            if v  $\in$   $\bigcup$ (set p) then do {
              — Back edge: Collapse path
              RETURN (collapse v (p,D,pE))
            } else if  $v \notin D$  then do {
              — Edge to new node. Append to path
              RETURN (push v (p,D,pE))
            } else do {
              — Edge to done node. Skip
              RETURN (p,D,pE)
            }
          }
          | None  $\Rightarrow$  do {
            ASSERT (pE  $\cap$  last p  $\times$  UNIV = {});
            — No more outgoing edges from current node on path
            RETURN (pop (p,D,pE))
          }
        }) s;
      ASSERT (p=[]  $\wedge$  pE={});
      RETURN D
    } else
      RETURN D0
  }) D;
  RETURN r
}

```

end

### 1.3.4 Invariant Preservation

context fr-graph begin

lemma set-collapse-aux[simp]:  $\bigcup$ (set (collapse-aux p i)) =  $\bigcup$ (set p)  
 apply (subst (2) append-take-drop-id[of - p,symmetric])  
 apply (simp del: append-take-drop-id)

**unfolding** *collapse-aux-def* **by** *auto*

**lemma** *touched-collapse[simp]*: *touched (collapse-aux p i) D = touched p D*  
**unfolding** *touched-def* **by** *simp*

**lemma** *vE-collapse-aux[simp]*: *vE (collapse-aux p i) D pE = vE p D pE*  
**unfolding** *vE-def* **by** *simp*

**lemma** *touched-push[simp]*: *touched (p @ [V]) D = touched p D ∪ V*  
**unfolding** *touched-def* **by** *auto*

**end**

**Corollaries of the invariant** In this section, we prove some more corollaries of the invariant, which are helpful to show invariant preservation

**context** *invar-loc*

**begin**

**lemma** *cnode-connectedI*:  
 $\llbracket i < \text{length } p; u \in p!i; v \in p!i \rrbracket \implies (u, v) \in (lvE \cap p!i \times p!i)^*$   
**using** *p-sc[of p!i]* **by** (*auto simp: in-set-conv-nth*)

**lemma** *cnode-connectedI'*:  $\llbracket i < \text{length } p; u \in p!i; v \in p!i \rrbracket \implies (u, v) \in (lvE)^*$   
**by** (*metis inf.cobounded1 rtrancl-mono-mp cnode-connectedI*)

**lemma** *p-no-empty*:  $\{\} \notin \text{set } p$

**proof**

**assume**  $\{\} \in \text{set } p$   
**then obtain** *i* **where** *IDX*:  $i < \text{length } p \quad p!i = \{\}$   
**by** (*auto simp add: in-set-conv-nth*)  
**show** *False* **proof** (*cases i*)  
**case 0** **with** *root-v0* *IDX* **show** *False* **by** (*cases p*) *auto*  
**next**  
**case** [*simp*]: (*Suc j*)  
**from** *p-connected'[of j]* *IDX* **show** *False* **by** *simp*  
**qed**

**qed**

**corollary** *p-no-empty-idx*:  $i < \text{length } p \implies p!i \neq \{\}$   
**using** *p-no-empty* **by** (*metis nth-mem*)

**lemma** *p-disjoint-sym*:  $\llbracket i < \text{length } p; j < \text{length } p; v \in p!i; v \in p!j \rrbracket \implies i = j$   
**by** (*metis disjoint-iff-not-equal linorder-neqE-nat p-disjoint*)

**lemma** *pi-ss-path-seg-eq[simp]*:  
**assumes** *A*:  $i < \text{length } p \quad u \leq \text{length } p$   
**shows**  $p!i \subseteq \text{path-seg } p \ l \ u \iff l \leq i \wedge i < u$

**proof**

**assume** *B*:  $p!i \subseteq \text{path-seg } p \ l \ u$

```

from  $A$  obtain  $x$  where  $x \in p!i$  by (blast dest: p-no-empty-idx)
with  $B$  obtain  $i'$  where  $C: x \in p!i' \quad l \leq i' \quad i' < u$ 
  by (auto simp: path-seg-def)
from  $p$ -disjoint-sym[OF  $\langle i < \text{length } p \rangle - \langle x \in p!i \rangle \langle x \in p!i' \rangle$ ]  $\langle i' < u \rangle \langle u \leq \text{length } p \rangle$ 
have  $i = i'$  by simp
with  $C$  show  $l \leq i \wedge i < u$  by auto
qed (auto simp: path-seg-def)

```

```

lemma path-seg-ss-eq[simp]:
  assumes  $A: l1 < u1 \quad u1 \leq \text{length } p \quad l2 < u2 \quad u2 \leq \text{length } p$ 
  shows  $\text{path-seg } p \ l1 \ u1 \subseteq \text{path-seg } p \ l2 \ u2 \iff l2 \leq l1 \wedge u1 \leq u2$ 
proof
  assume  $S: \text{path-seg } p \ l1 \ u1 \subseteq \text{path-seg } p \ l2 \ u2$ 
  have  $p!l1 \subseteq \text{path-seg } p \ l1 \ u1$  using  $A$  by simp
  also note  $S$  finally have  $1: l2 \leq l1$  using  $A$  by simp
  have  $p!(u1 - 1) \subseteq \text{path-seg } p \ l1 \ u1$  using  $A$  by simp
  also note  $S$  finally have  $2: u1 \leq u2$  using  $A$  by auto
  from  $1 \ 2$  show  $l2 \leq l1 \wedge u1 \leq u2$  ..
next
  assume  $l2 \leq l1 \wedge u1 \leq u2$  thus  $\text{path-seg } p \ l1 \ u1 \subseteq \text{path-seg } p \ l2 \ u2$ 
  using  $A$ 
  apply (clarsimp simp: path-seg-def) []
  apply (metis dual-order.strict-trans1 dual-order.trans)
  done

```

**qed**

```

lemma pathI:
  assumes  $x \in p!i \quad y \in p!j$ 
  assumes  $i \leq j \quad j < \text{length } p$ 
  defines  $\text{seg} \equiv \text{path-seg } p \ i \ (\text{Suc } j)$ 
  shows  $(x, y) \in (lvE \cap \text{seg} \times \text{seg})^*$ 
  — We can obtain a path between cnodes on path
  using assms(3,1,2,4) unfolding seg-def
proof (induction arbitrary: y rule: dec-induct)
  case base thus  $?case$  by (auto intro!: cnode-connectedI)
next
  case (step j)

```

```

  let  $?seg = \text{path-seg } p \ i \ (\text{Suc } j)$ 
  let  $?seg' = \text{path-seg } p \ i \ (\text{Suc } (\text{Suc } j))$ 

```

```

  have  $SSS: ?seg \subseteq ?seg'$ 
  apply (subst path-seg-ss-eq)
  using step.hyps step.premis by auto

```

```

from  $p$ -connected'[OF  $\langle \text{Suc } j < \text{length } p \rangle$ ] obtain  $u \ v$  where
   $UV: (u, v) \in lvE \quad u \in p!j \quad v \in p!\text{Suc } j$  by auto

```

```

have  $ISS: p!j \subseteq ?seg' \quad p!\text{Suc } j \subseteq ?seg'$ 

```

**using** *step.hyps step.premis by simp-all*

**from** *p-no-empty-idx*[*of j*]  $\langle \text{Suc } j < \text{length } p \rangle$  **obtain**  $x'$  **where**  $x' \in p!j$   
**by** *auto*

**with** *step.IH*[*of x'*]  $\langle x \in p!i \rangle \langle \text{Suc } j < \text{length } p \rangle$   
**have**  $t: (x, x') \in (lvE \cap ?seg \times ?seg)^*$  **by** *auto*  
**have**  $(x, x') \in (lvE \cap ?seg' \times ?seg')^*$  **using** *SSS*  
**by** (*auto intro: rtrancl-mono-mp*[*OF - t*])

**also**

**from** *cnode-connectedI*[*OF - x' \in p!j*]  $\langle u \in p!j \rangle \langle \text{Suc } j < \text{length } p \rangle$  **have**  
 $t: (x', u) \in (lvE \cap p!j \times p!j)^*$  **by** *auto*  
**have**  $(x', u) \in (lvE \cap ?seg' \times ?seg')^*$  **using** *ISS*  
**by** (*auto intro: rtrancl-mono-mp*[*OF - t*])

**also have**  $(u, v) \in lvE \cap ?seg' \times ?seg'$  **using** *UV ISS by auto*

**also from** *cnode-connectedI*[*OF x' \in p!j*]  $\langle v \in p! \text{Suc } j \rangle \langle y \in p! \text{Suc } j \rangle$   
**have**  $t: (v, y) \in (lvE \cap p! \text{Suc } j \times p! \text{Suc } j)^*$  **by** *auto*  
**have**  $(v, y) \in (lvE \cap ?seg' \times ?seg')^*$  **using** *ISS*  
**by** (*auto intro: rtrancl-mono-mp*[*OF - t*])

**finally show**  $(x, y) \in (lvE \cap ?seg' \times ?seg')^*$  .

**qed**

**lemma** *p-reachable*:  $\bigcup (\text{set } p) \subseteq E^* \{v0\}$  — Nodes on path are reachable  
**proof**

**fix**  $v$   
**assume**  $A: v \in \bigcup (\text{set } p)$   
**then obtain**  $i$  **where**  $i < \text{length } p$  **and**  $v \in p!i$   
**by** (*metis UnionE in-set-conv-nth*)  
**moreover from**  $A$  *root-v0* **have**  $v0 \in p!0$  **by** (*cases p*) *auto*  
**ultimately have**  
 $t: (v0, v) \in (lvE \cap \text{path-seg } p \ 0 \ (\text{Suc } i) \times \text{path-seg } p \ 0 \ (\text{Suc } i))^*$   
**by** (*auto intro: pathI*)  
**from** *lvE-ss-E* **have**  $(v0, v) \in E^*$  **by** (*auto intro: rtrancl-mono-mp*[*OF - t*])  
**thus**  $v \in E^* \{v0\}$  **by** *auto*

**qed**

**lemma** *touched-reachable*:  $ltouched \subseteq E^* V0$  — Touched nodes are reachable  
**unfolding** *touched-def* **using** *p-reachable D-reachable by blast*

**lemma** *vE-reachable*:  $lvE \subseteq E^* V0 \times E^* V0$   
**apply** (*rule order-trans*[*OF vE-touched*])  
**using** *touched-reachable by blast*

**lemma** *pE-reachable*:  $pE \subseteq E^* \{v0\} \times E^* \{v0\}$

**proof** *safe*

**fix**  $u \ v$   
**assume**  $E: (u, v) \in pE$   
**with** *pE-from-p p-reachable* **have**  $(v0, u) \in E^*$      $(u, v) \in E$  **by** *blast+*  
**thus**  $(v0, u) \in E^*$      $(v0, v) \in E^*$  **by** *auto*

**qed**

**lemma** *D-closed-vE-rtrancl*:  $lvE^* \text{“} D \subseteq D$   
**by** (*metis D-closed Image-closed-trancl eq-iff reachable-mono lvE-ss-E*)

**lemma** *D-closed-path*:  $\llbracket path\ E\ u\ q\ w; u \in D \rrbracket \implies set\ q \subseteq D$

**proof** –

**assume**  $a1: path\ E\ u\ q\ w$   
**assume**  $u \in D$   
**hence**  $f1: \{u\} \subseteq D$   
**using** *bot.extremum* **by** *force*  
**have**  $set\ q \subseteq E^* \text{“} \{u\}$   
**using**  $a1$  **by** (*metis insert-subset path-nodes-reachable*)  
**thus**  $set\ q \subseteq D$   
**using**  $f1$  **by** (*metis D-closed rtrancl-reachable-induct subset-trans*)

**qed**

**lemma** *D-closed-path-vE*:  $\llbracket path\ lvE\ u\ q\ w; u \in D \rrbracket \implies set\ q \subseteq D$

**by** (*metis D-closed-path path-mono lvE-ss-E*)

**lemma** *path-in-lastnode*:

**assumes**  $P: path\ lvE\ u\ q\ v$   
**assumes**  $[simp]: p \neq []$   
**assumes**  $ND: u \in last\ p \quad v \in last\ p$   
**shows**  $set\ q \subseteq last\ p$

– A path from the last Cnode to the last Cnode remains in the last Cnode

**using**  $P\ ND$

**proof** (*induction*)

**case** (*path-prepend*  $u\ v\ l\ w$ )

**from**  $\langle u, v \rangle \in lvE$  *vE-touched* **have**  $v \in ltouched$  **by** *auto*

**hence**  $v \in \bigcup (set\ p)$

**unfolding** *touched-def*

**proof**

**assume**  $v \in D$

**moreover from**  $\langle path\ lvE\ v\ l\ w \rangle$  **have**  $(v, w) \in lvE^*$  **by** (*rule path-is-rtrancl*)

**ultimately have**  $w \in D$  **using** *D-closed-vE-rtrancl* **by** *auto*

**with**  $\langle w \in last\ p \rangle$  *p-not-D* **have** *False*

**by** (*metis IntI Misc.last-in-set Sup-inf-eq-bot-iff assms(2)*)

*bex-empty path-prepend.hyps(2)*)

**thus** *?thesis ..*

**qed**

**then obtain**  $i$  **where**  $i < length\ p \quad v \in p!i$

**by** (*metis UnionE in-set-conv-nth*)

**have**  $i = length\ p - 1$

**proof** (*rule ccontr*)

**assume**  $i \neq length\ p - 1$

**with**  $\langle i < length\ p \rangle$  **have**  $i < length\ p - 1$  **by** *simp*

**with** *vE-no-back*[*of*  $i\ length\ p - 1$ ]  $\langle i < length\ p \rangle$

**have**  $lvE \cap last\ p \times p!i = \{\}$

```

    by (simp add: last-conv-nth)
  with ⟨(u,v)∈lvE⟩ ⟨u∈last p⟩ ⟨v∈p!i⟩ show False by auto
qed
with ⟨v∈p!i⟩ have v∈last p by (simp add: last-conv-nth)
with path-prepend.IH ⟨w∈last p⟩ ⟨u∈last p⟩ show ?case by auto
qed simp

```

**lemma** *loop-in-lastnode*:

**assumes**  $P$ : path lvE u q u

**assumes** [simp]:  $p \neq []$

**assumes**  $ND$ : set  $q \cap \text{last } p \neq \{\}$

**shows**  $u \in \text{last } p$  and set  $q \subseteq \text{last } p$

— A loop that touches the last node is completely inside the last node

**proof** —

**from**  $ND$  **obtain**  $v$  **where**  $v \in \text{set } q$   $v \in \text{last } p$  **by** auto

**then obtain**  $q1$   $q2$  **where** [simp]:  $q = q1 @ v \# q2$

**by** (auto simp: in-set-conv-decomp)

**from**  $P$  **have** path lvE v (v#q2@q1) v

**by** (auto simp: path-conc-conv path-cons-conv)

**from** path-in-lastnode[OF this ⟨ $p \neq []$ ⟩] ⟨ $v \in \text{last } p$ ⟩ ⟨ $v \in \text{last } p$ ⟩

**show** set  $q \subseteq \text{last } p$  **by** simp

**from**  $P$  **show**  $u \in \text{last } p$

**apply** (cases q, simp)

**apply** simp

**using** ⟨set  $q \subseteq \text{last } p$ ⟩

**apply** (auto simp: path-cons-conv)

**done**

**qed**

**lemma** *no-D-p-edges*:  $E \cap D \times \bigcup (\text{set } p) = \{\}$

**using**  $D$ -closed  $p$ -not- $D$  **by** auto

**lemma** *idx-of-props*:

**assumes**  $ON$ -STACK:  $v \in \bigcup (\text{set } p)$

**shows**

$\text{idx-of } p \ v < \text{length } p$  and

$v \in p \ ! \ \text{idx-of } p \ v$

**using**  $\text{idx-of-props}$ [OF - assms]  $p$ -disjoint-sym **by** blast+

**end**

**Auxiliary Lemmas Regarding the Operations** lemma (in *fr-graph*)

$vE$ -initial[simp]:  $vE \ [\{v0\}] \ \{\} \ (E \cap \{v0\} \times UNIV) = \{\}$

**unfolding**  $vE$ -def touched-def **by** auto

**context** *invar-loc*

**begin**

**lemma** *vE-push*:  $\llbracket (u,v) \in pE; u \in \text{last } p; v \notin \bigcup (\text{set } p); v \notin D \rrbracket$   
 $\implies vE (p @ \{\{v\}\}) D ((pE - \{(u,v)\}) \cup E \cap \{v\} \times UNIV) = \text{insert } (u,v) \text{ } lvE$   
**unfolding** *vE-def touched-def* **using** *pE-E-from-p*  
**by** *auto*

**lemma** *vE-remove[simp]*:  
 $\llbracket p \neq \llbracket; (u,v) \in pE \rrbracket \implies vE p D (pE - \{(u,v)\}) = \text{insert } (u,v) \text{ } lvE$   
**unfolding** *vE-def touched-def* **using** *pE-E-from-p* **by** *blast*

**lemma** *vE-pop[simp]*:  $p \neq \llbracket \implies vE (\text{butlast } p) (\text{last } p \cup D) pE = lvE$   
**unfolding** *vE-def touched-def*  
**by** (*cases p rule: rev-cases*) *auto*

**lemma** *pE-fin*:  $p = \llbracket \implies pE = \{\}$   
**using** *pE-by-vE* **by** *auto*

**lemma** (**in** *invar-loc*) *lastp-un-D-closed*:  
**assumes** *NE*:  $p \neq \llbracket$   
**assumes** *NO'*:  $pE \cap (\text{last } p \times UNIV) = \{\}$   
**shows**  $E''(\text{last } p \cup D) \subseteq (\text{last } p \cup D)$   
— On pop, the popped CNode and D are closed under transitions  
**proof** (*intro subsetI, elim ImageE*)  
**from** *NO'* **have** *NO*:  $(E - lvE) \cap (\text{last } p \times UNIV) = \{\}$   
**by** (*simp add: pick-pending[OF NE]*)

**let**  $?i = \text{length } p - 1$   
**from** *NE* **have** [*simp*]:  $\text{last } p = p! ?i$  **by** (*metis last-conv-nth*)

**fix**  $u v$   
**assume** *E*:  $(u,v) \in E$   
**assume** *UI*:  $u \in \text{last } p \cup D$  **hence**  $u \in p! ?i \cup D$  **by** *simp*

{  
**assume**  $u \in \text{last } p \quad v \notin \text{last } p$   
**moreover from** *E NO*  $\langle u \in \text{last } p \rangle$  **have**  $(u,v) \in lvE$  **by** *auto*  
**ultimately have**  $v \in D \vee v \in \bigcup (\text{set } p)$   
**using** *vE-touched unfolding touched-def* **by** *auto*  
**moreover** {  
**assume**  $v \in \bigcup (\text{set } p)$   
**then obtain**  $j$  **where**  $V: j < \text{length } p \quad v \in p! j$   
**by** (*metis UnionE in-set-conv-nth*)  
**with**  $\langle v \notin \text{last } p \rangle$  **have**  $j < ?i$  **by** (*cases j=?i*) *auto*  
**from** *vE-no-back[OF*  $\langle j < ?i \rangle$   $\langle (u,v) \in lvE \rangle$   $V \langle u \in \text{last } p \rangle$  **have** *False* **by** *auto*  
**} ultimately have**  $v \in D$  **by** *blast*  
**} with** *E UI D-closed* **show**  $v \in \text{last } p \cup D$  **by** *auto*  
**qed**

end

**Preservation of Invariant by Operations** context *fr-graph*

begin

lemma (in *outer-invar-loc*) *invar-initial-aux*:

assumes  $v0 \in it - D$

shows *invar*  $v0 D$  (*initial*  $v0 D$ )

unfolding *invar-def initial-def*

apply *simp*

apply *unfold-locales*

apply *simp-all*

using *assms it-initial* apply auto []

using *D-reachable it-initial assms* apply auto []

using *D-closed* apply auto []

using *assms* apply auto []

done

lemma *invar-initial*:

$\llbracket \text{outer-invar } it D0; v0 \in it; v0 \notin D0 \rrbracket \implies \text{invar } v0 D0$  (*initial*  $v0 D0$ )

unfolding *outer-invar-def*

apply (*drule outer-invar-loc.invar-initial-aux*)

by *auto*

lemma *outer-invar-initial*[*simp, intro!*]: *outer-invar*  $V0 \{\}$

unfolding *outer-invar-def*

apply *unfold-locales*

by *auto*

lemma *invar-pop*:

assumes *INV*: *invar*  $v0 D0$  ( $p, D, pE$ )

assumes *NE*[*simp*]:  $p \neq []$

assumes *NO'*:  $pE \cap (\text{last } p \times \text{UNIV}) = \{\}$

shows *invar*  $v0 D0$  (*pop* ( $p, D, pE$ ))

unfolding *invar-def pop-def*

apply *simp*

proof –

from *INV* interpret *invar-loc*  $G v0 D0 p D pE$  unfolding *invar-def* by *simp*

have [*simp*]:  $\text{set } p = \text{insert } (\text{last } p) (\text{set } (\text{butlast } p))$

using *NE* by (*cases p rule: rev-cases*) *auto*

from *p-disjoint* have *lp-dj-blp*:  $\text{last } p \cap \bigcup (\text{set } (\text{butlast } p)) = \{\}$

apply (*cases p rule: rev-cases*)

apply *simp*

apply (*fastforce simp: in-set-conv-nth nth-append*)

done

{

```

fix  $i$ 
assume  $A$ :  $Suc\ i < length\ (butlast\ p)$ 
hence  $A'$ :  $Suc\ i < length\ p$  by auto

from  $nth-butlast[of\ i\ p]\ A$  have  $[simp]$ :  $butlast\ p\ !\ i = p\ !\ i$  by auto
from  $nth-butlast[of\ Suc\ i\ p]\ A$ 
have  $[simp]$ :  $butlast\ p\ !\ Suc\ i = p\ !\ Suc\ i$  by auto

from  $p-connected[OF\ A']$ 
have  $butlast\ p\ !\ i \times butlast\ p\ !\ Suc\ i \cap (E - pE) \neq \{\}$ 
by simp
} note  $AUX-p-connected = this$ 

show  $invar-loc\ G\ v0\ D0\ (butlast\ p)\ (last\ p \cup D)\ pE$ 
apply unfold-locales

unfolding  $vE-pop[OF\ NE]$ 

apply simp

using  $D-incr$  apply auto []

using  $pE-E-from-p\ NO'$  apply auto []

using  $E-from-p-touched$  apply (auto simp: touched-def) []

using  $D-reachable\ p-reachable\ NE$  apply auto []

apply (rule AUX-p-connected, assumption+) []

using  $p-disjoint$  apply (simp add: nth-butlast)

using  $p-sc$  apply simp

using  $root-v0$  apply (cases p rule: rev-cases) apply auto [2]

using  $root-v0\ p-empty-v0$  apply (cases p rule: rev-cases) apply auto [2]

apply (rule lastp-un-D-closed, insert NO', auto) []

using  $vE-no-back$  apply (auto simp: nth-butlast) []

using  $p-not-D\ lp-dj-blp$  apply auto []
done
qed

thm  $invar-pop[of\ v-0\ D-0, no-vars]$ 

```

**lemma** *invar-collapse*:  
**assumes** *INV*: *invar v0 D0 (p,D,pE)*  
**assumes** *NE[simp]*:  $p \neq []$   
**assumes** *E*:  $(u,v) \in pE$  **and**  $u \in \text{last } p$   
**assumes** *BACK*:  $v \in \bigcup (\text{set } p)$   
**defines**  $i \equiv \text{idx-of } p \ v$   
**defines**  $p' \equiv \text{collapse-aux } p \ i$   
**shows** *invar v0 D0 (collapse v (p,D,pE - {(u,v)}))*  
**unfolding** *invar-def collapse-def*  
**apply** *simp*  
**unfolding** *i-def[symmetric] p'-def[symmetric]*  
**proof** –  
**from** *INV* **interpret** *invar-loc G v0 D0 p D pE* **unfolding** *invar-def* **by** *simp*  
  
**let**  $?thesis = \text{invar-loc } G \ v0 \ D0 \ p' \ D \ (pE - \{(u,v)\})$   
  
**have** *SETP'[simp]*:  $\bigcup (\text{set } p') = \bigcup (\text{set } p)$  **unfolding** *p'-def* **by** *simp*  
  
**have** *IL*:  $i < \text{length } p$  **and** *VMEM*:  $v \in p!i$   
**using** *idx-of-props[OF BACK]* **unfolding** *i-def* **by** *auto*  
  
**have** *[simp]*:  $\text{length } p' = \text{Suc } i$   
**unfolding** *p'-def collapse-aux-def* **using** *IL* **by** *auto*  
  
**have** *P'-IDX-SS*:  $\forall j < \text{Suc } i. p!j \subseteq p^!j$   
**unfolding** *p'-def collapse-aux-def* **using** *IL*  
**by** (*auto simp add: nth-append path-seg-drop*)  
  
**from**  $\langle u \in \text{last } p \rangle$  **have**  $u \in p!(\text{length } p - 1)$  **by** (*auto simp: last-conv-nth*)  
  
**have** *defs-fold*:  
 $vE \ p' \ D \ (pE - \{(u,v)\}) = \text{insert } (u,v) \ lwE$   
 $\text{touched } p' \ D = \text{ltouched}$   
**by** (*simp-all add: p'-def E*)  
  
{  
**fix**  $j$   
**assume** *A*:  $\text{Suc } j < \text{length } p'$   
**hence**  $\text{Suc } j < \text{length } p$  **using** *IL* **by** *simp*  
**from** *p-connected[OF this]* **have**  $p!j \times p!\text{Suc } j \cap (E - pE) \neq \{\}$ .  
**moreover** **from** *P'-IDX-SS A* **have**  $p!j \subseteq p^!j$  **and**  $p!\text{Suc } j \subseteq p^!\text{Suc } j$   
**by** *auto*  
**ultimately** **have**  $p^!j \times p^!\text{Suc } j \cap (E - (pE - \{(u,v)\})) \neq \{\}$   
**by** *blast*  
} **note** *AUX-p-connected = this*  
  
**have** *P-IDX-EQ[simp]*:  $\forall j. j < i \longrightarrow p^!j = p!j$   
**unfolding** *p'-def collapse-aux-def* **using** *IL*

```

by (auto simp: nth-append)

have P'-LAST[simp]: p!i = path-seg p i (length p) (is - = ?last-cnode)
  unfolding p'-def collapse-aux-def using IL
  by (auto simp: nth-append path-seg-drop)

{
  fix j k
  assume A: j < k    k < length p'
  have p' ! j ∩ p' ! k = {}
  proof (safe, simp)
    fix v
    assume v ∈ p!j and v ∈ p!k
    with A have v ∈ p!j by simp
    show False proof (cases)
      assume k=i
      with ⟨v ∈ p!k⟩ obtain k' where v ∈ p!k'    i ≤ k'    k' < length p
        by (auto simp: path-seg-def)
      hence p ! j ∩ p ! k' = {}
        using A by (auto intro!: p-disjoint)
      with ⟨v ∈ p!j⟩ ⟨v ∈ p!k'⟩ show False by auto
    next
      assume k ≠ i with A have k < i by simp
      hence k < length p using IL by simp
      note p-disjoint[OF ⟨j < k⟩ this]
      also have p!j = p!j using ⟨j < k⟩ ⟨k < i⟩ by simp
      also have p!k = p!k using ⟨k < i⟩ by simp
      finally show False using ⟨v ∈ p!j⟩ ⟨v ∈ p!k⟩ by auto
    qed
  qed
} note AUX-p-disjoint = this

{
  fix U
  assume A: U ∈ set p'
  then obtain j where j < Suc i and [simp]: U = p!j
    by (auto simp: in-set-conv-nth)
  hence U × U ⊆ (insert (u, v) lvE ∩ U × U)*
  proof cases
    assume [simp]: j=i
    show ?thesis proof (clarsimp)
      fix x y
      assume x ∈ path-seg p i (length p)    y ∈ path-seg p i (length p)
      then obtain ix iy where
        IX: x ∈ p!ix    i ≤ ix    ix < length p and
        IY: y ∈ p!iy    i ≤ iy    iy < length p
      by (auto simp: path-seg-def)
    
```

```

from IX have SS1: path-seg p ix (length p) ⊆ ?last-cnode
  by (subst path-seg-ss-eq) auto

from IY have SS2: path-seg p i (Suc iy) ⊆ ?last-cnode
  by (subst path-seg-ss-eq) auto

let ?rE =  $\lambda R. (lvE \cap R \times R)$ 
let ?E = (insert (u,v) lvE  $\cap$  ?last-cnode  $\times$  ?last-cnode)

from pathI[OF  $\langle x \in p!ix \rangle \langle u \in p!(length\ p - 1) \rangle$ ] have
  (x,u)  $\in$  (?rE (path-seg p ix (Suc (length p - 1))))* using IX by auto
hence (x,u)  $\in$  ?E*
  apply (rule rtrancl-mono-mp[rotated])
  using SS1
  by auto

also have (u,v)  $\in$  ?E using  $\langle i < length\ p \rangle$ 
  apply (clarsimp)
  apply (intro conjI)
  apply (rule rev-subsetD[OF  $\langle u \in p!(length\ p - 1) \rangle$ ])
  apply (simp)
  apply (rule rev-subsetD[OF VMEM])
  apply (simp)
  done
also
from pathI[OF  $\langle v \in p!i \rangle \langle y \in p!iy \rangle$ ] have
  (v,y)  $\in$  (?rE (path-seg p i (Suc iy)))* using IY by auto
hence (v,y)  $\in$  ?E*
  apply (rule rtrancl-mono-mp[rotated])
  using SS2
  by auto
finally show (x,y)  $\in$  ?E* .
qed
next
assume j  $\neq$  i
with  $\langle j < Suc\ i \rangle$  have [simp]: j  $<$  i by simp
with  $\langle i < length\ p \rangle$  have p!j  $\in$  set p
  by (metis Suc-lessD in-set-conv-nth less-trans-Suc)

thus ?thesis using p-sc[of U]  $\langle p!$ j  $\in$  set p  $\rangle$ 
  apply (clarsimp)
  apply (subgoal-tac (a,b)  $\in$  (lvE  $\cap$  p ! j  $\times$  p ! j))*
  apply (erule rtrancl-mono-mp[rotated])
  apply auto
  done
qed
} note AUX-p-sc = this

{ fix j k

```

```

assume  $A: j < k \quad k < \text{length } p'$ 
hence  $j < i$  by simp
have  $\text{insert } (u, v) \text{ lvE} \cap p' ! k \times p' ! j = \{\}$ 
proof –
  have  $\{(u, v)\} \cap p' ! k \times p' ! j = \{\}$ 
    apply auto
    by (metis IL P-IDX-EQ Suc-lessD VMEM <j < i>
      less-irrefl-nat less-trans-Suc p-disjoint-sym)
  moreover have  $\text{lvE} \cap p' ! k \times p' ! j = \{\}$ 
  proof (cases k < i)
    case True thus ?thesis
      using vE-no-back[of j k] A <i < length p> by auto
  next
    case False with  $A$  have [simp]:  $k = i$  by simp
    show ?thesis proof (rule disjointI, clarsimp simp: <j < i>)
      fix  $x y$ 
      assume  $B: (x, y) \in \text{lvE} \quad x \in \text{path-seg } p \ i \ (\text{length } p) \quad y \in p ! j$ 
      then obtain  $ix$  where  $x \in p ! ix \quad i \leq ix \quad ix < \text{length } p$ 
        by (auto simp: path-seg-def)
      moreover with  $A$  have  $j < ix$  by simp
      ultimately show False using vE-no-back[of j ix] B by auto
    qed
  qed
  ultimately show ?thesis by blast
qed
} note AUX-vE-no-back = this

show ?thesis
apply unfold-locales
unfolding defs-fold

apply simp

using D-incr apply auto []

using pE-E-from-p apply auto []

using E-from-p-touched BACK apply (simp add: touched-def) apply blast

apply (rule D-reachable)

apply (rule AUX-p-connected, assumption+) []

apply (rule AUX-p-disjoint, assumption+) []

apply (rule AUX-p-sc, assumption+) []

using root-v0
apply (cases i)

```

```

apply (simp add: p'-def collapse-aux-def)
apply (metis NE hd-in-set)
apply (cases p, simp-all add: p'-def collapse-aux-def) []

apply (simp add: p'-def collapse-aux-def)

apply (rule D-closed)

apply (drule (1) AUX-vE-no-back, auto) []

using p-not-D apply simp
done
qed

lemma invar-push:
assumes INV: invar v0 D0 (p,D,pE)
assumes NE[simp]: p≠[]
assumes E: (u,v)∈pE and UIL: u∈last p
assumes VNE: v∉(∪(set p) v∉D
shows invar v0 D0 (push v (p,D,pE - {(u,v)}))
unfolding invar-def push-def
apply simp
proof -
from INV interpret invar-loc G v0 D0 p D pE unfolding invar-def by simp

let ?thesis
  = invar-loc G v0 D0 (p @ [{v}]) D (pE - {(u, v)} ∪ E ∩ {v} × UNIV)

note defs-fold = vE-push[OF E UIL VNE] touched-push

{
fix i
assume SILL: Suc i < length (p @ [{v}])
have (p @ [{v}]) ! i × (p @ [{v}]) ! Suc i
  ∩ (E - (pE - {(u, v)} ∪ E ∩ {v} × UNIV)) ≠ {}
proof (cases i = length p - 1)
  case True thus ?thesis using SILL E pE-E-from-p UIL VNE
    by (simp add: nth-append last-conv-nth) fast
next
  case False
  with SILL have SILL': Suc i < length p by simp

  with SILL' VNE have X1: v∉p!i v∉p!Suc i by auto

from p-connected[OF SILL'] obtain a b where
  a∈p!i b∈p!Suc i (a,b)∈E (a,b)∉pE
  by auto
with X1 have a≠v b≠v by auto
with ⟨(a,b)∈E⟩ ⟨(a,b)∉pE⟩ have (a,b)∈(E - (pE - {(u, v)} ∪ E ∩ {v} ×

```

```

UNIV))
  by auto
  with ⟨a∈p!i⟩ ⟨b∈p!Suc i⟩
  show ?thesis using SILL'
    by (simp add: nth-append; blast)
qed
} note AUX-p-connected = this

{
  fix U
  assume A: U ∈ set (p @ [{v}])
  have U × U ⊆ (insert (u, v) lvE ∩ U × U)*
  proof cases
    assume U∈set p
    with p-sc have U×U ⊆ (lvE ∩ U×U)* .
    thus ?thesis
      by (metis (lifting, no-types) Int-insert-left-if0 Int-insert-left-if1
          in-mono insert-subset rtrancl-mono-mp subsetI)
  next
    assume U∉set p with A have U={v} by simp
    thus ?thesis by auto
  qed
} note AUX-p-sc = this

{
  fix i j
  assume A: i < j    j < length (p @ [{v}])
  have insert (u, v) lvE ∩ (p @ [{v}]) ! j × (p @ [{v}]) ! i = {}
  proof (cases j=length p)
    case False with A have j<length p by simp
    from vE-no-back ⟨i<j⟩ this VNE show ?thesis
      by (auto simp add: nth-append)
  next
    from p-not-D A have PDDJ: p!i ∩ D = {}
      by (auto simp: Sup-inf-eq-bot-iff)
    case True thus ?thesis
      using A apply (simp add: nth-append)
      apply (rule conjI)
      using UIL A p-disjoint-sym
      apply (metis Misc.last-in-set NE UnionI VNE(1))

      using vE-touched VNE PDDJ apply (auto simp: touched-def) []
    done
  qed
} note AUX-vE-no-back = this

show ?thesis
  apply unfold-locales
  unfolding defs-fold

```

```

apply simp

using D-incr apply auto []

using pE-E-from-p apply auto []

using E-from-p-touched VNE apply (auto simp: touched-def) []

apply (rule D-reachable)

apply (rule AUX-p-connected, assumption+) []

using p-disjoint  $\langle v \notin \bigcup (\text{set } p) \rangle$  apply (auto simp: nth-append) []

apply (rule AUX-p-sc, assumption+) []

using root-v0 apply simp

apply simp

apply (rule D-closed)

apply (rule AUX-vE-no-back, assumption+) []

using p-not-D VNE apply auto []
done
qed

```

**lemma** *invar-skip*:

```

assumes INV: invar v0 D0 (p, D, pE)
assumes NE[simp]: p ≠ []
assumes E:  $(u, v) \in pE$  and UIL: u ∈ last p
assumes VNP:  $v \notin \bigcup (\text{set } p)$  and VD: v ∈ D
shows invar v0 D0 (p, D, pE - {(u, v)})
unfolding invar-def
apply simp

```

**proof** –

```

from INV interpret invar-loc G v0 D0 p D pE unfolding invar-def by simp
let ?thesis = invar-loc G v0 D0 p D (pE - {(u, v)})
note defs-fold = vE-remove[OF NE E]

```

```

show ?thesis
apply unfold-locales
unfolding defs-fold

```

```

apply simp

```

```

using D-incr apply auto []

```

```

using pE-E-from-p apply auto []

using E-from-p-touched VD apply (auto simp: touched-def) []

apply (rule D-reachable)

using p-connected apply auto []

apply (rule p-disjoint, assumption+) []

apply (drule p-sc)
apply (erule order-trans)
apply (rule rtrancl-mono)
apply blast []

apply (rule root-v0, assumption+) []

apply (rule p-empty-v0, assumption+) []

apply (rule D-closed)

using vE-no-back VD p-not-D
apply clarsimp
apply (metis Suc-lessD UnionI VNP less-trans-Suc nth-mem)

apply (rule p-not-D)
done
qed

lemma fin-D-is-reachable:
  — When inner loop terminates, all nodes reachable from start node are finished
  assumes INV: invar v0 D0 ( $\square$ , D, pE)
  shows  $D \supseteq E^* \{v0\}$ 
proof —
  from INV interpret invar-loc G v0 D0 ( $\square$  D pE) unfolding invar-def by auto

  from p-empty-v0 rtrancl-reachable-induct[OF order-refl D-closed] D-reachable
  show ?thesis by auto
qed

lemma fin-reachable-path:
  — When inner loop terminates, nodes reachable from start node are reachable
  over visited edges
  assumes INV: invar v0 D0 ( $\square$ , D, pE)
  assumes UR: u ∈ E* {v0}
  shows path (vE ( $\square$  D pE) u q v)  $\longleftrightarrow$  path E u q v
proof —

```

**from** *INV* **interpret** *invar-loc G v0 D0 [] D pE* **unfolding** *invar-def* **by** *auto*

**show** *?thesis*

**proof**

**assume** *path lvE u q v*

**thus** *path E u q v* **using** *path-mono[OF lvE-ss-E]* **by** *blast*

**next**

**assume** *path E u q v*

**thus** *path lvE u q v* **using** *UR*

**proof** *induction*

**case** (*path-prepend u v p w*)

**with** *fin-D-is-reachable[OF INV]* **have** *u ∈ D* **by** *auto*

**with** *D-closed ⟨(u,v) ∈ E⟩* **have** *v ∈ D* **by** *auto*

**from** *path-prepend.prem*s *path-prepend.hyps* **have** *v ∈ E\** “*{v0}*” **by** *auto*

**with** *path-prepend.IH* *fin-D-is-reachable[OF INV]* **have** *path lvE v p w*  
    **by** *simp*

**moreover** **from** *⟨u ∈ D⟩ ⟨v ∈ D⟩ ⟨(u,v) ∈ E⟩ D-vis* **have** *(u,v) ∈ lvE* **by** *auto*

**ultimately** **show** *?case* **by** (*auto simp: path-cons-conv*)

**qed** *simp*

**qed**

**qed**

**lemma** *invar-outer-newnode:*

**assumes** *A: v0 ∉ D0 v0 ∈ it*

**assumes** *OINV: outer-invar it D0*

**assumes** *INV: invar v0 D0 ([], D', pE)*

**shows** *outer-invar (it - {v0}) D'*

**proof** –

**from** *OINV* **interpret** *outer-invar-loc G it D0* **unfolding** *outer-invar-def* .

**from** *INV* **interpret** *inv: invar-loc G v0 D0 [] D' pE*

**unfolding** *invar-def* **by** *simp*

**from** *fin-D-is-reachable[OF INV]* **have** [*simp*]: *v0 ∈ D'* **by** *auto*

**show** *?thesis*

**unfolding** *outer-invar-def*

**apply** *unfold-locales*

**using** *it-initial* **apply** *auto* []

**using** *it-done inv.D-incr* **apply** *auto* []

**using** *inv.D-reachable* **apply** *assumption*

**using** *inv.D-closed* **apply** *assumption*

**done**

**qed**

**lemma** *invar-outer-Dnode:*

**assumes** *A: v0 ∈ D0 v0 ∈ it*

**assumes** *OINV: outer-invar it D0*

**shows** *outer-invar (it - {v0}) D0*

**proof** –

from *OINV* interpret *outer-invar-loc* *G* it *D0* **unfolding** *outer-invar-def* .

show *?thesis*  
**unfolding** *outer-invar-def*  
**apply** *unfold-locales*  
**using** *it-initial* **apply** *auto* []  
**using** *it-done* *A* **apply** *auto* []  
**using** *D-reachable* **apply** *assumption*  
**using** *D-closed* **apply** *assumption*  
**done**  
**qed**

**lemma** *pE-fin'*: *invar* *x*  $\sigma$  ([], *D*, *pE*)  $\implies$  *pE* = {}  
**unfolding** *invar-def* **by** (*simp* *add*: *invar-loc.pE-fin*)

**end**

**Termination context** *invar-loc*

**begin**

**lemma** *unproc-finite*[*simp*, *intro!*]: *finite* (*unproc-edges* *v0* *p* *D* *pE*)  
— The set of unprocessed edges is finite

**proof** —

**have** *unproc-edges* *v0* *p* *D* *pE*  $\subseteq$   $E^* \{v0\} \times E^* \{v0\}$   
**unfolding** *unproc-edges-def*  
**using** *pE-reachable*  
**by** *auto*  
**thus** *?thesis*  
**by** (*rule* *finite-subset*) *simp*

**qed**

**lemma** *unproc-decreasing*:

— As effect of selecting a pending edge, the set of unprocessed edges decreases

**assumes** [*simp*]: *p*  $\neq$  [] **and** *A*:  $(u,v) \in pE \quad u \in last \ p$   
**shows** *unproc-edges* *v0* *p* *D* (*pE* - {(*u*,*v*)})  $\subset$  *unproc-edges* *v0* *p* *D* *pE*  
**using** *A* **unfolding** *unproc-edges-def*  
**by** *fastforce*

**end**

**context** *fr-graph*

**begin**

**lemma** *abs-wf-pop*:

**assumes** *INV*: *invar* *v0* *D0* (*p*, *D*, *pE*)  
**assumes** *NE*[*simp*]: *p*  $\neq$  []  
**assumes** *NO*:  $pE \cap last \ aba \times UNIV = \{\}$   
**shows** (*pop* (*p*, *D*, *pE*), (*p*, *D*, *pE*))  $\in$  *abs-wf-rel* *v0*  
**unfolding** *pop-def*  
**apply** *simp*

**proof** —

**from** *INV* **interpret** *invar-loc* *G v0 D0 p D pE* **unfolding** *invar-def* **by** *simp*

**let** *?thesis* = ((*butlast p*, *last p*  $\cup$  *D*, *pE*), *p*, *D*, *pE*)  $\in$  *abs-wf-rel v0*  
**have** *unproc-edges v0* (*butlast p*) (*last p*  $\cup$  *D*) *pE* = *unproc-edges v0 p D pE*  
**unfolding** *unproc-edges-def*  
**apply** (*cases p rule: rev-cases, simp*)  
**apply** *auto*  
**done**  
**thus** *?thesis*  
**by** (*auto simp: abs-wf-rel-def*)

**qed**

**lemma** *abs-wf-collapse:*

**assumes** *INV: invar v0 D0 (p,D,pE)*  
**assumes** *NE[simp]: p $\neq$ []*  
**assumes** *E: (u,v) $\in$ pE* *u $\in$ last p*  
**shows** (*collapse v (p,D,pE-{(u,v)}*), (*p*, *D*, *pE*))  $\in$  *abs-wf-rel v0*  
**unfolding** *collapse-def*  
**apply** *simp*

**proof** –

**from** *INV* **interpret** *invar-loc* *G v0 D0 p D pE* **unfolding** *invar-def* **by** *simp*

**define** *i* **where** *i = idx-of p v*

**let** *?thesis*  
= ((*collapse-aux p i*, *D*, *pE-{(u,v)}*), (*p*, *D*, *pE*))  $\in$  *abs-wf-rel v0*

**have** *unproc-edges v0* (*collapse-aux p i*) *D* (*pE-{(u,v)}*)

= *unproc-edges v0 p D* (*pE-{(u,v)}*)

**unfolding** *unproc-edges-def* **by** (*auto*)

**also note** *unproc-decreasing[OF NE E]*

**finally show** *?thesis*

**by** (*auto simp: abs-wf-rel-def*)

**qed**

**lemma** *abs-wf-push:*

**assumes** *INV: invar v0 D0 (p,D,pE)*  
**assumes** *NE[simp]: p $\neq$ []*  
**assumes** *E: (u,v) $\in$ pE* *u $\in$ last p* **and** *A: v $\notin$ D* *v $\notin$  $\bigcup$ (set p)*  
**shows** (*push v (p,D,pE-{(u,v)}*), (*p*, *D*, *pE*))  $\in$  *abs-wf-rel v0*  
**unfolding** *push-def*  
**apply** *simp*

**proof** –

**from** *INV* **interpret** *invar-loc* *G v0 D0 p D pE* **unfolding** *invar-def* **by** *simp*

**let** *?thesis*

= ((*p@[{v}]*, *D*, *pE-{(u,v)}*  $\cup$  *E $\cap$ {v}*  $\times$  *UNIV*), (*p*, *D*, *pE*))  $\in$  *abs-wf-rel v0*

**have** *unproc-edges v0* (*p@[{v}]*) *D* (*pE-{(u,v)}*  $\cup$  *E $\cap$ {v}*  $\times$  *UNIV*)

= *unproc-edges v0 p D* (*pE-{(u,v)}*)

```

unfolding unproc-edges-def
using E A pE-reachable
by auto
also note unproc-decreasing[OF NE E]
finally show ?thesis
by (auto simp: abs-wf-rel-def)
qed

```

```

lemma abs-wf-skip:
assumes INV: invar v0 D0 (p,D,pE)
assumes NE[simp]: p≠[]
assumes E: (u,v)∈pE u∈last p
shows  $((p, D, pE - \{(u,v)\}), (p, D, pE)) \in \text{abs-wf-rel } v0$ 
proof –
from INV interpret invar-loc G v0 D0 p D pE unfolding invar-def by simp
from unproc-decreasing[OF NE E] show ?thesis
by (auto simp: abs-wf-rel-def)
qed
end

```

**Main Correctness Theorem** **context** *fr-graph*  
**begin**

```

lemmas invar-preserve =
invar-initial
invar-pop invar-push invar-skip invar-collapse
abs-wf-pop abs-wf-collapse abs-wf-push abs-wf-skip
outer-invar-initial invar-outer-newnode invar-outer-Dnode

```

The main correctness theorem for the dummy-algorithm just states that it satisfies the invariant when finished, and the path is empty.

```

theorem skeleton-spec: skeleton ≤ SPEC (λD. outer-invar {} D)
proof –
note [simp del] = Union-iff
note [[goals-limit = 4]]

show ?thesis
unfolding skeleton-def select-edge-def select-def
apply (refine-vcg WHILEIT-rule[OF abs-wf-rel-wf])
apply (vc-solve solve: invar-preserve simp: pE-fin' finite-V0)
apply auto
done
qed

```

Short proof, as presented in the paper

```

context
notes [refine] = refine-vcg
begin
theorem skeleton ≤ SPEC (λD. outer-invar {} D)

```

```

    unfolding skeleton-def select-edge-def select-def
    by (refine-vcg WHILEIT-rule[OF abs-wf-rel-wf])
      (auto intro: invar-preserve simp: pE-fin' finite-V0)
  end
end

```

### 1.3.5 Consequences of Invariant when Finished

```

context fr-graph
begin
  lemma fin-outer-D-is-reachable:
    — When outer loop terminates, exactly the reachable nodes are finished
    assumes INV: outer-invar {} D
    shows D = E* “V0
  proof —
    from INV interpret outer-invar-loc G {} D unfolding outer-invar-def by
    auto

    from it-done rtrancl-reachable-induct[OF order-refl D-closed] D-reachable
    show ?thesis by auto
  qed
end

```

## 1.4 Refinement to Gabow’s Data Structure

The implementation due to Gabow [5] represents a path as a stack  $S$  of single nodes, and a stack  $B$  that contains the boundaries of the collapsed segments. Moreover, a map  $I$  maps nodes to their stack indices.

As we use a tail-recursive formulation, we use another stack  $P :: (\text{nat} \times 'v \text{ set}) \text{ list}$  to represent the pending edges. The entries in  $P$  are sorted by ascending first component, and  $P$  only contains entries with non-empty second component. An entry  $(i, l)$  means that the edges from the node at  $S[i]$  to the nodes stored in  $l$  are pending.

### 1.4.1 Preliminaries

```

primrec find-max-nat :: nat  $\Rightarrow$  (nat  $\Rightarrow$  bool)  $\Rightarrow$  nat
  — Find the maximum number below an upper bound for which a predicate holds
  where
    find-max-nat 0 - = 0
  | find-max-nat (Suc n) P = (if (P n) then n else find-max-nat n P)

```

```

lemma find-max-nat-correct:
   $\llbracket P \ 0; \ 0 < u \rrbracket \Longrightarrow \text{find-max-nat } u \ P = \text{Max } \{i. \ i < u \wedge P \ i\}$ 
  apply (induction u)
  apply auto

```

```

apply (rule Max-eqI[THEN sym])
apply auto [3]

apply (case-tac u)
apply simp
apply clarsimp
by (metis less-SucI less-antisym)

lemma find-max-nat-param[param]:
  assumes (n,n')∈nat-rel
  assumes  $\bigwedge j j'. \llbracket (j,j') \in \text{nat-rel}; j' < n \rrbracket \implies (P j, P' j') \in \text{bool-rel}$ 
  shows (find-max-nat n P, find-max-nat n' P') ∈ nat-rel
  using assms
  by (induction n arbitrary: n') auto

context begin interpretation autoref-syn .
  lemma find-max-nat-autoref[autoref-rules]:
    assumes (n,n')∈nat-rel
    assumes  $\bigwedge j j'. \llbracket (j,j') \in \text{nat-rel}; j' < n \rrbracket \implies (P j, P' \$j') \in \text{bool-rel}$ 
    shows (find-max-nat n P,
      (OP find-max-nat ::: nat-rel → (nat-rel → bool-rel) → nat-rel) $n' $P'
    ) ∈ nat-rel
    using find-max-nat-param[OF assms]
    by simp
end

1.4.2 Gabow's Datastructure

Definition and Invariant datatype node-state = STACK nat | DONE

type-synonym 'v oGS = 'v → node-state

definition oGS-α :: 'v oGS ⇒ 'v set where oGS-α I ≡ {v. I v = Some DONE}

locale oGS-invar =
  fixes I :: 'v oGS
  assumes I-no-stack: I v ≠ Some (STACK j)

type-synonym 'a GS
  = 'a list × nat list × ('a → node-state) × (nat × 'a set) list
locale GS =
  fixes SBIP :: 'a GS
begin
  definition S ≡ (λ(S,B,I,P). S) SBIP
  definition B ≡ (λ(S,B,I,P). B) SBIP
  definition I ≡ (λ(S,B,I,P). I) SBIP

```

**definition**  $P \equiv (\lambda(S,B,I,P). P) SBIP$

**definition**  $seg\text{-}start :: nat \Rightarrow nat$  — Start index of segment, inclusive  
**where**  $seg\text{-}start\ i \equiv B!i$

**definition**  $seg\text{-}end :: nat \Rightarrow nat$  — End index of segment, exclusive  
**where**  $seg\text{-}end\ i \equiv \text{if } i+1 = \text{length } B \text{ then length } S \text{ else } B!(i+1)$

**definition**  $seg :: nat \Rightarrow 'a\ set$  — Collapsed set at index  
**where**  $seg\ i \equiv \{S!j \mid j. seg\text{-}start\ i \leq j \wedge j < seg\text{-}end\ i\}$

**definition**  $p\text{-}\alpha \equiv \text{map } seg\ [0..<\text{length } B]$  — Collapsed path

**definition**  $D\text{-}\alpha \equiv \{v. I\ v = \text{Some } DONE\}$  — Done nodes

**definition**  $pE\text{-}\alpha \equiv \{(u,v) . \exists j\ I. (j,I) \in \text{set } P \wedge u = S!j \wedge v \in I\}$   
— Pending edges

**definition**  $\alpha \equiv (p\text{-}\alpha, D\text{-}\alpha, pE\text{-}\alpha)$  — Abstract state

**end**

**lemma**  $GS\text{-}sel\text{-}simps[simp]$ :

$GS.S\ (S,B,I,P) = S$

$GS.B\ (S,B,I,P) = B$

$GS.I\ (S,B,I,P) = I$

$GS.P\ (S,B,I,P) = P$

**unfolding**  $GS.S\text{-}def\ GS.B\text{-}def\ GS.I\text{-}def\ GS.P\text{-}def$

**by**  $auto$

**context**  $GS\ \text{begin}$

**lemma**  $seg\text{-}start\text{-}indep[simp]$ :  $GS.seg\text{-}start\ (S',B',I',P') = seg\text{-}start$

**unfolding**  $GS.seg\text{-}start\text{-}def[abs\text{-}def]$  **by**  $(auto)$

**lemma**  $seg\text{-}end\text{-}indep[simp]$ :  $GS.seg\text{-}end\ (S',B',I',P') = seg\text{-}end$

**unfolding**  $GS.seg\text{-}end\text{-}def[abs\text{-}def]$  **by**  $auto$

**lemma**  $seg\text{-}indep[simp]$ :  $GS.seg\ (S',B',I',P') = seg$

**unfolding**  $GS.seg\text{-}def[abs\text{-}def]$  **by**  $auto$

**lemma**  $p\text{-}\alpha\text{-}indep[simp]$ :  $GS.p\text{-}\alpha\ (S',B',I',P') = p\text{-}\alpha$

**unfolding**  $GS.p\text{-}\alpha\text{-}def$  **by**  $auto$

**lemma**  $D\text{-}\alpha\text{-}indep[simp]$ :  $GS.D\text{-}\alpha\ (S',B',I',P') = D\text{-}\alpha$

**unfolding**  $GS.D\text{-}\alpha\text{-}def$  **by**  $auto$

**lemma**  $pE\text{-}\alpha\text{-}indep[simp]$ :  $GS.pE\text{-}\alpha\ (S',B',I',P') = pE\text{-}\alpha$

**unfolding**  $GS.pE\text{-}\alpha\text{-}def$  **by**  $auto$

**definition**  $find\text{-}seg$  — Abs-path index for stack index

**where**  $find\text{-}seg\ j \equiv \text{Max } \{i. i < \text{length } B \wedge B!i \leq j\}$

```

definition S-idx-of — Stack index for node
  where S-idx-of  $v \equiv \text{case } I \ v \text{ of } \text{Some } (STACK \ i) \Rightarrow i$ 

end

locale GS-invar = GS +
  assumes B-in-bound:  $set \ B \subseteq \{0..<length \ S\}$ 
  assumes B-sorted:  $sorted \ B$ 
  assumes B-distinct:  $distinct \ B$ 
  assumes B0:  $S \neq [] \Longrightarrow B \neq [] \wedge B!0=0$ 
  assumes S-distinct:  $distinct \ S$ 

  assumes I-consistent:  $(I \ v = \text{Some } (STACK \ j)) \longleftrightarrow (j < length \ S \wedge v = S!j)$ 

  assumes P-sorted:  $sorted \ (\text{map } fst \ P)$ 
  assumes P-distinct:  $distinct \ (\text{map } fst \ P)$ 
  assumes P-bound:  $set \ P \subseteq \{0..<length \ S\} \times Collect \ ((\neq) \ \{\})$ 
begin
  lemma locale-this: GS-invar SBIP by unfold-locales
end

definition oGS-rel  $\equiv br \ oGS.\alpha \ oGS.invar$ 
lemma oGS-rel-sv[intro!,simp,relator-props]: single-valued oGS-rel
  unfolding oGS-rel-def by auto

definition GS-rel  $\equiv br \ GS.\alpha \ GS.invar$ 
lemma GS-rel-sv[intro!,simp,relator-props]: single-valued GS-rel
  unfolding GS-rel-def by auto

context GS-invar
begin
  lemma empty-eq:  $S=[] \longleftrightarrow B=[]$ 
    using B-in-bound B0 by auto

  lemma B-in-bound':  $i < length \ B \Longrightarrow B!i < length \ S$ 
    using B-in-bound nth-mem by fastforce

  lemma seg-start-bound:
    assumes A:  $i < length \ B$  shows  $seg-start \ i < length \ S$ 
    using B-in-bound nth-mem[OF A] unfolding seg-start-def by auto

  lemma seg-end-bound:
    assumes A:  $i < length \ B$  shows  $seg-end \ i \leq length \ S$ 
proof (cases  $i+1=length \ B$ )
  case True thus ?thesis by (simp add: seg-end-def)
next
  case False with A have  $i+1 < length \ B$  by simp
  from nth-mem[OF this] B-in-bound have  $B!(i+1) < length \ S$  by auto

```

**thus** *?thesis* **using** *False* **by** (*simp add: seg-end-def*)  
**qed**

**lemma** *seg-start-less-end*:  $i < \text{length } B \implies \text{seg-start } i < \text{seg-end } i$   
**unfolding** *seg-start-def seg-end-def*  
**using** *B-in-bound' distinct-sorted-mono[OF B-sorted B-distinct]*  
**by** *auto*

**lemma** *seg-end-less-start*:  $\llbracket i < j; j < \text{length } B \rrbracket \implies \text{seg-end } i \leq \text{seg-start } j$   
**unfolding** *seg-start-def seg-end-def*  
**by** (*auto simp: distinct-sorted-mono-iff[OF B-distinct B-sorted]*)

**lemma** *find-seg-bounds*:  
**assumes** *A: j < length S*  
**shows**  $\text{seg-start } (\text{find-seg } j) \leq j$   
**and**  $j < \text{seg-end } (\text{find-seg } j)$   
**and**  $\text{find-seg } j < \text{length } B$

**proof** –  
**let**  $?M = \{i. i < \text{length } B \wedge B!i \leq j\}$   
**from** *A* **have** [*simp*]:  $B \neq \{\}$  **using** *empty-eq* **by** (*cases S*) *auto*  
**have** *NE*:  $?M \neq \{\}$  **using** *A B0* **by** (*cases B*) *auto*

**have** *F*: *finite ?M* **by** *auto*

**from** *Max-in[OF F NE]*  
**have** *LEN*:  $\text{find-seg } j < \text{length } B$  **and** *LB*:  $B! \text{find-seg } j \leq j$   
**unfolding** *find-seg-def*  
**by** *auto*

**thus**  $\text{find-seg } j < \text{length } B$  **by** –

**from** *LB* **show** *LB'*:  $\text{seg-start } (\text{find-seg } j) \leq j$   
**unfolding** *seg-start-def* **by** *simp*

**moreover** **show** *UB'*:  $j < \text{seg-end } (\text{find-seg } j)$   
**unfolding** *seg-end-def*  
**proof** (*split if-split, intro impI conjI*)  
**show**  $j < \text{length } S$  **using** *A* .

**assume**  $\text{find-seg } j + 1 \neq \text{length } B$   
**with** *LEN* **have** *P1*:  $\text{find-seg } j + 1 < \text{length } B$  **by** *simp*

**show**  $j < B! (\text{find-seg } j + 1)$   
**proof** (*rule ccontr, simp only: linorder-not-less*)  
**assume** *P2*:  $B! (\text{find-seg } j + 1) \leq j$   
**with** *P1 Max-ge[OF F, of find-seg j + 1, folded find-seg-def]*  
**show** *False* **by** *simp*

**qed**  
**qed**

qed

lemma *find-seg-correct*:

assumes  $A: j < \text{length } S$   
shows  $S!j \in \text{seg } (\text{find-seg } j)$  and  $\text{find-seg } j < \text{length } B$   
using *find-seg-bounds*[*OF A*]  
unfolding *seg-def* by *auto*

lemma *set-p- $\alpha$ -is-set-S*:

$\bigcup (\text{set } p\text{-}\alpha) = \text{set } S$   
apply *rule*  
unfolding *p- $\alpha$ -def seg-def*[*abs-def*]  
using *seg-end-bound* apply *fastforce* []

apply (*auto simp: in-set-conv-nth*)

using *find-seg-bounds*  
apply (*fastforce simp: in-set-conv-nth*)  
done

lemma *S-idx-uniq*:

$\llbracket i < \text{length } S; j < \text{length } S \rrbracket \implies S!i = S!j \longleftrightarrow i = j$   
using *S-distinct*  
by (*simp add: nth-eq-iff-index-eq*)

lemma *S-idx-of-correct*:

assumes  $A: v \in \bigcup (\text{set } p\text{-}\alpha)$   
shows  $S\text{-idx-of } v < \text{length } S$  and  $S!S\text{-idx-of } v = v$

proof –

from  $A$  have  $v \in \text{set } S$  by (*simp add: set-p- $\alpha$ -is-set-S*)  
then obtain  $j$  where  $G1: j < \text{length } S \quad v = S!j$  by (*auto simp: in-set-conv-nth*)  
with *I-consistent* have  $I v = \text{Some } (STACK j)$  by *simp*  
hence  $S\text{-idx-of } v = j$  by (*simp add: S-idx-of-def*)  
with  $G1$  show  $S\text{-idx-of } v < \text{length } S$  and  $S!S\text{-idx-of } v = v$  by *simp-all*

qed

lemma *p- $\alpha$ -disjoint-sym*:

shows  $\forall i j v. i < \text{length } p\text{-}\alpha \wedge j < \text{length } p\text{-}\alpha \wedge v \in p\text{-}\alpha!i \wedge v \in p\text{-}\alpha!j \longrightarrow i = j$

proof (*intro allI impI, elim conjE*)

fix  $i j v$

assume  $A: i < \text{length } p\text{-}\alpha \quad j < \text{length } p\text{-}\alpha \quad v \in p\text{-}\alpha!i \quad v \in p\text{-}\alpha!j$   
from  $A$  have  $LI: i < \text{length } B$  and  $LJ: j < \text{length } B$  by (*simp-all add: p- $\alpha$ -def*)

from  $A$  have  $B1: \text{seg-start } j < \text{seg-end } i$  and  $B2: \text{seg-start } i < \text{seg-end } j$

unfolding *p- $\alpha$ -def seg-def*[*abs-def*]  
apply *clarsimp-all*  
apply (*subst (asm) S-idx-uniq*)  
apply (*metis dual-order.strict-trans1 seg-end-bound*)  
apply (*metis dual-order.strict-trans1 seg-end-bound*)

```

apply simp
apply (subst (asm) S-idx-uniq)
apply (metis dual-order.strict-trans1 seg-end-bound)
apply (metis dual-order.strict-trans1 seg-end-bound)
apply simp
done

```

```

from B1 have B1: ( $B!j < B!Suc\ i \wedge Suc\ i < length\ B$ )  $\vee\ i=length\ B - 1$ 
using LI unfolding seg-start-def seg-end-def by (auto split: if-split-asm)

```

```

from B2 have B2: ( $B!i < B!Suc\ j \wedge Suc\ j < length\ B$ )  $\vee\ j=length\ B - 1$ 
using LJ unfolding seg-start-def seg-end-def by (auto split: if-split-asm)

```

```

from B1 have B1:  $j < Suc\ i \vee i=length\ B - 1$ 
using LI LJ distinct-sorted-strict-mono-iff[OF B-distinct B-sorted]
by auto

```

```

from B2 have B2:  $i < Suc\ j \vee j=length\ B - 1$ 
using LI LJ distinct-sorted-strict-mono-iff[OF B-distinct B-sorted]
by auto

```

```

from B1 B2 show  $i=j$ 
using LI LJ
by auto

```

**qed**

**end**

### 1.4.3 Refinement of the Operations

**definition** *GS-initial-impl* ::  $'a\ oGS \Rightarrow 'a \Rightarrow 'a\ set \Rightarrow 'a\ GS$  **where**

```

GS-initial-impl I v0 succs  $\equiv$  (
  [v0],
  [0],
   $I(v0 \mapsto (STACK\ 0))$ ,
  if succs={} then [] else [(0,succs)]
)

```

**context** *GS*

**begin**

**definition** *push-impl* *v succs*  $\equiv$

```

let
  - = stat-newnode ();
  j = length S;
  S = S@[v];
  B = B@[j];
  I =  $I(v \mapsto STACK\ j)$ ;
  P = if succs={} then P else  $P@[j,succs]$ 
in
  (S,B,I,P)

```

**definition** *mark-as-done*

```
where  $\bigwedge l u I. \text{mark-as-done } l u I \equiv \text{do } \{$   
   $(-,I) \leftarrow \text{WHILET}$   
     $(\lambda(l,I). l < u)$   
     $(\lambda(l,I). \text{do } \{ \text{ASSERT } (l < \text{length } S); \text{RETURN } (\text{Suc } l, I(S!l \mapsto \text{DONE})) \})$   
     $(l,I);$   
   $\text{RETURN } I$   
 $\}$ 
```

**definition** *mark-as-done-abs where*

```
 $\bigwedge l u I. \text{mark-as-done-abs } l u I$   
 $\equiv (\lambda v. \text{if } v \in \{S!j \mid j. l \leq j \wedge j < u\} \text{ then Some DONE else } I v)$ 
```

**lemma** *mark-as-done-aux:*

```
fixes  $l u I$   
shows  $\llbracket l < u; u \leq \text{length } S \rrbracket \implies \text{mark-as-done } l u I$   
 $\leq \text{SPEC } (\lambda r. r = \text{mark-as-done-abs } l u I)$   
unfolding mark-as-done-def mark-as-done-abs-def  
apply (refine-rcg)  
  WHILET-rule[where  
     $I = \lambda(l',I').$   
     $I' = (\lambda v. \text{if } v \in \{S!j \mid j. l \leq j \wedge j < l'\} \text{ then Some DONE else } I v)$   
     $\wedge l \leq l' \wedge l' \leq u$   
    and  $R = \text{measure } (\lambda(l',-). u - l')$   
  ]  
refine-vcg)
```

apply (*auto intro!: ext simp: less-Suc-eq*)

done

**definition** *pop-impl*  $\equiv$

```
do {  
  let  $lsi = \text{length } B - 1;$   
   $\text{ASSERT } (lsi < \text{length } B);$   
   $I \leftarrow \text{mark-as-done } (\text{seg-start } lsi) (\text{seg-end } lsi) I;$   
   $\text{ASSERT } (B \neq []);$   
  let  $S = \text{take } (\text{last } B) S;$   
   $\text{ASSERT } (B \neq []);$   
  let  $B = \text{butlast } B;$   
   $\text{RETURN } (S, B, I, P)$   
}
```

**definition** *sel-rem-last*  $\equiv$

```
if  $P = []$  then  
   $\text{RETURN } (\text{None}, (S, B, I, P))$   
else do {  
  let  $(j, \text{succs}) = \text{last } P;$ 
```

```

  ASSERT (length B - 1 < length B);
  if j ≥ seg-start (length B - 1) then do {
    ASSERT (succs≠{});
    v ← SPEC (λx. x∈succs);
    let succs = succs - {v};
    ASSERT (P≠[] ∧ length P - 1 < length P);
    let P = (if succs={ } then butlast P else P[length P - 1 := (j,succs)]);
    RETURN (Some v,(S,B,I,P))
  } else RETURN (None,(S,B,I,P))
}

```

**definition** *find-seg-impl*  $j \equiv \text{find-max-nat } (\text{length } B) (\lambda i. B!i \leq j)$

**lemma** (in *GS-invar*) *find-seg-impl*:  
 $j < \text{length } S \implies \text{find-seg-impl } j = \text{find-seg } j$   
**unfolding** *find-seg-impl-def*  
**thm** *find-max-nat-correct*  
**apply** (subst *find-max-nat-correct*)  
**apply** (simp add: *B0*)  
**apply** (simp add: *B0*)  
**apply** (simp add: *find-seg-def*)  
**done**

**definition** *idx-of-impl*  $v \equiv \text{do } \{$   
 ASSERT ( $\exists i. I v = \text{Some } (\text{STACK } i)$ );  
 let  $j = S\text{-idx-of } v$ ;  
 ASSERT ( $j < \text{length } S$ );  
 let  $i = \text{find-seg-impl } j$ ;  
 RETURN  $i$   
 $\}$

**definition** *collapse-impl*  $v \equiv$   
 do {  
 $i \leftarrow \text{idx-of-impl } v$ ;  
 ASSERT ( $i+1 \leq \text{length } B$ );  
 let  $B = \text{take } (i+1) B$ ;  
 RETURN ( $S,B,I,P$ )  
 $\}$

**end**

**lemma** (in  $-$ ) *GS-initial-correct*:  
**assumes** *REL*:  $(I,D) \in \text{oGS-rel}$   
**assumes** *A*:  $v0 \notin D$   
**shows**  $\text{GS.}\alpha$  (*GS-initial-impl*  $I v0 \text{ succs}$ ) =  $([\{v0\}], D, \{v0\} \times \text{succs})$  (**is** ?*G1*)  
**and** *GS-invar* (*GS-initial-impl*  $I v0 \text{ succs}$ ) (**is** ?*G2*)  
**proof**  $-$

```

from REL have [simp]:  $D = oGS\text{-}\alpha\ I$  and  $I: oGS\text{-}invar\ I$ 
  by (simp-all add: oGS-rel-def br-def)

from  $I$  have [simp]:  $\bigwedge j\ v.\ I\ v \neq Some\ (STACK\ j)$ 
  by (simp add: oGS-invar-def)

show ?G1
  unfolding GS.alpha-def GS-initial-impl-def
  apply (simp split del: if-split) apply (intro conjI)

  unfolding GS.p-alpha-def GS.seg-def[abs-def] GS.seg-start-def GS.seg-end-def
  apply (auto) []

  using  $A$  unfolding GS.D-alpha-def apply (auto simp: oGS-alpha-def) []

  unfolding GS.pE-alpha-def apply auto []
  done

show ?G2
  unfolding GS-initial-impl-def
  apply unfold-locales
  apply auto
  done
qed

context GS-invar
begin
  lemma push-correct:
    assumes  $A: v \notin \bigcup (set\ p\text{-}\alpha)$  and  $B: v \notin D\text{-}\alpha$ 
    shows  $GS.\alpha\ (push\text{-}impl\ v\ succs) = (p\text{-}\alpha @ [\{v\}], D\text{-}\alpha, pE\text{-}\alpha \cup \{v\}) \times succs$ 
      (is ?G1)
    and  $GS\text{-}invar\ (push\text{-}impl\ v\ succs)$  (is ?G2)
  proof –

    note [simp] = Let-def

    have  $A1: GS.D\text{-}\alpha\ (push\text{-}impl\ v\ succs) = D\text{-}\alpha$ 
      using  $B$ 
      by (auto simp: push-impl-def GS.D-alpha-def)

    have  $iexI: \bigwedge a\ b\ j\ P.\ \llbracket a!j = b!j; P\ j \rrbracket \implies \exists j'. a!j = b!j' \wedge P\ j'$ 
      by blast

    have  $A2: GS.p\text{-}\alpha\ (push\text{-}impl\ v\ succs) = p\text{-}\alpha @ [\{v\}]$ 
      unfolding push-impl-def GS.p-alpha-def GS.seg-def[abs-def]
        GS.seg-start-def GS.seg-end-def
      apply (clarsimp split del: if-split)

    apply clarsimp

```

```

apply safe
apply (((rule iexI)?,
  (auto
    simp: nth-append nat-in-between-eq
    dest: order.strict-trans[OF - B-in-bound]
  )) []
) +
done

have iexI2:  $\bigwedge j I Q. \llbracket (j,I) \in \text{set } P; (j,I) \in \text{set } P \implies Q j \rrbracket \implies \exists j. Q j$ 
by blast

have A3:  $GS.pE\text{-}\alpha$  (push-impl v succs) =  $pE\text{-}\alpha \cup \{v\} \times \text{succs}$ 
unfolding push-impl-def GS.pE- $\alpha$ -def
using P-bound
apply (force simp: nth-append elim!: iexI2)
done

show ?G1
unfolding GS. $\alpha$ -def
by (simp add: A1 A2 A3)

show ?G2
apply unfold-locales
unfolding push-impl-def
apply simp-all

using B-in-bound B-sorted B-distinct apply (auto simp: sorted-append) [3]
using B-in-bound B0 apply (cases S) apply (auto simp: nth-append) [2]

using S-distinct A apply (simp add: set-p- $\alpha$ -is-set-S)

using A I-consistent
apply (auto simp: nth-append set-p- $\alpha$ -is-set-S split: if-split-asm) []

using P-sorted P-distinct P-bound apply (auto simp: sorted-append) [3]
done
qed

lemma no-last-out-P-aux:
  assumes NE:  $p\text{-}\alpha \neq []$  and NS:  $pE\text{-}\alpha \cap \text{last } p\text{-}\alpha \times UNIV = \{\}$ 
  shows  $\text{set } P \subseteq \{0..<\text{last } B\} \times UNIV$ 
proof –
  {
    fix j I
    assume jII:  $(j,I) \in \text{set } P$ 
    and JL:  $\text{last } B \leq j$ 
    with P-bound have JU:  $j < \text{length } S$  and INE:  $I \neq \{\}$  by auto
    with JL JU have S!j  $\in \text{last } p\text{-}\alpha$ 
  }

```

```

using NE
unfolding p- $\alpha$ -def
apply (auto
  simp: last-map seg-def seg-start-def seg-end-def last-conv-nth)
done
moreover from jII have  $\{S!j\} \times I \subseteq pE-\alpha$  unfolding pE- $\alpha$ -def
  by auto
moreover note INE NS
ultimately have False by blast
} thus ?thesis by fastforce
qed

```

**lemma** *pop-correct*:

```

assumes NE: p- $\alpha$  $\neq$ [] and NS: pE- $\alpha$   $\cap$  last p- $\alpha$   $\times$  UNIV = {}
shows pop-impl
   $\leq \Downarrow GS-rel (SPEC (\lambda r. r=(butlast p-\alpha, D-\alpha \cup last p-\alpha, pE-\alpha)))$ 

```

**proof** –

```

have ixI:  $\bigwedge a b j P. \llbracket a!j = b!j; P j \rrbracket \implies \exists j'. a!j = b!j' \wedge P j'$ 
  by blast

```

```

have [simp]:  $\bigwedge n. n - Suc\ 0 \neq n \longleftrightarrow n \neq 0$  by auto

```

```

from NE have BNE: B $\neq$ []
  unfolding p- $\alpha$ -def by auto

```

```

{
  fix i j
  assume B: j < B!i and A: i < length B
  note B
  also from sorted-nth-mono[OF B-sorted, of i length B - 1] A
  have B!i  $\leq$  last B
    by (simp add: last-conv-nth)
  finally have j < last B .
  hence take (last B) S ! j = S ! j
    and take (B!(length B - Suc 0)) S ! j = S!j
    by (simp-all add: last-conv-nth BNE)
} note AUX1=this

```

```

{
  fix v j
  have (mark-as-done-abs
    (seg-start (length B - Suc 0))
    (seg-end (length B - Suc 0)) I v = Some (STACK j))
     $\longleftrightarrow (j < length\ S \wedge j < last\ B \wedge v = take\ (last\ B)\ S\ !\ j)$ 
  apply (simp add: mark-as-done-abs-def)
  apply safe []
  using I-consistent
  apply (clarsimp-all
    simp: seg-start-def seg-end-def last-conv-nth BNE)

```

```

    simp: S-idx-uniq)

apply (force)
apply (subst nth-take)
apply force
apply force
done
} note AUX2 = this

define ci where ci = (
  take (last B) S,
  butlast B,
  mark-as-done-abs
  (seg-start (length B - Suc 0)) (seg-end (length B - Suc 0)) I,
  P)

have ABS: GS.α ci = (butlast p-α, D-α ∪ last p-α, pE-α)
apply (simp add: GS.α-def ci-def)
apply (intro conjI)
apply (auto
  simp del: map-butlast
  simp add: map-butlast[symmetric] butlast-upt
  simp add: GS.p-α-def GS.seg-def[abs-def] GS.seg-start-def GS.seg-end-def
  simp: nth-butlast last-conv-nth nth-take AUX1
  cong: if-cong
  intro!: iexI
  dest: order.strict-trans[OF - B-in-bound']
) []

apply (auto
  simp: GS.D-α-def p-α-def last-map BNE seg-def mark-as-done-abs-def) []

using AUX1 no-last-out-P-aux[OF NE NS]
apply (auto simp: GS.pE-α-def mark-as-done-abs-def elim!: be2I) []
done

have INV: GS-invar ci
apply unfold-locales
apply (simp-all add: ci-def)

using B-in-bound B-sorted B-distinct
apply (cases B rule: rev-cases, simp)
apply (auto simp: sorted-append order.strict-iff-order) []

using B-sorted BNE apply (auto simp: sorted-butlast) []

using B-distinct BNE apply (auto simp: distinct-butlast) []

using B0 apply (cases B rule: rev-cases, simp add: BNE)

```

```

apply (auto simp: nth-append split: if-split-asm) []

using S-distinct apply (auto) []

apply (rule AUX2)

using P-sorted P-distinct
apply (auto) [2]

using P-bound no-last-out-P-aux[OF NE NS]
apply (auto simp: in-set-conv-decomp)
done

show ?thesis
  unfolding pop-impl-def
  apply (refine-rcg
    SPEC-refine refine-vcg order-trans[OF mark-as-done-aux])
  apply (simp-all add: BNE seg-start-less-end seg-end-bound)
  apply (fold ci-def)
  unfolding GS-rel-def
  apply (rule brI)
  apply (simp-all add: ABS INV)
  done
qed

lemma sel-rem-last-correct:
  assumes NE:  $p \neq \alpha$ 
  shows
     $sel\text{-rem}\text{-last} \leq \Downarrow (Id \times_r GS\text{-rel}) (select\text{-edge} (p, \alpha, D, \alpha, pE, \alpha))$ 
  proof -
  {
    fix l i a b b'
    have  $\llbracket i < length\ l; !i = (a, b) \rrbracket \implies map\ fst\ (l[i := (a, b')]) = map\ fst\ l$ 
      by (induct l arbitrary: i) (auto split: nat.split)
    } note map-fst-upd-snd-eq = this

  from NE have BNE[simp]:  $B \neq \alpha$  unfolding p- $\alpha$ -def by simp

  have INVAR:  $sel\text{-rem}\text{-last} \leq SPEC (GS\text{-invar}\ o\ snd)$ 
    unfolding sel-rem-last-def
    apply (refine-rcg refine-vcg)
    using locale-this apply (cases SBIP) apply simp

  apply simp

  using P-bound apply (cases P rule: rev-cases, auto) []

```

```

apply simp

apply simp apply (intro impI conjI)

apply (unfold-locales, simp-all) []
using B-in-bound B-sorted B-distinct B0 S-distinct I-consistent
apply auto [6]

using P-sorted P-distinct
apply (auto simp: map-butlast sorted-butlast distinct-butlast) [2]

using P-bound apply (auto dest: in-set-butlastD) []

apply (unfold-locales, simp-all) []
using B-in-bound B-sorted B-distinct B0 S-distinct I-consistent
apply auto [6]

using P-sorted P-distinct
apply (auto simp: last-conv-nth map-fst-upd-snd-eq) [2]

using P-bound
apply (cases P rule: rev-cases, simp)
apply (auto) []

using locale-this apply (cases SBIP) apply simp
done

{
assume NS: pE- $\alpha$   $\cap$  last p- $\alpha$   $\times$  UNIV = {}
hence sel-rem-last
   $\leq$  SPEC ( $\lambda r. \text{case } r \text{ of } (None, SBIP') \Rightarrow SBIP' = SBIP \mid - \Rightarrow False$ )
unfolding sel-rem-last-def
apply (refine-rcg refine-vcg)
apply (cases SBIP)
apply simp

apply simp
using P-bound apply (cases P rule: rev-cases, auto) []
apply simp

using no-last-out-P-aux[OF NE NS]
apply (auto simp: seg-start-def last-conv-nth) []

apply (cases SBIP)
apply simp
done
} note SPEC-E = this

```

```

{
  assume NON-EMPTY:  $pE\text{-}\alpha \cap \text{last } p\text{-}\alpha \times UNIV \neq \{\}$ 

  then obtain  $j$  succs  $P'$  where
    EFMT:  $P = P' @ [(j, \text{succs})]$ 
    unfolding  $pE\text{-}\alpha\text{-def}$ 
    by (cases  $P$  rule: rev-cases) auto

  with  $P$ -bound have  $J$ -UPPER:  $j < \text{length } S$  and  $SNE: \text{succs} \neq \{\}$ 
    by auto

  have  $J$ -LOWER:  $\text{seg-start } (\text{length } B - \text{Suc } 0) \leq j$ 
  proof (rule ccontr)
    assume  $\neg(\text{seg-start } (\text{length } B - \text{Suc } 0) \leq j)$ 
    hence  $j < \text{seg-start } (\text{length } B - 1)$  by simp
    with  $P$ -sorted EFMT
    have  $P$ -bound':  $\text{set } P \subseteq \{0..<\text{seg-start } (\text{length } B - 1)\} \times UNIV$ 
      by (auto simp: sorted-append)
    hence  $pE\text{-}\alpha \cap \text{last } p\text{-}\alpha \times UNIV = \{\}$ 
      by (auto)
      simp: p\text{-}\alpha\text{-def last-conv-nth seg-def pE\text{-}\alpha\text{-def S-idx-uniq seg-end-def)
    thus False using NON-EMPTY by simp
  qed

  from  $J$ -UPPER  $J$ -LOWER have  $SJL: S!j \in \text{last } p\text{-}\alpha$ 
    unfolding  $p\text{-}\alpha\text{-def seg-def[abs-def] seg-end-def$ 
    by (auto simp: last-map)

  from EFMT have  $SSS: \{S!j\} \times \text{succs} \subseteq pE\text{-}\alpha$ 
    unfolding  $pE\text{-}\alpha\text{-def}$ 
    by auto

  {
    fix  $v$ 
    assume  $v \in \text{succs}$ 
    with  $SJL$   $SSS$  have  $G: (S!j, v) \in pE\text{-}\alpha \cap \text{last } p\text{-}\alpha \times UNIV$  by auto

    {
      fix  $j'$  succs'
      assume  $S!j' = S!j \quad (j', \text{succs}') \in \text{set } P'$ 
      with  $J$ -UPPER  $P$ -bound  $S$ -idx-uniq EFMT have  $j' = j$  by auto
      with  $P$ -distinct  $\langle (j', \text{succs}') \in \text{set } P' \rangle$  EFMT have False by auto
    } note AUX3=this

    have  $G1: GS.pE\text{-}\alpha (S, B, I, P' @ [(j, \text{succs} - \{v\}]]) = pE\text{-}\alpha - \{(S!j, v)\}$ 
      unfolding  $GS.pE\text{-}\alpha\text{-def}$  using AUX3
      by (auto simp: EFMT)
  }

```

```

{
  assume  $succs \sqsubseteq \{v\}$ 
  hence  $GS.pE-\alpha (S,B,I,P' @ [(j, succs - \{v\})]) = GS.pE-\alpha (S,B,I,P')$ 
  unfolding  $GS.pE-\alpha-def$  by auto

  with  $G1$  have  $GS.pE-\alpha (S,B,I,P') = pE-\alpha - \{(S!j, v)\}$  by simp
} note  $G2 = this$ 

note  $G G1 G2$ 
} note  $AUX3 = this$ 

have  $sel-rem-last \leq SPEC (\lambda r. case\ r\ of$ 
  ( $Some\ v, SBIP'$ )  $\Rightarrow \exists u.$ 
    ( $u,v \in (pE-\alpha \cap last\ p-\alpha \times UNIV)$ 
       $\wedge GS.\alpha\ SBIP' = (p-\alpha, D-\alpha, pE-\alpha - \{(u,v)\})$ )
  |  $- \Rightarrow False$ )
  unfolding  $sel-rem-last-def$ 
  apply ( $refine-rcg\ refine-vcg$ )

  using  $SNE$  apply ( $vc-solve\ simp: J-LOWER\ EFMT$ )

  apply ( $frule\ AUX3(1)$ )

  apply safe

  apply ( $drule\ (1)\ AUX3(3)$ ) apply ( $auto\ simp: EFMT\ GS.\alpha-def$ ) []
  apply ( $drule\ AUX3(2)$ ) apply ( $auto\ simp: GS.\alpha-def$ ) []
  done
} note  $SPEC-NE=this$ 

have  $SPEC: sel-rem-last \leq SPEC (\lambda r. case\ r\ of$ 
  ( $None, SBIP'$ )  $\Rightarrow SBIP' = SBIP \wedge pE-\alpha \cap last\ p-\alpha \times UNIV = \{\}$   $\wedge$ 
   $GS-invar\ SBIP$ 
  | ( $Some\ v, SBIP'$ )  $\Rightarrow \exists u. (u, v) \in pE-\alpha \cap last\ p-\alpha \times UNIV$ 
     $\wedge GS.\alpha\ SBIP' = (p-\alpha, D-\alpha, pE-\alpha - \{(u, v)\})$ 
     $\wedge GS-invar\ SBIP'$ 
  )
  using  $INVAR$ 
  apply ( $cases\ pE-\alpha \cap last\ p-\alpha \times UNIV = \{\}$ )
  apply ( $frule\ SPEC-E$ )
  apply ( $auto\ split: option.splits\ simp: pw-le-iff; blast; fail$ )
  apply ( $frule\ SPEC-NE$ )
  apply ( $auto\ split: option.splits\ simp: pw-le-iff; blast; fail$ )
  done

have  $X1: (\exists y. (y=None \longrightarrow \Phi\ y) \wedge (\forall a\ b. y=Some\ (a,b) \longrightarrow \Psi\ y\ a\ b)) \longleftrightarrow$ 
  ( $\Phi\ None \vee (\exists a\ b. \Psi\ (Some\ (a,b))\ a\ b)$ ) for  $\Phi\ \Psi$ 
  by auto

```

```

show ?thesis
  apply (rule order-trans[OF SPEC])
  unfolding select-edge-def select-def
  apply (simp
    add: pw-le-iff refine-pw-simps prod-rel-sv
    del: SELECT-pw
    split: option.splits prod.splits)
  apply (fastforce simp: br-def GS-rel-def GS.alpha-def)
  done
qed

```

```

lemma find-seg-idx-of-correct:
  assumes A:  $v \in \bigcup (\text{set } p\text{-}\alpha)$ 
  shows (find-seg (S-idx-of v)) = idx-of p- $\alpha$  v
proof –
  note S-idx-of-correct[OF A] idx-of-props[OF p- $\alpha$ -disjoint-sym A]
  from find-seg-correct[OF  $\langle S\text{-idx-of } v < \text{length } S \rangle$ ] have
    find-seg (S-idx-of v) < length p- $\alpha$ 
    and S!S-idx-of v  $\in$  p- $\alpha$ !find-seg (S-idx-of v)
    unfolding p- $\alpha$ -def by auto
  from idx-of-uniq[OF p- $\alpha$ -disjoint-sym this]  $\langle S ! S\text{-idx-of } v = v \rangle$ 
  show ?thesis by auto
qed

```

```

lemma idx-of-correct:
  assumes A:  $v \in \bigcup (\text{set } p\text{-}\alpha)$ 
  shows idx-of-impl v  $\leq$  SPEC ( $\lambda x. x = \text{idx-of } p\text{-}\alpha \ v \wedge x < \text{length } B$ )
  using assms
  unfolding idx-of-impl-def
  apply (refine-rcg refine-vcg)
  apply (metis I-consistent in-set-conv-nth set-p- $\alpha$ -is-set-S)
  apply (erule S-idx-of-correct)
  apply (simp add: find-seg-impl find-seg-idx-of-correct)
  by (metis find-seg-correct(2) find-seg-impl)

```

```

lemma collapse-correct:
  assumes A:  $v \in \bigcup (\text{set } p\text{-}\alpha)$ 
  shows collapse-impl v  $\leq$  GS-rel (SPEC ( $\lambda r. r = \text{collapse } v \ \alpha$ ))
proof –
  {
    fix i
    assume i < length p- $\alpha$ 
    hence ILEN: i < length B by (simp add: p- $\alpha$ -def)

    let ?SBIP' = (S, take (Suc i) B, I, P)
  }

```

```

{
  have [simp]: GS.seg-start ?SBIP' i = seg-start i
    by (simp add: GS.seg-start-def)

  have [simp]: GS.seg-end ?SBIP' i = seg-end (length B - 1)
    using ILEN by (simp add: GS.seg-end-def min-absorb2)

  {
    fix j
    assume B: seg-start i ≤ j    j < seg-end (length B - Suc 0)
    hence j < length S using ILEN seg-end-bound
    proof -
      note B(2)
      also from ⟨i < length B⟩ have (length B - Suc 0) < length B by auto
      from seg-end-bound[OF this]
      have seg-end (length B - Suc 0) ≤ length S .
      finally show ?thesis .
    qed

    have i ≤ find-seg j ∧ find-seg j < length B
      ∧ seg-start (find-seg j) ≤ j ∧ j < seg-end (find-seg j)
    proof (intro conjI)
      show i ≤ find-seg j
        by (metis le-trans not-less B(1) find-seg-bounds(2)
            seg-end-less-start ILEN ⟨j < length S⟩)
      qed (simp-all add: find-seg-bounds[OF ⟨j < length S⟩])
    } note AUX1 = this

  {
    fix Q and j::nat
    assume Q j
    hence ∃ i. S!j = S!i ∧ Q i
      by blast
    } note AUX-ex-conj-SeqSI = this

  have GS.seg ?SBIP' i = ⋃ (seg ‘ {i..<length B})
    unfolding GS.seg-def[abs-def]
    apply simp
    apply (rule)
    apply (auto dest!: AUX1) []

  apply (auto
    simp: seg-start-def seg-end-def
    split: if-split-asm
    intro!: AUX-ex-conj-SeqSI
  )

```

```

apply (metis diff-diff-cancel le-diff-conv le-eq-less-or-eq
  lessI trans-le-add1
  distinct-sorted-mono[OF B-sorted B-distinct, of i])

apply (metis diff-diff-cancel le-diff-conv le-eq-less-or-eq
  trans-le-add1 distinct-sorted-mono[OF B-sorted B-distinct, of i])

apply (metis (opaque-lifting, no-types) Suc-lessD Suc-lessI less-trans-Suc
  B-in-bound')
done
} note AUX2 = this

from ILEN have GS.p- $\alpha$  (S, take (Suc i) B, I, P) = collapse-aux p- $\alpha$  i
unfolding GS.p- $\alpha$ -def collapse-aux-def
apply (simp add: min-absorb2 drop-map)
apply (rule conjI)
apply (auto
  simp: GS.seg-def[abs-def] GS.seg-start-def GS.seg-end-def take-map) []

apply (simp add: AUX2)
done
} note AUX1 = this

from A obtain i where [simp]: I v = Some (STACK i)
using I-consistent set-p- $\alpha$ -is-set-S
by (auto simp: in-set-conv-nth)

{
have (collapse-aux p- $\alpha$  (idx-of p- $\alpha$  v), D- $\alpha$ , pE- $\alpha$ ) =
  GS. $\alpha$  (S, take (Suc (idx-of p- $\alpha$  v)) B, I, P)
unfolding GS. $\alpha$ -def
using idx-of-props[OF p- $\alpha$ -disjoint-sym A]
by (simp add: AUX1)
} note ABS=this

{
have GS-invar (S, take (Suc (idx-of p- $\alpha$  v)) B, I, P)
apply unfold-locales
apply simp-all

using B-in-bound B-sorted B-distinct
apply (auto simp: sorted-wrt-take dest: in-set-takeD) [3]

using B0 S-distinct apply auto [2]

using I-consistent apply simp

using P-sorted P-distinct P-bound apply auto [3]
done
}

```

} **note**  $INV=this$

**show** *?thesis*

**unfolding** *collapse-impl-def*

**apply** (*refine-rcg SPEC-refine refine-vcg order-trans[OF idx-of-correct]*)

**apply** *fact*

**apply** *simp*

**apply** (*simp add: collapse-def  $\alpha$ -def find-seg-impl GS-rel-def*)

**apply** (*rule brI*)

**apply** (*rule ABS*)

**apply** (*rule INV*)

**done**

**qed**

**end**

Technical adjustment for avoiding case-splits for definitions extracted from GS-locale

**lemma** *opt-GSdef*:  $f \equiv g \implies f s \equiv \text{case } s \text{ of } (S,B,I,P) \Rightarrow g (S,B,I,P)$  **by** *auto*

**lemma** *ext-def*:  $f \equiv g \implies f x \equiv g x$  **by** *auto*

**context** *fr-graph* **begin**

**definition** *push-impl*  $v s \equiv GS.\text{push-impl } s v (E'\{v\})$

**lemmas** *push-impl-def-opt* =

*push-impl-def[abs-def,*

*THEN ext-def, THEN opt-GSdef, unfolded GS.push-impl-def GS-sel-simps]*

Definition for presentation

**lemma** *push-impl*  $v (S,B,I,P) \equiv (S@[v], B@[length S], I(v \rightarrow STACK (length S)),$   
*if*  $E'\{v\}=\{\}$  *then*  $P$  *else*  $P@[length S, E'\{v\}]$ )

**unfolding** *push-impl-def* *GS.push-impl-def* *GS.P-def* *GS.S-def*

**by** (*auto simp: Let-def*)

**lemma** *GS- $\alpha$ -split*:

$GS.\alpha s = (p,D,pE) \longleftrightarrow (p=GS.p-\alpha s \wedge D=GS.D-\alpha s \wedge pE=GS.pE-\alpha s)$

$(p,D,pE) = GS.\alpha s \longleftrightarrow (p=GS.p-\alpha s \wedge D=GS.D-\alpha s \wedge pE=GS.pE-\alpha s)$

**by** (*auto simp add: GS- $\alpha$ -def*)

**lemma** *push-refine*:

**assumes**  $A: (s,(p,D,pE)) \in GS-rel \quad (v,v') \in Id$

**assumes**  $B: v \notin \bigcup (set p) \quad v \notin D$

**shows**  $(\text{push-impl } v s, \text{push } v' (p,D,pE)) \in GS-rel$

**proof** –

**from**  $A$  **have** [*simp*]:  $p=GS.p-\alpha s \wedge D=GS.D-\alpha s \wedge pE=GS.pE-\alpha s \quad v'=v$

**and**  $INV: GS-invar s$

**by** (*auto simp add: GS-rel-def br-def GS- $\alpha$ -split*)

**from**  $INV B$  **show** *?thesis*

by (auto  
 simp: GS-rel-def br-def GS-invar.push-correct push-impl-def push-def)  
 qed

**definition** pop-impl  $s \equiv GS.pop-impl\ s$

**lemmas** pop-impl-def-opt =

pop-impl-def[abs-def, THEN opt-GSdef, unfolded GS.pop-impl-def  
 GS.mark-as-done-def GS.seq-start-def GS.seq-end-def  
 GS-sel-simps]

**lemma** pop-refine:

assumes  $A: (s, (p, D, pE)) \in GS-rel$

assumes  $B: p \neq [] \quad pE \cap last\ p \times UNIV = \{\}$

shows  $pop-impl\ s \leq \Downarrow GS-rel (RETURN (pop (p, D, pE)))$

**proof** –

from  $A$  have [simp]:  $p = GS.p-\alpha\ s \wedge D = GS.D-\alpha\ s \wedge pE = GS.pE-\alpha\ s$

and INV:  $GS-invar\ s$

by (auto simp add: GS-rel-def br-def GS- $\alpha$ -split)

show ?thesis

unfolding pop-impl-def[abs-def] pop-def

apply (rule order-trans[OF GS-invar.pop-correct])

using INV B

apply (simp-all add: Un-commute RETURN-def)

done

qed

**thm** pop-refine[no-vars]

**definition** collapse-impl  $v\ s \equiv GS.collapse-impl\ s\ v$

**lemmas** collapse-impl-def-opt =

collapse-impl-def[abs-def,  
 THEN ext-def, THEN opt-GSdef, unfolded GS.collapse-impl-def GS-sel-simps]

**lemma** collapse-refine:

assumes  $A: (s, (p, D, pE)) \in GS-rel \quad (v, v') \in Id$

assumes  $B: v' \in \bigcup (set\ p)$

shows  $collapse-impl\ v\ s \leq \Downarrow GS-rel (RETURN (collapse\ v' (p, D, pE)))$

**proof** –

from  $A$  have [simp]:  $p = GS.p-\alpha\ s \wedge D = GS.D-\alpha\ s \wedge pE = GS.pE-\alpha\ s \quad v' = v$

and INV:  $GS-invar\ s$

by (auto simp add: GS-rel-def br-def GS- $\alpha$ -split)

show ?thesis

unfolding collapse-impl-def[abs-def]

apply (rule order-trans[OF GS-invar.collapse-correct])

using INV B by (simp-all add: GS- $\alpha$ -def RETURN-def)

qed

**definition** *select-edge-impl*  $s \equiv GS.sel\text{-rem-last } s$

**lemmas** *select-edge-impl-def-opt* =

*select-edge-impl-def*[*abs-def*,

*THEN opt-GSdef*,

*unfolded GS.sel-rem-last-def GS(seg-start-def GS-sel-simps)*]

**lemma** *select-edge-refine*:

**assumes**  $A: (s, (p, D, pE)) \in GS\text{-rel}$

**assumes**  $NE: p \neq []$

**shows** *select-edge-impl*  $s \leq \Downarrow (Id \times_r GS\text{-rel})$  (*select-edge*  $(p, D, pE)$ )

**proof** –

**from**  $A$  **have** [*simp*]:  $p = GS.p\text{-}\alpha \ s \wedge D = GS.D\text{-}\alpha \ s \wedge pE = GS.pE\text{-}\alpha \ s$

**and**  $INV: GS\text{-invar } s$

**by** (*auto simp add: GS-rel-def br-def GS- $\alpha$ -split*)

**from**  $INV$   $NE$  **show** *?thesis*

**unfolding** *select-edge-impl-def*

**using**  $GS\text{-invar.sel-rem-last-correct}$ [*OF INV*]  $NE$

**by** (*simp*)

**qed**

**definition** *initial-impl*  $v0 \ I \equiv GS\text{-initial-impl } I \ v0 \ (E^{\{\{v0\}\}})$

**lemma** *initial-refine*:

$\llbracket v0 \notin D0; (I, D0) \in oGS\text{-rel}; (v0i, v0) \in Id \rrbracket$

$\implies (initial\text{-impl } v0i \ I, initial \ v0 \ D0) \in GS\text{-rel}$

**unfolding** *initial-impl-def*  $GS\text{-rel-def}$   $br\text{-def}$

**apply** (*simp-all add: GS-initial-correct*)

**apply** (*auto simp: initial-def*)

**done**

**definition** *path-is-empty-impl*  $s \equiv GS.S \ s = []$

**lemma** *path-is-empty-refine*:

$GS\text{-invar } s \implies path\text{-is-empty-impl } s \longleftrightarrow GS.p\text{-}\alpha \ s = []$

**unfolding** *path-is-empty-impl-def*  $GS.p\text{-}\alpha\text{-def}$   $GS\text{-invar.empty-eq}$

**by** *auto*

**definition** (**in**  $GS$ ) *is-on-stack-impl*  $v$

$\equiv case \ I \ v \ of \ Some \ (STACK \ -) \ \Rightarrow \ True \ | \ - \ \Rightarrow \ False$

**lemma** (**in**  $GS\text{-invar}$ ) *is-on-stack-impl-correct*:

**shows** *is-on-stack-impl*  $v \longleftrightarrow v \in \bigcup (set \ p\text{-}\alpha)$

**unfolding** *is-on-stack-impl-def*

**using**  $I\text{-consistent}$ [*of v*]

**apply** (*force*)

*simp: set-p- $\alpha$ -is-set-S in-set-conv-nth*

*split: option.split node-state.split*)

**done**

**definition** *is-on-stack-impl*  $v\ s \equiv GS.is-on-stack-impl\ s\ v$

**lemmas** *is-on-stack-impl-def-opt* =

*is-on-stack-impl-def*[*abs-def*, *THEN ext-def*, *THEN opt-GSdef*,  
*unfolded GS.is-on-stack-impl-def GS-sel-simps*]

**lemma** *is-on-stack-refine*:

$\llbracket GS-invar\ s \rrbracket \implies is-on-stack-impl\ v\ s \longleftrightarrow v \in \bigcup (set\ (GS.p-\alpha\ s))$

**unfolding** *is-on-stack-impl-def GS-rel-def br-def*

**by** (*simp add: GS-invar.is-on-stack-impl-correct*)

**definition** (**in** *GS*) *is-done-impl*  $v$

$\equiv case\ I\ v\ of\ Some\ DONE \Rightarrow True\ |\ - \Rightarrow False$

**lemma** (**in** *GS-invar*) *is-done-impl-correct*:

**shows** *is-done-impl*  $v \longleftrightarrow v \in D-\alpha$

**unfolding** *is-done-impl-def D- $\alpha$ -def*

**apply** (*auto split: option.split node-state.split*)

**done**

**definition** *is-done-oimpl*  $v\ I \equiv case\ I\ v\ of\ Some\ DONE \Rightarrow True\ |\ - \Rightarrow False$

**definition** *is-done-impl*  $v\ s \equiv GS.is-done-impl\ s\ v$

**lemma** *is-done-orefine*:

$\llbracket oGS-invar\ s \rrbracket \implies is-done-oimpl\ v\ s \longleftrightarrow v \in oGS-\alpha\ s$

**unfolding** *is-done-oimpl-def oGS-rel-def br-def*

**by** (*auto*

*simp: oGS-invar-def oGS- $\alpha$ -def*

*split: option.splits node-state.split*)

**lemma** *is-done-refine*:

$\llbracket GS-invar\ s \rrbracket \implies is-done-impl\ v\ s \longleftrightarrow v \in GS.D-\alpha\ s$

**unfolding** *is-done-impl-def GS-rel-def br-def*

**by** (*simp add: GS-invar.is-done-impl-correct*)

**lemma** *oinitial-refine*:  $(Map.empty, \{\}) \in oGS-rel$

**by** (*auto simp: oGS-rel-def br-def oGS- $\alpha$ -def oGS-invar-def*)

**end**

#### 1.4.4 Refined Skeleton Algorithm

**context** *fr-graph* **begin**

**lemma** *I-to-outer*:

**assumes**  $((S, B, I, P), (\llbracket, D, \{\})) \in GS-rel$

**shows**  $(I, D) \in oGS-rel$

```

using assms
unfolding GS-rel-def oGS-rel-def br-def oGS- $\alpha$ -def GS. $\alpha$ -def GS.D- $\alpha$ -def GS-invar-def
oGS-invar-def
apply (auto simp: GS.p- $\alpha$ -def)
done

```

**definition** *skeleton-impl* :: '*v* *oGS nres* **where**

```

skeleton-impl  $\equiv$  do {
  stat-start-nres;
  let I=Map.empty;
  r  $\leftarrow$  FOREACHi ( $\lambda$ it I. outer-invar it (oGS- $\alpha$  I)) V0 ( $\lambda$ v0 I0. do {
    if  $\neg$ is-done-oimpl v0 I0 then do {
      let s = initial-impl v0 I0;

      (S,B,I,P) $\leftarrow$  WHILEIT (invar v0 (oGS- $\alpha$  I0) o GS. $\alpha$ )
        ( $\lambda$ s.  $\neg$ path-is-empty-impl s) ( $\lambda$ s.
        do {
          — Select edge from end of path
          (vo,s)  $\leftarrow$  select-edge-impl s;

          case vo of
            Some v  $\Rightarrow$  do {
              if is-on-stack-impl v s then do {
                collapse-impl v s
              } else if  $\neg$ is-done-impl v s then do {
                — Edge to new node. Append to path
                RETURN (push-impl v s)
              } else do {
                — Edge to done node. Skip
                RETURN s
              }
            }
          | None  $\Rightarrow$  do {
            — No more outgoing edges from current node on path
            pop-impl s
          }
        }) s;
        RETURN I
      } else
        RETURN I0
      }) I;
  stat-stop-nres;
  RETURN r
}

```

**Correctness Theorem** lemma *skeleton-impl*  $\leq$   $\Downarrow$ *oGS-rel skeleton*

```

using [[goals-limit = 1]]
unfolding skeleton-impl-def skeleton-def

```

```

apply (refine-rcg
  bind-refine'
  select-edge-refine push-refine
  pop-refine
  collapse-refine
  initial-refine
  oinitial-refine
  inj-on-id
)
using [[goals-limit = 5]]
apply refine-dref-type

apply (vc-solve (nopre) solve: asm-rl I-to-outer
  simp: GS-rel-def br-def GS.α-def oGS-rel-def oGS-α-def
  is-on-stack-refine path-is-empty-refine is-done-refine is-done-orefine
)

done

lemmas skeleton-refines
= select-edge-refine push-refine pop-refine collapse-refine
  initial-refine oinitial-refine
lemmas skeleton-refine-simps
= GS-rel-def br-def GS.α-def oGS-rel-def oGS-α-def
  is-on-stack-refine path-is-empty-refine is-done-refine is-done-orefine

```

Short proof, for presentation

```

context
  notes [[goals-limit = 1]]
  notes [refine] = inj-on-id bind-refine'
begin
lemma skeleton-impl ≤  $\Downarrow$ oGS-rel skeleton
  unfolding skeleton-impl-def skeleton-def
  by (refine-rcg skeleton-refines, refine-dref-type)
  (vc-solve (nopre) solve: asm-rl I-to-outer simp: skeleton-refine-simps)

```

**end**

**end**

**end**

## 1.5 Enumerating the SCCs of a Graph

```

theory Gabow-SCC
imports Gabow-Skeleton
begin

```

As a first variant, we implement an algorithm that computes a list of SCCs of a graph, in topological order. This is the standard variant described by

Gabow [5].

## 1.6 Specification

**context** *fr-graph*  
**begin**

We specify a distinct list that covers all reachable nodes and contains SCCs in topological order

**definition** *compute-SCC-spec*  $\equiv$  *SPEC* ( $\lambda l.$   
 $distinct\ l \wedge \bigcup (set\ l) = E^* \wedge (\forall U \in set\ l. is-scc\ E\ U)$   
 $\wedge (\forall i\ j. i < j \wedge j < length\ l \longrightarrow !j \times !i \cap E^* = \{\})$ )  
**end**

## 1.7 Extended Invariant

**locale** *csc-cc-invar-ext* = *fr-graph* *G*  
**for** *G* :: ('v,'more) *graph-rec-scheme* +  
**fixes** *l* :: 'v *set list* **and** *D* :: 'v *set*  
**assumes** *l-is-D*:  $\bigcup (set\ l) = D$  — The output contains all done CNodes  
**assumes** *l-scc*:  $set\ l \subseteq Collect\ (is-scc\ E)$  — The output contains only SCCs  
**assumes** *l-no-fwd*:  $\bigwedge i\ j. \llbracket i < j; j < length\ l \rrbracket \implies !j \times !i \cap E^* = \{\}$   
— The output contains no forward edges  
**begin**  
**lemma** *l-no-empty*:  $\{\} \notin set\ l$  **using** *l-scc* **by** (*auto simp: in-set-conv-decomp*)  
**end**

**locale** *csc-cc-outer-invar-loc* = *outer-invar-loc* *G* *it* *D* + *csc-cc-invar-ext* *G* *l* *D*  
**for** *G* :: ('v,'more) *graph-rec-scheme* **and** *it* *l* *D*  
**begin**  
**lemma** *locale-this*: *csc-cc-outer-invar-loc* *G* *it* *l* *D* **by** *unfold-locales*  
**lemma** *abs-outer-this*: *outer-invar-loc* *G* *it* *D* **by** *unfold-locales*  
**end**

**locale** *csc-cc-invar-loc* = *invar-loc* *G* *v0* *D0* *p* *D* *pE* + *csc-cc-invar-ext* *G* *l* *D*  
**for** *G* :: ('v,'more) *graph-rec-scheme* **and** *v0* *D0* **and** *l* :: 'v *set list*  
**and** *p* *D* *pE*  
**begin**  
**lemma** *locale-this*: *csc-cc-invar-loc* *G* *v0* *D0* *l* *p* *D* *pE* **by** *unfold-locales*  
**lemma** *invar-this*: *invar-loc* *G* *v0* *D0* *p* *D* *pE* **by** *unfold-locales*  
**end**

**context** *fr-graph*  
**begin**  
**definition** *csc-cc-outer-invar*  $\equiv$   $\lambda it\ (l,D). csc-cc-outer-invar-loc\ G\ it\ l\ D$   
**definition** *csc-cc-invar*  $\equiv$   $\lambda v0\ D0\ (l,p,D,pE). csc-cc-invar-loc\ G\ v0\ D0\ l\ p\ D\ pE$   
**end**

## 1.8 Definition of the SCC-Algorithm

**context** *fr-graph*

**begin**

**definition** *compute-SCC* :: '*v set list nres* **where**

*compute-SCC*  $\equiv$  *do* {

*let* *so* = ( $\square$ , { $\}$ );

  (*l*, *D*)  $\leftarrow$  *FOREACH**i* *csc**c*-*outer-invar* *V0* ( $\lambda v0$  (*l*, *D0*)). *do* {

*if*  $v0 \notin D0$  *then do* {

*let* *s* = (*l*, *initial* *v0* *D0*);

      (*l*, *p*, *D*, *pE*)  $\leftarrow$

$WHILEIT$  (*csc**c*-*invar* *v0* *D0*)

      ( $\lambda(l, p, D, pE). p \neq \square$ ) ( $\lambda(l, p, D, pE).$

*do* {

        — Select edge from end of path

        (*vo*, (*p*, *D*, *pE*))  $\leftarrow$  *select-edge* (*p*, *D*, *pE*);

$ASSERT$  ( $p \neq \square$ );

*case* *vo* *of*

*Some* *v*  $\Rightarrow$  *do* {

*if*  $v \in \bigcup(\text{set } p)$  *then do* {

            — Collapse

$RETURN$  (*l*, *collapse* *v* (*p*, *D*, *pE*))

          } *else if*  $v \notin D$  *then do* {

            — Edge to new node. Append to path

$RETURN$  (*l*, *push* *v* (*p*, *D*, *pE*))

          } *else*  $RETURN$  (*l*, *p*, *D*, *pE*)

        }

        | *None*  $\Rightarrow$  *do* {

          — No more outgoing edges from current node on path

$ASSERT$  ( $pE \cap \text{last } p \times UNIV = \{\}$ );

*let* *V* = *last* *p*;

*let* (*p*, *D*, *pE*) = *pop* (*p*, *D*, *pE*);

*let* *l* = *V* # *l*;

$RETURN$  (*l*, *p*, *D*, *pE*)

        }

      }) *s*;

$ASSERT$  ( $p = \square \wedge pE = \{\}$ );

$RETURN$  (*l*, *D*)

    } *else*

$RETURN$  (*l*, *D0*)

  }) *so*;

$RETURN$  *l*

}

**end**

## 1.9 Preservation of Invariant Extension

**context** *csc**c*-*invar-ext*

```

begin
  lemma l-disjoint:
    assumes A:  $i < j$     $j < \text{length } l$ 
    shows  $!!i \cap !!j = \{\}$ 
  proof (rule disjointI)
    fix u
    assume  $u \in !!i$     $u \in !!j$ 
    with l-no-fwd A show False by auto
  qed

  corollary l-distinct: distinct l
    using l-disjoint l-no-empty
    by (metis distinct-conv-nth inf-idem linorder-cases nth-mem)
end

context fr-graph
begin
  definition csc c cc invar-part  $\equiv \lambda(l,p,D,pE). \text{csc} \text{c} \text{cc} \text{invar-ext } G \ l \ D$ 

  lemma csc c cc invarI [intro?]:
    assumes invar v0 D0 PDPE
    assumes invar v0 D0 PDPE  $\implies \text{csc} \text{c} \text{cc} \text{invar-part } (l,PDPE)$ 
    shows csc c cc invar v0 D0  $(l,PDPE)$ 
    using assms
    unfolding initial-def csc c cc invar-def invar-def
    apply (simp split: prod.split-asm)
    apply intro-locales
    apply (simp add: invar-loc-def)
    apply (simp add: csc c cc invar-part-def csc c cc invar-ext-def)
    done

  thm csc c cc invarI [of v-0 D-0 s l]

  lemma csc c cc outer-invarI [intro?]:
    assumes outer-invar it D
    assumes outer-invar it D  $\implies \text{csc} \text{c} \text{cc} \text{invar-ext } G \ l \ D$ 
    shows csc c cc outer-invar it  $(l,D)$ 
    using assms
    unfolding initial-def csc c cc outer-invar-def outer-invar-def
    apply (simp split: prod.split-asm)
    apply intro-locales
    apply (simp add: outer-invar-loc-def)
    apply (simp add: csc c cc invar-ext-def)
    done

  lemma csc c cc invar-initial [simp, intro!]:
    assumes A:  $v0 \in it$     $v0 \notin D0$ 
    assumes INV: csc c cc outer-invar it  $(l,D0)$ 
    shows csc c cc invar-part  $(l,\text{initial } v0 \ D0)$ 

```

```

proof –
  from INV interpret csc-outer-invar-loc G it l D0
    unfolding csc-outer-invar-def by simp

  show ?thesis
    unfolding csc-invar-part-def initial-def
    apply simp
    by unfold-locales
qed

lemma csc-invar-pop:
  assumes INV: csc-invar v0 D0 (l,p,D,pE)
  assumes invar v0 D0 (pop (p,D,pE))
  assumes NE[simp]: p≠[]
  assumes NO':  $pE \cap (\text{last } p \times UNIV) = \{\}$ 
  shows csc-invar-part (last p \# l, pop (p,D,pE))
proof –
  from INV interpret csc-invar-loc G v0 D0 l p D pE
    unfolding csc-invar-def by simp

  have AUX-l-scc: is-scc E (last p)
    unfolding is-scc-pointwise
proof safe
  {
    assume last p = {} thus False
    using p-no-empty by (cases p rule: rev-cases) auto
  }

  fix u v
  assume  $u \in \text{last } p \quad v \in \text{last } p$ 
  with p-sc[of last p] have  $(u,v) \in (lvE \cap \text{last } p \times \text{last } p)^*$  by auto
  with lvE-ss-E show  $(u,v) \in (E \cap \text{last } p \times \text{last } p)^*$ 
    by (metis Int-mono equalityE rtrancl-mono-mp)

  fix u'
  assume  $u' \notin \text{last } p \quad (u,u') \in E^* \quad (u',v) \in E^*$ 

  from  $\langle u' \notin \text{last } p \rangle \langle u \in \text{last } p \rangle \langle (u,u') \in E^* \rangle$ 
    and rtrancl-reachable-induct[OF order-refl lastp-un-D-closed[OF NE NO']]
  have  $u' \in D$  by auto
  with  $\langle (u',v) \in E^* \rangle$  and rtrancl-reachable-induct[OF order-refl D-closed]
  have  $v \in D$  by auto
  with  $\langle v \in \text{last } p \rangle$  p-not-D show False by (cases p rule: rev-cases) auto
qed

{
  fix i j
  assume A:  $i < j \quad j < \text{Suc } (\text{length } l)$ 
  have  $l ! (j - \text{Suc } 0) \times (\text{last } p \# l) ! i \cap E^* = \{\}$ 

```

```

proof (rule disjointI, safe)
  fix u v
  assume  $(u, v) \in E^*$   $u \in l ! (j - \text{Suc } 0)$   $v \in (\text{last } p \# l) ! i$ 
  from  $\langle u \in l ! (j - \text{Suc } 0) \rangle A$  have  $u \in \bigcup (\text{set } l)$ 
    by (metis Ex-list-of-length Suc-pred UnionI length-greater-0-conv
      less-nat-zero-code not-less-eq nth-mem)
  with l-is-D have  $u \in D$  by simp
  with rtrancl-reachable-induct[OF order-refl D-closed]  $\langle (u, v) \in E^* \rangle$ 
  have  $v \in D$  by auto

  show False proof cases
    assume  $i=0$  hence  $v \in \text{last } p$  using  $\langle v \in (\text{last } p \# l) ! i \rangle$  by simp
    with p-not-D  $\langle v \in D \rangle$  show False by (cases p rule: rev-cases) auto
  next
    assume  $i \neq 0$  with  $\langle v \in (\text{last } p \# l) ! i \rangle$  have  $v \in l!(i - 1)$  by auto
    with l-no-fwd[of  $i - 1$   $j - 1$ ]
      and  $\langle u \in l ! (j - \text{Suc } 0) \rangle \langle (u, v) \in E^* \rangle \langle i \neq 0 \rangle A$ 
    show False by fastforce
  qed
qed
} note AUX-l-no-fwd = this

show ?thesis
  unfolding csccl-invar-part-def pop-def apply simp
  apply unfold-locales
  apply clarsimp-all
  using l-is-D apply auto []

  using l-scc AUX-l-scc apply auto []

  apply (rule AUX-l-no-fwd, assumption+) []
  done
qed

thm csccl-invar-pop[of v-0 D-0 l p D pE]

lemma csccl-invar-unchanged:
  assumes INV: csccl-invar v0 D0 (l,p,D,pE)
  shows csccl-invar-part (l,p',D,pE')
  using INV unfolding csccl-invar-def csccl-invar-part-def csccl-invar-loc-def
  by simp

corollary csccl-invar-collapse:
  assumes INV: csccl-invar v0 D0 (l,p,D,pE)
  shows csccl-invar-part (l,collapse v (p',D,pE'))
  unfolding collapse-def
  by (simp add: csccl-invar-unchanged[OF INV])

corollary csccl-invar-push:

```

**assumes**  $INV: csc\text{-}invar\ v0\ D0\ (l,p,D,pE)$   
**shows**  $csc\text{-}invar\text{-}part\ (l,push\ v\ (p',D,pE'))$   
**unfolding**  $push\text{-}def$   
**by** ( $simp\ add: csc\text{-}invar\text{-}unchanged[OF\ INV]$ )

**lemma**  $csc\text{-}outer\text{-}invar\text{-}initial: csc\text{-}invar\text{-}ext\ G\ []\ \{\}$   
**by**  $unfold\text{-}locales\ auto$

**lemma**  $csc\text{-}invar\text{-}outer\text{-}newnode:$

**assumes**  $A: v0 \notin D0\ \ v0 \in it$   
**assumes**  $OINV: csc\text{-}outer\text{-}invar\ it\ (l,D0)$   
**assumes**  $INV: csc\text{-}invar\ v0\ D0\ (l',[],D',pE)$   
**shows**  $csc\text{-}invar\text{-}ext\ G\ l'\ D'$

**proof** –

**from**  $OINV$  **interpret**  $csc\text{-}outer\text{-}invar\text{-}loc\ G\ it\ l\ D0$   
**unfolding**  $csc\text{-}outer\text{-}invar\text{-}def$  **by**  $simp$   
**from**  $INV$  **interpret**  $inv: csc\text{-}invar\text{-}loc\ G\ v0\ D0\ l'\ []\ D'\ pE$   
**unfolding**  $csc\text{-}invar\text{-}def$  **by**  $simp$

**show**  $?thesis$   
**by**  $unfold\text{-}locales$

**qed**

**lemma**  $csc\text{-}invar\text{-}outer\text{-}Dnode:$

**assumes**  $csc\text{-}outer\text{-}invar\ it\ (l, D)$   
**shows**  $csc\text{-}invar\text{-}ext\ G\ l\ D$   
**using**  $assms$   
**by** ( $simp\ add: csc\text{-}outer\text{-}invar\text{-}def\ csc\text{-}outer\text{-}invar\text{-}loc\text{-}def$ )

**lemmas**  $csc\text{-}invar\text{-}preserve = invar\text{-}preserve$

$csc\text{-}invar\text{-}initial$   
 $csc\text{-}invar\text{-}pop\ csc\text{-}invar\text{-}collapse\ csc\text{-}invar\text{-}push\ csc\text{-}invar\text{-}unchanged$   
 $csc\text{-}outer\text{-}invar\text{-}initial\ csc\text{-}invar\text{-}outer\text{-}newnode\ csc\text{-}invar\text{-}outer\text{-}Dnode$

On termination, the invariant implies the specification

**lemma**  $csc\text{-}finI:$

**assumes**  $INV: csc\text{-}outer\text{-}invar\ \{\}\ (l,D)$   
**shows**  $fin\text{-}l\text{-}is\text{-}scc: \llbracket U \in set\ l \rrbracket \implies is\text{-}scc\ E\ U$   
**and**  $fin\text{-}l\text{-}distinct: distinct\ l$   
**and**  $fin\text{-}l\text{-}is\text{-}reachable: \bigcup (set\ l) = E^*$  “  $V0$   
**and**  $fin\text{-}l\text{-}no\text{-}fwd: \llbracket i < j; j < length\ l \rrbracket \implies l[j] \times l[i] \cap E^* = \{\}$

**proof** –

**from**  $INV$  **interpret**  $csc\text{-}outer\text{-}invar\text{-}loc\ G\ \{\}\ l\ D$   
**unfolding**  $csc\text{-}outer\text{-}invar\text{-}def$  **by**  $simp$

**show**  $\llbracket U \in set\ l \rrbracket \implies is\text{-}scc\ E\ U$  **using**  $l\text{-}scc$  **by**  $auto$

```

show distinct l by (rule l-distinct)

show  $\bigcup (\text{set } l) = E^* \text{ “ } V0$ 
  using fin-outer-D-is-reachable[OF outer-invar-this] l-is-D
  by auto

show  $\llbracket i < j; j < \text{length } l \rrbracket \implies l!j \times l!i \cap E^* = \{\}$ 
  by (rule l-no-fwd)

qed

end

```

## 1.10 Main Correctness Proof

```

context fr-graph
begin
  lemma invar-from-cscC-invarI: cscC-invar v0 D0 (L,PDPE)  $\implies$  invar v0 D0
  PDPE
    unfolding cscC-invar-def invar-def
    apply (simp split: prod.splits)
    unfolding cscC-invar-loc-def by simp

  lemma outer-invar-from-cscC-invarI:
    cscC-outer-invar it (L,D)  $\implies$  outer-invar it D
    unfolding cscC-outer-invar-def outer-invar-def
    apply (simp split: prod.splits)
    unfolding cscC-outer-invar-loc-def by simp

```

With the extended invariant and the auxiliary lemmas, the actual correctness proof is straightforward:

```

theorem compute-SCC-correct: compute-SCC  $\leq$  compute-SCC-spec
proof –
  note  $\llbracket \text{goals-limit} = 2 \rrbracket$ 
  note  $\llbracket \text{simp del} \rrbracket = \text{Union-iff}$ 

  show ?thesis
    unfolding compute-SCC-def compute-SCC-spec-def select-edge-def select-def
    apply (refine-rcg
      WHILEIT-rule[where R=inv-image (abs-wf-rel v0) snd for v0]
      refine-vcg
    )

    apply (vc-solve
      rec: cscC-invarI cscC-outer-invarI
      solve: cscC-invar-preserve cscC-finI
      intro: invar-from-cscC-invarI outer-invar-from-cscC-invarI
      dest!: sym[of pop A for A]
    )

```

```

    simp: pE-fin'[OF invar-from-cscc-invarI] finite-V0
  )
  apply auto
  done
qed

```

Simple proof, for presentation

```

context
  notes [refine]=refine-vcg
  notes [[goals-limit = 1]]
begin
  theorem compute-SCC ≤ compute-SCC-spec
  unfolding compute-SCC-def compute-SCC-spec-def select-edge-def select-def
  by (refine-rcg
    WHILEIT-rule[where R=inv-image (abs-wf-rel v0) snd for v0])
  (vc-solve
    rec: cscc-invarI cscc-outer-invarI solve: cscc-invar-preserve cscc-finI
    intro: invar-from-cscc-invarI outer-invar-from-cscc-invarI
    dest!: sym[of pop A for A]
    simp: pE-fin'[OF invar-from-cscc-invarI] finite-V0, auto)
end
end

```

## 1.11 Refinement to Gabow's Data Structure

```

context GS begin
  definition seg-set-impl l u ≡ do {
    (-,res) ← WHILET
      (λ(l,-). l < u)
      (λ(l,res). do {
        ASSERT (l < length S);
        let x = S!l;
        ASSERT (x ∉ res);
        RETURN (Suc l,insert x res)
      })
    (l,{});

    RETURN res
  }

  lemma seg-set-impl-aux:
  fixes l u
  shows [[l < u; u ≤ length S; distinct S] ⇒ seg-set-impl l u
  ≤ SPEC (λr. r = {S!j | j. l ≤ j ∧ j < u})]
  unfolding seg-set-impl-def
  apply (refine-rcg
    WHILET-rule[where
      I=λ(l',res). res = {S!j | j. l ≤ j ∧ j < l'} ∧ l ≤ l' ∧ l' ≤ u

```

```

    and R=measure ( $\lambda(l',-). u-l'$ )
  ]
  refine-vcg)

```

```

apply (auto simp: less-Suc-eq nth-eq-iff-index-eq)
done

```

```

lemma (in GS-invar) seg-set-impl-correct:
assumes  $i < \text{length } B$ 
shows  $\text{seg-set-impl } (\text{seg-start } i) (\text{seg-end } i) \leq \text{SPEC } (\lambda r. r = p-\alpha!i)$ 
apply (refine-rcg order-trans[OF seg-set-impl-aux] refine-vcg)

using assms
apply (simp-all add: seg-start-less-end seg-end-bound S-distinct) [3]

apply (auto simp: p- $\alpha$ -def assms seg-def) []
done

```

```

definition last-seg-impl
 $\equiv$  do {
  ASSERT ( $\text{length } B - 1 < \text{length } B$ );
  seg-set-impl (seg-start ( $\text{length } B - 1$ )) (seg-end ( $\text{length } B - 1$ ))
}

```

```

lemma (in GS-invar) last-seg-impl-correct:
assumes  $p-\alpha \neq []$ 
shows  $\text{last-seg-impl} \leq \text{SPEC } (\lambda r. r = \text{last } p-\alpha)$ 
unfolding last-seg-impl-def
apply (refine-rcg order-trans[OF seg-set-impl-correct] refine-vcg)
using assms apply (auto simp add: p- $\alpha$ -def last-conv-nth)
done

```

**end**

```

context fr-graph
begin

```

```

definition last-seg-impl  $s \equiv \text{GS.last-seg-impl } s$ 
lemmas last-seg-impl-def-opt =
  last-seg-impl-def[abs-def, THEN opt-GSdef,
  unfolded GS.last-seg-impl-def GS.seg-set-impl-def
  GS.seg-start-def GS.seg-end-def GS.sel-simps]

```

```

lemma last-seg-impl-refine:
assumes  $A: (s, (p, D, pE)) \in \text{GS-rel}$ 
assumes  $NE: p \neq []$ 
shows  $\text{last-seg-impl } s \leq \Downarrow \text{Id } (\text{RETURN } (\text{last } p))$ 
proof -

```

**from** *A* **have**  
 [*simp*]:  $p = GS.p\text{-}\alpha\ s \wedge D = GS.D\text{-}\alpha\ s \wedge pE = GS.pE\text{-}\alpha\ s$   
**and** *INV*: *GS-invar* *s*  
**by** (*auto simp add: GS-rel-def br-def GS- $\alpha$ -split*)

**show** *?thesis*  
**unfolding** *last-seg-impl-def[abs-def]*  
**apply** (*rule order-trans[OF GS-invar.last-seg-impl-correct]*)  
**using** *INV NE*  
**apply** (*simp-all*)  
**done**  
**qed**

**definition** *compute-SCC-impl* :: '*v* set list *nres* **where**

```

compute-SCC-impl  $\equiv$  do {
  stat-start-nres;
  let so = ( $\square$ , Map.empty);
  (l, D)  $\leftarrow$  FOREACHi ( $\lambda$ it (l, s). csc-outer-invar it (l, oGS- $\alpha$  s))
  V0 ( $\lambda$ v0 (l, IO). do {
    if  $\neg$ is-done-impl v0 IO then do {
      let ls = (l, initial-impl v0 IO);

      (l, (S, B, I, P))  $\leftarrow$  WHILEIT ( $\lambda$ (l, s). csc-invar v0 (oGS- $\alpha$  IO) (l, GS. $\alpha$  s))
        ( $\lambda$ (l, s).  $\neg$ path-is-empty-impl s) ( $\lambda$ (l, s).
      do {
        — Select edge from end of path
        (vo, s)  $\leftarrow$  select-edge-impl s;

        case vo of
        Some v  $\Rightarrow$  do {
          if is-on-stack-impl v s then do {
            s  $\leftarrow$  collapse-impl v s;
            RETURN (l, s)
          } else if  $\neg$ is-done-impl v s then do {
            — Edge to new node. Append to path
            RETURN (l, push-impl v s)
          } else do {
            — Edge to done node. Skip
            RETURN (l, s)
          }
        }
      }
    }
  }
  | None  $\Rightarrow$  do {
    — No more outgoing edges from current node on path
    scc  $\leftarrow$  last-seg-impl s;
    s  $\leftarrow$  pop-impl s;
    let l = scc#l;
    RETURN (l, s)
  }
} (ls);

```

```

      RETURN (l,I)
    } else RETURN (l,I0)
  }) so;
  stat-stop-nres;
  RETURN l
}

```

**lemma** *compute-SCC-impl-refine: compute-SCC-impl  $\leq \Downarrow Id$  compute-SCC*

**proof** –

**note** [*refine2*] = *bind-Let-refine2*[*OF last-seg-impl-refine*]

```

have [refine2]:  $\bigwedge s' p D pE l' l v' v.$  [
  (s',(p,D,pE)) $\in$ GS-rel;
  (l',l) $\in$ Id;
  (v',v) $\in$ Id;
  v $\in$  $\bigcup$ (set p)
]  $\implies$  do { s' $\leftarrow$ collapse-impl v' s'; RETURN (l',s') }
 $\leq \Downarrow (Id \times_r GS\text{-rel}) (RETURN (l, collapse v (p,D,pE)))$ 
apply (refine-rcg order-trans[OF collapse-refine] refine-vcg)
apply assumption+
apply (auto simp add: pw-le-iff refine-pw-simps)
done

```

**note** [[*goals-limit* = 1]]

**show** ?*thesis*

**unfolding** *compute-SCC-impl-def compute-SCC-def*

**apply** (*refine-rcg*  
*bind-refine'*  
*select-edge-refine push-refine*  
*pop-refine*

*initial-refine*  
*oinitial-refine*

*prod-relI IdI*  
*inj-on-id*

)

**apply** *refine-dref-type*  
**apply** (*vc-solve (nopre) solve: asm-rl I-to-outer*  
*simp: GS-rel-def br-def GS.alpha-def oGS-rel-def oGS.alpha-def*  
*is-on-stack-refine path-is-empty-refine is-done-refine is-done-orefine*  
 )

**done**

**qed**

**end**

end

## 1.12 Safety-Property Model-Checker

```
theory Find-Path
imports
  CAVA-Automata.Digraph
  CAVA-Base.CAVA-Code-Target
begin
```

### 1.13 Finding Path to Error

This function searches a graph and a set of start nodes for a reachable node that satisfies some property, and returns a path to such a node iff it exists.

```
definition find-path E U0 P  $\equiv$  do {
  ASSERT (finite U0);
  ASSERT (finite (E*“U0));
  SPEC ( $\lambda p$ . case p of
    Some (p,v)  $\Rightarrow$   $\exists u0 \in U0$ . path E u0 p v  $\wedge$  P v  $\wedge$  ( $\forall v \in \text{set } p$ .  $\neg$  P v)
  | None  $\Rightarrow$   $\forall u0 \in U0$ .  $\forall v \in E^*“\{u0\}$ .  $\neg$  P v)
}
```

```
lemma find-path-ex-rule:
assumes finite U0
assumes finite (E*“U0)
assumes  $\exists v \in E^*“U0$ . P v
shows find-path E U0 P  $\leq$  SPEC ( $\lambda r$ .
   $\exists p v$ .  $r = \text{Some } (p,v) \wedge P v \wedge (\forall v \in \text{set } p$ .  $\neg$  P v)  $\wedge$  ( $\exists u0 \in U0$ . path E u0 p v))
unfolding find-path-def
using assms
by (fastforce split: option.splits)
```

#### 1.13.1 Nontrivial Paths

```
definition find-path1 E u0 P  $\equiv$  do {
  ASSERT (finite (E*“{u0}));
  SPEC ( $\lambda p$ . case p of
    Some (p,v)  $\Rightarrow$  path E u0 p v  $\wedge$  P v  $\wedge$   $p \neq []$ 
  | None  $\Rightarrow$   $\forall v \in E^+“\{u0\}$ .  $\neg$  P v))
```

```
lemma (in -) find-path1-ex-rule:
assumes finite (E*“{u0})
assumes  $\exists v \in E^+“\{u0\}$ . P v
shows find-path1 E u0 P  $\leq$  SPEC ( $\lambda r$ .
   $\exists p v$ .  $r = \text{Some } (p,v) \wedge p \neq [] \wedge P v \wedge \text{path } E u0 p v$ )
unfolding find-path1-def
using assms
by (fastforce split: option.splits)
```

end

## 1.14 Lasso Finding Algorithm for Generalized Büchi Graphs

**theory** *Gabow-GBG*

**imports**

*Gabow-Skeleton*

*CAVA-Automata.Lasso*

*Find-Path*

**begin**

**locale** *igb-fr-graph* =

*igb-graph*  $G$  + *fr-graph*  $G$

**for**  $G :: ('Q, 'more)$  *igb-graph-rec-scheme*

**lemma** *igb-fr-graphI*:

**assumes** *igb-graph*  $G$

**assumes** *finite*  $((g-E\ G)^* \text{ `` } g-V0\ G)$

**shows** *igb-fr-graph*  $G$

**proof** –

**interpret** *igb-graph*  $G$  **by** *fact*

**show** *?thesis* **using** *assms(2)* **by** *unfold-locales*

**qed**

We implement an algorithm that computes witnesses for the non-emptiness of Generalized Büchi Graphs (GBG).

## 1.15 Specification

**context** *igb-graph*

**begin**

**definition** *ce-correct*

— Specifies a correct counter-example

**where**

*ce-correct*  $Vr\ Vl \equiv (\exists pr\ pl.$

$Vr \subseteq E^* \text{ `` } V0 \wedge Vl \subseteq E^* \text{ `` } V0$  — Only reachable nodes are covered

$\wedge set\ pr \subseteq Vr \wedge set\ pl \subseteq Vl$  — The paths are inside the specified sets

$\wedge Vl \times Vl \subseteq (E \cap Vl \times Vl)^*$  —  $Vl$  is mutually connected

$\wedge Vl \times Vl \cap E \neq \{\}$  —  $Vl$  is non-trivial

$\wedge is-lasso-prpl\ (pr, pl)$  — Paths form a lasso

**definition** *find-ce-spec*  $:: ('Q\ set \times 'Q\ set)\ option\ nres$  **where**

*find-ce-spec*  $\equiv SPEC\ (\lambda r. case\ r\ of$

*None*  $\Rightarrow (\forall prpl. \neg is-lasso-prpl\ prpl)$

| *Some*  $(Vr, Vl) \Rightarrow ce-correct\ Vr\ Vl$

)

**definition** *find-lasso-spec* :: ('Q list × 'Q list) option nres **where**  
*find-lasso-spec* ≡ SPEC (λr. case r of  
 None ⇒ (∀ prpl. ¬is-lasso-prpl prpl)  
 | Some prpl ⇒ is-lasso-prpl prpl  
 )

**end**

## 1.16 Invariant Extension

Extension of the outer invariant:

**context** *igb-fr-graph*

**begin**

**definition** *no-acc-over*

— Specifies that there is no accepting cycle touching a set of nodes

**where**

*no-acc-over* D ≡ ¬(∃ v ∈ D. ∃ pl. pl ≠ [] ∧ path E v pl v ∧  
 (∀ i < num-acc. ∃ q ∈ set pl. i ∈ acc q))

**definition** *fgl-outer-invar-ext* ≡ λit (brk, D).

case brk of None ⇒ *no-acc-over* D | Some (Vr, Vl) ⇒ *ce-correct* Vr Vl

**definition** *fgl-outer-invar* ≡ λit (brk, D). case brk of

None ⇒ *outer-invar* it D ∧ *no-acc-over* D

| Some (Vr, Vl) ⇒ *ce-correct* Vr Vl

**end**

Extension of the inner invariant:

**locale** *fgl-invar-loc* =

*invar-loc* G v0 D0 p D pE

+ *igb-graph* G

**for** G :: ('Q, 'more) *igb-graph-rec-scheme*

**and** v0 D0 **and** brk :: ('Q set × 'Q set) option **and** p D pE +

**assumes** *no-acc*: brk = None ⇒ ¬(∃ v pl. pl ≠ [] ∧ path lvE v pl v ∧  
 (∀ i < num-acc. ∃ q ∈ set pl. i ∈ acc q)) — No accepting cycle over visited edges

**assumes** *acc*: brk = Some (Vr, Vl) ⇒ *ce-correct* Vr Vl

**begin**

**lemma** *locale-this*: *fgl-invar-loc* G v0 D0 brk p D pE

**by** *unfold-locales*

**lemma** *invar-loc-this*: *invar-loc* G v0 D0 p D pE **by** *unfold-locales*

**lemma** *eas-gba-graph-this*: *igb-graph* G **by** *unfold-locales*

**end**

**definition** (**in** *igb-graph*) *fgl-invar* v0 D0 ≡

λ(brk, p, D, pE). *fgl-invar-loc* G v0 D0 brk p D pE

## 1.17 Definition of the Lasso-Finding Algorithm

context *igb-fr-graph*

begin

**definition** *find-ce* :: ('Q set × 'Q set) option nres **where**

*find-ce* ≡ do {

let  $D = \{\}$ ;

(*brk*,-) ← FOREACHci *fgl-outer-invar*  $V0$

( $\lambda$ (*brk*,-). *brk*=None)

( $\lambda v0$  (*brk*, $D0$ ). do {

if  $v0 \notin D0$  then do {

let  $s = (\text{None}, \text{initial } v0 \ D0)$ ;

(*brk*, $p$ , $D$ , $pE$ ) ← WHILEIT (*fgl-invar*  $v0 \ D0$ )

( $\lambda$ (*brk*, $p$ , $D$ , $pE$ ). *brk*=None  $\wedge p \neq []$ ) ( $\lambda(-,p,D,pE)$ .

do {

— Select edge from end of path

( $vo$ ,( $p$ , $D$ , $pE$ )) ← *select-edge* ( $p$ , $D$ , $pE$ );

ASSERT ( $p \neq []$ );

case  $vo$  of

Some  $v \Rightarrow$  do {

if  $v \in \bigcup(\text{set } p)$  then do {

— Collapse

let ( $p$ , $D$ , $pE$ ) = *collapse*  $v$  ( $p$ , $D$ , $pE$ );

ASSERT ( $p \neq []$ );

if  $\forall i < \text{num-acc}. \exists q \in \text{last } p. i \in \text{acc } q$  then

RETURN (Some ( $\bigcup(\text{set } (\text{butlast } p))$ ), $\text{last } p$ ), $p$ , $D$ , $pE$ )

else

RETURN (None, $p$ , $D$ , $pE$ )

} else if  $v \notin D$  then do {

— Edge to new node. Append to path

RETURN (None,*push*  $v$  ( $p$ , $D$ , $pE$ ))

} else RETURN (None, $p$ , $D$ , $pE$ )

}

| None  $\Rightarrow$  do {

— No more outgoing edges from current node on path

ASSERT ( $pE \cap \text{last } p \times \text{UNIV} = \{\}$ );

RETURN (None,*pop* ( $p$ , $D$ , $pE$ ))

}

})  $s$ ;

ASSERT (*brk*=None  $\longrightarrow (p=[] \wedge pE=\{\})$ );

RETURN (*brk*, $D$ )

} else

RETURN (*brk*, $D0$ )

}) (None, $D$ );

RETURN *brk*

}

end

## 1.18 Invariant Preservation

context *igb-fr-graph*

begin

**definition** *fgl-invar-part*  $\equiv \lambda(\text{brk}, p, D, pE).$   
*fgl-invar-loc-axioms* *G brk p D pE*

**lemma** *fgl-outer-invarI*[*intro?*]:

[[  
  *brk=None*  $\implies$  *outer-invar it D*;  
  [[*brk=None*  $\implies$  *outer-invar it D*]]  $\implies$  *fgl-outer-invar-ext it (brk,D)*]]  
 $\implies$  *fgl-outer-invar it (brk,D)*

**unfolding** *outer-invar-def fgl-outer-invar-ext-def fgl-outer-invar-def*

**apply** (*auto split: prod.splits option.splits*)

**done**

**lemma** *fgl-invarI*[*intro?*]:

[[  
  *invar v0 D0 PDPE*;  
  *invar v0 D0 PDPE*  $\implies$  *fgl-invar-part (B,PDPE)*]]  
 $\implies$  *fgl-invar v0 D0 (B,PDPE)*

**unfolding** *invar-def fgl-invar-part-def fgl-invar-def*

**apply** (*simp split: prod.split-asm*)

**apply** *intro-locales*

**apply** (*simp add: invar-loc-def*)

**apply** *assumption*

**done**

**lemma** *fgl-invar-initial*:

**assumes** *OINV: fgl-outer-invar it (None,D0)*

**assumes** *A: v0 ∈ it    v0 ∉ D0*

**shows** *fgl-invar-part (None, initial v0 D0)*

**proof** –

**from** *OINV interpret outer-invar-loc G it D0*

**by** (*simp add: fgl-outer-invar-def outer-invar-def*)

**from** *OINV have no-acc: no-acc-over D0*

**by** (*simp add: fgl-outer-invar-def fgl-outer-invar-ext-def*)

{

**fix** *v pl*

**assume** *pl ≠ [] and P: path (vE [{v0}] D0 (E ∩ {v0} × UNIV)) v pl v*

**hence** *1: v ∈ D0*

**by** (*cases pl (auto simp: path-cons-conv vE-def touched-def)*)

**have** *2: path E v pl v using path-mono[OF vE-ss-E P] .*

```

note 1 2
} note AUX1=this

```

```

show ?thesis
unfolding fgl-invar-part-def
apply (simp split: prod.splits add: initial-def)
apply unfold-locales
using ⟨v0 ∉ D0⟩
using AUX1 no-acc unfolding no-acc-over-def apply blast
by simp
qed

```

```

lemma fgl-invar-pop:
assumes INV: fgl-invar v0 D0 (None,p,D,pE)
assumes INV': invar v0 D0 (pop (p,D,pE))
assumes NE[simp]: p≠[]
assumes NO': pE ∩ last p × UNIV = {}
shows fgl-invar-part (None, pop (p,D,pE))
proof –
from INV interpret fgl-invar-loc G v0 D0 None p D pE
by (simp add: fgl-invar-def)

```

```

show ?thesis
apply (unfold fgl-invar-part-def pop-def)
apply (simp split: prod.splits)
apply unfold-locales
unfolding vE-pop[OF NE]

using no-acc apply auto []
apply simp
done

```

**qed**

```

lemma fgl-invar-collapse-ce-aux:
assumes INV: invar v0 D0 (p, D, pE)
assumes NE[simp]: p≠[]
assumes NONTRIV: vE p D pE ∩ (last p × last p) ≠ {}
assumes ACC: ∀ i < num-acc. ∃ q ∈ last p. i ∈ acc q
shows fgl-invar-part (Some (⋃ (set (butlast p)), last p), p, D, pE)
proof –
from INV interpret invar-loc G v0 D0 p D pE by (simp add: invar-def)

```

The last collapsed node on the path contains states from all accepting sets. As it is strongly connected and reachable, we get a counter-example. Here, we explicitly construct the lasso.

```

let ?Er = E ∩ (⋃ (set (butlast p)) × UNIV)

```

We choose a node in the last Cnode, that is reachable only using former Cnodes.

```

obtain w where (v0,w) ∈ ?Er*   w ∈ last p

```

```

proof cases
  assume  $\text{length } p = 1$ 
  hence  $v0 \in \text{last } p$ 
    using root-v0
    by (cases p) auto
  thus thesis by (auto intro: that)
next
  assume  $\text{length } p \neq 1$ 
  hence  $\text{length } p > 1$  by (cases p) auto
  hence  $\text{Suc } (\text{length } p - 2) < \text{length } p$  by auto
  from p-connected [OF this] obtain  $u v$  where
     $UIP: u \in p!(\text{length } p - 2)$  and  $VIP: v \in p!(\text{length } p - 1)$  and  $(u, v) \in lvE$ 
    using  $\langle \text{length } p > 1 \rangle$  by auto
  from root-v0 have  $V0IP: v0 \in p!0$  by (cases p) auto

from VIP have  $v \in \text{last } p$  by (cases p rule: rev-cases) auto

from pathI [OF V0IP UIP]  $\langle \text{length } p > 1 \rangle$  have
   $(v0, u) \in (lvE \cap \bigcup (\text{set } (\text{butlast } p)) \times \bigcup (\text{set } (\text{butlast } p)))^*$ 
  (is - \in ...*)
  by (simp add: path-seg-butlast)
  also have  $\dots \subseteq ?Er$  using lvE-ss-E by auto
  finally (rtrancl-mono-mp [rotated]) have  $(v0, u) \in ?Er^*$  .
  also note  $\langle (u, v) \in lvE \rangle$  UIP hence  $(u, v) \in ?Er$  using lvE-ss-E  $\langle \text{length } p > 1 \rangle$ 
    apply (auto simp: Bex-def in-set-conv-nth)
    by (metis One-nat-def Suc-lessE  $\langle \text{Suc } (\text{length } p - 2) < \text{length } p \rangle$ 
      diff-Suc-1 length-butlast nth-butlast)
  finally show ?thesis by (rule that) fact
qed
then obtain  $pr$  where
  P-REACH: path E v0 pr w and
  R-SS: set pr \subseteq \bigcup (\text{set } (\text{butlast } p))
  apply -
  apply (erule rtrancl-is-path)
  apply (frule path-nodes-edges)
  apply (auto)
  dest!: order-trans [OF - image-Int-subset]
  dest: path-mono [of - E, rotated]
done

have [simp]:  $\text{last } p = p!(\text{length } p - 1)$  by (cases p rule: rev-cases) auto

```

From that node, we construct a lasso by inductively appending a path for each accepting set

```

{
  fix  $na$ 
  assume na-def:  $na = \text{num-acc}$ 

  have  $\exists pl. pl \neq []$ 

```

```

 $\wedge$  path (lvE  $\cap$  last p  $\times$  last p) w pl w
 $\wedge$  ( $\forall i < \text{num-acc. } \exists q \in \text{set pl. } i \in \text{acc } q$ )
using ACC
unfolding na-def[symmetric]
proof (induction na)
case 0

from NONTRIV obtain u v
  where (u,v)  $\in$  lvE  $\cap$  last p  $\times$  last p   u  $\in$  last p   v  $\in$  last p
  by auto
from cnode-connectedI  $\langle w \in \text{last } p \rangle \langle u \in \text{last } p \rangle$ 
have (w,u)  $\in$  (lvE  $\cap$  last p  $\times$  last p)*
  by auto
also note  $\langle (u,v) \in \text{lvE} \cap \text{last } p \times \text{last } p \rangle$ 
also (rtrancl-into-trancl1) from cnode-connectedI  $\langle v \in \text{last } p \rangle \langle w \in \text{last } p \rangle$ 
have (v,w)  $\in$  (lvE  $\cap$  last p  $\times$  last p)*
  by auto
finally obtain pl where pl  $\neq []$    path (lvE  $\cap$  last p  $\times$  last p) w pl w
  by (rule trancl-is-path)
thus ?case by auto
next
case (Suc n)
from Suc.prem have  $\forall i < n. \exists q \in \text{last } p. i \in \text{acc } q$  by auto
with Suc.IH obtain pl where IH:
  pl  $\neq []$ 
  path (lvE  $\cap$  last p  $\times$  last p) w pl w
   $\forall i < n. \exists q \in \text{set pl. } i \in \text{acc } q$ 
  by blast

from Suc.prem obtain v where v  $\in$  last p and n  $\in$  acc v by auto
from cnode-connectedI  $\langle w \in \text{last } p \rangle \langle v \in \text{last } p \rangle$ 
have (w,v)  $\in$  (lvE  $\cap$  last p  $\times$  last p)* by auto
then obtain pl1 where P1: path (lvE  $\cap$  last p  $\times$  last p) w pl1 v
  by (rule rtrancl-is-path)
also from cnode-connectedI  $\langle w \in \text{last } p \rangle \langle v \in \text{last } p \rangle$ 
have (v,w)  $\in$  (lvE  $\cap$  last p  $\times$  last p)* by auto
then obtain pl2 where P2: path (lvE  $\cap$  last p  $\times$  last p) v pl2 w
  by (rule rtrancl-is-path)
also (path-conc) note IH(2)
finally (path-conc) have
  P: path (lvE  $\cap$  last p  $\times$  last p) w (pl1@pl2@pl) w
  by simp
moreover from IH(1) have pl1@pl2@pl  $\neq []$  by simp
moreover have  $\forall i' < n. \exists q \in \text{set } (pl1@pl2@pl). i' \in \text{acc } q$  using IH(3) by
auto
moreover have v  $\in$  set (pl1@pl2@pl) using P1 P2 P IH(1)
  apply (cases pl2, simp-all add: path-cons-conv path-conc-conv)
  apply (cases pl, simp-all add: path-cons-conv)
  apply (cases pl1, simp-all add: path-cons-conv)

```

```

    done
  with ⟨ $n \in acc\ v$ ⟩ have  $\exists q \in set\ (pl1 @ pl2 @ pl)$ .  $n \in acc\ q$  by auto
  ultimately show ?case
    apply (intro exI conjI)
    apply assumption+
    apply (auto elim: less-SucE)
  done
qed
}
then obtain  $pl$  where  $pl: pl \neq []$  path  $(lvE \cap last\ p \times last\ p)$  w  $pl$  w
   $\forall i < num\ acc$ .  $\exists q \in set\ pl$ .  $i \in acc\ q$  by blast
hence path  $E$  w  $pl$  w and  $L\text{-}SS$ :  $set\ pl \subseteq last\ p$ 
  apply -
  apply (frule path-mono[of -  $E$ , rotated])
  using  $lvE\text{-}ss\text{-}E$ 
  apply auto [2]

  apply (drule path-nodes-edges)
  apply (drule order-trans[OF - image-Int-subset])
  apply auto []
done

have  $LASSO$ :  $is\text{-}lasso\text{-}prpl\ (pr, pl)$ 
  unfolding  $is\text{-}lasso\text{-}prpl\text{-}def\ is\text{-}lasso\text{-}prpl\text{-}pre\text{-}def$ 
  using ⟨ $path\ E\ w\ pl\ w$ ⟩  $P\text{-}REACH\ pl$  by auto

from  $p\text{-}sc$  have  $last\ p \times last\ p \subseteq (lvE \cap last\ p \times last\ p)^*$  by auto
with  $lvE\text{-}ss\text{-}E$  have  $VL\text{-}CLOSED$ :  $last\ p \times last\ p \subseteq (E \cap last\ p \times last\ p)^*$ 
  apply (erule-tac order-trans)
  apply (rule rtrancl-mono)
  by blast

have  $NONTRIV'$ :  $last\ p \times last\ p \cap E \neq \{\}$ 
  by (metis Int-commute  $NONTRIV\ disjoint\text{-}mono\ lvE\text{-}ss\text{-}E\ subset\text{-}refl$ )

from order-trans[OF path-touched touched-reachable]
have  $LP\text{-}REACH$ :  $last\ p \subseteq E^* \text{“}V0$ 
  and  $BLP\text{-}REACH$ :  $\bigcup (set\ (butlast\ p)) \subseteq E^* \text{“}V0$ 
  apply -
  apply (cases p rule: rev-cases)
  apply simp
  apply auto []

  apply (cases p rule: rev-cases)
  apply simp
  apply auto []
done

show ?thesis

```

**apply** (*simp add: fgl-invar-part-def*)  
**apply** *unfold-locales*  
**apply** *simp*

**using** *LASSO R-SS L-SS VL-CLOSED NONTRIV' LP-REACH BLP-REACH*  
**unfolding** *ce-correct-def*  
**apply** *simp*  
**apply** *blast*  
**done**

**qed**

**lemma** *fgl-invar-collapse-ce:*

**fixes** *u v*  
**assumes** *INV: fgl-invar v0 D0 (None,p,D,pE)*  
**defines**  $pE' \equiv pE - \{(u,v)\}$   
**assumes** *CFMT: (p',D',pE'') = collapse v (p,D,pE')*  
**assumes** *INV': invar v0 D0 (p',D',pE'')*  
**assumes** *NE[simp]: p ≠ []*  
**assumes** *E: (u,v) ∈ pE and u ∈ last p*  
**assumes** *BACK: v ∈ ⋃ (set p)*  
**assumes** *ACC: ∀ i < num-acc. ∃ q ∈ last p'. i ∈ acc q*  
**defines** *i-def: i ≡ idx-of p v*  
**shows** *fgl-invar-part* (  
*Some (⋃ (set (butlast p')), last p'),*  
*collapse v (p,D,pE')*)

**proof** –

**from** *CFMT* **have** *p'-def: p' = collapse-aux p i* **and** *[simp]: D' = D pE'' = pE'*  
**by** (*simp-all add: collapse-def i-def*)

**from** *INV* **interpret** *fgl-invar-loc G v0 D0 None p D pE*  
**by** (*simp add: fgl-invar-def*)

**from** *idx-of-props[OF BACK]* **have** *i < length p* **and** *v ∈ p!i*  
**by** (*simp-all add: i-def*)

**have** *u ∈ last p'*  
**using**  $\langle u \in \text{last } p \rangle \langle i < \text{length } p \rangle$   
**unfolding** *p'-def collapse-aux-def*  
**apply** (*simp add: last-drop last-snoc*)  
**by** (*metis Misc.last-in-set drop-eq-Nil last-drop not-le*)

**moreover** **have** *v ∈ last p'*  
**using**  $\langle v \in p!i \rangle \langle i < \text{length } p \rangle$   
**unfolding** *p'-def collapse-aux-def*  
**by** (*metis UnionI append-Nil Cons-nth-drop-Suc in-set-conv-decomp last-snoc*)  
**ultimately** **have**  $v \in p' D pE' \cap \text{last } p' \times \text{last } p' \neq \{\}$   
**unfolding** *p'-def pE'-def* **by** (*auto simp: E*)

**have**  $p' \neq []$  **by** (*simp add: p'-def collapse-aux-def*)

**have** [*simp*]:  $\text{collapse } v (p, D, pE') = (p', D, pE')$   
**unfolding** *collapse-def p'-def i-def*  
**by** *simp*

**show** *?thesis*  
**apply** *simp*  
**apply** (*rule fgl-invar-collapse-ce-aux*)  
**using** *INV'* **apply** *simp*  
**apply** *fact+*  
**done**

**qed**

**lemma** *fgl-invar-collapse-nce*:

**fixes**  $u v$   
**assumes** *INV*: *fgl-invar v0 D0 (None, p, D, pE)*  
**defines**  $pE' \equiv pE - \{(u, v)\}$   
**assumes** *CFMT*:  $(p', D', pE'') = \text{collapse } v (p, D, pE')$   
**assumes** *INV'*: *invar v0 D0 (p', D', pE'')*  
**assumes** *NE*[*simp*]:  $p' \neq []$   
**assumes** *E*:  $(u, v) \in pE$  **and**  $u \in \text{last } p$   
**assumes** *BACK*:  $v \in \bigcup (\text{set } p)$   
**assumes** *NACC*:  $j < \text{num-acc} \quad \forall q \in \text{last } p'. j \notin \text{acc } q$   
**defines**  $i \equiv \text{idx-of } p v$   
**shows** *fgl-invar-part (None, collapse v (p, D, pE'))*

**proof** –

**from** *CFMT* **have** *p'-def*:  $p' = \text{collapse-aux } p i$  **and** [*simp*]:  $D' = D \quad pE'' = pE'$   
**by** (*simp-all add: collapse-def i-def*)

**have** [*simp*]:  $\text{collapse } v (p, D, pE') = (p', D, pE')$   
**by** (*simp add: collapse-def p'-def i-def*)

**from** *INV* **interpret** *fgl-invar-loc G v0 D0 None p D pE*  
**by** (*simp add: fgl-invar-def*)

**from** *INV'* **interpret** *inv': invar-loc G v0 D0 p' D pE'* **by** (*simp add: invar-def*)

**define**  $vE'$  **where**  $vE' = vE p' D pE'$

**have** *vE'-alt*:  $vE' = \text{insert } (u, v) \text{ } vE$   
**by** (*simp add: vE'-def p'-def pE'-def E*)

**from** *idx-of-props[OF BACK]* **have**  $i < \text{length } p$  **and**  $v \in p!i$   
**by** (*simp-all add: i-def*)

**have**  $u \in \text{last } p'$   
**using**  $\langle u \in \text{last } p \rangle \langle i < \text{length } p \rangle$   
**unfolding** *p'-def collapse-aux-def*

```

apply (simp add: last-drop last-snoc)
by (metis Misc.last-in-set drop-eq-Nil last-drop leD)
moreover have  $v \in \text{last } p'$ 
using  $\langle v \in p!i \rangle \langle i < \text{length } p \rangle$ 
unfolding  $p'\text{-def collapse-aux-def}$ 
by (metis UnionI append-Nil Cons-nth-drop-Suc in-set-conv-decomp last-snoc)
ultimately have  $vE' \cap \text{last } p' \times \text{last } p' \neq \{\}$ 
unfolding  $vE'\text{-alt}$  by (auto)

```

```

have  $p' \neq []$  by (simp add:  $p'\text{-def collapse-aux-def}$ )

```

```

{

```

We show that no visited strongly connected component contains states from all acceptance sets.

```

fix  $w \text{ pl}$ 

```

For this, we chose a non-trivial loop inside the visited edges

```

assume  $P$ :  $\text{path } vE' \ w \ \text{pl} \ w$  and  $NT$ :  $\text{pl} \neq []$ 

```

And show that there is one acceptance set disjoint with the nodes of the loop

```

have  $\exists i < \text{num-acc}. \forall q \in \text{set } \text{pl}. i \notin \text{acc } q$ 
proof cases
assume  $\text{set } \text{pl} \cap \text{last } p' = \{\}$ 
  — Case: The loop is outside the last Cnode
with  $\text{path-restrict}[OF \ P] \langle u \in \text{last } p' \rangle \langle v \in \text{last } p' \rangle$  have  $\text{path } lvE \ w \ \text{pl} \ w$ 
apply —
apply (drule  $\text{path-mono}[of \ - \ lvE, \ \text{rotated}]$ )
unfolding  $vE'\text{-alt}$ 
by  $\text{auto}$ 
with  $\text{no-acc } NT$  show  $?thesis$  by  $\text{auto}$ 
next
assume  $\text{set } \text{pl} \cap \text{last } p' \neq \{\}$ 
  — Case: The loop touches the last Cnode

```

Then, the loop must be completely inside the last CNode

```

from  $\text{inv}'.\text{loop-in-lastnode}[\text{folded } vE'\text{-def}, \ OF \ P \ \langle p' \neq [] \rangle \ \text{this}]$ 
have  $w \in \text{last } p' \quad \text{set } \text{pl} \subseteq \text{last } p' .$ 
with  $NACC$  show  $?thesis$  by  $\text{blast}$ 
qed
} note  $AUX\text{-no-acc} = \text{this}$ 

```

```

show  $?thesis$ 

```

```

apply (simp add:  $\text{fgl-invar-part-def}$ )
apply  $\text{unfold-locales}$ 
using  $AUX\text{-no-acc}[\text{unfolded } vE'\text{-def}]$  apply  $\text{auto}$  []

```

```

apply  $\text{simp}$ 
done

```

qed

**lemma** *collapse-ne*:  $([], D', pE') \neq \text{collapse } v (p, D, pE)$   
**by** (*simp add: collapse-def collapse-aux-def Let-def*)

**lemma** *fgl-invar-push*:

**assumes** *INV*: *fgl-invar*  $v0 D0 (None, p, D, pE)$   
**assumes** *BRK*[*simp*]: *brk*=*None*  
**assumes** *NE*[*simp*]:  $p \neq []$   
**assumes** *E*:  $(u, v) \in pE$  **and** *UIL*:  $u \in \text{last } p$   
**assumes** *VNE*:  $v \notin \bigcup (\text{set } p) \quad v \notin D$   
**assumes** *INV'*: *invar*  $v0 D0 (\text{push } v (p, D, pE - \{(u, v)\}))$   
**shows** *fgl-invar-part*  $(None, \text{push } v (p, D, pE - \{(u, v)\}))$

**proof** –

**from** *INV* **interpret** *fgl-invar-loc*  $G v0 D0 None p D pE$   
**by** (*simp add: fgl-invar-def*)

**define**  $pE'$  **where**  $pE' = (pE - \{(u, v)\} \cup E \cap \{v\} \times UNIV)$

**have** [*simp*]:  $\text{push } v (p, D, pE - \{(u, v)\}) = (p@[v], D, pE')$   
**by** (*simp add: push-def pE'-def*)

**from** *INV'* **interpret** *inv'*: *invar-loc*  $G v0 D0 (p@[v]) D pE'$   
**by** (*simp add: invar-def*)

**note** *defs-fold* = *vE-push*[*OF E UIL VNE, folded pE'-def*]

{

We show that there still is no loop that contains all accepting nodes. For this, we choose some loop.

**fix**  $w pl$   
**assume** *P*: *path*  $(\text{insert } (u, v) lvE) w pl w$  **and** [*simp*]:  $pl \neq []$   
**have**  $\exists i < \text{num-acc}. \forall q \in \text{set } pl. i \notin \text{acc } q$   
**proof** *cases*

**assume**  $v \in \text{set } pl$  — Case: The newly pushed last cnode is on the loop

Then the loop is entirely on the last cnode

**with** *inv'.loop-in-lastnode*[*unfolded defs-fold, OF P*]  
**have** [*simp*]:  $w = v$  **and** *SPL*:  $\text{set } pl = \{v\}$  **by** *auto*

However, we then either have that the last cnode is contained in the last but one cnode, or that there is a visited edge inside the last cnode.

**from** *P SPL* **have**  $u = v \vee (v, v) \in lvE$   
**apply** (*cases pl*) **apply** (*auto simp: path-cons-conv*)  
**apply** (*case-tac list*)  
**apply** (*auto simp: path-cons-conv*)  
**done**

Both leads to a contradiction

**hence *False* proof**  
**assume  $u=v$**  — This is impossible, as  $u$  was on the original path, but  $v$  was not  
**with  $UIL\ VNE$  show *False* by *auto***  
**next**  
**assume  $(v,v)\in lvE$**  — This is impossible, as all visited edges are from touched nodes, but  $v$  was untouched  
**with  $vE$ -touched  $VNE$  show *False* unfolding *touched-def* by *auto***  
**qed**  
**thus *?thesis ..***  
**next**  
**assume  $A: v\notin set\ pl$**   
 — Case: The newly pushed last cnode is not on the loop

Then, the path lays inside the old visited edges

**have  $path\ lvE\ w\ pl\ w$**   
**proof** —  
**have  $w\in set\ pl$  using  $P$  by (*cases pl*) (*auto simp: path-cons-conv*)**  
**with  $A$  show *?thesis* using *path-restrict[OF P]***  
**apply** —  
**apply (*drule path-mono[of - lvE, rotated]*)**  
**apply (*cases pl, auto*) []**  
  
**apply *assumption***  
**done**  
**qed**

And thus, the proposition follows from the invariant on the old state

**with *no-acc* show *?thesis***  
**apply *simp***  
**using  $\langle pl\neq [] \rangle$**   
**by *blast***  
**qed**  
**} note *AUX-no-acc = this***  
  
**show *?thesis***  
**unfolding *fgl-invar-part-def***  
**apply *simp***  
**apply *unfold-locales***  
**unfolding *defs-fold***  
  
**using *AUX-no-acc* apply *auto* []**  
  
**apply *simp***  
**done**  
**qed**

**lemma *fgl-invar-skip*:**

```

assumes INV: fgl-invar v0 D0 (None,p,D,pE)
assumes BRK[simp]: brk=None
assumes NE[simp]: p≠[]
assumes E:  $(u,v) \in pE$  and UIL: u ∈ last p
assumes VID: v ∈ D
assumes INV': invar v0 D0 (p, D, (pE - {(u,v)}))
shows fgl-invar-part (None, p, D, (pE - {(u,v)}))
proof -
from INV interpret fgl-invar-loc G v0 D0 None p D pE
  by (simp add: fgl-invar-def)
from INV' interpret inv': invar-loc G v0 D0 p D (pE - {(u,v)})
  by (simp add: invar-def)

{

```

We show that there still is no loop that contains all accepting nodes. For this, we choose some loop.

```

fix w pl
assume P: path (insert (u,v) lvE) w pl w and [simp]: pl≠[]
from P have  $\exists i < \text{num-acc}. \forall q \in \text{set } pl. i \notin \text{acc } q$ 
proof (cases rule: path-edge-rev-cases)
  case no-use — Case: The loop does not use the new edge

```

The proposition follows from the invariant for the old state

```

with no-acc show ?thesis
  apply simp
  using  $\langle pl \neq [] \rangle$ 
  by blast
next
  case (split p1 p2) — Case: The loop uses the new edge

```

As done is closed under transitions, the nodes of the edge have already been visited

```

from split(2) D-closed-vE-rtrancl
have WID: w ∈ D
  using VID by (auto dest!: path-is-rtrancl)
from split(1) WID D-closed-vE-rtrancl have u ∈ D
  apply (cases rule: path-edge-cases)
  apply (auto dest!: path-is-rtrancl)
done

```

Which is a contradiction to the assumptions

```

with UIL p-not-D have False by (cases p rule: rev-cases) auto
thus ?thesis ..
qed
} note AUX-no-acc = this

```

```

show ?thesis
  apply (simp add: fgl-invar-part-def)

```

**apply** *unfold-locales*  
**unfolding** *vE-remove*[*OF NE E*]

**using** *AUX-no-acc* **apply** *auto* []

**apply** *simp*  
**done**

**qed**

**lemma** *fgl-outer-invar-initial*:

*outer-invar V0 {}*  $\implies$  *fgl-outer-invar-ext V0 (None, {})*

**unfolding** *fgl-outer-invar-ext-def*

**apply** (*simp add: no-acc-over-def*)

**done**

**lemma** *fgl-outer-invar-brk*:

**assumes** *INV: fgl-invar v0 D0 (Some (Vr, Vl), p, D, pE)*

**shows** *fgl-outer-invar-ext anyIt (Some (Vr, Vl), anyD)*

**proof** –

**from** *INV* **interpret** *fgl-invar-loc G v0 D0 Some (Vr, Vl) p D pE*

**by** (*simp add: fgl-invar-def*)

**from** *acc* **show** *?thesis* **by** (*simp add: fgl-outer-invar-ext-def*)

**qed**

**lemma** *fgl-outer-invar-newnode-nobrk*:

**assumes** *A: v0 ∉ D0 v0 ∈ it*

**assumes** *OINV: fgl-outer-invar it (None, D0)*

**assumes** *INV: fgl-invar v0 D0 (None, [], D', pE)*

**shows** *fgl-outer-invar-ext (it - {v0}) (None, D')*

**proof** –

**from** *OINV* **interpret** *outer-invar-loc G it D0*

**unfolding** *fgl-outer-invar-def outer-invar-def* **by** *simp*

**from** *INV* **interpret** *inv: fgl-invar-loc G v0 D0 None [] D' pE*

**unfolding** *fgl-invar-def* **by** *simp*

**from** *inv.pE-fin* **have** [*simp*]: *pE = {}* **by** *simp*

{ **fix** *v pl*

**assume** *A: v ∈ D' path E v pl v*

**have** *path (E ∩ D' × UNIV) v pl v*

**apply** (*rule path-mono*[*OF - path-restrict-closed*[*OF inv.D-closed A*]])

**by** *auto*

} **note** *AUX1=this*

**show** *?thesis*

**unfolding** *fgl-outer-invar-ext-def*

```

apply simp
using inv.no-acc AUX1
apply (auto simp add: vE-def touched-def no-acc-over-def) []
done
qed

lemma fgl-outer-invar-newnode:
assumes A: v0 ∉ D0 v0 ∈ it
assumes OINV: fgl-outer-invar it (None, D0)
assumes INV: fgl-invar v0 D0 (brk, p, D', pE)
assumes CASES: (∃ Vr Vl. brk = Some (Vr, Vl)) ∨ p = []
shows fgl-outer-invar-ext (it - {v0}) (brk, D')
using CASES
apply (elim disjE1)
using fgl-outer-invar-brk[of v0 D0 - - p D' pE] INV
apply -
apply (auto, assumption) []
using fgl-outer-invar-newnode-nobrck[OF A] OINV INV apply auto []
done

```

```

lemma fgl-outer-invar-Dnode:
assumes fgl-outer-invar it (None, D) v ∈ D
shows fgl-outer-invar-ext (it - {v}) (None, D)
using assms
by (auto simp: fgl-outer-invar-def fgl-outer-invar-ext-def)

```

```

lemma fgl-fin-no-lasso:
assumes A: fgl-outer-invar {} (None, D)
assumes B: is-lasso-prpl prpl
shows False
proof -
obtain pr pl where [simp]: prpl = (pr, pl) by (cases prpl)
from A have NA: no-acc-over D
by (simp add: fgl-outer-invar-def fgl-outer-invar-ext-def)

from A have outer-invar {} D by (simp add: fgl-outer-invar-def)
with fin-outer-D-is-reachable have [simp]: D = E* "V0 by simp

from NA B show False
apply (simp add: no-acc-over-def is-lasso-prpl-def is-lasso-prpl-pre-def)
apply clarsimp
apply (blast dest: path-is-rtrancl)
done
qed

```

```

lemma fgl-fin-lasso:
assumes A: fgl-outer-invar it (Some (Vr, Vl), D)
shows ce-correct Vr Vl

```

**using** *A* **by** (*simp add: fgl-outer-invar-def fgl-outer-invar-ext-def*)

**lemmas** *fgl-invar-preserve* =  
*fgl-invar-initial fgl-invar-push fgl-invar-pop*  
*fgl-invar-collapse-ce fgl-invar-collapse-nce fgl-invar-skip*  
*fgl-outer-invar-newnode fgl-outer-invar-Dnode*  
*invar-initial outer-invar-initial fgl-invar-initial fgl-outer-invar-initial*  
*fgl-fin-no-lasso fgl-fin-lasso*

**end**

## 1.19 Main Correctness Proof

**context** *igb-fr-graph*

**begin**

**lemma** *outer-invar-from-fgl-invarI*:  
*fgl-outer-invar it (None,D)  $\implies$  outer-invar it D*  
**unfolding** *fgl-outer-invar-def outer-invar-def*  
**by** (*simp split: prod.splits*)

**lemma** *invar-from-fgl-invarI*: *fgl-invar v0 D0 (B,PDPE)  $\implies$  invar v0 D0 PDPE*  
**unfolding** *fgl-invar-def invar-def*  
**apply** (*simp split: prod.splits*)  
**unfolding** *fgl-invar-loc-def* **by** *simp*

**theorem** *find-ce-correct*: *find-ce  $\leq$  find-ce-spec*

**proof** –

**note** [*simp del*] = *Union-iff*

**show** *?thesis*

**unfolding** *find-ce-def find-ce-spec-def select-edge-def select-def*  
**apply** (*refine-rcg*  
*WHILEIT-rule[where R=inv-image (abs-wf-rel v0) snd for v0]*  
*refine-vcg*  
)

**using** [[*goals-limit = 5*]]

**apply** (*vc-solve*  
*rec: fgl-invarI fgl-outer-invarI*  
*intro: invar-from-fgl-invarI outer-invar-from-fgl-invarI*  
*dest!: sym[of collapse a b for a b]*  
*simp: collapse-ne*  
*simp: pE-fin'[OF invar-from-fgl-invarI] finite-V0*  
*solve: invar-preserve*  
*solve: asm-rl[of -  $\cap$  - = {}]*  
*solve: fgl-invar-preserve*)  
**done**

**qed**  
**end**

## 1.20 Emptiness Check

Using the lasso-finding algorithm, we can define an emptiness check

**context** *igb-fr-graph*

**begin**

**definition** *abs-is-empty*  $\equiv$  *do* {  
 $ce \leftarrow find\_ce;$   
 $RETURN (ce = None)$   
}

**theorem** *abs-is-empty-correct*:

$abs-is-empty \leq SPEC (\lambda res. res \longleftrightarrow (\forall r. \neg is-acc-run r))$

**unfolding** *abs-is-empty-def*

**apply** (*refine-rcg refine-vcg*  
*order-trans[OF find-ce-correct, unfolded find-ce-spec-def]*)

**unfolding** *ce-correct-def*

**using** *lasso-accepted accepted-lasso*

**apply** (*clarsimp split: option.splits*)

**apply** (*metis is-lasso-prpl-of-lasso surj-pair*)

**by** (*metis is-lasso-prpl-conv*)

**definition** *abs-is-empty-ce*  $\equiv$  *do* {

$ce \leftarrow find\_ce;$

*case* *ce* *of*

*None*  $\Rightarrow RETURN None$

| *Some* (*Vr*, *Vl*)  $\Rightarrow$  *do* {

$ASSERT (\exists pr\ pl. set\ pr \subseteq Vr \wedge set\ pl \subseteq Vl \wedge Vl \times Vl \subseteq (E \cap Vl \times Vl)^*$

$\wedge is-lasso-prpl (pr, pl));$

$(pr, pl) \leftarrow SPEC (\lambda(pr, pl).$

$set\ pr \subseteq Vr$

$\wedge set\ pl \subseteq Vl$

$\wedge Vl \times Vl \subseteq (E \cap Vl \times Vl)^*$

$\wedge is-lasso-prpl (pr, pl));$

$RETURN (Some (pr, pl))$

}

}

**theorem** *abs-is-empty-ce-correct*:  $abs-is-empty-ce \leq SPEC (\lambda res. case\ res\ of$

*None*  $\Rightarrow (\forall r. \neg is-acc-run r)$

| *Some* (*pr*, *pl*)  $\Rightarrow is-acc-run (pr \frown pl^\omega)$

)

**unfolding** *abs-is-empty-ce-def*

**apply** (*refine-rcg refine-vcg*  
*order-trans[OF find-ce-correct, unfolded find-ce-spec-def]*)

**apply** (*clarsimp-all simp: ce-correct-def*)

```

    using accepted-lasso finite-reachableE-V0 apply (metis is-lasso-prpl-of-lasso
surj-pair)
    apply blast
    apply (simp add: lasso-prpl-acc-run)
    done

end

```

## 1.21 Refinement

In this section, we refine the lasso finding algorithm to use efficient data structures. First, we explicitly keep track of the set of acceptance classes for every c-node on the path. Second, we use Gabow's data structure to represent the path.

### 1.21.1 Addition of Explicit Accepting Sets

In a first step, we explicitly keep track of the current set of acceptance classes for every c-node on the path.

```

type-synonym 'a abs-gstate = nat set list × 'a abs-state
type-synonym 'a ce = ('a set × 'a set) option
type-synonym 'a abs-gstate = 'a ce × 'a set

```

```

context igb-fr-graph
begin

```

```

definition gstate-invar :: 'Q abs-gstate ⇒ bool where
  gstate-invar ≡ λ(a,p,D,pE). a = map (λV. ⋃(acc'V)) p

```

```

definition gstate-rel ≡ br snd gstate-invar

```

```

lemma gstate-rel-sv[relator-props,simp,intro!]: single-valued gstate-rel
by (simp add: gstate-rel-def)

```

```

definition (in -) gcollapse-aux
  :: nat set list ⇒ 'a set list ⇒ nat ⇒ nat set list × 'a set list
where gcollapse-aux a p i ≡
  (take i a @ [⋃(set (drop i a)),take i p @ [⋃(set (drop i p))])

```

```

definition (in -) gcollapse :: 'a ⇒ 'a abs-gstate ⇒ 'a abs-gstate
where gcollapse v APDPE ≡
  let
    (a,p,D,pE)=APDPE;
    i=idx-of p v;
    (a,p) = gcollapse-aux a p i
  in (a,p,D,pE)

```

**definition** *gpush*  $v\ s \equiv$

```
let
  (a,s) = s
in
  (a@[acc v],push v s)
```

**definition** *gpop*  $s \equiv$

```
let (a,s) = s in (butlast a,pop s)
```

**definition** *ginitial*  $:: 'Q \Rightarrow 'Q\ abs-gostate \Rightarrow 'Q\ abs-gstate$

**where** *ginitial*  $v0\ s0 \equiv ([acc\ v0],\ initial\ v0\ (snd\ s0))$

**definition** *goinitial*  $:: 'Q\ abs-gostate\ \mathbf{where}\ goinitial \equiv (None,\{\})$

**definition** *go-is-no-brk*  $:: 'Q\ abs-gostate \Rightarrow bool$

**where** *go-is-no-brk*  $s \equiv fst\ s = None$

**definition** *goD*  $:: 'Q\ abs-gostate \Rightarrow 'Q\ set\ \mathbf{where}\ goD\ s \equiv snd\ s$

**definition** *goBrk*  $:: 'Q\ abs-gostate \Rightarrow 'Q\ ce\ \mathbf{where}\ goBrk\ s \equiv fst\ s$

**definition** *gto-outer*  $:: 'Q\ ce \Rightarrow 'Q\ abs-gstate \Rightarrow 'Q\ abs-gostate$

**where** *gto-outer*  $brk\ s \equiv let\ (A,p,D,pE)=s\ in\ (brk,D)$

**definition** *gselect-edge*  $s \equiv do\ \{$

```
let (a,s)=s;
(r,s)←select-edge s;
RETURN (r,a,s)
```

$\}$

**definition** *gfind-ce*  $:: ('Q\ set \times 'Q\ set)\ option\ nres\ \mathbf{where}$

```
gfind-ce  $\equiv do\ \{$ 
  let  $os = goinitial;$ 
   $os \leftarrow FOREACHci\ fgl\ outer\ invar\ V0\ (go-is-no-brk)\ (\lambda v0\ s0.\ do\ \{$ 
    if  $v0 \notin goD\ s0$  then do {
      let  $s = (None,ginitial\ v0\ s0);$ 
```

```
(brk,a,p,D,pE) ← WHILEIT ( $\lambda(brk,a,s).\ fgl\ invar\ v0\ (goD\ s0)\ (brk,s)$ )
  ( $\lambda(brk,a,p,D,pE).\ brk= None \wedge p \neq []$ ) ( $\lambda(-,a,p,D,pE).$ 
```

```
do {
```

```
— Select edge from end of path
```

```
(vo,(a,p,D,pE)) ← gselect-edge (a,p,D,pE);
```

```
ASSERT (p≠[]);
```

```
case vo of
```

```
Some v ⇒ do {
```

```
if  $v \in \bigcup (set\ p)$  then do {
```

```
— Collapse
```

```
let (a,p,D,pE) = gcollapse v (a,p,D,pE);
```

```
ASSERT (p≠[]);
```

```
ASSERT (a≠[]);
```

```

    if last a = {0..<num-acc} then
      RETURN (Some (∪(set (butlast p)),last p),a,p,D,pE)
    else
      RETURN (None,a,p,D,pE)
  } else if v∉D then do {
    — Edge to new node. Append to path
    RETURN (None,gpush v (a,p,D,pE))
  } else RETURN (None,a,p,D,pE)
}
| None ⇒ do {
  — No more outgoing edges from current node on path
  ASSERT (pE ∩ last p × UNIV = {});
  RETURN (None,gpop (a,p,D,pE))
}
}) s;
ASSERT (brk=None → (p=[] ∧ pE={}));
RETURN (gto-outer brk (a,p,D,pE))
} else RETURN s0
}) os;
RETURN (goBrk os)
}

```

**lemma** *gcollapse-refine*:

```

[[ (v',v)∈Id; (s',s)∈gstate-rel ]
  ⇒ (gcollapse v' s',collapse v s)∈gstate-rel
unfolding gcollapse-def collapse-def collapse-aux-def gcollapse-aux-def
apply (simp add: gstate-rel-def br-def Let-def)
unfolding gstate-invar-def[abs-def]
apply (auto split: prod.splits simp: take-map drop-map)
done

```

**lemma** *gpush-refine*:

```

[[ (v',v)∈Id; (s',s)∈gstate-rel ] ⇒ (gpush v' s',push v s)∈gstate-rel
unfolding gpush-def push-def
apply (simp add: gstate-rel-def br-def)
unfolding gstate-invar-def[abs-def]
apply (auto split: prod.splits)
done

```

**lemma** *gpob-refine*:

```

[[ (s',s)∈gstate-rel ] ⇒ (gpob s',pob s)∈gstate-rel
unfolding gpob-def pob-def
apply (simp add: gstate-rel-def br-def)
unfolding gstate-invar-def[abs-def]
apply (auto split: prod.splits simp: map-butlast)
done

```

**lemma** *ginitial-refine*:

```

(ginitial x (None, b), initial x b) ∈ gstate-rel

```

**unfolding** *ginitial-def gstate-rel-def br-def gstate-invar-def initial-def*  
**by** *auto*

**lemma** *oinitial-b-refine*:  $((None, \{\}), (None, \{\})) \in Id \times_r Id$  **by** *simp*

**lemma** *gselect-edge-refine*:  $\llbracket (s', s) \in gstate-rel \rrbracket \implies gselect-edge\ s'$   
 $\leq \Downarrow (\langle Id \rangle option-rel \times_r gstate-rel)\ (select-edge\ s)$

**unfolding** *gselect-edge-def select-edge-def*

**apply** (*simp add: pw-le-iff refine-pw-simps prod-rel-sv*

*split: prod.splits option.splits*)

**apply** (*auto simp: gstate-rel-def br-def gstate-invar-def*)

**done**

**lemma** *last-acc-impl*:

**assumes**  $p \neq []$

**assumes**  $((a', p', D', pE'), (p, D, pE)) \in gstate-rel$

**shows**  $(last\ a' = \{0..<num-acc\}) = (\forall i < num-acc. \exists q \in last\ p. i \in acc\ q)$

**using** *assms acc-bound* **unfolding** *gstate-rel-def br-def gstate-invar-def*

**by** (*auto simp: last-map*)

**lemma** *fglr-aux1*:

**assumes**  $V: (v', v) \in Id$  **and**  $S: (s', s) \in gstate-rel$

**and**  $P: \bigwedge a' p' D' pE' p D pE. ((a', p', D', pE'), (p, D, pE)) \in gstate-rel$

$\implies f' a' p' D' pE' \leq \Downarrow R (f p D pE)$

**shows**  $(let\ (a', p', D', pE') = gcollapse\ v'\ s'\ in\ f'\ a'\ p'\ D'\ pE')$

$\leq \Downarrow R (let\ (p, D, pE) = collapse\ v\ s\ in\ f\ p\ D\ pE)$

**apply** (*auto split: prod.splits*)

**apply** (*rule P*)

**using** *gcollapse-refine[OF V S]*

**apply** *simp*

**done**

**lemma** *gstate-invar-empty*:

*gstate-invar*  $(a, [], D, pE) \implies a = []$

*gstate-invar*  $([], p, D, pE) \implies p = []$

**by** (*auto simp add: gstate-invar-def*)

**lemma** *find-ce-refine*: *gfind-ce*  $\leq \Downarrow Id$  *find-ce*

**unfolding** *gfind-ce-def find-ce-def*

**unfolding** *goinitial-def go-is-no-brk-def[abs-def] goD-def goBrk-def*

*gto-outer-def*

**using**  $[[goals-limit = 1]]$

**apply** (*refine-rcg*

*gselect-edge-refine prod-rell[OF IdI gpop-refine]*

*prod-rell[OF IdI gpush-refine]*

*fglr-aux1 last-acc-impl oinitial-b-refine*

*inj-on-id*

)

```

apply refine-dref-type
apply (simp-all add: ginitial-refine)
apply (vc-solve (nopre)
  solve: asm-rl
  simp: gstate-rel-def br-def gstate-invar-empty)
done
end

```

### 1.21.2 Refinement to Gabow's Data Structure

**Preliminaries definition** *Un-set-drop-impl* :: *nat*  $\Rightarrow$  '*a set list*  $\Rightarrow$  '*a set nres*

— Executable version of  $\bigcup \text{set } (\text{drop } i \ A)$ , using indexing to access *A*

**where** *Un-set-drop-impl* *i A*  $\equiv$

```

do {
  (-,res)  $\leftarrow$  WHILET ( $\lambda(i, \text{res}). i < \text{length } A$ ) ( $\lambda(i, \text{res}). \text{do } \{$ 
    ASSERT ( $i < \text{length } A$ );
    let res =  $A!i \cup \text{res}$ ;
    let  $i = i + 1$ ;
    RETURN ( $i, \text{res}$ )
  }) ( $i, \{\}$ );
  RETURN res
}

```

**lemma** *Un-set-drop-impl-correct*:

*Un-set-drop-impl* *i A*  $\leq$  SPEC ( $\lambda r. r = \bigcup (\text{set } (\text{drop } i \ A))$ )

**unfolding** *Un-set-drop-impl-def*

**apply** (refine-rcg

WHILET-rule[**where**  $I = \lambda(i', \text{res}). \text{res} = \bigcup (\text{set } ((\text{drop } i \ (\text{take } i' \ A)))) \wedge i \leq i'$

**and**  $R = \text{measure } (\lambda(i', -). \text{length } A - i')$ ]

refine-vcg)

**apply** (auto simp: take-Suc-conv-app-nth)

**done**

**schematic-goal** *Un-set-drop-code-aux*:

**assumes** [autoref-rules]: (*es-impl*,  $\{\}$ )  $\in \langle R \rangle Rs$

**assumes** [autoref-rules]: (*un-impl*,  $\cup$ )  $\in \langle R \rangle Rs \rightarrow \langle R \rangle Rs \rightarrow \langle R \rangle Rs$

**shows** ( $?c, \text{Un-set-drop-impl}$ )  $\in \text{nat-rel} \rightarrow \langle \langle R \rangle Rs \rangle \text{as-rel} \rightarrow \langle \langle R \rangle Rs \rangle \text{nres-rel}$

**unfolding** *Un-set-drop-impl-def* [abs-def]

**apply** (autoref (trace, keep-goal))

**done**

**concrete-definition** *Un-set-drop-code* **uses** *Un-set-drop-code-aux*

**schematic-goal** *Un-set-drop-tr-aux*:

RETURN  $?c \leq \text{Un-set-drop-code } \text{es-impl } \text{un-impl } i \ A$

**unfolding** *Un-set-drop-code-def*

**by** refine-transfer

**concrete-definition** *Un-set-drop-tr* **for** *es-impl un-impl* *i A*

**uses** *Un-set-drop-tr-aux*

**lemma** *Un-set-drop-autoref*[*autoref-rules*]:  
**assumes** *GEN-OP es-impl* {} ( $\langle R \rangle Rs$ )  
**assumes** *GEN-OP un-impl* ( $\cup$ ) ( $\langle R \rangle Rs \rightarrow \langle R \rangle Rs \rightarrow \langle R \rangle Rs$ )  
**shows** ( $\lambda i A. RETURN (Un-set-drop-tr es-impl un-impl i A), Un-set-drop-impl$ )  
 $\in nat-rel \rightarrow \langle \langle R \rangle Rs \rangle as-rel \rightarrow \langle \langle R \rangle Rs \rangle nres-rel$   
**apply** (*intro fun-rell nres-rell*)  
**apply** (*rule order-trans[OF Un-set-drop-tr.refine]*)  
**using** *Un-set-drop-code.refine*[*of es-impl Rs R un-impl,*  
*param-fo, THEN nres-reld*]  
**using** *assms*  
**by** *simp*

**Actual Refinement** **type-synonym**  $'Q gGS = nat\ set\ list \times 'Q\ GS$

**type-synonym**  $'Q goGS = 'Q\ ce \times 'Q\ oGS$

**context** *igb-graph*  
**begin**

**definition** *gGS-invar* ::  $'Q\ gGS \Rightarrow bool$   
**where** *gGS-invar*  $s \equiv$   
 $let\ (a,S,B,I,P) = s\ in$   
 $GS-invar\ (S,B,I,P)$   
 $\wedge\ length\ a = length\ B$   
 $\wedge\ \cup(set\ a) \subseteq \{0..<num-acc\}$

**definition** *gGS- $\alpha$*  ::  $'Q\ gGS \Rightarrow 'Q\ abs-gstate$   
**where** *gGS- $\alpha$*   $s \equiv let\ (a,s)=s\ in\ (a,GS.\alpha\ s)$

**definition** *gGS-rel*  $\equiv br\ gGS-\alpha\ gGS-invar$

**lemma** *gGS-rel-sv*[*relator-props,intro!,simp*]: *single-valued gGS-rel*  
**unfolding** *gGS-rel-def* **by** *auto*

**definition** *goGS-invar* ::  $'Q\ goGS \Rightarrow bool$  **where**  
 $goGS-invar\ s \equiv let\ (brk,ogs)=s\ in\ brk=None \longrightarrow oGS-invar\ ogs$

**definition** *goGS- $\alpha$*   $s \equiv let\ (brk,ogs)=s\ in\ (brk,oGS-\alpha\ ogs)$

**definition** *goGS-rel*  $\equiv br\ goGS-\alpha\ goGS-invar$

**lemma** *goGS-rel-sv*[*relator-props,intro!,simp*]: *single-valued goGS-rel*  
**unfolding** *goGS-rel-def* **by** *auto*

**end**

**context** *igb-fr-graph*

**begin**

**lemma** *gGS-relE*:

**assumes**  $(s', (a, p, D, pE)) \in gGS\text{-rel}$

**obtains**  $S' B' I' P'$  **where**  $s' = (a, S', B', I', P')$

**and**  $((S', B', I', P'), (p, D, pE)) \in GS\text{-rel}$

**and**  $\text{length } a = \text{length } B'$

**and**  $\bigcup (\text{set } a) \subseteq \{0..<\text{num-acc}\}$

**using** *assms*

**apply** (*cases s'*)

**apply** (*simp add: gGS-rel-def br-def gGS- $\alpha$ -def GS. $\alpha$ -def*)

**apply** (*rule that*)

**apply** (*simp only:*)

**apply** (*auto simp: GS-rel-def br-def gGS-invar-def GS. $\alpha$ -def*)

**done**

**definition** *goinitial-impl* ::  $'Q$  *goGS*

**where** *goinitial-impl*  $\equiv$  (*None, Map.empty*)

**lemma** *goinitial-impl-refine*:  $(\text{goinitial-impl}, \text{goinitial}) \in \text{goGS-rel}$

**by** (*auto*)

*simp: goinitial-impl-def goinitial-def goGS-rel-def br-def*

*simp: goGS- $\alpha$ -def goGS-invar-def oGS- $\alpha$ -def oGS-invar-def*)

**definition** *gto-outer-impl* ::  $'Q$  *ce*  $\Rightarrow$   $'Q$  *gGS*  $\Rightarrow$   $'Q$  *goGS*

**where** *gto-outer-impl* *brk s*  $\equiv$  *let* (*A, S, B, I, P*) = *s* *in* (*brk, I*)

**lemma** *gto-outer-refine*:

**assumes** *A*: *brk* = *None*  $\longrightarrow$  (*p* = []  $\wedge$  *pE* = {})

**assumes** *B*:  $(s, (A, p, D, pE)) \in gGS\text{-rel}$

**assumes** *C*:  $(\text{brk}', \text{brk}) \in \text{Id}$

**shows**  $(\text{gto-outer-impl } \text{brk}' s, \text{gto-outer } \text{brk } (A, p, D, pE)) \in \text{goGS-rel}$

**proof** (*cases s*)

**fix** *A S B I P*

**assume** [*simp*]:  $s = (A, S, B, I, P)$

**show** *?thesis*

**using** *C*

**apply** (*cases brk*)

**using** *assms I-to-outer*[*of S B I P D*]

**apply** (*auto*)

*simp: goGS-rel-def br-def goGS- $\alpha$ -def gto-outer-def*

*gto-outer-impl-def goGS-invar-def*

*simp: gGS-rel-def oGS-rel-def GS-rel-def gGS- $\alpha$ -def gGS-invar-def*

*GS. $\alpha$ -def*) []

**using** *B* **apply** (*auto*)

*simp: gto-outer-def gto-outer-impl-def*

*simp: br-def goGS-rel-def goGS-invar-def goGS- $\alpha$ -def oGS- $\alpha$ -def*

*simp: gGS-rel-def gGS- $\alpha$ -def GS. $\alpha$ -def GS.D- $\alpha$ -def*

)

**done**  
**qed**

**definition** *gpush-impl*  $v\ s \equiv \text{let } (a,s)=s \text{ in } (a@[acc\ v], \text{push-impl } v\ s)$

**lemma** *gpush-impl-refine*:

**assumes**  $B: (s',(a,p,D,pE)) \in gGS\text{-rel}$   
**assumes**  $A: (v',v) \in Id$   
**assumes**  $PRE: v' \notin \bigcup(\text{set } p) \quad v' \notin D$   
**shows**  $(\text{gpush-impl } v' s', \text{gpush } v (a,p,D,pE)) \in gGS\text{-rel}$

**proof** –

**from**  $B$  **obtain**  $S' B' I' P'$  **where**  $[simp]: s'=(a,S',B',I',P')$   
**and**  $OSR: ((S',B',I',P'),(p,D,pE)) \in GS\text{-rel}$  **and**  $L: \text{length } a = \text{length } B'$   
**and**  $R: \bigcup(\text{set } a) \subseteq \{0..<num\text{-acc}\}$   
**by** (*rule gGS-relE*)

{

**fix**  $S\ B\ I\ P\ S'\ B'\ I'\ P'$   
**assume**  $\text{push-impl } v (S, B, I, P) = (S', B', I', P')$   
**hence**  $\text{length } B' = \text{Suc } (\text{length } B)$   
**by** (*auto simp add: push-impl-def GS.push-impl-def Let-def*)

} **note**  $AUX1=this$

**from**  $\text{push-refine}[OF\ OSR\ A\ PRE]\ A\ L\ \text{acc-bound } R$  **show** *?thesis*  
**unfolding** *gpush-impl-def gpush-def*  
*gGS-rel-def gGS-invar-def gGS- $\alpha$ -def GS-rel-def br-def*  
**apply** (*auto dest: AUX1*)  
**done**

**qed**

**definition** *gpop-impl*  $:: 'Q\ gGS \Rightarrow 'Q\ gGS\ nres$

**where**  $\text{gpop-impl } s \equiv \text{do } \{$   
 $\text{let } (a,s)=s;$   
 $s \leftarrow \text{pop-impl } s;$   
 $ASSERT (a \neq []);$   
 $\text{let } a = \text{butlast } a;$   
 $RETURN (a,s)$   
 $\}$

**lemma** *gpop-impl-refine*:

**assumes**  $A: (s',(a,p,D,pE)) \in gGS\text{-rel}$   
**assumes**  $PRE: p \neq [] \quad pE \cap \text{last } p \times UNIV = \{\}$   
**shows**  $\text{gpop-impl } s' \leq \Downarrow gGS\text{-rel } (RETURN (\text{gpop } (a,p,D,pE)))$

**proof** –

**from**  $A$  **obtain**  $S' B' I' P'$  **where**  $[simp]: s'=(a,S',B',I',P')$   
**and**  $OSR: ((S',B',I',P'),(p,D,pE)) \in GS\text{-rel}$  **and**  $L: \text{length } a = \text{length } B'$   
**and**  $R: \bigcup(\text{set } a) \subseteq \{0..<num\text{-acc}\}$

```

by (rule gGS-relE)

from PRE OSR have [simp]: a≠[] using L
by (auto simp add: GS-rel-def br-def GS.α-def GS.p-α-def)

{
  fix S B I P S' B' I' P'
  assume nofail (pop-impl ((S, B, I, P)::'a GS))
  inres (pop-impl ((S, B, I, P)::'a GS)) (S', B', I', P')
  hence length B' = length B - Suc 0
  apply (simp add: pop-impl-def GS.pop-impl-def Let-def
    refine-pw-simps)
  apply auto
  done
} note AUX1=this

from A L show ?thesis
  unfolding gpop-impl-def gpop-def gGS-rel-def gGS-α-def br-def
  apply (simp add: Let-def)
  using pop-refine[OF OSR PRE]
  apply (simp add: pw-le-iff refine-pw-simps split: prod.splits)
  unfolding gGS-rel-def gGS-invar-def gGS-α-def GS-rel-def GS.α-def br-def
  apply (auto dest!: AUX1 in-set-butlastD iff: Sup-le-iff)
  done
qed

definition gselect-edge-impl :: 'Q gGS ⇒ ('Q option × 'Q gGS) nres
where gselect-edge-impl s ≡
do {
  let (a,s)=s;
  (vo,s)←select-edge-impl s;
  RETURN (vo,a,s)
}

thm select-edge-refine
lemma gselect-edge-impl-refine:
  assumes A: (s', a, p, D, pE) ∈ gGS-rel
  assumes PRE: p ≠ []
  shows gselect-edge-impl s' ≤ ↓(Id ×r gGS-rel) (gselect-edge (a, p, D, pE))
proof –
from A obtain S' B' I' P' where [simp]: s'=(a,S',B',I',P')
  and OSR: ((S',B',I',P'),(p,D,pE))∈GS-rel and L: length a = length B'
  and R: ⋃(set a) ⊆ {0..num-acc}
  by (rule gGS-relE)

{
  fix S B I P S' B' I' P' vo
  assume nofail (select-edge-impl ((S, B, I, P)::'a GS))
  inres (select-edge-impl ((S, B, I, P)::'a GS)) (vo, (S', B', I', P'))
}

```

```

hence  $length\ B' = length\ B$ 
  apply (simp add: select-edge-impl-def GS.sel-rem-last-def refine-pw-simps
    split: if-split-asm prod.splits)
  apply auto
  done
} note AUX1=this

```

```

show ?thesis
  using select-edge-refine[OF OSR PRE]
  unfolding gselect-edge-impl-def gselect-edge-def
  apply (simp add: refine-pw-simps pw-le-iff prod-rel-sv)

```

```

unfolding gGS-rel-def br-def gGS- $\alpha$ -def gGS-invar-def GS-rel-def GS. $\alpha$ -def
apply (simp split: prod.splits)
apply clarsimp
using R
apply (auto simp: L dest: AUX1)
done

```

**qed**

**term** *GS.idx-of-impl*

**thm** *GS-invar.idx-of-correct*

**definition** *gcollapse-impl-aux* :: '*Q*  $\Rightarrow$  '*Q* *gGS*  $\Rightarrow$  '*Q* *gGS* *nres* **where**

```

gcollapse-impl-aux v s  $\equiv$ 
do {
  let (A,s)=s;
  ASSERT (( $\exists$  i  $\in$  UNSET (GS.idx-of-impl s))
  i  $\leftarrow$  GS.idx-of-impl s v;
  s  $\leftarrow$  collapse-impl v s;
  ASSERT (i < length A);
  us  $\leftarrow$  Un-set-drop-impl i A;
  let A = take i A @ [us];
  RETURN (A,s)
}

```

**term** *collapse*

**lemma** *gcollapse-alt*:

```

gcollapse v APDPE = (
  let
    (a,p,D,pE)=APDPE;
    i=idx-of p v;
    s=collapse v (p,D,pE);
    us= $\bigcup$ (set (drop i a));
    a = take i a @ [us]
  in (a,s)

```

**unfolding** *gcollapse-def gcollapse-aux-def collapse-def collapse-aux-def*  
**by** *auto*

**thm** *collapse-refine*

**lemma** *gcollapse-impl-aux-refine*:

**assumes** *A*:  $(s', a, p, D, pE) \in gGS\text{-rel}$

**assumes** *B*:  $(v', v) \in Id$

**assumes** *PRE*:  $v \in \bigcup (set\ p)$

**shows** *gcollapse-impl-aux*  $v' s'$

$\leq \Downarrow gGS\text{-rel} (RETURN (gcollapse\ v\ (a, p, D, pE)))$

**proof**  $-$

**note**  $[simp] = Let\text{-def}$

**from** *A* **obtain**  $S' B' I' P'$  **where**  $[simp]: s' = (a, S', B', I', P')$

**and** *OSR*:  $((S', B', I', P'), (p, D, pE)) \in GS\text{-rel}$  **and** *L*:  $length\ a = length\ B'$

**and** *R*:  $\bigcup (set\ a) \subseteq \{0..<num\text{-acc}\}$

**by**  $(rule\ gGS\text{-rel}E)$

**from** *B* **have**  $[simp]: v' = v$  **by** *simp*

**from** *OSR* **have**  $[simp]: GS.p\text{-}\alpha\ (S', B', I', P') = p$

**by**  $(simp\ add: GS\text{-rel}\text{-def}\ br\text{-def}\ GS.\alpha\text{-def})$

**from** *OSR PRE* **have** *PRE'*:  $v \in \bigcup (set\ (GS.p\text{-}\alpha\ (S', B', I', P')))$

**by**  $(simp\ add: GS\text{-rel}\text{-def}\ br\text{-def}\ GS.\alpha\text{-def})$

**from** *OSR* **have** *GS-invar*:  $GS\text{-invar}\ (S', B', I', P')$

**by**  $(simp\ add: GS\text{-rel}\text{-def}\ br\text{-def})$

**term** *GS.B*

{

**fix** *s*

**assume** *collapse v*  $(p, D, pE) = (GS.p\text{-}\alpha\ s, GS.D\text{-}\alpha\ s, GS.pE\text{-}\alpha\ s)$

**hence**  $length\ (GS.B\ s) = Suc\ (idx\text{-of}\ p\ v)$

**unfolding** *collapse-def collapse-aux-def Let-def*

**apply**  $(cases\ s)$

**apply**  $(auto\ simp: GS.p\text{-}\alpha\text{-def})$

**apply**  $(drule\ arg\text{-cong}[where\ f=length])$

**using** *GS-invar.p-α-disjoint-sym*  $[OF\ GS\text{-invar}]$

**and** *PRE*  $\langle GS.p\text{-}\alpha\ (S', B', I', P') = p \rangle\ idx\text{-of}\text{-props}(1)[of\ p\ v]$

**by** *simp*

} **note** *AUX1 = this*

**show** *?thesis*

**unfolding** *gcollapse-alt gcollapse-impl-aux-def*

**apply** *simp*

**apply**  $(rule\ RETURN\text{-as}\text{-SPEC}\text{-refine})$

**apply**  $(refine\ rcg)$

$order\text{-trans}[OF\ GS\text{-invar}.idx\text{-of}\text{-correct}[OF\ GS\text{-invar}\ PRE']]$

```

    order-trans[OF collapse-refine[OF OSR B PRE, simplified]]
    refine-vcg
  )
  using PRE' apply simp

  apply (simp add: L)

  using Un-set-drop-impl-correct acc-bound R
  apply (simp add: refine-pw-simps pw-le-iff)
  unfolding gGS-rel-def GS-rel-def GS.alpha-def br-def gGS.alpha-def gGS-invar-def
  apply (clarsimp simp: L dest!: AUX1)
  apply (auto dest!: AUX1 simp: L)
  apply (force dest!: in-set-dropD) []
  apply (force dest!: in-set-takeD) []
  done
qed

```

```

definition gcollapse-impl :: 'Q ⇒ 'Q gGS ⇒ 'Q gGS nres
where gcollapse-impl v s ≡
  do {
    let (A,S,B,I,P)=s;
    i ← GS.idx-of-impl (S,B,I,P) v;
    ASSERT (i+1 ≤ length B);
    let B = take (i+1) B;
    ASSERT (i < length A);
    us ← Un-set-drop-impl i A;
    let A = take i A @ [us];
    RETURN (A,S,B,I,P)
  }

```

```

lemma gcollapse-impl-aux-opt-refine:
  gcollapse-impl v s ≤ gcollapse-impl-aux v s
unfolding gcollapse-impl-def gcollapse-impl-aux-def collapse-impl-def
  GS.collapse-impl-def
apply (simp add: refine-pw-simps pw-le-iff split: prod.splits)
apply blast
done

```

```

lemma gcollapse-impl-refine:
assumes A: (s', a, p, D, pE) ∈ gGS-rel
assumes B: (v',v) ∈ Id
assumes PRE: v ∈ ⋃ (set p)
shows gcollapse-impl v' s'
  ≤ ↓ gGS-rel (RETURN (gcollapse v (a, p, D, pE)))
using order-trans[OF
  gcollapse-impl-aux-opt-refine
  gcollapse-impl-aux-refine[OF assms]]
.

```

**definition** *ginitial-impl* :: 'Q ⇒ 'Q goGS ⇒ 'Q gGS  
**where** *ginitial-impl* v0 s0 ≡ ([acc v0],initial-impl v0 (snd s0))

**lemma** *ginitial-impl-refine*:  
**assumes** A: v0 ∉ goD s0 go-is-no-brk s0  
**assumes** REL: (s0i,s0) ∈ goGS-rel (v0i,v0) ∈ Id  
**shows** (ginitial-impl v0i s0i,ginitial v0 s0) ∈ gGS-rel  
**unfolding** *ginitial-impl-def* *ginitial-def*  
**using** REL *initial-refine*[OF A(1) - REL(2), of snd s0i] A(2)  
**apply** (auto  
simp: gGS-rel-def br-def gGS-α-def gGS-invar-def goGS-rel-def goGS-α-def  
simp: go-is-no-brk-def goD-def oGS-rel-def GS-rel-def goGS-invar-def  
split: prod.splits  
)  
**using** acc-bound  
**apply** (fastforce simp: initial-impl-def GS-initial-impl-def)+  
**done**

**definition** *gpath-is-empty-impl* :: 'Q gGS ⇒ bool  
**where** *gpath-is-empty-impl* s = *path-is-empty-impl* (snd s)

**lemma** *gpath-is-empty-refine*:  
((s,(a,p,D,pE)) ∈ gGS-rel) ⇒ *gpath-is-empty-impl* s ↔ p = []  
**unfolding** *gpath-is-empty-impl-def*  
**using** *path-is-empty-refine*  
**by** (fastforce simp: gGS-rel-def br-def gGS-invar-def gGS-α-def GS.α-def)

**definition** *gis-on-stack-impl* :: 'Q ⇒ 'Q gGS ⇒ bool  
**where** *gis-on-stack-impl* v s = *is-on-stack-impl* v (snd s)

**lemma** *gis-on-stack-refine*:  
(((s,(a,p,D,pE)) ∈ gGS-rel) ⇒ *gis-on-stack-impl* v s ↔ v ∈ ⋃ (set p)  
**unfolding** *gis-on-stack-impl-def*  
**using** *is-on-stack-refine*  
**by** (fastforce simp: gGS-rel-def br-def gGS-invar-def gGS-α-def GS.α-def)

**definition** *gis-done-impl* :: 'Q ⇒ 'Q gGS ⇒ bool  
**where** *gis-done-impl* v s ≡ *is-done-impl* v (snd s)

**thm** *is-done-refine*

**lemma** *gis-done-refine*: (s,(a,p,D,pE)) ∈ gGS-rel  
⇒ *gis-done-impl* v s ↔ (v ∈ D)  
**using** *is-done-refine*[of (snd s) v]  
**by** (auto  
simp: gGS-rel-def br-def gGS-α-def gGS-invar-def GS.α-def  
*gis-done-impl-def*)

**definition** (in -) *on-stack-less* I u v ≡  
case I v of

*Some (STACK j) ⇒ j < u*  
 | - ⇒ *False*

**definition** (in -) *on-stack-ge I l v* ≡  
*case I v of*  
*Some (STACK j) ⇒ l ≤ j*  
 | - ⇒ *False*

**lemma** (in *GS-invar*) *set-butlast-p-refine*:

**assumes** *PRE*:  $p\text{-}\alpha \neq []$

**shows**  $\text{Collect } (\text{on-stack-less } I \text{ (last } B)) = \bigcup (\text{set } (\text{butlast } p\text{-}\alpha))$  (is ?L=?R)

**proof** (intro *equalityI subsetI*)

**from** *PRE* **have** [*simp*]:  $B \neq []$  **by** (*auto simp: p-α-def*)

**have** [*simp*]:  $S \neq []$

**by** (*simp add: empty-eq*)

{

**fix** *v*

**assume**  $v \in ?L$

**then obtain** *j* **where** [*simp*]:  $I \ v = \text{Some } (\text{STACK } j)$  **and**  $j < \text{last } B$

**by** (*auto simp: on-stack-less-def split: option.splits node-state.splits*)

**from** *I-consistent*[*of v j*] **have** [*simp*]:  $j < \text{length } S \quad v = S!j$  **by** *auto*

**from** *B0* **have**  $B!0 = 0$  **by** *simp*

**from**  $\langle j < \text{last } B \rangle$  **have**  $j < B!(\text{length } B - 1)$  **by** (*simp add: last-conv-nth*)

**from** *find-seg-bounds*[*OF*  $\langle j < \text{length } S \rangle$ ] *find-seg-correct*[*OF*  $\langle j < \text{length } S \rangle$ ]

**have**  $v \in \text{seg } (\text{find-seg } j) \quad \text{find-seg } j < \text{length } B$  **by** *auto*

**moreover with**  $\langle j < B!(\text{length } B - 1) \rangle$  **have**  $\text{find-seg } j < \text{length } B - 1$

**proof** -

**have** *f1*:  $\bigwedge x_1 \ x. \neg (x_1::\text{nat}) < x_1 - x$

**using** *less-imp-diff-less* **by** *blast*

**have**  $j \leq \text{last } B$

**by** (*metis*  $\langle j < \text{last } B \rangle$  *less-le*)

**hence** *f2*:  $\bigwedge x_1. \neg \text{last } B < x_1 \vee \neg x_1 \leq j$

**using** *f1* **by** (*metis* *diff-diff-cancel le-trans*)

**have**  $\bigwedge x_1. \text{seg-end } x_1 \leq j \vee \neg x_1 < \text{find-seg } j$

**by** (*metis*  $\langle \text{seg-start } (\text{find-seg } j) \leq j \rangle$  *calculation(2)*)

*le-trans seg-end-less-start*)

**thus**  $\text{find-seg } j < \text{length } B - 1$

**using** *f1 f2*

**by** (*metis* *GS.seg-start-def*  $\langle B \neq [] \rangle \langle j < B!(\text{length } B - 1) \rangle$

$\langle \text{seg-start } (\text{find-seg } j) \leq j \rangle$  *calculation(2)* *diff-diff-cancel*

*last-conv-nth nat-neq-iff seg-start-less-end*)

**qed**

**ultimately show**  $v \in ?R$

```

    by (auto simp: p- $\alpha$ -def map-butlast[symmetric] butlast-upt)
  }

{
  fix v
  assume v $\in$ ?R
  then obtain i where i < length B - 1 and v $\in$ seg i
    by (auto simp: p- $\alpha$ -def map-butlast[symmetric] butlast-upt)
  then obtain j where j < seg-end i and v=S!j
    by (auto simp: seg-def)
  hence j < B!(i+1) and i+1  $\leq$  length B - 1 using <i < length B - 1>
    by (auto simp: seg-end-def last-conv-nth split: if-split-asm)
  with sorted-nth-mono[OF B-sorted <i+1  $\leq$  length B - 1>] have j < last B
    by (auto simp: last-conv-nth)
  moreover from <j < seg-end i> have j < length S
    by (metis GS.seg-end-def add-diff-inverse-nat <i + 1  $\leq$  length B - 1>
      add-lessD1 less-imp-diff-less less-le-not-le nat-neq-iff
      seg-end-bound)

  with I-consistent <v=S!j> have I v = Some (STACK j) by auto
  ultimately show v $\in$ ?L
    by (auto simp: on-stack-less-def)
}
qed

```

```

lemma (in GS-invar) set-last-p-refine:
  assumes PRE: p- $\alpha$  $\neq$ []
  shows Collect (on-stack-ge I (last B)) = last p- $\alpha$  (is ?L=?R)
proof (intro equalityI subsetI)
  from PRE have [simp]: B $\neq$ [] by (auto simp: p- $\alpha$ -def)

```

```

  have [simp]: S $\neq$ [] by (simp add: empty-eq)

```

```

{
  fix v
  assume v $\in$ ?L
  then obtain j where [simp]: I v = Some (STACK j) and j $\geq$ last B
    by (auto simp: on-stack-ge-def split: option.splits node-state.splits)

  from I-consistent[of v j] have [simp]: j < length S v=S!j by auto
  hence v $\in$ seg (length B - 1) using <j $\geq$ last B>
    by (auto simp: seg-def last-conv-nth seg-start-def seg-end-def)
  thus v $\in$ last p- $\alpha$  by (auto simp: p- $\alpha$ -def last-map)
}

```

```

{
  fix v
  assume v $\in$ ?R
  hence v $\in$ seg (length B - 1)

```

```

    by (auto simp: p- $\alpha$ -def last-map)
  then obtain j where v=S!j  j $\geq$ last B  j<length S
    by (auto simp: seg-def last-conv-nth seg-start-def seg-end-def)
  with I-consistent have I v = Some (STACK j) by simp
  with <j $\geq$ last B> show v $\in$ ?L by (auto simp: on-stack-ge-def)
}
qed

```

```

definition ce-impl :: 'Q gGS  $\Rightarrow$  (('Q set  $\times$  'Q set) option  $\times$  'Q gGS) nres
where ce-impl s  $\equiv$ 
  do {
    let (a,S,B,I,P) = s;
    ASSERT (B $\neq$ []);
    let bls = Collect (on-stack-less I (last B));
    let ls = Collect (on-stack-ge I (last B));
    RETURN (Some (bls, ls),a,S,B,I,P)
  }

```

```

lemma ce-impl-refine:
assumes A: (s,(a,p,D,pE)) $\in$ gGS-rel
assumes PRE: p $\neq$ []
shows ce-impl s  $\leq$   $\Downarrow$ (Id $\times_r$ gGS-rel)
  (RETURN (Some ( $\bigcup$ (set (butlast p)),last p),a,p,D,pE))

```

```

proof -
from A obtain S' B' I' P' where [simp]: s=(a,S',B',I',P')
and OSR: ((S',B',I',P'),(p,D,pE)) $\in$ GS-rel and L: length a = length B'
by (rule gGS-reLE)

```

```

from OSR have [simp]: GS.p- $\alpha$  (S',B',I',P') = p
by (simp add: GS-rel-def br-def GS. $\alpha$ -def)

```

```

from PRE have NE': GS.p- $\alpha$  (S', B', I', P')  $\neq$  [] by simp
hence BNE[simp]: B' $\neq$ [] by (simp add: GS.p- $\alpha$ -def)

```

```

from OSR have GS-invar: GS-invar (S',B',I',P')
by (simp add: GS-rel-def br-def)

```

```

show ?thesis
using GS-invar.set-butlast-p-refine[OF GS-invar NE']
using GS-invar.set-last-p-refine[OF GS-invar NE']
unfolding ce-impl-def
using A
by auto

```

qed

```

definition last-is-acc-impl s  $\equiv$ 
  do {
    let (a,-)=s;
    ASSERT (a $\neq$ []);
  }

```

```

    RETURN ( $\forall i < \text{num-acc}. i \in \text{last } a$ )
  }

```

**lemma** *last-is-acc-impl-refine*:

**assumes** *A*:  $(s, (a, p, D, pE)) \in \text{gGS-rel}$

**assumes** *PRE*:  $a \neq []$

**shows** *last-is-acc-impl*  $s \leq \text{RETURN } (\text{last } a = \{0..<\text{num-acc}\})$

**proof** –

**from** *A PRE* **have**  $\text{last } a \subseteq \{0..<\text{num-acc}\}$

**unfolding** *gGS-rel-def gGS-invar-def br-def gGS- $\alpha$ -def* **by** *auto*

**hence** *C*:  $(\forall i < \text{num-acc}. i \in \text{last } a) \longleftrightarrow (\text{last } a = \{0..<\text{num-acc}\})$

**by** *auto*

**from** *A* **obtain** *gs* **where**  $[simp]: s = (a, gs)$

**by** (*auto simp: gGS-rel-def gGS- $\alpha$ -def br-def split: prod.splits*)

**show** *?thesis*

**unfolding** *last-is-acc-impl-def*

**by** (*auto simp: gGS-rel-def br-def gGS- $\alpha$ -def C PRE split: prod.splits*)

**qed**

**definition** *go-is-no-brk-impl* ::  $'Q \text{ goGS} \Rightarrow \text{bool}$

**where** *go-is-no-brk-impl*  $s \equiv \text{fst } s = \text{None}$

**lemma** *go-is-no-brk-refine*:

$(s, s') \in \text{goGS-rel} \Longrightarrow \text{go-is-no-brk-impl } s \longleftrightarrow \text{go-is-no-brk } s'$

**unfolding** *go-is-no-brk-def go-is-no-brk-impl-def*

**by** (*auto simp: goGS-rel-def br-def goGS- $\alpha$ -def split: prod.splits*)

**definition** *goD-impl* ::  $'Q \text{ goGS} \Rightarrow 'Q \text{ oGS}$  **where** *goD-impl*  $s \equiv \text{snd } s$

**lemma** *goD-refine*:

$\text{go-is-no-brk } s' \Longrightarrow (s, s') \in \text{goGS-rel} \Longrightarrow (\text{goD-impl } s, \text{goD } s') \in \text{oGS-rel}$

**unfolding** *goD-impl-def goD-def*

**by** (*auto*)

*simp: goGS-rel-def br-def goGS- $\alpha$ -def goGS-invar-def oGS-rel-def go-is-no-brk-def*)

**definition** *go-is-done-impl* ::  $'Q \Rightarrow 'Q \text{ goGS} \Rightarrow \text{bool}$

**where** *go-is-done-impl*  $v s \equiv \text{is-done-oimpl } v (\text{snd } s)$

**thm** *is-done-orefine*

**lemma** *go-is-done-impl-refine*:  $\llbracket \text{go-is-no-brk } s'; (s, s') \in \text{goGS-rel}; (v, v') \in \text{Id} \rrbracket$

$\Longrightarrow \text{go-is-done-impl } v s \longleftrightarrow (v' \in \text{goD } s')$

**using** *is-done-orefine*

**unfolding** *go-is-done-impl-def goD-def go-is-no-brk-def*

**apply** (*fastforce simp: goGS-rel-def br-def goGS-invar-def goGS- $\alpha$ -def*)

**done**

**definition** *goBrk-impl* ::  $'Q \text{ goGS} \Rightarrow 'Q \text{ ce}$  **where** *goBrk-impl*  $\equiv \text{fst}$

**lemma** *goBrk-refine*:  $(s,s') \in \text{goGS-rel} \implies (\text{goBrk-impl } s, \text{goBrk } s') \in \text{Id}$   
**unfolding** *goBrk-impl-def goBrk-def*  
**by** (*auto simp: goGS-rel-def br-def goGS- $\alpha$ -def split: prod.splits*)

**definition** *find-ce-impl* :: ('Q set  $\times$  'Q set) option nres **where**

```

find-ce-impl  $\equiv$  do {
  stat-start-nres;
  let os = goinitial-impl;
  os  $\leftarrow$  FOREACHci ( $\lambda$ it os. fgl-outer-invar it (goGS- $\alpha$  os)) V0
    (go-is-no-brk-impl) ( $\lambda$ v0 s0).
  do {
    if  $\neg$ go-is-done-impl v0 s0 then do {

      let s = (None, ginitial-impl v0 s0);

      (brk, s)  $\leftarrow$  WHILEIT
        ( $\lambda$ (brk, s). fgl-invar v0 (oGS- $\alpha$  (goD-impl s0)) (brk, snd (gGS- $\alpha$  s)))
        ( $\lambda$ (brk, s). brk = None  $\wedge$   $\neg$ gpath-is-empty-impl s) ( $\lambda$ (l, s).
      do {
        — Select edge from end of path
        (vo, s)  $\leftarrow$  gselect-edge-impl s;

        case vo of
          Some v  $\Rightarrow$  do {
            if gis-on-stack-impl v s then do {
              s  $\leftarrow$  gcollapse-impl v s;
              b  $\leftarrow$  last-is-acc-impl s;
              if b then
                ce-impl s
              else
                RETURN (None, s)
            } else if  $\neg$ gis-done-impl v s then do {
              — Edge to new node. Append to path
              RETURN (None, gpush-impl v s)
            } else do {
              — Edge to done node. Skip
              RETURN (None, s)
            }
          }
        | None  $\Rightarrow$  do {
          — No more outgoing edges from current node on path
          s  $\leftarrow$  gpop-impl s;
          RETURN (None, s)
        }
      }) (s);
      RETURN (gto-outer-impl brk s)
    } else RETURN s0
  }) os;
  stat-stop-nres;

```

```

    RETURN (goBrk-impl os)
  }

```

**lemma** *find-ce-impl-refine*:  $find\text{-}ce\text{-}impl \leq \Downarrow Id\ gfind\text{-}ce$

**proof** –

```

note [refine2] = prod-relI[OF IdI[of None] ginitial-impl-refine]

```

```

have [refine]:  $\bigwedge s\ a\ p\ D\ pE.$   $\llbracket$ 
  (s,(a,p,D,pE)) $\in$ gGS-rel;
  p  $\neq$  []; pE  $\cap$  last p  $\times$  UNIV = {}
 $\rrbracket \implies$ 
  gpop-impl s  $\gg$  (λs. RETURN (None, s))
  ≤ SPEC (λc. (c, None, gpop (a,p,D,pE))  $\in$  Id  $\times_r$  gGS-rel)
apply (drule (2) gpop-impl-refine)
apply (fastforce simp add: pw-le-iff refine-pw-simps)
done

```

```

note [[goals-limit = 1]]

```

```

note FOREACHci-refine-rcg'[refine del]

```

**show** ?thesis

```

unfolding find-ce-impl-def gfind-ce-def

```

```

apply (refine-rcg

```

```

  bind-refine'

```

```

  prod-relI IdI

```

```

  inj-on-id

```

```

  gselect-edge-impl-refine gpush-impl-refine

```

```

  oinitial-refine ginitial-impl-refine

```

```

  bind-Let-refine2[OF gcollapse-impl-refine]

```

```

  if-bind-cond-refine[OF last-is-acc-impl-refine]

```

```

  ce-impl-refine

```

```

  goinitial-impl-refine

```

```

  gto-outer-refine

```

```

  goBrk-refine

```

```

  FOREACHci-refine-rcg'[where R=goGS-rel, OF inj-on-id]

```

```

)

```

```

apply refine-dref-type

```

```

apply (simp-all add: go-is-no-brk-refine go-is-done-impl-refine)

```

```

apply (auto simp: goGS-rel-def br-def)  $\llbracket$ 

```

```

apply (auto simp: goGS-rel-def br-def goGS- $\alpha$ -def gGS- $\alpha$ -def gGS-rel-def
  goD-def goD-impl-def)  $\llbracket$ 

```

```

apply (auto dest: gpath-is-empty-refine )  $\llbracket$ 

```

```

apply (auto dest: gis-on-stack-refine)  $\llbracket$ 

```

```

apply (auto dest: gis-done-refine)  $\llbracket$ 

```

done  
qed

end

## 1.22 Constructing a Lasso from Counterexample

### 1.22.1 Lassos in GBAs

context *igb-fr-graph* begin

**definition** *reconstruct-reach* :: 'Q set  $\Rightarrow$  'Q set  $\Rightarrow$  ('Q list  $\times$  'Q) nres  
 — Reconstruct the reaching path of a lasso  
**where** *reconstruct-reach* Vr Vl  $\equiv$  do {  
   res  $\leftarrow$  *find-path* (E  $\cap$  Vr  $\times$  UNIV) V0 ( $\lambda v. v \in Vl$ );  
   ASSERT (res  $\neq$  None);  
   RETURN (the res)  
 }

**lemma** *reconstruct-reach-correct*:  
**assumes** CEC: *ce-correct* Vr Vl  
**shows** *reconstruct-reach* Vr Vl  
 $\leq$  SPEC ( $\lambda(pr, va). \exists v0 \in V0. \text{path } E \ v0 \ pr \ va \wedge va \in Vl$ )

**proof** —

**have** *FIN-aux*: *finite* ((E  $\cap$  Vr  $\times$  UNIV)\* “ V0)  
**by** (*metis finite-reachableE-V0 finite-subset inf-sup-ord(1) inf-sup-ord(2) inf-top.left-neutral reachable-mono*)

{  
**fix** u p v  
**assume** P: *path* E u p v **and** SS: *set* p  $\subseteq$  Vr  
**have** *path* (E  $\cap$  Vr  $\times$  UNIV) u p v  
**apply** (*rule path-mono[OF - path-restrict[OF P]]*)  
**using** SS **by** *auto*  
**}** **note** P-CONV=*this*

**from** CEC **obtain** v0 pr va **where** v0  $\in$  V0    *set* pr  $\subseteq$  Vr    va  $\in$  Vl  
*path* (E  $\cap$  Vr  $\times$  UNIV) v0 pr va  
**unfolding** *ce-correct-def is-lasso-prpl-def is-lasso-prpl-pre-def*  
**by** (*force simp: neq-Nil-conv path-simps dest: P-CONV*)  
**hence** 1: va  $\in$  (E  $\cap$  Vr  $\times$  UNIV)\* “ V0  
**by** (*auto dest: path-is-rtrancl*)

**show** ?thesis

**using** *assms unfolding reconstruct-reach-def*  
**apply** (*refine-rcg refine-vcg order-trans[OF find-path-ex-rule]*)  
**apply** (*clarsimp-all simp: FIN-aux finite-V0*)

**using**  $\langle va \in Vl \rangle$  1 **apply** *auto* []

**apply** (*auto dest: path-mono*[of  $E \cap Vr \times UNIV E$ , *simplified*]) []  
**done**  
**qed**

**definition** *rec-loop-invar*  $Vl va s \equiv let (v,p,cS) = s in$   
 $va \in E^* \wedge V0 \wedge$   
 $path E va p v \wedge$   
 $cS = acc v \cup (\bigcup (acc\ set\ p)) \wedge$   
 $va \in Vl \wedge v \in Vl \wedge set\ p \subseteq Vl$

**definition** *reconstruct-lasso* ::  $'Q\ set \Rightarrow 'Q\ set \Rightarrow ('Q\ list \times 'Q\ list)\ nres$   
— Reconstruct lasso  
**where** *reconstruct-lasso*  $Vr Vl \equiv do \{$   
 $(pr,va) \leftarrow reconstruct-reach\ Vr\ Vl;$

$let\ cS-full = \{0..<num-acc\};$   
 $let\ E = E \cap UNIV \times Vl;$

$(vd,p,-) \leftarrow WHILEIT (rec-loop-invar\ Vl\ va)$   
 $(\lambda(-,-,cS). cS \neq cS-full)$   
 $(\lambda(v,p,cS). do \{$   
 $ASSERT (\exists v'. (v,v') \in E^* \wedge \neg (acc\ v' \subseteq cS));$   
 $sr \leftarrow find-path\ E\ \{v\} (\lambda v. \neg (acc\ v \subseteq cS));$   
 $ASSERT (sr \neq None);$   
 $let (p-seg,v) = the\ sr;$   
 $RETURN (v,p@p-seg,cS \cup acc\ v)$   
 $\}) (va,[],acc\ va);$

$p-close-r \leftarrow (if\ p=[]\ then$   
 $find-path1\ E\ vd\ ((=)\ va)$   
 $else$   
 $find-path\ E\ \{vd\} ((=)\ va));$

$ASSERT (p-close-r \neq None);$   
 $let (p-close,-) = the\ p-close-r;$

$RETURN (pr, p@p-close)$   
 $\}$

**lemma** (*in igb-fr-graph*) *reconstruct-lasso-correct*:  
**assumes** *CEC: ce-correct*  $Vr Vl$   
**shows** *reconstruct-lasso*  $Vr Vl \leq SPEC (is-lasso-prpl)$   
**proof** —

**let**  $?E = E \cap UNIV \times Vl$

**have** *E-SS*:  $E \cap Vl \times Vl \subseteq ?E$  **by** *auto*

```

from CEC have
  REACH:  $Vl \subseteq E^* \text{“} V0$ 
  and CONN:  $Vl \times Vl \subseteq (E \cap Vl \times Vl)^*$ 
  and NONTRIV:  $Vl \times Vl \cap E \neq \{\}$ 
  and NES[simp]:  $Vl \neq \{\}$ 
  and ALL:  $\bigcup (acc \text{‘} Vl) = \{0..<num-acc\}$ 
  unfolding ce-correct-def is-lasso-prpl-def
  apply clarsimp-all
  apply auto []
  apply force
  done

define term-rel
  where term-rel = (inv-image (finite-psupset  $\{0..<num-acc\}$ ) ( $\lambda(-::'Q,-::'Q$ 
list,cS). cS))
  hence WF: wf term-rel by simp

{ fix va
  assume  $va \in Vl$ 
  hence rec-loop-invar  $Vl \text{ } va \text{ } (va, [], acc \text{ } va)$ 
  unfolding rec-loop-invar-def using REACH by auto
} note INVAR-INITIAL = this

{
  fix v p cS va
  assume rec-loop-invar  $Vl \text{ } va \text{ } (v, p, cS)$ 
  hence finite ( $(?E)^* \text{“} \{v\}$ )
  apply –
  apply (rule finite-subset[where  $B = E^* \text{“} V0$ ])
  unfolding rec-loop-invar-def
  using REACH
  apply (clarsimp-all dest!: path-is-rtrancl)
  apply (drule rtrancl-mono-mp[where  $U = ?E$  and  $V = E, rotated$ ], (auto) [])
  by (metis rev-ImageI rtrancl-trans)
} note FIN1 = this

{
  fix  $va \text{ } v :: 'Q$  and  $p \text{ } cS$ 
  assume INV: rec-loop-invar  $Vl \text{ } va \text{ } (v,p,cS)$ 
  and NC:  $cS \neq \{0..<num-acc\}$ 

from NC INV obtain i where  $i < num-acc \quad i \notin cS$ 
  unfolding rec-loop-invar-def by auto blast

with ALL obtain  $v'$  where  $v': v' \in Vl \quad \neg acc \text{ } v' \subseteq cS$ 
  by (metis NES atLeastLessThan-iff cSUP-least in-mono zero-le)

with CONN INV have  $(v,v') \in (E \cap Vl \times Vl)^*$ 
  unfolding rec-loop-invar-def by auto

```

```

  hence  $(v, v') \in ?E^*$  using rtrancl-mono-mp[OF E-SS] by blast
  with  $v'$  have  $\exists v'. (v, v') \in (?E)^* \wedge \neg \text{acc } v' \subseteq cS$  by auto
} note ASSERT1 = this

{
  fix  $va\ v\ p\ cS\ v'\ p'$ 
  assume rec-loop-invar  $Vl\ va\ (v, p, cS)$ 
  and path  $(?E)\ v\ p'\ v'$ 
  and  $\neg (\text{acc } v' \subseteq cS)$ 
  and  $\forall v \in \text{set } p'. \text{acc } v \subseteq cS$ 
  hence rec-loop-invar  $Vl\ va\ (v', p@p', cS \cup \text{acc } v')$ 
  unfolding rec-loop-invar-def
  apply simp
  apply (intro conjI)
  apply (auto simp: path-simps dest: path-mono[of ?E E, simplified]) []

  apply (cases p')
  apply (auto simp: path-simps) [2]

  apply (cases p' rule: rev-cases)
  apply (auto simp: path-simps) [2]

  apply (erule path-set-induct)
  apply auto [2]
  done
} note INV-PRES = this

{
  fix  $va\ v\ p\ cS\ v'\ p'$ 
  assume rec-loop-invar  $Vl\ va\ (v, p, cS)$ 
  and path  $?E\ v\ p'\ v'$ 
  and  $\neg (\text{acc } v' \subseteq cS)$ 
  and  $\forall v \in \text{set } p'. \text{acc } v \subseteq cS$ 
  hence  $((v', p@p', cS \cup \text{acc } v'), (v, p, cS)) \in \text{term-rel}$ 
  unfolding term-rel-def rec-loop-invar-def
  by (auto simp: finite-psupset-def)
} note VAR = this

have CONN1:  $Vl \times Vl \subseteq (?E)^+$ 
proof clarify
  fix  $a\ b$ 
  assume  $a \in Vl\ b \in Vl$ 
  from NONTRIV obtain  $u\ v$  where  $E: (u, v) \in (E \cap Vl \times Vl)$  by auto
  from CONN  $\langle a \in Vl \rangle E$  have  $(a, u) \in (E \cap Vl \times Vl)^*$  by auto
  also note  $E$ 
  also (rtrancl-into-trancl1) from CONN  $\langle b \in Vl \rangle E$  have  $(v, b) \in (E \cap Vl \times Vl)^*$ 
  by auto
  finally show  $(a, b) \in (?E)^+$  using trancl-mono[OF - E-SS] by auto
qed

```

```

{
  fix va v p cS
  assume rec-loop-invar Vl va (v, p, cS)
  hence (v,va) ∈ (?E)+
    unfolding rec-loop-invar-def
    using CONN1
    by auto
} note CLOSE1 = this

{
  fix va v p cS
  assume rec-loop-invar Vl va (v, p, cS)
  hence (v,va) ∈ (?E)*
    unfolding rec-loop-invar-def
    using CONN rtrancl-mono[OF E-SS]
    by auto
} note CLOSE2 = this

{
  fix pr vd pl va v0
  assume rec-loop-invar Vl va (vd, [], {0..

```

```

    dest!: path-mono[of ?E E, simplified]) []

  have is-lasso-prpl (pr,p@p')
    unfolding is-lasso-prpl-def is-lasso-prpl-pre-def
    apply (clarsimp simp: ACC)
    using PR PL ⟨p≠[]⟩ by auto
  } note INV-POST2 = this

show ?thesis
  unfolding reconstruct-lasso-def
  apply (refine-rcg
    WF
    order-trans[OF reconstruct-reach-correct]
    order-trans[OF find-path-ex-rule]
    order-trans[OF find-path1-ex-rule]
    refine-vcg
  )

  apply (vc-solve
    del: subsetI
    solve: ASSERT1 INV-PRES asm-rl VAR CLOSE1 CLOSE2 INV-POST1
    INV-POST2
    simp: INVAR-INITIAL FIN1 CEC)
  done
qed

definition find-lasso where find-lasso ≡ do {
  ce ← find-ce-spec;
  case ce of
  None ⇒ RETURN None
| Some (Vr, Vl) ⇒ do {
  l ← reconstruct-lasso Vr Vl;
  RETURN (Some l)
}
}

lemma (in igb-fr-graph) find-lasso-correct: find-lasso ≤ find-lasso-spec
  unfolding find-lasso-spec-def find-lasso-def find-ce-spec-def
  apply (refine-rcg refine-vcg order-trans[OF reconstruct-lasso-correct])
  apply auto
  done

end

end

```

### 1.23 Code Generation for the Skeleton Algorithm

**theory** Gabow-Skeleton-Code

```

imports
  Gabow-Skeleton
  CAVA-Automata.Digraph-Impl
  CAVA-Base.CAVA-Code-Target
begin

```

## 1.24 Statistics

In this section, we do the ML setup that gathers statistics about the algorithm's execution.

### code-printing

```

code-module Gabow-Skeleton-Statistics  $\rightarrow$  (SML)  $\langle$ 
  structure Gabow-Skeleton-Statistics = struct
    val active = Unsynchronized.ref false
    val num-vis = Unsynchronized.ref 0

    val time = Unsynchronized.ref Time.zeroTime

    fun is-active () = !active
    fun newnode () =
      (
        num-vis := !num-vis + 1;
        if !num-vis mod 10000 = 0 then tracing (IntInf.toString (!num-vis) ^ \n)
      )
    else ()
      )

    fun start () = (active := true; time := Time.now ())
    fun stop () = (time := Time.- (Time.now (), !time))

    fun to-string () = let
      val t = Time.toMilliseconds (!time)
      val states-per-ms = real (!num-vis) / real t
      val realStr = Real.fmt (StringCvt.FIX (SOME 2))
    in
      Required time: ^ IntInf.toString (t) ^ ms\n
      ^ States per ms: ^ realStr states-per-ms ^ \n
      ^ # states: ^ IntInf.toString (!num-vis) ^ \n
    end

    val - = Statistics.register-stat (Gabow-Skeleton,is-active,to-string)

  end
   $\rangle$ 
code-reserved (SML) Gabow-Skeleton-Statistics

```

### code-printing

```

constant stat-newnode  $\rightarrow$  (SML) Gabow'-Skeleton'-Statistics.newnode
| constant stat-start  $\rightarrow$  (SML) Gabow'-Skeleton'-Statistics.start
| constant stat-stop  $\rightarrow$  (SML) Gabow'-Skeleton'-Statistics.stop

```

## 1.25 Automatic Refinement Setup

**consts** *i-node-state* :: *interface*

**definition** *node-state-rel*  $\equiv \{(-1::\text{int},\text{DONE})\} \cup \{(int\ k,\text{STACK}\ k) \mid k.\ \text{True}\}$

**lemma** *node-state-rel-simps*[*simp*]:

$(i,\text{DONE}) \in \text{node-state-rel} \longleftrightarrow i = -1$

$(i,\text{STACK}\ n) \in \text{node-state-rel} \longleftrightarrow i = \text{int}\ n$

**unfolding** *node-state-rel-def*

**by** *auto*

**lemma** *node-state-rel-sv*[*simp,intro!,relator-props*]:

*single-valued node-state-rel*

**unfolding** *node-state-rel-def*

**by** (*auto intro: single-valuedI*)

**lemmas** [*autoref-rel-intf*] = *REL-INTFI*[*of node-state-rel i-node-state*]

**primrec** *is-DONE* **where**

*is-DONE* *DONE* = *True*

| *is-DONE* (*STACK* -) = *False*

**lemma** *node-state-rel-refine*[*autoref-rules*]:

$(-1,\text{DONE}) \in \text{node-state-rel}$

$(int,\text{STACK}) \in \text{nat-rel} \rightarrow \text{node-state-rel}$

$(\lambda i.\ i < 0, \text{is-DONE}) \in \text{node-state-rel} \rightarrow \text{bool-rel}$

$((\lambda f\ g\ i.\ \text{if}\ i \geq 0\ \text{then}\ f\ (\text{nat}\ i)\ \text{else}\ g), \text{case-node-state})$

$\in (\text{nat-rel} \rightarrow R) \rightarrow R \rightarrow \text{node-state-rel} \rightarrow R$

**unfolding** *node-state-rel-def*

**apply** *auto* [*3*]

**apply** (*fastforce dest: fun-relD*)

**done**

**lemma** [*autoref-op-pat*]:

$(x = \text{DONE}) \equiv \text{is-DONE}\ x$

$(\text{DONE} = x) \equiv \text{is-DONE}\ x$

**apply** (*auto intro!: eq-reflection*)

**apply** ((*cases* *x*, *simp-all*) [])+

**done**

**consts** *i-node* :: *interface*

**locale** *fr-graph-impl-loc* = *fr-graph* *G*

**for** *mrel* **and** *node-rel* :: (*'vi*  $\times$  *'v*) *set*

**and** *node-eq-impl* :: *'vi*  $\Rightarrow$  *'vi*  $\Rightarrow$  *bool*

**and** *node-hash-impl* :: *nat*  $\Rightarrow$  *'vi*  $\Rightarrow$  *nat*

**and** *node-def-hash-size* :: *nat*

**and** *G-impl* **and** *G* :: (*'v*, *'more*) *graph-rec-scheme*

```

+
assumes G-refine:  $(G\text{-impl}, G) \in \langle mrel, node\text{-rel} \rangle g\text{-impl}\text{-rel}\text{-ext}$ 
  and node-eq-refine:  $(node\text{-eq}\text{-impl}, (=)) \in node\text{-rel} \rightarrow node\text{-rel} \rightarrow bool\text{-rel}$ 
  and node-hash: is-bounded-hashcode node-rel node-eq-impl node-hash-impl
  and node-hash-def-size:  $(is\text{-valid}\text{-def}\text{-hm}\text{-size } TYPE('vi) \text{ node}\text{-def}\text{-hash}\text{-size})$ 
begin

  lemmas [autoref-rel-intf] = REL-INTFI[of node-rel i-node]

  lemmas [autoref-rules] = G-refine node-eq-refine

  lemmas [autoref-ga-rules] = node-hash node-hash-def-size

  lemma locale-this: fr-graph-impl-loc mrel node-rel node-eq-impl node-hash-impl
node-def-hash-size G-impl G
    by unfold-locales

  abbreviation oGSi-rel  $\equiv \langle node\text{-rel}, node\text{-state}\text{-rel} \rangle (ahm\text{-rel } node\text{-hash}\text{-impl})$ 

  abbreviation GSi-rel  $\equiv$ 
     $\langle node\text{-rel} \rangle as\text{-rel}$ 
     $\times_r \langle nat\text{-rel} \rangle as\text{-rel}$ 
     $\times_r oGSi\text{-rel}$ 
     $\times_r \langle nat\text{-rel} \times_r \langle node\text{-rel} \rangle list\text{-set}\text{-rel} \rangle as\text{-rel}$ 

  lemmas [autoref-op-pat] = GS.S-def GS.B-def GS.I-def GS.P-def

end

```

## 1.26 Generating the Code

**thm** *autoref-ga-rules*

**context** *fr-graph-impl-loc*

**begin**

**schematic-goal** *push-code-aux*:  $(?c, push\text{-impl}) \in node\text{-rel} \rightarrow GSi\text{-rel} \rightarrow GSi\text{-rel}$

**unfolding** *push-impl-def-opt*[*abs-def*]

**using** [[*autoref-trace-failed-id*]]

**apply** (*autoref* (*keep-goal*))

**done**

**concrete-definition** (**in**  $-$ ) *push-code* **uses** *fr-graph-impl-loc.push-code-aux*

**lemmas** [*autoref-rules*] = *push-code.refine*[*OF locale-this*]

**schematic-goal** *pop-code-aux*:  $(?c, pop\text{-impl}) \in GSi\text{-rel} \rightarrow \langle GSi\text{-rel} \rangle nres\text{-rel}$

**unfolding** *pop-impl-def-opt*[*abs-def*]

**unfolding** *GS.mark-as-done-def*

**using**  $[[\text{autoref-trace-failed-id}]]$   
**apply**  $(\text{autoref } (\text{keep-goal}))$   
**done**  
**concrete-definition** **(in -)** *pop-code* **uses** *fr-graph-impl-loc.pop-code-aux*  
**lemmas**  $[\text{autoref-rules}] = \text{pop-code.refine}[OF \text{ locale-this}]$

**schematic-goal** *S-idx-of-code-aux*:  
**notes**  $[\text{autoref-rules}] = IdI[\text{of undefined}::\text{nat}]$   
**shows**  $(?c, GS.S\text{-idx-of}) \in GSi\text{-rel} \rightarrow \text{node-rel} \rightarrow \text{nat-rel}$   
**unfolding** *GS.S-idx-of-def* $[\text{abs-def}]$   
**using**  $[[\text{autoref-trace-failed-id}]]$   
**apply**  $(\text{autoref } (\text{keep-goal}))$   
**done**  
**concrete-definition** **(in -)** *S-idx-of-code*  
**uses** *fr-graph-impl-loc.S-idx-of-code-aux*  
**lemmas**  $[\text{autoref-rules}] = S\text{-idx-of-code.refine}[OF \text{ locale-this}]$

**schematic-goal** *idx-of-code-aux*:  
**notes**  $[\text{autoref-rules}] = IdI[\text{of undefined}::\text{nat}]$   
**shows**  $(?c, GS.\text{idx-of-impl}) \in GSi\text{-rel} \rightarrow \text{node-rel} \rightarrow \langle \text{nat-rel} \rangle \text{nres-rel}$   
**unfolding**  
*GS.idx-of-impl-def* $[\text{abs-def}, \text{unfolded } GS.\text{find-seg-impl-def } GS.S\text{-idx-of-def},$   
*THEN opt-GSdef, unfolded GS-sel-simps, abs-def]*  
**using**  $[[\text{autoref-trace-failed-id}]]$   
**apply**  $(\text{autoref } (\text{keep-goal}))$   
**done**  
**concrete-definition** **(in -)** *idx-of-code* **uses** *fr-graph-impl-loc.idx-of-code-aux*  
**lemmas**  $[\text{autoref-rules}] = \text{idx-of-code.refine}[OF \text{ locale-this}]$

**schematic-goal** *collapse-code-aux*:  
 $(?c, \text{collapse-impl}) \in \text{node-rel} \rightarrow GSi\text{-rel} \rightarrow \langle GSi\text{-rel} \rangle \text{nres-rel}$   
**unfolding** *collapse-impl-def-opt* $[\text{abs-def}]$   
**using**  $[[\text{autoref-trace-failed-id}]]$   
**apply**  $(\text{autoref } (\text{keep-goal}))$   
**done**  
**concrete-definition** **(in -)** *collapse-code*  
**uses** *fr-graph-impl-loc.collapse-code-aux*  
**lemmas**  $[\text{autoref-rules}] = \text{collapse-code.refine}[OF \text{ locale-this}]$

**term** *select-edge-impl*

**schematic-goal** *select-edge-code-aux*:  
 $(?c, \text{select-edge-impl})$   
 $\in GSi\text{-rel} \rightarrow \langle \langle \text{node-rel} \rangle \text{option-rel} \times_r GSi\text{-rel} \rangle \text{nres-rel}$   
**unfolding** *select-edge-impl-def-opt* $[\text{abs-def}]$   
  
**using**  $[[\text{autoref-trace-failed-id}]]$   
**using**  $[[\text{goals-limit}=1]]$   
**apply**  $(\text{autoref } (\text{keep-goal}, \text{trace}))$   
**done**

**concrete-definition** (in  $-$ ) *select-edge-code*  
**uses** *fr-graph-impl-loc.select-edge-code-aux*  
**lemmas** [*autoref-rules*] = *select-edge-code.refine[OF locale-this]*

**context begin interpretation** *autoref-syn* .

**term** *fr-graph.pop-impl*  
**lemma** [*autoref-op-pat*]:  
*push-impl*  $\equiv$  *OP push-impl*  
*collapse-impl*  $\equiv$  *OP collapse-impl*  
*select-edge-impl*  $\equiv$  *OP select-edge-impl*  
*pop-impl*  $\equiv$  *OP pop-impl*  
**by** *simp-all*

**end**

**schematic-goal** *skeleton-code-aux*:  
 $(?c, skeleton-impl) \in \langle oGSi-rel \rangle nres-rel$   
**unfolding** *skeleton-impl-def[abs-def] initial-impl-def GS-initial-impl-def*  
**unfolding** *path-is-empty-impl-def is-on-stack-impl-def is-done-impl-def*  
*is-done-oimpl-def*  
**unfolding** *GS.is-on-stack-impl-def GS.is-done-impl-def*  
**using** [[*autoref-trace-failed-id*]]  
**apply** (*autoref (keep-goal, trace)*)  
**done**

**concrete-definition** (in  $-$ ) *skeleton-code*  
**for** *node-eq-impl G-impl*  
**uses** *fr-graph-impl-loc.skeleton-code-aux*

**thm** *skeleton-code.refine*

**lemmas** [*autoref-rules*] = *skeleton-code.refine[OF locale-this]*

**schematic-goal** *pop-tr-aux*: *RETURN ?c  $\leq$  pop-code node-eq-impl node-hash-impl*  
<sup>s</sup>  
**unfolding** *pop-code-def* **by** *refine-transfer*  
**concrete-definition** (in  $-$ ) *pop-tr* **uses** *fr-graph-impl-loc.pop-tr-aux*  
**lemmas** [*refine-transfer*] = *pop-tr.refine[OF locale-this]*

**schematic-goal** *select-edge-tr-aux*: *RETURN ?c  $\leq$  select-edge-code node-eq-impl*  
<sup>s</sup>  
**unfolding** *select-edge-code-def* **by** *refine-transfer*  
**concrete-definition** (in  $-$ ) *select-edge-tr*  
**uses** *fr-graph-impl-loc.select-edge-tr-aux*  
**lemmas** [*refine-transfer*] = *select-edge-tr.refine[OF locale-this]*

**schematic-goal** *idx-of-tr-aux*: RETURN ?c ≤ *idx-of-code node-eq-impl node-hash-impl*  
*v s*

**unfolding** *idx-of-code-def* **by** *refine-transfer*

**concrete-definition** (in *−*) *idx-of-tr* **uses** *fr-graph-impl-loc.idx-of-tr-aux*

**lemmas** [*refine-transfer*] = *idx-of-tr.refine[OF locale-this]*

**schematic-goal** *collapse-tr-aux*: RETURN ?c ≤ *collapse-code node-eq-impl node-hash-impl*  
*v s*

**unfolding** *collapse-code-def* **by** *refine-transfer*

**concrete-definition** (in *−*) *collapse-tr* **uses** *fr-graph-impl-loc.collapse-tr-aux*

**lemmas** [*refine-transfer*] = *collapse-tr.refine[OF locale-this]*

**schematic-goal** *skeleton-tr-aux*: RETURN ?c ≤ *skeleton-code node-hash-impl*  
*node-def-hash-size node-eq-impl g*

**unfolding** *skeleton-code-def* **by** *refine-transfer*

**concrete-definition** (in *−*) *skeleton-tr* **uses** *fr-graph-impl-loc.skeleton-tr-aux*

**lemmas** [*refine-transfer*] = *skeleton-tr.refine[OF locale-this]*

**end**

**term** *skeleton-tr*

**export-code** *skeleton-tr* **checking** *SML*

**end**

## 1.27 Code Generation for SCC-Computation

**theory** *Gabow-SCC-Code*

**imports**

*Gabow-SCC*

*Gabow-Skeleton-Code*

*CAVA-Base.CAVA-Code-Target*

**begin**

## 1.28 Automatic Refinement to Efficient Data Structures

**context** *fr-graph-impl-loc*

**begin**

**schematic-goal** *last-seg-code-aux*:

(?c, *last-seg-impl*) ∈ *GSi-rel* → ⟨⟨*node-rel*⟩*list-set-rel*⟩*nres-rel*

**unfolding** *last-seg-impl-def-opt*[*abs-def*]

**using** [[*autoref-trace-failed-id*]]

**apply** (*autoref* (*keep-goal*, *trace*))

**done**

**concrete-definition** (in *−*) *last-seg-code*

**uses** *fr-graph-impl-loc.last-seg-code-aux*

**lemmas** [*autoref-rules*] = *last-seg-code.refine[OF locale-this]*

**context begin interpretation** *autoref-syn* .

**lemma** [*autoref-op-pat*]:  
*last-seg-impl*  $\equiv$  *OP last-seg-impl*  
**by** *simp-all*

**end**

**schematic-goal** *compute-SCC-code-aux*:

(*?c, compute-SCC-impl*)  $\in$   $\langle\langle\langle$ node-rel $\rangle$ list-set-rel $\rangle$ list-rel $\rangle$ nres-rel  
**unfolding** *compute-SCC-impl-def*[*abs-def*] *initial-impl-def* *GS-initial-impl-def*  
**unfolding** *path-is-empty-impl-def* *is-on-stack-impl-def* *is-done-impl-def*  
*is-done-oimpl-def*  
**unfolding** *GS.is-on-stack-impl-def* *GS.is-done-impl-def*  
**using** [[*autoref-trace-failed-id*]]  
**apply** (*autoref (keep-goal, trace)*)  
**done**

**concrete-definition** (**in**  $-$ ) *compute-SCC-code*

**uses** *fr-graph-impl-loc.compute-SCC-code-aux*

**lemmas** [*autoref-rules*] = *compute-SCC-code.refine*[*OF locale-this*]

**schematic-goal** *last-seg-tr-aux*: RETURN  $?c \leq$  *last-seg-code s*

**unfolding** *last-seg-code-def* **by** *refine-transfer*

**concrete-definition** (**in**  $-$ ) *last-seg-tr* **uses** *fr-graph-impl-loc.last-seg-tr-aux*

**lemmas** [*refine-transfer*] = *last-seg-tr.refine*[*OF locale-this*]

**schematic-goal** *compute-SCC-tr-aux*: RETURN  $?c \leq$  *compute-SCC-code node-eq-impl*  
*node-hash-impl node-def-hash-size g*

**unfolding** *compute-SCC-code-def* **by** *refine-transfer*

**concrete-definition** (**in**  $-$ ) *compute-SCC-tr*

**uses** *fr-graph-impl-loc.compute-SCC-tr-aux*

**lemmas** [*refine-transfer*] = *compute-SCC-tr.refine*[*OF locale-this*]

**end**

**export-code** *compute-SCC-tr* **checking** *SML*

## 1.29 Correctness Theorem

**theorem** *compute-SCC-tr-correct*:

— Correctness theorem for the constant we extracted to SML

**fixes** *Re* **and** *node-rel* :: ( $'vi \times 'v$ ) set

**fixes** *G* :: ( $'v, 'more$ ) graph-rec-scheme

**assumes** *A*:

(*G-impl, G*)  $\in$  (*Re, node-rel*) *g-impl-rel-ext*

(*node-eq-impl, (=)*)  $\in$  *node-rel*  $\rightarrow$  *node-rel*  $\rightarrow$  *bool-rel*

*is-bounded-hashcode node-rel node-eq-impl node-hash-impl*

(*is-valid-def-hm-size TYPE('vi) node-def-hash-size*)

**assumes** *C*: *fr-graph G*

```

shows RETURN (compute-SCC-tr node-eq-impl node-hash-impl node-def-hash-size
G-impl)
  ≤  $\Downarrow$ (( $\langle$ node-rel $\rangle$ list-set-rel)list-rel) (fr-graph.compute-SCC-spec G)
proof –
  from C interpret fr-graph G .
  have I: fr-graph-impl-loc Re node-rel node-eq-impl node-hash-impl node-def-hash-size
G-impl G
    apply unfold-locales using A .
  then interpret fr-graph-impl-loc Re node-rel node-eq-impl node-hash-impl node-def-hash-size
G-impl G .

  note compute-SCC-tr.refine[OF I]
  also note compute-SCC-code.refine[OF I, THEN nres-relD]
  also note compute-SCC-impl-refine
  also note compute-SCC-correct
  finally show ?thesis using A by simp
qed

```

### 1.30 Extraction of Benchmark Code

```

schematic-goal list-set-of-list-aux:
  (?c,set) ∈ (nat-rel)list-rel →  $\langle$ nat-rel $\rangle$ list-set-rel
  by autoref
concrete-definition list-set-of-list uses list-set-of-list-aux

```

**term** compute-SCC-tr

```

definition compute-SCC-tr-nat :: - ⇒ - ⇒ - ⇒ - ⇒ nat list list
  where compute-SCC-tr-nat ≡ compute-SCC-tr

```

**end**

### 1.31 Implementation of Safety Property Model Checker

```

theory Find-Path-Impl
imports
  Find-Path
  CAVA-Automata.Digraph-Impl
begin

```

### 1.32 Workset Algorithm

A simple implementation is by a workset algorithm.

```

definition ws-update E u p V ws ≡ RETURN (
  V ∪ E“{u},
  ws ++ (λv. if (u,v) ∈ E ∧ v ∉ V then Some (u#p) else None))

```

**definition**  $s\text{-init } U0 \equiv \text{RETURN } (None, U0, \lambda u. \text{ if } u \in U0 \text{ then Some } [] \text{ else None})$

**definition**  $wset\text{-find-path } E \ U0 \ P \equiv \text{do } \{$   
 $\text{ASSERT } (\text{finite } U0);$   
 $s0 \leftarrow s\text{-init } U0;$   
 $(res, -, -) \leftarrow \text{WHILET}$   
 $(\lambda(res, V, ws). \text{ res} = None \wedge ws \neq \text{Map.empty})$   
 $(\lambda(res, V, ws). \text{ do } \{$   
 $\text{ASSERT } (ws \neq \text{Map.empty});$   
 $(u, p) \leftarrow \text{SPEC } (\lambda(u, p). \text{ ws } u = \text{Some } p);$   
 $\text{let } ws = ws \mid' (-\{u\});$   
  
 $\text{if } P \ u \ \text{then}$   
 $\text{RETURN } (\text{Some } (\text{rev } p, u), V, ws)$   
 $\text{else do } \{$   
 $\text{ASSERT } (\text{finite } (E''\{u\}));$   
 $\text{ASSERT } (\text{dom } ws \subseteq V);$   
 $(V, ws) \leftarrow \text{ws-update } E \ u \ p \ V \ ws;$   
 $\text{RETURN } (None, V, ws)$   
 $\}$   
 $\}) \ s0;$   
 $\text{RETURN } res$   
 $\}$

**lemma**  $wset\text{-find-path-correct}$ :

**fixes**  $E :: ('v \times 'v) \ \text{set}$

**shows**  $wset\text{-find-path } E \ U0 \ P \leq \text{find-path } E \ U0 \ P$

**proof** –

**define**  $inv \ \text{where } inv = (\lambda(res, V, ws). \text{ case } res \ \text{of}$

$None \Rightarrow$

$\text{dom } ws \subseteq V$

$\wedge \text{finite } (\text{dom } ws) \quad \text{— Derived}$

$\wedge V \subseteq E^*''U0$

$\wedge E''(V - \text{dom } ws) \subseteq V$

$\wedge (\forall v \in V - \text{dom } ws. \neg P \ v)$

$\wedge U0 \subseteq V$

$\wedge (\forall v \ p. \text{ ws } v = \text{Some } p$

$\longrightarrow ((\forall v \in \text{set } p. \neg P \ v) \wedge (\exists u0 \in U0. \text{ path } E \ u0 \ (\text{rev } p) \ v)))$

$\mid \text{Some } (p, v) \Rightarrow (\exists u0 \in U0. \text{ path } E \ u0 \ p \ v \wedge P \ v \wedge (\forall v \in \text{set } p. \neg P \ v)))$

**define**  $var \ \text{where } var = \text{inv-image}$

$(\text{brk-rel } (\text{finite-psupset } (E^*''U0) \ <*\text{lex}*\> \text{measure } (\text{card } o \ \text{dom})))$

$(\lambda(res::('v \ \text{list} \times 'v) \ \text{option}, V::'v \ \text{set}, ws::'v \ \longrightarrow 'v \ \text{list}).$

$(res \neq None, V, ws))$

**have**  $[\text{simp}]$ :  $\bigwedge u \ p \ V. \ \text{dom } (\lambda v. \text{ if } (u, v) \in E \wedge v \notin V \ \text{then } \text{Some } (u \ \# \ p))$

$else\ None) = E^{\{\!|\!|u|\!\}} - V$   
**by** (*auto split: if-split-asm*)

```

{
  fix  $V\ ws\ u\ p$ 
  assume  $INV: inv\ (None, V, ws)$ 
  assume  $WSU: ws\ u = Some\ p$ 

  from  $INV\ WSU$  have
    [simp]:  $V \subseteq E^{\{\!|\!|U0|\!\}}$ 
    and [simp]:  $u \in V$ 
    and  $UREACH: \exists u0 \in U0. (u0, u) \in E^*$ 
    and [simp]:  $finite\ (dom\ ws)$ 
    unfolding inv-def
    apply simp-all
    apply auto []
    apply clarsimp
    apply blast
    done
  have  $(V \cup E^{\{\!|\!|u|\!\}}, V) \in finite-psupset\ (E^*\ \{\!|\!|U0|\!\}) \vee$ 
     $V \cup E^{\{\!|\!|u|\!\}} = V \wedge$ 
     $card\ (E^{\{\!|\!|u|\!\}} - V \cup (dom\ ws - \{u\})) < card\ (dom\ ws)$ 
  proof (subst disj-commute, intro disjCI conjI)
    assume  $(V \cup E^{\{\!|\!|u|\!\}}, V) \notin finite-psupset\ (E^*\ \{\!|\!|U0|\!\})$ 
    thus  $V \cup E^{\{\!|\!|u|\!\}} = V$  using  $UREACH$ 
    by (auto simp: finite-psupset-def intro: rev-ImageI)

    hence [simp]:  $E^{\{\!|\!|u|\!\}} - V = \{\}$  by force
    show  $card\ (E^{\{\!|\!|u|\!\}} - V \cup (dom\ ws - \{u\})) < card\ (dom\ ws)$ 
    using  $WSU$ 
    by (auto intro: card-Diff1-less)
  qed
} note wf-aux=this

```

```

{
  fix  $V\ ws\ u\ p$ 
  assume  $FIN: finite\ (E^*\ \{\!|\!|U0|\!\})$ 
  assume  $inv\ (None, V, ws)$   $ws\ u = Some\ p$ 
  then obtain  $u0$  where  $u0 \in U0$   $(u0, u) \in E^*$  unfolding inv-def
    by clarsimp blast
  hence  $E^{\{\!|\!|u|\!\}} \subseteq E^*\ \{\!|\!|U0|\!\}$  by (auto intro: rev-ImageI)
  hence  $finite\ (E^{\{\!|\!|u|\!\}})$  using  $FIN(1)$  by (rule finite-subset)
} note succs-finite=this

```

```

{
  fix  $V\ ws\ u\ p$ 
  assume  $FIN: finite\ (E^*\ \{\!|\!|U0|\!\})$ 
  assume  $INV: inv\ (None, V, ws)$ 

```

```

assume WSU:  $ws\ u = Some\ p$ 
assume NVD:  $\neg P\ u$ 

have inv (None,  $V \cup E \text{ `` } \{u\}$ ,
   $ws \mid' (- \{u\}) ++$ 
  ( $\lambda v. \text{if } (u, v) \in E \wedge v \notin V \text{ then } Some\ (u \# p)$ 
   $\text{else } None$ ))
unfolding inv-def

apply (simp, intro conjI)
using INV WSU apply (auto simp: inv-def) []
using INV WSU apply (auto simp: inv-def) []
using INV WSU apply (auto simp: succs-finite FIN) []
using INV apply (auto simp: inv-def) []
using INV apply (auto simp: inv-def) []

using INV WSU apply (auto
  simp: inv-def
  intro: rtrancl-image-advance
) []

using INV WSU apply (auto simp: inv-def) []

using INV NVD apply (auto simp: inv-def) []
using INV NVD apply (auto simp: inv-def) []

using INV WSU NVD apply (fastforce
  simp: inv-def restrict-map-def
  intro!: path-conc path1
  split: if-split-asm
) []
done
} note ip-aux=this

show ?thesis
unfolding wset-find-path-def find-path-def ws-update-def s-init-def

apply (refine-rcg refine-vcg le-ASSERTI
  WHILET-rule[where
     $R = var$  and  $I = inv$ ]
)

using [[goals-limit = 1]]

apply (auto simp: var-def) []

apply (auto
  simp: inv-def dom-def
  split: if-split-asm) []

```

```

apply simp
apply (auto simp: inv-def) []
apply (auto simp: var-def brk-rel-def) []

apply (simp add: succs-finite)

apply (auto simp: inv-def) []

apply clarsimp
apply (simp add: ip-aux)

apply clarsimp
apply (simp add: var-def brk-rel-def wf-aux) []

apply (fastforce
  simp: inv-def
  split: option.splits
  intro: rev-ImageI
  dest: Image-closed-trancl) []
done
qed

```

We refine the algorithm to use a foreach-loop

```

definition ws-update-foreach  $E\ u\ p\ V\ ws \equiv$ 
  FOREACH (LIST-SET-REV-TAG ( $E\ \{u\}$ )) ( $\lambda v\ (V, ws)$ ).
  if  $v \in V$  then
    RETURN ( $V, ws$ )
  else do {
    ASSERT ( $v \notin \text{dom } ws$ );
    RETURN (insert  $v\ V, ws\ (v \mapsto u\ \#p)$ )
  }
) ( $V, ws$ )

```

```

lemma ws-update-foreach-refine[refine]:
assumes FIN: finite ( $E\ \{u\}$ )
assumes WSS:  $\text{dom } ws \subseteq V$ 
assumes ID: ( $E', E$ ) $\in Id$  ( $u', u$ ) $\in Id$  ( $p', p$ ) $\in Id$  ( $V', V$ ) $\in Id$  ( $ws', ws$ ) $\in Id$ 
shows ws-update-foreach  $E'\ u'\ p'\ V'\ ws' \leq \Downarrow Id$  (ws-update  $E\ u\ p\ V\ ws$ )
unfolding ID[simplified]
unfolding ws-update-foreach-def ws-update-def LIST-SET-REV-TAG-def
apply (refine-rcg refine-vcg FIN
  FOREACH-rule[where  $I = \lambda it\ (V', ws')$ .
     $V' = V \cup (E'\ \{u\} - it)$ 
     $\wedge \text{dom } ws' \subseteq V'$ 
     $\wedge ws' = ws ++ (\lambda v. \text{if } (u, v) \in E \wedge v \notin it \wedge v \notin V \text{ then } \text{Some } (u\ \#p) \text{ else } \text{None})$ ]
)
using WSS
apply (auto
  simp: Map.map-add-def

```

*split*: *option.splits if-split-asm*  
*intro!*: *ext[where 'a='a and 'b='b list option]*)

**done**

**definition** *s-init-foreach*  $U0 \equiv do \{$   
 $(U0,ws) \leftarrow FOREACH\ U0\ (\lambda x\ (U0,ws)).$   
 $RETURN\ (insert\ x\ U0,ws(x \mapsto []))\ (\{\}, Map.empty);$   
 $RETURN\ (None, U0,ws)$   
 $\}$

**lemma** *s-init-foreach-refine*[*refine*]:

**assumes** *FIN*: *finite U0*  
**assumes** *ID*:  $(U0', U0) \in Id$   
**shows** *s-init-foreach*  $U0' \leq \Downarrow Id\ (s-init\ U0)$   
**unfolding** *s-init-foreach-def s-init-def ID*[*simplified*]

**apply** (*refine-rcg refine-vcg*  
 $FOREACH-rule[where$   
 $I = \lambda it\ (U,ws).$   
 $U = U0-it$   
 $\wedge ws = (\lambda x. if\ x \in U0-it\ then\ Some\ []\ else\ None)]$   
 $)$

**apply** (*auto*  
*simp: FIN*  
*intro!: ext*  
 $)$

**done**

**definition** *wset-find-path'*  $E\ U0\ P \equiv do \{$   
 $ASSERT\ (finite\ U0);$   
 $s0 \leftarrow s-init-foreach\ U0;$   
 $(res, -, -) \leftarrow WHILET$   
 $(\lambda(res, V, ws). res = None \wedge ws \neq Map.empty)$   
 $(\lambda(res, V, ws). do \{$   
 $ASSERT\ (ws \neq Map.empty);$   
 $((u, p), ws) \leftarrow op-map-pick-remove\ ws;$   
  
 $if\ P\ u\ then$   
 $RETURN\ (Some\ (rev\ p, u), V, ws)$   
 $else\ do \{$   
 $(V, ws) \leftarrow ws-update-foreach\ E\ u\ p\ V\ ws;$   
 $RETURN\ (None, V, ws)$   
 $\}$   
 $\})$   
 $s0;$   
 $RETURN\ res$   
 $\}$

```

lemma wset-find-path'-refine:
  wset-find-path' E U0 P ≤ ↓Id (wset-find-path E U0 P)
  unfolding wset-find-path'-def wset-find-path-def
  unfolding op-map-pick-remove-alt
  apply (refine-rcg IdI)
  apply assumption
  apply simp-all
done

```

### 1.33 Refinement to efficient data structures

```

schematic-goal wset-find-path'-refine-aux:
  fixes U0::'a set and P::'a ⇒ bool and E::('a×'a) set
    and Pimpl :: 'ai ⇒ bool
    and node-rel :: ('ai × 'a) set
    and node-eq-impl :: 'ai ⇒ 'ai ⇒ bool
    and node-hash-impl
    and node-def-hash-size

  assumes [autoref-rules]:
    (succi,E)∈⟨node-rel⟩slg-rel
    (Pimpl,P)∈node-rel → bool-rel
    (node-eq-impl, (=)) ∈ node-rel → node-rel → bool-rel
    (U0',U0)∈⟨node-rel⟩list-set-rel

  assumes [autoref-ga-rules]:
    is-bounded-hashcode node-rel node-eq-impl node-hash-impl
    is-valid-def-hm-size TYPE('ai) node-def-hash-size

  notes [autoref-tyrel] =
    TYRELI[where
      R=⟨node-rel,⟨node-rel⟩list-rel⟩list-map-rel]
    TYRELI[where R=⟨node-rel⟩map2set-rel (ahm-rel node-hash-impl)]

  shows (?c::?'c,wset-find-path' E U0 P) ∈ ?R
  unfolding wset-find-path'-def ws-update-foreach-def s-init-foreach-def
  using [[autoref-trace-failed-id]]
  using [[autoref-trace-intf-unif]]
  using [[autoref-trace-pat]]
  apply (autoref (keep-goal))
done

```

```

concrete-definition wset-find-path-impl for node-eq-impl succi U0' Pimpl
  uses wset-find-path'-refine-aux

```

### 1.34 Autoref Setup

```

context begin interpretation autoref-syn .
  lemma [autoref-itype]:

```

*find-path* ::<sub>i</sub> ⟨I⟩<sub>i</sub>i-slg →<sub>i</sub> ⟨I⟩<sub>i</sub>i-set →<sub>i</sub> (I→<sub>i</sub>i-bool)  
→<sub>i</sub> ⟨⟨⟨I⟩<sub>i</sub>i-list, I⟩<sub>i</sub>i-prod⟩<sub>i</sub>i-option⟩<sub>i</sub>i-nres **by** *simp*

**lemma** *wset-find-path-autoref*[*autoref-rules*]:

**fixes** *node-rel* :: ('a i × 'a) set

**assumes** *eq*: GEN-OP *node-eq-impl* (=) (*node-rel*→*node-rel*→*bool-rel*)

**assumes** *hash*: SIDE-GEN-ALGO (*is-bounded-hashcode* *node-rel* *node-eq-impl*  
*node-hash-impl*)

**assumes** *hash-dsz*: SIDE-GEN-ALGO (*is-valid-def-hm-size* TYPE('a i) *node-def-hash-size*)

**shows** (

*wset-find-path-impl* *node-hash-impl* *node-def-hash-size* *node-eq-impl*,

*find-path*)

∈ ⟨*node-rel*⟩*slg-rel* → ⟨*node-rel*⟩*list-set-rel* → (*node-rel*→*bool-rel*)

→ ⟨⟨⟨*node-rel*⟩*list-rel*×<sub>r</sub>*node-rel*⟩*option-rel*⟩*nres-rel*

**proof** –

**note** *EQI* = GEN-OP-D[*OF eq*]

**note** *HASHI* = SIDE-GEN-ALGO-D[*OF hash*]

**note** *DSZI* = SIDE-GEN-ALGO-D[*OF hash-dsz*]

**note** *wset-find-path-impl.refine*[*THEN nres-relD, OF - - EQI - HASHI DSZI*]

**also note** *wset-find-path'-refine*

**also note** *wset-find-path-correct*

**finally show** *?thesis*

**by** (*fastforce intro!*: *nres-relI*)

**qed**

**end**

**schematic-goal** *wset-find-path-transfer-aux*:

*RETURN ?c* ≤ *wset-find-path-impl* *hashi dszi eqi E U0 P*

**unfolding** *wset-find-path-impl-def*

**by** (*refine-transfer* (*post*))

**concrete-definition** *wset-find-path-code*

**for** *E ?U0.0 P* **uses** *wset-find-path-transfer-aux*

**lemmas** [*refine-transfer*] = *wset-find-path-code.refine*

**export-code** *wset-find-path-code* **checking** *SML*

### 1.35 Nontrivial paths

**definition** *find-path1-gen* *E u0 P* ≡ *do* {

*res* ← *find-path* *E* (*E*'{*u0*}) *P*;

*case res* of *None* ⇒ *RETURN None*

| *Some* (*p,v*) ⇒ *RETURN* (*Some* (*u0*#*p,v*))

}

**lemma** *find-path1-gen-correct*: *find-path1-gen* *E u0 P* ≤ *find-path1* *E u0 P*

**unfolding** *find-path1-gen-def* *find-path-def* *find-path1-def*

**apply** (*refine-rcg* *refine-vcg* *le-ASSERTI*)

```

apply (auto
  intro: path-prepend
  dest: tranclD
  elim: finite-subset[rotated]
)
done

```

**schematic-goal** *find-path1-impl-aux*:

```

fixes node-rel :: ('ai × 'a) set
assumes [autoref-rules]: (node-eq-impl, (=)) ∈ node-rel → node-rel → bool-rel
assumes [autoref-ga-rules]:
  is-bounded-hashcode node-rel node-eq-impl node-hash-impl
  is-valid-def-hm-size TYPE('ai) node-def-hash-size

```

```

shows (?c,find-path1-gen::(-×-) set ⇒ -) ∈ ⟨node-rel⟩slg-rel → node-rel → (node-rel
→ bool-rel) → ⟨⟨node-rel⟩list-rel ×r node-rel⟩option-rel⟩nres-rel
unfolding find-path1-gen-def[abs-def]
apply (autoref (trace,keep-goal))
done

```

**lemma** [autoref-itype]:

```

find-path1 ::i ⟨I⟩ii-slg →i I →i (I→ii-bool)
→i ⟨⟨⟨I⟩ii-list, I⟩ii-prod⟩ii-option⟩ii-nres by simp

```

**concrete-definition** *find-path1-impl* uses *find-path1-impl-aux*

**lemma** *find-path1-autoref*[autoref-rules]:

```

fixes node-rel :: ('ai × 'a) set
assumes eq: GEN-OP node-eq-impl (=) (node-rel→node-rel→bool-rel)
assumes hash: SIDE-GEN-ALGO (is-bounded-hashcode node-rel node-eq-impl
node-hash-impl)
assumes hash-dsz: SIDE-GEN-ALGO (is-valid-def-hm-size TYPE('ai) node-def-hash-size)

```

**shows** (*find-path1-impl* node-eq-impl node-hash-impl node-def-hash-size,*find-path1*)

```

∈ ⟨node-rel⟩slg-rel →node-rel → (node-rel → bool-rel) →
⟨⟨⟨node-rel⟩list-rel ×r node-rel⟩Relators.option-rel⟩nres-rel

```

**proof** –

```

note EQI = GEN-OP-D[OF eq]
note HASHI = SIDE-GEN-ALGO-D[OF hash]
note DSZI = SIDE-GEN-ALGO-D[OF hash-dsz]

```

**note** *R* = *find-path1-impl.refine*[param-fo, THEN *nres-rel*D, OF EQI HASHI DSZI]

**note** *R*

**also note** *find-path1-gen-correct*

**finally show** ?thesis **by** (blast intro: *nres-rel*I)

**qed**

**schematic-goal** *find-path1-transfer-aux*:  
  *RETURN ?c ≤ find-path1-impl eqi hashi dszi E u P*  
  **unfolding** *find-path1-impl-def*  
  **by** *refine-transfer*  
**concrete-definition** *find-path1-code* **for** *E u P* **uses** *find-path1-transfer-aux*  
**lemmas** [*refine-transfer*] = *find-path1-code.refine*  
**end**

### 1.36 Code Generation for GBG Lasso Finding Algorithm

**theory** *Gabow-GBG-Code*  
**imports**  
  *Gabow-GBG*  
  *Gabow-Skeleton-Code*  
  *CAVA-Automata.Automata-Impl*  
  *Find-Path-Impl*  
  *CAVA-Base.CAVA-Code-Target*  
**begin**

### 1.37 Autoref Setup

**locale** *impl-lasso-loc* = *igb-fr-graph G*  
  + *fr-graph-impl-loc* (*mrel,node-rel*)*igbg-impl-rel-ecxt node-rel node-eq-impl node-hash-impl*  
  *node-def-hash-size G-impl G*  
  **for** *mrel* **and** *node-rel* **and** *node-eq-impl* *node-hash-impl* *node-def-hash-size* **and**  
  *G-impl* **and** *G* :: ('q,'more) *igb-graph-rec-scheme*  
**begin**

**lemma** *locale-this: impl-lasso-loc mrel node-rel node-eq-impl node-hash-impl node-def-hash-size*  
*G-impl G*  
  **by** *unfold-locales*

**context begin interpretation** *autoref-syn* .

**lemma** [*autoref-op-pat*]:  
  *goinitial-impl* ≡ *OP goinitial-impl*  
  *ginitial-impl* ≡ *OP ginitial-impl*  
  *gpath-is-empty-impl* ≡ *OP gpath-is-empty-impl*  
  *gselect-edge-impl* ≡ *OP gselect-edge-impl*  
  *gis-on-stack-impl* ≡ *OP gis-on-stack-impl*  
  *gcollapse-impl* ≡ *OP gcollapse-impl*  
  *last-is-acc-impl* ≡ *OP last-is-acc-impl*  
  *ce-impl* ≡ *OP ce-impl*  
  *gis-done-impl* ≡ *OP gis-done-impl*  
  *gpush-impl* ≡ *OP gpush-impl*  
  *gpop-impl* ≡ *OP gpop-impl*  
  *goBrk-impl* ≡ *OP goBrk-impl*

*gto-outer-impl*  $\equiv$  *OP gto-outer-impl*  
*go-is-done-impl*  $\equiv$  *OP go-is-done-impl*  
*is-done-oimpl*  $\equiv$  *OP is-done-oimpl*  
*go-is-no-brk-impl*  $\equiv$  *OP go-is-no-brk-impl*  
 by *simp-all*  
**end**

**abbreviation** *gGSi-rel*  $\equiv$   $\langle\langle$ *nat-rel* $\rangle$ *bs-set-rel* $\rangle$ *as-rel*  $\times_r$  *GSi-rel*  
**abbreviation** (in  $-$ ) *ce-rel node-rel*  $\equiv$   $\langle\langle$ *node-rel* $\rangle$ *fun-set-rel*  $\times_r$   $\langle$ *node-rel* $\rangle$ *fun-set-rel* $\rangle$ *option-rel*  
**abbreviation** *goGSi-rel*  $\equiv$  *ce-rel node-rel*  $\times_r$  *oGSi-rel*  
**end**

### 1.38 Automatic Refinement

**context** *impl-lasso-loc*  
**begin**

**schematic-goal** *goinitial-code-aux*: ( $?c, goinitial-impl$ )  $\in goGSi-rel$   
**unfolding** *goinitial-impl-def*[*abs-def*]  
**using** [[*autoref-trace-failed-id*]]  
**by** (*autoref* (*trace, keep-goal*))  
**concrete-definition** (in  $-$ ) *goinitial-code*  
**uses** *impl-lasso-loc.goinitial-code-aux*  
**lemmas** [*autoref-rules*] = *goinitial-code.refine*[*OF locale-this*]

**term** *ginitial-impl*

**schematic-goal** *ginitial-code-aux*:  
 ( $?c, ginitial-impl$ )  $\in node-rel \rightarrow goGSi-rel \rightarrow gGSi-rel$   
**unfolding** *ginitial-impl-def*[*abs-def*] *initial-impl-def GS-initial-impl-def*  
  
**using** [[*autoref-trace-failed-id*]]  
**by** (*autoref* (*trace, keep-goal*))  
**concrete-definition** (in  $-$ ) *ginitial-code* **uses** *impl-lasso-loc.ginitial-code-aux*  
**lemmas** [*autoref-rules*] = *ginitial-code.refine*[*OF locale-this*]

**schematic-goal** *gpath-is-empty-code-aux*:

( $?c, gpath-is-empty-impl$ )  $\in gGSi-rel \rightarrow bool-rel$   
**unfolding** *gpath-is-empty-impl-def*[*abs-def*] *path-is-empty-impl-def*  
  
**using** [[*autoref-trace-failed-id*]]  
**by** (*autoref* (*trace, keep-goal*))  
**concrete-definition** (in  $-$ ) *gpath-is-empty-code*  
**uses** *impl-lasso-loc.gpath-is-empty-code-aux*  
**lemmas** [*autoref-rules*] = *gpath-is-empty-code.refine*[*OF locale-this*]

**term** *goBrk*

**schematic-goal** *goBrk-code-aux*: ( $?c, goBrk-impl$ )  $\in goGSi-rel \rightarrow ce-rel node-rel$   
**unfolding** *goBrk-impl-def*[*abs-def*] *goBrk-impl-def*

**using**  $[[\text{autoref-trace-failed-id}]]$   
**by**  $(\text{autoref } (\text{trace,keep-goal}))$   
**concrete-definition** **(in -)**  $\text{goBrk-code}$  **uses**  $\text{impl-lasso-loc.goBrk-code-aux}$   
**lemmas**  $[\text{autoref-rules}] = \text{goBrk-code.refine}[OF \text{ locale-this}]$   
**thm**  $\text{autoref-itype}(1)$

**term**  $\text{gto-outer-impl}$   
**schematic-goal**  $\text{gto-outer-code-aux}$ :  
 $(?c, \text{gto-outer-impl}) \in \text{ce-rel node-rel} \rightarrow \text{gGSi-rel} \rightarrow \text{goGSi-rel}$   
**unfolding**  $\text{gto-outer-impl-def}[\text{abs-def}] \text{gto-outer-impl-def}$   
**using**  $[[\text{autoref-trace-failed-id}]]$   
**by**  $(\text{autoref } (\text{trace,keep-goal}))$   
**concrete-definition** **(in -)**  $\text{gto-outer-code}$   
**uses**  $\text{impl-lasso-loc.gto-outer-code-aux}$   
**lemmas**  $[\text{autoref-rules}] = \text{gto-outer-code.refine}[OF \text{ locale-this}]$

**term**  $\text{go-is-done-impl}$   
**schematic-goal**  $\text{go-is-done-code-aux}$ :  
 $(?c, \text{go-is-done-impl}) \in \text{node-rel} \rightarrow \text{goGSi-rel} \rightarrow \text{bool-rel}$   
**unfolding**  $\text{go-is-done-impl-def}[\text{abs-def}] \text{is-done-oimpl-def}$   
**using**  $[[\text{autoref-trace-failed-id}]]$   
**by**  $(\text{autoref } (\text{trace,keep-goal}))$   
**concrete-definition** **(in -)**  $\text{go-is-done-code}$   
**uses**  $\text{impl-lasso-loc.go-is-done-code-aux}$   
**lemmas**  $[\text{autoref-rules}] = \text{go-is-done-code.refine}[OF \text{ locale-this}]$

**schematic-goal**  $\text{go-is-no-brk-code-aux}$ :  
 $(?c, \text{go-is-no-brk-impl}) \in \text{goGSi-rel} \rightarrow \text{bool-rel}$   
**unfolding**  $\text{go-is-no-brk-impl-def}[\text{abs-def}] \text{go-is-no-brk-impl-def}$   
**using**  $[[\text{autoref-trace-failed-id}]]$   
**by**  $(\text{autoref } (\text{trace,keep-goal}))$   
**concrete-definition** **(in -)**  $\text{go-is-no-brk-code}$   
**uses**  $\text{impl-lasso-loc.go-is-no-brk-code-aux}$   
**lemmas**  $[\text{autoref-rules}] = \text{go-is-no-brk-code.refine}[OF \text{ locale-this}]$

**schematic-goal**  $\text{gselect-edge-code-aux}$ :  $(?c, \text{gselect-edge-impl})$   
 $\in \text{gGSi-rel} \rightarrow \langle \langle \text{node-rel} \rangle \text{option-rel} \times_r \text{gGSi-rel} \rangle \text{nres-rel}$   
**unfolding**  $\text{gselect-edge-impl-def}[\text{abs-def}]$   
**using**  $[[\text{autoref-trace-failed-id}]]$   
**by**  $(\text{autoref } (\text{trace,keep-goal}))$   
**concrete-definition** **(in -)**  $\text{gselect-edge-code}$   
**uses**  $\text{impl-lasso-loc.gselect-edge-code-aux}$   
**lemmas**  $[\text{autoref-rules}] = \text{gselect-edge-code.refine}[OF \text{ locale-this}]$

**term**  $\text{gis-on-stack-impl}$   
**schematic-goal**  $\text{gis-on-stack-code-aux}$ :  
 $(?c, \text{gis-on-stack-impl}) \in \text{node-rel} \rightarrow \text{gGSi-rel} \rightarrow \text{bool-rel}$

**unfolding** *gis-on-stack-impl-def*[*abs-def*] *is-on-stack-impl-def*[*abs-def*]  
*GS.is-on-stack-impl-def*[*abs-def*]

**using** [[*autoref-trace-failed-id*]]  
**by** (*autoref* (*trace,keep-goal*))  
**concrete-definition** (**in**  $-$ ) *gis-on-stack-code*  
**uses** *impl-lasso-loc.gis-on-stack-code-aux*  
**lemmas** [*autoref-rules*] = *gis-on-stack-code.refine*[*OF locale-this*]

**term** *gcollapse-impl*  
**schematic-goal** *gcollapse-code-aux*: ( $?c,gcollapse-impl$ ) $\in$ *node-rel*  $\rightarrow$  *gGSi-rel*  
 $\rightarrow$   $\langle$ *gGSi-rel* $\rangle$ *nres-rel*  
**unfolding** *gcollapse-impl-def*[*abs-def*]  
**using** [[*autoref-trace-failed-id*]]  
**by** (*autoref* (*trace,keep-goal*))  
**concrete-definition** (**in**  $-$ ) *gcollapse-code*  
**uses** *impl-lasso-loc.gcollapse-code-aux*  
**lemmas** [*autoref-rules*] = *gcollapse-code.refine*[*OF locale-this*]

**schematic-goal** *last-is-acc-code-aux*:  
( $?c,last-is-acc-impl$ ) $\in$ *gGSi-rel* $\rightarrow$ (*bool-rel*)*nres-rel*  
**unfolding** *last-is-acc-impl-def*[*abs-def*]  
**using** [[*autoref-trace-failed-id*]]  
**by** (*autoref* (*trace,keep-goal*))  
**concrete-definition** (**in**  $-$ ) *last-is-acc-code*  
**uses** *impl-lasso-loc.last-is-acc-code-aux*  
**lemmas** [*autoref-rules*] = *last-is-acc-code.refine*[*OF locale-this*]

**schematic-goal** *ce-code-aux*: ( $?c,ce-impl$ )  
 $\in$ *gGSi-rel* $\rightarrow$ (*ce-rel node-rel*  $\times_r$  *gGSi-rel*)*nres-rel*  
**unfolding** *ce-impl-def*[*abs-def*] *on-stack-less-def*[*abs-def*]  
*on-stack-ge-def*[*abs-def*]  
**using** [[*autoref-trace-failed-id*]]  
**by** (*autoref* (*trace,keep-goal*))  
**concrete-definition** (**in**  $-$ ) *ce-code* **uses** *impl-lasso-loc.ce-code-aux*  
**lemmas** [*autoref-rules*] = *ce-code.refine*[*OF locale-this*]

**schematic-goal** *gis-done-code-aux*:  
( $?c,gis-done-impl$ ) $\in$ *node-rel* $\rightarrow$ *gGSi-rel* $\rightarrow$ *bool-rel*  
**unfolding** *gis-done-impl-def*[*abs-def*] *is-done-impl-def* *GS.is-done-impl-def*  
  
**using** [[*autoref-trace-failed-id*]]  
**by** (*autoref* (*trace,keep-goal*))  
**concrete-definition** (**in**  $-$ ) *gis-done-code* **uses** *impl-lasso-loc.gis-done-code-aux*  
**lemmas** [*autoref-rules*] = *gis-done-code.refine*[*OF locale-this*]

**schematic-goal** *gpush-code-aux*:  
( $?c,gpush-impl$ ) $\in$ *node-rel*  $\rightarrow$  *gGSi-rel* $\rightarrow$ *gGSi-rel*  
**unfolding** *gpush-impl-def*[*abs-def*]

**using**  $[[\text{autoref-trace-failed-id}]]$   
**by** (*autoref* (*trace,keep-goal*))  
**concrete-definition** (**in**  $-$ ) *gpush-code* **uses** *impl-lasso-loc.gpush-code-aux*  
**lemmas**  $[\text{autoref-rules}] = \text{gpush-code.refine}[OF \text{ locale-this}]$

**schematic-goal** *gpop-code-aux*:  $(?c, \text{gpop-impl}) \in gGSi\text{-rel} \rightarrow \langle gGSi\text{-rel} \rangle nres\text{-rel}$   
**unfolding** *gpop-impl-def* $[\text{abs-def}]$   
**using**  $[[\text{autoref-trace-failed-id}]]$   
**by** (*autoref* (*trace,keep-goal*))  
**concrete-definition** (**in**  $-$ ) *gpop-code* **uses** *impl-lasso-loc.gpop-code-aux*  
**lemmas**  $[\text{autoref-rules}] = \text{gpop-code.refine}[OF \text{ locale-this}]$

**schematic-goal** *find-ce-code-aux*:  $(?c, \text{find-ce-impl}) \in \langle ce\text{-rel node-rel} \rangle nres\text{-rel}$   
**unfolding** *find-ce-impl-def* $[\text{abs-def}]$   
**using**  $[[\text{autoref-trace-failed-id}]]$   
**apply** (*autoref* (*trace,keep-goal*))  
**done**  
**concrete-definition** (**in**  $-$ ) *find-ce-code*  
**uses** *impl-lasso-loc.find-ce-code-aux*  
**lemmas**  $[\text{autoref-rules}] = \text{find-ce-code.refine}[OF \text{ locale-this}]$

**schematic-goal** *find-ce-tr-aux*: *RETURN*  $?c \leq \text{find-ce-code node-eq-impl node-hash-impl}$   
*node-def-hash-size G-impl*

**unfolding**  
*find-ce-code-def*  
*ginitial-code-def*  
*gpath-is-empty-code-def*  
*gselect-edge-code-def*  
*gis-on-stack-code-def*  
*gcollapse-code-def*  
*last-is-acc-code-def*  
*ce-code-def*  
*gis-done-code-def*  
*gpush-code-def*  
*gpop-code-def*  
**apply** *refine-transfer*  
**done**

**concrete-definition** (**in**  $-$ ) *find-ce-tr* **for** *G-impl*  
**uses** *impl-lasso-loc.find-ce-tr-aux*  
**lemmas**  $[\text{refine-transfer}] = \text{find-ce-tr.refine}[OF \text{ locale-this}]$

**context begin interpretation** *autoref-syn* .  
**lemma**  $[\text{autoref-op-pat}]$ :  
*find-ce-spec*  $\equiv OP \text{ find-ce-spec}$   
**by** *auto*  
**end**

```

theorem find-ce-autoref[autoref-rules]:
  — Main Correctness theorem (inside locale)
  shows (find-ce-code node-eq-impl node-hash-impl node-def-hash-size G-impl,
find-ce-spec) ∈ ⟨ce-rel node-rel⟩nres-rel
proof —
  note find-ce-code.refine[OF locale-this, THEN nres-relD]
  also note find-ce-impl-refine
  also note find-ce-refine
  also note find-ce-correct
  finally show ?thesis by (auto intro: nres-relI)
qed

```

```

end

```

```

context impl-lasso-loc
begin

```

```

  context begin interpretation autoref-syn .

```

```

  lemma [autoref-op-pat]:
    reconstruct-reach ≡ OP reconstruct-reach
    reconstruct-lasso ≡ OP reconstruct-lasso
    by auto
end

```

```

schematic-goal reconstruct-reach-code-aux:
  shows (?c, reconstruct-reach) ∈ ⟨node-rel⟩fun-set-rel →
  ⟨node-rel⟩fun-set-rel →
  ⟨⟨node-rel⟩list-rel ×r node-rel⟩nres-rel
  unfolding reconstruct-lasso-def[abs-def] reconstruct-reach-def[abs-def]
  using [[autoref-trace-failed-id]]
  apply (autoref (keep-goal, trace))
  done

```

```

concrete-definition (in —) reconstruct-reach-code
  uses impl-lasso-loc.reconstruct-reach-code-aux
lemmas [autoref-rules] = reconstruct-reach-code.refine[OF locale-this]

```

```

schematic-goal reconstruct-lasso-code-aux:
  shows (?c, reconstruct-lasso) ∈ ⟨node-rel⟩fun-set-rel →
  ⟨node-rel⟩fun-set-rel →
  ⟨⟨node-rel⟩list-rel ×r ⟨node-rel⟩list-rel⟩nres-rel
  unfolding reconstruct-lasso-def[abs-def]
  using [[autoref-trace-failed-id]]

  apply (autoref (keep-goal, trace))
  done

```

**concrete-definition** (in  $-$ ) *reconstruct-lasso-code*  
 uses *impl-lasso-loc.reconstruct-lasso-code-aux*  
 lemmas [*autoref-rules*] = *reconstruct-lasso-code.refine[OF locale-this]*

**schematic-goal** *reconstruct-lasso-tr-aux*:  
 RETURN  $?c \leq \text{reconstruct-lasso-code eqi hi dszi } G\text{-impl } Vr Vl$   
 unfolding *reconstruct-lasso-code-def reconstruct-reach-code-def*  
 apply (*refine-transfer (post)*)  
 done

**concrete-definition** (in  $-$ ) *reconstruct-lasso-tr* for *G-impl*  
 uses *impl-lasso-loc.reconstruct-lasso-tr-aux*  
 lemmas [*refine-transfer*] = *reconstruct-lasso-tr.refine[OF locale-this]*

**schematic-goal** *find-lasso-code-aux*:  
 shows  $(?c::?'c, \text{find-lasso}) \in ?R$   
 unfolding *find-lasso-def[abs-def]*  
 using [*autoref-trace-failed-id*]  
 apply (*autoref (keep-goal, trace)*)  
 done

**concrete-definition** (in  $-$ ) *find-lasso-code*  
 uses *impl-lasso-loc.find-lasso-code-aux*  
 lemmas [*autoref-rules*] = *find-lasso-code.refine[OF locale-this]*

**schematic-goal** *find-lasso-tr-aux*:  
 RETURN  $?c \leq \text{find-lasso-code node-eq-impl node-hash-impl node-def-hash-size}$   
*G-impl*  
 unfolding *find-lasso-code-def*  
 apply (*refine-transfer (post)*)  
 done

**concrete-definition** (in  $-$ ) *find-lasso-tr* for *G-impl*  
 uses *impl-lasso-loc.find-lasso-tr-aux*  
 lemmas [*refine-transfer*] = *find-lasso-tr.refine[OF locale-this]*

end

export-code *find-lasso-tr* checking *SML*

### 1.39 Main Correctness Theorem

**abbreviation** *fl-rel* ::  $- \Rightarrow (- \times ('a \text{ list} \times 'a \text{ list}) \text{ option}) \text{ set}$  where  
*fl-rel node-rel*  $\equiv \langle \langle \text{node-rel} \rangle \text{list-rel} \times_r \langle \text{node-rel} \rangle \text{list-rel} \rangle \text{Relators.option-rel}$

**theorem** *find-lasso-tr-correct*:

— Correctness theorem for the constant we extracted to SML

**fixes** *Re* and *node-rel* ::  $('vi \times 'v) \text{ set}$

**assumes** *A*:  $(G\text{-impl}, G) \in \text{igbg-impl-rel-ext } Re \text{ node-rel}$

**and** *node-eq-refine*:  $(\text{node-eq-impl}, (=)) \in \text{node-rel} \rightarrow \text{node-rel} \rightarrow \text{bool-rel}$

**and** *node-hash*: *is-bounded-hashcode node-rel node-eq-impl node-hash-impl*

**and** *node-hash-def-size*:  $(\text{is-valid-def-hm-size } \text{TYPE}('vi) \text{ node-def-hash-size})$

```

assumes B: igb-fr-graph G
shows RETURN (find-lasso-tr node-eq-impl node-hash-impl node-def-hash-size
G-impl)
  ≤  $\Downarrow$ (fl-rel node-rel) (igb-graph.find-lasso-spec G)
proof –
  from B interpret igb-fr-graph G .

  have I: impl-lasso-loc Re node-rel node-eq-impl node-hash-impl node-def-hash-size
G-impl G
    apply unfold-locales
    by fact+

  then interpret impl-lasso-loc Re node-rel node-eq-impl node-hash-impl node-def-hash-size
G-impl G .

  note find-lasso-tr.refine[OF I]
  also note find-lasso-code.refine[OF I, THEN nres-relD]
  also note find-lasso-correct
  finally show ?thesis .
qed

```

#### 1.40 Autoref Setup for *igb-graph.find-lasso-spec*

Setup for Autoref, such that *igb-graph.find-lasso-spec* can be used

**definition** [*simp*]: *op-find-lasso-spec*  $\equiv$  *igb-graph.find-lasso-spec*

**context begin interpretation** *autoref-syn* .

**lemma** [*autoref-op-pat*]: *igb-graph.find-lasso-spec*  $\equiv$  *op-find-lasso-spec*  
**by** *simp*

**term** *op-find-lasso-spec*

**lemma** [*autoref-itype*]:  
*op-find-lasso-spec*  
 $::_i$  *i-igbg* *Ie I*  $\rightarrow_i$   $\langle\langle\langle I \rangle_i i\text{-list}, \langle I \rangle_i i\text{-list}\rangle_i i\text{-prod}\rangle_i i\text{-option}\rangle_i i\text{-nres}$   
**by** *simp*

**lemma** *find-lasso-spec-autoref[autoref-rules-raw]*:  
**fixes** *Re* **and** *node-rel*  $::$  (*'vi*  $\times$  *'v*) *set*  
**assumes** *GR*: *SIDE-PRECOND* (*igb-fr-graph* *G*)  
**assumes** *eq*: *GEN-OP* *node-eq-impl* (=) (*node-rel*  $\rightarrow$  *node-rel*  $\rightarrow$  *bool-rel*)  
**assumes** *hash*: *SIDE-GEN-ALGO* (*is-bounded-hashcode* *node-rel* *node-eq-impl*  
*node-hash-impl*)  
**assumes** *hash-dsz*: *SIDE-GEN-ALGO* (*is-valid-def-hm-size* *TYPE('vi)* *node-def-hash-size*)  
**assumes** *Gi*: (*G-impl*, *G*)  $\in$  *igbg-impl-rel-ext* *Re* *node-rel*  
**shows** (*RETURN* (*find-lasso-tr node-eq-impl node-hash-impl node-def-hash-size*  
*G-impl*),

```

    (OP op-find-lasso-spec
     :: igbg-impl-rel-ext Re node-rel →  $\langle \textit{fl-rel node-rel} \rangle \textit{nres-rel}$ )$G) ∈  $\langle \textit{fl-rel node-rel} \rangle \textit{nres-rel}$ 
    using find-lasso-tr-correct[OF Gi GEN-OP-D[OF eq] SIDE-GEN-ALGO-D[OF hash] SIDE-GEN-ALGO-D[OF hash-dsz]] using GR
    apply (fastforce intro!: nres-relI)
    done

```

**end**

**end**

## 2 Conclusion

We have presented a verification of two variants of Gabow’s algorithm: Computation of the strongly connected components of a graph, and emptiness check of a generalized Büchi automaton. We have extracted efficient code with a performance comparable to a reference implementation in Java.

We have modularized the formalization in two directions: First, we share most of the proofs between the two variants of the algorithm. Second, we use a stepwise refinement approach to separate the algorithmic ideas and the correctness proof from implementation details. Sharing of the proofs reduced the overall effort of developing both algorithms. Using a stepwise refinement approach allowed us to formalize an efficient implementation, without making the correctness proof complex and unmanageable by cluttering it with implementation details.

Our development approach is independent of Gabow’s algorithm, and can be re-used for the verification of other algorithms.

**Current and Future Work** An important direction of future work is to fine-tune the implementation of the emptiness check algorithm for speed, as speed of the checking algorithm directly influences the performance of the modelchecker.

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