# Fun With Functions 

Tobias Nipkow

September 13， 2023


#### Abstract

This is a collection of cute puzzles of the form＂Show that if a function satisfies the following constraints，it must be ．．．＂Please add further examples to this collection！


Apart from the one about factorial，they all come from the delightful booklet by Terence Tao［1］but go back to Math Olympiads and similar events．

Please add further examples of this kind，either directly or by sending them to me．Let us make this a growing body of fun！

## theory FunWithFunctions imports Complex－Main begin

See［1］．Was first brought to our attention by Herbert Ehler who provided a similar proof．
theorem identity1：fixes $f::$ nat $\Rightarrow$ nat
assumes fff：$\wedge n . f(f(n))<f(S u c(n))$
shows $f(n)=n$
〈proof〉
See［1］．Possible extension：Should also hold if the range of $f$ is the reals！
lemma identity2：fixes $f::$ nat $\Rightarrow$ nat
assumes $f(k)=k$ and $k \geq 2$
and $f$－times：$\bigwedge m n . f(m * n)=f(m) * f(n)$
and $f$－mono：$\wedge m n . m<n \Longrightarrow f m<f n$
shows $f(n)=n$
$\langle p r o o f\rangle$
One more from Tao＇s booklet．If $f$ is also assumed to be continuous，$f x$ $=x+1$ holds for all reals，not only rationals．Extend the proof！
theorem plus1：
fixes $f::$ real $\Rightarrow$ real
assumes $0: f 0=1$ and $f$－add：$\bigwedge x y \cdot f(x+y+1)=f x+f y$
assumes $r: \mathbb{Q}$ shows $f(r)=r+1$
〈proof〉

The only total model of a naive recursion equation of factorial on integers is 0 for all negative arguments. Probably folklore.
theorem ifac-neg0: fixes ifac :: int $\Rightarrow$ int
assumes ifac-rec: $\backslash$. ifac $i=($ if $i=0$ then 1 else $i * i f a c(i-1))$
shows $i<0 \Longrightarrow$ ifac $i=0$
〈proof〉
end

## References

[1] Terence Tao. Solving Mathematical Problems. Oxford University Press, 2006.

