Fun With Functions

Tobias Nipkow

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Abstract

This is a collection of cute puzzles of the form "Show that if a function satisfies the following constraints, it must be . . . " Please add further examples to this collection!

Apart from the one about factorial, they all come from the delightful booklet by Terence Tao [1] but go back to Math Olympiads and similar events.

Please add further examples of this kind, either directly or by sending them to me. Let us make this a growing body of *fun*!

theory *FunWithFunctions* **imports** *Complex-Main* **begin**

See [1]. Was first brought to our attention by Herbert Ehler who provided a similar proof.

theorem *identity1*: **fixes** $f :: nat \Rightarrow nat$
assumes *fff*: $\bigwedge n.\ f(f(n)) < f(Suc(n))$
shows $f(n) = n$
⟨*proof*⟩

See [1]. Possible extension: Should also hold if the range of f is the reals!

lemma *identity2*: **fixes** $f :: nat \Rightarrow nat$
assumes $f(k) = k$ **and** $k \geq 2$
and *f-times*: $\bigwedge m\ n.\ f(m{*}n) = f(m){*}f(n)$
and *f-mono*: $\bigwedge m\ n.\ m{<}n \Longrightarrow f\ m < f\ n$
shows $f(n) = n$
⟨*proof*⟩

One more from Tao's booklet. If f is also assumed to be continuous, $f\ x = x + 1$ holds for all reals, not only rationals. Extend the proof!

theorem *plus1*:
fixes $f :: real \Rightarrow real$
assumes *0*: $f\ 0 = 1$ **and** *f-add*: $\bigwedge x\ y.\ f(x{+}y{+}1) = f\ x + f\ y$

assumes $r : \mathbb{Q}$ **shows** $f(r) = r + 1$
⟨*proof*⟩

The only total model of a naive recursion equation of factorial on integers is 0 for all negative arguments. Probably folklore.

theorem *ifac-neg0*: **fixes** *ifac* :: *int* ⇒ *int*
assumes *ifac-rec*: $\bigwedge i.$ *ifac i* = (*if i=0 then 1 else i∗ifac*(*i* − *1*))
shows *i<0* ⟹ *ifac i* = *0*
⟨*proof*⟩

end

References

[1] Terence Tao. *Solving Mathematical Problems.* Oxford University Press, 2006.