Fun With Tilings

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Abstract

Tilings are defined inductively. It is shown that one form of mutilated chess board cannot be tiled with dominoes, while another one can be tiled with L-shaped tiles.

Sections 1 and 2 are by Paulson and described elsewhere [1]. Section 3 is by Nipkow and formalizes a well-known argument from the literature [2].

Please add further fun examples of this kind!

theory *Tilings* **imports** *Main* **begin**

1 Inductive Tiling

inductive-set
  $tiling :: 'a\ set\ set \Rightarrow 'a\ set\ set$
  for $A :: 'a\ set\ set$ **where**
    $empty\ [simp,\ intro]: \{\} \in tiling\ A\ |$
    $Un\ [simp,\ intro]: \quad [\![\ a \in A;\ t \in tiling\ A;\ a \cap t = \{\}\ ]\!]$
      $\Longrightarrow a \cup t \in tiling\ A$

lemma *tiling-UnI* [*intro*]:
  $[\![\ t \in tiling\ A;\ u \in tiling\ A;\ t \cap u = \{\}\ ]\!] \Longrightarrow\ t \cup u \in tiling\ A$
  ⟨*proof*⟩

lemma *tiling-Diff1E*:
  assumes $t{-}a \in tiling\ A$ **and** $a \in A$ **and** $a \subseteq t$
  shows $t \in tiling\ A$
  ⟨*proof*⟩

lemma *tiling-finite*:
  assumes $\bigwedge a.\ a \in A \Longrightarrow finite\ a$
  shows $t \in tiling\ A \Longrightarrow finite\ t$
  ⟨*proof*⟩

2 The Mutilated Chess Board Cannot be Tiled by Dominoes

The originator of this problem is Max Black, according to J A Robinson. It was popularized as the *Mutilated Checkerboard Problem* by J McCarthy.

inductive-set *domino* :: (*nat* × *nat*) *set set* **where**
horiz [*simp*]: {(*i*, *j*), (*i*, *Suc j*)} ∈ *domino* |
vertl [*simp*]: {(*i*, *j*), (*Suc i*, *j*)} ∈ *domino*

lemma *domino-finite*: *d* ∈ *domino* ⟹ *finite d*
⟨*proof*⟩

declare *tiling-finite*[*OF domino-finite*, *simp*]

Sets of squares of the given colour

definition
coloured :: *nat* ⇒ (*nat* × *nat*) *set* **where**
coloured b = {(*i*, *j*). (*i* + *j*) *mod 2* = *b*}

abbreviation
whites :: (*nat* × *nat*) *set* **where**
whites ≡ *coloured 0*

abbreviation
blacks :: (*nat* × *nat*) *set* **where**
blacks ≡ *coloured* (*Suc 0*)

Chess boards

lemma *Sigma-Suc1* [*simp*]:
{*0*..< *Suc n*} × *B* = ({*n*} × *B*) ∪ ({*0*..<*n*} × *B*)
⟨*proof*⟩

lemma *Sigma-Suc2* [*simp*]:
A × {*0*..< *Suc n*} = (*A* × {*n*}) ∪ (*A* × {*0*..<*n*})
⟨*proof*⟩

lemma *dominoes-tile-row* [*intro*!]: {*i*} × {*0*..< *2*∗*n*} ∈ *tiling domino*
⟨*proof*⟩

lemma *dominoes-tile-matrix*: {*0*..<*m*} × {*0*..< *2*∗*n*} ∈ *tiling domino*
⟨*proof*⟩

coloured and Dominoes

lemma *coloured-insert* [*simp*]:
coloured b ∩ (*insert* (*i*, *j*) *t*) =
(*if* (*i* + *j*) *mod 2* = *b then insert* (*i*, *j*) (*coloured b* ∩ *t*)
else coloured b ∩ *t*)

⟨*proof*⟩

lemma *domino-singletons*:
  d ∈ *domino* ⟹
   (∃ *i j*. *whites* ∩ *d* = {(*i*, *j*)}) ∧
   (∃ *m n*. *blacks* ∩ *d* = {(*m*, *n*)})
  ⟨*proof*⟩

Tilings of dominoes

declare
  Int-Un-distrib [*simp*]
  Diff-Int-distrib [*simp*]

lemma *tiling-domino-0-1*:
  t ∈ *tiling domino* ⟹ *card*(*whites* ∩ *t*) = *card*(*blacks* ∩ *t*)
⟨*proof*⟩

Final argument is surprisingly complex

theorem *gen-mutil-not-tiling*:
  assumes *t* ∈ *tiling domino* (*i* + *j*) *mod 2* = *0*
   (*m* + *n*) *mod 2* = *0* {(*i*, *j*), (*m*, *n*)} ⊆ *t*
  shows *t* − {(*i*, *j*)} − {(*m*, *n*)} ∉ *tiling domino*
⟨*proof*⟩

Apply the general theorem to the well-known case

theorem *mutil-not-tiling*:
  assumes *t* = {*0*..< *2* ∗ *Suc m*} × {*0*..< *2* ∗ *Suc n*}
  shows *t* − {(*0*,*0*)} − {(*Suc*(*2* ∗ *m*), *Suc*(*2* ∗ *n*))} ∉ *tiling domino*
⟨*proof*⟩

3 The Mutilated Chess Board Can be Tiled by Ls

Remove a arbitrary square from a chess board of size $2^n \times 2^n$. The result can be tiled by L-shaped tiles: . The four possible L-shaped tiles are obtained by dropping one of the four squares from $\{(x, y), (x + 1, y), (x, y + 1), (x + 1, y + 1)\}$:

definition *L2* (*x*::*nat*) (*y*::*nat*) = {(*x*,*y*), (*x*+*1*,*y*), (*x*, *y*+*1*)}
definition *L3* (*x*::*nat*) (*y*::*nat*) = {(*x*,*y*), (*x*+*1*,*y*), (*x*+*1*, *y*+*1*)}
definition *L0* (*x*::*nat*) (*y*::*nat*) = {(*x*+*1*,*y*), (*x*,*y*+*1*), (*x*+*1*, *y*+*1*)}
definition *L1* (*x*::*nat*) (*y*::*nat*) = {(*x*,*y*), (*x*,*y*+*1*), (*x*+*1*, *y*+*1*)}

All tiles:

definition *Ls* :: (*nat* ∗ *nat*) *set set* **where**
  Ls ≡ { *L0 x y* | *x y*. *True*} ∪ { *L1 x y* | *x y*. *True*} ∪
     { *L2 x y* | *x y*. *True*} ∪ { *L3 x y* | *x y*. *True*}

lemma *LinLs*: *L0 i j* : *Ls* & *L1 i j* : *Ls* & *L2 i j* : *Ls* & *L3 i j* : *Ls*

⟨*proof*⟩

Square $2^n \times 2^n$ grid, shifted by i and j:

definition *square2* (*n*::*nat*) (*i*::*nat*) (*j*::*nat*) = {*i*..< *2*^*n*+*i*} × {*j*..< *2*^*n*+*j*}

lemma *in-square2*[*simp*]:
(*a*,*b*) : *square2 n i j* ⟷ *i*≤*a* ∧ *a*<*2*^*n*+*i* ∧ *j*≤*b* ∧ *b*<*2*^*n*+*j*
⟨*proof*⟩

lemma *square2-Suc*: *square2* (*Suc n*) *i j* =
square2 n i j ∪ *square2 n* (*2*^*n* + *i*) *j* ∪ *square2 n i* (*2*^*n* + *j*) ∪
square2 n (*2*^*n* + *i*) (*2*^*n* + *j*)
⟨*proof*⟩

lemma *square2-disj*: *square2 n i j* ∩ *square2 n x y* = {} ⟷
(*2*^*n*+*i* ≤ *x* ∨ *2*^*n*+*x* ≤ *i*) ∨ (*2*^*n*+*j* ≤ *y* ∨ *2*^*n*+*y* ≤ *j*) (**is** *?A* = *?B*)
⟨*proof*⟩

Some specific lemmas:

lemma *pos-pow2*: (*0*::*nat*) < *2*^*n*
⟨*proof*⟩

declare *nat-zero-less-power-iff*[*simp del*] *zero-less-power*[*simp del*]

lemma *Diff-insert-if*: **shows**
B ≠ {} ⟹ *a* ∈ *A* ⟹ *A* − *insert a B* = (*A*−*B* − {*a*}) **and**
B ≠ {} ⟹ *a* ∉ *A* ⟹ *A* − *insert a B* = *A*−*B*
⟨*proof*⟩

lemma *DisjI1*: *A* ∩ *B* = {} ⟹ (*A*−*X*) ∩ *B* = {}
⟨*proof*⟩

lemma *DisjI2*: *A* ∩ *B* = {} ⟹ *A* ∩ (*B*−*X*) = {}
⟨*proof*⟩

The main theorem:

theorem *Ls-can-tile*: *i* ≤ *a* ⟹ *a* < *2*^*n* + *i* ⟹ *j* ≤ *b* ⟹ *b* < *2*^*n* + *j*
⟹ *square2 n i j* − {(*a*,*b*)} ∈ *tiling Ls*
⟨*proof*⟩

corollary *Ls-can-tile00*:
a < *2*^*n* ⟹ *b* < *2*^*n* ⟹ *square2 n 0 0* − {(*a*, *b*)} ∈ *tiling Ls*
⟨*proof*⟩

end

References

[1] Lawrence C. Paulson. A simple formalization and proof for the mutilated chess board. *Logic J. of the IGPL*, 9(3), 2001.

[2] Velleman. *How to Prove it.* Cambridge University Press, 1994.