Fun With Functions

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Abstract

This is a collection of cute puzzles of the form “Show that if a function satisfies the following constraints, it must be . . .” Please add further examples to this collection!

Apart from the one about factorial, they all come from the delightful booklet by Terence Tao [1] but go back to Math Olympiads and similar events.

Please add further examples of this kind, either directly or by sending them to me. Let us make this a growing body of fun!

theory FunWithFunctions imports Complex-Main begin

See [1]. Was first brought to our attention by Herbert Ehler who provided a similar proof.

theorem identity1: fixes f :: nat ⇒ nat
assumes fff: ∀n. f(f(n)) < f(Suc(n))
shows f(n) = n
⟨proof⟩

See [1]. Possible extension: Should also hold if the range of f is the reals!

lemma identity2: fixes f :: nat ⇒ nat
assumes f(k) = k and k ≥ 2
and f-times: ∀m n. f(m*n) = f(m)*f(n)
and f-mono: ∀m n. m<n ⇒ f m < f n
shows f(n) = n
⟨proof⟩

One more from Tao’s booklet. If f is also assumed to be continuous, f x = x + 1 holds for all reals, not only rationals. Extend the proof!

theorem plus1:
fixes f :: real ⇒ real
assumes 0: f 0 = 1 and f-add: ∀x y. f(x+y+1) = f x + f y
assumes r : ℚ shows f(r) = r + 1
⟨proof⟩
The only total model of a naive recursion equation of factorial on integers is 0 for all negative arguments. Probably folklore.

\textbf{theorem} \ ifac-neg0: \ \textbf{fixes} \ ifac :: \ int \Rightarrow \ int
\textbf{assumes} \ ifac-rec: \ \forall i. \ ifac \ i = (if \ i=0 \ then \ 1 \ else \ i*ifac(i-1))
\textbf{shows} \ i<0 \ \Rightarrow \ \text{ifac} \ i = 0
\langle \text{proof} \rangle
end

\textbf{References}