Fourier Series

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Abstract

This development formalises the square integrable functions over the reals and the basics of Fourier series. It culminates with a proof that every well-behaved periodic function can be approximated by a Fourier series. The material is ported from HOL Light.[1]

Contents

[1] https://github.com/jrh13/hol-light/blob/master/100/fourier.ml

1 Shifting the origin for integration of periodic functions

theory *Periodic*
imports *HOL−Analysis.Analysis*
begin

lemma *has-bochner-integral-null* [*intro*]:
fixes f :: *'a::euclidean-space* ⇒ *'b::euclidean-space*
assumes $N \in$ *null-sets lebesgue*
shows *has-bochner-integral* (*lebesgue-on* N) f *0*
⟨*proof*⟩

lemma *has-bochner-integral-null-eq*[*simp*]:
fixes f :: *'a::euclidean-space* ⇒ *'b::euclidean-space*
assumes $N \in$ *null-sets lebesgue*
shows *has-bochner-integral* (*lebesgue-on* N) f $i \longleftrightarrow i = 0$
⟨*proof*⟩

lemma *periodic-integer-multiple*:
$(\forall x.\ f(x + a) = f\ x) \longleftrightarrow (\forall x.\ \forall n \in \mathbb{Z}.\ f(x + n * a) = f\ x)$ (**is** *?lhs = ?rhs*)
⟨*proof*⟩

lemma *has-integral-offset*:
fixes f :: *real* ⇒ *'a::euclidean-space*
assumes *has-bochner-integral* (*lebesgue-on* $\{a..b\}$) f i
shows *has-bochner-integral* (*lebesgue-on* $\{a-c..b-c\}$) $(\lambda x.\ f(x + c))$ i
⟨*proof*⟩

lemma *has-integral-periodic-offset-lemma*:

fixes f :: *real* $\Rightarrow$ *'a::euclidean-space*
assumes *periodic*: $\bigwedge x.\ f(x + (b-a)) = f\,x$ **and** f: *has-bochner-integral* (*lebesgue-on* $\{a..a+c\}$) $f\ i$
shows *has-bochner-integral* (*lebesgue-on* $\{b..b+c\}$) $f\ i$
⟨*proof*⟩

lemma *has-integral-periodic-offset-pos*:
fixes f :: *real* $\Rightarrow$ *real*
assumes f: *has-bochner-integral* (*lebesgue-on* $\{a..b\}$) $f\ i$ **and** *periodic*: $\bigwedge x.\ f(x + (b-a)) = f\ x$
and c: $c \geq 0$ $a + c \leq b$
shows *has-bochner-integral* (*lebesgue-on* $\{a..b\}$) $(\lambda x.\ f(x + c))$ i
⟨*proof*⟩

lemma *has-integral-periodic-offset-weak*:
fixes f :: *real* $\Rightarrow$ *real*
assumes f: *has-bochner-integral* (*lebesgue-on* $\{a..b\}$) $f\ i$ **and** *periodic*: $\bigwedge x.\ f(x + (b-a)) = f\ x$ **and** c: $|c| \leq b-a$
shows *has-bochner-integral* (*lebesgue-on* $\{a..b\}$) $(\lambda x.\ f(x + c))$ i
⟨*proof*⟩

lemma *has-integral-periodic-offset*:
fixes f :: *real* $\Rightarrow$ *real*
assumes f: *has-bochner-integral* (*lebesgue-on* $\{a..b\}$) $f\ i$ **and** *periodic*: $\bigwedge x.\ f(x + (b-a)) = f\ x$
shows *has-bochner-integral* (*lebesgue-on* $\{a..b\}$) $(\lambda x.\ f(x + c))$ i
⟨*proof*⟩

lemma *integrable-periodic-offset*:
fixes f :: *real* $\Rightarrow$ *real*
assumes f: *integrable* (*lebesgue-on* $\{a..b\}$) f **and** *periodic*: $\bigwedge x.\ f(x + (b-a)) = f\ x$
shows *integrable* (*lebesgue-on* $\{a..b\}$) $(\lambda x.\ f(x + c))$
⟨*proof*⟩

lemma *absolutely-integrable-periodic-offset*:
fixes f :: *real* $\Rightarrow$ *real*
assumes f: f *absolutely-integrable-on* $\{a..b\}$ **and** *periodic*: $\bigwedge x.\ f(x + (b-a)) = f\ x$
shows $(\lambda x.\ f(x + c))$ *absolutely-integrable-on* $\{a..b\}$ $(\lambda x.\ f(c + x))$ *absolutely-integrable-on* $\{a..b\}$
⟨*proof*⟩

lemma *integral-periodic-offset*:
fixes f :: *real* $\Rightarrow$ *real*
assumes *periodic*: $\bigwedge x.\ f(x + (b-a)) = f\ x$
shows $integral^L$ (*lebesgue-on* $\{a..b\}$) $(\lambda x.\ f(x + c))$ = $integral^L$ (*lebesgue-on*

$\{a..b\}$) *f*
⟨*proof*⟩

end

2 Lspace as it is in HOL Light

Mainly a repackaging of existing material from Lp

theory *Lspace*
imports *Lp.Lp*
begin

abbreviation *lspace* :: *'a measure* ⇒ *ennreal* ⇒ (*'a* ⇒ *real*) *set*
where *lspace M p* ≡ $space_N$ ($\mathfrak{L}$ *p M*)

lemma *lspace-1*:
assumes *S* ∈ *sets lebesgue*
shows *f* ∈ *lspace* (*lebesgue-on S*) *1* ⟷ *f absolutely-integrable-on S*
⟨*proof*⟩

lemma *lspace-ennreal-iff*:
assumes *p* > *0*
shows *lspace* (*lebesgue-on S*) (*ennreal p*) = {*f* ∈ *borel-measurable* (*lebesgue-on S*). *integrable* (*lebesgue-on S*) (λ*x*. (*norm*(*f x*) *powr p*))}
⟨*proof*⟩

lemma *lspace-iff*:
assumes ∞ > *p p* > *0*
shows *lspace* (*lebesgue-on S*) *p* = {*f* ∈ *borel-measurable* (*lebesgue-on S*). *integrable* (*lebesgue-on S*) (λ*x*. (*norm*(*f x*) *powr* (*enn2real p*)))}
⟨*proof*⟩

lemma *lspace-iff'*:
assumes *p*: ∞ > *p p* > *0* **and** *S*: *S* ∈ *sets lebesgue*
shows *lspace* (*lebesgue-on S*) *p* = {*f* ∈ *borel-measurable* (*lebesgue-on S*). (λ*x*. (*norm*(*f x*) *powr* (*enn2real p*))) *integrable-on S*}
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *lspace-mono*:
assumes *f* ∈ *lspace* (*lebesgue-on S*) *q* **and** *S*: *S* ∈ *lmeasurable* **and** *p* > *0 p* ≤ *q q* < ∞
shows *f* ∈ *lspace* (*lebesgue-on S*) *p*
⟨*proof*⟩

lemma *lspace-inclusion*:
assumes *S* ∈ *lmeasurable* **and** *p* > *0 p* ≤ *q q* < ∞
shows *lspace* (*lebesgue-on S*) *q* ⊆ *lspace* (*lebesgue-on S*) *p*

⟨*proof*⟩

lemma *lspace-const*:
fixes *p*::*real*
assumes $p > 0S \in$ *lmeasurable*
shows ($\lambda x.$ *c*) $\in$ *lspace* (*lebesgue-on S*) *p*
⟨*proof*⟩

lemma *lspace-max*:
fixes *p*::*real*
assumes $f \in$ *lspace* (*lebesgue-on S*) *p* $g \in$ *lspace* (*lebesgue-on S*) *p* $p > 0$
shows ($\lambda x.$ *max* (*f x*) (*g x*)) $\in$ *lspace* (*lebesgue-on S*) *p*
⟨*proof*⟩

lemma *lspace-min*:
fixes *p*::*real*
assumes $f \in$ *lspace* (*lebesgue-on S*) *p* $g \in$ *lspace* (*lebesgue-on S*) *p* $p > 0$
shows ($\lambda x.$ *min* (*f x*) (*g x*)) $\in$ *lspace* (*lebesgue-on S*) *p*
⟨*proof*⟩

lemma *Lp-space-numeral*:
assumes *numeral n* > (*0*::*int*)
shows $space_N$ ($\mathfrak{L}$ (*numeral n*) *M*) = {*f* $\in$ *borel-measurable M*. *integrable M* ($\lambda x.$ |*f x*| ^ *numeral n*)}
⟨*proof*⟩

end

3 Square integrable functions over the reals

theory *Square-Integrable*
imports *Lspace*
begin

3.1 Basic definitions

definition *square-integrable*:: (*real* ⇒ *real*) ⇒ *real set* ⇒ *bool* (**infixr** ‹*square′-integrable*› *46*)
where *f square-integrable S* ≡ *S* ∈ *sets lebesgue* ∧ *f* ∈ *borel-measurable* (*lebesgue-on S*) ∧ *integrable* (*lebesgue-on S*) ($\lambda x.$ *f x* ^ *2*)

lemma *square-integrable-imp-measurable*:
f square-integrable S ⟹ *f* ∈ *borel-measurable* (*lebesgue-on S*)
⟨*proof*⟩

lemma *square-integrable-imp-lebesgue*:
f square-integrable S ⟹ *S* ∈ *sets lebesgue*
⟨*proof*⟩

lemma *square-integrable-imp-lspace*:
assumes *f square-integrable S* **shows** $f \in$ *lspace* (*lebesgue-on S*) *2*
⟨*proof*⟩

lemma *square-integrable-iff-lspace*:
assumes $S \in$ *sets lebesgue*
shows *f square-integrable S* $\longleftrightarrow$ $f \in$ *lspace* (*lebesgue-on S*) *2* (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *square-integrable-0* [*simp*]:
$S \in$ *sets lebesgue* $\Longrightarrow$ ($\lambda x.$ *0*) *square-integrable S*
⟨*proof*⟩

lemma *square-integrable-neg-eq* [*simp*]:
($\lambda x.$ $-(f\ x)$) *square-integrable S* $\longleftrightarrow$ *f square-integrable S*
⟨*proof*⟩

lemma *square-integrable-lmult* [*simp*]:
assumes *f square-integrable S*
shows ($\lambda x.$ $c * f\ x$) *square-integrable S*
⟨*proof*⟩

lemma *square-integrable-rmult* [*simp*]:
f square-integrable S $\Longrightarrow$ ($\lambda x.$ $f\ x * c$) *square-integrable S*
⟨*proof*⟩

lemma *square-integrable-imp-absolutely-integrable-product*:
assumes *f*: *f square-integrable S* **and** *g*: *g square-integrable S*
shows ($\lambda x.$ $f\ x * g\ x$) *absolutely-integrable-on S*
⟨*proof*⟩

lemma *square-integrable-imp-integrable-product*:
assumes *f square-integrable S g square-integrable S*
shows *integrable* (*lebesgue-on S*) ($\lambda x.$ $f\ x * g\ x$)
⟨*proof*⟩

lemma *square-integrable-add* [*simp*]:
assumes *f*: *f square-integrable S* **and** *g*: *g square-integrable S*
shows ($\lambda x.$ $f\ x + g\ x$) *square-integrable S*
⟨*proof*⟩

lemma *square-integrable-diff* [*simp*]:
⟦*f square-integrable S*; *g square-integrable S*⟧ $\Longrightarrow$ ($\lambda x.$ $f\ x - g\ x$) *square-integrable S*
⟨*proof*⟩

lemma *square-integrable-abs* [*simp*]:
f square-integrable S $\Longrightarrow$ ($\lambda x.$ $|f\ x|$) *square-integrable S*
⟨*proof*⟩

lemma *square-integrable-sum* [*simp*]:
assumes *I*: *finite I* $\bigwedge i.\ i \in I \Longrightarrow f\ i$ *square-integrable S* **and** *S*: $S \in$ *sets lebesgue*
shows $(\lambda x.\ \sum i{\in}I.\ f\ i\ x)$ *square-integrable S*
⟨*proof*⟩

lemma *continuous-imp-square-integrable* [*simp*]:
continuous-on $\{a..b\}\ f \Longrightarrow f$ *square-integrable* $\{a..b\}$
⟨*proof*⟩

lemma *square-integrable-imp-absolutely-integrable*:
assumes *f*: *f square-integrable S* **and** *S*: $S \in$ *lmeasurable*
shows *f absolutely-integrable-on S*
⟨*proof*⟩

lemma *square-integrable-imp-integrable*:
assumes *f*: *f square-integrable S* **and** *S*: $S \in$ *lmeasurable*
shows *integrable* (*lebesgue-on S*) *f*
⟨*proof*⟩

3.2 The norm and inner product in L2

definition *l2product* :: *'a::euclidean-space set* ⇒ (*'a* ⇒ *real*) ⇒ (*'a* ⇒ *real*) ⇒ *real*
where *l2product S f g* ≡ ($\int x.\ f\ x * g\ x\ \partial$(*lebesgue-on S*))

definition *l2norm* :: [*'a::euclidean-space set*, *'a* ⇒ *real*] ⇒ *real*
where *l2norm S f* ≡ *sqrt* (*l2product S f f*)

definition *lnorm* :: [*'a measure*, *real*, *'a* ⇒ *real*] ⇒ *real*
where *lnorm M p f* ≡ ($\int x.\ |f\ x|$ *powr p* ∂M) *powr* (*1/p*)

corollary *Holder-inequality-lnorm*:
assumes $p > (0{::}real)\ q > 0\ 1/p{+}1/q = 1$
and $f \in$ *borel-measurable M* $g \in$ *borel-measurable M*
integrable M ($\lambda x.\ |f\ x|$ *powr p*)
integrable M ($\lambda x.\ |g\ x|$ *powr q*)
shows ($\int x.\ |f\ x * g\ x|\ \partial M$) ≤ *lnorm M p f* ∗ *lnorm M q g*
$|\int x.\ f\ x * g\ x\ \partial M\ |$ ≤ *lnorm M p f* ∗ *lnorm M q g*
⟨*proof*⟩

lemma *l2norm-lnorm*: *l2norm S f* = *lnorm* (*lebesgue-on S*) *2 f*
⟨*proof*⟩

lemma *lnorm-nonneg*: *lnorm M p f* ≥ *0*
⟨*proof*⟩

lemma *lnorm-minus-commute*: *lnorm M p* (*g* − *f*) = *lnorm M p* (*f* − *g*)
⟨*proof*⟩

Extending a continuous function in a periodic way

proposition *continuous-on-compose-frac*:
fixes f:: *real* ⇒ *real*
assumes *contf*: *continuous-on* $\{0..1\}$ f **and** *f10*: $f\ 1 = f\ 0$
shows *continuous-on UNIV* $(f \circ frac)$
⟨*proof*⟩

proposition *Tietze-periodic-interval*:
fixes f:: *real* ⇒ *real*
assumes *contf*: *continuous-on* $\{a..b\}$ f **and** *fab*: $f\ a = f\ b$
obtains g **where** *continuous-on UNIV* g $\bigwedge x.\ x \in \{a..b\} \Longrightarrow g\ x = f\ x$
$\bigwedge x.\ g(x + (b-a)) = g\ x$
⟨*proof*⟩

3.3 Lspace stuff

lemma *eNorm-triangle-eps*:
assumes *eNorm* $N\ (x' - x) < a$ *defect* $N = 1$
obtains e **where** $e > 0$ $\bigwedge y.$ *eNorm* $N\ (y - x') < e \Longrightarrow$ *eNorm* $N\ (y - x) < a$
⟨*proof*⟩

lemma *topspace-topology*$_N$ [*simp*]:
assumes *defect* $N = 1$ **shows** *topspace* (*topology*$_N$ N) = *UNIV*
⟨*proof*⟩

lemma *tendsto-ine*$_N$*-iff-limitin*:
assumes *defect* $N = 1$
shows *tendsto-ine*$_N$ $N\ u\ x$ = *limitin* (*topology*$_N$ N) $u\ x$ *sequentially*
⟨*proof*⟩

corollary *tendsto-ine*$_N$*-iff-limitin-ge1*:
fixes p :: *ennreal*
assumes $p \geq 1$
shows *tendsto-ine*$_N$ $(\mathfrak{L}\ p\ M)\ u\ x$ = *limitin* (*topology*$_N$ $(\mathfrak{L}\ p\ M)$) $u\ x$ *sequentially*
⟨*proof*⟩

corollary *tendsto-in*$_N$*-iff-limitin*:
assumes *defect* $N = 1$ $x \in$ *space*$_N$ N $\bigwedge n.\ u\ n \in$ *space*$_N$ N
shows *tendsto-in*$_N$ $N\ u\ x$ = *limitin* (*topology*$_N$ N) $u\ x$ *sequentially*
⟨*proof*⟩

corollary *tendsto-in*$_N$*-iff-limitin-ge1*:
fixes p :: *ennreal*
assumes $p \geq 1$ $x \in$ *lspace* $M\ p$ $\bigwedge n.\ u\ n \in$ *lspace* $M\ p$
shows *tendsto-in*$_N$ $(\mathfrak{L}\ p\ M)\ u\ x$ = *limitin* (*topology*$_N$ $(\mathfrak{L}\ p\ M)$) $u\ x$ *sequentially*
⟨*proof*⟩

lemma *l2product-sym*: *l2product* $S\ f\ g$ = *l2product* $S\ g\ f$

⟨*proof*⟩

lemma *l2product-pos-le*:
f square-integrable S $\Longrightarrow$ *0* $\leq$ *l2product S f f*
⟨*proof*⟩

lemma *l2norm-pow-2*:
f square-integrable S $\Longrightarrow$ (*l2norm S f*) ^ *2* = *l2product S f f*
⟨*proof*⟩

lemma *l2norm-pos-le*:
f square-integrable S $\Longrightarrow$ *0* $\leq$ *l2norm S f*
⟨*proof*⟩

lemma *l2norm-le*: (*l2norm S f* $\leq$ *l2norm S g* $\longleftrightarrow$ *l2product S f f* $\leq$ *l2product S g g*)
⟨*proof*⟩

lemma *l2norm-eq*:
(*l2norm S f* = *l2norm S g* $\longleftrightarrow$ *l2product S f f* = *l2product S g g*)
⟨*proof*⟩

lemma *Schwartz-inequality-strong*:
assumes *f square-integrable S g square-integrable S*
shows *l2product S* (λx. |*f x*|) (λx. |*g x*|) $\leq$ *l2norm S f* $*$ *l2norm S g*
⟨*proof*⟩

lemma *Schwartz-inequality-abs*:
assumes *f square-integrable S g square-integrable S*
shows |*l2product S f g*| $\leq$ *l2norm S f* $*$ *l2norm S g*
⟨*proof*⟩

lemma *Schwartz-inequality*:
assumes *f square-integrable S g square-integrable S*
shows *l2product S f g* $\leq$ *l2norm S f* $*$ *l2norm S g*
⟨*proof*⟩

lemma *lnorm-triangle*:
assumes *f*: *f* $\in$ *lspace M p* **and** *g*: *g* $\in$ *lspace M p* **and** *p* $\geq$ *1*
shows *lnorm M p* (λx. *f x* + *g x*) $\leq$ *lnorm M p f* + *lnorm M p g*
⟨*proof*⟩

lemma *lnorm-triangle-fun*:
assumes *f*: *f* $\in$ *lspace M p* **and** *g*: *g* $\in$ *lspace M p* **and** *p* $\geq$ *1*
shows *lnorm M p* (*f* + *g*) $\leq$ *lnorm M p f* + *lnorm M p g*
⟨*proof*⟩

lemma *l2norm-triangle*:

assumes *f square-integrable S g square-integrable S*
shows *l2norm S* ($\lambda x.$ *f x* + *g x*) $\leq$ *l2norm S f* + *l2norm S g*
⟨*proof*⟩

lemma *l2product-ladd*:
⟦*f square-integrable S*; *g square-integrable S*; *h square-integrable S*⟧
$\Longrightarrow$ *l2product S* ($\lambda x.$ *f x* + *g x*) *h* = *l2product S f h* + *l2product S g h*
⟨*proof*⟩

lemma *l2product-radd*:
⟦*f square-integrable S*; *g square-integrable S*; *h square-integrable S*⟧
$\Longrightarrow$ *l2product S f*($\lambda x.$ *g x* + *h x*) = *l2product S f g* + *l2product S f h*
⟨*proof*⟩

lemma *l2product-ldiff*:
⟦*f square-integrable S*; *g square-integrable S*; *h square-integrable S*⟧
$\Longrightarrow$ *l2product S* ($\lambda x.$ *f x* − *g x*) *h* = *l2product S f h* − *l2product S g h*
⟨*proof*⟩

lemma *l2product-rdiff*:
⟦*f square-integrable S*; *g square-integrable S*; *h square-integrable S*⟧
$\Longrightarrow$ *l2product S f*($\lambda x.$ *g x* − *h x*) = *l2product S f g* − *l2product S f h*
⟨*proof*⟩

lemma *l2product-lmult*:
⟦*f square-integrable S*; *g square-integrable S*⟧
$\Longrightarrow$ *l2product S* ($\lambda x.$ *c* ∗ *f x*) *g* = *c* ∗ *l2product S f g*
⟨*proof*⟩

lemma *l2product-rmult*:
⟦*f square-integrable S*; *g square-integrable S*⟧
$\Longrightarrow$ *l2product S f*($\lambda x.$ *c* ∗ *g x*) = *c* ∗ *l2product S f g*
⟨*proof*⟩

lemma *l2product-lzero* [*simp*]: *l2product S* ($\lambda x.$ *0*) *f* = *0*
⟨*proof*⟩

lemma *l2product-rzero* [*simp*]: *l2product S f*($\lambda x.$ *0*) = *0*
⟨*proof*⟩

lemma *l2product-lsum*:
assumes *I*: *finite I* $\bigwedge i.$ *i* ∈ *I* $\Longrightarrow$ (*f i*) *square-integrable S* **and** *S*: *g square-integrable S*
shows *l2product S* ($\lambda x.$ $\sum$ *i*∈*I*. *f i x*) *g* = ($\sum$ *i*∈*I*. *l2product S* (*f i*) *g*)
⟨*proof*⟩

lemma *l2product-rsum*:
assumes *I*: *finite I* $\bigwedge i.$ *i* ∈ *I* $\Longrightarrow$ (*f i*) *square-integrable S* **and** *S*: *g square-integrable*

S
shows *l2product S g* $(\lambda x. \sum i{\in}I.\ f\ i\ x) = (\sum i{\in}I.\ l2product\ S\ g\ (f\ i))$
⟨*proof*⟩

lemma *l2norm-lmult*:
f square-integrable S $\Longrightarrow$ *l2norm S* $(\lambda x.\ c * f\ x) = |c| * l2norm\ S\ f$
⟨*proof*⟩

lemma *l2norm-rmult*:
f square-integrable S $\Longrightarrow$ *l2norm S* $(\lambda x.\ f\ x * c) = l2norm\ S\ f * |c|$
⟨*proof*⟩

lemma *l2norm-neg*:
f square-integrable S $\Longrightarrow$ *l2norm S* $(\lambda x.\ - f\ x) = l2norm\ S\ f$
⟨*proof*⟩

lemma *l2norm-diff*:
assumes *f square-integrable S g square-integrable S*
shows *l2norm S* $(\lambda x.\ f\ x - g\ x) = l2norm\ S\ (\lambda x.\ g\ x - f\ x)$
⟨*proof*⟩

3.4 Completeness (Riesz-Fischer)

lemma *eNorm-eq-lnorm*: $[\![f \in lspace\ M\ p;\ p > 0]\!] \Longrightarrow eNorm\ (\mathfrak{L}\ (ennreal\ p)\ M)$ $f = ennreal\ (lnorm\ M\ p\ f)$
⟨*proof*⟩

lemma *Norm-eq-lnorm*: $[\![f \in lspace\ M\ p;\ p > 0]\!] \Longrightarrow Norm\ (\mathfrak{L}\ (ennreal\ p)\ M)\ f$ $= lnorm\ M\ p\ f$
⟨*proof*⟩

lemma *eNorm-ge1-triangular-ineq*:
assumes $p \geq (1{::}real)$
shows *eNorm* $(\mathfrak{L}\ p\ M)\ (x + y) \leq eNorm\ (\mathfrak{L}\ p\ M)\ x + eNorm\ (\mathfrak{L}\ p\ M)\ y$
⟨*proof*⟩

A mere repackaging of the theorem $complete_N\ (\mathfrak{L}\ ?p\ ?M)$, but nearly as much work again.

proposition *l2-complete*:
assumes *f*: $\bigwedge i{::}nat.\ f\ i$ *square-integrable S*
and *cauchy*: $\bigwedge e.\ 0 < e \Longrightarrow \exists N.\ \forall m{\geq}N.\ \forall n{\geq}N.\ l2norm\ S\ (\lambda x.\ f\ m\ x - f\ n\ x) < e$
obtains *g* **where** *g square-integrable S* $((\lambda n.\ l2norm\ S\ (\lambda x.\ f\ n\ x - g\ x)) \longrightarrow 0)$
⟨*proof*⟩

3.5 Approximation of functions in Lp by bounded and continuous ones

lemma *lspace-bounded-measurable*:
fixes *p*::*real*
assumes *f*: *f* ∈ *borel-measurable* (*lebesgue-on S*) **and** *g*: *g* ∈ *lspace* (*lebesgue-on S*) *p* **and** *p* > *0*
and *le*: *AE x in lebesgue-on S*. *norm* (|*f x*| *powr p*) ≤ *norm* (|*g x*| *powr p*)
shows *f* ∈ *lspace* (*lebesgue-on S*) *p*
⟨*proof*⟩

lemma *lspace-approximate-bounded*:
assumes *f*: *f* ∈ *lspace* (*lebesgue-on S*) *p* **and** *S*: *S* ∈ *lmeasurable* **and** *p* > *0 e* > *0*
obtains *g* **where** *g* ∈ *lspace* (*lebesgue-on S*) *p bounded* (*g* ‘ *S*)
lnorm (*lebesgue-on S*) *p* (*f* − *g*) < *e*
⟨*proof*⟩

lemma *borel-measurable-imp-continuous-limit*:
fixes *h* :: ′*a*::*euclidean-space* ⇒ ′*b*::*euclidean-space*
assumes *h*: *h* ∈ *borel-measurable* (*lebesgue-on S*) **and** *S*: *S* ∈ *sets lebesgue*
obtains *g* **where** ⋀*n*. *continuous-on UNIV* (*g n*) *AE x in lebesgue-on S*. (λ*n*::*nat*. *g n x*) ⟶ *h x*
⟨*proof*⟩

proposition *lspace-approximate-continuous*:
assumes *f*: *f* ∈ *lspace* (*lebesgue-on S*) *p* **and** *S*: *S* ∈ *lmeasurable* **and** *1* ≤ *p e* > *0*
obtains *g* **where** *continuous-on UNIV g g* ∈ *lspace* (*lebesgue-on S*) *p lnorm* (*lebesgue-on S*) *p* (*f* − *g*) < *e*
⟨*proof*⟩

proposition *square-integrable-approximate-continuous*:
assumes *f*: *f square-integrable S* **and** *S*: *S* ∈ *lmeasurable* **and** *e* > *0*
obtains *g* **where** *continuous-on UNIV g g square-integrable S l2norm S* (λ*x*. *f x* − *g x*) < *e*
⟨*proof*⟩

lemma *absolutely-integrable-approximate-continuous*:
fixes *f* :: *real* ⇒ *real*
assumes *f*: *f absolutely-integrable-on S* **and** *S*: *S* ∈ *lmeasurable* **and** *0* < *e*
obtains *g* **where** *continuous-on UNIV g g absolutely-integrable-on S* $integral^L$ (*lebesgue-on S*) (λ*x*. |*f x* − *g x*|) < *e*
⟨*proof*⟩

end

4 Confining a series to a set

theory *Confine*
imports *Complex-Main*
begin

definition $confine :: ('a \Rightarrow 'b::zero) \Rightarrow 'a\ set \Rightarrow 'a \Rightarrow 'b$
where $confine\ f\ A = (\lambda x.\ if\ x \in A\ then\ f\ x\ else\ 0)$

lemma *confine-UNIV* [*simp*]: $confine\ f\ UNIV = f$
⟨*proof*⟩

lemma *sum-confine-eq-Int*:
assumes *finite I*
shows $sum\ (confine\ f\ A)\ I = (\sum i \in I \cap A.\ f\ i)$
⟨*proof*⟩

lemma *sums-confine-add*:
fixes $f :: nat \Rightarrow 'a::real\text{-}normed\text{-}vector$
assumes *confine f N sums a confine g N sums b*
shows $confine\ (\lambda i.\ f\ i + g\ i)\ N\ sums\ (a+b)$
⟨*proof*⟩

lemma *sums-confine-minus*:
fixes $f :: nat \Rightarrow 'a::real\text{-}normed\text{-}vector$
shows $confine\ f\ N\ sums\ a \Longrightarrow confine\ (uminus \circ f)\ N\ sums\ (-a)$
⟨*proof*⟩

lemma *sums-confine-mult*:
fixes $f :: nat \Rightarrow 'a::real\text{-}normed\text{-}algebra$
shows $confine\ f\ N\ sums\ a \Longrightarrow confine\ (\lambda n.\ c * f\ n)\ N\ sums\ (c * a)$
⟨*proof*⟩

lemma *sums-confine-divide*:
fixes $f :: nat \Rightarrow 'a::real\text{-}normed\text{-}field$
shows $confine\ f\ N\ sums\ a \Longrightarrow confine\ (\lambda n.\ f\ n\ /\ c)\ N\ sums\ (a/c)$
⟨*proof*⟩

lemma *sums-confine-divide-iff*:
fixes $f :: nat \Rightarrow 'a::real\text{-}normed\text{-}field$
assumes $c \neq 0$
shows $confine\ (\lambda n.\ f\ n\ /\ c)\ N\ sums\ (a/c) \longleftrightarrow confine\ f\ N\ sums\ a$
⟨*proof*⟩

lemma *sums-confine*:
fixes $f :: nat \Rightarrow 'a::real\text{-}normed\text{-}vector$
shows $confine\ f\ N\ sums\ l \longleftrightarrow ((\lambda n.\ \sum i \in \{..<n\} \cap N.\ f\ i) \longlongrightarrow l)$
⟨*proof*⟩

lemma *sums-confine-le*:
fixes f :: $nat \Rightarrow$ $'a$::*real-normed-vector*
shows *confine f N sums l* $\longleftrightarrow$ $((\lambda n. \sum i \in \{..n\} \cap N.\ f\ i) \longrightarrow l)$
⟨*proof*⟩

end

5 Lemmas possibly destined for future Isabelle releases

theory *Fourier-Aux2*
imports *HOL−Analysis.Analysis*
begin

lemma *integral-sin-Z* [*simp*]:
assumes $n \in \mathbb{Z}$
shows $integral^L$ (*lebesgue-on* $\{-pi..pi\}$) $(\lambda x.\ sin(x * n)) = 0$
⟨*proof*⟩

lemma *integral-sin-Z′* [*simp*]:
assumes $n \in \mathbb{Z}$
shows $integral^L$ (*lebesgue-on* $\{-pi..pi\}$) $(\lambda x.\ sin(n * x)) = 0$
⟨*proof*⟩

lemma *integral-cos-Z* [*simp*]:
assumes $n \in \mathbb{Z}$
shows $integral^L$ (*lebesgue-on* $\{-pi..pi\}$) $(\lambda x.\ cos(x * n)) = (if\ n = 0\ then\ 2 * pi\ else\ 0)$
⟨*proof*⟩

lemma *integral-cos-Z′* [*simp*]:
assumes $n \in \mathbb{Z}$
shows $integral^L$ (*lebesgue-on* $\{-pi..pi\}$) $(\lambda x.\ cos(n * x)) = (if\ n = 0\ then\ 2 * pi\ else\ 0)$
⟨*proof*⟩

lemma *odd-even-cases* [*case-names 0 odd even*]:
assumes *P 0* **and** *odd*: $\bigwedge n.\ P(Suc\ (2 * n))$ **and** *even*: $\bigwedge n.\ P(2 * n + 2)$
shows *P n*
⟨*proof*⟩

end

6 The basics of Fourier series

Ported from HOL Light; thanks to Manuel Eberl for help with the real asymp proof methods

theory *Fourier*
imports *Periodic Square-Integrable HOL−Real-Asymp.Real-Asymp Confine Fourier-Aux2*
begin

6.1 Orthonormal system of L2 functions and their Fourier coefficients

definition *orthonormal-system* :: $'a$::*euclidean-space set* $\Rightarrow$ ($'b \Rightarrow$ $'a \Rightarrow$ *real*) $\Rightarrow$ *bool*
where *orthonormal-system S w* $\equiv \forall$ *m n. l2product S* (*w m*) (*w n*) = (*if m* = *n then 1 else 0*)

definition *orthonormal-coeff* :: $'a$::*euclidean-space set* $\Rightarrow$ (*nat* $\Rightarrow$ $'a \Rightarrow$ *real*) $\Rightarrow$ ($'a \Rightarrow$ *real*) $\Rightarrow$ *nat* $\Rightarrow$ *real*
where *orthonormal-coeff S w f n* = *l2product S* (*w n*) *f*

lemma *orthonormal-system-eq*: *orthonormal-system S w* $\Longrightarrow$ *l2product S* (*w m*) (*w n*) = (*if m* = *n then 1 else 0*)
⟨*proof*⟩

lemma *orthonormal-system-l2norm*:
orthonormal-system S w $\Longrightarrow$ *l2norm S* (*w i*) = *1*
⟨*proof*⟩

lemma *orthonormal-partial-sum-diff*:
assumes *os*: *orthonormal-system S w* **and** *w*: $\bigwedge$*i.* (*w i*) *square-integrable S*
and *f*: *f square-integrable S* **and** *finite I*
shows (*l2norm S* (λ*x. f x* − ($\sum$ *i*$\in$*I. a i* ∗ *w i x*)))^2 =
(*l2norm S f*)^2 + ($\sum$ *i*$\in$*I.* (*a i*)^2) − *2* ∗ ($\sum$ *i*$\in$*I. a i* ∗ *orthonormal-coeff S w f i*)
⟨*proof*⟩

lemma *orthonormal-optimal-partial-sum*:
assumes *orthonormal-system S w* $\bigwedge$*i.* (*w i*) *square-integrable S*
f square-integrable S finite I
shows *l2norm S* (λ*x. f x* − ($\sum$ *i*$\in$*I. orthonormal-coeff S w f i* ∗ *w i x*))
$\le$ *l2norm S* (λ*x. f x* − ($\sum$ *i*$\in$*I. a i* ∗ *w i x*))
⟨*proof*⟩

lemma *Bessel-inequality*:
assumes *orthonormal-system S w* $\bigwedge$*i.* (*w i*) *square-integrable S*
f square-integrable S finite I
shows ($\sum$ *i*$\in$*I.* (*orthonormal-coeff S w f i*)^2) $\le$ (*l2norm S f*)^2
⟨*proof*⟩

lemma *Fourier-series-square-summable*:
assumes *os*: *orthonormal-system S w* **and** *w*: $\bigwedge$*i.* (*w i*) *square-integrable S*
and *f*: *f square-integrable S*
shows *summable* (*confine* (λ*i.* (*orthonormal-coeff S w f i*) ^ *2*) *I*)

$\langle proof \rangle$

lemma *orthonormal-Fourier-partial-sum-diff-squared*:
assumes *os*: *orthonormal-system S w* **and** *w*: $\bigwedge i.$ (*w i*) *square-integrable S*
and *f*: *f square-integrable S* **and** *finite I*
shows (*l2norm S* ($\lambda x.$ *f x* $-(\sum i{\in}I.$ *orthonormal-coeff S w f i* $*$ *w i x*)))^2 =
(*l2norm S f*)^2 $-$ ($\sum i{\in}I.$ (*orthonormal-coeff S w f i*)^2)
$\langle proof \rangle$

lemma *Fourier-series-l2-summable*:
assumes *os*: *orthonormal-system S w* **and** *w*: $\bigwedge i.$ (*w i*) *square-integrable S*
and *f*: *f square-integrable S*
obtains *g* **where** *g square-integrable S*
($\lambda n.$ *l2norm S* ($\lambda x.$ ($\sum i{\in}I \cap \{..n\}.$ *orthonormal-coeff S w f i* $*$ *w i x*) $-$ *g x*))
$\longrightarrow$ *0*
$\langle proof \rangle$

lemma *Fourier-series-l2-summable-strong*:
assumes *os*: *orthonormal-system S w* **and** *w*: $\bigwedge i.$ (*w i*) *square-integrable S*
and *f*: *f square-integrable S*
obtains *g* **where** *g square-integrable S*
$\bigwedge i.$ $i \in I \Longrightarrow$ *orthonormal-coeff S w* ($\lambda x.$ *f x* $-$ *g x*) *i* = *0*
($\lambda n.$ *l2norm S* ($\lambda x.$ ($\sum i{\in}I \cap \{..n\}.$ *orthonormal-coeff S w f i* $*$ *w i x*) $-$ *g x*))
$\longrightarrow$ *0*
$\langle proof \rangle$

6.2 Actual trigonometric orthogonality relations

lemma *integrable-sin-cx*:
integrable (*lebesgue-on* $\{-pi..pi\}$) ($\lambda x.$ *sin*($x * c$))
$\langle proof \rangle$

lemma *integrable-cos-cx*:
integrable (*lebesgue-on* $\{-pi..pi\}$) ($\lambda x.$ *cos*($x * c$))
$\langle proof \rangle$

lemma *integral-cos-Z'* [*simp*]:
assumes $n \in \mathbb{Z}$
shows *integral*^{L} (*lebesgue-on* $\{-pi..pi\}$) ($\lambda x.$ *cos*($n * x$)) = (*if n = 0 then 2* $*$ *pi else 0*)
$\langle proof \rangle$

lemma *integral-sin-and-cos*:
fixes *m n*::*int*
shows
integral^L (*lebesgue-on* $\{-pi..pi\}$) ($\lambda x.$ *cos*($m * x$) $*$ *cos*($n * x$)) = (*if* $|m| = |n|$

then if $n = 0$ *then* $2 * pi$ *else pi else* 0)
  $integral^L$ (*lebesgue-on* $\{-pi..pi\}$) ($\lambda x.$ $cos(m * x) * sin(n * x)) = 0$
  $integral^L$ (*lebesgue-on* $\{-pi..pi\}$) ($\lambda x.$ $sin(m * x) * cos(n * x)) = 0$
  $[\![m \geq 0; n \geq 0]\!] \Longrightarrow$ $integral^L$ (*lebesgue-on* $\{-pi..pi\}$) ($\lambda x.$ $sin\ (m * x) * sin\ (n * x)) = ($*if* $m = n \wedge n \neq 0$ *then pi else* 0)
  $|integral^L$ (*lebesgue-on* $\{-pi..pi\}$) ($\lambda x.$ $sin\ (m * x) * sin\ (n * x))| = ($*if* $|m| = |n| \wedge n \neq 0$ *then pi else* 0)
  ⟨*proof*⟩

lemma *integral-sin-and-cos-Z* [*simp*]:
  fixes *m n*::*real*
  assumes $m \in \mathbb{Z}$ $n \in \mathbb{Z}$
  shows
  $integral^L$ (*lebesgue-on* $\{-pi..pi\}$) ($\lambda x.$ $cos(m * x) * cos(n * x)) = ($*if* $|m| = |n|$ *then if* $n = 0$ *then* $2 * pi$ *else pi else* 0)
  $integral^L$ (*lebesgue-on* $\{-pi..pi\}$) ($\lambda x.$ $cos(m * x) * sin(n * x)) = 0$
  $integral^L$ (*lebesgue-on* $\{-pi..pi\}$) ($\lambda x.$ $sin(m * x) * cos(n * x)) = 0$
  $|integral^L$ (*lebesgue-on* $\{-pi..pi\}$) ($\lambda x.$ $sin\ (m * x) * sin\ (n * x))| = ($*if* $|m| = |n| \wedge n \neq 0$ *then pi else* 0)
  ⟨*proof*⟩

lemma *integral-sin-and-cos-N* [*simp*]:
  fixes *m n*::*real*
  assumes $m \in \mathbb{N}$ $n \in \mathbb{N}$
  shows $integral^L$ (*lebesgue-on* $\{-pi..pi\}$) ($\lambda x.$ $sin\ (m * x) * sin\ (n * x)) = ($*if* $m = n \wedge n \neq 0$ *then pi else* 0)
  ⟨*proof*⟩

lemma *integrable-sin-and-cos*:
  fixes *m n*::*int*
  shows *integrable* (*lebesgue-on* $\{a..b\}$) ($\lambda x.$ $cos(x * m) * cos(x * n)$)
    integrable (*lebesgue-on* $\{a..b\}$) ($\lambda x.$ $cos(x * m) * sin(x * n)$)
    integrable (*lebesgue-on* $\{a..b\}$) ($\lambda x.$ $sin(x * m) * cos(x * n)$)
    integrable (*lebesgue-on* $\{a..b\}$) ($\lambda x.$ $sin(x * m) * sin(x * n)$)
  ⟨*proof*⟩

lemma *sqrt-pi-ge1*: *sqrt pi* ≥ 1
  ⟨*proof*⟩

definition *trigonometric-set* :: *nat* $\Rightarrow$ *real* $\Rightarrow$ *real*
  where *trigonometric-set n* $\equiv$
    if $n = 0$ *then* $\lambda x.$ $1\ /\ sqrt(2 * pi)$
    else if odd n then $\lambda x.$ $sin(real(Suc\ (n\ div\ 2)) * x)\ /\ sqrt(pi)$
    else ($\lambda x.$ $cos((n\ div\ 2) * x)\ /\ sqrt\ pi$)

lemma *trigonometric-set*:
  trigonometric-set 0 $x = 1\ /\ sqrt(2 * pi)$
  trigonometric-set $(Suc\ (2 * n))$ $x = sin(real(Suc\ n) * x)\ /\ sqrt(pi)$

trigonometric-set (*2* ∗ *n* + *2*) *x* = *cos*(*real*(*Suc n*) ∗ *x*) / *sqrt*(*pi*)
trigonometric-set (*Suc* (*Suc* (*2* ∗ *n*))) *x* = *cos*(*real*(*Suc n*) ∗ *x*) / *sqrt*(*pi*)
⟨*proof*⟩

lemma *trigonometric-set-even*:
trigonometric-set(*2*∗*k*) = (*if k* = *0 then* (λ*x*. *1* / *sqrt*(*2* ∗ *pi*)) *else* (λ*x*. *cos*(*k* ∗ *x*) / *sqrt pi*))
⟨*proof*⟩

lemma *orthonormal-system-trigonometric-set*:
orthonormal-system {−*pi*..*pi*} *trigonometric-set*
⟨*proof*⟩

lemma *square-integrable-trigonometric-set*:
(*trigonometric-set i*) *square-integrable* {−*pi*..*pi*}
⟨*proof*⟩

6.3 Weierstrass for trigonometric polynomials

lemma *Weierstrass-trig-1*:
fixes *g* :: *real* ⇒ *real*
assumes *contf*: *continuous-on UNIV g* **and** *periodic*: ⋀*x*. *g*(*x* + *2* ∗ *pi*) = *g x* **and** *1*: *norm z* = *1*
shows *continuous* (*at z within* (*sphere 0 1*)) (*g* ∘ *Im* ∘ *Ln*)
⟨*proof*⟩

inductive-set *cx-poly* :: (*complex* ⇒ *real*) *set* **where**
Re: *Re* ∈ *cx-poly*
| *Im*: *Im* ∈ *cx-poly*
| *const*: (λ*x*. *c*) ∈ *cx-poly*
| *add*: ⟦*f* ∈ *cx-poly*; *g* ∈ *cx-poly*⟧ ⟹ (λ*x*. *f x* + *g x*) ∈ *cx-poly*
| *mult*: ⟦*f* ∈ *cx-poly*; *g* ∈ *cx-poly*⟧ ⟹ (λ*x*. *f x* ∗ *g x*) ∈ *cx-poly*

declare *cx-poly.intros* [*intro*]

lemma *Weierstrass-trig-polynomial*:
assumes *contf*: *continuous-on* {−*pi*..*pi*} *f* **and** *fpi*: *f*(−*pi*) = *f pi* **and** *0* < *e*
obtains *n*::*nat* **and** *a b* **where**
⋀*x*::*real*. *x* ∈ {−*pi*..*pi*} ⟹ |*f x* − (∑*k*≤*n*. *a k* ∗ *sin* (*k* ∗ *x*) + *b k* ∗ *cos* (*k* ∗ *x*))| < *e*
⟨*proof*⟩

6.4 A bit of extra hacking round so that the ends of a function are OK

lemma *integral-tweak-ends*:
fixes *a b* :: *real*

assumes $a < b$ $e > 0$
obtains f **where** *continuous-on* $\{a..b\}$ f $f\ a = d$ $f\ b = 0$ *l2norm* $\{a..b\}$ $f < e$
⟨*proof*⟩

lemma *square-integrable-approximate-continuous-ends*:
assumes f: f *square-integrable* $\{a..b\}$ **and** $a < b$ $0 < e$
obtains g **where** *continuous-on* $\{a..b\}$ g $g\ b = g\ a$ g *square-integrable* $\{a..b\}$ *l2norm* $\{a..b\}$ $(\lambda x.\ f\ x - g\ x) < e$
⟨*proof*⟩

6.5 Hence the main approximation result

lemma *Weierstrass-l2-trig-polynomial*:
assumes f: f *square-integrable* $\{-pi..pi\}$ **and** $0 < e$
obtains $n\ a\ b$ **where**
l2norm $\{-pi..pi\}$ $(\lambda x.\ f\ x - (\sum k{\le}n.\ a\ k * sin(real\ k * x) + b\ k * cos(real\ k * x))) < e$
⟨*proof*⟩

proposition *Weierstrass-l2-trigonometric-set*:
assumes f: f *square-integrable* $\{-pi..pi\}$ **and** $0 < e$
obtains $n\ a$ **where** *l2norm* $\{-pi..pi\}$ $(\lambda x.\ f\ x - (\sum k{\le}n.\ a\ k * \textit{trigonometric-set}\ k\ x)) < e$
⟨*proof*⟩

6.6 Convergence wrt the L2 norm of trigonometric Fourier series

definition *Fourier-coefficient*
where *Fourier-coefficient* ≡ *orthonormal-coeff* $\{-pi..pi\}$ *trigonometric-set*

lemma *Fourier-series-l2*:
assumes f *square-integrable* $\{-pi..pi\}$
shows $(\lambda n.\ l2norm\ \{-pi..pi\}\ (\lambda x.\ f\ x - (\sum i{\le}n.\ \textit{Fourier-coefficient}\ f\ i * \textit{trigonometric-set}\ i\ x)))$
$\longrightarrow 0$
⟨*proof*⟩

6.7 Fourier coefficients go to 0 (weak form of Riemann-Lebesgue)

lemma *trigonometric-set-mul-absolutely-integrable*:
assumes f *absolutely-integrable-on* $\{-pi..pi\}$
shows $(\lambda x.\ \textit{trigonometric-set}\ n\ x * f\ x)$ *absolutely-integrable-on* $\{-pi..pi\}$
⟨*proof*⟩

lemma *trigonometric-set-mul-integrable*:

f absolutely-integrable-on {−*pi*..*pi*} ⟹ *integrable* (*lebesgue-on* {−*pi*..*pi*}) (λ*x*. *trigonometric-set n x* ∗ *f x*)
⟨*proof*⟩

lemma *trigonometric-set-integrable* [*simp*]: *integrable* (*lebesgue-on* {−*pi*..*pi*}) (*trigonometric-set n*)
⟨*proof*⟩

lemma *absolutely-integrable-sin-product*:
assumes *f absolutely-integrable-on* {−*pi*..*pi*}
shows (λ*x*. *sin*(*k* ∗ *x*) ∗ *f x*) *absolutely-integrable-on* {−*pi*..*pi*}
⟨*proof*⟩

lemma *absolutely-integrable-cos-product*:
assumes *f absolutely-integrable-on* {−*pi*..*pi*}
shows (λ*x*. *cos*(*k* ∗ *x*) ∗ *f x*) *absolutely-integrable-on* {−*pi*..*pi*}
⟨*proof*⟩

lemma
assumes *f absolutely-integrable-on* {−*pi*..*pi*}
shows *Fourier-products-integrable-cos*: *integrable* (*lebesgue-on* {−*pi*..*pi*}) (λ*x*. *cos*(*k* ∗ *x*) ∗ *f x*)
and *Fourier-products-integrable-sin*: *integrable* (*lebesgue-on* {−*pi*..*pi*}) (λ*x*. *sin*(*k* ∗ *x*) ∗ *f x*)
⟨*proof*⟩

lemma *Riemann-lebesgue-square-integrable*:
assumes *orthonormal-system S w* ⋀*i*. *w i square-integrable S f square-integrable S*
shows *orthonormal-coeff S w f* ⟶ *0*
⟨*proof*⟩

proposition *Riemann-lebesgue*:
assumes *f absolutely-integrable-on* {−*pi*..*pi*}
shows *Fourier-coefficient f* ⟶ *0*
⟨*proof*⟩

lemma *Riemann-lebesgue-sin*:
assumes *f absolutely-integrable-on* {−*pi*..*pi*}
shows (λ*n*. *integral*^{L} (*lebesgue-on* {−*pi*..*pi*}) (λ*x*. *sin*(*real n* ∗ *x*) ∗ *f x*)) ⟶ *0*
⟨*proof*⟩

lemma *Riemann-lebesgue-cos*:
assumes *f absolutely-integrable-on* {−*pi*..*pi*}
shows (λ*n*. *integral*^{L} (*lebesgue-on* {−*pi*..*pi*}) (λ*x*. *cos*(*real n* ∗ *x*) ∗ *f x*)) ⟶ *0*

⟨proof⟩

lemma *Riemann-lebesgue-sin-half*:
assumes f *absolutely-integrable-on* $\{-pi..pi\}$
shows $(\lambda n.\ LINT\ x|lebesgue\text{-}on\ \{-pi..pi\}.\ sin\ ((real\ n + 1/2) * x) * f\ x) \longrightarrow 0$
⟨proof⟩

lemma *Fourier-sum-limit-pair*:
assumes f *absolutely-integrable-on* $\{-pi..pi\}$
shows $(\lambda n.\ \sum k{\le}2 * n.\ Fourier\text{-}coefficient\ f\ k * trigonometric\text{-}set\ k\ t) \longrightarrow l$
$\longleftrightarrow (\lambda n.\ \sum k{\le}n.\ Fourier\text{-}coefficient\ f\ k * trigonometric\text{-}set\ k\ t) \longrightarrow l$
(**is** *?lhs = ?rhs*)
⟨proof⟩

6.8 Express Fourier sum in terms of the special expansion at the origin

lemma *Fourier-sum-0*:
$(\sum k \le n.\ Fourier\text{-}coefficient\ f\ k * trigonometric\text{-}set\ k\ 0) =$
$(\sum k \le n\ div\ 2.\ Fourier\text{-}coefficient\ f(2{*}k) * trigonometric\text{-}set\ (2{*}k)\ 0)$
(**is** *?lhs = ?rhs*)
⟨proof⟩

lemma *Fourier-sum-0-explicit*:
$(\sum k{\le}n.\ Fourier\text{-}coefficient\ f\ k * trigonometric\text{-}set\ k\ 0)$
$= (Fourier\text{-}coefficient\ f\ 0\ /\ sqrt\ 2 + (\sum k = 1..n\ div\ 2.\ Fourier\text{-}coefficient\ f(2{*}k)))\ /\ sqrt\ pi$
(**is** *?lhs = ?rhs*)
⟨proof⟩

lemma *Fourier-sum-0-integrals*:
assumes f *absolutely-integrable-on* $\{-pi..pi\}$
shows $(\sum k{\le}n.\ Fourier\text{-}coefficient\ f\ k * trigonometric\text{-}set\ k\ 0) =$
$(integral^L\ (lebesgue\text{-}on\ \{-pi..pi\})\ f\ /\ 2 +$
$(\sum k = Suc\ 0..n\ div\ 2.\ integral^L\ (lebesgue\text{-}on\ \{-pi..pi\})\ (\lambda x.\ cos(k * x) * f\ x)))\ /\ pi$
⟨proof⟩

lemma *Fourier-sum-0-integral*:
assumes f *absolutely-integrable-on* $\{-pi..pi\}$
shows $(\sum k{\le}n.\ Fourier\text{-}coefficient\ f\ k * trigonometric\text{-}set\ k\ 0) =$
$integral^L\ (lebesgue\text{-}on\ \{-pi..pi\})\ (\lambda x.\ (1/2 + (\sum k = Suc\ 0..n\ div\ 2.\ cos(k * x))) * f\ x)\ /\ pi$
⟨proof⟩

6.9 How Fourier coefficients behave under addition etc

lemma *Fourier-coefficient-add*:
assumes *f absolutely-integrable-on* $\{-pi..pi\}$ *g absolutely-integrable-on* $\{-pi..pi\}$
shows *Fourier-coefficient* $(\lambda x.\ f\ x + g\ x)\ i =$
Fourier-coefficient f i + *Fourier-coefficient g i*
⟨*proof*⟩

lemma *Fourier-coefficient-minus*:
assumes *f absolutely-integrable-on* $\{-pi..pi\}$
shows *Fourier-coefficient* $(\lambda x.\ -\ f\ x)\ i = -$ *Fourier-coefficient f i*
⟨*proof*⟩

lemma *Fourier-coefficient-diff*:
assumes *f*: *f absolutely-integrable-on* $\{-pi..pi\}$ **and** *g*: *g absolutely-integrable-on* $\{-pi..pi\}$
shows *Fourier-coefficient* $(\lambda x.\ f\ x - g\ x)\ i =$ *Fourier-coefficient f i* $-$ *Fourier-coefficient g i*
⟨*proof*⟩

lemma *Fourier-coefficient-const*:
Fourier-coefficient $(\lambda x.\ c)\ i = (if\ i = 0\ then\ c * sqrt(2 * pi)\ else\ 0)$
⟨*proof*⟩

lemma *Fourier-offset-term*:
fixes $f :: real \Rightarrow real$
assumes *f*: *f absolutely-integrable-on* $\{-pi..pi\}$ **and** *periodic*: $\bigwedge x.\ f(x + 2{*}pi) = f\ x$
shows *Fourier-coefficient* $(\lambda x.\ f(x{+}t))\ (2 * n + 2) *$ *trigonometric-set* $(2 * n + 2)\ 0$
$=$ *Fourier-coefficient* $f(2 * n{+}1) *$ *trigonometric-set* $(2 * n{+}1)\ t$
$+$ *Fourier-coefficient* $f(2 * n + 2) *$ *trigonometric-set* $(2 * n + 2)\ t$
⟨*proof*⟩

lemma *Fourier-sum-offset*:
fixes $f :: real \Rightarrow real$
assumes *f*: *f absolutely-integrable-on* $\{-pi..pi\}$ **and** *periodic*: $\bigwedge x.\ f(x + 2{*}pi) = f\ x$
shows $(\sum k{\le}2{*}n.$ *Fourier-coefficient f k* $*$ *trigonometric-set k t*$) =$
$(\sum k{\le}2{*}n.$ *Fourier-coefficient* $(\lambda x.\ f(x{+}t))\ k *$ *trigonometric-set k 0*$)$ (**is** *?lhs = ?rhs*)
⟨*proof*⟩

lemma *Fourier-sum-offset-unpaired*:
fixes $f :: real \Rightarrow real$
assumes *f*: *f absolutely-integrable-on* $\{-pi..pi\}$ **and** *periodic*: $\bigwedge x.\ f(x + 2{*}pi) = f\ x$
shows $(\sum k{\le}2{*}n.$ *Fourier-coefficient f k* $*$ *trigonometric-set k t*$) =$

($\sum k \le n$. *Fourier-coefficient* (λx. $f(x+t)$) ($2*k$) $*$ *trigonometric-set* ($2*k$) 0)
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

6.10 Express partial sums using Dirichlet kernel

definition *Dirichlet-kernel*
where *Dirichlet-kernel* ≡
$\lambda n\ x.$ *if* $x = 0$ *then real* $n + 1/2$
else $sin((real\ n + 1/2) * x) / (2 * sin(x/2))$

lemma *Dirichlet-kernel-0* [*simp*]:
$|x| < 2 * pi \Longrightarrow$ *Dirichlet-kernel* $0\ x = 1/2$
⟨*proof*⟩

lemma *Dirichlet-kernel-minus* [*simp*]: *Dirichlet-kernel* $n\ (-x)$ = *Dirichlet-kernel* $n\ x$
⟨*proof*⟩

lemma *Dirichlet-kernel-continuous-strong*:
continuous-on $\{-(2 * pi)<..<2 * pi\}$ (*Dirichlet-kernel* n)
⟨*proof*⟩

lemma *Dirichlet-kernel-continuous*: *continuous-on* $\{-pi..pi\}$ (*Dirichlet-kernel* n)
⟨*proof*⟩

lemma *absolutely-integrable-mult-Dirichlet-kernel*:
assumes f *absolutely-integrable-on* $\{-pi..pi\}$
shows (λx. *Dirichlet-kernel* $n\ x * f\ x$) *absolutely-integrable-on* $\{-pi..pi\}$
⟨*proof*⟩

lemma *cosine-sum-lemma*:
$(1/2 + (\sum k = Suc\ 0..n.\ cos(real\ k * x))) * sin(x/2) = sin((real\ n + 1/2) * x) / 2$
⟨*proof*⟩

lemma *Dirichlet-kernel-cosine-sum*:
assumes $|x| < 2 * pi$
shows *Dirichlet-kernel* $n\ x = 1/2 + (\sum k = Suc\ 0..n.\ cos(real\ k * x))$
⟨*proof*⟩

lemma *integrable-Dirichlet-kernel*: *integrable* (*lebesgue-on* $\{-pi..pi\}$) (*Dirichlet-kernel* n)
⟨*proof*⟩

lemma *integral-Dirichlet-kernel* [*simp*]:
integral^L (*lebesgue-on* {−*pi*..*pi*}) (*Dirichlet-kernel n*) = *pi*
⟨*proof*⟩

lemma *integral-Dirichlet-kernel-half* [*simp*]:
integral^L (*lebesgue-on* {*0*..*pi*}) (*Dirichlet-kernel n*) = *pi*/*2*
⟨*proof*⟩

lemma *Fourier-sum-offset-Dirichlet-kernel*:
assumes *f*: *f absolutely-integrable-on* {−*pi*..*pi*} **and** *periodic*: ⋀*x*. *f*(*x* + *2*∗*pi*) = *f x*
shows
(∑ *k*≤*2*∗*n*. *Fourier-coefficient f k* ∗ *trigonometric-set k t*) =
integral^L (*lebesgue-on* {−*pi*..*pi*}) (λ*x*. *Dirichlet-kernel n x* ∗ *f*(*x*+*t*)) / *pi*
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *Fourier-sum-limit-Dirichlet-kernel*:
assumes *f*: *f absolutely-integrable-on* {−*pi*..*pi*} **and** *periodic*: ⋀*x*. *f*(*x* + *2*∗*pi*) = *f x*
shows ((λ*n*. (∑ *k*≤*n*. *Fourier-coefficient f k* ∗ *trigonometric-set k t*)) ⟶ *l*)
⟷ (λ*n*. *LINT x*|*lebesgue-on* {−*pi*..*pi*}. *Dirichlet-kernel n x* ∗ *f*(*x* + *t*)) ⟶ *pi* ∗ *l*
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

6.11 A directly deduced sufficient condition for convergence at a point

lemma *simple-Fourier-convergence-periodic*:
assumes *f*: *f absolutely-integrable-on* {−*pi*..*pi*}
and *ft*: (λ*x*. (*f*(*x*+*t*) − *f t*) / *sin*(*x*/*2*)) *absolutely-integrable-on* {−*pi*..*pi*}
and *periodic*: ⋀*x*. *f*(*x* + *2*∗*pi*) = *f x*
shows (λ*n*. (∑ *k*≤*n*. *Fourier-coefficient f k* ∗ *trigonometric-set k t*)) ⟶ *f t*
⟨*proof*⟩

6.12 A more natural sufficient Hoelder condition at a point

lemma *bounded-inverse-sin-half*:
assumes *d* > *0*
obtains *B* **where** *B*>*0* ⋀*x*. *x* ∈ ({−*pi*..*pi*} − {−*d*<..<*d*}) ⟹ |*inverse* (*sin* (*x*/*2*))| ≤ *B*
⟨*proof*⟩

proposition *Hoelder-Fourier-convergence-periodic*:
assumes *f*: *f absolutely-integrable-on* {−*pi*..*pi*} **and** *d* > *0 a* > *0*

and *ft*: $\bigwedge x.\ |x-t| < d \Longrightarrow |f\ x - f\ t| \le M * |x-t|$ *powr a*
and *periodic*: $\bigwedge x.\ f(x + 2*pi) = f\ x$
shows ($\lambda n.$ ($\sum k \le n.$ *Fourier-coefficient f k* $*$ *trigonometric-set k t*)) $\longrightarrow$ *f t*
⟨*proof*⟩

In particular, a Lipschitz condition at the point

corollary *Lipschitz-Fourier-convergence-periodic*:
assumes *f*: *f absolutely-integrable-on* $\{-pi..pi\}$ **and** $d > 0$
and *ft*: $\bigwedge x.\ |x-t| < d \Longrightarrow |f\ x - f\ t| \le M * |x-t|$
and *periodic*: $\bigwedge x.\ f(x + 2*pi) = f\ x$
shows ($\lambda n.$ ($\sum k \le n.$ *Fourier-coefficient f k* $*$ *trigonometric-set k t*)) $\longrightarrow$ *f t*
⟨*proof*⟩

In particular, if left and right derivatives both exist

proposition *bi-differentiable-Fourier-convergence-periodic*:
assumes *f*: *f absolutely-integrable-on* $\{-pi..pi\}$
and *f-lt*: *f differentiable at-left t*
and *f-gt*: *f differentiable at-right t*
and *periodic*: $\bigwedge x.\ f(x + 2*pi) = f\ x$
shows ($\lambda n.$ ($\sum k \le n.$ *Fourier-coefficient f k* $*$ *trigonometric-set k t*)) $\longrightarrow$ *f t*
⟨*proof*⟩

And in particular at points where the function is differentiable

lemma *differentiable-Fourier-convergence-periodic*:
assumes *f*: *f absolutely-integrable-on* $\{-pi..pi\}$
and *fdif*: *f differentiable* (*at t*)
and *periodic*: $\bigwedge x.\ f(x + 2*pi) = f\ x$
shows ($\lambda n.$ ($\sum k \le n.$ *Fourier-coefficient f k* $*$ *trigonometric-set k t*)) $\longrightarrow$ *f t*
⟨*proof*⟩

Use reflection to halve the region of integration

lemma *absolutely-integrable-mult-Dirichlet-kernel-reflected*:
assumes *f*: *f absolutely-integrable-on* $\{-pi..pi\}$
and *periodic*: $\bigwedge x.\ f(x + 2*pi) = f\ x$
shows ($\lambda x.$ *Dirichlet-kernel n x* $*$ $f(t+x)$) *absolutely-integrable-on* $\{-pi..pi\}$
($\lambda x.$ *Dirichlet-kernel n x* $*$ $f(t-x)$) *absolutely-integrable-on* $\{-pi..pi\}$
($\lambda x.$ *Dirichlet-kernel n x* $*$ *c*) *absolutely-integrable-on* $\{-pi..pi\}$
⟨*proof*⟩

lemma *absolutely-integrable-mult-Dirichlet-kernel-reflected-part*:
assumes *f*: *f absolutely-integrable-on* $\{-pi..pi\}$
and *periodic*: $\bigwedge x.\ f(x + 2*pi) = f\ x$ **and** $d \le pi$
shows ($\lambda x.$ *Dirichlet-kernel n x* $*$ $f(t+x)$) *absolutely-integrable-on* $\{0..d\}$
($\lambda x.$ *Dirichlet-kernel n x* $*$ $f(t-x)$) *absolutely-integrable-on* $\{0..d\}$
($\lambda x.$ *Dirichlet-kernel n x* $*$ *c*) *absolutely-integrable-on* $\{0..d\}$
⟨*proof*⟩

lemma *absolutely-integrable-mult-Dirichlet-kernel-reflected-part2*:
assumes *f*: *f absolutely-integrable-on* {−*pi*..*pi*}
and *periodic*: $\bigwedge x$. *f*(*x* + *2*∗*pi*) = *f x* **and** *d* ≤ *pi*
shows (λ*x*. *Dirichlet-kernel n x* ∗ (*f*(*t*+*x*) + *f*(*t*−*x*))) *absolutely-integrable-on* {*0*..*d*}
(λ*x*. *Dirichlet-kernel n x* ∗ ((*f*(*t*+*x*) + *f*(*t*−*x*)) − *c*)) *absolutely-integrable-on* {*0*..*d*}
⟨*proof*⟩

lemma *integral-reflect-and-add*:
fixes *f* :: *real* ⇒ ′*b*::*euclidean-space*
assumes *integrable* (*lebesgue-on* {−*a*..*a*}) *f*
shows $integral^L$ (*lebesgue-on* {−*a*..*a*}) *f* = $integral^L$ (*lebesgue-on* {*0*..*a*}) (λ*x*. *f x* + *f*(−*x*))
⟨*proof*⟩

lemma *Fourier-sum-offset-Dirichlet-kernel-half*:
assumes *f*: *f absolutely-integrable-on* {−*pi*..*pi*}
and *periodic*: $\bigwedge x$. *f*(*x* + *2*∗*pi*) = *f x*
shows ($\sum$ *k*≤*2*∗*n*. *Fourier-coefficient f k* ∗ *trigonometric-set k t*) − *l*
= (*LINT x*|*lebesgue-on* {*0*..*pi*}. *Dirichlet-kernel n x* ∗ (*f*(*t*+*x*) + *f*(*t*−*x*) − *2*∗*l*)) / *pi*
⟨*proof*⟩

lemma *Fourier-sum-limit-Dirichlet-kernel-half*:
assumes *f*: *f absolutely-integrable-on* {−*pi*..*pi*}
and *periodic*: $\bigwedge x$. *f*(*x* + *2*∗*pi*) = *f x*
shows (λ*n*. ($\sum$ *k*≤*n*. *Fourier-coefficient f k* ∗ *trigonometric-set k t*)) ⟶ *l*
⟷ (λ*n*. (*LINT x*|*lebesgue-on* {*0*..*pi*}. *Dirichlet-kernel n x* ∗ (*f*(*t*+*x*) + *f*(*t*−*x*) − *2*∗*l*))) ⟶ *0*
⟨*proof*⟩

6.13 Localization principle: convergence only depends on values "nearby"

proposition *Riemann-localization-integral*:
assumes *f*: *f absolutely-integrable-on* {−*pi*..*pi*} **and** *g*: *g absolutely-integrable-on* {−*pi*..*pi*}
and *d* > *0* **and** *d*: $\bigwedge x$. |*x*| < *d* ⟹ *f x* = *g x*
shows (λ*n*. $integral^L$ (*lebesgue-on* {−*pi*..*pi*}) (λ*x*. *Dirichlet-kernel n x* ∗ *f x*)
− $integral^L$ (*lebesgue-on* {−*pi*..*pi*}) (λ*x*. *Dirichlet-kernel n x* ∗ *g x*))
⟶ *0* (**is** *?a* ⟶ *0*)
⟨*proof*⟩

lemma *Riemann-localization-integral-range*:
assumes *f*: *f absolutely-integrable-on* {−*pi*..*pi*}
and *0* < *d d* ≤ *pi*
shows (λ*n*. $integral^L$ (*lebesgue-on* {−*pi*..*pi*}) (λ*x*. *Dirichlet-kernel n x* ∗ *f x*)
− $integral^L$ (*lebesgue-on* {−*d*..*d*}) (λ*x*. *Dirichlet-kernel n x* ∗ *f x*))

⟶ *0*
⟨*proof*⟩

lemma *Riemann-localization*:
assumes *f*: *f absolutely-integrable-on* {−*pi..pi*} **and** *g*: *g absolutely-integrable-on* {−*pi..pi*}
and *perf*: ⋀*x*. *f*(*x* + *2*∗*pi*) = *f x*
and *perg*: ⋀*x*. *g*(*x* + *2*∗*pi*) = *g x*
and *d* > *0* **and** *d*: ⋀*x*. |*x*−*t*| < *d* ⟹ *f x* = *g x*
shows (λ*n*. ∑*k*≤*n*. *Fourier-coefficient f k* ∗ *trigonometric-set k t*) ⟶ *c*
⟷ (λ*n*. ∑*k*≤*n*. *Fourier-coefficient g k* ∗ *trigonometric-set k t*) ⟶ *c*
⟨*proof*⟩

6.14 Localize the earlier integral

lemma *Riemann-localization-integral-range-half*:
assumes *f*: *f absolutely-integrable-on* {−*pi..pi*}
and *0* < *d d* ≤ *pi*
shows (λ*n*. (*LINT x*|*lebesgue-on* {*0..pi*}. *Dirichlet-kernel n x* ∗ (*f x* + *f*(−*x*)))
− (*LINT x*|*lebesgue-on* {*0..d*}. *Dirichlet-kernel n x* ∗ (*f x* + *f*(−*x*))))
⟶ *0*
⟨*proof*⟩

lemma *Fourier-sum-limit-Dirichlet-kernel-part*:
assumes *f*: *f absolutely-integrable-on* {−*pi..pi*}
and *periodic*: ⋀*x*. *f*(*x* + *2*∗*pi*) = *f x*
and *d*: *0* < *d d* ≤ *pi*
shows (λ*n*. ∑*k*≤*n*. *Fourier-coefficient f k* ∗ *trigonometric-set k t*) ⟶ *l*
⟷ (λ*n*. (*LINT x*|*lebesgue-on* {*0..d*}. *Dirichlet-kernel n x* ∗ ((*f*(*t*+*x*) + *f*(*t*−*x*)) − *2*∗*l*))) ⟶ *0*
⟨*proof*⟩

6.15 Make a harmless simplifying tweak to the Dirichlet kernel

lemma *inte-Dirichlet-kernel-mul-expand*:
assumes *f*: *f* ∈ *borel-measurable* (*lebesgue-on S*) **and** *S*: *S* ∈ *sets lebesgue*
shows (*LINT x*|*lebesgue-on S*. *Dirichlet-kernel n x* ∗ *f x*
= *LINT x*|*lebesgue-on S*. *sin*((*n*+*1*/*2*) ∗ *x*) ∗ *f x* / (*2* ∗ *sin*(*x*/*2*)))
∧ (*integrable* (*lebesgue-on S*) (λ*x*. *Dirichlet-kernel n x* ∗ *f x*)
⟷ *integrable* (*lebesgue-on S*) (λ*x*. *sin*((*n*+*1*/*2*) ∗ *x*) ∗ *f x* / (*2* ∗ *sin*(*x*/*2*))))
⟨*proof*⟩

lemma
assumes *f*: *f* ∈ *borel-measurable* (*lebesgue-on S*) **and** *S*: *S* ∈ *sets lebesgue*
shows *integral-Dirichlet-kernel-mul-expand*:
(*LINT x*|*lebesgue-on S*. *Dirichlet-kernel n x* ∗ *f x*)
= (*LINT x*|*lebesgue-on S*. *sin*((*n*+*1*/*2*) ∗ *x*) ∗ *f x* / (*2* ∗ *sin*(*x*/*2*))) (**is** *?th1*)

and *integrable-Dirichlet-kernel-mul-expand*:
integrable (*lebesgue-on S*) (λ*x*. *Dirichlet-kernel n x* ∗ *f x*)
⟷ *integrable* (*lebesgue-on S*) (λ*x*. *sin*((*n*+*1*/*2*) ∗ *x*) ∗ *f x* / (*2* ∗ *sin*(*x*/*2*)))
(**is** *?th2*)
⟨*proof*⟩

proposition *Fourier-sum-limit-sine-part*:
assumes *f*: *f absolutely-integrable-on* {−*pi*..*pi*}
and *periodic*: ⋀*x*. *f*(*x* + *2*∗*pi*) = *f x*
and *d*: *0* < *d d* ≤ *pi*
shows (λ*n*. (∑ *k*≤*n*. *Fourier-coefficient f k* ∗ *trigonometric-set k t*)) ⟶ *l*
⟷ (λ*n*. *LINT x*|*lebesgue-on* {*0*..*d*}. *sin*((*n* + *1*/*2*) ∗ *x*) ∗ ((*f*(*t*+*x*) + *f*(*t*−*x*) − *2*∗*l*) / *x*)) ⟶ *0*
(**is** *?lhs* ⟷ *?*Ψ ⟶ *0*)
⟨*proof*⟩

6.16 Dini's test for the convergence of a Fourier series

proposition *Fourier-Dini-test*:
assumes *f*: *f absolutely-integrable-on* {−*pi*..*pi*}
and *periodic*: ⋀*x*. *f*(*x* + *2*∗*pi*) = *f x*
and *int0d*: *integrable* (*lebesgue-on* {*0*..*d*}) (λ*x*. |*f*(*t*+*x*) + *f*(*t*−*x*) − *2*∗*l*| / *x*)
and *0* < *d*
shows (λ*n*. (∑ *k*≤*n*. *Fourier-coefficient f k* ∗ *trigonometric-set k t*)) ⟶ *l*
⟨*proof*⟩

6.17 Cesaro summability of Fourier series using Fejér kernel

definition *Fejer-kernel* :: *nat* ⇒ *real* ⇒ *real*
where
Fejer-kernel ≡ λ*n x*. *if n* = *0 then 0 else* (∑ *r*<*n*. *Dirichlet-kernel r x*) / *n*

lemma *Fejer-kernel*:
Fejer-kernel n x =
(*if n* = *0 then 0*
else if x = *0 then n*/*2*
else sin(*n* / *2* ∗ *x*) ^ *2* / (*2* ∗ *n* ∗ *sin*(*x*/*2*) ^ *2*))
⟨*proof*⟩

lemma *Fejer-kernel-0* [*simp*]: *Fejer-kernel 0 x* = *0* *Fejer-kernel n 0* = *n*/*2*
⟨*proof*⟩

lemma *Fejer-kernel-continuous-strong*:
continuous-on {−(*2* ∗ *pi*)<..<*2* ∗ *pi*} (*Fejer-kernel n*)
⟨*proof*⟩

lemma *Fejer-kernel-continuous*:
continuous-on {−*pi*..*pi*} (*Fejer-kernel n*)
⟨*proof*⟩

lemma *absolutely-integrable-mult-Fejer-kernel*:
assumes *f absolutely-integrable-on* {−*pi*..*pi*}
shows (λ*x*. *Fejer-kernel n x* ∗ *f x*) *absolutely-integrable-on* {−*pi*..*pi*}
⟨*proof*⟩

lemma *absolutely-integrable-mult-Fejer-kernel-reflected1*:
assumes *f*: *f absolutely-integrable-on* {−*pi*..*pi*}
and *periodic*: ⋀*x*. *f*(*x* + *2*∗*pi*) = *f x*
shows (λ*x*. *Fejer-kernel n x* ∗ *f*(*t* + *x*)) *absolutely-integrable-on* {−*pi*..*pi*}
⟨*proof*⟩

lemma *absolutely-integrable-mult-Fejer-kernel-reflected2*:
assumes *f*: *f absolutely-integrable-on* {−*pi*..*pi*}
and *periodic*: ⋀*x*. *f*(*x* + *2*∗*pi*) = *f x*
shows (λ*x*. *Fejer-kernel n x* ∗ *f*(*t* − *x*)) *absolutely-integrable-on* {−*pi*..*pi*}
⟨*proof*⟩

lemma *absolutely-integrable-mult-Fejer-kernel-reflected3*:
shows (λ*x*. *Fejer-kernel n x* ∗ *c*) *absolutely-integrable-on* {−*pi*..*pi*}
⟨*proof*⟩

lemma *absolutely-integrable-mult-Fejer-kernel-reflected-part1*:
assumes *f*: *f absolutely-integrable-on* {−*pi*..*pi*}
and *periodic*: ⋀*x*. *f*(*x* + *2*∗*pi*) = *f x* **and** *d* ≤ *pi*
shows (λ*x*. *Fejer-kernel n x* ∗ *f*(*t* + *x*)) *absolutely-integrable-on* {*0*..*d*}
⟨*proof*⟩

lemma *absolutely-integrable-mult-Fejer-kernel-reflected-part2*:
assumes *f*: *f absolutely-integrable-on* {−*pi*..*pi*}
and *periodic*: ⋀*x*. *f*(*x* + *2*∗*pi*) = *f x* **and** *d* ≤ *pi*
shows (λ*x*. *Fejer-kernel n x* ∗ *f*(*t* − *x*)) *absolutely-integrable-on* {*0*..*d*}
⟨*proof*⟩

lemma *absolutely-integrable-mult-Fejer-kernel-reflected-part3*:
assumes *d* ≤ *pi*
shows (λ*x*. *Fejer-kernel n x* ∗ *c*) *absolutely-integrable-on* {*0*..*d*}
⟨*proof*⟩

lemma *absolutely-integrable-mult-Fejer-kernel-reflected-part4*:
assumes *f*: *f absolutely-integrable-on* {−*pi*..*pi*}
and *periodic*: ⋀*x*. *f*(*x* + *2*∗*pi*) = *f x* **and** *d* ≤ *pi*
shows (λ*x*. *Fejer-kernel n x* ∗ (*f*(*t* + *x*) + *f*(*t* − *x*))) *absolutely-integrable-on* {*0*..*d*}
⟨*proof*⟩

lemma *absolutely-integrable-mult-Fejer-kernel-reflected-part5*:
  assumes *f*: *f absolutely-integrable-on* $\{-pi..pi\}$
    and *periodic*: $\bigwedge x.\ f(x + 2{*}pi) = f\ x$ **and** $d \leq pi$
  shows (λx. *Fejer-kernel n x* $*$ $((f(t + x) + f(t - x)) - c))$ *absolutely-integrable-on* $\{0..d\}$
  ⟨*proof*⟩

lemma *Fourier-sum-offset-Fejer-kernel-half*:
  fixes *n*::*nat*
  assumes *f*: *f absolutely-integrable-on* $\{-pi..pi\}$
    and *periodic*: $\bigwedge x.\ f(x + 2{*}pi) = f\ x$ **and** $n > 0$
  shows ($\sum r{<}n.\ \sum k{\leq}2{*}r.$ *Fourier-coefficient f k* $*$ *trigonometric-set k t*) / $n - l$
      = (*LINT x*|*lebesgue-on* $\{0..pi\}$. *Fejer-kernel n x* $*$ $(f(t + x) + f(t - x) - 2 * l))$ / *pi*
⟨*proof*⟩

lemma *Fourier-sum-limit-Fejer-kernel-half*:
  fixes *n*::*nat*
  assumes *f*: *f absolutely-integrable-on* $\{-pi..pi\}$
    and *periodic*: $\bigwedge x.\ f(x + 2{*}pi) = f\ x$
  shows (λn. (($\sum r{<}n.\ \sum k{\leq}2{*}r.$ *Fourier-coefficient f k* $*$ *trigonometric-set k t*)) / $n) \longrightarrow l$
      $\longleftrightarrow$
      ((λn. $integral^L$ (*lebesgue-on* $\{0..pi\}$) (λx. *Fejer-kernel n x* $*$ $((f(t + x) + f(t - x)) - 2{*}l)))\ \longrightarrow 0$)
    (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *has-integral-Fejer-kernel*:
  has-bochner-integral (*lebesgue-on* $\{-pi..pi\}$) (*Fejer-kernel n*) (*if* $n = 0$ *then* 0 *else* *pi*)
⟨*proof*⟩

lemma *has-integral-Fejer-kernel-half*:
  has-bochner-integral (*lebesgue-on* $\{0..pi\}$) (*Fejer-kernel n*) (*if* $n = 0$ *then* 0 *else* *pi*/2)
⟨*proof*⟩

lemma *Fejer-kernel-pos-le* [*simp*]: *Fejer-kernel n x* ≥ 0
  ⟨*proof*⟩

theorem *Fourier-Fejer-Cesaro-summable*:
  assumes *f*: *f absolutely-integrable-on* $\{-pi..pi\}$
    and *periodic*: $\bigwedge x.\ f(x + 2{*}pi) = f\ x$

and *fl*: $(f \longrightarrow l)$ (*at t within atMost t*)
and *fr*: $(f \longrightarrow r)$ (*at t within atLeast t*)
shows ($\lambda n.$ ($\sum$ *m*<*n*. $\sum$ *k*≤*2**m*. Fourier-coefficient f k* ∗ *trigonometric-set k t*) / *n*) $\longrightarrow$ (*l*+*r*) / *2*
⟨*proof*⟩

corollary *Fourier-Fejer-Cesaro-summable-simple*:
assumes *f*: *continuous-on UNIV f*
and *periodic*: $\bigwedge x.\ f(x + 2*pi) = f\ x$
shows ($\lambda n.$ ($\sum$ *m*<*n*. $\sum$ *k*≤*2**m*. Fourier-coefficient f k* ∗ *trigonometric-set k x*) / *n*) $\longrightarrow$ *f x*
⟨*proof*⟩

end

7 Acknowledgements

The author was supported by the ERC Advanced Grant ALEXANDRIA (Project 742178) funded by the European Research Council.