Fun With Functions

Tobias Nipkow

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Abstract

This is a collection of cute puzzles of the form "Show that if a function satisfies the following constraints, it must be ... " Please add further examples to this collection!

Apart from the one about factorial, they all come from the delightful booklet by Terence Tao [1] but go back to Math Olympiads and similar events.

Please add further examples of this kind, either directly or by sending them to me. Let us make this a growing body of fun!

theory FunWithFunctions imports Complex-Main begin

See [1]. Was first brought to our attention by Herbert Ehler who provided a similar proof.

theorem *identity1*: **fixes** $f :: nat \Rightarrow nat$ **assumes** *fff*: $\bigwedge n$. f(f(n)) < f(Suc(n))**shows** f(n) = n $\langle proof \rangle$

See [1]. Possible extension: Should also hold if the range of f is the reals!

lemma *identity2*: **fixes** $f :: nat \Rightarrow nat$ **assumes** f(k) = k **and** $k \ge 2$ **and** f-times: $\bigwedge m \ n. \ f(m*n) = f(m)*f(n)$ **and** f-mono: $\bigwedge m \ n. \ m < n \Longrightarrow f \ m < f \ n$ **shows** f(n) = n $\langle proof \rangle$

One more from Tao's booklet. If f is also assumed to be continuous, f x = x + 1 holds for all reals, not only rationals. Extend the proof!

theorem plus1: **fixes** $f :: real \Rightarrow real$ **assumes** $0: f \ 0 = 1$ and f-add: $\bigwedge x \ y. \ f(x+y+1) = f \ x + f \ y$

assumes $r : \mathbb{Q}$ shows f(r) = r + 1 $\langle proof \rangle$ The only total model of a naive recursion equation of factorial on integers is 0 for all negative arguments. Probably folklore.

theorem *ifac-neg0*: **fixes** *ifac* :: *int* \Rightarrow *int* **assumes** *ifac-rec*: $\bigwedge i$. *ifac* i = (if i=0 then 1 else i*ifac(i-1))**shows** $i < 0 \implies ifac \ i = 0$ $\langle proof \rangle$

 \mathbf{end}

References

[1] Terence Tao. Solving Mathematical Problems. Oxford University Press, 2006.