

Foundation of geometry in planes, and some complements : Excluding the parallel axioms

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Abstract

“Foundations of Geometry” is a mathematical book written by Hilbert in 1899. This entry is a complete formalization of “Incidence” (excluding cubic axioms), “Order” and “Congruence” (excluding point sequences) of the axioms constructed in this book. In addition, the theorem of the problem about the part that is treated implicitly and is not clearly stated in it is being carried out in parallel.

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1 Incidence

D.Hilbert made a rigorous reconstruction of Euclidean geometry in Chapter 1 of [1]. There, five types of axioms are listed and 32 theorems are proved. In Hilbert’s axiom system, basic concepts such as points and lines are treated as undefined terms, and only their relationships are defined by axioms. In addition, the continuity axiom stipulates that the Euclidean plane is essentially equivalent to the real plane \mathbb{R}^2 , ensuring that the axiom system is categorical.

Implement each axiom and definition and prove the theorem (Coupling axioms related to space geometry axiom 4 to 8 are excluded).

datatype *Point* = *char*
datatype *Segment* = *Se Point Point*
datatype *Line* = *Li Point Point*

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datatype Angle = An Point Point Point
datatype Triangle = Tr Point Point Point
datatype Geo-object =
  Poi Point
  | Seg Segment
  | Lin Line
  | Ang Angle
  | Tri Triangle
datatype sign = add | sub
datatype Geo-objects = Emp | Geos Geo-object sign Geo-objects

locale Eq-relation =
  fixes Eq :: Geo-objects  $\Rightarrow$  Geo-objects  $\Rightarrow$  bool
  and Inv :: bool  $\Rightarrow$  bool
  assumes Eq-refl [simp,intro] : Eq obs obs
  and Eq-rev :  $\llbracket$ Eq obs1 obs2 $\rrbracket \Longrightarrow$  Eq obs2 obs1
  and Eq-trans :  $\llbracket$ Eq obs1 obs3; Eq obs2 obs3 $\rrbracket \Longrightarrow$  Eq obs1 obs2
  and Inv-def : Inv b1  $\longleftrightarrow$   $\neg$  b1

locale Definition-1 = Eq-relation +
  fixes Line-on :: Line  $\Rightarrow$  Point  $\Rightarrow$  bool

locale Axiom-1 = Definition-1 +
  assumes Line-exist :  $\llbracket$  $\neg$  Eq (Geos (Poi p1) add Emp) (Geos (Poi p2) add Emp) $\rrbracket$ 
     $\Longrightarrow$   $\exists$  l. Line-on l p1  $\wedge$  Line-on l p2
  and Line-unique :  $\llbracket$ Line-on l1 p1; Line-on l1 p2; Line-on l2 p1; Line-on l2 p2;
     $\neg$  Eq (Geos (Poi p1) add Emp) (Geos (Poi p2) add Emp) $\rrbracket \Longrightarrow$  Eq (Geos
(Lin l1) add Emp) (Geos (Lin l2) add Emp)
  and Line-on-exist :  $\exists$  p q. Line-on l1 p  $\wedge$  Line-on l1 q
     $\wedge$   $\neg$  Eq (Geos (Poi p) add Emp) (Geos (Poi q) add Emp)
  and Line-not-on-exist :  $\exists$  p q r.  $\neg$  Line-on l1 p  $\wedge$   $\neg$  Line-on l1 q  $\wedge$   $\neg$  Line-on
l1 r
     $\wedge$   $\neg$  Eq (Geos (Poi p) add Emp) (Geos (Poi q) add Emp)
     $\wedge$   $\neg$  Eq (Geos (Poi q) add Emp) (Geos (Poi r) add Emp)
     $\wedge$   $\neg$  Eq (Geos (Poi r) add Emp) (Geos (Poi p) add Emp)

locale Incidence-Rule = Axiom-1 +
  assumes Point-Eq :  $\llbracket$ P1(p1); Eq (Geos (Poi p1) add Emp) (Geos (Poi p2) add
Emp) $\rrbracket \Longrightarrow$  P1(p2)
  and Line-on-trans :  $\llbracket$ Eq (Geos (Lin l1) add Emp) (Geos (Lin l2) add Emp);
Line-on l1 p1 $\rrbracket$ 
     $\Longrightarrow$  Line-on l2 p1
  and Line-on-rule : Line-on (Li p1 p2) p1  $\wedge$  Line-on (Li p1 p2) p2

lemma(in Incidence-Rule) Eq-not-trans :
  assumes N :
     $\neg$  Eq obs1 obs2
    Eq obs2 obs3

```

shows $\neg \text{Eq } \text{obs1 } \text{obs3}$
proof
assume $W : \text{Eq } \text{obs1 } \text{obs3}$
from assms **have** $P1 : \text{Eq } \text{obs3 } \text{obs2}$ **by** (simp add:Eq-rev)
from $W P1$ **have** $P2 : \text{Eq } \text{obs1 } \text{obs2}$ **by** $(\text{blast intro:Eq-trans})$
from $N P2$ **show** False **by** simp
qed

lemma(**in** *Incidence-Rule*) *Line-rev* :
assumes $\neg \text{Eq } (\text{Geos } (\text{Poi } p1) \text{ add Emp}) (\text{Geos } (\text{Poi } p2) \text{ add Emp})$
shows $\text{Eq } (\text{Geos } (\text{Lin } (\text{Li } p1 p2)) \text{ add Emp}) (\text{Geos } (\text{Lin } (\text{Li } p2 p1)) \text{ add Emp})$
proof –
from assms **have** $P1 : \text{Line-on } (\text{Li } p1 p2) p1 \wedge \text{Line-on } (\text{Li } p1 p2) p2$ **by** $(\text{simp add:Line-on-rule})$
have $P2 : \text{Line-on } (\text{Li } p2 p1) p1 \wedge \text{Line-on } (\text{Li } p2 p1) p2$ **by** $(\text{simp add:Line-on-rule})$
from $\text{assms } P1 P2$ **show** $\text{Eq } (\text{Geos } (\text{Lin } (\text{Li } p1 p2)) \text{ add Emp}) (\text{Geos } (\text{Lin } (\text{Li } p2 p1)) \text{ add Emp})$ **by** $(\text{blast intro:Line-unique})$
qed

lemma(**in** *Incidence-Rule*) *Line-not-on-Point* :
assumes $N :$
 $\neg \text{Line-on } (\text{Li } p1 p2) p3$
shows $\neg \text{Eq } (\text{Geos } (\text{Poi } p1) \text{ add Emp}) (\text{Geos } (\text{Poi } p3) \text{ add Emp})$
proof
assume $W : \text{Eq } (\text{Geos } (\text{Poi } p1) \text{ add Emp}) (\text{Geos } (\text{Poi } p3) \text{ add Emp})$
have $P1 : \text{Line-on } (\text{Li } p1 p2) p1$ **by** $(\text{simp add:Line-on-rule})$
from $W P1$ **have** $P2 : \text{Line-on } (\text{Li } p1 p2) p3$ **by** $(\text{simp add:Point-Eq})$
from $N P2$ **show** False **by** simp
qed

lemma(**in** *Incidence-Rule*) *Line-not-on-trans* :
assumes
 $\text{Eq } (\text{Geos } (\text{Lin } l1) \text{ add Emp}) (\text{Geos } (\text{Lin } l2) \text{ add Emp})$
 $\neg \text{Line-on } l1 p1$
shows $\neg \text{Line-on } l2 p1$
proof –
from assms **have** $P1 : \text{Eq } (\text{Geos } (\text{Lin } l2) \text{ add Emp}) (\text{Geos } (\text{Lin } l1) \text{ add Emp})$ **by**
 (simp add:Eq-rev)
from $P1$ **have** $P2 : \text{Line-on } l2 p1 \implies \text{Line-on } l1 p1$ **by** $(\text{simp add:Line-on-trans})$
from $\text{assms } P2$ **show** $\neg \text{Line-on } l2 p1$ **by** blast
qed

lemma(**in** *Incidence-Rule*) *Line-on-rev* :
assumes
 $\neg \text{Eq } (\text{Geos } (\text{Poi } p1) \text{ add Emp}) (\text{Geos } (\text{Poi } p2) \text{ add Emp})$
 $\neg \text{Eq } (\text{Geos } (\text{Poi } p1) \text{ add Emp}) (\text{Geos } (\text{Poi } p3) \text{ add Emp})$
 $\text{Line-on } (\text{Li } p1 p2) p3$
shows $\text{Line-on } (\text{Li } p1 p3) p2$
proof –

have $P1 : \text{Line-on } (Li\ p1\ p2)\ p1$ **by** (*simp add:Line-on-rule*)
have $P2 : \text{Line-on } (Li\ p1\ p3)\ p1$ **by** (*simp add:Line-on-rule*)
have $P3 : \text{Line-on } (Li\ p1\ p3)\ p3$ **by** (*simp add:Line-on-rule*)
from *assms* $P1\ P2\ P3$ **have** $P4 : \text{Eq } (\text{Geos } (\text{Lin } (Li\ p1\ p2)))\ \text{add Emp}$ (*Geos*
(Lin (Li p1 p3)) add Emp) **by** (*simp add:Line-unique*)
have $P5 : \text{Line-on } (Li\ p1\ p2)\ p2$ **by** (*simp add:Line-on-rule*)
from $P4\ P5$ **show** $\text{Line-on } (Li\ p1\ p3)\ p2$ **by** (*simp add:Line-on-trans*)
qed

lemma(*in Incidence-Rule*) *Line-not-Eq* :

assumes
 $\neg \text{Eq } (\text{Geos } (\text{Poi } p1)\ \text{add Emp})\ (\text{Geos } (\text{Poi } p2)\ \text{add Emp})$
 $\neg \text{Line-on } (Li\ p1\ p2)\ p3$
shows $\neg \text{Eq } (\text{Geos } (\text{Lin } (Li\ p1\ p2)))\ \text{add Emp}$ (*Geos (Lin (Li p1 p3)) add Emp*)
proof –
have $P1 : \text{Line-on } (Li\ p1\ p3)\ p3$ **by** (*simp add:Line-on-rule*)
have $P2 : \text{Eq } (\text{Geos } (\text{Lin } (Li\ p1\ p2)))\ \text{add Emp}$ (*Geos (Lin (Li p1 p3)) add Emp*)
 \implies
 $\text{Eq } (\text{Geos } (\text{Lin } (Li\ p1\ p3)))\ \text{add Emp}$ (*Geos (Lin (Li p1 p2)) add Emp*) **by**
(*simp add:Eq-rev*)
from $P1\ P2$ **have** $P3 : \text{Eq } (\text{Geos } (\text{Lin } (Li\ p1\ p2)))\ \text{add Emp}$ (*Geos (Lin (Li p1*
p3)) add Emp) \implies
 $\text{Line-on } (Li\ p1\ p2)\ p3$ **by** (*simp add:Line-on-trans*)
from *assms* $P3$ **show** $\neg \text{Eq } (\text{Geos } (\text{Lin } (Li\ p1\ p2)))\ \text{add Emp}$ (*Geos (Lin (Li p1*
p3)) add Emp) **by** *blast*
qed

lemma(*in Incidence-Rule*) *Line-not-Eq-on* :

assumes N :
 $\neg \text{Eq } (\text{Geos } (\text{Poi } p1)\ \text{add Emp})\ (\text{Geos } (\text{Poi } p2)\ \text{add Emp})$
 $\neg \text{Eq } (\text{Geos } (\text{Poi } p1)\ \text{add Emp})\ (\text{Geos } (\text{Poi } p3)\ \text{add Emp})$
 $\neg \text{Eq } (\text{Geos } (\text{Lin } (Li\ p1\ p2)))\ \text{add Emp}$ (*Geos (Lin (Li p1 p3)) add Emp*)
shows $\neg \text{Line-on } (Li\ p1\ p2)\ p3$
proof
assume $W : \text{Line-on } (Li\ p1\ p2)\ p3$
have $P1 : \text{Line-on } (Li\ p1\ p2)\ p1$ **by** (*simp add:Line-on-rule*)
have $P2 : \text{Line-on } (Li\ p1\ p3)\ p1$ **by** (*simp add:Line-on-rule*)
have $P3 : \text{Line-on } (Li\ p1\ p3)\ p3$ **by** (*simp add:Line-on-rule*)
from $N\ W\ P1\ P2\ P3$ **have** $P4 : \text{Eq } (\text{Geos } (\text{Lin } (Li\ p1\ p2)))\ \text{add Emp}$ (*Geos (Lin*
(Li p1 p3)) add Emp) **by** (*simp add:Line-unique*)
from $N\ P4$ **show** *False* **by** *simp*
qed

lemma(*in Incidence-Rule*) *Line-unique-Point* :

assumes
 $\neg \text{Eq } (\text{Geos } (\text{Lin } l1)\ \text{add Emp})\ (\text{Geos } (\text{Lin } l2)\ \text{add Emp})$
 $\text{Line-on } l1\ p1\ \text{Line-on } l1\ p2$
 $\text{Line-on } l2\ p1\ \text{Line-on } l2\ p2$
shows $\text{Eq } (\text{Geos } (\text{Poi } p1)\ \text{add Emp})\ (\text{Geos } (\text{Poi } p2)\ \text{add Emp})$

proof –
from *assms* **have** $P1 : \neg \text{Eq} (\text{Geos} (\text{Poi } p1) \text{ add Emp}) (\text{Geos} (\text{Poi } p2) \text{ add Emp})$
 \implies
 $\text{Eq} (\text{Geos} (\text{Lin } l1) \text{ add Emp}) (\text{Geos} (\text{Lin } l2) \text{ add Emp})$ **by** (*simp add:Line-unique*)
from *assms* $P1$ **show** $\text{Eq} (\text{Geos} (\text{Poi } p1) \text{ add Emp}) (\text{Geos} (\text{Poi } p2) \text{ add Emp})$ **by**
blast
qed

lemma(*in Incidence-Rule*) *Line-not-on-Eq* :
assumes $N :$
 $\neg \text{Line-on } l1 \ p1$
 $\text{Line-on } l2 \ p1$
shows $\neg \text{Eq} (\text{Geos} (\text{Lin } l1) \text{ add Emp}) (\text{Geos} (\text{Lin } l2) \text{ add Emp})$
proof
assume $W : \text{Eq} (\text{Geos} (\text{Lin } l1) \text{ add Emp}) (\text{Geos} (\text{Lin } l2) \text{ add Emp})$
from $N \ W$ **have** $P1 : \text{Line-on } l1 \ p1$ **by** (*blast intro:Line-on-trans Eq-rev*)
from $N \ P1$ **show** *False* **by** *simp*
qed

lemma(*in Incidence-Rule*) *Line-cross-not-on* :
assumes
 $\neg \text{Eq} (\text{Geos} (\text{Poi } p1) \text{ add Emp}) (\text{Geos} (\text{Poi } p2) \text{ add Emp})$
 $\neg \text{Eq} (\text{Geos} (\text{Poi } p2) \text{ add Emp}) (\text{Geos} (\text{Poi } p4) \text{ add Emp})$
 $\neg \text{Line-on} (\text{Li } p1 \ p2) \ p3$
 $\text{Line-on} (\text{Li } p2 \ p3) \ p4$
shows $\neg \text{Line-on} (\text{Li } p1 \ p2) \ p4$
proof –
have $P1 : \text{Line-on} (\text{Li } p1 \ p2) \ p2$ **by** (*simp add:Line-on-rule*)
have $P2 : \text{Line-on} (\text{Li } p2 \ p3) \ p2$ **by** (*simp add:Line-on-rule*)
from *assms* $P1 \ P2$ **have** $P3 : \text{Line-on} (\text{Li } p1 \ p2) \ p4 \implies \text{Eq} (\text{Geos} (\text{Lin} (\text{Li } p1 \ p2)) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li } p2 \ p3)) \text{ add Emp})$ **by** (*simp add:Line-unique*)
have $P4 : \text{Line-on} (\text{Li } p2 \ p3) \ p3$ **by** (*simp add:Line-on-rule*)
from $P3 \ P4$ **have** $P5 : \text{Line-on} (\text{Li } p1 \ p2) \ p4 \implies \text{Line-on} (\text{Li } p1 \ p2) \ p3$ **by**
(*blast intro:Line-on-trans Eq-rev*)
from *assms* $P5$ **show** $\neg \text{Line-on} (\text{Li } p1 \ p2) \ p4$ **by** *blast*
qed

end

2 Order

locale *Definition-2 = Incidence-Rule* +
fixes *Line-on-Seg* :: $\text{Line} \Rightarrow \text{Segment} \Rightarrow \text{bool}$
and *Bet-Point* :: $\text{Segment} \Rightarrow \text{Point} \Rightarrow \text{bool}$
and *Seg-on-Seg* :: $\text{Segment} \Rightarrow \text{Segment} \Rightarrow \text{bool}$
and *Line-on-Line* :: $\text{Line} \Rightarrow \text{Line} \Rightarrow \text{bool}$
and *Plane-sameside* :: $\text{Line} \Rightarrow \text{Point} \Rightarrow \text{Point} \Rightarrow \text{bool}$
and *Plane-diffside* :: $\text{Line} \Rightarrow \text{Point} \Rightarrow \text{Point} \Rightarrow \text{bool}$
assumes *Bet-Point-def* : $\llbracket \text{Bet-Point} (\text{Se } p1 \ p2) \ p3 \rrbracket \implies \neg \text{Eq} (\text{Geos} (\text{Poi } p1)$

$add\ Emp) (Geos (Poi\ p2) add\ Emp)$
 $\wedge \neg Eq (Geos (Poi\ p2) add\ Emp) (Geos (Poi\ p3) add\ Emp) \wedge \neg Eq (Geos$
 $(Poi\ p3) add\ Emp) (Geos (Poi\ p1) add\ Emp)$
and $Bet\text{-}rev : \llbracket Bet\text{-}Point (Se\ p1\ p2) p3 \rrbracket \implies Bet\text{-}Point (Se\ p2\ p1) p3$
and $Line\text{-}Bet\text{-}exist : \llbracket Bet\text{-}Point (Se\ p1\ p2) p3 \rrbracket \implies \exists l. Line\text{-}on\ l\ p1 \wedge Line\text{-}on$
 $l\ p2 \wedge Line\text{-}on\ l\ p3$
and $Seg\text{-}rev : Eq (Geos (Seg (Se\ p1\ p2)) add\ Emp) (Geos (Seg (Se\ p2\ p1)) add$
 $Emp)$
and $Plane\text{-}sameside\text{-}def : Plane\text{-}sameside\ l1\ p1\ p2 \longleftrightarrow \neg Line\text{-}on\text{-}Seg\ l1 (Se$
 $p1\ p2) \wedge \neg Line\text{-}on\ l1\ p1 \wedge \neg Line\text{-}on\ l1\ p2 \wedge \neg Eq (Geos (Poi\ p1) add\ Emp)$
 $(Geos (Poi\ p2) add\ Emp)$
and $Plane\text{-}diffside\text{-}def : Plane\text{-}diffside\ l1\ p1\ p2 \longleftrightarrow (\exists p. Bet\text{-}Point (Se\ p1\ p2)$
 $p \wedge Line\text{-}on\ l1\ p \wedge \neg Line\text{-}on\ l1\ p1 \wedge \neg Line\text{-}on\ l1\ p2)$

locale $Axiom\text{-}2 = Definition\text{-}2 +$

assumes $Bet\text{-}extension : \llbracket Line\text{-}on\ l1\ p1; Line\text{-}on\ l1\ p2; \neg Eq (Geos (Poi\ p1) add$
 $Emp) (Geos (Poi\ p2) add\ Emp) \rrbracket \implies \exists p. Bet\text{-}Point (Se\ p1\ p) p2 \wedge Line\text{-}on\ l1\ p$
and $Bet\text{-}iff : \llbracket Bet\text{-}Point (Se\ p1\ p2) p3 \rrbracket \implies Inv (Bet\text{-}Point (Se\ p2\ p3) p1) \wedge$
 $Inv (Bet\text{-}Point (Se\ p3\ p1) p2)$
and $Pachets\text{-}axiom : \llbracket \neg Line\text{-}on (Li\ p1\ p2) p3; Bet\text{-}Point (Se\ p1\ p2) p4; Line\text{-}on$
 $l1\ p4;$
 $\neg Line\text{-}on\ l1\ p1; \neg Line\text{-}on\ l1\ p2; \neg Line\text{-}on\ l1\ p3 \rrbracket \implies$
 $Line\text{-}on\text{-}Seg\ l1 (Se\ p1\ p3) \wedge \neg Line\text{-}on\text{-}Seg\ l1 (Se\ p2\ p3)$
 $\vee Line\text{-}on\text{-}Seg\ l1 (Se\ p2\ p3) \wedge \neg Line\text{-}on\text{-}Seg\ l1 (Se\ p1\ p3)$
and $Seg\text{-}move\text{-}sameside : \llbracket Line\text{-}on\ l1\ p1; Line\text{-}on\ l1\ p2; \neg Eq (Geos (Poi\ p1)$
 $add\ Emp) (Geos (Poi\ p2) add\ Emp);$
 $\neg Eq (Geos (Poi\ p3) add\ Emp) (Geos (Poi\ p4) add\ Emp) \rrbracket \implies$
 $\exists p. Eq (Geos (Seg (Se\ p3\ p4)) add\ Emp) (Geos (Seg (Se\ p1\ p)) add\ Emp) \wedge$
 $\neg Bet\text{-}Point (Se\ p\ p2) p1 \wedge Line\text{-}on\ l1\ p \wedge \neg Eq (Geos (Poi\ p1) add\ Emp) (Geos$
 $(Poi\ p) add\ Emp)$
and $Seg\text{-}move\text{-}diffside : \llbracket Line\text{-}on\ l1\ p1; Line\text{-}on\ l1\ p2; \neg Eq (Geos (Poi\ p1) add$
 $Emp) (Geos (Poi\ p2) add\ Emp);$
 $\neg Eq (Geos (Poi\ p3) add\ Emp) (Geos (Poi\ p4) add\ Emp) \rrbracket \implies$
 $\exists p. Eq (Geos (Seg (Se\ p3\ p4)) add\ Emp) (Geos (Seg (Se\ p1\ p)) add\ Emp)$
 $\wedge Bet\text{-}Point (Se\ p\ p2) p1 \wedge Line\text{-}on\ l1\ p \wedge \neg Eq (Geos (Poi\ p1) add\ Emp) (Geos$
 $(Poi\ p) add\ Emp)$

locale $Order\text{-}Rule = Axiom\text{-}2 +$

assumes $Bet\text{-}Point\text{-}Eq : \llbracket Bet\text{-}Point (Se\ p1\ p2) p3; Eq (Geos (Poi\ p1) add\ Emp)$
 $(Geos (Poi\ p4) add\ Emp) \rrbracket \implies Bet\text{-}Point (Se\ p4\ p2) p3$
and $Line\text{-}on\text{-}Seg\text{-}rule : Line\text{-}on\text{-}Seg\ l1 (Se\ p1\ p2) \longleftrightarrow (\exists p. Line\text{-}on\ l1\ p \wedge$
 $Bet\text{-}Point (Se\ p1\ p2) p)$
and $Seg\text{-}on\text{-}Seg\text{-}rule : Seg\text{-}on\text{-}Seg (Se\ p1\ p2) (Se\ p3\ p4) \longleftrightarrow (\exists p. Bet\text{-}Point$
 $(Se\ p1\ p2) p \wedge Bet\text{-}Point (Se\ p3\ p4) p)$
and $Line\text{-}on\text{-}Line\text{-}rule : Line\text{-}on\text{-}Line\ l1\ l2 \longleftrightarrow (\exists p. Line\text{-}on\ l1\ p \wedge Line\text{-}on$
 $l2\ p)$
and $Seg\text{-}Point\text{-}Eq : \llbracket Eq (Geos (Poi\ p1) add\ Emp) (Geos (Poi\ p2) add\ Emp) \rrbracket$
 $\implies Eq (Geos (Seg (Se\ p3\ p1)) add\ Emp) (Geos (Seg (Se\ p3\ p2)) add\ Emp)$

lemma(in *Order-Rule*) *Line-Bet-on* :
assumes
Bet-Point (*Se* $p1$ $p2$) $p3$
shows *Line-on* (*Li* $p1$ $p2$) $p3$ **and** *Line-on* (*Li* $p2$ $p1$) $p3$
and *Line-on* (*Li* $p2$ $p3$) $p1$ **and** *Line-on* (*Li* $p3$ $p2$) $p1$
and *Line-on* (*Li* $p1$ $p3$) $p2$ **and** *Line-on* (*Li* $p3$ $p1$) $p2$
proof –
from *assms* **have** $\exists l. \text{Line-on } l \ p1 \wedge \text{Line-on } l \ p2 \wedge \text{Line-on } l \ p3$ **by** (*blast intro:Line-Bet-exist*)
then obtain $l1 :: \text{Line}$ **where** $P1 : \text{Line-on } l1 \ p1 \wedge \text{Line-on } l1 \ p2 \wedge \text{Line-on } l1 \ p3$ **by** *blast*
from *assms* **have** $P2 : \neg \text{Eq} (\text{Geos} (\text{Poi } p1) \text{ add Emp}) (\text{Geos} (\text{Poi } p2) \text{ add Emp})$
by (*simp add:Bet-Point-def*)
have $P3 : \text{Line-on} (\text{Li } p1 \ p2) \ p1 \wedge \text{Line-on} (\text{Li } p1 \ p2) \ p2$ **by** (*simp add:Line-on-rule*)
from $P1$ **have** $P4 : \text{Line-on } l1 \ p1$ **by** *simp*
from $P1$ **have** $P5 : \text{Line-on } l1 \ p2$ **by** *simp*
from $P2 \ P3 \ P4 \ P5$ **have** $P6 : \text{Eq} (\text{Geos} (\text{Lin } l1) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li } p1 \ p2)) \text{ add Emp})$ **by** (*simp add:Line-unique*)
from $P1 \ P6$ **show** $P7 : \text{Line-on} (\text{Li } p1 \ p2) \ p3$ **by** (*simp add:Line-on-trans*)
from *assms* **have** $P8 : \neg \text{Eq} (\text{Geos} (\text{Poi } p3) \text{ add Emp}) (\text{Geos} (\text{Poi } p1) \text{ add Emp})$
by (*simp add:Bet-Point-def*)
from $P2 \ P7 \ P8$ **show** *Line-on* (*Li* $p1$ $p3$) $p2$ **by** (*blast intro:Line-on-rev Eq-rev*)
from $P2 \ P7 \ P8$ **show** *Line-on* (*Li* $p3$ $p1$) $p2$ **by** (*blast intro:Line-on-trans Line-on-rev Eq-rev Line-rev*)
from $P2 \ P7$ **show** *Line-on* (*Li* $p2$ $p1$) $p3$ **by** (*blast intro:Line-on-trans Line-rev*)
from *assms* **have** $P9 : \neg \text{Eq} (\text{Geos} (\text{Poi } p2) \text{ add Emp}) (\text{Geos} (\text{Poi } p3) \text{ add Emp})$
by (*simp add:Bet-Point-def*)
from $P2 \ P7 \ P9$ **show** *Line-on* (*Li* $p2$ $p3$) $p1$ **by** (*blast intro:Line-on-rev Line-on-trans Line-rev Eq-rev*)
from $P9$ **have** $P10 : \neg \text{Eq} (\text{Geos} (\text{Poi } p3) \text{ add Emp}) (\text{Geos} (\text{Poi } p2) \text{ add Emp})$
by (*blast intro:Eq-rev*)
from *assms* $P2 \ P7 \ P8 \ P10$ **show** *Line-on* (*Li* $p3$ $p2$) $p1$ **by** (*blast intro:Line-on-rev Bet-Point-def Line-on-trans Eq-rev Line-rev*)
qed

lemma(in *Order-Rule*) *Line-Bet-not-Eq* :
assumes N :
Bet-Point (*Se* $p1$ $p2$) $p3$
 $\neg \text{Line-on} (\text{Li } p1 \ p2) \ p4$
shows $\neg \text{Eq} (\text{Geos} (\text{Lin} (\text{Li } p4 \ p3)) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li } p4 \ p2)) \text{ add Emp})$
proof
assume $W : \text{Eq} (\text{Geos} (\text{Lin} (\text{Li } p4 \ p3)) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li } p4 \ p2)) \text{ add Emp})$
have $P1 : \text{Line-on} (\text{Li } p4 \ p3) \ p3$ **by** (*simp add:Line-on-rule*)
from $W \ P1$ **have** $P2 : \text{Line-on} (\text{Li } p4 \ p2) \ p3$ **by** (*simp add:Line-on-trans*)
have $P3 : \text{Line-on} (\text{Li } p4 \ p2) \ p2$ **by** (*simp add:Line-on-rule*)
from N **have** $P4 : \text{Line-on} (\text{Li } p1 \ p2) \ p3$ **by** (*simp add:Line-Bet-on*)
have $P5 : \text{Line-on} (\text{Li } p1 \ p2) \ p2$ **by** (*simp add:Line-on-rule*)
from *assms* **have** $P6 : \neg \text{Eq} (\text{Geos} (\text{Poi } p2) \text{ add Emp}) (\text{Geos} (\text{Poi } p3) \text{ add Emp})$

by (*simp add:Bet-Point-def*)
from $P2\ P3\ P4\ P5\ P6$ **have** $P7 : Eq\ (Geos\ (Lin\ (Li\ p4\ p2)))\ add\ Emp$ (*Geos*
(Lin (Li p1 p2)) add Emp) **by** (*simp add:Line-unique*)
have $P8 : Line-on\ (Li\ p4\ p2)\ p4$ **by** (*simp add:Line-on-rule*)
from $P7\ P8$ **have** $P9 : Line-on\ (Li\ p1\ p2)\ p4$ **by** (*simp add:Line-on-trans*)
from $N\ P9$ **show** *False* **by** *simp*
qed

Theorem3

theorem(*in Order-Rule*) *Seg-density* :
assumes $\neg Eq\ (Geos\ (Poi\ A)\ add\ Emp)\ (Geos\ (Poi\ C)\ add\ Emp)$
shows $\exists p. Bet-Point\ (Se\ A\ C)\ p$
proof –
have $\exists p\ q\ r. \neg Line-on\ (Li\ A\ C)\ p \wedge \neg Line-on\ (Li\ A\ C)\ q \wedge \neg Line-on\ (Li\ A\ C)\ r$
 $\wedge \neg Eq\ (Geos\ (Poi\ p)\ add\ Emp)\ (Geos\ (Poi\ q)\ add\ Emp) \wedge \neg Eq\ (Geos\ (Poi\ q)\ add\ Emp)\ (Geos\ (Poi\ r)\ add\ Emp)$
 $\wedge \neg Eq\ (Geos\ (Poi\ r)\ add\ Emp)\ (Geos\ (Poi\ p)\ add\ Emp)$ **by** (*blast*
intro:Line-not-on-exist)
then obtain $E :: Point$ **where** $P1 : \neg Line-on\ (Li\ A\ C)\ E$ **by** *blast*
then have $P2 : \neg Eq\ (Geos\ (Poi\ A)\ add\ Emp)\ (Geos\ (Poi\ E)\ add\ Emp)$ **by**
(simp add:Line-not-on-Point)
have $P3 : Line-on\ (Li\ A\ E)\ A \wedge Line-on\ (Li\ A\ E)\ E$ **by** (*simp add:Line-on-rule*)
from $P2\ P3$ **have** $\exists p. Bet-Point\ (Se\ A\ p)\ E \wedge Line-on\ (Li\ A\ E)\ p$ **by** (*simp*
add:Bet-extension)
then obtain $F :: Point$ **where** $P4 : Bet-Point\ (Se\ A\ F)\ E \wedge Line-on\ (Li\ A\ E)\ F$
by *blast*
then have $P5 : Line-on\ (Li\ A\ F)\ E$ **by** (*simp add:Line-Bet-on*)
from $P4$ **have** $P6 : Bet-Point\ (Se\ A\ F)\ E$ **by** *simp*
from $P6$ **have** $P7 : \neg Eq\ (Geos\ (Poi\ A)\ add\ Emp)\ (Geos\ (Poi\ F)\ add\ Emp)$ **by**
(simp add:Bet-Point-def)
from $P2\ P4\ P6\ P7$ **have** $P8 : Line-on\ (Li\ A\ E)\ F$ **by** (*simp add:Line-on-rev*)
from *assms* $P1$ **have** $P9 : \neg Eq\ (Geos\ (Lin\ (Li\ A\ C))\ add\ Emp)\ (Geos\ (Lin\ (Li\ A\ E))\ add\ Emp)$ **by**
(simp add:Line-not-Eq)
have $P10 : Line-on\ (Li\ A\ F)\ A$ **by** (*simp add:Line-on-rule*)
from $P2\ P3\ P5\ P10$ **have** $P11 : Eq\ (Geos\ (Lin\ (Li\ A\ E))\ add\ Emp)\ (Geos\ (Lin\ (Li\ A\ F))\ add\ Emp)$ **by**
(blast intro:Line-unique)
from $P9\ P11$ **have** $P12 : \neg Eq\ (Geos\ (Lin\ (Li\ A\ C))\ add\ Emp)\ (Geos\ (Lin\ (Li\ A\ F))\ add\ Emp)$ **by**
(simp add:Eq-not-trans)
from *assms* $P7\ P12$ **have** $P13 : \neg Line-on\ (Li\ A\ C)\ F$ **by** (*simp add:Line-not-Eq-on*)
from *assms* $P7\ P13$ **have** $P14 : \neg Line-on\ (Li\ A\ F)\ C$ **by** (*blast intro:Line-on-rev*)
have $Line-on\ (Li\ A\ F)\ F$ **by** (*simp add:Line-on-rule*)
then have $P15 : Eq\ (Geos\ (Poi\ F)\ add\ Emp)\ (Geos\ (Poi\ C)\ add\ Emp) \implies Line-on\ (Li\ A\ F)\ C$ **by**
(simp add:Point-Eq)
from $P14\ P15$ **have** $P16 : \neg Eq\ (Geos\ (Poi\ F)\ add\ Emp)\ (Geos\ (Poi\ C)\ add\ Emp)$ **by** *blast*
have $P17 : Line-on\ (Li\ F\ C)\ F \wedge Line-on\ (Li\ F\ C)\ C$ **by** (*simp add:Line-on-rule*)
from $P16\ P17$ **have** $\exists p. Bet-Point\ (Se\ F\ p)\ C \wedge Line-on\ (Li\ F\ C)\ p$ **by** (*simp*
add:Bet-extension)

then obtain $G :: \text{Point}$ **where** $P18 : \text{Bet-Point } (Se\ F\ G)\ C \wedge \text{Line-on } (Li\ F\ C)\ G$ **by** *blast*
from $P18$ **have** $P19 : \text{Line-on } (Li\ F\ G)\ C$ **by** *(simp add:Line-Bet-on)*
from $P18$ **have** $P20 : \text{Bet-Point } (Se\ F\ G)\ C$ **by** *simp*
then have $P21 : \neg \text{Eq } (Geos\ (Poi\ F)\ \text{add}\ \text{Emp})\ (Geos\ (Poi\ G)\ \text{add}\ \text{Emp})$ **by**
(simp add:Bet-Point-def)
from $P20$ **have** $P22 : \text{Line-on } (Li\ F\ C)\ G$ **by** *(simp add:Line-Bet-on)*
from $P7\ P14\ P21\ P22$ **have** $P23 : \neg \text{Line-on } (Li\ A\ F)\ G$ **by** *(simp add:Line-cross-not-on)*
from $P6\ P23$ **have** $P24 : \neg \text{Eq } (Geos\ (Lin\ (Li\ G\ E))\ \text{add}\ \text{Emp})\ (Geos\ (Lin\ (Li\ G\ F))\ \text{add}\ \text{Emp})$ **by** *(simp add:Line-Bet-not-Eq)*
from $P5$ **have** $P25 : \text{Eq } (Geos\ (Poi\ E)\ \text{add}\ \text{Emp})\ (Geos\ (Poi\ G)\ \text{add}\ \text{Emp}) \implies$
 $\text{Line-on } (Li\ A\ F)\ G$ **by** *(simp add:Point-Eq)*
from $P23\ P25$ **have** $P26 : \neg \text{Eq } (Geos\ (Poi\ G)\ \text{add}\ \text{Emp})\ (Geos\ (Poi\ E)\ \text{add}\ \text{Emp})$ **by**
(blast intro:Eq-rev)
from $P21$ **have** $P27 : \neg \text{Eq } (Geos\ (Poi\ G)\ \text{add}\ \text{Emp})\ (Geos\ (Poi\ F)\ \text{add}\ \text{Emp})$
by *(blast intro:Eq-rev)*
from $P24\ P26\ P27$ **have** $P28 : \neg \text{Line-on } (Li\ G\ E)\ F$ **by** *(simp add:Line-not-Eq-on)*
from $P26\ P28$ **have** $P29 : \neg \text{Line-on } (Li\ E\ G)\ F$ **by** *(blast intro:Line-rev Line-on-trans Eq-rev)*
have $P30 : \text{Line-on } (Li\ E\ G)\ E$ **by** *(simp add:Line-on-rule)*
have $P31 : \text{Line-on } (Li\ A\ E)\ E$ **by** *(simp add:Line-on-rule)*
have $P32 : \text{Line-on } (Li\ A\ E)\ A$ **by** *(simp add:Line-on-rule)*
from $P2\ P30\ P31\ P32$ **have** $P33 : \text{Line-on } (Li\ E\ G)\ A \implies \text{Eq } (Geos\ (Lin\ (Li\ A\ E))\ \text{add}\ \text{Emp})\ (Geos\ (Lin\ (Li\ E\ G))\ \text{add}\ \text{Emp})$ **by** *(simp add:Line-unique)*
from $P8\ P33$ **have** $P34 : \text{Line-on } (Li\ E\ G)\ A \implies \text{Line-on } (Li\ E\ G)\ F$ **by** *(simp add:Line-on-trans)*
from $P29\ P34$ **have** $P35 : \neg \text{Line-on } (Li\ E\ G)\ A$ **by** *blast*
have $P36 : \text{Line-on } (Li\ E\ G)\ G$ **by** *(simp add:Line-on-rule)*
have $P37 : \text{Line-on } (Li\ F\ G)\ G$ **by** *(simp add:Line-on-rule)*
from $P20$ **have** $P38 : \neg \text{Eq } (Geos\ (Poi\ G)\ \text{add}\ \text{Emp})\ (Geos\ (Poi\ C)\ \text{add}\ \text{Emp})$
by *(simp add:Bet-Point-def)*
from $P19\ P36\ P37\ P38$ **have** $P39 : \text{Line-on } (Li\ E\ G)\ C \implies \text{Eq } (Geos\ (Lin\ (Li\ F\ G))\ \text{add}\ \text{Emp})\ (Geos\ (Lin\ (Li\ E\ G))\ \text{add}\ \text{Emp})$ **by** *(simp add:Line-unique)*
have $P40 : \text{Line-on } (Li\ F\ G)\ F$ **by** *(simp add:Line-on-rule)*
from $P39\ P40$ **have** $P41 : \text{Line-on } (Li\ E\ G)\ C \implies \text{Line-on } (Li\ E\ G)\ F$ **by**
(simp add:Line-on-trans)
from $P29\ P41$ **have** $P42 : \neg \text{Line-on } (Li\ E\ G)\ C$ **by** *blast*
from $P6\ P14\ P29\ P30\ P35\ P42$ **have** $P43 : \text{Line-on-Seg } (Li\ E\ G)\ (Se\ A\ C) \wedge \neg$
 $\text{Line-on-Seg } (Li\ E\ G)\ (Se\ F\ C) \vee \text{Line-on-Seg } (Li\ E\ G)\ (Se\ F\ C) \wedge \neg \text{Line-on-Seg}$
 $(Li\ E\ G)\ (Se\ A\ C)$ **by** *(simp add:Pachets-axiom)*
then have $\text{Line-on-Seg } (Li\ E\ G)\ (Se\ F\ C) \implies \exists p. \text{Line-on } (Li\ E\ G)\ p \wedge$
 $\text{Bet-Point } (Se\ F\ C)\ p$ **by** *(simp add:Line-on-Seg-rule)*
then obtain $D :: \text{Point}$ **where** $P44 : \text{Line-on-Seg } (Li\ E\ G)\ (Se\ F\ C) \implies \text{Line-on}$
 $(Li\ E\ G)\ D \wedge \text{Bet-Point } (Se\ F\ C)\ D$ **by** *blast*
from $P44$ **have** $P46 : \text{Line-on-Seg } (Li\ E\ G)\ (Se\ F\ C) \implies \text{Bet-Point } (Se\ F\ C)$
 D **by** *simp*
from $P46$ **have** $\text{Line-on-Seg } (Li\ E\ G)\ (Se\ F\ C) \implies \neg \text{Eq } (Geos\ (Poi\ D)\ \text{add}\ \text{Emp})\ (Geos\ (Poi\ F)\ \text{add}\ \text{Emp})$ **by** *(simp add:Bet-Point-def)*
from $P46$ **have** $P47 : \text{Line-on-Seg } (Li\ E\ G)\ (Se\ F\ C) \implies \text{Line-on } (Li\ F\ D)\ C$

by (*simp add:Line-Bet-on*)
have $P_{48} : \text{Line-on } (Li F D) F$ **by** (*simp add:Line-on-rule*)
have $P_{49} : \text{Line-on } (Li F G) F$ **by** (*simp add:Line-on-rule*)
from $P_{16} P_{19} P_{47} P_{48} P_{49}$ **have** $P_{50} : \text{Line-on-Seg } (Li E G) (Se F C) \implies$
 $\text{Eq } (\text{Geos } (\text{Lin } (Li F D)) \text{ add Emp}) (\text{Geos } (\text{Lin } (Li F G)) \text{ add Emp})$ **by** (*simp*
add:Line-unique)
have $P_{51} : \text{Line-on } (Li F D) D$ **by** (*simp add:Line-on-rule*)
from $P_{50} P_{51}$ **have** $P_{52} : \text{Line-on-Seg } (Li E G) (Se F C) \implies \text{Line-on } (Li F$
 $G) D$ **by** (*simp add:Line-on-trans*)
have $P_{53} : \text{Line-on } (Li F G) G$ **by** (*simp add:Line-on-rule*)
have $P_{54} : \text{Line-on } (Li E G) G$ **by** (*simp add:Line-on-rule*)
from P_{46} **have** $P_{55} : \text{Line-on-Seg } (Li E G) (Se F C) \implies \text{Eq } (\text{Geos } (Poi D)$
 $\text{add Emp}) (\text{Geos } (Poi G) \text{ add Emp})$
 $\implies \text{Bet-Point } (Se F C) G$ **by** (*simp add:Point-Eq*)
from P_{20} **have** $\text{Inv } (\text{Bet-Point } (Se G C) F) \wedge \text{Inv } (\text{Bet-Point } (Se C F) G)$ **by**
(*simp add:Bet-iff*)
then have $\neg \text{Bet-Point } (Se C F) G$ **by** (*simp add:Inv-def*)
then have $P_{56} : \neg \text{Bet-Point } (Se F C) G$ **by** (*blast intro:Bet-rev*)
from $P_{55} P_{56}$ **have** $P_{57} : \text{Line-on-Seg } (Li E G) (Se F C) \implies \neg \text{Eq } (\text{Geos } (Poi$
 $D) \text{ add Emp}) (\text{Geos } (Poi G) \text{ add Emp})$ **by** *blast*
from $P_{44} P_{52} P_{53} P_{54} P_{57}$ **have** $P_{58} : \text{Line-on-Seg } (Li E G) (Se F C) \implies$
 $\text{Eq } (\text{Geos } (\text{Lin } (Li E G)) \text{ add Emp}) (\text{Geos } (\text{Lin } (Li F G)) \text{ add Emp})$ **by** (*blast*
intro:Line-unique)
from P_{26} **have** $P_{59} : \text{Eq } (\text{Geos } (\text{Lin } (Li E G)) \text{ add Emp}) (\text{Geos } (\text{Lin } (Li G E))$
 $\text{add Emp})$ **by** (*simp add:Line-rev Eq-rev*)
from P_{27} **have** $P_{60} : \text{Eq } (\text{Geos } (\text{Lin } (Li F G)) \text{ add Emp}) (\text{Geos } (\text{Lin } (Li G F))$
 $\text{add Emp})$ **by** (*simp add:Line-rev Eq-rev*)
from $P_{58} P_{59} P_{60}$ **have** $P_{61} : \text{Line-on-Seg } (Li E G) (Se F C) \implies$
 $\text{Eq } (\text{Geos } (\text{Lin } (Li G E)) \text{ add Emp}) (\text{Geos } (\text{Lin } (Li G F)) \text{ add Emp})$ **by** (*blast*
intro:Eq-trans Eq-rev)
from $P_{24} P_{61}$ **have** $P_{62} : \neg \text{Line-on-Seg } (Li E G) (Se F C)$ **by** *blast*
from $P_{43} P_{62}$ **have** $\text{Line-on-Seg } (Li E G) (Se A C) \wedge \neg \text{Line-on-Seg } (Li E G)$
 $(Se F C)$ **by** *blast*
then have $\exists p. \text{Line-on } (Li E G) p \wedge \text{Bet-Point } (Se A C) p$ **by** (*simp add:Line-on-Seg-rule*)
thus $\exists p. \text{Bet-Point } (Se A C) p$ **by** *blast*
qed

lemma(*in Order-Rule*) *Line-Bet-not-on* :

assumes

$\text{Line-on } (Li p1 p2) p3$

$\neg \text{Line-on } (Li p1 p2) p4$

$\text{Bet-Point } (Se p3 p4) p5$

shows $\text{Inv } (\text{Line-on } (Li p1 p2) p5)$

proof –

from *assms* **have** $\neg \text{Eq } (\text{Geos } (Poi p5) \text{ add Emp}) (\text{Geos } (Poi p3) \text{ add Emp})$ **by**
(*simp add:Bet-Point-def*)

then have $P1 : \neg \text{Eq } (\text{Geos } (Poi p3) \text{ add Emp}) (\text{Geos } (Poi p5) \text{ add Emp})$ **by**
(*blast intro:Eq-rev*)

from *assms* **have** $P2 : \text{Line-on } (Li p3 p5) p4$ **by** (*simp add:Line-Bet-on*)

have $P3 : \text{Line-on } (Li\ p3\ p5)\ p3$ **by** (*simp add:Line-on-rule*)
have $P4 : \text{Line-on } (Li\ p3\ p5)\ p5$ **by** (*simp add:Line-on-rule*)
from *assms P1 P3 P4* **have** $P5 : \text{Line-on } (Li\ p1\ p2)\ p5 \implies \text{Eq } (\text{Geos } (\text{Lin } (Li\ p3\ p5))\ \text{add } \text{Emp})\ (\text{Geos } (\text{Lin } (Li\ p1\ p2))\ \text{add } \text{Emp})$ **by** (*simp add:Line-unique*)
from $P2\ P5$ **have** $P6 : \text{Line-on } (Li\ p1\ p2)\ p5 \implies \text{Line-on } (Li\ p1\ p2)\ p4$ **by** (*simp add:Line-on-trans*)
from *assms P6* **have** $\neg \text{Line-on } (Li\ p1\ p2)\ p5$ **by** *blast*
thus $\text{Inv } (\text{Line-on } (Li\ p1\ p2)\ p5)$ **by** (*simp add:Inv-def*)
qed

lemma(*in Order-Rule*) *Line-not-on-ex* :

assumes $N :$
 $\neg \text{Eq } (\text{Geos } (\text{Poi } p1)\ \text{add } \text{Emp})\ (\text{Geos } (\text{Poi } p2)\ \text{add } \text{Emp})$
 $\neg \text{Line-on } (Li\ p1\ p2)\ p3$
 $\text{Line-on } (Li\ p1\ p4)\ p2$
shows $\neg \text{Line-on } (Li\ p1\ p4)\ p3$

proof

assume $W : \text{Line-on } (Li\ p1\ p4)\ p3$
have $P1 : \text{Line-on } (Li\ p1\ p2)\ p2$ **by** (*simp add:Line-on-rule*)
have $P2 : \text{Line-on } (Li\ p1\ p2)\ p1$ **by** (*simp add:Line-on-rule*)
have $P3 : \text{Line-on } (Li\ p1\ p4)\ p1$ **by** (*simp add:Line-on-rule*)
from $N\ P1\ P2\ P3$ **have** $P4 : \text{Eq } (\text{Geos } (\text{Lin } (Li\ p1\ p4))\ \text{add } \text{Emp})\ (\text{Geos } (\text{Lin } (Li\ p1\ p2))\ \text{add } \text{Emp})$ **by** (*simp add:Line-unique*)
from $W\ P4$ **have** $P5 : \text{Line-on } (Li\ p1\ p2)\ p3$ **by** (*simp add:Line-on-trans*)
from $N\ P5$ **show** *False* **by** *simp*
qed

lemma(*in Order-Rule*) *Line-on-dens* :

assumes
 $\neg \text{Eq } (\text{Geos } (\text{Poi } p1)\ \text{add } \text{Emp})\ (\text{Geos } (\text{Poi } p3)\ \text{add } \text{Emp})$
 $\neg \text{Eq } (\text{Geos } (\text{Poi } p2)\ \text{add } \text{Emp})\ (\text{Geos } (\text{Poi } p4)\ \text{add } \text{Emp})$
 $\text{Line-on } (Li\ p1\ p2)\ p3$
 $\text{Line-on } (Li\ p1\ p4)\ p3$
shows $\text{Line-on } (Li\ p2\ p4)\ p3$

proof –

have $P1 : \text{Line-on } (Li\ p1\ p2)\ p1$ **by** (*simp add:Line-on-rule*)
have $P2 : \text{Line-on } (Li\ p1\ p4)\ p1$ **by** (*simp add:Line-on-rule*)
from *assms P1 P2* **have** $P3 : \text{Eq } (\text{Geos } (\text{Lin } (Li\ p1\ p2))\ \text{add } \text{Emp})\ (\text{Geos } (\text{Lin } (Li\ p1\ p4))\ \text{add } \text{Emp})$ **by** (*simp add:Line-unique*)
have $P4 : \text{Line-on } (Li\ p1\ p2)\ p2$ **by** (*simp add:Line-on-rule*)
from $P3\ P4$ **have** $P5 : \text{Line-on } (Li\ p1\ p4)\ p2$ **by** (*simp add:Line-on-trans*)
have $P6 : \text{Line-on } (Li\ p1\ p4)\ p4$ **by** (*simp add:Line-on-rule*)
have $P7 : \text{Line-on } (Li\ p2\ p4)\ p2$ **by** (*simp add:Line-on-rule*)
have $P8 : \text{Line-on } (Li\ p2\ p4)\ p4$ **by** (*simp add:Line-on-rule*)
from *assms P5 P6 P7 P8* **have** $P9 : \text{Eq } (\text{Geos } (\text{Lin } (Li\ p1\ p4))\ \text{add } \text{Emp})\ (\text{Geos } (\text{Lin } (Li\ p2\ p4))\ \text{add } \text{Emp})$ **by** (*simp add:Line-unique*)
from *assms P9* **show** $\text{Line-on } (Li\ p2\ p4)\ p3$ **by** (*simp add:Line-on-trans*)
qed

lemma(in *Order-Rule*) *Bet-case-lemma1* :

assumes

Line-on *l1* *A*

Line-on *l1* *B*

Line-on *l1* *C*

\neg *Bet-Point* (*Se B A*) *C*

\neg *Bet-Point* (*Se C B*) *A*

\neg *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi B*) *add Emp*)

\neg *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*)

\neg *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi A*) *add Emp*)

\neg *Line-on* (*Li A C*) *D*

Bet-Point (*Se B G*) *D*

shows $\exists p.$ *Line-on* (*Li A D*) *p* \wedge *Bet-Point* (*Se G C*) *p*

proof –

have *P1* : *Line-on* (*Li A C*) *A* **by** (*simp add:Line-on-rule*)

have *P2* : *Line-on* (*Li A C*) *C* **by** (*simp add:Line-on-rule*)

from *assms P1 P2* **have** *P3* : *Eq* (*Geos* (*Lin l1*) *add Emp*) (*Geos* (*Lin* (*Li A C*)) *add Emp*) **by** (*simp add:Line-unique*)

from *assms P3* **have** *P4* : *Line-on* (*Li A C*) *B* **by** (*simp add:Line-on-trans*)

have *P11* : *Line-on* (*Li B G*) *B* **by** (*simp add:Line-on-rule*)

from *assms P2 P4 P11* **have** *P12* : *Line-on* (*Li B G*) *C* \implies *Eq* (*Geos* (*Lin* (*Li B G*)) *add Emp*) (*Geos* (*Lin* (*Li A C*)) *add Emp*) **by** (*simp add:Line-unique*)

from *assms* **have** *P13* : *Line-on* (*Li B G*) *D* **by** (*simp add:Line-Bet-on*)

from *P12 P13* **have** *P14* : *Line-on* (*Li B G*) *C* \implies *Line-on* (*Li A C*) *D* **by** (*simp add:Line-on-trans*)

from *assms P14* **have** *P15* : \neg *Line-on* (*Li B G*) *C* **by** *blast*

have *P16* : *Line-on* (*Li A D*) *A* **by** (*simp add:Line-on-rule*)

from *assms P1 P4 P16* **have** *P17* : *Line-on* (*Li A D*) *B* \implies *Eq* (*Geos* (*Lin* (*Li A D*)) *add Emp*) (*Geos* (*Lin* (*Li A C*)) *add Emp*) **by** (*simp add:Line-unique*)

have *P18* : *Line-on* (*Li A D*) *D* **by** (*simp add:Line-on-rule*)

from *P17 P18* **have** *P19* : *Line-on* (*Li A D*) *B* \implies *Line-on* (*Li A C*) *D* **by** (*simp add:Line-on-trans*)

from *assms P19* **have** *P20* : \neg *Line-on* (*Li A D*) *B* **by** *blast*

from *assms P1 P2 P16* **have** *P21* : *Line-on* (*Li A D*) *C* \implies *Eq* (*Geos* (*Lin* (*Li A D*)) *add Emp*) (*Geos* (*Lin* (*Li A C*)) *add Emp*) **by** (*simp add:Line-unique*)

from *P18 P21* **have** *P22* : *Line-on* (*Li A D*) *C* \implies *Line-on* (*Li A C*) *D* **by** (*simp add:Line-on-trans*)

from *assms P22* **have** *P23* : \neg *Line-on* (*Li A D*) *C* **by** *blast*

from *assms P1 P4 P11* **have** *P24* : *Line-on* (*Li B G*) *A* \implies *Eq* (*Geos* (*Lin* (*Li B G*)) *add Emp*) (*Geos* (*Lin* (*Li A C*)) *add Emp*) **by** (*simp add:Line-unique*)

from *P13 P24* **have** *P25* : *Line-on* (*Li B G*) *A* \implies *Line-on* (*Li A C*) *D* **by** (*simp add:Line-on-trans*)

from *assms P25* **have** *P26* : \neg *Line-on* (*Li B G*) *A* **by** *blast*

have *P27* : *Line-on* (*Li B G*) *G* **by** (*simp add:Line-on-rule*)

from *assms* **have** *P28* : \neg *Eq* (*Geos* (*Poi G*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) **by** (*simp add:Bet-Point-def*)

from *P13 P18 P27 P28* **have** *P29* : *Line-on* (*Li A D*) *G* \implies *Eq* (*Geos* (*Lin* (*Li A D*)) *add Emp*) (*Geos* (*Lin* (*Li B G*)) *add Emp*) **by** (*simp add:Line-unique*)

from *P16 P29* **have** *P30* : *Line-on* (*Li A D*) *G* \implies *Line-on* (*Li B G*) *A* **by**

(simp add:Line-on-trans)
from $P26\ P30$ **have** $P31 : \neg \text{Line-on } (Li\ A\ D)\ G$ **by** *blast*
from *assms* $P15\ P18\ P20\ P23\ P31$ **have** $P32 : \text{Line-on-Seg } (Li\ A\ D)\ (Se\ B\ C)$
 $\wedge \neg \text{Line-on-Seg } (Li\ A\ D)\ (Se\ G\ C)$
 $\vee \text{Line-on-Seg } (Li\ A\ D)\ (Se\ G\ C) \wedge \neg \text{Line-on-Seg } (Li\ A\ D)\ (Se\ B\ C)$ **by**
(simp add:Pachets-axiom)
have $\text{Line-on-Seg } (Li\ A\ D)\ (Se\ B\ C) \implies \exists p. \text{Line-on } (Li\ A\ D)\ p \wedge \text{Bet-Point}$
 $(Se\ B\ C)\ p$ **by** *(simp add:Line-on-Seg-rule)*
then obtain $A2 :: \text{Point}$ **where** $P33 : \text{Line-on-Seg } (Li\ A\ D)\ (Se\ B\ C) \implies$
 $\text{Line-on } (Li\ A\ D)\ A2 \wedge \text{Bet-Point } (Se\ B\ C)\ A2$ **by** *blast*
from *assms* **have** $P34 : \neg \text{Eq } (\text{Geos } (Poi\ A)\ \text{add Emp})\ (\text{Geos } (Poi\ C)\ \text{add Emp})$
by *(blast intro:Eq-rev)*
from *assms* $P34$ **have** $P35 : \neg \text{Eq } (\text{Geos } (Lin\ (Li\ A\ C))\ \text{add Emp})\ (\text{Geos } (Lin$
 $(Li\ A\ D))\ \text{add Emp})$ **by** *(simp add:Line-not-Eq)*
have $P36 : \text{Line-on } (Li\ B\ C)\ B$ **by** *(simp add:Line-on-rule)*
have $P37 : \text{Line-on } (Li\ B\ C)\ C$ **by** *(simp add:Line-on-rule)*
from *assms* $P2\ P4\ P36\ P37$ **have** $P38 : \text{Eq } (\text{Geos } (Lin\ (Li\ B\ C))\ \text{add Emp})$
 $(\text{Geos } (Lin\ (Li\ A\ C))\ \text{add Emp})$ **by** *(simp add:Line-unique)*
from $P33$ **have** $P39 : \text{Line-on-Seg } (Li\ A\ D)\ (Se\ B\ C) \implies \text{Bet-Point } (Se\ B\ C)$
 $A2$ **by** *simp*
then have $P40 : \text{Line-on-Seg } (Li\ A\ D)\ (Se\ B\ C) \implies \text{Line-on } (Li\ B\ C)\ A2$ **by**
(simp add:Line-Bet-on)
from $P38\ P40$ **have** $P41 : \text{Line-on-Seg } (Li\ A\ D)\ (Se\ B\ C) \implies \text{Line-on } (Li\ A$
 $C)\ A2$ **by** *(simp add:Line-on-trans)*
from $P1\ P16\ P33\ P35\ P41$ **have** $P42 : \text{Line-on-Seg } (Li\ A\ D)\ (Se\ B\ C) \implies \text{Eq}$
 $(\text{Geos } (Poi\ A2)\ \text{add Emp})\ (\text{Geos } (Poi\ A)\ \text{add Emp})$ **by** *(simp add:Line-unique-Point)*
from $P39\ P42$ **have** $P43 : \text{Line-on-Seg } (Li\ A\ D)\ (Se\ B\ C) \implies \text{Bet-Point } (Se\ B$
 $C)\ A$ **by** *(simp add:Point-Eq)*
from *assms* **have** $P44 : \neg \text{Bet-Point } (Se\ B\ C)\ A$ **by** *(blast intro:Bet-rev)*
from $P43\ P44$ **have** $P45 : \neg \text{Line-on-Seg } (Li\ A\ D)\ (Se\ B\ C)$ **by** *blast*
from $P32\ P45$ **have** $\text{Line-on-Seg } (Li\ A\ D)\ (Se\ G\ C) \wedge \neg \text{Line-on-Seg } (Li\ A\ D)$
 $(Se\ B\ C)$ **by** *blast*
thus $\exists p. \text{Line-on } (Li\ A\ D)\ p \wedge \text{Bet-Point } (Se\ G\ C)\ p$ **by** *(simp add:Line-on-Seg-rule)*
qed

lemma(in *Order-Rule*) *Bet-case-lemma2* :

assumes

Line-on $l1\ A$

Line-on $l1\ B$

Line-on $l1\ C$

$\neg \text{Bet-Point } (Se\ B\ A)\ C$

$\neg \text{Bet-Point } (Se\ C\ B)\ A$

$\neg \text{Eq } (\text{Geos } (Poi\ A)\ \text{add Emp})\ (\text{Geos } (Poi\ B)\ \text{add Emp})$

$\neg \text{Eq } (\text{Geos } (Poi\ B)\ \text{add Emp})\ (\text{Geos } (Poi\ C)\ \text{add Emp})$

$\neg \text{Eq } (\text{Geos } (Poi\ C)\ \text{add Emp})\ (\text{Geos } (Poi\ A)\ \text{add Emp})$

shows *Bet-Point* $(Se\ A\ C)\ B$

proof –

have $P1 : \text{Line-on } (Li\ A\ C)\ A$ **by** *(simp add:Line-on-rule)*

have $P2 : \text{Line-on } (Li\ A\ C)\ C$ **by** *(simp add:Line-on-rule)*

from *assms* $P1\ P2$ **have** $P3 : Eq\ (Geos\ (Lin\ l1)\ add\ Emp)\ (Geos\ (Lin\ (Li\ A\ C))\ add\ Emp)$ **by** (*simp add:Line-unique*)
from *assms* $P3$ **have** $P4 : Line-on\ (Li\ A\ C)\ B$ **by** (*simp add:Line-on-trans*)
have $\exists p\ q\ r. \neg Line-on\ (Li\ A\ C)\ p \wedge \neg Line-on\ (Li\ A\ C)\ q \wedge \neg Line-on\ (Li\ A\ C)\ r$
 $\wedge \neg Eq\ (Geos\ (Poi\ p)\ add\ Emp)\ (Geos\ (Poi\ q)\ add\ Emp) \wedge \neg Eq\ (Geos\ (Poi\ q)\ add\ Emp)\ (Geos\ (Poi\ r)\ add\ Emp)$
 $\wedge \neg Eq\ (Geos\ (Poi\ r)\ add\ Emp)\ (Geos\ (Poi\ p)\ add\ Emp)$ **by** (*blast intro:Line-not-on-exist*)
then obtain $D :: Point$ **where** $P5 : \neg Line-on\ (Li\ A\ C)\ D$ **by** *blast*
from $P4$ **have** $P6 : Eq\ (Geos\ (Poi\ B)\ add\ Emp)\ (Geos\ (Poi\ D)\ add\ Emp) \implies Line-on\ (Li\ A\ C)\ D$ **by** (*simp add:Point-Eq*)
from $P5\ P6$ **have** $P7 : \neg Eq\ (Geos\ (Poi\ B)\ add\ Emp)\ (Geos\ (Poi\ D)\ add\ Emp)$ **by** *blast*
have $P8 : Line-on\ (Li\ B\ D)\ B$ **by** (*simp add:Line-on-rule*)
have $P9 : Line-on\ (Li\ B\ D)\ D$ **by** (*simp add:Line-on-rule*)
from $P7\ P8\ P9$ **have** $\exists p. Bet-Point\ (Se\ B\ p)\ D \wedge Line-on\ (Li\ B\ D)\ p$ **by** (*simp add:Bet-extension*)
then obtain $G :: Point$ **where** $P10 : Bet-Point\ (Se\ B\ G)\ D$ **by** *blast*
from *assms* $P5\ P10$ **have** $\exists p. Line-on\ (Li\ A\ D)\ p \wedge Bet-Point\ (Se\ G\ C)\ p$ **by** (*simp add:Bet-case-lemma1*)
then obtain $E :: Point$ **where** $P11 : Line-on\ (Li\ A\ D)\ E \wedge Bet-Point\ (Se\ G\ C)\ E$ **by** *blast*
from *assms* **have** $P12 : \neg Bet-Point\ (Se\ B\ C)\ A$ **by** (*blast intro:Bet-rev*)
from *assms* **have** $P13 : \neg Bet-Point\ (Se\ A\ B)\ C$ **by** (*blast intro:Bet-rev*)
from *assms* **have** $P14 : \neg Eq\ (Geos\ (Poi\ A)\ add\ Emp)\ (Geos\ (Poi\ C)\ add\ Emp)$ **by** (*blast intro:Eq-rev*)
from *assms* **have** $P15 : \neg Eq\ (Geos\ (Poi\ B)\ add\ Emp)\ (Geos\ (Poi\ A)\ add\ Emp)$ **by** (*blast intro:Eq-rev*)
from *assms* **have** $P16 : \neg Eq\ (Geos\ (Poi\ C)\ add\ Emp)\ (Geos\ (Poi\ B)\ add\ Emp)$ **by** (*blast intro:Eq-rev*)
from $P14$ **have** $P17 : Eq\ (Geos\ (Lin\ (Li\ A\ C))\ add\ Emp)\ (Geos\ (Lin\ (Li\ C\ A))\ add\ Emp)$ **by** (*simp add:Line-rev*)
from $P5\ P17$ **have** $P18 : \neg Line-on\ (Li\ C\ A)\ D$ **by** (*simp add:Line-not-on-trans*)
from *assms* $P10\ P12\ P13\ P14\ P15\ P16\ P18$ **have** $\exists p. Line-on\ (Li\ C\ D)\ p \wedge Bet-Point\ (Se\ G\ A)\ p$ **by** (*simp add:Bet-case-lemma1*)
then obtain $F :: Point$ **where** $P19 : Line-on\ (Li\ C\ D)\ F \wedge Bet-Point\ (Se\ G\ A)\ F$ **by** *blast*
have $P20 : Line-on\ (Li\ B\ G)\ B$ **by** (*simp add:Line-on-rule*)
have $P21 : Line-on\ (Li\ B\ G)\ G$ **by** (*simp add:Line-on-rule*)
from $P10$ **have** $P22 : \neg Eq\ (Geos\ (Poi\ B)\ add\ Emp)\ (Geos\ (Poi\ G)\ add\ Emp)$ **by** (*simp add:Bet-Point-def*)
from $P4\ P20\ P21\ P22$ **have** $P23 : Line-on\ (Li\ A\ C)\ G \implies Eq\ (Geos\ (Lin\ (Li\ B\ G))\ add\ Emp)\ (Geos\ (Lin\ (Li\ A\ C))\ add\ Emp)$ **by** (*simp add:Line-unique*)
from $P10$ **have** $P24 : Line-on\ (Li\ B\ G)\ D$ **by** (*simp add:Line-Bet-on*)
from $P23\ P24$ **have** $P25 : Line-on\ (Li\ A\ C)\ G \implies Line-on\ (Li\ A\ C)\ D$ **by** (*simp add:Line-on-trans*)
from $P5\ P25$ **have** $P26 : \neg Line-on\ (Li\ A\ C)\ G$ **by** *blast*
from $P11$ **have** $P27 : Bet-Point\ (Se\ C\ G)\ E$ **by** (*blast intro:Bet-rev*)

have $P28 : \text{Line-on } (Li\ C\ G)\ C$ **by** (*simp add:Line-on-rule*)
from *assms P1 P2 P28* **have** $P29 : \text{Line-on } (Li\ C\ G)\ A \implies \text{Eq } (\text{Geos } (\text{Lin } (Li\ C\ G))\ \text{add Emp})\ (\text{Geos } (\text{Lin } (Li\ A\ C))\ \text{add Emp})$ **by** (*simp add:Line-unique*)
have $P30 : \text{Line-on } (Li\ C\ G)\ G$ **by** (*simp add:Line-on-rule*)
from $P29\ P30$ **have** $P31 : \text{Line-on } (Li\ C\ G)\ A \implies \text{Line-on } (Li\ A\ C)\ G$ **by** (*simp add:Line-on-trans*)
from $P26\ P31$ **have** $P32 : \neg \text{Line-on } (Li\ C\ G)\ A$ **by** *blast*
from $P27\ P32$ **have** $\neg \text{Eq } (\text{Geos } (\text{Lin } (Li\ A\ E))\ \text{add Emp})\ (\text{Geos } (\text{Lin } (Li\ A\ G))\ \text{add Emp})$ **by** (*simp add:Line-Bet-not-Eq*)
then **have** $P33 : \neg \text{Eq } (\text{Geos } (\text{Lin } (Li\ A\ G))\ \text{add Emp})\ (\text{Geos } (\text{Lin } (Li\ A\ E))\ \text{add Emp})$ **by** (*blast intro:Eq-rev*)
from $P19$ **have** $P34 : \text{Bet-Point } (Se\ A\ G)\ F$ **by** (*blast intro:Bet-rev*)
then **have** $P35 : \neg \text{Eq } (\text{Geos } (Poi\ A)\ \text{add Emp})\ (\text{Geos } (Poi\ G)\ \text{add Emp})$ **by** (*simp add:Bet-Point-def*)
from $P27$ **have** $P36 : \text{Line-on } (Li\ C\ G)\ E$ **by** (*simp add:Line-Bet-on*)
then **have** $P37 : \text{Eq } (\text{Geos } (Poi\ E)\ \text{add Emp})\ (\text{Geos } (Poi\ A)\ \text{add Emp}) \implies \text{Line-on } (Li\ C\ G)\ A$ **by** (*simp add:Point-Eq*)
from $P32\ P37$ **have** $P38 : \neg \text{Eq } (\text{Geos } (Poi\ A)\ \text{add Emp})\ (\text{Geos } (Poi\ E)\ \text{add Emp})$ **by** (*blast intro:Eq-rev*)
from $P33\ P35\ P38$ **have** $P39 : \neg \text{Line-on } (Li\ A\ G)\ E$ **by** (*simp add:Line-not-Eq-on*)
from $P14\ P26\ P35$ **have** $P40 : \neg \text{Line-on } (Li\ A\ G)\ C$ **by** (*blast intro:Line-on-rev*)
from $P34\ P40$ **have** $P41 : \neg \text{Eq } (\text{Geos } (\text{Lin } (Li\ C\ F))\ \text{add Emp})\ (\text{Geos } (\text{Lin } (Li\ C\ G))\ \text{add Emp})$ **by** (*simp add:Line-Bet-not-Eq*)
from $P34$ **have** $P42 : \text{Line-on } (Li\ A\ G)\ F$ **by** (*simp add:Line-Bet-on*)
then **have** $P43 : \text{Eq } (\text{Geos } (Poi\ F)\ \text{add Emp})\ (\text{Geos } (Poi\ C)\ \text{add Emp}) \implies \text{Line-on } (Li\ A\ G)\ C$ **by** (*simp add:Point-Eq*)
from $P40\ P43$ **have** $P44 : \neg \text{Eq } (\text{Geos } (Poi\ C)\ \text{add Emp})\ (\text{Geos } (Poi\ F)\ \text{add Emp})$ **by** (*blast intro:Eq-rev*)
from $P27$ **have** $P45 : \neg \text{Eq } (\text{Geos } (Poi\ C)\ \text{add Emp})\ (\text{Geos } (Poi\ G)\ \text{add Emp})$ **by** (*simp add:Bet-Point-def*)
from $P41\ P44\ P45$ **have** $P46 : \neg \text{Line-on } (Li\ C\ F)\ G$ **by** (*simp add:Line-not-Eq-on*)
from $P35$ **have** $P47 : \text{Eq } (\text{Geos } (\text{Lin } (Li\ A\ G))\ \text{add Emp})\ (\text{Geos } (\text{Lin } (Li\ G\ A))\ \text{add Emp})$ **by** (*simp add:Line-rev*)
from $P40\ P47$ **have** $P48 : \neg \text{Line-on } (Li\ G\ A)\ C$ **by** (*simp add:Line-not-on-trans*)
from $P19$ **have** $P49 : \text{Bet-Point } (Se\ G\ A)\ F$ **by** *simp*
from $P48\ P49$ **have** $P50 : \neg \text{Eq } (\text{Geos } (\text{Lin } (Li\ C\ F))\ \text{add Emp})\ (\text{Geos } (\text{Lin } (Li\ C\ A))\ \text{add Emp})$ **by** (*simp add:Line-Bet-not-Eq*)
from *assms P44 P50* **have** $P51 : \neg \text{Line-on } (Li\ C\ F)\ A$ **by** (*simp add:Line-not-Eq-on*)
have $P52 : \text{Line-on } (Li\ C\ F)\ C$ **by** (*simp add:Line-on-rule*)
from $P27$ **have** $P53 : \neg \text{Eq } (\text{Geos } (Poi\ E)\ \text{add Emp})\ (\text{Geos } (Poi\ C)\ \text{add Emp})$ **by** (*simp add:Bet-Point-def*)
from $P28\ P36\ P52\ P53$ **have** $P54 : \text{Line-on } (Li\ C\ F)\ E \implies \text{Eq } (\text{Geos } (\text{Lin } (Li\ C\ G))\ \text{add Emp})\ (\text{Geos } (\text{Lin } (Li\ C\ F))\ \text{add Emp})$ **by** (*simp add:Line-unique*)
from $P30\ P54$ **have** $P55 : \text{Line-on } (Li\ C\ F)\ E \implies \text{Line-on } (Li\ C\ F)\ G$ **by** (*simp add:Line-on-trans*)
from $P46\ P55$ **have** $P56 : \neg \text{Line-on } (Li\ C\ F)\ E$ **by** *blast*
have $P57 : \text{Line-on } (Li\ C\ F)\ F$ **by** (*simp add:Line-on-rule*)
from $P34\ P39\ P46\ P51\ P56\ P57$ **have** $P58 : \text{Line-on-Seg } (Li\ C\ F)\ (Se\ A\ E) \wedge \neg \text{Line-on-Seg } (Li\ C\ F)\ (Se\ G\ E)$

\vee *Line-on-Seg* (*Li C F*) (*Se G E*) \wedge \neg *Line-on-Seg* (*Li C F*) (*Se A E*) **by** (*simp add:Pachets-axiom*)
have *Line-on-Seg* (*Li C F*) (*Se G E*) \implies $\exists p.$ *Line-on* (*Li C F*) *p* \wedge *Bet-Point* (*Se G E*) *p* **by** (*simp add:Line-on-Seg-rule*)
then obtain *D2* :: *Point* **where** *P59* : *Line-on-Seg* (*Li C F*) (*Se G E*) \implies *Line-on* (*Li C F*) *D2* \wedge *Bet-Point* (*Se G E*) *D2* **by** *blast*
then have *P60* : *Line-on-Seg* (*Li C F*) (*Se G E*) \implies *Bet-Point* (*Se G E*) *D2* **by** *simp*
then have *P61* : *Line-on-Seg* (*Li C F*) (*Se G E*) \implies *Line-on* (*Li G E*) *D2* **by** (*simp add:Line-Bet-on*)
have *P62* : *Line-on* (*Li G E*) *G* **by** (*simp add:Line-on-rule*)
have *P63* : *Line-on* (*Li G E*) *E* **by** (*simp add:Line-on-rule*)
from *P27* **have** *P64* : \neg *Eq* (*Geos* (*Poi G*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) **by** (*simp add:Bet-Point-def*)
from *P27* **have** *P66* : *Line-on* (*Li G E*) *C* **by** (*simp add:Line-Bet-on*)
from *P59* **have** *P67* : *Line-on-Seg* (*Li C F*) (*Se G E*) \implies *Line-on* (*Li C F*) *D2* **by** *simp*
from *P60* **have** *P68* : *Line-on-Seg* (*Li C F*) (*Se G E*) \implies *Eq* (*Geos* (*Poi D2*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) \implies *Bet-Point* (*Se G E*) *C* **by** (*simp add:Point-Eq*)
from *P27* **have** *Inv* (*Bet-Point* (*Se G E*) *C*) \wedge *Inv* (*Bet-Point* (*Se E C*) *G*) **by** (*simp add:Bet-iff*)
then have *P69* : \neg *Bet-Point* (*Se G E*) *C* \wedge \neg *Bet-Point* (*Se E C*) *G* **by** (*simp add:Inv-def*)
from *P68* *P69* **have** *P70* : *Line-on-Seg* (*Li C F*) (*Se G E*) \implies \neg *Eq* (*Geos* (*Poi D2*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) **by** *blast*
from *P52* *P61* *P66* *P67* *P70* **have** *P71* : *Line-on-Seg* (*Li C F*) (*Se G E*) \implies *Eq* (*Geos* (*Lin* (*Li G E*)) *add Emp*) (*Geos* (*Lin* (*Li C F*)) *add Emp*) **by** (*simp add:Line-unique*)
from *P63* *P71* **have** *P72* : *Line-on-Seg* (*Li C F*) (*Se G E*) \implies *Line-on* (*Li C F*) *E* **by** (*simp add:Line-on-trans*)
from *P56* *P72* **have** *P73* : \neg *Line-on-Seg* (*Li C F*) (*Se G E*) **by** *blast*
from *P58* *P73* **have** *Line-on-Seg* (*Li C F*) (*Se A E*) \wedge \neg *Line-on-Seg* (*Li C F*) (*Se G E*) **by** *blast*
then have $\exists p.$ *Line-on* (*Li C F*) *p* \wedge *Bet-Point* (*Se A E*) *p* **by** (*simp add:Line-on-Seg-rule*)
then obtain *D3* :: *Point* **where** *P74* : *Line-on* (*Li C F*) *D3* \wedge *Bet-Point* (*Se A E*) *D3* **by** *blast*
then have *P75* : *Line-on* (*Li C F*) *D3* **by** *simp*
from *P74* **have** *P76* : *Bet-Point* (*Se A E*) *D3* **by** *simp*
then have *P77* : *Line-on* (*Li A E*) *D3* **by** (*simp add:Line-Bet-on*)
from *P19* **have** *P78* : *Line-on* (*Li C D*) *F* **by** *simp*
from *P2* **have** *P79* : *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) \implies *Line-on* (*Li A C*) *D* **by** (*simp add:Point-Eq*)
from *P5* *P79* **have** *P80* : \neg *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) **by** *blast*
from *P44* *P78* *P80* **have** *P81* : *Line-on* (*Li C F*) *D* **by** (*simp add:Line-on-rev*)
from *P11* **have** *P82* : *Line-on* (*Li A D*) *E* **by** *simp*
from *P1* **have** *P83* : *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) \implies *Line-on* (*Li A C*) *D* **by** (*simp add:Point-Eq*)

from $P5\ P83$ **have** $P84 : \neg \text{Eq} (\text{Geos} (\text{Poi } A) \text{ add Emp}) (\text{Geos} (\text{Poi } D) \text{ add Emp})$ **by** *blast*
from $P38\ P82\ P84$ **have** $P85 : \text{Line-on} (\text{Li } A\ E)\ D$ **by** (*simp add:Line-on-rev*)
have $P86 : \text{Line-on} (\text{Li } A\ E)\ E$ **by** (*simp add:Line-on-rule*)
then have $P87 : \text{Eq} (\text{Geos} (\text{Lin} (\text{Li } A\ E)) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li } C\ F)) \text{ add Emp}) \implies \text{Line-on} (\text{Li } C\ F)\ E$ **by** (*simp add:Line-on-trans*)
from $P56\ P87$ **have** $P88 : \neg \text{Eq} (\text{Geos} (\text{Lin} (\text{Li } A\ E)) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li } C\ F)) \text{ add Emp})$ **by** *blast*
from $P75\ P77\ P81\ P85\ P88$ **have** $P89 : \text{Eq} (\text{Geos} (\text{Poi } D3) \text{ add Emp}) (\text{Geos} (\text{Poi } D) \text{ add Emp})$ **by** (*simp add:Line-unique-Point*)
from $P76\ P89$ **have** $P90 : \text{Bet-Point} (\text{Se } A\ E)\ D$ **by** (*simp add:Point-Eq*)
have $P91 : \text{Line-on} (\text{Li } A\ E)\ A$ **by** (*simp add:Line-on-rule*)
from *assms* $P1\ P2\ P91$ **have** $P92 : \text{Line-on} (\text{Li } A\ E)\ C \implies \text{Eq} (\text{Geos} (\text{Lin} (\text{Li } A\ E)) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li } A\ C)) \text{ add Emp})$ **by** (*simp add:Line-unique*)
from $P85\ P92$ **have** $P93 : \text{Line-on} (\text{Li } A\ E)\ C \implies \text{Line-on} (\text{Li } A\ C)\ D$ **by** (*simp add:Line-on-trans*)
from $P5\ P93$ **have** $P94 : \neg \text{Line-on} (\text{Li } A\ E)\ C$ **by** *blast*
from *assms* $P1\ P4\ P20$ **have** $P95 : \text{Line-on} (\text{Li } B\ G)\ A \implies \text{Eq} (\text{Geos} (\text{Lin} (\text{Li } B\ G)) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li } A\ C)) \text{ add Emp})$ **by** (*simp add:Line-unique*)
from $P24\ P95$ **have** $P96 : \text{Line-on} (\text{Li } B\ G)\ A \implies \text{Line-on} (\text{Li } A\ C)\ D$ **by** (*simp add:Line-on-trans*)
from $P5\ P96$ **have** $P97 : \neg \text{Line-on} (\text{Li } B\ G)\ A$ **by** *blast*
from *assms* $P2\ P4\ P20$ **have** $P98 : \text{Line-on} (\text{Li } B\ G)\ C \implies \text{Eq} (\text{Geos} (\text{Lin} (\text{Li } B\ G)) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li } A\ C)) \text{ add Emp})$ **by** (*simp add:Line-unique*)
from $P24\ P98$ **have** $P99 : \text{Line-on} (\text{Li } B\ G)\ C \implies \text{Line-on} (\text{Li } A\ C)\ D$ **by** (*simp add:Line-on-trans*)
from $P5\ P99$ **have** $P100 : \neg \text{Line-on} (\text{Li } B\ G)\ C$ **by** *blast*
from $P21\ P62\ P63\ P64$ **have** $P101 : \text{Line-on} (\text{Li } B\ G)\ E \implies \text{Eq} (\text{Geos} (\text{Lin} (\text{Li } G\ E)) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li } B\ G)) \text{ add Emp})$ **by** (*simp add:Line-unique*)
from $P66\ P101$ **have** $P102 : \text{Line-on} (\text{Li } B\ G)\ E \implies \text{Line-on} (\text{Li } B\ G)\ C$ **by** (*simp add:Line-on-trans*)
from $P100\ P102$ **have** $P103 : \neg \text{Line-on} (\text{Li } B\ G)\ E$ **by** *blast*
from $P24\ P90\ P94\ P97\ P100\ P103$ **have** $P104 : \text{Line-on-Seg} (\text{Li } B\ G) (\text{Se } A\ C) \wedge \neg \text{Line-on-Seg} (\text{Li } B\ G) (\text{Se } E\ C) \vee \text{Line-on-Seg} (\text{Li } B\ G) (\text{Se } E\ C) \wedge \neg \text{Line-on-Seg} (\text{Li } B\ G) (\text{Se } A\ C)$ **by** (*simp add:Pachets-axiom*)
have $\text{Line-on-Seg} (\text{Li } B\ G) (\text{Se } E\ C) \implies \exists p. \text{Line-on} (\text{Li } B\ G)\ p \wedge \text{Bet-Point} (\text{Se } E\ C)\ p$ **by** (*simp add:Line-on-Seg-rule*)
then obtain $B2 :: \text{Point}$ **where** $P105 : \text{Line-on-Seg} (\text{Li } B\ G) (\text{Se } E\ C) \implies \text{Line-on} (\text{Li } B\ G)\ B2 \wedge \text{Bet-Point} (\text{Se } E\ C)\ B2$ **by** *blast*
then have $P106 : \text{Line-on-Seg} (\text{Li } B\ G) (\text{Se } E\ C) \implies \text{Bet-Point} (\text{Se } E\ C)\ B2$ **by** *simp*
then have $P107 : \text{Line-on-Seg} (\text{Li } B\ G) (\text{Se } E\ C) \implies \text{Line-on} (\text{Li } E\ C)\ B2$ **by** (*simp add:Line-Bet-on*)
from $P105$ **have** $P108 : \text{Line-on-Seg} (\text{Li } B\ G) (\text{Se } E\ C) \implies \text{Line-on} (\text{Li } B\ G)\ B2$ **by** *simp*
have $P109 : \text{Line-on} (\text{Li } E\ C)\ E$ **by** (*simp add:Line-on-rule*)
have $P110 : \text{Line-on} (\text{Li } E\ C)\ C$ **by** (*simp add:Line-on-rule*)
from $P28\ P36\ P53\ P109\ P110$ **have** $P111 : \text{Eq} (\text{Geos} (\text{Lin} (\text{Li } C\ G)) \text{ add Emp})$

(*Geos (Lin (Li E C)) add Emp*) **by** (*simp add:Line-unique*)
from *P30 P111* **have** *P112 : Line-on (Li E C) G* **by** (*simp add:Line-on-trans*)
from *P106* **have** *P113 : Line-on-Seg (Li B G) (Se E C) \implies Eq (Geos (Poi B2) add Emp) (Geos (Poi G) add Emp) \implies*
Bet-Point (Se E C) G **by** (*simp add:Point-Eq*)
from *P69 P113* **have** *P114 : Line-on-Seg (Li B G) (Se E C) \implies \neg Eq (Geos (Poi B2) add Emp) (Geos (Poi G) add Emp)* **by** *blast*
from *P21 P107 P108 P112 P114* **have** *P115 : Line-on-Seg (Li B G) (Se E C) \implies Eq (Geos (Lin (Li E C)) add Emp) (Geos (Lin (Li B G)) add Emp)* **by** (*simp add:Line-unique*)
from *P109 P115* **have** *P116 : Line-on-Seg (Li B G) (Se E C) \implies Line-on (Li B G) E* **by** (*simp add:Line-on-trans*)
from *P103 P116* **have** *P117 : \neg Line-on-Seg (Li B G) (Se E C)* **by** *blast*
from *P104 P117* **have** *Line-on-Seg (Li B G) (Se A C)* **by** *blast*
then **have** $\exists p.$ *Line-on (Li B G) p \wedge Bet-Point (Se A C) p* **by** (*simp add:Line-on-Seg-rule*)
then **obtain** *B3 :: Point* **where** *P118 : Line-on (Li B G) B3 \wedge Bet-Point (Se A C) B3* **by** *blast*
from *P24* **have** *P119 : Eq (Geos (Lin (Li B G)) add Emp) (Geos (Lin (Li A C)) add Emp) \implies Line-on (Li A C) D* **by** (*simp add:Line-on-trans*)
from *P5 P119* **have** *P120 : \neg Eq (Geos (Lin (Li B G)) add Emp) (Geos (Lin (Li A C)) add Emp)* **by** *blast*
from *P118* **have** *P121 : Line-on (Li B G) B3* **by** *simp*
from *P118* **have** *P122 : Bet-Point (Se A C) B3* **by** *simp*
then **have** *P123 : Line-on (Li A C) B3* **by** (*simp add:Line-Bet-on*)
from *P4 P20 P120 P121 P123* **have** *P124 : Eq (Geos (Poi B3) add Emp) (Geos (Poi B) add Emp)* **by** (*simp add:Line-unique-Point*)
from *P122 P124* **show** *Bet-Point (Se A C) B* **by** (*simp add:Point-Eq*)
qed

Theorem4

lemma(in *Order-Rule*) *Bet-case* :

assumes

Line-on l1 A

Line-on l1 B

Line-on l1 C

\neg *Eq (Geos (Poi A) add Emp) (Geos (Poi B) add Emp)*

\neg *Eq (Geos (Poi B) add Emp) (Geos (Poi C) add Emp)*

\neg *Eq (Geos (Poi C) add Emp) (Geos (Poi A) add Emp)*

shows *Bet-Point (Se A C) B \vee Bet-Point (Se C B) A \vee Bet-Point (Se B A) C*

proof –

from *assms* **have** *P1 : \neg Bet-Point (Se B A) C \wedge \neg Bet-Point (Se C B) A \implies Bet-Point (Se A C) B* **by** (*simp add:Bet-case-lemma2*)

from *assms* **have** *P2 : \neg Bet-Point (Se C B) A \wedge \neg Bet-Point (Se A C) B \implies Bet-Point (Se B A) C* **by** (*simp add:Bet-case-lemma2*)

from *assms* **have** *P3 : \neg Bet-Point (Se A C) B \wedge \neg Bet-Point (Se B A) C \implies Bet-Point (Se C B) A* **by** (*simp add:Bet-case-lemma2*)

from *P1 P2 P3* **show** *Bet-Point (Se A C) B \vee Bet-Point (Se C B) A \vee Bet-Point (Se B A) C* **by** *blast*

qed

lemma(in *Order-Rule*) *Bet-case-fact* :

assumes

$Bet\text{-}Point (Se\ A\ C)\ B \vee Bet\text{-}Point (Se\ C\ B)\ A \vee Bet\text{-}Point (Se\ B\ A)\ C$

shows

$Bet\text{-}Point (Se\ A\ C)\ B \wedge \neg Bet\text{-}Point (Se\ C\ B)\ A \wedge \neg Bet\text{-}Point (Se\ B\ A)\ C$
 $\vee \neg Bet\text{-}Point (Se\ A\ C)\ B \wedge Bet\text{-}Point (Se\ C\ B)\ A \wedge \neg Bet\text{-}Point (Se\ B\ A)\ C$
 $\vee \neg Bet\text{-}Point (Se\ A\ C)\ B \wedge \neg Bet\text{-}Point (Se\ C\ B)\ A \wedge Bet\text{-}Point (Se\ B\ A)\ C$

proof –

have $Bet\text{-}Point (Se\ A\ C)\ B \implies Inv (Bet\text{-}Point (Se\ C\ B)\ A) \wedge Inv (Bet\text{-}Point (Se\ B\ A)\ C)$ **by** (*simp add:Bet-iff*)

then have $P1 : Bet\text{-}Point (Se\ A\ C)\ B \implies \neg Bet\text{-}Point (Se\ C\ B)\ A \wedge \neg Bet\text{-}Point (Se\ B\ A)\ C$ **by** (*simp add:Inv-def*)

have $Bet\text{-}Point (Se\ C\ B)\ A \implies Inv (Bet\text{-}Point (Se\ A\ C)\ B) \wedge Inv (Bet\text{-}Point (Se\ B\ A)\ C)$ **by** (*simp add:Bet-iff*)

then have $P2 : Bet\text{-}Point (Se\ C\ B)\ A \implies \neg Bet\text{-}Point (Se\ A\ C)\ B \wedge \neg Bet\text{-}Point (Se\ B\ A)\ C$ **by** (*simp add:Inv-def*)

have $Bet\text{-}Point (Se\ B\ A)\ C \implies Inv (Bet\text{-}Point (Se\ A\ C)\ B) \wedge Inv (Bet\text{-}Point (Se\ C\ B)\ A)$ **by** (*simp add:Bet-iff*)

then have $P3 : Bet\text{-}Point (Se\ B\ A)\ C \implies \neg Bet\text{-}Point (Se\ A\ C)\ B \wedge \neg Bet\text{-}Point (Se\ C\ B)\ A$ **by** (*simp add:Inv-def*)

from *assms* $P1\ P2\ P3$ **show** $Bet\text{-}Point (Se\ A\ C)\ B \wedge \neg Bet\text{-}Point (Se\ C\ B)\ A$
 $\wedge \neg Bet\text{-}Point (Se\ B\ A)\ C$

$\vee \neg Bet\text{-}Point (Se\ A\ C)\ B \wedge Bet\text{-}Point (Se\ C\ B)\ A \wedge \neg Bet\text{-}Point (Se\ B\ A)\ C$

$\vee \neg Bet\text{-}Point (Se\ A\ C)\ B \wedge \neg Bet\text{-}Point (Se\ C\ B)\ A \wedge Bet\text{-}Point (Se\ B\ A)\ C$

C by *blast*

qed

lemma(in *Order-Rule*) *Bet-swap-lemma-1* :

assumes

$\neg Eq (Geos (Poi\ A)\ add\ Emp) (Geos (Poi\ D)\ add\ Emp)$

$Bet\text{-}Point (Se\ A\ C)\ B$

$Bet\text{-}Point (Se\ B\ D)\ C$

shows $Line\text{-}on (Li\ A\ D)\ B \wedge Line\text{-}on (Li\ A\ D)\ C$

proof –

from *assms* **have** $P1 : Line\text{-}on (Li\ A\ B)\ C$ **by** (*simp add:Line-Bet-on*)

have $P2 : Line\text{-}on (Li\ A\ B)\ B$ **by** (*simp add:Line-on-rule*)

have $P3 : Line\text{-}on (Li\ B\ C)\ C$ **by** (*simp add:Line-on-rule*)

have $P4 : Line\text{-}on (Li\ B\ C)\ B$ **by** (*simp add:Line-on-rule*)

from *assms* **have** $P5 : \neg Eq (Geos (Poi\ C)\ add\ Emp) (Geos (Poi\ B)\ add\ Emp)$ **by** (*simp add:Bet-Point-def*)

from $P1\ P2\ P3\ P4\ P5$ **have** $P6 : Eq (Geos (Lin (Li\ B\ C))\ add\ Emp) (Geos (Lin (Li\ A\ B))\ add\ Emp)$ **by** (*simp add:Line-unique*)

from *assms* **have** $P7 : Line\text{-}on (Li\ B\ C)\ D$ **by** (*simp add:Line-Bet-on*)

from $P6\ P7$ **have** $P8 : Line\text{-}on (Li\ A\ B)\ D$ **by** (*simp add:Line-on-trans*)

from *assms* **have** $\neg Eq (Geos (Poi\ B)\ add\ Emp) (Geos (Poi\ A)\ add\ Emp)$ **by** (*simp add:Bet-Point-def*)

then have $P9 : \neg Eq (Geos (Poi\ A)\ add\ Emp) (Geos (Poi\ B)\ add\ Emp)$ **by** (*blast intro:Eq-rev*)

from *assms* $P8\ P9$ **have** $P10 : \text{Line-on } (Li\ A\ D)\ B$ **by** (*simp add:Line-on-rev*)
have $P11 : \text{Line-on } (Li\ A\ D)\ D$ **by** (*simp add:Line-on-rule*)
from *assms* **have** $P12 : \neg \text{Eq } (\text{Geos } (Poi\ B)\ \text{add Emp})\ (\text{Geos } (Poi\ D)\ \text{add Emp})$
by (*simp add:Bet-Point-def*)
from $P4\ P7\ P10\ P11\ P12$ **have** $P13 : \text{Eq } (\text{Geos } (\text{Lin } (Li\ B\ C))\ \text{add Emp})\ (\text{Geos } (\text{Lin } (Li\ A\ D))\ \text{add Emp})$ **by** (*simp add:Line-unique*)
from $P3\ P13$ **have** $P14 : \text{Line-on } (Li\ A\ D)\ C$ **by** (*simp add:Line-on-trans*)
from $P10\ P14$ **show** $\text{Line-on } (Li\ A\ D)\ B \wedge \text{Line-on } (Li\ A\ D)\ C$ **by** *simp*
qed

lemma(in *Order-Rule*) *Bet-swap-lemma-2* :

assumes

$\text{Bet-Point } (Se\ p1\ p3)\ p2$
 $\neg \text{Line-on } (Li\ p1\ p3)\ p4$
 $\neg \text{Line-on } (Li\ p2\ p5)\ p3$
 $\neg \text{Line-on } (Li\ p2\ p5)\ p1$
 $\neg \text{Line-on } (Li\ p2\ p5)\ p4$
 $\text{Bet-Point } (Se\ p3\ p5)\ p4$

shows $\exists p. \text{Line-on } (Li\ p2\ p5)\ p \wedge \text{Bet-Point } (Se\ p1\ p4)\ p$

proof –

have $P1 : \text{Line-on } (Li\ p2\ p5)\ p2$ **by** (*simp add:Line-on-rule*)

from *assms* $P1$ **have** $P2 : \text{Line-on-Seg } (Li\ p2\ p5)\ (Se\ p1\ p4) \wedge \neg \text{Line-on-Seg } (Li\ p2\ p5)\ (Se\ p3\ p4) \vee \text{Line-on-Seg } (Li\ p2\ p5)\ (Se\ p3\ p4) \wedge \neg \text{Line-on-Seg } (Li\ p2\ p5)\ (Se\ p1\ p4)$ **by** (*simp add:Pachets-axiom*)

then **have** $\text{Line-on-Seg } (Li\ p2\ p5)\ (Se\ p3\ p4) \implies \exists p. \text{Line-on } (Li\ p2\ p5)\ p \wedge \text{Bet-Point } (Se\ p3\ p4)\ p$ **by** (*simp add:Line-on-Seg-rule*)

then **obtain** $p6 :: \text{Point}$ **where** $P3 : \text{Line-on-Seg } (Li\ p2\ p5)\ (Se\ p3\ p4) \implies \text{Line-on } (Li\ p2\ p5)\ p6 \wedge \text{Bet-Point } (Se\ p3\ p4)\ p6$ **by** *blast*

from *assms* **have** $\neg \text{Eq } (\text{Geos } (Poi\ p4)\ \text{add Emp})\ (\text{Geos } (Poi\ p3)\ \text{add Emp})$ **by** (*simp add:Bet-Point-def*)

then **have** $P4 : \neg \text{Eq } (\text{Geos } (Poi\ p3)\ \text{add Emp})\ (\text{Geos } (Poi\ p4)\ \text{add Emp})$ **by** (*blast intro:Eq-rev*)

from $P3$ **have** $P5 : \text{Line-on-Seg } (Li\ p2\ p5)\ (Se\ p3\ p4) \implies \text{Bet-Point } (Se\ p3\ p4)\ p6$ **by** *simp*

from $P3$ **have** $P6 : \text{Line-on-Seg } (Li\ p2\ p5)\ (Se\ p3\ p4) \implies \text{Line-on } (Li\ p3\ p6)\ p4$ **by** (*simp add:Line-Bet-on*)

from *assms* **have** $P7 : \text{Line-on } (Li\ p3\ p5)\ p4$ **by** (*simp add:Line-Bet-on*)

have $P8 : \text{Line-on } (Li\ p3\ p6)\ p3$ **by** (*simp add:Line-on-rule*)

have $P9 : \text{Line-on } (Li\ p3\ p5)\ p3$ **by** (*simp add:Line-on-rule*)

from $P4\ P6\ P7\ P8\ P9$ **have** $P10 : \text{Line-on-Seg } (Li\ p2\ p5)\ (Se\ p3\ p4) \implies$

$\text{Eq } (\text{Geos } (\text{Lin } (Li\ p3\ p5))\ \text{add Emp})\ (\text{Geos } (\text{Lin } (Li\ p3\ p6))\ \text{add Emp})$ **by** (*simp add:Line-unique*)

have $P11 : \text{Line-on } (Li\ p3\ p5)\ p5$ **by** (*simp add:Line-on-rule*)

from $P10\ P11$ **have** $P12 : \text{Line-on-Seg } (Li\ p2\ p5)\ (Se\ p3\ p4) \implies \text{Line-on } (Li\ p3\ p6)\ p5$ **by** (*simp add:Line-on-trans*)

have $P13 : \text{Line-on } (Li\ p2\ p5)\ p5$ **by** (*simp add:Line-on-rule*)

have $P14 : \text{Line-on } (Li\ p3\ p6)\ p6$ **by** (*simp add:Line-on-rule*)

from $P5$ **have** $P15 : \text{Line-on-Seg } (Li\ p2\ p5)\ (Se\ p3\ p4) \implies \text{Eq } (\text{Geos } (Poi\ p6)\ \text{add Emp})\ (\text{Geos } (Poi\ p5)\ \text{add Emp}) \implies$

Bet-Point (Se p3 p4) p5 **by** (*simp add:Point-Eq*)
from *assms* **have** *Inv (Bet-Point (Se p5 p4) p3) ∧ Inv (Bet-Point (Se p4 p3) p5)* **by** (*simp add:Bet-iff*)
then **have** \neg *Bet-Point (Se p4 p3) p5* **by** (*simp add:Inv-def*)
then **have** *P16 : \neg Bet-Point (Se p3 p4) p5* **by** (*blast intro:Bet-rev*)
from *P15 P16* **have** *P17 : Line-on-Seg (Li p2 p5) (Se p3 p4) \implies \neg Eq (Geos (Poi p6) add Emp) (Geos (Poi p5) add Emp)* **by** *blast*
from *P3 P12 P13 P14 P17* **have** *P18 : Line-on-Seg (Li p2 p5) (Se p3 p4) \implies Eq (Geos (Lin (Li p3 p6)) add Emp) (Geos (Lin (Li p2 p5)) add Emp)* **by** (*simp add:Line-unique*)
from *P8 P18* **have** *P19 : Line-on-Seg (Li p2 p5) (Se p3 p4) \implies Line-on (Li p2 p5) p3* **by** (*simp add:Line-on-trans*)
from *assms P19* **have** *P20 : \neg Line-on-Seg (Li p2 p5) (Se p3 p4)* **by** *blast*
from *P2 P3 P20* **have** *Line-on-Seg (Li p2 p5) (Se p1 p4)* **by** *blast*
thus $\exists p.$ *Line-on (Li p2 p5) p ∧ Bet-Point (Se p1 p4) p* **by** (*simp add:Line-on-Seg-rule*)
qed

lemma(*in Order-Rule*) *Bet-swap-lemma-3* :

assumes

Bet-Point (Se p1 p3) p2

Bet-Point (Se p3 p5) p4

\neg *Line-on (Li p1 p3) p5*

shows $\exists p.$ *Bet-Point (Se p1 p4) p ∧ Bet-Point (Se p5 p2) p*

proof –

from *assms* **have** *P1 : \neg Eq (Geos (Poi p1) add Emp) (Geos (Poi p3) add Emp)* **by** (*simp add:Bet-Point-def*)

then **have** *P2 : Eq (Geos (Lin (Li p1 p3)) add Emp) (Geos (Lin (Li p3 p1)) add Emp)* **by** (*simp add:Line-rev*)

from *assms P2* **have** *P3 : \neg Line-on (Li p3 p1) p5* **by** (*simp add:Line-not-on-trans*)

from *assms* **have** *P4 : \neg Eq (Geos (Poi p3) add Emp) (Geos (Poi p5) add Emp)* **by** (*simp add:Bet-Point-def*)

from *P1* **have** *P5 : \neg Eq (Geos (Poi p3) add Emp) (Geos (Poi p1) add Emp)* **by** (*blast intro:Eq-rev*)

from *P3 P4 P5* **have** *P6 : \neg Line-on (Li p3 p5) p1* **by** (*blast intro:Line-on-rev*)

from *assms* **have** *P7 : Bet-Point (Se p5 p3) p4* **by** (*simp add:Bet-rev*)

from *P4* **have** *P8 : Eq (Geos (Lin (Li p3 p5)) add Emp) (Geos (Lin (Li p5 p3)) add Emp)* **by** (*simp add:Line-rev*)

from *P8 P6* **have** *P9 : \neg Line-on (Li p5 p3) p1* **by** (*simp add:Line-not-on-trans*)

from *P7 P9* **have** *P10 : \neg Eq (Geos (Lin (Li p1 p4)) add Emp) (Geos (Lin (Li p1 p3)) add Emp)* **by** (*simp add:Line-Bet-not-Eq*)

from *assms* **have** *Line-on (Li p3 p5) p4* **by** (*simp add:Line-Bet-on*)

then **have** *P11 : Eq (Geos (Poi p4) add Emp) (Geos (Poi p1) add Emp) \implies Line-on (Li p3 p5) p1* **by** (*simp add:Point-Eq*)

from *P6 P11* **have** \neg *Eq (Geos (Poi p4) add Emp) (Geos (Poi p1) add Emp)* **by** *blast*

then **have** *P12 : \neg Eq (Geos (Poi p1) add Emp) (Geos (Poi p4) add Emp)* **by** (*blast intro:Eq-rev*)

from *P1 P10 P12* **have** *P13 : \neg Line-on (Li p1 p4) p3* **by** (*simp add:Line-not-Eq-on*)

from *P1 P12 P13* **have** *P14 : \neg Line-on (Li p1 p3) p4* **by** (*blast intro:Line-on-rev*)

from *assms* **have** $P15 : \neg \text{Eq} (\text{Geos} (\text{Lin} (\text{Li} \ p5 \ p2)) \ \text{add} \ \text{Emp}) (\text{Geos} (\text{Lin} (\text{Li} \ p5 \ p3)) \ \text{add} \ \text{Emp})$ **by** (*simp add:Line-Bet-not-Eq*)
from *assms* **have** $P16 : \text{Line-on} (\text{Li} \ p1 \ p3) \ p2$ **by** (*simp add:Line-Bet-on*)
then **have** $P17 : \text{Eq} (\text{Geos} (\text{Poi} \ p2) \ \text{add} \ \text{Emp}) (\text{Geos} (\text{Poi} \ p5) \ \text{add} \ \text{Emp}) \implies$
 $\text{Line-on} (\text{Li} \ p1 \ p3) \ p5$ **by** (*simp add:Point-Eq*)
from *assms* $P17$ **have** $\neg \text{Eq} (\text{Geos} (\text{Poi} \ p2) \ \text{add} \ \text{Emp}) (\text{Geos} (\text{Poi} \ p5) \ \text{add} \ \text{Emp})$
by *blast*
then **have** $P18 : \neg \text{Eq} (\text{Geos} (\text{Poi} \ p5) \ \text{add} \ \text{Emp}) (\text{Geos} (\text{Poi} \ p2) \ \text{add} \ \text{Emp})$ **by**
(*blast intro:Eq-rev*)
from $P4$ **have** $P19 : \neg \text{Eq} (\text{Geos} (\text{Poi} \ p5) \ \text{add} \ \text{Emp}) (\text{Geos} (\text{Poi} \ p3) \ \text{add} \ \text{Emp})$
by (*blast intro:Eq-rev*)
from $P15 \ P18 \ P19$ **have** $P20 : \neg \text{Line-on} (\text{Li} \ p5 \ p2) \ p3$ **by** (*simp add:Line-not-Eq-on*)
from $P18$ **have** $P21 : \text{Eq} (\text{Geos} (\text{Lin} (\text{Li} \ p5 \ p2)) \ \text{add} \ \text{Emp}) (\text{Geos} (\text{Lin} (\text{Li} \ p2 \ p5)) \ \text{add} \ \text{Emp})$ **by** (*simp add:Line-rev*)
from $P20 \ P21$ **have** $P22 : \neg \text{Line-on} (\text{Li} \ p2 \ p5) \ p3$ **by** (*simp add:Line-not-on-trans*)

from *assms* **have** $P23 : \text{Bet-Point} (\text{Se} \ p3 \ p1) \ p2$ **by** (*blast intro:Bet-rev*)
from $P3 \ P23$ **have** $P24 : \neg \text{Eq} (\text{Geos} (\text{Lin} (\text{Li} \ p5 \ p2)) \ \text{add} \ \text{Emp}) (\text{Geos} (\text{Lin} (\text{Li} \ p5 \ p1)) \ \text{add} \ \text{Emp})$ **by** (*simp add:Line-Bet-not-Eq*)
have $\text{Line-on} (\text{Li} \ p3 \ p1) \ p1$ **by** (*simp add:Line-on-rule*)
then **have** $P25 : \text{Eq} (\text{Geos} (\text{Poi} \ p1) \ \text{add} \ \text{Emp}) (\text{Geos} (\text{Poi} \ p5) \ \text{add} \ \text{Emp}) \implies$
 $\text{Line-on} (\text{Li} \ p3 \ p1) \ p5$ **by** (*simp add:Point-Eq*)
from $P3 \ P25$ **have** $P26 : \neg \text{Eq} (\text{Geos} (\text{Poi} \ p1) \ \text{add} \ \text{Emp}) (\text{Geos} (\text{Poi} \ p5) \ \text{add} \ \text{Emp})$ **by** *blast*
then **have** $P27 : \neg \text{Eq} (\text{Geos} (\text{Poi} \ p5) \ \text{add} \ \text{Emp}) (\text{Geos} (\text{Poi} \ p1) \ \text{add} \ \text{Emp})$ **by**
(*blast intro:Eq-rev*)
from $P18 \ P24 \ P27$ **have** $P28 : \neg \text{Line-on} (\text{Li} \ p5 \ p2) \ p1$ **by** (*simp add:Line-not-Eq-on*)
from $P21 \ P28$ **have** $P29 : \neg \text{Line-on} (\text{Li} \ p2 \ p5) \ p1$ **by** (*simp add:Line-not-on-trans*)

from *assms* **have** $P31 : \text{Line-on} (\text{Li} \ p3 \ p4) \ p5$ **by** (*simp add:Line-Bet-on*)
have $P32 : \text{Line-on} (\text{Li} \ p3 \ p4) \ p4$ **by** (*simp add:Line-on-rule*)
have $P33 : \text{Line-on} (\text{Li} \ p2 \ p5) \ p5$ **by** (*simp add:Line-on-rule*)
from *assms* **have** $P34 : \neg \text{Eq} (\text{Geos} (\text{Poi} \ p5) \ \text{add} \ \text{Emp}) (\text{Geos} (\text{Poi} \ p4) \ \text{add} \ \text{Emp})$
by (*simp add:Bet-Point-def*)
from $P31 \ P32 \ P33 \ P34$ **have** $P35 : \text{Line-on} (\text{Li} \ p2 \ p5) \ p4 \implies \text{Eq} (\text{Geos} (\text{Lin} (\text{Li} \ p3 \ p4)) \ \text{add} \ \text{Emp}) (\text{Geos} (\text{Lin} (\text{Li} \ p2 \ p5)) \ \text{add} \ \text{Emp})$ **by** (*simp add:Line-unique*)
have $P36 : \text{Line-on} (\text{Li} \ p3 \ p4) \ p3$ **by** (*simp add:Line-on-rule*)
from $P35 \ P36$ **have** $P37 : \text{Line-on} (\text{Li} \ p2 \ p5) \ p4 \implies \text{Line-on} (\text{Li} \ p2 \ p5) \ p3$ **by**
(*simp add:Line-on-trans*)
from $P22 \ P37$ **have** $P38 : \neg \text{Line-on} (\text{Li} \ p2 \ p5) \ p4$ **by** *blast*
from *assms* $P14 \ P22 \ P29 \ P38$ **have** $\exists p. \text{Line-on} (\text{Li} \ p2 \ p5) \ p \wedge \text{Bet-Point} (\text{Se} \ p1 \ p4) \ p$ **by** (*simp add:Bet-swap-lemma-2*)
then **obtain** $p6 :: \text{Point}$ **where** $P39 : \text{Line-on} (\text{Li} \ p2 \ p5) \ p6 \wedge \text{Bet-Point} (\text{Se} \ p1 \ p4) \ p6$ **by** *blast*
from $P12$ **have** $P40 : \text{Eq} (\text{Geos} (\text{Lin} (\text{Li} \ p1 \ p4)) \ \text{add} \ \text{Emp}) (\text{Geos} (\text{Lin} (\text{Li} \ p4 \ p1)) \ \text{add} \ \text{Emp})$ **by** (*simp add:Line-rev*)
from $P13 \ P40$ **have** $P41 : \neg \text{Line-on} (\text{Li} \ p4 \ p1) \ p3$ **by** (*simp add:Line-not-on-trans*)

from *assms* $P6$ **have** $P42 : \neg Eq (Geos (Lin (Li p1 p4)) add Emp) (Geos (Lin (Li p1 p5)) add Emp)$ **by** (*simp add:Line-Bet-not-Eq*)
from $P12 P26 P42$ **have** $P43 : \neg Line-on (Li p1 p4) p5$ **by** (*simp add:Line-not-Eq-on*)
from $P40 P43$ **have** $P44 : \neg Line-on (Li p4 p1) p5$ **by** (*simp add:Line-not-on-trans*)
from *assms* **have** $\neg Eq (Geos (Poi p2) add Emp) (Geos (Poi p1) add Emp)$ **by** (*simp add:Bet-Point-def*)
then **have** $P45 : \neg Eq (Geos (Poi p1) add Emp) (Geos (Poi p2) add Emp)$ **by** (*blast intro:Eq-rev*)
from $P1 P16 P45$ **have** $P47 : Line-on (Li p1 p2) p3$ **by** (*simp add:Line-on-rev*)
from $P47$ **have** $P48 : Eq (Geos (Lin (Li p1 p2)) add Emp) (Geos (Lin (Li p1 p4)) add Emp) \implies Line-on (Li p1 p4) p3$ **by** (*simp add:Line-on-trans*)
from $P13 P48$ **have** $P49 : \neg Eq (Geos (Lin (Li p1 p2)) add Emp) (Geos (Lin (Li p1 p4)) add Emp)$ **by** *blast*
from $P12 P45 P49$ **have** $P50 : \neg Line-on (Li p1 p2) p4$ **by** (*simp add:Line-not-Eq-on*)
from $P12 P45 P50$ **have** $P51 : \neg Line-on (Li p1 p4) p2$ **by** (*blast intro:Line-on-rev*)
from $P40 P51$ **have** $P52 : \neg Line-on (Li p4 p1) p2$ **by** (*simp add:Line-not-on-trans*)

from $P18 P19 P20$ **have** $P53 : \neg Line-on (Li p5 p3) p2$ **by** (*blast intro:Line-on-rev*)
from $P7 P23 P41 P44 P52 P53$ **have** $\exists p. Line-on (Li p4 p1) p \wedge Bet-Point (Se p5 p2) p$ **by** (*simp add:Bet-swap-lemma-2*)
then **obtain** $p7 :: Point$ **where** $P54 : Line-on (Li p4 p1) p7 \wedge Bet-Point (Se p5 p2) p7$ **by** *blast*
from $P33 P44$ **have** $P55 : \neg Eq (Geos (Lin (Li p4 p1)) add Emp) (Geos (Lin (Li p2 p5)) add Emp)$ **by** (*simp add:Line-not-on-Eq*)
from $P39$ **have** $P56 : Line-on (Li p4 p1) p6$ **by** (*simp add:Line-Bet-on*)
from $P54$ **have** $P57 : Line-on (Li p2 p5) p7$ **by** (*simp add:Line-Bet-on*)
from $P39 P54 P55 P56 P57$ **have** $P58 : Eq (Geos (Poi p7) add Emp) (Geos (Poi p6) add Emp)$ **by** (*blast intro:Line-unique-Point*)
from $P54$ **have** $P59 : Bet-Point (Se p5 p2) p7$ **by** *simp*
from $P58 P59$ **have** $P60 : Bet-Point (Se p5 p2) p6$ **by** (*simp add:Point-Eq*)
from $P39 P60$ **show** $\exists p. Bet-Point (Se p1 p4) p \wedge Bet-Point (Se p5 p2) p$ **by** *blast*
qed

lemma(*in Order-Rule*) *Bet-swap-lemma-4* :

assumes

$\neg Eq (Geos (Poi A) add Emp) (Geos (Poi D) add Emp)$

$Bet-Point (Se A E) G$

$Bet-Point (Se D G) H$

$\neg Line-on (Li A D) E$

shows $\exists p. Line-on (Li H E) p \wedge Bet-Point (Se D A) p$

proof –

from *assms* **have** $P1 : \neg Eq (Geos (Poi A) add Emp) (Geos (Poi E) add Emp)$ **by** (*simp add:Bet-Point-def*)

from *assms* $P1$ **have** $P2 : \neg Line-on (Li A E) D$ **by** (*blast intro:Line-on-rev*)

from $P1$ **have** $P3 : Eq (Geos (Lin (Li A E)) add Emp) (Geos (Lin (Li E A)) add Emp)$ **by** (*simp add:Line-rev*)

from $P2 P3$ **have** $P4 : \neg Line-on (Li E A) D$ **by** (*simp add:Line-not-on-trans*)

from *assms* **have** $P5 : Bet-Point (Se E A) G$ **by** (*simp add:Bet-rev*)

from $P_4 P_5$ **have** $P_6 : \neg \text{Eq} (\text{Geos} (\text{Lin} (\text{Li} D G)) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li} D A)) \text{ add Emp})$ **by** (*simp add:Line-Bet-not-Eq*)
from *assms* **have** $P_7 : \neg \text{Eq} (\text{Geos} (\text{Poi} D) \text{ add Emp}) (\text{Geos} (\text{Poi} G) \text{ add Emp})$ **by** (*simp add:Bet-Point-def*)
from *assms* **have** $P_8 : \neg \text{Eq} (\text{Geos} (\text{Poi} D) \text{ add Emp}) (\text{Geos} (\text{Poi} A) \text{ add Emp})$ **by** (*blast intro:Eq-rev*)
from $P_6 P_7 P_8$ **have** $P_9 : \neg \text{Line-on} (\text{Li} D G) A$ **by** (*simp add:Line-not-Eq-on*)
from *assms* P_2 **have** $P_{10} : \neg \text{Eq} (\text{Geos} (\text{Lin} (\text{Li} D G)) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li} D E)) \text{ add Emp})$ **by** (*simp add:Line-Bet-not-Eq*)
have $\text{Line-on} (\text{Li} A D) D$ **by** (*simp add:Line-on-rule*)
then **have** $P_{11} : \text{Eq} (\text{Geos} (\text{Poi} D) \text{ add Emp}) (\text{Geos} (\text{Poi} E) \text{ add Emp}) \implies \text{Line-on} (\text{Li} A D) E$ **by** (*simp add:Point-Eq*)
from *assms* P_{11} **have** $P_{12} : \neg \text{Eq} (\text{Geos} (\text{Poi} D) \text{ add Emp}) (\text{Geos} (\text{Poi} E) \text{ add Emp})$ **by** *blast*
from $P_7 P_{10} P_{12}$ **have** $P_{13} : \neg \text{Line-on} (\text{Li} D G) E$ **by** (*simp add:Line-not-Eq-on*)
from *assms* P_{13} **have** $P_{14} : \neg \text{Eq} (\text{Geos} (\text{Lin} (\text{Li} E H)) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li} E G)) \text{ add Emp})$ **by** (*simp add:Line-Bet-not-Eq*)
from *assms* **have** $\text{Line-on} (\text{Li} D G) H$ **by** (*simp add:Line-Bet-on*)
then **have** $P_{15} : \text{Eq} (\text{Geos} (\text{Poi} H) \text{ add Emp}) (\text{Geos} (\text{Poi} E) \text{ add Emp}) \implies \text{Line-on} (\text{Li} D G) E$ **by** (*simp add:Point-Eq*)
from $P_{13} P_{15}$ **have** $P_{16} : \neg \text{Eq} (\text{Geos} (\text{Poi} E) \text{ add Emp}) (\text{Geos} (\text{Poi} H) \text{ add Emp})$ **by** (*blast intro:Eq-rev*)
from *assms* **have** $\text{Line-on} (\text{Li} D G) G$ **by** (*simp add:Line-on-rule*)
then **have** $P_{17} : \text{Eq} (\text{Geos} (\text{Poi} G) \text{ add Emp}) (\text{Geos} (\text{Poi} E) \text{ add Emp}) \implies \text{Line-on} (\text{Li} D G) E$ **by** (*simp add:Point-Eq*)
from $P_{13} P_{17}$ **have** $P_{18} : \neg \text{Eq} (\text{Geos} (\text{Poi} E) \text{ add Emp}) (\text{Geos} (\text{Poi} G) \text{ add Emp})$ **by** (*blast intro:Eq-rev*)
from $P_{14} P_{16} P_{18}$ **have** $P_{19} : \neg \text{Line-on} (\text{Li} E H) G$ **by** (*simp add:Line-not-Eq-on*)
from P_7 **have** $P_{20} : \text{Eq} (\text{Geos} (\text{Lin} (\text{Li} D G)) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li} G D)) \text{ add Emp})$ **by** (*simp add:Line-rev*)
from $P_{13} P_{20}$ **have** $P_{21} : \neg \text{Line-on} (\text{Li} G D) E$ **by** (*simp add:Line-not-on-trans*)
from *assms* **have** $P_{22} : \text{Bet-Point} (\text{Se} G D) H$ **by** (*simp add:Bet-rev*)
from $P_{21} P_{22}$ **have** $P_{23} : \neg \text{Eq} (\text{Geos} (\text{Lin} (\text{Li} E H)) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li} E D)) \text{ add Emp})$ **by** (*simp add:Line-Bet-not-Eq*)
from P_{12} **have** $P_{24} : \neg \text{Eq} (\text{Geos} (\text{Poi} E) \text{ add Emp}) (\text{Geos} (\text{Poi} D) \text{ add Emp})$ **by** (*blast intro:Eq-rev*)
from $P_{16} P_{23} P_{24}$ **have** $P_{25} : \neg \text{Line-on} (\text{Li} E H) D$ **by** (*simp add:Line-not-Eq-on*)
from P_{16} **have** $P_{26} : \text{Eq} (\text{Geos} (\text{Lin} (\text{Li} E H)) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li} H E)) \text{ add Emp})$ **by** (*simp add:Line-rev*)
from $P_{25} P_{26}$ **have** $P_{27} : \neg \text{Line-on} (\text{Li} H E) D$ **by** (*simp add:Line-not-on-trans*)
have $P_{28} : \text{Line-on} (\text{Li} A E) A$ **by** (*simp add:Line-on-rule*)
have $P_{29} : \text{Line-on} (\text{Li} A E) E$ **by** (*simp add:Line-on-rule*)
have $P_{30} : \text{Line-on} (\text{Li} E H) E$ **by** (*simp add:Line-on-rule*)
from $P_1 P_{28} P_{29} P_{30}$ **have** $P_{31} : \text{Line-on} (\text{Li} E H) A \implies \text{Eq} (\text{Geos} (\text{Lin} (\text{Li} A E)) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li} E H)) \text{ add Emp})$ **by** (*simp add:Line-unique*)
from *assms* **have** $P_{32} : \text{Line-on} (\text{Li} A E) G$ **by** (*simp add:Line-Bet-on*)
from *assms* **have** $\neg \text{Eq} (\text{Geos} (\text{Poi} G) \text{ add Emp}) (\text{Geos} (\text{Poi} A) \text{ add Emp})$ **by** (*simp add:Bet-Point-def*)
then **have** $P_{33} : \neg \text{Eq} (\text{Geos} (\text{Poi} A) \text{ add Emp}) (\text{Geos} (\text{Poi} G) \text{ add Emp})$ **by**

(blast intro:Eq-rev)
from $P31 P32$ **have** $P34 : \text{Line-on } (Li E H) A \implies \text{Line-on } (Li E H) G$ **by**
 (simp add:Line-on-trans)
from $P19 P34$ **have** $P35 : \neg \text{Line-on } (Li E H) A$ **by** blast
from $P26 P35$ **have** $P36 : \neg \text{Line-on } (Li H E) A$ **by** (simp add:Line-not-on-trans)
from $P26 P19$ **have** $P37 : \neg \text{Line-on } (Li H E) G$ **by** (simp add:Line-not-on-trans)
have $P38 : \text{Line-on } (Li H E) H$ **by** (simp add:Line-on-rule)
from *assms* $P9 P27 P36 P37 P38$ **have** $P39 : \text{Line-on-Seg } (Li H E) (Se D A) \wedge \neg$
 $\text{Line-on-Seg } (Li H E) (Se G A) \vee \text{Line-on-Seg } (Li H E) (Se G A) \wedge \neg \text{Line-on-Seg}$
 $(Li H E) (Se D A)$ **by** (simp add:Pachets-axiom)
then **have** $\text{Line-on-Seg } (Li H E) (Se G A) \implies \exists p. \text{Line-on } (Li H E) p \wedge$
 $\text{Bet-Point } (Se G A) p$ **by** (simp add:Line-on-Seg-rule)
then **obtain** $C2 :: \text{Point}$ **where** $P40 : \text{Line-on-Seg } (Li H E) (Se G A) \implies$
 $\text{Line-on } (Li H E) C2 \wedge \text{Bet-Point } (Se G A) C2$ **by** blast
from *assms* **have** $P41 : \text{Line-on } (Li G A) E$ **by** (simp add:Line-Bet-on)
from $P40$ **have** $P42 : \text{Line-on-Seg } (Li H E) (Se G A) \implies \text{Line-on } (Li G A) C2$
by (simp add:Line-Bet-on)
have $P43 : \text{Line-on } (Li H E) E$ **by** (simp add:Line-on-rule)
from $P40$ **have** $\text{Line-on-Seg } (Li H E) (Se G A) \implies \text{Bet-Point } (Se G A) C2$ **by**
simp
then **have** $P44 : \text{Line-on-Seg } (Li H E) (Se G A) \implies \text{Eq } (\text{Geos } (Poi C2) \text{ add}$
 $\text{Emp}) (\text{Geos } (Poi E) \text{ add Emp}) \implies$
 $\text{Bet-Point } (Se G A) E$ **by** (simp add:Point-Eq)
from *assms* **have** $\text{Inv } (\text{Bet-Point } (Se E G) A) \wedge \text{Inv } (\text{Bet-Point } (Se G A) E)$
by (simp add:Bet-iff)
then **have** $P45 : \neg \text{Bet-Point } (Se G A) E$ **by** (simp add:Inv-def)
from $P44 P45$ **have** $P46 : \text{Line-on-Seg } (Li H E) (Se G A) \implies \neg \text{Eq } (\text{Geos } (Poi$
 $C2) \text{ add Emp}) (\text{Geos } (Poi E) \text{ add Emp})$ **by** blast
from $P40 P41 P42 P43 P46$ **have** $P47 : \text{Line-on-Seg } (Li H E) (Se G A) \implies$
 $\text{Eq } (\text{Geos } (Lin (Li G A)) \text{ add Emp}) (\text{Geos } (Lin (Li H E)) \text{ add Emp})$ **by** (simp
add:Line-unique)
have $P48 : \text{Line-on } (Li G A) G$ **by** (simp add:Line-on-rule)
from $P47 P48$ **have** $P49 : \text{Line-on-Seg } (Li H E) (Se G A) \implies \text{Line-on } (Li H$
 $E) G$ **by** (simp add:Line-on-trans)
from $P37 P49$ **have** $P50 : \neg \text{Line-on-Seg } (Li H E) (Se G A)$ **by** blast
from $P39 P40 P50$ **have** $\text{Line-on-Seg } (Li H E) (Se D A)$ **by** blast
thus $\exists p. \text{Line-on } (Li H E) p \wedge \text{Bet-Point } (Se D A) p$ **by** (simp add:Line-on-Seg-rule)
qed

lemma(in *Order-Rule*) *Bet-swap-lemma-5* :

assumes

$\text{Bet-Point } (Se A C) B$

$\text{Bet-Point } (Se B D) C$

$\text{Bet-Point } (Se C F) E$

$\neg \text{Line-on } (Li A D) F$

$\neg \text{Line-on } (Li A C) F$

shows $\text{Bet-Point } (Se A D) C$

proof –

from *assms* **have** $P1 : \text{Eq } (\text{Geos } (Poi A) \text{ add Emp}) (\text{Geos } (Poi D) \text{ add Emp})$

\implies *Bet-Point* (Se D C) B **by** (simp add:Bet-Point-Eq)
from *assms* **have** *Inv* (Bet-Point (Se D C) B) \wedge *Inv* (Bet-Point (Se C B) D)
by (simp add:Bet-iff)
then **have** *P2* : \neg *Bet-Point* (Se D C) B **by** (simp add:Inv-def)
from *P1 P2* **have** *P3* : \neg *Eq* (Geos (Poi A) add Emp) (Geos (Poi D) add Emp)
by *blast*
from *assms P3* **have** *P4* : *Line-on* (Li A D) B \wedge *Line-on* (Li A D) C **by** (simp
add:Bet-swap-lemma-1)
then **have** *P5* : *Line-on* (Li A D) C **by** *simp*
from *assms* **have** $\exists p$. *Bet-Point* (Se A E) p \wedge *Bet-Point* (Se F B) p **by** (simp
add:Bet-swap-lemma-3)
then **obtain** *G* :: *Point* **where** *P6* : *Bet-Point* (Se A E) G \wedge *Bet-Point* (Se F
B) G **by** *blast*
from *P3* **have** *P7* : *Eq* (Geos (Lin (Li A D)) add Emp) (Geos (Lin (Li D A))
add Emp) **by** (simp add:Line-rev)
from *P4 P7* **have** *P8* : *Line-on* (Li D A) B **by** (*blast intro:Line-on-trans*)
from *assms P7* **have** *P9* : \neg *Line-on* (Li D A) F **by** (simp add:Line-not-on-trans)
have *P10* : *Line-on* (Li D A) D **by** (simp add:Line-on-rule)
have *P11* : *Line-on* (Li D B) D **by** (simp add:Line-on-rule)
have *P12* : *Line-on* (Li D B) B **by** (simp add:Line-on-rule)
from *assms* **have** *P13* : \neg *Eq* (Geos (Poi B) add Emp) (Geos (Poi D) add Emp)
by (simp add:Bet-Point-def)
from *P8 P10 P11 P12 P13* **have** *P14* : *Eq* (Geos (Lin (Li D A)) add Emp) (Geos
(Lin (Li D B)) add Emp) **by** (simp add:Line-unique)
from *P9 P14* **have** *P15* : \neg *Line-on* (Li D B) F **by** (simp add:Line-not-on-trans)

from *assms* **have** *P16* : *Bet-Point* (Se D B) C **by** (simp add:Bet-rev)
from *P6* **have** *P17* : *Bet-Point* (Se B F) G **by** (simp add:Bet-rev)
from *P15 P16 P17* **have** $\exists p$. *Bet-Point* (Se D G) p \wedge *Bet-Point* (Se F C) p **by**
(simp add:Bet-swap-lemma-3)
then **obtain** *H* :: *Point* **where** *P18* : *Bet-Point* (Se D G) H \wedge *Bet-Point* (Se F
C) H **by** *blast*
from *assms* **have** *P19* : \neg *Eq* (Geos (Poi A) add Emp) (Geos (Poi C) add Emp)
by (simp add:Bet-Point-def)
then **have** *P20* : *Eq* (Geos (Lin (Li A C)) add Emp) (Geos (Lin (Li C A)) add
Emp) **by** (simp add:Line-rev)
from *assms P20* **have** *P21* : \neg *Line-on* (Li C A) F **by** (simp add:Line-not-on-trans)
from *P19* **have** *P22* : \neg *Eq* (Geos (Poi C) add Emp) (Geos (Poi A) add Emp)
by (*blast intro:Eq-rev*)
from *assms* **have** *P23* : \neg *Eq* (Geos (Poi C) add Emp) (Geos (Poi F) add Emp)
by (simp add:Bet-Point-def)
from *P21 P22 P23* **have** *P24* : \neg *Line-on* (Li C F) A **by** (*blast intro:Line-on-rev*)
from *assms* **have** *P25* : *Bet-Point* (Se F C) E **by** (simp add:Bet-rev)
from *P23* **have** *P26* : *Eq* (Geos (Lin (Li C F)) add Emp) (Geos (Lin (Li F C))
add Emp) **by** (simp add:Line-rev)
from *P24 P26* **have** *P27* : \neg *Line-on* (Li F C) A **by** (simp add:Line-not-on-trans)
from *P25 P27* **have** *P28* : \neg *Eq* (Geos (Lin (Li A E)) add Emp) (Geos (Lin (Li
A C)) add Emp) **by** (simp add:Line-Bet-not-Eq)
from *P25* **have** *Line-on* (Li F C) E **by** (simp add:Line-Bet-on)

then have $P29 : Eq (Geos (Poi E) add Emp) (Geos (Poi A) add Emp) \implies$
Line-on (Li F C) A **by** (*simp add:Point-Eq*)
from $P27 P29$ **have** $\neg Eq (Geos (Poi E) add Emp) (Geos (Poi A) add Emp)$ **by**
blast
then have $P30 : \neg Eq (Geos (Poi A) add Emp) (Geos (Poi E) add Emp)$ **by**
(*blast intro:Eq-rev*)
from $P19 P28 P30$ **have** $P31 : \neg Line-on (Li A E) C$ **by** (*simp add:Line-not-Eq-on*)
from $P19 P30 P31$ **have** $P32 : \neg Line-on (Li A C) E$ **by** (*blast intro:Line-on-rev*)

from $P5 P19 P32$ **have** $P33 : \neg Line-on (Li A D) E$ **by** (*simp add:Line-not-on-ex*)

from $P3 P30 P33$ **have** $P34 : \neg Line-on (Li A E) D$ **by** (*blast intro:Line-on-rev*)
from $P30$ **have** $P35 : Eq (Geos (Lin (Li A E)) add Emp) (Geos (Lin (Li E A))$
add Emp) **by** (*simp add:Line-rev*)
from $P18$ **have** $P36 : Bet-Point (Se D G) H$ **by** *simp*
from $P6$ **have** $P37 : Bet-Point (Se A E) G$ **by** *simp*
from $P3 P18 P33 P37$ **have** $\exists p. Line-on (Li H E) p \wedge Bet-Point (Se D A) p$
by (*simp add:Bet-swap-lemma-4*)
then obtain $C2 :: Point$ **where** $P38 : Line-on (Li H E) C2 \wedge Bet-Point (Se D$
 $A) C2$ **by** *blast*
have $Line-on (Li H E) E$ **by** (*simp add:Line-on-rule*)
then have $P39 : Eq (Geos (Lin (Li H E)) add Emp) (Geos (Lin (Li A D)) add$
 $Emp) \implies Line-on (Li A D) E$ **by** (*simp add:Line-on-trans*)
from $P33 P39$ **have** $P40 : \neg Eq (Geos (Lin (Li H E)) add Emp) (Geos (Lin (Li$
 $A D)) add Emp)$ **by** *blast*
from $P23$ **have** $P41 : \neg Eq (Geos (Poi F) add Emp) (Geos (Poi C) add Emp)$
by (*blast intro:Eq-rev*)
from $P25$ **have** $P42 : Line-on (Li F E) C$ **by** (*simp add:Line-Bet-on*)
from $P18$ **have** $P43 : Line-on (Li F H) C$ **by** (*simp add:Line-Bet-on*)
from $P36$ **have** $P44 : Eq (Geos (Poi H) add Emp) (Geos (Poi E) add Emp) \implies$
 $Bet-Point (Se D G) E$ **by** (*simp add:Point-Eq*)
then have $P45 : Eq (Geos (Poi H) add Emp) (Geos (Poi E) add Emp) \implies$
 $Line-on (Li D G) E$ **by** (*simp add:Line-Bet-on*)
have $P46 : Line-on (Li D G) G$ **by** (*simp add:Line-on-rule*)
have $P47 : Line-on (Li A E) E$ **by** (*simp add:Line-on-rule*)
from $P37$ **have** $P48 : Line-on (Li A E) G$ **by** (*simp add:Line-Bet-on*)
from $P44$ **have** $P49 : Eq (Geos (Poi H) add Emp) (Geos (Poi E) add Emp) \implies \neg$
 $Eq (Geos (Poi G) add Emp) (Geos (Poi E) add Emp)$ **by** (*simp add:Bet-Point-def*)
from $P45 P46 P47 P48 P49$ **have** $P50 : Eq (Geos (Poi H) add Emp) (Geos (Poi$
 $E) add Emp) \implies$
 $Eq (Geos (Lin (Li D G)) add Emp) (Geos (Lin (Li A E)) add Emp)$ **by** (*simp*
add:Line-unique)
have $P51 : Line-on (Li D G) D$ **by** (*simp add:Line-on-rule*)
from $P50 P51$ **have** $P52 : Eq (Geos (Poi H) add Emp) (Geos (Poi E) add Emp)$
 $\implies Line-on (Li A E) D$ **by** (*simp add:Line-on-trans*)
from $P34 P52$ **have** $P53 : \neg Eq (Geos (Poi E) add Emp) (Geos (Poi H) add$
 $Emp)$ **by** (*blast intro:Eq-rev*)
from $P41 P42 P43 P53$ **have** $P54 : Line-on (Li E H) C$ **by** (*blast intro:Line-on-dens*)
from $P53$ **have** $P55 : Eq (Geos (Lin (Li E H)) add Emp) (Geos (Lin (Li H E))$

add Emp) **by** (*simp add:Line-rev*)
from $P54$ $P55$ **have** $P56 : \text{Line-on } (Li\ H\ E)\ C$ **by** (*blast intro:Line-on-trans*)
from $P38$ **have** $P57 : \text{Line-on } (Li\ A\ D)\ C2$ **by** (*simp add:Line-Bet-on*)
from $P5\ P38\ P40\ P56\ P57$ **have** $P58 : \text{Eq } (\text{Geos } (Poi\ C2)\ \text{add Emp})\ (\text{Geos } (Poi\ C)\ \text{add Emp})$ **by** (*blast intro:Line-unique-Point*)
from $P38$ **have** $P59 : \text{Bet-Point } (Se\ D\ A)\ C2$ **by** *simp*
from $P58\ P59$ **have** $\text{Bet-Point } (Se\ D\ A)\ C$ **by** (*simp add:Point-Eq*)
thus $\text{Bet-Point } (Se\ A\ D)\ C$ **by** (*simp add:Bet-rev*)
qed

theorem(in *Order-Rule*) *Bet-swap-234-134* :

assumes

$\text{Bet-Point } (Se\ A\ C)\ B$

$\text{Bet-Point } (Se\ B\ D)\ C$

shows $\text{Bet-Point } (Se\ A\ D)\ C$

proof –

from *assms* **have** $P1 : \text{Eq } (\text{Geos } (Poi\ A)\ \text{add Emp})\ (\text{Geos } (Poi\ D)\ \text{add Emp})$
 $\implies \text{Bet-Point } (Se\ D\ C)\ B$ **by** (*simp add:Bet-Point-Eq*)

from *assms* **have** $\text{Inv } (\text{Bet-Point } (Se\ D\ C)\ B) \wedge \text{Inv } (\text{Bet-Point } (Se\ C\ B)\ D)$
by (*simp add:Bet-iff*)

then **have** $P2 : \neg \text{Bet-Point } (Se\ D\ C)\ B$ **by** (*simp add:Inv-def*)

from $P1\ P2$ **have** $P3 : \neg \text{Eq } (\text{Geos } (Poi\ A)\ \text{add Emp})\ (\text{Geos } (Poi\ D)\ \text{add Emp})$
by *blast*

from *assms* $P3$ **have** $\text{Line-on } (Li\ A\ D)\ B \wedge \text{Line-on } (Li\ A\ D)\ C$ **by** (*simp add:Bet-swap-lemma-1*)

then **have** $P4 : \text{Line-on } (Li\ A\ D)\ C$ **by** *simp*

have $\exists p\ q\ r. \neg \text{Line-on } (Li\ A\ D)\ p \wedge \neg \text{Line-on } (Li\ A\ D)\ q \wedge \neg \text{Line-on } (Li\ A\ D)\ r$

$\wedge \neg \text{Eq } (\text{Geos } (Poi\ p)\ \text{add Emp})\ (\text{Geos } (Poi\ q)\ \text{add Emp}) \wedge \neg \text{Eq } (\text{Geos } (Poi\ q)\ \text{add Emp})\ (\text{Geos } (Poi\ r)\ \text{add Emp})$

$\wedge \neg \text{Eq } (\text{Geos } (Poi\ r)\ \text{add Emp})\ (\text{Geos } (Poi\ p)\ \text{add Emp})$ **by** (*blast intro:Line-not-on-exist*)

then **obtain** $F :: \text{Point}$ **where** $P5 : \neg \text{Line-on } (Li\ A\ D)\ F$ **by** *blast*

from $P4$ **have** $P6 : \text{Eq } (\text{Geos } (Poi\ C)\ \text{add Emp})\ (\text{Geos } (Poi\ F)\ \text{add Emp}) \implies \text{Line-on } (Li\ A\ D)\ F$ **by** (*simp add:Point-Eq*)

from $P5\ P6$ **have** $\neg \text{Eq } (\text{Geos } (Poi\ C)\ \text{add Emp})\ (\text{Geos } (Poi\ F)\ \text{add Emp})$ **by** *blast*

then **have** $\exists p. \text{Bet-Point } (Se\ C\ F)\ p$ **by** (*simp add:Seg-density*)

then **obtain** $E :: \text{Point}$ **where** $P7 : \text{Bet-Point } (Se\ C\ F)\ E$ **by** *blast*

have $P8 : \text{Line-on } (Li\ A\ D)\ A$ **by** (*simp add:Line-on-rule*)

have $P9 : \text{Line-on } (Li\ A\ C)\ C$ **by** (*simp add:Line-on-rule*)

have $P10 : \text{Line-on } (Li\ A\ C)\ A$ **by** (*simp add:Line-on-rule*)

from *assms* **have** $P11 : \neg \text{Eq } (\text{Geos } (Poi\ A)\ \text{add Emp})\ (\text{Geos } (Poi\ C)\ \text{add Emp})$
by (*simp add:Bet-Point-def*)

from $P4\ P8\ P9\ P10\ P11$ **have** $\text{Eq } (\text{Geos } (Lin\ (Li\ A\ C))\ \text{add Emp})\ (\text{Geos } (Lin\ (Li\ A\ D))\ \text{add Emp})$ **by** (*simp add:Line-unique*)

then **have** $P12 : \text{Line-on } (Li\ A\ C)\ F \implies \text{Line-on } (Li\ A\ D)\ F$ **by** (*simp add:Line-on-trans*)

from $P5\ P12$ **have** $P13 : \neg \text{Line-on } (Li\ A\ C)\ F$ **by** *blast*

from *assms P5 P7 P13* **show** *Bet-Point (Se A D) C* **by** (*blast intro:Bet-swap-lemma-5*)
qed

theorem(in *Order-Rule*) *Bet-swap-234-124* :

assumes

Bet-Point (Se A C) B

Bet-Point (Se B D) C

shows *Bet-Point (Se A D) B*

proof –

from *assms* **have** *P1 : Eq (Geos (Poi A) add Emp) (Geos (Poi D) add Emp)*
 \implies *Bet-Point (Se D C) B* **by** (*simp add:Bet-Point-Eq*)

from *assms* **have** *Inv (Bet-Point (Se D C) B) \wedge Inv (Bet-Point (Se C B) D)*
by (*simp add:Bet-iff*)

then **have** *P2 : \neg Bet-Point (Se D C) B* **by** (*simp add:Inv-def*)

from *P1 P2* **have** *P3 : \neg Eq (Geos (Poi A) add Emp) (Geos (Poi D) add Emp)*
by *blast*

from *assms P3* **have** *Line-on (Li A D) B \wedge Line-on (Li A D) C* **by** (*simp add:Bet-swap-lemma-1*)

then **have** *P4 : Line-on (Li A D) B* **by** *simp*

have $\exists p q r. \neg$ *Line-on (Li A D) p \wedge \neg Line-on (Li A D) q \wedge \neg Line-on (Li A D) r*

\wedge \neg *Eq (Geos (Poi p) add Emp) (Geos (Poi q) add Emp) \wedge \neg Eq (Geos (Poi q) add Emp) (Geos (Poi r) add Emp)*

\wedge \neg *Eq (Geos (Poi r) add Emp) (Geos (Poi p) add Emp)* **by** (*blast intro:Line-not-on-exist*)

then **obtain** *F :: Point* **where** *P5 : \neg Line-on (Li A D) F* **by** *blast*

from *P4* **have** *P6 : Eq (Geos (Poi B) add Emp) (Geos (Poi F) add Emp)* \implies
Line-on (Li A D) F **by** (*simp add:Point-Eq*)

from *P5 P6* **have** \neg *Eq (Geos (Poi B) add Emp) (Geos (Poi F) add Emp)* **by**
blast

then **have** $\exists p. \text{Bet-Point (Se B F) p}$ **by** (*simp add:Seg-density*)

then **obtain** *E :: Point* **where** *P7 : Bet-Point (Se B F) E* **by** *blast*

from *assms* **have** *P8 : Bet-Point (Se D B) C* **by** (*simp add:Bet-rev*)

from *assms* **have** *P9 : Bet-Point (Se C A) B* **by** (*simp add:Bet-rev*)

from *P3* **have** *P10 : Eq (Geos (Lin (Li A D)) add Emp) (Geos (Lin (Li D A)) add Emp)* **by** (*simp add:Line-rev*)

from *P5 P10* **have** *P11 : \neg Line-on (Li D A) F* **by** (*simp add:Line-not-on-trans*)

from *P4 P10* **have** *P12 : Line-on (Li D A) B* **by** (*simp add:Line-on-trans*)

have *P13 : Line-on (Li D A) D* **by** (*simp add:Line-on-rule*)

have *P14 : Line-on (Li D B) D* **by** (*simp add:Line-on-rule*)

have *P15 : Line-on (Li D B) B* **by** (*simp add:Line-on-rule*)

from *assms* **have** *P16 : \neg Eq (Geos (Poi B) add Emp) (Geos (Poi D) add Emp)*
by (*simp add:Bet-Point-def*)

from *P12 P13 P14 P15 P16* **have** *Eq (Geos (Lin (Li D B)) add Emp) (Geos (Lin (Li D A)) add Emp)* **by** (*simp add:Line-unique*)

then **have** *P17 : Line-on (Li D B) F* \implies *Line-on (Li D A) F* **by** (*simp add:Line-on-trans*)

from *P11 P17* **have** *P18 : \neg Line-on (Li D B) F* **by** *blast*

from *P7 P8 P9 P11 P18* **have** *Bet-Point (Se D A) B* **by** (*blast intro:Bet-swap-lemma-5*)

thus *Bet-Point* (*Se A D*) *B* by (*blast intro:Bet-rev*)
qed

theorem(in *Order-Rule*) *Bet-swap-134-234* :

assumes

Bet-Point (*Se A C*) *B*

Bet-Point (*Se A D*) *C*

shows *Bet-Point* (*Se B D*) *C*

proof –

from *assms* have *P2* : $\neg \text{Eq} (\text{Geos} (\text{Poi } A) \text{ add Emp}) (\text{Geos} (\text{Poi } C) \text{ add Emp})$
by (*simp add:Bet-Point-def*)

from *assms* have *P3* : $\neg \text{Eq} (\text{Geos} (\text{Poi } B) \text{ add Emp}) (\text{Geos} (\text{Poi } A) \text{ add Emp})$
by (*simp add:Bet-Point-def*)

then have *P4* : $\neg \text{Eq} (\text{Geos} (\text{Poi } A) \text{ add Emp}) (\text{Geos} (\text{Poi } B) \text{ add Emp})$ by
(*blast intro:Eq-rev*)

from *assms* have *P5* : *Line-on* (*Li A B*) *C* by (*simp add:Line-Bet-on*)

have *P6* : *Line-on* (*Li A B*) *A* by (*simp add:Line-on-rule*)

from *assms* have *P7* : *Line-on* (*Li A D*) *C* by (*simp add:Line-Bet-on*)

have *P8* : *Line-on* (*Li A D*) *A* by (*simp add:Line-on-rule*)

from *P2 P5 P6 P7 P8* have *P9* : *Eq* (*Geos* (*Lin* (*Li A B*)) *add Emp*) (*Geos* (*Lin* (*Li A D*)) *add Emp*) by (*simp add:Line-unique*)

have *P10* : *Line-on* (*Li A B*) *B* by (*simp add:Line-on-rule*)

from *P9 P10* have *P11* : *Line-on* (*Li A D*) *B* by (*simp add:Line-on-trans*)

from *assms* have *P12* : $\neg \text{Eq} (\text{Geos} (\text{Poi } A) \text{ add Emp}) (\text{Geos} (\text{Poi } D) \text{ add Emp})$
by (*simp add:Bet-Point-def*)

then have *P13* : *Eq* (*Geos* (*Lin* (*Li A D*)) *add Emp*) (*Geos* (*Lin* (*Li D A*)) *add Emp*) by (*simp add:Line-rev*)

from *P11 P13* have *P14* : *Line-on* (*Li D A*) *B* by (*simp add:Line-on-trans*)

from *P12* have *P15* : $\neg \text{Eq} (\text{Geos} (\text{Poi } D) \text{ add Emp}) (\text{Geos} (\text{Poi } A) \text{ add Emp})$
by (*blast intro:Eq-rev*)

from *assms* have *P16* : *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi D*) *add Emp*)
 $\implies \text{Bet-Point} (\text{Se } A \text{ C}) \text{ D}$ by (*simp add:Point-Eq*)

from *assms* have *Inv* (*Bet-Point* (*Se D C*) *A*) \wedge *Inv* (*Bet-Point* (*Se C A*) *D*)
by (*simp add:Bet-iff*)

then have $\neg \text{Bet-Point} (\text{Se } C \text{ A}) \text{ D}$ by (*simp add:Inv-def*)

then have *P17* : $\neg \text{Bet-Point} (\text{Se } A \text{ C}) \text{ D}$ by (*blast intro:Bet-rev*)

from *P16 P17* have *P18* : $\neg \text{Eq} (\text{Geos} (\text{Poi } D) \text{ add Emp}) (\text{Geos} (\text{Poi } B) \text{ add Emp})$
by (*blast intro:Eq-rev*)

from *P14 P15 P18* have *P19* : *Line-on* (*Li D B*) *A* by (*simp add:Line-on-rev*)

from *P18* have *P20* : *Eq* (*Geos* (*Lin* (*Li D B*)) *add Emp*) (*Geos* (*Lin* (*Li B D*)) *add Emp*) by (*simp add:Line-rev*)

from *P19 P20* have *P21* : *Line-on* (*Li B D*) *A* by (*simp add:Line-on-trans*)

have *P22* : *Line-on* (*Li B D*) *B* by (*simp add:Line-on-rule*)

from *P4 P8 P11 P21 P22* have *P23* : *Eq* (*Geos* (*Lin* (*Li A D*)) *add Emp*) (*Geos* (*Lin* (*Li B D*)) *add Emp*) by (*simp add:Line-unique*)

from *P7 P23* have *P24* : *Line-on* (*Li B D*) *C* by (*simp add:Line-on-trans*)

have $\exists p \ q \ r. \neg \text{Line-on} (\text{Li } A \text{ D}) \ p \wedge \neg \text{Line-on} (\text{Li } A \text{ D}) \ q \wedge \neg \text{Line-on} (\text{Li } A \text{ D}) \ r$

$\wedge \neg \text{Eq} (\text{Geos} (\text{Poi } p) \text{ add Emp}) (\text{Geos} (\text{Poi } q) \text{ add Emp}) \wedge \neg \text{Eq} (\text{Geos} (\text{Poi } r)$

$q) \text{ add Emp) (Geos (Poi } r) \text{ add Emp)}$
 $\wedge \neg \text{Eq (Geos (Poi } r) \text{ add Emp) (Geos (Poi } p) \text{ add Emp)}$ **by** (*blast intro:Line-not-on-exist*)
then obtain $F :: \text{Point}$ **where** $P25 : \neg \text{Line-on (Li A D) F}$ **by** *blast*
from $P11$ **have** $P26 : \text{Eq (Geos (Poi B) add Emp) (Geos (Poi F) add Emp)}$ \implies
 $\text{Line-on (Li A D) F}$ **by** (*simp add:Point-Eq*)
from $P25 P26$ **have** $\neg \text{Eq (Geos (Poi B) add Emp) (Geos (Poi F) add Emp)}$ **by**
blast
then have $\exists p. \text{Bet-Point (Se B F) } p$ **by** (*simp add:Seg-density*)
then obtain $G :: \text{Point}$ **where** $P27 : \text{Bet-Point (Se B F) } G$ **by** *blast*
from $P11 P25 P27$ **have** $\text{Inv (Line-on (Li A D) } G)$ **by** (*simp add:Line-Bet-not-on*)
then have $P28 : \neg \text{Line-on (Li A D) } G$ **by** (*simp add:Inv-def*)
from *assms* $P25$ **have** $P29 : \neg \text{Eq (Geos (Lin (Li F C)) add Emp) (Geos (Lin$
 $(\text{Li F D})) \text{ add Emp)}$ **by** (*simp add:Line-Bet-not-Eq*)
from $P7$ **have** $P30 : \text{Eq (Geos (Poi C) add Emp) (Geos (Poi F) add Emp)}$ \implies
 $\text{Line-on (Li A D) } F$ **by** (*simp add:Point-Eq*)
from $P25 P30$ **have** $P31 : \neg \text{Eq (Geos (Poi F) add Emp) (Geos (Poi C) add$
 Emp) **by** (*blast intro:Eq-rev*)
have $P32 : \text{Line-on (Li A D) } D$ **by** (*simp add:Line-on-rule*)
then have $P33 : \text{Eq (Geos (Poi D) add Emp) (Geos (Poi F) add Emp)}$ \implies
 $\text{Line-on (Li A D) } F$ **by** (*simp add:Point-Eq*)
from $P25 P33$ **have** $P34 : \neg \text{Eq (Geos (Poi F) add Emp) (Geos (Poi D) add$
 Emp) **by** (*blast intro:Eq-rev*)
from $P29 P31 P34$ **have** $P35 : \neg \text{Line-on (Li F C) } D$ **by** (*simp add:Line-not-Eq-on*)
from $P31$ **have** $P36 : \text{Eq (Geos (Lin (Li F C)) add Emp) (Geos (Lin (Li C F))$
 add Emp) **by** (*simp add:Line-rev*)
from $P35 P36$ **have** $P37 : \neg \text{Line-on (Li C F) } D$ **by** (*simp add:Line-not-on-trans*)

from *assms* **have** $P38 : \text{Bet-Point (Se D A) } C$ **by** (*simp add:Bet-rev*)
from $P13 P25$ **have** $P39 : \neg \text{Line-on (Li D A) } F$ **by** (*simp add:Line-not-on-trans*)
from $P38 P39$ **have** $P40 : \neg \text{Eq (Geos (Lin (Li F C)) add Emp) (Geos (Lin (Li$
 $(\text{Li F A})) \text{ add Emp)}$ **by** (*simp add:Line-Bet-not-Eq*)
from $P8$ **have** $P41 : \text{Eq (Geos (Poi A) add Emp) (Geos (Poi F) add Emp)}$ \implies
 $\text{Line-on (Li A D) } F$ **by** (*simp add:Point-Eq*)
from $P25 P41$ **have** $P42 : \neg \text{Eq (Geos (Poi F) add Emp) (Geos (Poi A) add$
 Emp) **by** (*blast intro:Eq-rev*)
from $P31 P40 P42$ **have** $P43 : \neg \text{Line-on (Li F C) } A$ **by** (*simp add:Line-not-Eq-on*)
from $P36 P43$ **have** $P44 : \neg \text{Line-on (Li C F) } A$ **by** (*simp add:Line-not-on-trans*)

from $P2$ **have** $P45 : \neg \text{Eq (Geos (Poi C) add Emp) (Geos (Poi A) add Emp)}$
by (*blast intro:Eq-rev*)
from $P31$ **have** $P46 : \neg \text{Eq (Geos (Poi C) add Emp) (Geos (Poi F) add Emp)}$
by (*blast intro:Eq-rev*)
from $P44 P45 P46$ **have** $P47 : \neg \text{Line-on (Li C A) } F$ **by** (*blast intro:Line-on-rev*)
from $P45$ **have** $P48 : \text{Eq (Geos (Lin (Li C A)) add Emp) (Geos (Lin (Li A C))$
 add Emp) **by** (*simp add:Line-rev*)
from $P47 P48$ **have** $P49 : \neg \text{Line-on (Li A C) } F$ **by** (*simp add:Line-not-on-trans*)
from *assms* $P49$ **have** $P50 : \neg \text{Eq (Geos (Lin (Li F B)) add Emp) (Geos (Lin$
 $(\text{Li F C})) \text{ add Emp)}$ **by** (*simp add:Line-Bet-not-Eq*)

from $P11$ **have** $P51 : Eq (Geos (Poi B) add Emp) (Geos (Poi F) add Emp) \implies$
 $Line-on (Li A D) F$ **by** (*simp add:Point-Eq*)
from $P25 P51$ **have** $P52 : \neg Eq (Geos (Poi F) add Emp) (Geos (Poi B) add$
 $Emp)$ **by** (*blast intro:Eq-rev*)
from $P31 P50 P52$ **have** $P53 : \neg Line-on (Li F B) C$ **by** (*simp add:Line-not-Eq-on*)
from $P27$ **have** $P54 : Line-on (Li B F) G$ **by** (*simp add:Line-Bet-on*)
from $P27$ **have** $P56 : Line-on (Li F B) G$ **by** (*simp add:Line-Bet-on*)
have $P57 : Line-on (Li F B) F$ **by** (*simp add:Line-on-rule*)
have $P58 : Line-on (Li C F) F$ **by** (*simp add:Line-on-rule*)
from $P27$ **have** $P59 : \neg Eq (Geos (Poi F) add Emp) (Geos (Poi G) add Emp)$
by (*simp add:Bet-Point-def*)
from $P56 P57 P58 P59$ **have** $P60 : Line-on (Li C F) G \implies Eq (Geos (Lin (Li$
 $C F)) add Emp) (Geos (Lin (Li F B)) add Emp)$ **by** (*simp add:Line-unique*)
have $P61 : Line-on (Li C F) C$ **by** (*simp add:Line-on-rule*)
from $P60 P61$ **have** $P62 : Line-on (Li C F) G \implies Line-on (Li F B) C$ **by**
(*simp add:Line-on-trans*)
from $P53 P62$ **have** $P63 : \neg Line-on (Li C F) G$ **by** *blast*
have $P64 : Line-on (Li C F) C$ **by** (*simp add:Line-on-rule*)
from *assms* $P28 P37 P44 P63 P64$ **have** $P65 : Line-on-Seg (Li C F) (Se A$
 $G) \wedge \neg Line-on-Seg (Li C F) (Se D G) \vee Line-on-Seg (Li C F) (Se D G) \wedge \neg$
 $Line-on-Seg (Li C F) (Se A G)$ **by** (*simp add:Pachets-axiom*)
then **have** $Line-on-Seg (Li C F) (Se A G) \implies \exists p. Line-on (Li C F) p \wedge$
 $Bet-Point (Se A G) p$ **by** (*simp add:Line-on-Seg-rule*)
then **obtain** $H :: Point$ **where** $P66 : Line-on-Seg (Li C F) (Se A G) \implies$
 $Line-on (Li C F) H \wedge Bet-Point (Se A G) H$ **by** *blast*
from $P9$ **have** $P67 : Line-on (Li A B) F \implies Line-on (Li A D) F$ **by** (*simp*
add:Line-on-trans)
from $P25 P67$ **have** $P68 : \neg Line-on (Li A B) F$ **by** *blast*
from $P4$ **have** $P69 : Eq (Geos (Lin (Li A B)) add Emp) (Geos (Lin (Li B A))$
 $add Emp)$ **by** (*simp add:Line-rev*)
from $P68 P69$ **have** $P70 : \neg Line-on (Li B A) F$ **by** (*simp add:Line-not-on-trans*)
from $P3 P27 P66 P70$ **have** $Line-on-Seg (Li C F) (Se A G) \implies \exists p. Line-on$
 $(Li H F) p \wedge Bet-Point (Se A B) p$ **by** (*simp add:Bet-swap-lemma-4*)
then **obtain** $E :: Point$ **where** $P71 : Line-on-Seg (Li C F) (Se A G) \implies Line-on$
 $(Li H F) E \wedge Bet-Point (Se A B) E$ **by** *blast*
then **have** $P72 : Line-on-Seg (Li C F) (Se A G) \implies Line-on (Li A B) E$ **by**
(*simp add:Line-Bet-on*)
from $P36$ **have** $P73 : Eq (Geos (Lin (Li C F)) add Emp) (Geos (Lin (Li F C))$
 $add Emp)$ **by** (*simp add:Eq-rev*)
from $P66 P73$ **have** $P74 : Line-on-Seg (Li C F) (Se A G) \implies Line-on (Li F$
 $C) H$ **by** (*simp add:Line-on-trans*)
from $P66$ **have** $Line-on-Seg (Li C F) (Se A G) \implies Bet-Point (Se A G) H$ **by**
simp
then **have** $Line-on-Seg (Li C F) (Se A G) \implies Eq (Geos (Poi H) add Emp)$
 $(Geos (Poi F) add Emp) \implies$
 $Bet-Point (Se A G) F$ **by** (*simp add:Point-Eq*)
then **have** $P75 : Line-on-Seg (Li C F) (Se A G) \implies Eq (Geos (Poi H) add$
 $Emp) (Geos (Poi F) add Emp) \implies$
 $Line-on (Li A G) F$ **by** (*simp add:Line-Bet-on*)

have $P76 : \text{Line-on } (Li A G) G$ **by** (*simp add:Line-on-rule*)
have $P77 : \text{Line-on } (Li B F) F$ **by** (*simp add:Line-on-rule*)
from $P54 P59 P75 P76 P77$ **have** $P78 : \text{Line-on-Seg } (Li C F) (Se A G) \implies Eq$
(Geos (Poi H) add Emp) (Geos (Poi F) add Emp) \implies
Eq (Geos (Lin (Li A G)) add Emp) (Geos (Lin (Li B F)) add Emp) **by** (*simp*
add:Line-unique)
have $P79 : \text{Line-on } (Li A G) A$ **by** (*simp add:Line-on-rule*)
from $P78 P79$ **have** $P80 : \text{Line-on-Seg } (Li C F) (Se A G) \implies Eq$ *(Geos (Poi*
H) add Emp) (Geos (Poi F) add Emp) \implies
Line-on (Li B F) A **by** (*simp add:Line-on-trans*)
have $P81 : \text{Line-on } (Li B F) B$ **by** (*simp add:Line-on-rule*)
from $P4 P6 P10 P80 P81$ **have** $P82 : \text{Line-on-Seg } (Li C F) (Se A G) \implies Eq$
(Geos (Poi H) add Emp) (Geos (Poi F) add Emp) \implies
Eq (Geos (Lin (Li B F)) add Emp) (Geos (Lin (Li A B)) add Emp) **by** (*simp*
add:Line-unique)
from $P77 P82$ **have** $P83 : \text{Line-on-Seg } (Li C F) (Se A G) \implies Eq$ *(Geos (Poi*
H) add Emp) (Geos (Poi F) add Emp) \implies
Line-on (Li A B) F **by** (*simp add:Line-on-trans*)
from $P68 P83$ **have** $P84 : \text{Line-on-Seg } (Li C F) (Se A G) \implies \neg Eq$ *(Geos (Poi*
F) add Emp) (Geos (Poi H) add Emp) **by** (*blast intro:Eq-rev*)
from $P46$ **have** $P85 : \neg Eq$ *(Geos (Poi F) add Emp) (Geos (Poi C) add Emp)*
by (*blast intro:Eq-rev*)
from $P74 P84 P85$ **have** $P86 : \text{Line-on-Seg } (Li C F) (Se A G) \implies \text{Line-on } (Li$
 $F H) C$ **by** (*blast intro:Line-on-rev*)
from $P84$ **have** $P87 : \text{Line-on-Seg } (Li C F) (Se A G) \implies Eq$ *(Geos (Lin (Li F*
 $H))$ *add Emp) (Geos (Lin (Li H F)) add Emp)* **by** (*simp add:Line-rev*)
from $P86 P87$ **have** $P88 : \text{Line-on-Seg } (Li C F) (Se A G) \implies \text{Line-on } (Li H$
 $F) C$ **by** (*simp add:Line-on-trans*)
from $P71$ **have** $\text{Line-on-Seg } (Li C F) (Se A G) \implies \text{Bet-Point } (Se A B) E$ **by**
simp
then have $P89 : \text{Line-on-Seg } (Li C F) (Se A G) \implies Eq$ *(Geos (Poi E) add*
 $Emp)$ *(Geos (Poi C) add Emp) \implies Bet-Point (Se A B) C* **by** (*simp add:Point-Eq*)
from *assms* **have** $\text{Inv } (\text{Bet-Point } (Se C B) A) \wedge \text{Inv } (\text{Bet-Point } (Se B A) C)$
by (*simp add:Bet-iff*)
then have $\neg \text{Bet-Point } (Se B A) C$ **by** (*simp add:Inv-def*)
then have $P90 : \neg \text{Bet-Point } (Se A B) C$ **by** (*blast intro:Bet-rev*)
from $P89 P90$ **have** $P91 : \text{Line-on-Seg } (Li C F) (Se A G) \implies \neg Eq$ *(Geos (Poi*
 $E)$ *add Emp) (Geos (Poi C) add Emp)* **by** *blast*
from $P5 P71 P72 P88 P91$ **have** $P92 : \text{Line-on-Seg } (Li C F) (Se A G) \implies$
Eq (Geos (Lin (Li A B)) add Emp) (Geos (Lin (Li H F)) add Emp) **by** (*simp*
add:Line-unique)
from $P4 P11 P12$ **have** $P93 : \text{Line-on } (Li A B) D$ **by** (*simp add:Line-on-rev*)
from $P92 P93$ **have** $P94 : \text{Line-on-Seg } (Li C F) (Se A G) \implies \text{Line-on } (Li H$
 $F) D$ **by** (*simp add:Line-on-trans*)
have $P95 : \text{Line-on } (Li C F) F$ **by** (*simp add:Line-on-rule*)
have $P96 : \text{Line-on } (Li H F) H$ **by** (*simp add:Line-on-rule*)
have $P97 : \text{Line-on } (Li H F) F$ **by** (*simp add:Line-on-rule*)
from $P66 P84 P95 P96 P97$ **have** $P98 : \text{Line-on-Seg } (Li C F) (Se A G) \implies$
Eq (Geos (Lin (Li H F)) add Emp) (Geos (Lin (Li C F)) add Emp) **by** (*simp*

add:Line-unique
from $P94$ $P98$ **have** $P99 : \text{Line-on-Seg } (Li\ C\ F)\ (Se\ A\ G) \implies \text{Line-on } (Li\ C\ F)\ D$ **by** (*simp add:Line-on-trans*)
from $P37$ $P99$ **have** $P100 : \neg \text{Line-on-Seg } (Li\ C\ F)\ (Se\ A\ G)$ **by** *blast*
from $P65$ $P100$ **have** $\text{Line-on-Seg } (Li\ C\ F)\ (Se\ D\ G)$ **by** *blast*
then have $\exists p. \text{Line-on } (Li\ C\ F)\ p \wedge \text{Bet-Point } (Se\ D\ G)\ p$ **by** (*simp add:Line-on-Seg-rule*)
then obtain $H2 :: \text{Point}$ **where** $P101 : \text{Line-on } (Li\ C\ F)\ H2 \wedge \text{Bet-Point } (Se\ D\ G)\ H2$ **by** *blast*
from $P23$ **have** $\text{Eq } (\text{Geos } (\text{Lin } (Li\ B\ D)))\ \text{add } \text{Emp} (\text{Geos } (\text{Lin } (Li\ A\ D)))\ \text{add } \text{Emp}$ **by** (*simp add:Eq-rev*)
then have $P102 : \text{Line-on } (Li\ B\ D)\ F \implies \text{Line-on } (Li\ A\ D)\ F$ **by** (*simp add:Line-on-trans*)
from $P25$ $P102$ **have** $P103 : \neg \text{Line-on } (Li\ B\ D)\ F$ **by** *blast*
from $P18$ **have** $P104 : \neg \text{Eq } (\text{Geos } (\text{Poi } B))\ \text{add } \text{Emp} (\text{Geos } (\text{Poi } D))\ \text{add } \text{Emp}$ **by** (*blast intro:Eq-rev*)
from $P27$ $P101$ $P103$ $P104$ **have** $\exists p. \text{Line-on } (Li\ H2\ F)\ p \wedge \text{Bet-Point } (Se\ D\ B)\ p$ **by** (*simp add:Bet-swap-lemma-4*)
then obtain $C2 :: \text{Point}$ **where** $P105 : \text{Line-on } (Li\ H2\ F)\ C2 \wedge \text{Bet-Point } (Se\ D\ B)\ C2$ **by** *blast*
have $\text{Line-on } (Li\ H2\ F)\ F$ **by** (*simp add:Line-on-rule*)
then have $P106 : \text{Eq } (\text{Geos } (\text{Lin } (Li\ H2\ F)))\ \text{add } \text{Emp} (\text{Geos } (\text{Lin } (Li\ B\ D)))\ \text{add } \text{Emp} \implies \text{Line-on } (Li\ B\ D)\ F$ **by** (*simp add:Line-on-trans*)
from $P103$ $P106$ **have** $P107 : \neg \text{Eq } (\text{Geos } (\text{Lin } (Li\ H2\ F)))\ \text{add } \text{Emp} (\text{Geos } (\text{Lin } (Li\ B\ D)))\ \text{add } \text{Emp}$ **by** *blast*
from $P73$ $P101$ **have** $P108 : \text{Line-on } (Li\ F\ C)\ H2$ **by** (*simp add:Line-on-trans*)
from $P101$ **have** $\text{Bet-Point } (Se\ D\ G)\ H2$ **by** *simp*
then have $\text{Eq } (\text{Geos } (\text{Poi } H2))\ \text{add } \text{Emp} (\text{Geos } (\text{Poi } F))\ \text{add } \text{Emp} \implies \text{Bet-Point } (Se\ D\ G)\ F$ **by** (*simp add:Point-Eq*)
then have $P109 : \text{Eq } (\text{Geos } (\text{Poi } H2))\ \text{add } \text{Emp} (\text{Geos } (\text{Poi } F))\ \text{add } \text{Emp} \implies \text{Line-on } (Li\ D\ G)\ F$ **by** (*simp add:Line-Bet-on*)
have $P110 : \text{Line-on } (Li\ D\ G)\ G$ **by** (*simp add:Line-on-rule*)
from $P54$ $P59$ $P77$ $P109$ $P110$ **have** $P111 : \text{Eq } (\text{Geos } (\text{Poi } H2))\ \text{add } \text{Emp} (\text{Geos } (\text{Poi } F))\ \text{add } \text{Emp} \implies \text{Eq } (\text{Geos } (\text{Lin } (Li\ B\ F)))\ \text{add } \text{Emp} (\text{Geos } (\text{Lin } (Li\ D\ G)))\ \text{add } \text{Emp}$ **by** (*simp add:Line-unique*)
from $P81$ $P111$ **have** $P112 : \text{Eq } (\text{Geos } (\text{Poi } H2))\ \text{add } \text{Emp} (\text{Geos } (\text{Poi } F))\ \text{add } \text{Emp} \implies \text{Line-on } (Li\ D\ G)\ B$ **by** (*simp add:Line-on-trans*)
have $P113 : \text{Line-on } (Li\ D\ G)\ D$ **by** (*simp add:Line-on-rule*)
from $P11$ $P18$ $P32$ $P112$ $P113$ **have** $P114 : \text{Eq } (\text{Geos } (\text{Poi } H2))\ \text{add } \text{Emp} (\text{Geos } (\text{Poi } F))\ \text{add } \text{Emp} \implies \text{Eq } (\text{Geos } (\text{Lin } (Li\ D\ G)))\ \text{add } \text{Emp} (\text{Geos } (\text{Lin } (Li\ A\ D)))\ \text{add } \text{Emp}$ **by** (*simp add:Line-unique*)
from $P110$ $P114$ **have** $P115 : \text{Eq } (\text{Geos } (\text{Poi } H2))\ \text{add } \text{Emp} (\text{Geos } (\text{Poi } F))\ \text{add } \text{Emp} \implies \text{Line-on } (Li\ A\ D)\ G$ **by** (*simp add:Line-on-trans*)
from $P28$ $P115$ **have** $P116 : \neg \text{Eq } (\text{Geos } (\text{Poi } F))\ \text{add } \text{Emp} (\text{Geos } (\text{Poi } H2))\ \text{add } \text{Emp}$ **by** (*blast intro:Eq-rev*)
from $P31$ $P108$ $P116$ **have** $P117 : \text{Line-on } (Li\ F\ H2)\ C$ **by** (*simp add:Line-on-rev*)
from $P116$ **have** $P118 : \text{Eq } (\text{Geos } (\text{Lin } (Li\ F\ H2)))\ \text{add } \text{Emp} (\text{Geos } (\text{Lin } (Li\ H2\ F)))\ \text{add } \text{Emp}$ **by** (*simp add:Line-rev*)

from $P117 P118$ **have** $P119 : \text{Line-on } (Li H2 F) C$ **by** (*simp add:Line-on-trans*)
from $P105$ **have** $P121 : \text{Line-on } (Li B D) C2$ **by** (*simp add:Line-Bet-on*)
from $P24 P105 P107 P119 P121$ **have** $P122 : \text{Eq } (Geos (Poi C2) add Emp)$
(*Geos (Poi C) add Emp*) **by** (*blast intro:Line-unique-Point*)
from $P105$ **have** $P123 : \text{Bet-Point } (Se D B) C2$ **by** *simp*
from $P122 P123$ **have** $\text{Bet-Point } (Se D B) C$ **by** (*simp add:Point-Eq*)
thus $\text{Bet-Point } (Se B D) C$ **by** (*blast intro:Bet-rev*)
qed

lemma(*in Order-Rule*) *Bet-swap-134-124* :

assumes

$\text{Bet-Point } (Se A C) B$

$\text{Bet-Point } (Se A D) C$

shows $\text{Bet-Point } (Se A D) B$

proof –

from *assms* **have** $P1 : \text{Bet-Point } (Se B D) C$ **by** (*blast intro:Bet-swap-134-234*)

from *assms* $P1$ **show** $\text{Bet-Point } (Se A D) B$ **by** (*blast intro:Bet-swap-234-124*)

qed

theorem(*in Order-Rule*) *Bet-swap-243-124* :

assumes

$\text{Bet-Point } (Se A D) B$

$\text{Bet-Point } (Se B D) C$

shows $\text{Bet-Point } (Se A C) B$

proof –

from *assms* **have** $P1 : \text{Bet-Point } (Se D B) C$ **by** (*simp add:Bet-rev*)

from *assms* **have** $P2 : \text{Bet-Point } (Se D A) B$ **by** (*simp add:Bet-rev*)

from $P1 P2$ **have** $\text{Bet-Point } (Se C A) B$ **by** (*blast intro:Bet-swap-134-234*)

thus $\text{Bet-Point } (Se A C) B$ **by** (*simp add:Bet-rev*)

qed

theorem(*in Order-Rule*) *Bet-swap-243-143* :

assumes

$\text{Bet-Point } (Se A D) B$

$\text{Bet-Point } (Se B D) C$

shows $\text{Bet-Point } (Se A D) C$

proof –

from *assms* **have** $P1 : \text{Bet-Point } (Se D B) C$ **by** (*simp add:Bet-rev*)

from *assms* **have** $P2 : \text{Bet-Point } (Se D A) B$ **by** (*simp add:Bet-rev*)

from $P1 P2$ **have** $\text{Bet-Point } (Se D A) C$ **by** (*blast intro:Bet-swap-134-124*)

thus $\text{Bet-Point } (Se A D) C$ **by** (*simp add:Bet-rev*)

qed

Theorem5

lemma(*in Order-Rule*) *Bet-four-Point-case* :

assumes

$\text{Line-on } l1 P$

$\text{Line-on } l1 Q$

$\text{Line-on } l1 R$

Line-on l1 S
Bet-Point (Se P R) Q
 $\neg \text{Eq (Geos (Poi P) add Emp) (Geos (Poi S) add Emp)}$
 $\neg \text{Eq (Geos (Poi Q) add Emp) (Geos (Poi S) add Emp)}$
 $\neg \text{Eq (Geos (Poi R) add Emp) (Geos (Poi S) add Emp)}$
shows *Bet-Point (Se P S) R* \vee *Bet-Point (Se R S) P*
 $\vee \text{Bet-Point (Se P R) S} \wedge \text{Bet-Point (Se P S) Q}$
 $\vee \text{Bet-Point (Se P Q) S} \vee \text{Bet-Point (Se Q S) P}$
proof –
from *assms* **have** $P1 : \neg \text{Eq (Geos (Poi P) add Emp) (Geos (Poi R) add Emp)}$
by (*simp add:Bet-Point-def*)
from *assms* **have** $P2 : \neg \text{Eq (Geos (Poi S) add Emp) (Geos (Poi P) add Emp)}$
by (*blast intro:Eq-rev*)
from *assms* $P1 P2$ **have** *Bet-Point (Se P S) R* \vee *Bet-Point (Se S R) P* \vee
Bet-Point (Se R P) S **by** (*simp add:Bet-case*)
then **have** $P3 : \text{Bet-Point (Se P S) R} \vee \text{Bet-Point (Se R S) P} \vee \text{Bet-Point (Se}$
P R) S **by** (*blast intro:Bet-rev*)
from *assms* **have** $P4 : \neg \text{Eq (Geos (Poi S) add Emp) (Geos (Poi Q) add Emp)}$
by (*blast intro:Eq-rev*)
from *assms* **have** $P5 : \neg \text{Eq (Geos (Poi Q) add Emp) (Geos (Poi P) add Emp)}$
by (*simp add:Bet-Point-def*)
from *assms* $P4 P5$ **have** *Bet-Point (Se P Q) S* \vee *Bet-Point (Se Q S) P* \vee
Bet-Point (Se S P) Q **by** (*simp add:Bet-case*)
then **have** $P6 : \text{Bet-Point (Se P Q) S} \vee \text{Bet-Point (Se Q S) P} \vee \text{Bet-Point (Se}$
P S) Q **by** (*blast intro:Bet-rev*)
from $P3 P6$ **show** *Bet-Point (Se P S) R* \vee *Bet-Point (Se R S) P*
 $\vee \text{Bet-Point (Se P R) S} \wedge \text{Bet-Point (Se P S) Q}$
 $\vee \text{Bet-Point (Se P Q) S} \vee \text{Bet-Point (Se Q S) P}$ **by** *blast*
qed

lemma(in *Order-Rule*) *Plane-diffside-rev* :

assumes

Plane-diffside l1 p1 p2

shows *Plane-diffside l1 p2 p1*

proof –

from *assms* **have** $\exists p. \text{Bet-Point (Se p1 p2) p} \wedge \text{Line-on l1 p} \wedge \neg \text{Line-on l1 p1}$
 $\wedge \neg \text{Line-on l1 p2}$ **by** (*simp add:Plane-diffside-def*)

then **obtain** $p3 :: \text{Point}$ **where** $P1 : \text{Bet-Point (Se p1 p2) p3} \wedge \text{Line-on l1 p3}$
 $\wedge \neg \text{Line-on l1 p1} \wedge \neg \text{Line-on l1 p2}$ **by** *blast*

then **have** $P2 : \text{Bet-Point (Se p2 p1) p3}$ **by** (*simp add:Bet-rev*)

from $P1 P2$ **have** $\exists p. \text{Bet-Point (Se p2 p1) p} \wedge \text{Line-on l1 p} \wedge \neg \text{Line-on l1 p2}$
 $\wedge \neg \text{Line-on l1 p1}$ **by** *blast*

thus *Plane-diffside l1 p2 p1* **by** (*simp add:Plane-diffside-def*)

qed

lemma(in *Order-Rule*) *Plane-sameside-rev* :

assumes

Plane-sameside l1 p1 p2

shows *Plane-sameside l1 p2 p1*

proof –

have $\text{Line-on-Seg } l1 \text{ (Se } p2 \text{ } p1) \implies \exists p. \text{Line-on } l1 \text{ } p \wedge \text{Bet-Point (Se } p2 \text{ } p1) \text{ } p$
by (*simp add:Line-on-Seg-rule*)
then obtain $p3 :: \text{Point}$ **where** $P1 : \text{Line-on-Seg } l1 \text{ (Se } p2 \text{ } p1) \implies$
 $\text{Line-on } l1 \text{ } p3 \wedge \text{Bet-Point (Se } p2 \text{ } p1) \text{ } p3$ **by** *blast*
then have $P2 : \text{Line-on-Seg } l1 \text{ (Se } p2 \text{ } p1) \implies \text{Bet-Point (Se } p1 \text{ } p2) \text{ } p3$ **by** (*simp*
add:Bet-rev)
from $P1 \text{ } P2$ **have** $\text{Line-on-Seg } l1 \text{ (Se } p2 \text{ } p1) \implies \exists p. \text{Line-on } l1 \text{ } p \wedge \text{Bet-Point$
 $(\text{Se } p1 \text{ } p2) \text{ } p$ **by** *blast*
then have $\text{Line-on-Seg } l1 \text{ (Se } p2 \text{ } p1) \implies \text{Line-on-Seg } l1 \text{ (Se } p1 \text{ } p2)$ **by** (*simp*
add:Line-on-Seg-rule)
then have $P3 : \neg \text{Line-on-Seg } l1 \text{ (Se } p1 \text{ } p2) \implies \neg \text{Line-on-Seg } l1 \text{ (Se } p2 \text{ } p1)$ **by**
blast
from *assms* **have** $P4 : \neg \text{Line-on-Seg } l1 \text{ (Se } p1 \text{ } p2) \wedge \neg \text{Line-on } l1 \text{ } p1$
 $\wedge \neg \text{Line-on } l1 \text{ } p2 \wedge \neg \text{Eq (Geos (Poi } p1) \text{ add Emp) (Geos (Poi } p2) \text{ add Emp)}$
by (*simp add:Plane-sameside-def*)
from $P3 \text{ } P4$ **have** $P5 : \neg \text{Line-on-Seg } l1 \text{ (Se } p2 \text{ } p1)$ **by** *blast*
from $P4$ **have** $P6 : \neg \text{Eq (Geos (Poi } p2) \text{ add Emp) (Geos (Poi } p1) \text{ add Emp)}$
by (*blast intro:Eq-rev*)
from $P4 \text{ } P5 \text{ } P6$ **show** $\text{Plane-sameside } l1 \text{ } p2 \text{ } p1$ **by** (*simp add:Plane-sameside-def*)
qed

lemma(*in Order-Rule*) *Plane-sameside-not-diffside* :

assumes $N :$

$\text{Plane-sameside } l1 \text{ } p1 \text{ } p2$

shows $\neg \text{Plane-diffside } l1 \text{ } p1 \text{ } p2$

proof

assume $W : \text{Plane-diffside } l1 \text{ } p1 \text{ } p2$

then have $\exists p. \text{Bet-Point (Se } p1 \text{ } p2) \text{ } p \wedge \text{Line-on } l1 \text{ } p \wedge \neg \text{Line-on } l1 \text{ } p1 \wedge \neg$
 $\text{Line-on } l1 \text{ } p2$ **by** (*simp add:Plane-diffside-def*)

then have $\exists p. \text{Line-on } l1 \text{ } p \wedge \text{Bet-Point (Se } p1 \text{ } p2) \text{ } p$ **by** *blast*

then have $P1 : \text{Line-on-Seg } l1 \text{ (Se } p1 \text{ } p2)$ **by** (*simp add:Line-on-Seg-rule*)

from N **have** $P2 : \neg \text{Line-on-Seg } l1 \text{ (Se } p1 \text{ } p2)$ **by** (*simp add:Plane-sameside-def*)

from $P1 \text{ } P2$ **show** False **by** *blast*

qed

lemma(*in Order-Rule*) *Plane-diffside-not-sameside* :

assumes $N :$

$\text{Plane-diffside } l1 \text{ } p1 \text{ } p2$

shows $\neg \text{Plane-sameside } l1 \text{ } p1 \text{ } p2$

proof

assume $W : \text{Plane-sameside } l1 \text{ } p1 \text{ } p2$

then have $P1 : \neg \text{Plane-diffside } l1 \text{ } p1 \text{ } p2$ **by** (*simp add:Plane-sameside-not-diffside*)

from $N \text{ } P1$ **show** False **by** *blast*

qed

lemma(*in Order-Rule*) *Plane-not-sameside-diffside* :

assumes $\neg \text{Plane-sameside } l1 \text{ } p1 \text{ } p2$

$\neg \text{Line-on } l1 \text{ } p1 \neg \text{Line-on } l1 \text{ } p2$

$\neg \text{Eq} (\text{Geos} (\text{Poi } p1) \text{ add Emp}) (\text{Geos} (\text{Poi } p2) \text{ add Emp})$
shows *Plane-diffside* $l1$ $p1$ $p2$
proof –
from *assms* **have** $P1 : \neg \text{Line-on-Seg } l1 (\text{Se } p1 \text{ } p2) \implies \text{Plane-sameside } l1 \text{ } p1 \text{ } p2$
by (*simp add:Plane-sameside-def*)
from *assms* $P1$ **have** $P2 : \text{Line-on-Seg } l1 (\text{Se } p1 \text{ } p2)$ **by** *blast*
from $P2$ **have** $P3 : \exists p. \text{Line-on } l1 \text{ } p \wedge \text{Bet-Point} (\text{Se } p1 \text{ } p2) \text{ } p$ **by** (*simp add:Line-on-Seg-rule*)
from *assms* $P3$ **have** $\exists p. \text{Bet-Point} (\text{Se } p1 \text{ } p2) \text{ } p$
 $\wedge \text{Line-on } l1 \text{ } p \wedge \neg \text{Line-on } l1 \text{ } p1 \wedge \neg \text{Line-on } l1 \text{ } p2$ **by** *blast*
thus *Plane-diffside* $l1$ $p1$ $p2$ **by** (*simp add:Plane-diffside-def*)
qed

lemma(*in Order-Rule*) *Plane-not-diffside-sameside* :
assumes $\neg \text{Plane-diffside } l1 \text{ } p1 \text{ } p2$
 $\neg \text{Line-on } l1 \text{ } p1 \neg \text{Line-on } l1 \text{ } p2$
 $\neg \text{Eq} (\text{Geos} (\text{Poi } p1) \text{ add Emp}) (\text{Geos} (\text{Poi } p2) \text{ add Emp})$
shows *Plane-sameside* $l1$ $p1$ $p2$
proof –
from *assms* **have** $P1 : \neg \text{Plane-sameside } l1 \text{ } p1 \text{ } p2 \implies \text{Plane-diffside } l1 \text{ } p1 \text{ } p2$ **by**
(*simp add:Plane-not-sameside-diffside*)
from *assms* $P1$ **show** *Plane-sameside* $l1$ $p1$ $p2$ **by** *blast*
qed

lemma(*in Order-Rule*) *Plane-Line-diff-trans* :
assumes
 $\text{Plane-diffside } l1 \text{ } p1 \text{ } p2$
 $\text{Eq} (\text{Geos} (\text{Lin } l1) \text{ add Emp}) (\text{Geos} (\text{Lin } l2) \text{ add Emp})$
shows *Plane-diffside* $l2$ $p1$ $p2$
proof –
from *assms* **have** $\exists p. \text{Bet-Point} (\text{Se } p1 \text{ } p2) \text{ } p \wedge \text{Line-on } l1 \text{ } p \wedge \neg \text{Line-on } l1 \text{ } p1$
 $\wedge \neg \text{Line-on } l1 \text{ } p2$ **by** (*simp add:Plane-diffside-def*)
then obtain $p3 :: \text{Point}$ **where** $P1 : \text{Bet-Point} (\text{Se } p1 \text{ } p2) \text{ } p3 \wedge \text{Line-on } l1 \text{ } p3$
 $\wedge \neg \text{Line-on } l1 \text{ } p1 \wedge \neg \text{Line-on } l1 \text{ } p2$ **by** *blast*
from *assms* $P1$ **have** $P2 : \text{Line-on } l2 \text{ } p3$ **by** (*simp add:Line-on-trans*)
from *assms* $P1$ **have** $P3 : \neg \text{Line-on } l2 \text{ } p1$ **by** (*simp add:Line-not-on-trans*)
from *assms* $P1$ **have** $P4 : \neg \text{Line-on } l2 \text{ } p2$ **by** (*simp add:Line-not-on-trans*)
from $P1$ $P2$ $P3$ $P4$ **have** $\exists p. \text{Bet-Point} (\text{Se } p1 \text{ } p2) \text{ } p \wedge \text{Line-on } l2 \text{ } p \wedge \neg \text{Line-on}$
 $l2 \text{ } p1 \wedge \neg \text{Line-on } l2 \text{ } p2$ **by** *blast*
thus *Plane-diffside* $l2$ $p1$ $p2$ **by** (*simp add:Plane-diffside-def*)
qed

lemma(*in Order-Rule*) *Plane-Line-trans* :
assumes
 $\text{Plane-sameside } l1 \text{ } p1 \text{ } p2$
 $\text{Eq} (\text{Geos} (\text{Lin } l1) \text{ add Emp}) (\text{Geos} (\text{Lin } l2) \text{ add Emp})$
shows *Plane-sameside* $l2$ $p1$ $p2$
proof –
have *Line-on-Seg* $l2$ $(\text{Se } p1 \text{ } p2) \implies \exists p. \text{Line-on } l2 \text{ } p \wedge \text{Bet-Point} (\text{Se } p1 \text{ } p2) \text{ } p$

by (*simp add:Line-on-Seg-rule*)
then obtain $p3 :: \text{Point}$ **where** $P1 : \text{Line-on-Seg } l2 \text{ (Se } p1 \text{ } p2) \implies \text{Line-on } l2$
 $p3 \wedge \text{Bet-Point (Se } p1 \text{ } p2) \text{ } p3$ **by** *blast*
from *assms* $P1$ **have** $P2 : \text{Line-on-Seg } l2 \text{ (Se } p1 \text{ } p2) \implies \text{Line-on } l1 \text{ } p3$ **by** (*blast*
intro:Line-on-trans Eq-rev)
from $P1 \text{ } P2$ **have** $\text{Line-on-Seg } l2 \text{ (Se } p1 \text{ } p2) \implies \exists p. \text{Line-on } l1 \text{ } p \wedge \text{Bet-Point}$
 $\text{(Se } p1 \text{ } p2) \text{ } p$ **by** *blast*
then have $P3 : \text{Line-on-Seg } l2 \text{ (Se } p1 \text{ } p2) \implies \text{Line-on-Seg } l1 \text{ (Se } p1 \text{ } p2)$ **by**
(*simp add:Line-on-Seg-rule*)
from *assms* **have** $P4 : \neg \text{Line-on-Seg } l1 \text{ (Se } p1 \text{ } p2) \wedge \neg \text{Line-on } l1 \text{ } p1$
 $\wedge \neg \text{Line-on } l1 \text{ } p2 \wedge \neg \text{Eq (Geos (Poi } p1) \text{ add Emp) (Geos (Poi } p2) \text{ add Emp)}$
by (*simp add:Plane-sameside-def*)
from $P3 \text{ } P4$ **have** $P5 : \neg \text{Line-on-Seg } l2 \text{ (Se } p1 \text{ } p2)$ **by** *blast*
from *assms* $P4$ **have** $P6 : \text{Line-on } l2 \text{ } p1 \implies \text{Line-on } l1 \text{ } p1$ **by** (*blast* *in-*
tro:Line-on-trans Eq-rev)
from $P4 \text{ } P6$ **have** $P7 : \neg \text{Line-on } l2 \text{ } p1$ **by** *blast*
from *assms* $P4$ **have** $P8 : \text{Line-on } l2 \text{ } p2 \implies \text{Line-on } l1 \text{ } p2$ **by** (*blast* *in-*
tro:Line-on-trans Eq-rev)
from $P4 \text{ } P8$ **have** $P9 : \neg \text{Line-on } l2 \text{ } p2$ **by** *blast*
from $P4 \text{ } P5 \text{ } P7 \text{ } P9$ **show** $\text{Plane-sameside } l2 \text{ } p1 \text{ } p2$ **by** (*simp add:Plane-sameside-def*)
qed

lemma(*in Order-Rule*) *Line-other-Point* :

assumes $\text{Line-on } l1 \text{ } p1$
shows $\exists p. \text{Line-on } l1 \text{ } p \wedge \neg \text{Eq (Geos (Poi } p1) \text{ add Emp) (Geos (Poi } p) \text{ add}$
 Emp)
proof –
have $\exists p \text{ } q. \text{Line-on } l1 \text{ } p \wedge \text{Line-on } l1 \text{ } q \wedge \neg \text{Eq (Geos (Poi } p) \text{ add Emp) (Geos}$
 $\text{(Poi } q) \text{ add Emp)}$ **by** (*blast intro:Line-on-exist*)
then obtain $p2 \text{ } p3 :: \text{Point}$ **where** $P1 : \text{Line-on } l1 \text{ } p2 \wedge \text{Line-on } l1 \text{ } p3 \wedge \neg \text{Eq}$
 $\text{(Geos (Poi } p2) \text{ add Emp) (Geos (Poi } p3) \text{ add Emp)}$ **by** *blast*
then have $P2 : \text{Eq (Geos (Poi } p1) \text{ add Emp) (Geos (Poi } p2) \text{ add Emp) } \wedge \text{Eq}$
 $\text{(Geos (Poi } p1) \text{ add Emp) (Geos (Poi } p3) \text{ add Emp)} \implies$
 $\text{Eq (Geos (Poi } p2) \text{ add Emp) (Geos (Poi } p3) \text{ add Emp)}$ **by** (*blast intro:Eq-trans*
Eq-rev)
from $P1 \text{ } P2$ **have** $\neg (\text{Eq (Geos (Poi } p1) \text{ add Emp) (Geos (Poi } p2) \text{ add Emp) } \wedge$
 $\text{Eq (Geos (Poi } p1) \text{ add Emp) (Geos (Poi } p3) \text{ add Emp)})$ **by** *blast*
then have $P3 : \text{Eq (Geos (Poi } p1) \text{ add Emp) (Geos (Poi } p2) \text{ add Emp) } \wedge \neg \text{Eq}$
 $\text{(Geos (Poi } p1) \text{ add Emp) (Geos (Poi } p3) \text{ add Emp)}$
 $\vee \neg \text{Eq (Geos (Poi } p1) \text{ add Emp) (Geos (Poi } p2) \text{ add Emp) } \wedge \text{Eq (Geos (Poi}$
 $\text{ } p1) \text{ add Emp) (Geos (Poi } p3) \text{ add Emp)}$
 $\vee \neg \text{Eq (Geos (Poi } p1) \text{ add Emp) (Geos (Poi } p2) \text{ add Emp) } \wedge \neg \text{Eq (Geos (Poi}$
 $\text{ } p1) \text{ add Emp) (Geos (Poi } p3) \text{ add Emp)}$ **by** *blast*
from $P1$ **have** $P4 : \text{Eq (Geos (Poi } p1) \text{ add Emp) (Geos (Poi } p2) \text{ add Emp) } \wedge \neg$
 $\text{Eq (Geos (Poi } p1) \text{ add Emp) (Geos (Poi } p3) \text{ add Emp)} \implies$
 $\exists p. \text{Line-on } l1 \text{ } p \wedge \neg \text{Eq (Geos (Poi } p1) \text{ add Emp) (Geos (Poi } p) \text{ add Emp)}$ **by**
blast
from $P1$ **have** $P5 : \neg \text{Eq (Geos (Poi } p1) \text{ add Emp) (Geos (Poi } p2) \text{ add Emp) } \wedge$
 $\text{Eq (Geos (Poi } p1) \text{ add Emp) (Geos (Poi } p3) \text{ add Emp)} \implies$

$\exists p. \text{Line-on } l1 \ p \wedge \neg \text{Eq} (\text{Geos} (\text{Poi } p1) \text{ add Emp}) (\text{Geos} (\text{Poi } p) \text{ add Emp})$ **by**
blast
from $P1$ **have** $P6 : \neg \text{Eq} (\text{Geos} (\text{Poi } p1) \text{ add Emp}) (\text{Geos} (\text{Poi } p2) \text{ add Emp}) \wedge$
 $\neg \text{Eq} (\text{Geos} (\text{Poi } p1) \text{ add Emp}) (\text{Geos} (\text{Poi } p3) \text{ add Emp}) \implies$
 $\exists p. \text{Line-on } l1 \ p \wedge \neg \text{Eq} (\text{Geos} (\text{Poi } p1) \text{ add Emp}) (\text{Geos} (\text{Poi } p) \text{ add Emp})$ **by**
blast
from $P3 \ P4 \ P5 \ P6$ **show** $\exists p. \text{Line-on } l1 \ p \wedge \neg \text{Eq} (\text{Geos} (\text{Poi } p1) \text{ add Emp})$
 $(\text{Geos} (\text{Poi } p) \text{ add Emp})$ **by** *blast*
qed

lemma(in *Order-Rule*) *Plane-Bet-sameside* :

assumes

$\text{Bet-Point} (\text{Se } p1 \ p3) \ p2$

$\text{Line-on } l1 \ p1$

$\neg \text{Eq} (\text{Geos} (\text{Lin} (\text{Li } p1 \ p3)) \text{ add Emp}) (\text{Geos} (\text{Lin } l1) \text{ add Emp})$

shows *Plane-sameside* $l1 \ p2 \ p3$

proof –

from *assms* **have** $\exists p. \text{Line-on } l1 \ p \wedge \neg \text{Eq} (\text{Geos} (\text{Poi } p1) \text{ add Emp}) (\text{Geos} (\text{Poi } p) \text{ add Emp})$ **by** (*simp add:Line-other-Point*)

then obtain $p4 :: \text{Point}$ **where** $P1 : \text{Line-on } l1 \ p4 \wedge \neg \text{Eq} (\text{Geos} (\text{Poi } p1) \text{ add Emp}) (\text{Geos} (\text{Poi } p4) \text{ add Emp})$ **by** *blast*

have $P2 : \text{Line-on} (\text{Li } p4 \ p1) \ p4$ **by** (*simp add:Line-on-rule*)

have $P3 : \text{Line-on} (\text{Li } p4 \ p1) \ p1$ **by** (*simp add:Line-on-rule*)

have *Plane-diffside* $(\text{Li } p4 \ p1) \ p2 \ p3 \implies$

$(\exists p. \text{Bet-Point} (\text{Se } p2 \ p3) \ p \wedge \text{Line-on} (\text{Li } p4 \ p1) \ p \wedge \neg \text{Line-on} (\text{Li } p4 \ p1) \ p2 \wedge \neg \text{Line-on} (\text{Li } p4 \ p1) \ p3)$ **by** (*simp add:Plane-diffside-def*)

then obtain $p5 :: \text{Point}$ **where** $P4 : \text{Plane-diffside} (\text{Li } p4 \ p1) \ p2 \ p3 \implies$

$\text{Bet-Point} (\text{Se } p2 \ p3) \ p5 \wedge \text{Line-on} (\text{Li } p4 \ p1) \ p5 \wedge \neg \text{Line-on} (\text{Li } p4 \ p1) \ p2 \wedge \neg \text{Line-on} (\text{Li } p4 \ p1) \ p3$ **by** *blast*

then have $P5 : \text{Plane-diffside} (\text{Li } p4 \ p1) \ p2 \ p3 \implies \text{Bet-Point} (\text{Se } p3 \ p2) \ p5$ **by** (*simp add:Bet-rev*)

from *assms* **have** $P6 : \text{Bet-Point} (\text{Se } p3 \ p1) \ p2$ **by** (*simp add:Bet-rev*)

from $P5 \ P6$ **have** *Plane-diffside* $(\text{Li } p4 \ p1) \ p2 \ p3 \implies \text{Bet-Point} (\text{Se } p3 \ p1) \ p5$ **by** (*blast intro:Bet-swap-134-124*)

then have $P7 : \text{Plane-diffside} (\text{Li } p4 \ p1) \ p2 \ p3 \implies \text{Line-on} (\text{Li } p3 \ p1) \ p5$ **by** (*simp add:Line-Bet-on*)

have $P8 : \text{Line-on} (\text{Li } p3 \ p1) \ p1$ **by** (*simp add:Line-on-rule*)

from $P4$ **have** *Plane-diffside* $(\text{Li } p4 \ p1) \ p2 \ p3 \implies \text{Bet-Point} (\text{Se } p2 \ p3) \ p5$ **by** *simp*

then have $P9 : \text{Plane-diffside} (\text{Li } p4 \ p1) \ p2 \ p3 \implies \text{Eq} (\text{Geos} (\text{Poi } p5) \text{ add Emp}) (\text{Geos} (\text{Poi } p1) \text{ add Emp}) \implies$

$\text{Bet-Point} (\text{Se } p2 \ p3) \ p1$ **by** (*simp add:Point-Eq*)

from *assms* **have** $\text{Inv} (\text{Bet-Point} (\text{Se } p3 \ p2) \ p1) \wedge \text{Inv} (\text{Bet-Point} (\text{Se } p2 \ p1) \ p3)$ **by** (*simp add:Bet-iff*)

then have $\neg \text{Bet-Point} (\text{Se } p3 \ p2) \ p1$ **by** (*simp add:Inv-def*)

then have $P10 : \neg \text{Bet-Point} (\text{Se } p2 \ p3) \ p1$ **by** (*blast intro:Bet-rev*)

from $P9 \ P10$ **have** $P11 : \text{Plane-diffside} (\text{Li } p4 \ p1) \ p2 \ p3 \implies \neg \text{Eq} (\text{Geos} (\text{Poi } p5) \text{ add Emp}) (\text{Geos} (\text{Poi } p1) \text{ add Emp})$ **by** *blast*

from $P3 \ P4 \ P7 \ P8 \ P11$ **have** $P12 : \text{Plane-diffside} (\text{Li } p4 \ p1) \ p2 \ p3 \implies$

$Eq (Geos (Lin (Li p3 p1)) add Emp) (Geos (Lin (Li p4 p1)) add Emp)$ **by** (*simp add:Line-unique*)
have $P13 : Line-on (Li p3 p1) p3$ **by** (*simp add:Line-on-rule*)
from $P12 P13$ **have** $P14 : Plane-diffside (Li p4 p1) p2 p3 \implies Line-on (Li p4 p1) p3$ **by** (*simp add:Line-on-trans*)
from $P4 P14$ **have** $P15 : \neg Plane-diffside (Li p4 p1) p2 p3$ **by** *blast*
from *assms* $P1 P2 P3$ **have** $Eq (Geos (Lin (Li p4 p1)) add Emp) (Geos (Lin l1) add Emp)$ **by** (*simp add:Line-unique*)
then **have** $P16 : Plane-diffside l1 p2 p3 \implies Plane-diffside (Li p4 p1) p2 p3$ **by** (*blast intro:Plane-Line-diff-trans Eq-rev*)
from $P15 P16$ **have** $P17 : \neg Plane-diffside l1 p2 p3$ **by** *blast*
from *assms* **have** $P18 : Line-on (Li p1 p3) p2$ **by** (*simp add:Line-Bet-on*)
have $P19 : Line-on (Li p1 p3) p1$ **by** (*simp add:Line-on-rule*)
have $P20 : Line-on (Li p1 p3) p3$ **by** (*simp add:Line-on-rule*)
from *assms* **have** $P21 : \neg Eq (Geos (Poi p1) add Emp) (Geos (Poi p3) add Emp)$ **by** (*simp add:Bet-Point-def*)
from *assms* $P19 P20 P21$ **have** $P22 : Line-on l1 p3 \implies Eq (Geos (Lin (Li p1 p3)) add Emp) (Geos (Lin l1) add Emp)$ **by** (*simp add:Line-unique*)
from *assms* $P22$ **have** $P23 : \neg Line-on l1 p3$ **by** *blast*
from *assms* **have** $P24 : \neg Eq (Geos (Poi p2) add Emp) (Geos (Poi p1) add Emp)$ **by** (*simp add:Bet-Point-def*)
from *assms* $P18 P19 P24$ **have** $P25 : Line-on l1 p2 \implies Eq (Geos (Lin (Li p1 p3)) add Emp) (Geos (Lin l1) add Emp)$ **by** (*simp add:Line-unique*)
from *assms* $P25$ **have** $P26 : \neg Line-on l1 p2$ **by** *blast*
from *assms* **have** $\neg Eq (Geos (Poi p3) add Emp) (Geos (Poi p2) add Emp)$ **by** (*simp add:Bet-Point-def*)
then **have** $P27 : \neg Eq (Geos (Poi p2) add Emp) (Geos (Poi p3) add Emp)$ **by** (*blast intro:Eq-rev*)
from $P17 P23 P26 P27$ **show** $Plane-sameside l1 p2 p3$ **by** (*simp add:Plane-not-diffside-sameside*)
qed

lemma(*in Order-Rule*) *Plane-Bet-diffside* :

assumes

Bet-Point (*Se* $p1 p3$) $p2$

Line-on $l1 p2$

$\neg Eq (Geos (Lin (Li p1 p3)) add Emp) (Geos (Lin l1) add Emp)$

shows *Plane-diffside* $l1 p1 p3$

proof –

from *assms* **have** $\exists p. Line-on l1 p \wedge \neg Eq (Geos (Poi p2) add Emp) (Geos (Poi p) add Emp)$ **by** (*simp add:Line-other-Point*)

then obtain $p4 :: Point$ **where** $P1 : Line-on l1 p4 \wedge \neg Eq (Geos (Poi p2) add Emp) (Geos (Poi p4) add Emp)$ **by** *blast*

from *assms* **have** $P2 : Line-on (Li p1 p3) p2$ **by** (*simp add:Line-Bet-on*)

from *assms* $P1 P2$ **have** $P3 : Line-on (Li p1 p3) p4 \implies Eq (Geos (Lin (Li p1 p3)) add Emp) (Geos (Lin l1) add Emp)$ **by** (*simp add:Line-unique*)

from *assms* $P3$ **have** $P4 : \neg Line-on (Li p1 p3) p4$ **by** *blast*

have $P5 : Line-on (Li p4 p2) p4$ **by** (*simp add:Line-on-rule*)

have $P6 : Line-on (Li p4 p2) p2$ **by** (*simp add:Line-on-rule*)

from *assms* $P4$ **have** $P7 : \neg Eq (Geos (Lin (Li p4 p2)) add Emp) (Geos (Lin$

(Li p4 p3)) *add Emp*) **by** (*simp add:Line-Bet-not-Eq*)
from *assms* **have** *Eq (Geos (Poi p2) add Emp) (Geos (Poi p4) add Emp) \implies*
Bet-Point (Se p1 p3) p4 **by** (*simp add:Point-Eq*)
then **have** *P8 : Eq (Geos (Poi p2) add Emp) (Geos (Poi p4) add Emp) \implies*
Line-on (Li p1 p3) p4 **by** (*simp add:Line-Bet-on*)
from *assms P4 P8* **have** *P9 : \neg Eq (Geos (Poi p4) add Emp) (Geos (Poi p2)*
add Emp) **by** (*blast intro:Eq-rev*)
have *Line-on (Li p1 p3) p3* **by** (*simp add:Line-on-rule*)
then **have** *P10 : Eq (Geos (Poi p3) add Emp) (Geos (Poi p4) add Emp) \implies*
Line-on (Li p1 p3) p4 **by** (*simp add:Point-Eq*)
from *assms P4 P10* **have** *P11 : \neg Eq (Geos (Poi p4) add Emp) (Geos (Poi p3)*
add Emp) **by** (*blast intro:Eq-rev*)
from *P7 P9 P11* **have** *P12 : \neg Line-on (Li p4 p2) p3* **by** (*simp add:Line-not-Eq-on*)
from *assms* **have** *P13 : Bet-Point (Se p3 p1) p2* **by** (*simp add:Bet-rev*)
from *assms* **have** *\neg Eq (Geos (Poi p1) add Emp) (Geos (Poi p3) add Emp)* **by**
(*simp add:Bet-Point-def*)
then **have** *P14 : Eq (Geos (Lin (Li p1 p3)) add Emp) (Geos (Lin (Li p3 p1))*
add Emp) **by** (*simp add:Line-rev*)
from *assms P4 P14* **have** *P15 : \neg Line-on (Li p3 p1) p4* **by** (*simp add:Line-not-on-trans*)
from *P13 P15* **have** *P16 : \neg Eq (Geos (Lin (Li p4 p2)) add Emp) (Geos (Lin*
(Li p4 p1)) add Emp) **by** (*simp add:Line-Bet-not-Eq*)
have *Line-on (Li p1 p3) p1* **by** (*simp add:Line-on-rule*)
then **have** *P17 : Eq (Geos (Poi p1) add Emp) (Geos (Poi p4) add Emp) \implies*
Line-on (Li p1 p3) p4 **by** (*simp add:Point-Eq*)
from *assms P4 P17* **have** *P18 : \neg Eq (Geos (Poi p4) add Emp) (Geos (Poi p1)*
add Emp) **by** (*blast intro:Eq-rev*)
from *P9 P16 P18* **have** *P19 : \neg Line-on (Li p4 p2) p1* **by** (*simp add:Line-not-Eq-on*)
from *assms P6 P12 P19* **have** *$\exists p. Bet-Point (Se p1 p3) p \wedge Line-on (Li p4 p2)$*
 $p \wedge \neg Line-on (Li p4 p2) p1 \wedge \neg Line-on (Li p4 p2) p3$ **by** *blast*
then **have** *P20 : Plane-diffside (Li p4 p2) p1 p3* **by** (*simp add:Plane-diffside-def*)
from *assms P1 P5 P6* **have** *P21 : Eq (Geos (Lin (Li p4 p2)) add Emp) (Geos*
(Lin l1) add Emp) **by** (*simp add:Line-unique*)
from *P20 P21* **show** *Plane-diffside l1 p1 p3* **by** (*simp add:Plane-Line-diff-trans*)
qed

lemma(*in Order-Rule*) *Plane-trans-inv* :

assumes

Plane-diffside l1 A B

Plane-diffside l1 A C

\neg Eq (Geos (Poi B) add Emp) (Geos (Poi C) add Emp)

shows *Plane-sameside l1 B C*

proof –

from *assms* **have** *$\exists p. Bet-Point (Se A B) p \wedge Line-on l1 p \wedge \neg Line-on l1 A \wedge$*
 $\neg Line-on l1 B$ **by** (*simp add:Plane-diffside-def*)

then **obtain** *D :: Point* **where** *P1 : Bet-Point (Se A B) D \wedge Line-on l1 D \wedge \neg*
Line-on l1 A \wedge \neg Line-on l1 B **by** *blast*

then **have** *P2 : Bet-Point (Se A B) D* **by** *simp*

from *assms* **have** *$\exists p. Bet-Point (Se A C) p \wedge Line-on l1 p \wedge \neg Line-on l1 A \wedge$*
 $\neg Line-on l1 C$ **by** (*simp add:Plane-diffside-def*)

then obtain $p2 :: \text{Point}$ **where** $P3 : \text{Bet-Point (Se A C) } p2 \wedge \text{Line-on l1 } p2 \wedge$
 $\neg \text{Line-on l1 A} \wedge \neg \text{Line-on l1 C}$ **by** *blast*
then have $\text{Bet-Point (Se A C) } p2$ **by** *simp*
then have $P4 : \neg \text{Eq (Geos (Poi A) add Emp) (Geos (Poi C) add Emp)}$ **by**
(simp add:Bet-Point-def)
from $P3$ **have** $P5 : \neg \text{Line-on l1 C}$ **by** *simp*
from $P1$ **have** $P6 : \text{Line-on l1 D}$ **by** *simp*
from $P1$ **have** $P7 : \neg \text{Line-on l1 A}$ **by** *simp*
from $P1$ **have** $P8 : \neg \text{Line-on l1 B}$ **by** *simp*
from $P2 P5 P6 P7 P8$ **have** $P9 : \neg \text{Line-on (Li A B) C} \implies \text{Line-on-Seg l1 (Se A C)} \wedge \neg \text{Line-on-Seg l1 (Se B C)}$
 $\vee \text{Line-on-Seg l1 (Se B C)} \wedge \neg \text{Line-on-Seg l1 (Se A C)}$ **by** *(simp add:Pachets-axiom)*
from $P3$ **have** $\text{Bet-Point (Se A C) } p2 \wedge \text{Line-on l1 } p2$ **by** *simp*
then have $\exists p. \text{Line-on l1 } p \wedge \text{Bet-Point (Se A C) } p$ **by** *blast*
then have $P10 : \text{Line-on-Seg l1 (Se A C)}$ **by** *(simp add:Line-on-Seg-rule)*
from $P9 P10$ **have** $P11 : \neg \text{Line-on (Li A B) C} \implies \neg \text{Line-on-Seg l1 (Se B C)}$
by *blast*
from *assms* $P5 P8 P11$ **have** $P12 : \neg \text{Line-on (Li A B) C} \implies \text{Plane-sameside l1 B C}$ **by** *(simp add:Plane-sameside-def)*
from $P6$ **have** $P13 : \text{Eq (Geos (Poi D) add Emp) (Geos (Poi C) add Emp)}$ \implies
 Line-on l1 C **by** *(simp add:Point-Eq)*
from $P5 P13$ **have** $P14 : \neg \text{Eq (Geos (Poi D) add Emp) (Geos (Poi C) add Emp)}$ **by** *blast*
from $P2$ **have** $P15 : \text{Line-on (Li A B) D}$ **by** *(simp add:Line-Bet-on)*
from $P2$ **have** $P16 : \text{Line-on (Li A B) A}$ **by** *(simp add:Line-on-rule)*
from $P2$ **have** $P17 : \text{Line-on (Li A B) B}$ **by** *(simp add:Line-on-rule)*
from *assms* $P2 P4 P14 P15 P16 P17$ **have** $P18 : \text{Line-on (Li A B) C} \implies$
 $\text{Bet-Point (Se A C) B} \vee \text{Bet-Point (Se B C) A}$
 $\vee \text{Bet-Point (Se A B) C} \wedge \text{Bet-Point (Se A C) D} \vee \text{Bet-Point (Se A D) C} \vee$
 $\text{Bet-Point (Se D C) A}$ **by** *(simp add:Bet-four-Point-case)*
from $P2$ **have** $P19 : \text{Line-on (Li A B) C} \implies \text{Bet-Point (Se A C) B} \implies$
 $\text{Bet-Point (Se D C) B}$ **by** *(blast intro:Bet-swap-134-234)*
have $\text{Line-on (Li D C) C}$ **by** *(simp add:Line-on-rule)*
then have $P20 : \text{Eq (Geos (Lin (Li D C)) add Emp) (Geos (Lin l1) add Emp)}$
 $\implies \text{Line-on l1 C}$ **by** *(simp add:Line-on-trans)*
from $P5 P20$ **have** $P21 : \neg \text{Eq (Geos (Lin (Li D C)) add Emp) (Geos (Lin l1) add Emp)}$ **by** *blast*
from $P6 P19 P21$ **have** $P22 : \text{Line-on (Li A B) C} \implies \text{Bet-Point (Se A C) B} \implies$
 $\text{Plane-sameside l1 B C}$ **by** *(simp add:Plane-Bet-sameside)*
from $P2$ **have** $\text{Bet-Point (Se B A) D}$ **by** *(simp add:Bet-rev)*
then have $P23 : \text{Bet-Point (Se B C) A} \implies \text{Bet-Point (Se D C) A}$ **by** *(blast intro:Bet-swap-134-234)*
from $P6 P21 P23$ **have** $P24 : \text{Bet-Point (Se B C) A} \implies \text{Plane-sameside l1 A C}$ **by** *(simp add:Plane-Bet-sameside)*
from *assms* **have** $P25 : \neg \text{Plane-sameside l1 A C}$ **by** *(simp add:Plane-diffside-not-sameside)*
from $P24 P25$ **have** $P26 : \neg \text{Bet-Point (Se B C) A}$ **by** *blast*
have $\text{Bet-Point (Se A B) C} \wedge \text{Bet-Point (Se A C) D} \implies \text{Bet-Point (Se B A) C} \wedge$
 $\text{Bet-Point (Se C A) D}$ **by** *(simp add:Bet-rev)*

then have $P27 : \text{Bet-Point } (Se\ A\ B)\ C \wedge \text{Bet-Point } (Se\ A\ C)\ D \implies \text{Bet-Point } (Se\ D\ B)\ C$ **by** (*blast intro:Bet-swap-243-124 Bet-rev*)
have $\text{Line-on } (Li\ D\ B)\ B$ **by** (*simp add:Line-on-rule*)
then have $P28 : \text{Eq } (\text{Geos } (\text{Lin } (Li\ D\ B))\ \text{add Emp})\ (\text{Geos } (\text{Lin } l1)\ \text{add Emp})$
 $\implies \text{Line-on } l1\ B$ **by** (*simp add:Line-on-trans*)
from $P8\ P28$ **have** $P29 : \neg \text{Eq } (\text{Geos } (\text{Lin } (Li\ D\ B))\ \text{add Emp})\ (\text{Geos } (\text{Lin } l1)\ \text{add Emp})$ **by** *blast*
from $P6\ P27\ P29$ **have** $P30 : \text{Bet-Point } (Se\ A\ B)\ C \wedge \text{Bet-Point } (Se\ A\ C)\ D$
 $\implies \text{Plane-sameside } l1\ B\ C$ **by** (*simp add:Plane-Bet-sameside Plane-sameside-rev*)
have $P31 : \text{Bet-Point } (Se\ A\ D)\ C \implies \text{Bet-Point } (Se\ D\ A)\ C$ **by** (*simp add:Bet-rev*)
have $\text{Line-on } (Li\ D\ A)\ A$ **by** (*simp add:Line-on-rule*)
then have $P32 : \text{Eq } (\text{Geos } (\text{Lin } (Li\ D\ A))\ \text{add Emp})\ (\text{Geos } (\text{Lin } l1)\ \text{add Emp})$
 $\implies \text{Line-on } l1\ A$ **by** (*simp add:Line-on-trans*)
from $P7\ P32$ **have** $P33 : \neg \text{Eq } (\text{Geos } (\text{Lin } (Li\ D\ A))\ \text{add Emp})\ (\text{Geos } (\text{Lin } l1)\ \text{add Emp})$ **by** *blast*
from $P6\ P31\ P33$ **have** $P34 : \text{Bet-Point } (Se\ A\ D)\ C \implies \text{Plane-sameside } l1\ A\ C$ **by** (*simp add:Plane-Bet-sameside Plane-sameside-rev*)
from $P25\ P34$ **have** $P35 : \neg \text{Bet-Point } (Se\ A\ D)\ C$ **by** *blast*
from $P6\ P21$ **have** $P36 : \text{Bet-Point } (Se\ D\ C)\ A \implies \text{Plane-sameside } l1\ A\ C$ **by** (*simp add:Plane-Bet-sameside*)
from $P25\ P36$ **have** $P37 : \neg \text{Bet-Point } (Se\ D\ C)\ A$ **by** *blast*
from $P18\ P22\ P26\ P30\ P35\ P37$ **have** $P38 : \text{Line-on } (Li\ A\ B)\ C \implies \text{Plane-sameside } l1\ B\ C$ **by** *blast*
from $P12\ P38$ **show** $\text{Plane-sameside } l1\ B\ C$ **by** *blast*
qed

lemma(*in Order-Rule*) *Plane-trans* :

assumes

$\text{Plane-sameside } l1\ A\ B$

$\text{Plane-diffside } l1\ A\ C$

shows $\text{Plane-diffside } l1\ B\ C$

proof –

from *assms* **have** $\exists p. \text{Bet-Point } (Se\ A\ C)\ p \wedge \text{Line-on } l1\ p \wedge \neg \text{Line-on } l1\ A \wedge \neg \text{Line-on } l1\ C$ **by** (*simp add:Plane-diffside-def*)

then obtain $D :: \text{Point}$ **where** $P1 : \text{Bet-Point } (Se\ A\ C)\ D \wedge \text{Line-on } l1\ D \wedge \neg \text{Line-on } l1\ A \wedge \neg \text{Line-on } l1\ C$ **by** *blast*

from *assms* **have** $P2 : \neg \text{Line-on } l1\ B$ **by** (*simp add:Plane-sameside-def*)

from $P1$ **have** $P3 : \text{Bet-Point } (Se\ A\ C)\ D$ **by** *simp*

from $P1$ **have** $P4 : \neg \text{Line-on } l1\ A$ **by** *simp*

from $P1$ **have** $P5 : \neg \text{Line-on } l1\ C$ **by** *simp*

from $P1$ **have** $P6 : \text{Line-on } l1\ D$ **by** *simp*

from $P2\ P3\ P4\ P5\ P6$ **have** $P7 : \neg \text{Line-on } (Li\ A\ C)\ B \implies \text{Line-on-Seg } l1\ (Se\ A\ B) \wedge \neg \text{Line-on-Seg } l1\ (Se\ C\ B)$

$\vee \text{Line-on-Seg } l1\ (Se\ C\ B) \wedge \neg \text{Line-on-Seg } l1\ (Se\ A\ B)$ **by** (*simp add:Pachets-axiom*)

have $P8 : \text{Line-on-Seg } l1\ (Se\ A\ B) \implies \exists p. \text{Line-on } l1\ p \wedge \text{Bet-Point } (Se\ A\ B)\ p$ **by** (*simp add:Line-on-Seg-rule*)

from $P2\ P4\ P8$ **have** $\text{Line-on-Seg } l1\ (Se\ A\ B) \implies \exists p. \text{Bet-Point } (Se\ A\ B)\ p \wedge \text{Line-on } l1\ p \wedge \neg \text{Line-on } l1\ A \wedge \neg \text{Line-on } l1\ B$ **by** *blast*

then have $\text{Line-on-Seg } l1 \text{ (Se } A \ B) \implies \text{Plane-diffside } l1 \ A \ B$ **by** (*simp add:Plane-diffside-def*)
then have $P9 : \text{Line-on-Seg } l1 \text{ (Se } A \ B) \implies \neg \text{Plane-sameside } l1 \ A \ B$ **by** (*simp add:Plane-diffside-not-sameside*)
from *assms* $P9$ **have** $P10 : \neg \text{Line-on-Seg } l1 \text{ (Se } A \ B)$ **by** *blast*
from $P7 \ P10$ **have** $\neg \text{Line-on } (Li \ A \ C) \ B \implies \text{Line-on-Seg } l1 \text{ (Se } C \ B)$ **by** *blast*
then have $P11 : \neg \text{Line-on } (Li \ A \ C) \ B \implies \exists p. \text{Line-on } l1 \ p \wedge \text{Bet-Point } (Se \ C \ B) \ p$ **by** (*simp add:Line-on-Seg-rule*)
from $P2 \ P5 \ P11$ **have** $\neg \text{Line-on } (Li \ A \ C) \ B \implies \exists p. \text{Bet-Point } (Se \ C \ B) \ p \wedge \text{Line-on } l1 \ p \wedge \neg \text{Line-on } l1 \ C \wedge \neg \text{Line-on } l1 \ B$ **by** *blast*
then have $\neg \text{Line-on } (Li \ A \ C) \ B \implies \text{Plane-diffside } l1 \ C \ B$ **by** (*simp add:Plane-diffside-def*)
then have $P12 : \neg \text{Line-on } (Li \ A \ C) \ B \implies \text{Plane-diffside } l1 \ B \ C$ **by** (*simp add:Plane-diffside-rev*)
have $P13 : \text{Line-on } (Li \ A \ C) \ A$ **by** (*simp add:Line-on-rule*)
have $P14 : \text{Line-on } (Li \ A \ C) \ C$ **by** (*simp add:Line-on-rule*)
from $P3$ **have** $P15 : \text{Line-on } (Li \ A \ C) \ D$ **by** (*simp add:Line-Bet-on*)
from *assms* **have** $\text{Eq } (\text{Geos } (Poi \ C) \ \text{add } \text{Emp}) \ (\text{Geos } (Poi \ B) \ \text{add } \text{Emp}) \implies \text{Plane-sameside } l1 \ A \ C$ **by** (*blast intro:Point-Eq Eq-rev*)
then have $P16 : \text{Eq } (\text{Geos } (Poi \ C) \ \text{add } \text{Emp}) \ (\text{Geos } (Poi \ B) \ \text{add } \text{Emp}) \implies \neg \text{Plane-diffside } l1 \ A \ C$ **by** (*simp add:Plane-sameside-not-diffside*)
from *assms* $P16$ **have** $P17 : \neg \text{Eq } (\text{Geos } (Poi \ C) \ \text{add } \text{Emp}) \ (\text{Geos } (Poi \ B) \ \text{add } \text{Emp})$ **by** *blast*
from $P6$ **have** $P18 : \text{Eq } (\text{Geos } (Poi \ D) \ \text{add } \text{Emp}) \ (\text{Geos } (Poi \ B) \ \text{add } \text{Emp}) \implies \text{Line-on } l1 \ B$ **by** (*simp add:Point-Eq*)
from $P2 \ P18$ **have** $P19 : \neg \text{Eq } (\text{Geos } (Poi \ D) \ \text{add } \text{Emp}) \ (\text{Geos } (Poi \ B) \ \text{add } \text{Emp})$ **by** *blast*
from *assms* **have** $P20 : \neg \text{Eq } (\text{Geos } (Poi \ A) \ \text{add } \text{Emp}) \ (\text{Geos } (Poi \ B) \ \text{add } \text{Emp})$ **by** (*simp add:Plane-sameside-def*)
from *assms* $P3 \ P13 \ P14 \ P15 \ P17 \ P19 \ P20$ **have** $P21 : \text{Line-on } (Li \ A \ C) \ B \implies \text{Bet-Point } (Se \ A \ B) \ C \vee \text{Bet-Point } (Se \ C \ B) \ A \vee \text{Bet-Point } (Se \ A \ C) \ B \wedge \text{Bet-Point } (Se \ A \ B) \ D \vee \text{Bet-Point } (Se \ A \ D) \ B \vee \text{Bet-Point } (Se \ D \ B) \ A$ **by** (*simp add:Bet-four-Point-case*)
from $P3$ **have** $P22 : \text{Bet-Point } (Se \ A \ B) \ C \implies \text{Bet-Point } (Se \ A \ B) \ D$ **by** (*blast intro:Bet-swap-134-124*)
have $\text{Line-on } (Li \ A \ B) \ A$ **by** (*simp add:Line-on-rule*)
then have $P23 : \text{Eq } (\text{Geos } (Lin \ (Li \ A \ B)) \ \text{add } \text{Emp}) \ (\text{Geos } (Lin \ l1) \ \text{add } \text{Emp}) \implies \text{Line-on } l1 \ A$ **by** (*simp add:Line-on-trans*)
from $P4 \ P23$ **have** $P24 : \neg \text{Eq } (\text{Geos } (Lin \ (Li \ A \ B)) \ \text{add } \text{Emp}) \ (\text{Geos } (Lin \ l1) \ \text{add } \text{Emp})$ **by** *blast*
from $P6 \ P22 \ P24$ **have** $\text{Bet-Point } (Se \ A \ B) \ C \implies \text{Plane-diffside } l1 \ A \ B$ **by** (*simp add:Plane-Bet-diffside*)
then have $P25 : \text{Bet-Point } (Se \ A \ B) \ C \implies \neg \text{Plane-sameside } l1 \ A \ B$ **by** (*simp add:Plane-diffside-not-sameside*)
from *assms* $P25$ **have** $P26 : \neg \text{Bet-Point } (Se \ A \ B) \ C$ **by** *blast*
from $P3$ **have** $P27 : \text{Bet-Point } (Se \ C \ A) \ D$ **by** (*simp add:Bet-rev*)
from $P27$ **have** $P28 : \text{Bet-Point } (Se \ C \ B) \ A \implies \text{Bet-Point } (Se \ C \ B) \ D$ **by** (*blast intro:Bet-swap-134-124*)
have $\text{Line-on } (Li \ C \ B) \ B$ **by** (*simp add:Line-on-rule*)
then have $P29 : \text{Eq } (\text{Geos } (Lin \ (Li \ C \ B)) \ \text{add } \text{Emp}) \ (\text{Geos } (Lin \ l1) \ \text{add } \text{Emp}) \implies \text{Line-on } l1 \ B$ **by** (*simp add:Line-on-trans*)

from $P2\ P29$ **have** $P30 : \neg Eq (Geos (Lin (Li\ C\ B))\ add\ Emp) (Geos (Lin\ l1)\ add\ Emp)$ **by** *blast*
from $P6\ P28\ P30$ **have** $Bet-Point (Se\ C\ B)\ A \implies Plane-diffside\ l1\ C\ B$ **by** *(simp add:Plane-Bet-diffside)*
then have $P31 : Bet-Point (Se\ C\ B)\ A \implies Plane-diffside\ l1\ B\ C$ **by** *(blast intro:Plane-diffside-rev)*
from $P6\ P24$ **have** $Bet-Point (Se\ A\ B)\ D \implies Plane-diffside\ l1\ A\ B$ **by** *(simp add:Plane-Bet-diffside)*
then have $P32 : Bet-Point (Se\ A\ B)\ D \implies \neg Plane-sameside\ l1\ A\ B$ **by** *(simp add:Plane-diffside-not-sameside)*
from *assms* $P32$ **have** $\neg Bet-Point (Se\ A\ B)\ D$ **by** *blast*
then have $P33 : \neg (Bet-Point (Se\ A\ C)\ B \wedge Bet-Point (Se\ A\ B)\ D)$ **by** *blast*
from $P3$ **have** $P34 : Bet-Point (Se\ A\ D)\ B \implies Bet-Point (Se\ C\ B)\ D$ **by** *(blast intro:Bet-swap-134-234 Bet-rev)*
from $P6\ P30\ P34$ **have** $Bet-Point (Se\ A\ D)\ B \implies Plane-diffside\ l1\ C\ B$ **by** *(simp add:Plane-Bet-diffside)*
then have $P35 : Bet-Point (Se\ A\ D)\ B \implies Plane-diffside\ l1\ B\ C$ **by** *(simp add:Plane-diffside-rev)*
from $P27$ **have** $P36 : Bet-Point (Se\ D\ B)\ A \implies Bet-Point (Se\ C\ B)\ D$ **by** *(blast intro:Bet-swap-234-124 Bet-rev)*
from $P6\ P30\ P36$ **have** $Bet-Point (Se\ D\ B)\ A \implies Plane-diffside\ l1\ C\ B$ **by** *(simp add:Plane-Bet-diffside)*
then have $P37 : Bet-Point (Se\ D\ B)\ A \implies Plane-diffside\ l1\ B\ C$ **by** *(simp add:Plane-diffside-rev)*
from $P21\ P26\ P31\ P33\ P35\ P37$ **have** $P38 : Line-on (Li\ A\ C)\ B \implies Plane-diffside\ l1\ B\ C$ **by** *blast*
from $P12\ P38$ **show** $Plane-diffside\ l1\ B\ C$ **by** *blast*
qed

lemma(in *Order-Rule*) *Plane-sameside-trans* :

assumes

Plane-sameside l1 A B

Plane-sameside l1 B C

$\neg Eq (Geos (Poi\ C)\ add\ Emp) (Geos (Poi\ A)\ add\ Emp)$

shows *Plane-sameside l1 A C*

proof –

from *assms* **have** $P1 : Plane-diffside\ l1\ A\ C \implies Plane-diffside\ l1\ B\ C$ **by** *(blast intro:Plane-trans)*

from *assms* **have** $P2 : \neg Plane-diffside\ l1\ B\ C$ **by** *(simp add:Plane-sameside-not-diffside)*

from $P1\ P2$ **have** $P3 : \neg Plane-diffside\ l1\ A\ C$ **by** *blast*

from *assms* **have** $P4 : \neg Line-on\ l1\ A$ **by** *(simp add:Plane-sameside-def)*

from *assms* **have** $P5 : \neg Line-on\ l1\ C$ **by** *(simp add:Plane-sameside-def)*

from *assms* **have** $P6 : \neg Eq (Geos (Poi\ A)\ add\ Emp) (Geos (Poi\ C)\ add\ Emp)$

by *(blast intro:Eq-rev)*

from $P3\ P4\ P5\ P6$ **show** $Plane-sameside\ l1\ A\ C$ **by** *(simp add:Plane-not-diffside-sameside)*

qed

lemma (in *Order-Rule*) *Seg-Bet-not-on* :

assumes

$Bet\text{-}Point (Se\ p1\ p3)\ p2$
shows $\neg Seg\text{-}on\text{-}Seg (Se\ p1\ p2)\ (Se\ p2\ p3)$
proof –
from *assms* **have** $\exists l. Line\text{-}on\ l\ p1 \wedge Line\text{-}on\ l\ p3 \wedge Line\text{-}on\ l\ p2$ **by** (*simp add:Line-Bet-exist*)
then obtain $l1 :: Line$ **where** $P1 : Line\text{-}on\ l1\ p1 \wedge Line\text{-}on\ l1\ p3 \wedge Line\text{-}on\ l1\ p2$ **by** *blast*
have $Seg\text{-}on\text{-}Seg (Se\ p1\ p2)\ (Se\ p2\ p3) \implies \exists p. Bet\text{-}Point (Se\ p1\ p2)\ p \wedge Bet\text{-}Point (Se\ p2\ p3)\ p$ **by** (*simp add:Seg-on-Seg-rule*)
then obtain $p4 :: Point$ **where** $P2 : Seg\text{-}on\text{-}Seg (Se\ p1\ p2)\ (Se\ p2\ p3) \implies Bet\text{-}Point (Se\ p1\ p2)\ p4 \wedge Bet\text{-}Point (Se\ p2\ p3)\ p4$ **by** *blast*
then have $P3 : Seg\text{-}on\text{-}Seg (Se\ p1\ p2)\ (Se\ p2\ p3) \implies Bet\text{-}Point (Se\ p2\ p1)\ p4$ **by** (*blast intro:Bet-rev*)
from *assms* **have** $P4 : Bet\text{-}Point (Se\ p3\ p1)\ p2$ **by** (*simp add:Bet-rev*)
from $P3\ P4$ **have** $P5 : Seg\text{-}on\text{-}Seg (Se\ p1\ p2)\ (Se\ p2\ p3) \implies Bet\text{-}Point (Se\ p3\ p1)\ p4$ **by** (*blast intro:Bet-swap-243-143*)
have $\exists p\ q\ r. \neg Line\text{-}on\ l1\ p \wedge \neg Line\text{-}on\ l1\ q \wedge \neg Line\text{-}on\ l1\ r$
 $\wedge \neg Eq (Geos (Poi\ p)\ add\ Emp)\ (Geos (Poi\ q)\ add\ Emp) \wedge \neg Eq (Geos (Poi\ q)\ add\ Emp)\ (Geos (Poi\ r)\ add\ Emp)$
 $\wedge \neg Eq (Geos (Poi\ r)\ add\ Emp)\ (Geos (Poi\ p)\ add\ Emp)$ **by** (*blast intro:Line-not-on-exist*)
then obtain $p5 :: Point$ **where** $P6 : \neg Line\text{-}on\ l1\ p5$ **by** *blast*
have $P7 : Line\text{-}on (Li\ p5\ p4)\ p5$ **by** (*simp add:Line-on-rule*)
have $P8 : Line\text{-}on (Li\ p3\ p1)\ p3$ **by** (*simp add:Line-on-rule*)
have $P9 : Line\text{-}on (Li\ p3\ p1)\ p1$ **by** (*simp add:Line-on-rule*)
from *assms* **have** $P10 : \neg Eq (Geos (Poi\ p1)\ add\ Emp)\ (Geos (Poi\ p3)\ add\ Emp)$ **by** (*simp add:Bet-Point-def*)
from $P1\ P8\ P9\ P10$ **have** $Eq (Geos (Lin (Li\ p3\ p1))\ add\ Emp)\ (Geos (Lin\ l1)\ add\ Emp)$ **by** (*simp add:Line-unique*)
then have $P11 : Line\text{-}on (Li\ p3\ p1)\ p5 \implies Line\text{-}on\ l1\ p5$ **by** (*simp add:Line-on-trans*)
from $P6\ P11$ **have** $P12 : \neg Line\text{-}on (Li\ p3\ p1)\ p5$ **by** *blast*
from $P7$ **have** $P13 : Eq (Geos (Lin (Li\ p5\ p4))\ add\ Emp)\ (Geos (Lin (Li\ p3\ p1))\ add\ Emp) \implies Line\text{-}on (Li\ p3\ p1)\ p5$ **by** (*simp add:Line-on-trans*)
from $P12\ P13$ **have** $P14 : \neg Eq (Geos (Lin (Li\ p3\ p1))\ add\ Emp)\ (Geos (Lin (Li\ p5\ p4))\ add\ Emp)$ **by** (*blast intro:Eq-rev*)
have $P15 : Line\text{-}on (Li\ p5\ p4)\ p4$ **by** (*simp add:Line-on-rule*)
from $P5\ P14\ P15$ **have** $P16 : Seg\text{-}on\text{-}Seg (Se\ p1\ p2)\ (Se\ p2\ p3) \implies Plane\text{-}diffside (Li\ p5\ p4)\ p3\ p1$ **by** (*simp add:Plane-Bet-diffside*)
have $P17 : Line\text{-}on (Li\ p1\ p2)\ p1$ **by** (*simp add:Line-on-rule*)
have $P18 : Line\text{-}on (Li\ p1\ p2)\ p2$ **by** (*simp add:Line-on-rule*)
from *assms* **have** $P19 : \neg Eq (Geos (Poi\ p2)\ add\ Emp)\ (Geos (Poi\ p1)\ add\ Emp)$ **by** (*simp add:Bet-Point-def*)
from $P1\ P17\ P18\ P19$ **have** $Eq (Geos (Lin (Li\ p1\ p2))\ add\ Emp)\ (Geos (Lin\ l1)\ add\ Emp)$ **by** (*simp add:Line-unique*)
then have $P20 : Line\text{-}on (Li\ p1\ p2)\ p5 \implies Line\text{-}on\ l1\ p5$ **by** (*simp add:Line-on-trans*)
from $P6\ P20$ **have** $P21 : \neg Line\text{-}on (Li\ p1\ p2)\ p5$ **by** *blast*
from $P7$ **have** $P22 : Eq (Geos (Lin (Li\ p5\ p4))\ add\ Emp)\ (Geos (Lin (Li\ p1\ p2))\ add\ Emp) \implies Line\text{-}on (Li\ p1\ p2)\ p5$ **by** (*simp add:Line-on-trans*)
from $P21\ P22$ **have** $P23 : \neg Eq (Geos (Lin (Li\ p1\ p2))\ add\ Emp)\ (Geos (Lin$

(Li p5 p4)) add Emp) by (blast intro:Eq-rev)
from *P2 have P24 : Seg-on-Seg (Se p1 p2) (Se p2 p3) \implies Bet-Point (Se p1 p2)*
p4 by simp
from *P15 P23 P24 have Seg-on-Seg (Se p1 p2) (Se p2 p3) \implies Plane-diffside*
(Li p5 p4) p1 p2 by (simp add:Plane-Bet-diffside)
then have *P25 : Seg-on-Seg (Se p1 p2) (Se p2 p3) \implies Plane-diffside (Li p5 p4)*
p2 p1 by (simp add:Plane-diffside-rev)
have *P26 : Line-on (Li p2 p3) p2 by (simp add:Line-on-rule)*
have *P27 : Line-on (Li p2 p3) p3 by (simp add:Line-on-rule)*
from *assms have P28 : \neg Eq (Geos (Poi p3) add Emp) (Geos (Poi p2) add*
Emp) by (simp add:Bet-Point-def)
from *P1 P26 P27 P28 have Eq (Geos (Lin (Li p2 p3)) add Emp) (Geos (Lin*
l1) add Emp) by (simp add:Line-unique)
then have *P29 : Line-on (Li p2 p3) p5 \implies Line-on l1 p5 by (simp add:Line-on-trans)*
from *P6 P29 have P30 : \neg Line-on (Li p2 p3) p5 by blast*
from *P7 have P31 : Eq (Geos (Lin (Li p5 p4)) add Emp) (Geos (Lin (Li p2*
p3)) add Emp) \implies Line-on (Li p2 p3) p5 by (simp add:Line-on-trans)
from *P30 P31 have P32 : \neg Eq (Geos (Lin (Li p2 p3)) add Emp) (Geos (Lin*
(Li p5 p4)) add Emp) by (blast intro:Eq-rev)
from *P2 have P33 : Seg-on-Seg (Se p1 p2) (Se p2 p3) \implies Bet-Point (Se p2 p3)*
p4 by simp
from *P15 P32 P33 have P34 : Seg-on-Seg (Se p1 p2) (Se p2 p3) \implies Plane-diffside*
(Li p5 p4) p2 p3 by (simp add:Plane-Bet-diffside)
from *P10 P25 P28 P34 have Seg-on-Seg (Se p1 p2) (Se p2 p3) \implies Plane-sameside*
(Li p5 p4) p1 p3 by (blast intro:Plane-trans-inv)
then have *Seg-on-Seg (Se p1 p2) (Se p2 p3) \implies Plane-sameside (Li p5 p4) p3*
p1 by (simp add:Plane-sameside-rev)
then have *P35 : Seg-on-Seg (Se p1 p2) (Se p2 p3) \implies \neg Plane-diffside (Li p5*
p4) p3 p1 by (simp add:Plane-sameside-not-diffside)
from *P16 P35 show \neg Seg-on-Seg (Se p1 p2) (Se p2 p3) by blast*
qed

end

3 Congruence

Of the equivalence relations for angles, only the transitive law is not included in the axiom, but is mentioned by the theorem. However, in the proofs before that, there are some scenes where it is regarded as congruence by the congruence relation with the same angle. Therefore, we add a weak transitive law that “when two angles are congruent, the same angle as one is congruent with the other”. Also, the uniqueness of the large and small relationship between the two angles and the transitive relation of three or more those have not been proved. Therefore, each proof regarding these is added to this section. Furthermore, regarding Theorem 23, the proof is omitted because the “large and small relationship of line segments”, which is treated as a premise, is undefined. As a result, the proof process of Theorem

24 is different from the existing one.

```

locale Definition-3 = Order-Rule +
  fixes Def :: Geo-object  $\Rightarrow$  bool
    and Cong :: Geo-objects  $\Rightarrow$  Geo-objects  $\Rightarrow$  bool
    and Gr :: Geo-objects  $\Rightarrow$  Geo-objects  $\Rightarrow$  bool
    and Ang-inside :: Angle  $\Rightarrow$  Point  $\Rightarrow$  bool
    and Right-angle :: Angle  $\Rightarrow$  bool
  assumes Tri-def : Def (Tri (Tr p1 p2 p3))  $\longleftrightarrow$   $\neg$  Eq (Geos (Poi p1) add Emp)
(Geos (Poi p2) add Emp)
   $\wedge$   $\neg$  Eq (Geos (Poi p2) add Emp) (Geos (Poi p3) add Emp)  $\wedge$   $\neg$  Eq (Geos
(Poi p3) add Emp) (Geos (Poi p1) add Emp)
   $\wedge$   $\neg$  Bet-Point (Se p1 p2) p3  $\wedge$   $\neg$  Bet-Point (Se p2 p3) p1  $\wedge$   $\neg$  Bet-Point
(Se p3 p1) p2
   $\wedge$   $\neg$  Seg-on-Seg (Se p1 p2) (Se p2 p3)  $\wedge$   $\neg$  Seg-on-Seg (Se p2 p3) (Se p3 p1)
 $\wedge$   $\neg$  Seg-on-Seg (Se p3 p1) (Se p1 p2)
  and Cong-refl [simp,intro] : Cong obs obs
  and Ang-def : Def (Ang (An p1 p2 p3))  $\longleftrightarrow$   $\neg$  Eq (Geos (Poi p1) add Emp)
(Geos (Poi p2) add Emp)
   $\wedge$   $\neg$  Eq (Geos (Poi p2) add Emp) (Geos (Poi p3) add Emp)  $\wedge$   $\neg$  Eq (Geos
(Poi p3) add Emp) (Geos (Poi p1) add Emp)
   $\wedge$   $\neg$  Eq (Geos (Lin (Li p2 p1)) add Emp) (Geos (Lin (Li p2 p3)) add Emp)
  and Ang-rev :  $\llbracket$  Cong (Geos (ang1) add Emp) (Geos (ang2) add Emp)  $\rrbracket \Longrightarrow$ 
  Cong (Geos (ang2) add Emp) (Geos (ang1) add Emp)
  and Ang-roll : Cong (Geos (Ang (An p1 p2 p3)) add Emp) (Geos (Ang (An p3
p2 p1)) add Emp)
   $\wedge$  Eq (Geos (Ang (An p1 p2 p3)) add Emp) (Geos (Ang (An p3 p2 p1)) add
Emp)
  and Ang-inside-def : Ang-inside (An p1 p2 p3) p  $\longleftrightarrow$  Def (Ang (An p1 p2
p3))  $\wedge$  Plane-sameside (Li p2 p1) p3 p  $\wedge$  Plane-sameside (Li p2 p3) p1 p
  and Ang-Point-swap :  $\llbracket$  Def (Ang (An p1 p2 p3)); Line-on (Li p2 p1) p4;  $\neg$ 
Bet-Point (Se p1 p4) p2;
  Line-on (Li p2 p3) p5;  $\neg$  Bet-Point (Se p3 p5) p2;  $\neg$  Eq (Geos (Poi p2) add
Emp) (Geos (Poi p4) add Emp);
   $\neg$  Eq (Geos (Poi p2) add Emp) (Geos (Poi p5) add Emp)  $\rrbracket \Longrightarrow$ 
  Eq (Geos (Ang (An p1 p2 p3)) add Emp) (Geos (Ang (An p4 p2 p5)) add
Emp)  $\wedge$  Def (Ang (An p4 p2 p5))
  and Ang-Right-angle-def : Right-angle (An p1 p2 p3)  $\longleftrightarrow$ 
  ( $\exists$  p. Cong (Geos (Ang (An p1 p2 p3)) add Emp) (Geos (Ang (An p1 p2 p))
add Emp)
   $\wedge$  Bet-Point (Se p3 p) p2  $\wedge$  Def (Ang (An p1 p2 p3))  $\wedge$  Def (Ang (An p1
p2 p)))
  and Tri-Cong-def : Cong (Geos (Tri (Tr p11 p12 p13)) add Emp) (Geos (Tri
(Tr p21 p22 p23)) add Emp)
   $\longleftrightarrow$  Eq (Geos (Seg (Se p11 p12)) add Emp) (Geos (Seg (Se p21 p22)) add
Emp)
   $\wedge$  Eq (Geos (Seg (Se p12 p13)) add Emp) (Geos (Seg (Se p22 p23)) add
Emp)
   $\wedge$  Eq (Geos (Seg (Se p13 p11)) add Emp) (Geos (Seg (Se p23 p21)) add
Emp)

```

$\wedge \text{Cong} (\text{Geos} (\text{Ang} (\text{An } p12 \ p11 \ p13)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } p22 \ p21 \ p23)) \text{ add Emp})$
 $\wedge \text{Cong} (\text{Geos} (\text{Ang} (\text{An } p13 \ p12 \ p11)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } p23 \ p22 \ p21)) \text{ add Emp})$
 $\wedge \text{Cong} (\text{Geos} (\text{Ang} (\text{An } p11 \ p13 \ p12)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } p21 \ p23 \ p22)) \text{ add Emp})$
and *Ang-greater-def* : $\llbracket \text{Cong} (\text{Geos} (\text{Ang } \text{an1}) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } p4 \ p2 \ p3)) \text{ add Emp})$
 $\text{Plane-sameside} (\text{Li } p2 \ p3) \ p4 \ p1 \rrbracket \implies$
 $\text{Ang-inside} (\text{An } p1 \ p2 \ p3) \ p4 \ \longleftrightarrow \ \text{Gr} (\text{Geos} (\text{Ang} (\text{An } p1 \ p2 \ p3)) \text{ add Emp})$
 $(\text{Geos} (\text{Ang } \text{an1}) \text{ add Emp})$
and *Ang-less-def* : $\llbracket \text{Cong} (\text{Geos} (\text{Ang } \text{an1}) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } p4 \ p2 \ p3)) \text{ add Emp})$
 $\text{Plane-sameside} (\text{Li } p2 \ p3) \ p4 \ p1; \neg \text{Ang-inside} (\text{An } p1 \ p2 \ p3) \ p4;$
 $\neg \text{Eq} (\text{Geos} (\text{Lin} (\text{Li } p2 \ p1)) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li } p2 \ p4)) \text{ add Emp}) \rrbracket$
 \implies
 $\text{Gr} (\text{Geos} (\text{Ang } \text{an1}) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } p1 \ p2 \ p3)) \text{ add Emp})$

locale *Axiom-3 = Definition-3 +*
assumes *Seg-add* : $\llbracket \text{Line-on } l1 \ p11; \text{Line-on } l1 \ p12; \text{Line-on } l1 \ p13; \neg \text{Seg-on-Seg} (\text{Se } p11 \ p12) (\text{Se } p12 \ p13);$
 $\text{Line-on } l2 \ p21; \text{Line-on } l2 \ p22; \text{Line-on } l2 \ p23; \neg \text{Seg-on-Seg} (\text{Se } p21 \ p22)$
 $(\text{Se } p22 \ p23);$
 $\text{Eq} (\text{Geos} (\text{Seg} (\text{Se } p11 \ p12)) \text{ add Emp}) (\text{Geos} (\text{Seg} (\text{Se } p21 \ p22)) \text{ add Emp});$
 $\text{Eq} (\text{Geos} (\text{Seg} (\text{Se } p12 \ p13)) \text{ add Emp}) (\text{Geos} (\text{Seg} (\text{Se } p22 \ p23)) \text{ add Emp}) \rrbracket$
 \implies
 $\text{Eq} (\text{Geos} (\text{Seg} (\text{Se } p11 \ p13)) \text{ add Emp}) (\text{Geos} (\text{Seg} (\text{Se } p21 \ p23)) \text{ add Emp})$
and *Seg-sub* : $\llbracket \text{Line-on } l1 \ p11; \text{Line-on } l1 \ p12; \text{Line-on } l1 \ p13; \neg \text{Seg-on-Seg} (\text{Se } p11 \ p12) (\text{Se } p12 \ p13);$
 $\text{Line-on } l2 \ p21; \text{Line-on } l2 \ p22; \text{Line-on } l2 \ p23; \neg \text{Seg-on-Seg} (\text{Se } p21 \ p22)$
 $(\text{Se } p22 \ p23);$
 $\text{Eq} (\text{Geos} (\text{Seg} (\text{Se } p11 \ p13)) \text{ add Emp}) (\text{Geos} (\text{Seg} (\text{Se } p21 \ p23)) \text{ add Emp}) \rrbracket$
 \implies
 $\text{Eq} (\text{Geos} (\text{Seg} (\text{Se } p11 \ p12)) \text{ add Emp}) (\text{Geos} (\text{Seg} (\text{Se } p21 \ p22)) \text{ add Emp})$
 $\wedge \text{Eq} (\text{Geos} (\text{Seg} (\text{Se } p12 \ p13)) \text{ add Emp}) (\text{Geos} (\text{Seg} (\text{Se } p22 \ p23)) \text{ add Emp})$
 $\text{Emp})$
and *Ang-move-sameside* : $\llbracket \neg \text{Line-on} (\text{Li } p1 \ p2) \ p3; \text{Def} (\text{Ang } a1) \rrbracket \implies \exists p. \text{Cong}$
 $(\text{Geos} (\text{Ang } a1) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } p \ p1 \ p2)) \text{ add Emp}) \wedge \text{Plane-sameside}$
 $(\text{Li } p1 \ p2) \ p \ p3$
and *Ang-move-diffside* : $\llbracket \neg \text{Line-on} (\text{Li } p1 \ p2) \ p3; \text{Def} (\text{Ang } a1) \rrbracket \implies \exists p. \text{Cong}$
 $(\text{Geos} (\text{Ang } a1) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } p \ p1 \ p2)) \text{ add Emp}) \wedge \text{Plane-diffside}$
 $(\text{Li } p1 \ p2) \ p \ p3$
and *Ang-move-unique* : $\llbracket \text{Cong} (\text{Geos} (\text{Ang } \text{an1}) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } p1 \ p2 \ p3)) \text{ add Emp});$
 $\text{Cong} (\text{Geos} (\text{Ang } \text{an1}) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } p4 \ p2 \ p3)) \text{ add Emp});$
 $\text{Plane-sameside} (\text{Li } p2 \ p3) \ p1 \ p4 \rrbracket \implies$
 $\text{Eq} (\text{Geos} (\text{Lin} (\text{Li } p1 \ p2)) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li } p4 \ p2)) \text{ add Emp}) \wedge \neg$
Bet-Point (*Se* *p1* *p4*) *p2*
and *Tri-week-SAS* : $\llbracket \text{Def} (\text{Tri} (\text{Tr } p11 \ p12 \ p13)); \text{Def} (\text{Tri} (\text{Tr } p21 \ p22 \ p23));$

$$\begin{aligned} & Eq (Geos (Seg (Se p11 p12)) add Emp) (Geos (Seg (Se p21 p22)) add Emp); \\ & Eq (Geos (Seg (Se p11 p13)) add Emp) (Geos (Seg (Se p21 p23)) add Emp); \\ & Cong (Geos (Ang (An p12 p11 p13)) add Emp) (Geos (Ang (An p22 p21 \\ & p23)) add Emp)] \\ & \implies Cong (Geos (Ang (An p13 p12 p11)) add Emp) (Geos (Ang (An p23 \\ & p22 p21)) add Emp) \end{aligned}$$

locale *Congruence-Rule* = *Axiom-3* +
assumes *Ang-weektrans* : $\llbracket Eq (Geos (Ang an1) add Emp) (Geos (Ang an2) add Emp);$
 $Cong (Geos (Ang an2) add Emp) (Geos (Ang an3) add Emp) \rrbracket \implies Cong$
 $(Geos (Ang an1) add Emp) (Geos (Ang an3) add Emp)$

lemma (in *Congruence-Rule*) *Seg-Bet-add* :

assumes

Bet-Point (Se p11 p13) p12

Bet-Point (Se p21 p23) p22

$Eq (Geos (Seg (Se p11 p12)) add Emp) (Geos (Seg (Se p21 p22)) add Emp)$

$Eq (Geos (Seg (Se p12 p13)) add Emp) (Geos (Seg (Se p22 p23)) add Emp)$

shows $Eq (Geos (Seg (Se p11 p13)) add Emp) (Geos (Seg (Se p21 p23)) add Emp)$

proof –

from *assms* **have** $\exists l. Line-on\ l\ p11 \wedge Line-on\ l\ p13 \wedge Line-on\ l\ p12$ **by** (*simp add:Line-Bet-exist*)

then obtain *l1* :: *Line* **where** $P1 : Line-on\ l1\ p11 \wedge Line-on\ l1\ p13 \wedge Line-on\ l1\ p12$ **by** *blast*

from *assms* **have** $\exists l. Line-on\ l\ p21 \wedge Line-on\ l\ p23 \wedge Line-on\ l\ p22$ **by** (*simp add:Line-Bet-exist*)

then obtain *l2* :: *Line* **where** $P2 : Line-on\ l2\ p21 \wedge Line-on\ l2\ p23 \wedge Line-on\ l2\ p22$ **by** *blast*

from *assms* **have** $P3 : \neg Seg-on-Seg (Se\ p11\ p12) (Se\ p12\ p13)$ **by** (*simp add:Seg-Bet-not-on*)

from *assms* **have** $P4 : \neg Seg-on-Seg (Se\ p21\ p22) (Se\ p22\ p23)$ **by** (*simp add:Seg-Bet-not-on*)

from *assms* *P1 P2 P3 P4* **show** $Eq (Geos (Seg (Se p11 p13)) add Emp) (Geos (Seg (Se p21 p23)) add Emp)$ **by** (*blast intro:Seg-add*)

qed

lemma (in *Congruence-Rule*) *Tri-simple-def* :

assumes

$\neg Eq (Geos (Poi A) add Emp) (Geos (Poi B) add Emp)$

$\neg Eq (Geos (Poi B) add Emp) (Geos (Poi C) add Emp)$

$\neg Eq (Geos (Poi C) add Emp) (Geos (Poi A) add Emp)$

$\neg Line-on (Li A B) C$

shows *Def* (*Tri* (*Tr* *A B C*))

proof –

have $P1 : Bet-Point (Se A B) C \implies Line-on (Li A B) C$ **by** (*simp add:Line-Bet-on*)

from *assms* *P1* **have** $P2 : \neg Bet-Point (Se A B) C$ **by** *blast*

from *assms* **have** $P3 : \neg Eq (Geos (Poi B) add Emp) (Geos (Poi A) add Emp)$

by (*blast intro:Eq-rev*)
from $P3$ **have** $P4 : Eq (Geos (Lin (Li B A)) add Emp) (Geos (Lin (Li A B)) add Emp)$ **by** (*simp add:Line-rev*)
have $P5 : Bet-Point (Se B C) A \implies Line-on (Li A B) C$ **by** (*simp add:Line-Bet-on*)
from *assms* $P5$ **have** $P6 : \neg Bet-Point (Se B C) A$ **by** *blast*
have $P7 : Bet-Point (Se C A) B \implies Line-on (Li A B) C$ **by** (*simp add:Line-Bet-on*)
from *assms* $P7$ **have** $P8 : \neg Bet-Point (Se C A) B$ **by** *blast*
have $Seg-on-Seg (Se A B) (Se B C) \implies \exists p. Bet-Point (Se A B) p \wedge Bet-Point (Se B C) p$ **by** (*simp add:Seg-on-Seg-rule*)
then obtain $D :: Point$ **where** $P9 : Seg-on-Seg (Se A B) (Se B C) \implies Bet-Point (Se A B) D \wedge Bet-Point (Se B C) D$ **by** *blast*
have $P10 : Line-on (Li A B) B$ **by** (*simp add:Line-on-rule*)
from $P9$ **have** $P11 : Seg-on-Seg (Se A B) (Se B C) \implies Line-on (Li A B) D$ **by** (*simp add:Line-Bet-on*)
have $P12 : Line-on (Li B C) B$ **by** (*simp add:Line-on-rule*)
from $P9$ **have** $P13 : Seg-on-Seg (Se A B) (Se B C) \implies Line-on (Li B C) D$ **by** (*simp add:Line-Bet-on*)
from $P9$ **have** $Seg-on-Seg (Se A B) (Se B C) \implies Bet-Point (Se A B) D$ **by** *simp*
then have $P14 : Seg-on-Seg (Se A B) (Se B C) \implies \neg Eq (Geos (Poi B) add Emp) (Geos (Poi D) add Emp)$ **by** (*simp add:Bet-Point-def*)
from $P10 P11 P12 P13 P14$ **have** $P15 : Seg-on-Seg (Se A B) (Se B C) \implies Eq (Geos (Lin (Li B C)) add Emp) (Geos (Lin (Li A B)) add Emp)$ **by** (*simp add:Line-unique*)
have $P16 : Line-on (Li B C) C$ **by** (*simp add:Line-on-rule*)
from $P15 P16$ **have** $P17 : Seg-on-Seg (Se A B) (Se B C) \implies Line-on (Li A B) C$ **by** (*simp add:Line-on-trans*)
from *assms* $P17$ **have** $P18 : \neg Seg-on-Seg (Se A B) (Se B C)$ **by** *blast*
have $Seg-on-Seg (Se B C) (Se C A) \implies \exists p. Bet-Point (Se B C) p \wedge Bet-Point (Se C A) p$ **by** (*simp add:Seg-on-Seg-rule*)
then obtain $E :: Point$ **where** $P19 : Seg-on-Seg (Se B C) (Se C A) \implies Bet-Point (Se B C) E \wedge Bet-Point (Se C A) E$ **by** *blast*
then have $P20 : Seg-on-Seg (Se B C) (Se C A) \implies Line-on (Li B C) E$ **by** (*simp add:Line-Bet-on*)
have $P21 : Line-on (Li C A) C$ **by** (*simp add:Line-on-rule*)
from $P19$ **have** $P22 : Seg-on-Seg (Se B C) (Se C A) \implies Line-on (Li C A) E$ **by** (*simp add:Line-Bet-on*)
from $P19$ **have** $Seg-on-Seg (Se B C) (Se C A) \implies Bet-Point (Se B C) E$ **by** *simp*
then have $P23 : Seg-on-Seg (Se B C) (Se C A) \implies \neg Eq (Geos (Poi C) add Emp) (Geos (Poi E) add Emp)$ **by** (*simp add:Bet-Point-def*)
from $P16 P20 P21 P22 P23$ **have** $P24 : Seg-on-Seg (Se B C) (Se C A) \implies Eq (Geos (Lin (Li C A)) add Emp) (Geos (Lin (Li B C)) add Emp)$ **by** (*simp add:Line-unique*)
have $P25 : Line-on (Li C A) A$ **by** (*simp add:Line-on-rule*)
from $P24 P25$ **have** $P26 : Seg-on-Seg (Se B C) (Se C A) \implies Line-on (Li B C) A$ **by** (*simp add:Line-on-trans*)
from *assms* $P3 P26$ **have** $P27 : Seg-on-Seg (Se B C) (Se C A) \implies Line-on (Li B A) C$ **by** (*blast intro:Line-on-rev*)

from P_4 P_{27} **have** $P_{28} : \text{Seg-on-Seg } (Se\ B\ C) (Se\ C\ A) \implies \text{Line-on } (Li\ A\ B)$
 C **by** (*simp add:Line-on-trans*)
from *assms* P_{28} **have** $P_{29} : \neg \text{Seg-on-Seg } (Se\ B\ C) (Se\ C\ A)$ **by** *blast*
have $\text{Seg-on-Seg } (Se\ C\ A) (Se\ A\ B) \implies \exists p. \text{Bet-Point } (Se\ C\ A)\ p \wedge \text{Bet-Point}$
 $(Se\ A\ B)\ p$ **by** (*simp add:Seg-on-Seg-rule*)
then obtain $F :: \text{Point}$ **where** $P_{30} : \text{Seg-on-Seg } (Se\ C\ A) (Se\ A\ B) \implies \text{Bet-Point}$
 $(Se\ C\ A)\ F \wedge \text{Bet-Point } (Se\ A\ B)\ F$ **by** *blast*
then have $P_{31} : \text{Seg-on-Seg } (Se\ C\ A) (Se\ A\ B) \implies \text{Line-on } (Li\ C\ A)\ F$ **by**
(simp add:Line-Bet-on)
have $P_{32} : \text{Line-on } (Li\ A\ B)\ A$ **by** (*simp add:Line-on-rule*)
from P_{30} **have** $P_{33} : \text{Seg-on-Seg } (Se\ C\ A) (Se\ A\ B) \implies \text{Line-on } (Li\ A\ B)\ F$
by (*simp add:Line-Bet-on*)
from P_{30} **have** $\text{Seg-on-Seg } (Se\ C\ A) (Se\ A\ B) \implies \text{Bet-Point } (Se\ C\ A)\ F$ **by**
simp
then have $P_{34} : \text{Seg-on-Seg } (Se\ C\ A) (Se\ A\ B) \implies \neg \text{Eq } (\text{Geos } (Poi\ A)\ \text{add}$
 $\text{Emp}) (\text{Geos } (Poi\ F)\ \text{add}\ \text{Emp})$ **by** (*simp add:Bet-Point-def*)
from P_{25} P_{31} P_{32} P_{33} P_{34} **have** $P_{35} : \text{Seg-on-Seg } (Se\ C\ A) (Se\ A\ B) \implies$
 $\text{Eq } (\text{Geos } (Lin\ (Li\ C\ A))\ \text{add}\ \text{Emp}) (\text{Geos } (Lin\ (Li\ A\ B))\ \text{add}\ \text{Emp})$ **by** (*simp*
add:Line-unique)
from P_{21} P_{35} **have** $P_{36} : \text{Seg-on-Seg } (Se\ C\ A) (Se\ A\ B) \implies \text{Line-on } (Li\ A\ B)$
 C **by** (*simp add:Line-on-trans*)
from *assms* P_{36} **have** $P_{37} : \neg \text{Seg-on-Seg } (Se\ C\ A) (Se\ A\ B)$ **by** *blast*
from *assms* P_2 P_6 P_8 P_{18} P_{29} P_{37} **show** $\text{Def } (\text{Tri } (\text{Tr } A\ B\ C))$ **by** (*simp*
add:Tri-def)
qed

lemma (in Congruence-Rule) Tri-def-Line :

assumes

$\text{Def } (\text{Tri } (\text{Tr } A\ B\ C))$

shows $\neg \text{Line-on } (Li\ A\ B)\ C \wedge \neg \text{Line-on } (Li\ B\ C)\ A \wedge \neg \text{Line-on } (Li\ C\ A)\ B$

proof –

from *assms* **have** $P_1 : \neg \text{Eq } (\text{Geos } (Poi\ A)\ \text{add}\ \text{Emp}) (\text{Geos } (Poi\ B)\ \text{add}\ \text{Emp})$
 $\wedge \neg \text{Eq } (\text{Geos } (Poi\ B)\ \text{add}\ \text{Emp}) (\text{Geos } (Poi\ C)\ \text{add}\ \text{Emp}) \wedge \neg \text{Eq } (\text{Geos}$
 $(Poi\ C)\ \text{add}\ \text{Emp}) (\text{Geos } (Poi\ A)\ \text{add}\ \text{Emp})$

$\wedge \neg \text{Bet-Point } (Se\ A\ B)\ C \wedge \neg \text{Bet-Point } (Se\ B\ C)\ A \wedge \neg \text{Bet-Point } (Se$
 $C\ A)\ B$ **by** (*simp add:Tri-def*)

have $P_2 : \text{Line-on } (Li\ A\ B)\ B$ **by** (*simp add:Line-on-rule*)

have $P_3 : \text{Line-on } (Li\ A\ B)\ A$ **by** (*simp add:Line-on-rule*)

from P_1 P_2 P_3 **have** $P_4 : \text{Line-on } (Li\ A\ B)\ C \implies \text{Bet-Point } (Se\ A\ C)\ B \vee$
 $\text{Bet-Point } (Se\ C\ B)\ A \vee \text{Bet-Point } (Se\ B\ A)\ C$ **by** (*simp add:Bet-case*)

from P_1 **have** $P_5 : \neg \text{Bet-Point } (Se\ A\ C)\ B$ **by** (*blast intro:Bet-rev*)

from P_1 **have** $P_6 : \neg \text{Bet-Point } (Se\ C\ B)\ A$ **by** (*blast intro:Bet-rev*)

from P_1 **have** $P_7 : \neg \text{Bet-Point } (Se\ B\ A)\ C$ **by** (*blast intro:Bet-rev*)

from P_4 P_5 P_6 P_7 **have** $P_8 : \neg \text{Line-on } (Li\ A\ B)\ C$ **by** *blast*

have $P_9 : \text{Line-on } (Li\ B\ C)\ B$ **by** (*simp add:Line-on-rule*)

have $P_{10} : \text{Line-on } (Li\ B\ C)\ C$ **by** (*simp add:Line-on-rule*)

from P_1 P_9 P_{10} **have** $P_{11} : \text{Line-on } (Li\ B\ C)\ A \implies \text{Bet-Point } (Se\ A\ C)\ B \vee$
 $\text{Bet-Point } (Se\ C\ B)\ A \vee \text{Bet-Point } (Se\ B\ A)\ C$ **by** (*simp add:Bet-case*)

from P_5 P_6 P_7 P_{11} **have** $P_{12} : \neg \text{Line-on } (Li\ B\ C)\ A$ **by** *blast*

have $P13 : \text{Line-on } (Li\ C\ A)\ C$ **by** (*simp add:Line-on-rule*)
have $P14 : \text{Line-on } (Li\ C\ A)\ A$ **by** (*simp add:Line-on-rule*)
from $P1\ P13\ P14$ **have** $P15 : \text{Line-on } (Li\ C\ A)\ B \implies \text{Bet-Point } (Se\ A\ C)\ B$
 $\vee \text{Bet-Point } (Se\ C\ B)\ A \vee \text{Bet-Point } (Se\ B\ A)\ C$ **by** (*simp add:Bet-case*)
from $P5\ P6\ P7\ P15$ **have** $P16 : \neg \text{Line-on } (Li\ C\ A)\ B$ **by** *blast*
from $P8\ P12\ P16$ **show** $\neg \text{Line-on } (Li\ A\ B)\ C \wedge \neg \text{Line-on } (Li\ B\ C)\ A \wedge \neg$
 $\text{Line-on } (Li\ C\ A)\ B$ **by** *simp*
qed

lemma (*in Congruence-Rule*) *Tri-def-trans* :

assumes
 $\text{Def } (Tri\ (Tr\ A\ B\ C))$
shows $\text{Def } (Tri\ (Tr\ B\ C\ A))$
proof –
from *assms* **have** $P1 : \neg \text{Eq } (Geos\ (Poi\ A)\ \text{add}\ \text{Emp})\ (Geos\ (Poi\ B)\ \text{add}\ \text{Emp})$
 $\wedge \neg \text{Eq } (Geos\ (Poi\ B)\ \text{add}\ \text{Emp})\ (Geos\ (Poi\ C)\ \text{add}\ \text{Emp}) \wedge \neg \text{Eq } (Geos$
 $(Poi\ C)\ \text{add}\ \text{Emp})\ (Geos\ (Poi\ A)\ \text{add}\ \text{Emp})$
 $\wedge \neg \text{Bet-Point } (Se\ A\ B)\ C \wedge \neg \text{Bet-Point } (Se\ B\ C)\ A \wedge \neg \text{Bet-Point } (Se$
 $C\ A)\ B$ **by** (*simp add:Tri-def*)
from *assms* **have** $P2 : \neg \text{Line-on } (Li\ B\ C)\ A$ **by** (*simp add:Tri-def-Line*)
from $P1\ P2$ **show** $\text{Def } (Tri\ (Tr\ B\ C\ A))$ **by** (*simp add:Tri-simple-def*)
qed

lemma (*in Congruence-Rule*) *Tri-def-rev* :

assumes
 $\text{Def } (Tri\ (Tr\ A\ B\ C))$
shows $\text{Def } (Tri\ (Tr\ C\ B\ A))$
proof –
from *assms* **have** $P1 : \neg \text{Eq } (Geos\ (Poi\ A)\ \text{add}\ \text{Emp})\ (Geos\ (Poi\ B)\ \text{add}\ \text{Emp})$
 $\wedge \neg \text{Eq } (Geos\ (Poi\ B)\ \text{add}\ \text{Emp})\ (Geos\ (Poi\ C)\ \text{add}\ \text{Emp}) \wedge \neg \text{Eq } (Geos$
 $(Poi\ C)\ \text{add}\ \text{Emp})\ (Geos\ (Poi\ A)\ \text{add}\ \text{Emp})$
 $\wedge \neg \text{Bet-Point } (Se\ A\ B)\ C \wedge \neg \text{Bet-Point } (Se\ B\ C)\ A \wedge \neg \text{Bet-Point } (Se$
 $C\ A)\ B$ **by** (*simp add:Tri-def*)
from *assms* **have** $P2 : \neg \text{Line-on } (Li\ B\ C)\ A$ **by** (*simp add:Tri-def-Line*)
from $P1$ **have** $P3 : \text{Eq } (Geos\ (Lin\ (Li\ B\ C))\ \text{add}\ \text{Emp})\ (Geos\ (Lin\ (Li\ C\ B))$
 $\text{add}\ \text{Emp})$ **by** (*simp add:Line-rev*)
from $P2\ P3$ **have** $P4 : \neg \text{Line-on } (Li\ C\ B)\ A$ **by** (*simp add:Line-not-on-trans*)
from $P1$ **have** $P5 : \neg \text{Eq } (Geos\ (Poi\ C)\ \text{add}\ \text{Emp})\ (Geos\ (Poi\ B)\ \text{add}\ \text{Emp})$ **by**
(blast intro:Eq-rev)
from $P1$ **have** $P6 : \neg \text{Eq } (Geos\ (Poi\ B)\ \text{add}\ \text{Emp})\ (Geos\ (Poi\ A)\ \text{add}\ \text{Emp})$ **by**
(blast intro:Eq-rev)
from $P1$ **have** $P7 : \neg \text{Eq } (Geos\ (Poi\ A)\ \text{add}\ \text{Emp})\ (Geos\ (Poi\ C)\ \text{add}\ \text{Emp})$ **by**
(blast intro:Eq-rev)
from $P4\ P5\ P6\ P7$ **show** $\text{Def } (Tri\ (Tr\ C\ B\ A))$ **by** (*simp add:Tri-simple-def*)
qed

lemma (*in Congruence-Rule*) *Tri-def-extension* :

assumes
 $\text{Def } (Tri\ (Tr\ A\ B\ C))$

$\neg \text{Eq} (\text{Geos} (\text{Poi } B) \text{ add Emp}) (\text{Geos} (\text{Poi } D) \text{ add Emp})$
 $\text{Line-on} (\text{Li } B \ C) \ D$
shows $\text{Def} (\text{Tri} (\text{Tr } A \ B \ D))$
proof –
from *assms* **have** $P1 : \neg \text{Eq} (\text{Geos} (\text{Poi } A) \text{ add Emp}) (\text{Geos} (\text{Poi } B) \text{ add Emp})$
by (*simp add:Tri-def*)
from *assms* **have** $P2 : \neg \text{Line-on} (\text{Li } B \ C) \ A$ **by** (*simp add:Tri-def-Line*)
from *assms* **have** $P3 : \text{Eq} (\text{Geos} (\text{Poi } D) \text{ add Emp}) (\text{Geos} (\text{Poi } A) \text{ add Emp})$
 $\implies \text{Line-on} (\text{Li } B \ C) \ A$ **by** (*simp add:Point-Eq*)
from $P2 \ P3$ **have** $P4 : \neg \text{Eq} (\text{Geos} (\text{Poi } D) \text{ add Emp}) (\text{Geos} (\text{Poi } A) \text{ add Emp})$
by *blast*
from *assms* **have** $P5 : \neg \text{Eq} (\text{Geos} (\text{Poi } B) \text{ add Emp}) (\text{Geos} (\text{Poi } C) \text{ add Emp})$
by (*simp add:Tri-def*)
from *assms* $P5$ **have** $P6 : \text{Line-on} (\text{Li } B \ D) \ C$ **by** (*simp add:Line-on-rev*)
have $P7 : \text{Line-on} (\text{Li } B \ D) \ B$ **by** (*simp add:Line-on-rule*)
have $P8 : \text{Line-on} (\text{Li } B \ C) \ B$ **by** (*simp add:Line-on-rule*)
have $P9 : \text{Line-on} (\text{Li } B \ C) \ C$ **by** (*simp add:Line-on-rule*)
from $P5 \ P6 \ P7 \ P8 \ P9$ **have** $\text{Eq} (\text{Geos} (\text{Lin} (\text{Li } B \ D)) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li } B \ C)) \text{ add Emp})$ **by** (*simp add:Line-unique*)
then **have** $P10 : \text{Line-on} (\text{Li } B \ D) \ A \implies \text{Line-on} (\text{Li } B \ C) \ A$ **by** (*simp add:Line-on-trans*)
from $P2 \ P10$ **have** $P11 : \neg \text{Line-on} (\text{Li } B \ D) \ A$ **by** *blast*
from *assms* $P1 \ P4 \ P11$ **have** $\text{Def} (\text{Tri} (\text{Tr } B \ D \ A))$ **by** (*simp add:Tri-simple-def*)
thus $\text{Def} (\text{Tri} (\text{Tr } A \ B \ D))$ **by** (*simp add:Tri-def-trans*)
qed

lemma (in Congruence-Rule) Ang-to-Tri :

assumes
 $\text{Def} (\text{Ang} (\text{An } A \ B \ C))$
shows $\text{Def} (\text{Tri} (\text{Tr } A \ B \ C))$
proof –
from *assms* **have** $P1 : \neg \text{Eq} (\text{Geos} (\text{Poi } A) \text{ add Emp}) (\text{Geos} (\text{Poi } B) \text{ add Emp})$
 $\wedge \neg \text{Eq} (\text{Geos} (\text{Poi } B) \text{ add Emp}) (\text{Geos} (\text{Poi } C) \text{ add Emp}) \wedge \neg \text{Eq} (\text{Geos} (\text{Poi } C) \text{ add Emp}) (\text{Geos} (\text{Poi } A) \text{ add Emp})$
 $\wedge \neg \text{Eq} (\text{Geos} (\text{Lin} (\text{Li } B \ A)) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li } B \ C)) \text{ add Emp})$ **by**
(*simp add:Ang-def*)
have $P2 : \text{Line-on} (\text{Li } B \ A) \ B$ **by** (*simp add:Line-on-rule*)
have $P3 : \text{Line-on} (\text{Li } B \ C) \ B$ **by** (*simp add:Line-on-rule*)
have $P4 : \text{Line-on} (\text{Li } B \ C) \ C$ **by** (*simp add:Line-on-rule*)
from $P1$ **have** $P5 : \neg \text{Eq} (\text{Geos} (\text{Poi } B) \text{ add Emp}) (\text{Geos} (\text{Poi } C) \text{ add Emp})$ **by**
simp
from $P1$ **have** $\text{Eq} (\text{Geos} (\text{Lin} (\text{Li } A \ B)) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li } B \ A)) \text{ add Emp})$ **by** (*simp add:Line-rev*)
then **have** $P6 : \text{Line-on} (\text{Li } A \ B) \ C \implies \text{Line-on} (\text{Li } B \ A) \ C$ **by** (*simp add:Line-on-trans*)
from $P2 \ P3 \ P4 \ P5 \ P6$ **have** $P7 : \text{Line-on} (\text{Li } A \ B) \ C \implies \text{Eq} (\text{Geos} (\text{Lin} (\text{Li } B \ A)) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li } B \ C)) \text{ add Emp})$ **by** (*simp add:Line-unique*)
from $P1 \ P7$ **have** $P8 : \neg \text{Line-on} (\text{Li } A \ B) \ C$ **by** *blast*
from $P1 \ P8$ **show** $\text{Def} (\text{Tri} (\text{Tr } A \ B \ C))$ **by** (*simp add:Tri-simple-def*)

qed

lemma (in *Congruence-Rule*) *Ang-simple-def* :

assumes

$\neg \text{Eq} (\text{Geos} (\text{Poi } A) \text{ add Emp}) (\text{Geos} (\text{Poi } B) \text{ add Emp})$

$\neg \text{Line-on} (\text{Li } A \ B) \ C$

shows $\text{Def} (\text{Ang} (\text{An } A \ B \ C))$

proof –

from *assms* **have** $P1 : \text{Eq} (\text{Geos} (\text{Lin} (\text{Li } A \ B)) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li } B \ A)) \text{ add Emp})$ **by** (*simp add:Line-rev*)

from *assms* $P1$ **have** $P2 : \neg \text{Line-on} (\text{Li } B \ A) \ C$ **by** (*simp add:Line-not-on-trans*)

have $\text{Line-on} (\text{Li } B \ A) \ B$ **by** (*simp add:Line-on-rule*)

then **have** $P3 : \text{Eq} (\text{Geos} (\text{Poi } B) \text{ add Emp}) (\text{Geos} (\text{Poi } C) \text{ add Emp}) \implies \text{Line-on} (\text{Li } B \ A) \ C$ **by** (*simp add:Point-Eq*)

from $P2 \ P3$ **have** $P4 : \neg \text{Eq} (\text{Geos} (\text{Poi } B) \text{ add Emp}) (\text{Geos} (\text{Poi } C) \text{ add Emp})$ **by** *blast*

have $\text{Line-on} (\text{Li } B \ A) \ A$ **by** (*simp add:Line-on-rule*)

then **have** $P5 : \text{Eq} (\text{Geos} (\text{Poi } A) \text{ add Emp}) (\text{Geos} (\text{Poi } C) \text{ add Emp}) \implies \text{Line-on} (\text{Li } B \ A) \ C$ **by** (*simp add:Point-Eq*)

from $P2 \ P5$ **have** $P6 : \neg \text{Eq} (\text{Geos} (\text{Poi } C) \text{ add Emp}) (\text{Geos} (\text{Poi } A) \text{ add Emp})$ **by** (*blast intro:Eq-rev*)

have $\text{Line-on} (\text{Li } B \ C) \ C$ **by** (*simp add:Line-on-rule*)

then **have** $P7 : \text{Eq} (\text{Geos} (\text{Lin} (\text{Li } B \ C)) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li } B \ A)) \text{ add Emp}) \implies \text{Line-on} (\text{Li } B \ A) \ C$ **by** (*simp add:Line-on-trans*)

from $P2 \ P7$ **have** $P8 : \neg \text{Eq} (\text{Geos} (\text{Lin} (\text{Li } B \ A)) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li } B \ C)) \text{ add Emp})$ **by** (*blast intro:Eq-rev*)

from *assms* $P4 \ P6 \ P8$ **show** $\text{Def} (\text{Ang} (\text{An } A \ B \ C))$ **by** (*simp add:Ang-def*)

qed

lemma (in *Congruence-Rule*) *Tri-to-Ang* :

assumes

$\text{Def} (\text{Tri} (\text{Tr } A \ B \ C))$

shows $\text{Def} (\text{Ang} (\text{An } A \ B \ C))$

proof –

from *assms* **have** $P1 : \neg \text{Eq} (\text{Geos} (\text{Poi } A) \text{ add Emp}) (\text{Geos} (\text{Poi } B) \text{ add Emp}) \wedge \neg \text{Eq} (\text{Geos} (\text{Poi } B) \text{ add Emp}) (\text{Geos} (\text{Poi } C) \text{ add Emp}) \wedge \neg \text{Eq} (\text{Geos} (\text{Poi } C) \text{ add Emp}) (\text{Geos} (\text{Poi } A) \text{ add Emp})$ **by** (*simp add:Tri-def*)

from *assms* **have** $P2 : \neg \text{Line-on} (\text{Li } A \ B) \ C$ **by** (*simp add:Tri-def-Line*)

from $P1 \ P2$ **show** $\text{Def} (\text{Ang} (\text{An } A \ B \ C))$ **by** (*simp add:Ang-simple-def*)

qed

lemma (in *Congruence-Rule*) *Ang-def-rev* :

assumes

$\text{Def} (\text{Ang} (\text{An } A \ B \ C))$

shows $\text{Def} (\text{Ang} (\text{An } C \ B \ A))$

proof –

from *assms* **have** $P1 : \neg \text{Eq} (\text{Geos} (\text{Poi } A) \text{ add Emp}) (\text{Geos} (\text{Poi } B) \text{ add Emp}) \wedge \neg \text{Eq} (\text{Geos} (\text{Poi } B) \text{ add Emp}) (\text{Geos} (\text{Poi } C) \text{ add Emp}) \wedge \neg \text{Eq} (\text{Geos} (\text{Poi } C) \text{ add Emp}) (\text{Geos} (\text{Poi } A) \text{ add Emp})$

$\wedge \neg Eq (Geos (Lin (Li B A)) add Emp) (Geos (Lin (Li B C)) add Emp)$ **by**
(simp add:Ang-def)
have $P2 : Line-on (Li B A) A$ **by** *(simp add:Line-on-rule)*
have $P3 : Line-on (Li B A) B$ **by** *(simp add:Line-on-rule)*
have $P4 : Line-on (Li B C) B$ **by** *(simp add:Line-on-rule)*
from $P1$ **have** $P5 : \neg Eq (Geos (Poi A) add Emp) (Geos (Poi B) add Emp)$ **by**
simp
from $P2 P3 P4 P5$ **have** $P6 : Line-on (Li B C) A \implies Eq (Geos (Lin (Li B A))$
 $add Emp) (Geos (Lin (Li B C)) add Emp)$ **by** *(simp add:Line-unique)*
from $P1 P6$ **have** $P7 : \neg Line-on (Li B C) A$ **by** *blast*
from $P1$ **have** $P8 : Eq (Geos (Lin (Li B C)) add Emp) (Geos (Lin (Li C B))$
 $add Emp)$ **by** *(simp add:Line-rev)*
from $P7 P8$ **have** $P9 : \neg Line-on (Li C B) A$ **by** *(simp add:Line-not-on-trans)*
from $P1$ **have** $P10 : \neg Eq (Geos (Poi C) add Emp) (Geos (Poi B) add Emp)$
by *(blast intro:Eq-rev)*
from $P9 P10$ **show** $Def (Ang (An C B A))$ **by** *(simp add:Ang-simple-def)*
qed

lemma (in Congruence-Rule) Ang-def-inv :

assumes

$Def (Ang (An A B C))$

shows $Def (Ang (An A C B))$

proof –

from *assms* **have** $P1 : \neg Eq (Geos (Poi A) add Emp) (Geos (Poi B) add Emp)$
 $\wedge \neg Eq (Geos (Poi B) add Emp) (Geos (Poi C) add Emp) \wedge \neg Eq (Geos (Poi$
 $C) add Emp) (Geos (Poi A) add Emp)$
 $\wedge \neg Eq (Geos (Lin (Li B A)) add Emp) (Geos (Lin (Li B C)) add Emp)$ **by**
(simp add:Ang-def)
have $P2 : Line-on (Li B A) A$ **by** *(simp add:Line-on-rule)*
have $P3 : Line-on (Li B A) B$ **by** *(simp add:Line-on-rule)*
have $P4 : Line-on (Li B C) B$ **by** *(simp add:Line-on-rule)*
from $P1$ **have** $P5 : \neg Eq (Geos (Poi A) add Emp) (Geos (Poi B) add Emp)$ **by**
simp
from $P2 P3 P4 P5$ **have** $P6 : Line-on (Li B C) A \implies Eq (Geos (Lin (Li B A))$
 $add Emp) (Geos (Lin (Li B C)) add Emp)$ **by** *(simp add:Line-unique)*
from $P1 P6$ **have** $P7 : \neg Line-on (Li B C) A$ **by** *blast*
have $P8 : Line-on (Li A C) C$ **by** *(simp add:Line-on-rule)*
have $P9 : Line-on (Li B C) C$ **by** *(simp add:Line-on-rule)*
from $P1 P4 P8 P9$ **have** $P10 : Line-on (Li A C) B \implies Eq (Geos (Lin (Li A$
 $C)) add Emp) (Geos (Lin (Li B C)) add Emp)$ **by** *(simp add:Line-unique)*
have $P11 : Line-on (Li A C) A$ **by** *(simp add:Line-on-rule)*
from $P10 P11$ **have** $P12 : Line-on (Li A C) B \implies Line-on (Li B C) A$ **by**
(simp add:Line-on-trans)
from $P7 P12$ **have** $P13 : \neg Line-on (Li A C) B$ **by** *blast*
from $P1$ **have** $P14 : \neg Eq (Geos (Poi A) add Emp) (Geos (Poi C) add Emp)$
by *(blast intro:Eq-rev)*
from $P13 P14$ **show** $Def (Ang (An A C B))$ **by** *(simp add:Ang-simple-def)*
qed

lemma (in *Congruence-Rule*) *Ang-def-extension* :

assumes

$Def (Ang (An A B C))$

$\neg Eq (Geos (Poi B) add Emp) (Geos (Poi D) add Emp)$

$Line-on (Li B C) D$

shows $Def (Ang (An A B D))$

proof –

from *assms* **have** $P1 : Def (Tri (Tr A B C))$ **by** (*simp add:Ang-to-Tri*)

from *assms* $P1$ **have** $Def (Tri (Tr A B D))$ **by** (*simp add:Tri-def-extension*)

thus $Def (Ang (An A B D))$ **by** (*simp add:Tri-to-Ang*)

qed

lemma (in *Congruence-Rule*) *Bet-end-Point* :

shows $\neg Bet-Point (Se p1 p1) p2$

proof

assume $W : Bet-Point (Se p1 p1) p2$

then **have** $P1 : \neg Eq (Geos (Poi p1) add Emp) (Geos (Poi p1) add Emp)$ **by** (*simp add:Bet-Point-def del:Eq-refl*)

have $P2 : Eq (Geos (Poi p1) add Emp) (Geos (Poi p1) add Emp)$ **by** *simp*

from $P1 P2$ **show** *False* **by** *blast*

qed

lemma (in *Congruence-Rule*) *Seg-Plane-sameside* :

assumes

$Line-on l1 A$

$Line-on l1 B$

$Line-on l1 C$

$\neg Line-on l1 D$

$\neg Eq (Geos (Poi A) add Emp) (Geos (Poi B) add Emp)$

$\neg Eq (Geos (Poi A) add Emp) (Geos (Poi C) add Emp)$

$\neg Bet-Point (Se B C) A$

shows $Plane-sameside (Li D A) B C \vee Eq (Geos (Poi B) add Emp) (Geos (Poi C) add Emp)$

proof –

have $Line-on (Li D A) D$ **by** (*simp add:Line-on-rule*)

then **have** $P1 : Eq (Geos (Lin (Li D A)) add Emp) (Geos (Lin l1) add Emp)$
 $\implies Line-on l1 D$ **by** (*simp add:Line-on-trans*)

from *assms* $P1$ **have** $P2 : \neg Eq (Geos (Lin (Li D A)) add Emp) (Geos (Lin l1) add Emp)$ **by** *blast*

have $Plane-diffside (Li D A) B C \implies \exists p. Bet-Point (Se B C) p \wedge Line-on (Li D A) p \wedge \neg Line-on (Li D A) B \wedge \neg Line-on (Li D A) C$ **by** (*simp add:Plane-diffside-def*)

then **obtain** $E :: Point$ **where** $P3 : Plane-diffside (Li D A) B C \implies Bet-Point (Se B C) E \wedge Line-on (Li D A) E$ **by** *blast*

then **have** $P4 : Plane-diffside (Li D A) B C \implies Line-on (Li B C) E$ **by** (*simp add:Line-Bet-on*)

have $P5 : Line-on (Li B C) B$ **by** (*simp add:Line-on-rule*)

have $P6 : Line-on (Li B C) C$ **by** (*simp add:Line-on-rule*)

from *assms* $P5 P6$ **have** $P7 : \neg Eq (Geos (Poi B) add Emp) (Geos (Poi C) add Emp)$

$Emp) \implies Eq (Geos (Lin (Li B C)) add Emp) (Geos (Lin l1) add Emp)$ **by** (*simp add:Line-unique*)
from $P_4 P_7$ **have** $P_8 : Plane-diffside (Li D A) B C \implies \neg Eq (Geos (Poi B) add Emp) (Geos (Poi C) add Emp) \implies$
Line-on l1 E **by** (*simp add:Line-on-trans*)
have $P_9 : Line-on (Li D A) A$ **by** (*simp add:Line-on-rule*)
from *assms* $P_2 P_3 P_8 P_9$ **have** $P_{10} : Plane-diffside (Li D A) B C \implies \neg Eq (Geos (Poi B) add Emp) (Geos (Poi C) add Emp) \implies$
 $Eq (Geos (Poi E) add Emp) (Geos (Poi A) add Emp)$ **by** (*simp add:Line-unique-Point*)
from P_3 **have** $P_{11} : Plane-diffside (Li D A) B C \implies Bet-Point (Se B C) E$ **by** *simp*
from $P_{10} P_{11}$ **have** $P_{12} : Plane-diffside (Li D A) B C \implies \neg Eq (Geos (Poi B) add Emp) (Geos (Poi C) add Emp) \implies$
 $Bet-Point (Se B C) A$ **by** (*simp add:Point-Eq*)
from *assms* P_{12} **have** $\neg Plane-diffside (Li D A) B C \vee Eq (Geos (Poi B) add Emp) (Geos (Poi C) add Emp)$ **by** *blast*
then **have** $P_{13} : \neg Plane-diffside (Li D A) B C \wedge \neg Eq (Geos (Poi B) add Emp) (Geos (Poi C) add Emp)$
 $\vee Eq (Geos (Poi B) add Emp) (Geos (Poi C) add Emp)$ **by** *blast*
from *assms* P_9 **have** $P_{14} : Line-on (Li D A) B \implies Eq (Geos (Lin (Li D A)) add Emp) (Geos (Lin l1) add Emp)$ **by** (*simp add:Line-unique*)
from $P_2 P_{14}$ **have** $P_{15} : \neg Line-on (Li D A) B$ **by** *blast*
from *assms* P_9 **have** $P_{16} : Line-on (Li D A) C \implies Eq (Geos (Lin (Li D A)) add Emp) (Geos (Lin l1) add Emp)$ **by** (*simp add:Line-unique*)
from $P_2 P_{16}$ **have** $P_{17} : \neg Line-on (Li D A) C$ **by** *blast*
from $P_{15} P_{17}$ **have** $P_{18} : \neg Plane-diffside (Li D A) B C \wedge \neg Eq (Geos (Poi B) add Emp) (Geos (Poi C) add Emp) \implies$
 $Plane-sameside (Li D A) B C$ **by** (*simp add:Plane-not-diffside-sameside*)
from $P_{13} P_{18}$ **show** $Plane-sameside (Li D A) B C \vee Eq (Geos (Poi B) add Emp) (Geos (Poi C) add Emp)$ **by** *blast*
qed

lemma (in Congruence-Rule) Seg-move-unique :

assumes

Line-on l1 A

Line-on l1 B

Line-on l1 C

$\neg Eq (Geos (Poi A) add Emp) (Geos (Poi B) add Emp)$

$\neg Eq (Geos (Poi A) add Emp) (Geos (Poi C) add Emp)$

$Eq (Geos (Seg (Se A B)) add Emp) (Geos (Seg (Se A C)) add Emp)$

$\neg Bet-Point (Se B C) A$

shows $Eq (Geos (Poi B) add Emp) (Geos (Poi C) add Emp)$

proof –

have $\exists p q r. \neg Line-on l1 p \wedge \neg Line-on l1 q \wedge \neg Line-on l1 r$

$\wedge \neg Eq (Geos (Poi p) add Emp) (Geos (Poi q) add Emp) \wedge \neg Eq (Geos (Poi q) add Emp) (Geos (Poi r) add Emp)$

$\wedge \neg Eq (Geos (Poi r) add Emp) (Geos (Poi p) add Emp)$ **by** (*blast intro:Line-not-on-exist*)

then obtain $D :: Point$ **where** $P_1 : \neg Line-on l1 D$ **by** *blast*

have $P2 : \text{Line-on } (Li\ A\ D)\ D$ **by** (*simp add:Line-on-rule*)
have $P3 : \text{Line-on } (Li\ A\ B)\ A$ **by** (*simp add:Line-on-rule*)
have $P4 : \text{Line-on } (Li\ A\ B)\ B$ **by** (*simp add:Line-on-rule*)
from *assms* $P3\ P4$ **have** $P5 : \text{Eq } (\text{Geos } (Lin\ l1)\ \text{add Emp})\ (\text{Geos } (Lin\ (Li\ A\ B))\ \text{add Emp})$ **by** (*simp add:Line-unique*)
from *assms* $P5$ **have** $P6 : \text{Line-on } (Li\ A\ B)\ C$ **by** (*simp add:Line-on-trans*)
from $P1\ P5$ **have** $P7 : \neg \text{Line-on } (Li\ A\ B)\ D$ **by** (*simp add:Line-not-on-trans*)
from *assms* $P7$ **have** $\text{Def } (\text{Ang } (An\ A\ B\ D))$ **by** (*simp add:Ang-simple-def*)
then **have** $P8 : \text{Def } (\text{Ang } (An\ D\ A\ B))$ **by** (*blast intro:Ang-def-rev Ang-def-inv*)
then **have** $\neg \text{Eq } (\text{Geos } (Poi\ D)\ \text{add Emp})\ (\text{Geos } (Poi\ A)\ \text{add Emp})$ **by** (*simp add:Ang-def*)
then **have** $P9 : \neg \text{Eq } (\text{Geos } (Poi\ A)\ \text{add Emp})\ (\text{Geos } (Poi\ D)\ \text{add Emp})$ **by** (*blast intro:Eq-rev*)
have $P10 : \neg \text{Bet-Point } (Se\ D\ D)\ A$ **by** (*simp add:Bet-end-Point*)
from *assms* $P2\ P6\ P8\ P9\ P10$ **have** $\text{Eq } (\text{Geos } (\text{Ang } (An\ D\ A\ B))\ \text{add Emp})\ (\text{Geos } (\text{Ang } (An\ D\ A\ C))\ \text{add Emp})$ **by** (*simp add:Ang-Point-swap*)
then **have** $P11 : \text{Cong } (\text{Geos } (\text{Ang } (An\ D\ A\ B))\ \text{add Emp})\ (\text{Geos } (\text{Ang } (An\ D\ A\ C))\ \text{add Emp})$ **by** (*simp add:Ang-weektrans*)
from *assms* $P7$ **have** $\text{Def } (\text{Tri } (Tr\ A\ B\ D))$ **by** (*blast intro:Ang-simple-def Ang-to-Tri*)
then **have** $\text{Def } (\text{Tri } (Tr\ D\ B\ A))$ **by** (*simp add:Tri-def-rev*)
then **have** $P12 : \text{Def } (\text{Tri } (Tr\ A\ D\ B))$ **by** (*simp add:Tri-def-trans*)
have $P13 : \text{Line-on } (Li\ A\ C)\ A$ **by** (*simp add:Line-on-rule*)
have $P14 : \text{Line-on } (Li\ A\ C)\ C$ **by** (*simp add:Line-on-rule*)
from *assms* $P13\ P14$ **have** $P15 : \text{Eq } (\text{Geos } (Lin\ l1)\ \text{add Emp})\ (\text{Geos } (Lin\ (Li\ A\ C))\ \text{add Emp})$ **by** (*simp add:Line-unique*)
from $P1\ P15$ **have** $P16 : \neg \text{Line-on } (Li\ A\ C)\ D$ **by** (*simp add:Line-not-on-trans*)
from *assms* $P16$ **have** $\text{Def } (\text{Tri } (Tr\ A\ C\ D))$ **by** (*blast intro:Ang-simple-def Ang-to-Tri*)
then **have** $\text{Def } (\text{Tri } (Tr\ D\ C\ A))$ **by** (*simp add:Tri-def-rev*)
then **have** $P17 : \text{Def } (\text{Tri } (Tr\ A\ D\ C))$ **by** (*simp add:Tri-def-trans*)
from *assms* $P11\ P12\ P17$ **have** $P18 : \text{Cong } (\text{Geos } (\text{Ang } (An\ B\ D\ A))\ \text{add Emp})\ (\text{Geos } (\text{Ang } (An\ C\ D\ A))\ \text{add Emp})$ **by** (*simp add:Tri-week-SAS*)
have $P19 : \text{Cong } (\text{Geos } (\text{Ang } (An\ A\ D\ B))\ \text{add Emp})\ (\text{Geos } (\text{Ang } (An\ B\ D\ A))\ \text{add Emp})$ **by** (*simp add:Ang-roll*)
have $P20 : \text{Eq } (\text{Geos } (\text{Ang } (An\ B\ D\ A))\ \text{add Emp})\ (\text{Geos } (\text{Ang } (An\ A\ D\ B))\ \text{add Emp})$ **by** (*simp add:Ang-roll*)
from $P18\ P20$ **have** $P21 : \text{Cong } (\text{Geos } (\text{Ang } (An\ A\ D\ B))\ \text{add Emp})\ (\text{Geos } (\text{Ang } (An\ C\ D\ A))\ \text{add Emp})$ **by** (*blast intro:Ang-weektrans Eq-rev*)
from *assms* $P1$ **have** $P22 : \text{Plane-sameside } (Li\ D\ A)\ B\ C \vee \text{Eq } (\text{Geos } (Poi\ B)\ \text{add Emp})\ (\text{Geos } (Poi\ C)\ \text{add Emp})$ **by** (*simp add:Seg-Plane-sameside*)
from $P19\ P21$ **have** $P23 : \text{Plane-sameside } (Li\ D\ A)\ B\ C \implies \text{Eq } (\text{Geos } (Lin\ (Li\ B\ D))\ \text{add Emp})\ (\text{Geos } (Lin\ (Li\ C\ D))\ \text{add Emp})$ **by** (*simp add:Ang-move-unique*)
have $P24 : \text{Line-on } (Li\ B\ D)\ B$ **by** (*simp add:Line-on-rule*)
from $P23\ P24$ **have** $P25 : \text{Plane-sameside } (Li\ D\ A)\ B\ C \implies \text{Line-on } (Li\ C\ D)\ B$ **by** (*simp add:Line-on-trans*)
have $P26 : \text{Line-on } (Li\ B\ C)\ B$ **by** (*simp add:Line-on-rule*)
have $P27 : \text{Line-on } (Li\ B\ C)\ C$ **by** (*simp add:Line-on-rule*)
have $P28 : \text{Line-on } (Li\ C\ D)\ C$ **by** (*simp add:Line-on-rule*)

from $P25 P26 P27 P28$ **have** $P29 : \text{Plane-sameside } (Li D A) B C \implies \neg Eq$
 $(Geos (Poi B) \text{ add Emp}) (Geos (Poi C) \text{ add Emp}) \implies$
 $Eq (Geos (Lin (Li C D)) \text{ add Emp}) (Geos (Lin (Li B C)) \text{ add Emp})$ **by** $(\text{simp}$
 $\text{add:Line-unique})$
have $P30 : \text{Line-on } (Li C D) D$ **by** $(\text{simp add:Line-on-rule})$
from $P29 P30$ **have** $P31 : \text{Plane-sameside } (Li D A) B C \implies \neg Eq (Geos (Poi$
 $B) \text{ add Emp}) (Geos (Poi C) \text{ add Emp}) \implies$
 $\text{Line-on } (Li B C) D$ **by** $(\text{simp add:Line-on-trans})$
from $\text{assms } P26 P27$ **have** $P32 : \neg Eq (Geos (Poi B) \text{ add Emp}) (Geos (Poi C)$
 $\text{add Emp}) \implies Eq (Geos (Lin (Li B C)) \text{ add Emp}) (Geos (Lin l1) \text{ add Emp})$ **by**
 $(\text{simp add:Line-unique})$
from $P31 P32$ **have** $P33 : \text{Plane-sameside } (Li D A) B C \implies \neg Eq (Geos (Poi$
 $B) \text{ add Emp}) (Geos (Poi C) \text{ add Emp}) \implies$
 $\text{Line-on } l1 D$ **by** $(\text{simp add:Line-on-trans})$
from $P1 P33$ **have** $P34 : \text{Plane-sameside } (Li D A) B C \implies Eq (Geos (Poi B)$
 $\text{add Emp}) (Geos (Poi C) \text{ add Emp})$ **by** blast
from $P22 P34$ **show** $Eq (Geos (Poi B) \text{ add Emp}) (Geos (Poi C) \text{ add Emp})$ **by**
 blast
qed

lemma (in Congruence-Rule) Seg-not-Eq-Point :

assumes
 $\neg Eq (Geos (Seg (Se A B)) \text{ add Emp}) (Geos (Seg (Se A C)) \text{ add Emp})$
shows $\neg Eq (Geos (Poi B) \text{ add Emp}) (Geos (Poi C) \text{ add Emp})$
proof –
have $P1 : Eq (Geos (Poi B) \text{ add Emp}) (Geos (Poi C) \text{ add Emp}) \implies$
 $Eq (Geos (Seg (Se A B)) \text{ add Emp}) (Geos (Seg (Se A C)) \text{ add Emp})$ **by** $(\text{simp}$
 $\text{add:Seg-Point-Eq})$
from $\text{assms } P1$ **show** $\neg Eq (Geos (Poi B) \text{ add Emp}) (Geos (Poi C) \text{ add Emp})$
by blast
qed

lemma (in Congruence-Rule) Ang-replace :

assumes
 $Def (Ang (An A B C))$
 $Def (Ang (An A1 B1 C1))$
 $Cong (Geos (Ang (An A B C)) \text{ add Emp}) (Geos (Ang (An A1 B1 C1)) \text{ add}$
 $\text{Emp})$
shows $\exists p. Cong (Geos (Ang (An A B C)) \text{ add Emp}) (Geos (Ang (An p B1 C1))$
 $\text{add Emp})$
 $\wedge Eq (Geos (Ang (An A1 B1 C1)) \text{ add Emp}) (Geos (Ang (An p B1 C1)) \text{ add}$
 $\text{Emp})$
 $\wedge Eq (Geos (Seg (Se B A)) \text{ add Emp}) (Geos (Seg (Se B1 p)) \text{ add Emp}) \wedge$
 $\text{Line-on } (Li B1 A1) p \wedge \neg \text{Bet-Point } (Se p A1) B1 \wedge Def (Ang (An p B1 C1))$
and $\exists p. Cong (Geos (Ang (An A B C)) \text{ add Emp}) (Geos (Ang (An A1 B1 p))$
 $\text{add Emp})$
 $\wedge Eq (Geos (Ang (An A1 B1 C1)) \text{ add Emp}) (Geos (Ang (An A1 B1 p)) \text{ add}$
 $\text{Emp})$
 $\wedge Eq (Geos (Seg (Se B C)) \text{ add Emp}) (Geos (Seg (Se B1 p)) \text{ add Emp}) \wedge$

Line-on (*Li B1 C1*) $p \wedge \neg \text{Bet-Point}$ (*Se p C1*) *B1* $\wedge \text{Def}$ (*Ang* (*An A1 B1 p*))
and $\exists p q. \text{Cong}$ (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An p B1 q*))
add Emp)
 $\wedge \text{Eq}$ (*Geos* (*Ang* (*An A1 B1 C1*)) *add Emp*) (*Geos* (*Ang* (*An p B1 q*)) *add*
Emp)
 $\wedge \text{Eq}$ (*Geos* (*Seg* (*Se B A*)) *add Emp*) (*Geos* (*Seg* (*Se B1 p*)) *add Emp*) \wedge
Line-on (*Li B1 A1*) $p \wedge \neg \text{Bet-Point}$ (*Se p A1*) *B1*
 $\wedge \text{Eq}$ (*Geos* (*Seg* (*Se B C*)) *add Emp*) (*Geos* (*Seg* (*Se B1 q*)) *add Emp*) \wedge
Line-on (*Li B1 C1*) $q \wedge \neg \text{Bet-Point}$ (*Se q C1*) *B1* $\wedge \text{Def}$ (*Ang* (*An p B1 q*))
proof –
from *assms* **have** $P1 : \neg \text{Eq}$ (*Geos* (*Poi A1*) *add Emp*) (*Geos* (*Poi B1*) *add Emp*)
by (*simp add:Ang-def*)
then **have** $P2 : \neg \text{Eq}$ (*Geos* (*Poi B1*) *add Emp*) (*Geos* (*Poi A1*) *add Emp*) **by**
(*blast intro:Eq-rev*)
have $P3 : \text{Line-on}$ (*Li B1 A1*) *A1* **by** (*simp add:Line-on-rule*)
have $P4 : \text{Line-on}$ (*Li B1 A1*) *B1* **by** (*simp add:Line-on-rule*)
from *assms* **have** $\neg \text{Eq}$ (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi B*) *add Emp*) **by**
(*simp add:Ang-def*)
then **have** $P5 : \neg \text{Eq}$ (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) **by**
(*blast intro:Eq-rev*)
from $P2 P3 P4 P5$ **have** $\exists p. \text{Eq}$ (*Geos* (*Seg* (*Se B A*)) *add Emp*) (*Geos* (*Seg* (*Se*
B1 p)) *add Emp*) $\wedge \neg \text{Bet-Point}$ (*Se p A1*) *B1* $\wedge \text{Line-on}$ (*Li B1 A1*) $p \wedge \neg \text{Eq}$ (*Geos*
(*Poi B1*) *add Emp*) (*Geos* (*Poi p*) *add Emp*) **by** (*simp add:Seg-move-sameside*)
then obtain $A2 :: \text{Point}$ **where** $P6 : \text{Eq}$ (*Geos* (*Seg* (*Se B A*)) *add Emp*) (*Geos*
(*Seg* (*Se B1 A2*)) *add Emp*) $\wedge \neg \text{Bet-Point}$ (*Se A2 A1*) *B1* $\wedge \text{Line-on}$ (*Li B1 A1*)
A2 $\wedge \neg \text{Eq}$ (*Geos* (*Poi B1*) *add Emp*) (*Geos* (*Poi A2*) *add Emp*) **by** *blast*
from *assms* **have** $P7 : \neg \text{Eq}$ (*Geos* (*Poi B1*) *add Emp*) (*Geos* (*Poi C1*) *add Emp*)
by (*simp add:Ang-def*)
have $P8 : \text{Line-on}$ (*Li B1 C1*) *B1* **by** (*simp add:Line-on-rule*)
have $P9 : \text{Line-on}$ (*Li B1 C1*) *C1* **by** (*simp add:Line-on-rule*)
from *assms* **have** $P10 : \neg \text{Eq}$ (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*)
by (*simp add:Ang-def*)
from $P7 P8 P9 P10$ **have** $\exists p. \text{Eq}$ (*Geos* (*Seg* (*Se B C*)) *add Emp*) (*Geos* (*Seg* (*Se*
B1 p)) *add Emp*) $\wedge \neg \text{Bet-Point}$ (*Se p C1*) *B1* $\wedge \text{Line-on}$ (*Li B1 C1*) $p \wedge \neg \text{Eq}$ (*Geos*
(*Poi B1*) *add Emp*) (*Geos* (*Poi p*) *add Emp*) **by** (*simp add:Seg-move-sameside*)
then obtain $C2 :: \text{Point}$ **where** $P11 : \text{Eq}$ (*Geos* (*Seg* (*Se B C*)) *add Emp*) (*Geos*
(*Seg* (*Se B1 C2*)) *add Emp*) $\wedge \neg \text{Bet-Point}$ (*Se C2 C1*) *B1* $\wedge \text{Line-on}$ (*Li B1 C1*)
C2 $\wedge \neg \text{Eq}$ (*Geos* (*Poi B1*) *add Emp*) (*Geos* (*Poi C2*) *add Emp*) **by** *blast*
have $P12 : \neg \text{Bet-Point}$ (*Se C1 C1*) *B1* **by** (*simp add:Bet-end-Point*)
from $P6$ **have** $P13 : \neg \text{Bet-Point}$ (*Se A1 A2*) *B1* **by** (*blast intro:Bet-rev*)
from *assms* $P3 P6 P7 P9 P12 P13$ **have** $P14 : \text{Eq}$ (*Geos* (*Ang* (*An A1 B1 C1*))
add Emp) (*Geos* (*Ang* (*An A2 B1 C1*)) *add Emp*) **by** (*simp add:Ang-Point-swap*)
from *assms* $P14$ **have** $P15 : \text{Cong}$ (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos*
(*Ang* (*An A2 B1 C1*)) *add Emp*) **by** (*blast intro:Ang-weektrans Ang-rev Eq-rev*)
from *assms* **have** $P16 : \neg \text{Eq}$ (*Geos* (*Lin* (*Li B1 A1*)) *add Emp*) (*Geos* (*Lin* (*Li*
B1 C1)) *add Emp*) **by** (*simp add:Ang-def*)
from $P6$ **have** $P17 : \text{Line-on}$ (*Li B1 A1*) *A2* **by** *simp*
from $P6$ **have** $P18 : \neg \text{Eq}$ (*Geos* (*Poi B1*) *add Emp*) (*Geos* (*Poi A2*) *add Emp*)
by *simp*

from $P_4 P_8 P_{17} P_{18}$ **have** $P_{19} : \text{Line-on } (Li B1 C1) A2 \implies Eq (\text{Geos } (Lin (Li B1 A1)) \text{ add Emp}) (\text{Geos } (Lin (Li B1 C1)) \text{ add Emp})$ **by** (*simp add:Line-unique*)
from $P_{16} P_{19}$ **have** $P_{20} : \neg \text{Line-on } (Li B1 C1) A2$ **by** *blast*
from $P_7 P_{20}$ **have** $Def (\text{Ang } (An B1 C1 A2))$ **by** (*simp add:Ang-simple-def*)
then have $P_{21} : Def (\text{Ang } (An A2 B1 C1))$ **by** (*blast intro:Ang-def-rev Ang-def-inv*)
from $P_6 P_{14} P_{15} P_{21}$ **show** $\exists p. Cong (\text{Geos } (\text{Ang } (An A B C)) \text{ add Emp}) (\text{Geos } (\text{Ang } (An p B1 C1)) \text{ add Emp})$
 $\wedge Eq (\text{Geos } (\text{Ang } (An A1 B1 C1)) \text{ add Emp}) (\text{Geos } (\text{Ang } (An p B1 C1)) \text{ add Emp})$
 $\wedge Eq (\text{Geos } (\text{Seg } (Se B A)) \text{ add Emp}) (\text{Geos } (\text{Seg } (Se B1 p)) \text{ add Emp}) \wedge$
 $Line-on (Li B1 A1) p \wedge \neg \text{Bet-Point } (Se p A1) B1 \wedge Def (\text{Ang } (An p B1 C1))$ **by** *blast*
have $P_{22} : \neg \text{Bet-Point } (Se A1 A1) B1$ **by** (*simp add:Bet-end-Point*)
from P_{11} **have** $P_{23} : \neg \text{Bet-Point } (Se C1 C2) B1$ **by** (*blast intro:Bet-rev*)
from *assms* $P_2 P_3 P_7 P_{11} P_{22} P_{23}$ **have** $P_{24} : Eq (\text{Geos } (\text{Ang } (An A1 B1 C1)) \text{ add Emp}) (\text{Geos } (\text{Ang } (An A1 B1 C2)) \text{ add Emp})$ **by** (*simp add:Ang-Point-swap*)
from *assms* P_{24} **have** $P_{25} : Cong (\text{Geos } (\text{Ang } (An A B C)) \text{ add Emp}) (\text{Geos } (\text{Ang } (An A1 B1 C2)) \text{ add Emp})$ **by** (*blast intro:Ang-weektrans Ang-rev Eq-rev*)
from P_{11} **have** $P_{26} : \text{Line-on } (Li B1 C1) C2$ **by** *simp*
from P_{11} **have** $P_{27} : \neg Eq (\text{Geos } (Poi B1) \text{ add Emp}) (\text{Geos } (Poi C2) \text{ add Emp})$
by *simp*
from $P_4 P_8 P_{26} P_{27}$ **have** $P_{28} : \text{Line-on } (Li B1 A1) C2 \implies Eq (\text{Geos } (Lin (Li B1 A1)) \text{ add Emp}) (\text{Geos } (Lin (Li B1 C1)) \text{ add Emp})$ **by** (*simp add:Line-unique*)
from $P_{16} P_{28}$ **have** $P_{29} : \neg \text{Line-on } (Li B1 A1) C2$ **by** *blast*
from $P_2 P_{29}$ **have** $Def (\text{Ang } (An B1 A1 C2))$ **by** (*simp add:Ang-simple-def*)
then have $P_{30} : Def (\text{Ang } (An A1 B1 C2))$ **by** (*blast intro:Ang-def-rev Ang-def-inv*)
from $P_{11} P_{24} P_{25} P_{30}$ **show** $\exists p. Cong (\text{Geos } (\text{Ang } (An A B C)) \text{ add Emp}) (\text{Geos } (\text{Ang } (An A1 B1 p)) \text{ add Emp})$
 $\wedge Eq (\text{Geos } (\text{Ang } (An A1 B1 C1)) \text{ add Emp}) (\text{Geos } (\text{Ang } (An A1 B1 p)) \text{ add Emp})$
 $\wedge Eq (\text{Geos } (\text{Seg } (Se B C)) \text{ add Emp}) (\text{Geos } (\text{Seg } (Se B1 p)) \text{ add Emp}) \wedge$
 $Line-on (Li B1 C1) p \wedge \neg \text{Bet-Point } (Se p C1) B1 \wedge Def (\text{Ang } (An A1 B1 p))$ **by** *blast*
from *assms* $P_6 P_{11} P_{13} P_{17} P_{23} P_{26}$ **have** $P_{31} : Eq (\text{Geos } (\text{Ang } (An A1 B1 C1)) \text{ add Emp}) (\text{Geos } (\text{Ang } (An A2 B1 C2)) \text{ add Emp})$ **by** (*simp add:Ang-Point-swap*)
from *assms* P_{31} **have** $P_{32} : Cong (\text{Geos } (\text{Ang } (An A B C)) \text{ add Emp}) (\text{Geos } (\text{Ang } (An A2 B1 C2)) \text{ add Emp})$ **by** (*blast intro:Ang-weektrans Ang-rev Eq-rev*)
have $P_{33} : \text{Line-on } (Li B1 C2) B1$ **by** (*simp add:Line-on-rule*)
have $P_{34} : \text{Line-on } (Li B1 C2) C2$ **by** (*simp add:Line-on-rule*)
from $P_8 P_{26} P_{27} P_{33} P_{34}$ **have** $Eq (\text{Geos } (Lin (Li B1 C2)) \text{ add Emp}) (\text{Geos } (Lin (Li B1 C1)) \text{ add Emp})$ **by** (*simp add:Line-unique*)
then have $P_{35} : \text{Line-on } (Li B1 C2) A2 \implies \text{Line-on } (Li B1 C1) A2$ **by** (*simp add:Line-on-trans*)
from $P_{20} P_{35}$ **have** $P_{36} : \neg \text{Line-on } (Li B1 C2) A2$ **by** *blast*
from $P_{11} P_{36}$ **have** $Def (\text{Ang } (An B1 C2 A2))$ **by** (*simp add:Ang-simple-def*)
then have $P_{37} : Def (\text{Ang } (An A2 B1 C2))$ **by** (*blast intro:Ang-def-rev Ang-def-inv*)
from $P_6 P_{11} P_{21} P_{30} P_{31} P_{32} P_{37}$ **show** $\exists p q. Cong (\text{Geos } (\text{Ang } (An A B C)) \text{ add Emp}) (\text{Geos } (\text{Ang } (An p B1 q)) \text{ add Emp})$
 $\wedge Eq (\text{Geos } (\text{Ang } (An A1 B1 C1)) \text{ add Emp}) (\text{Geos } (\text{Ang } (An p B1 q)) \text{ add Emp})$

Emp)
 \wedge *Eq* (*Geos* (*Seg* (*Se* *B A*)) *add Emp*) (*Geos* (*Seg* (*Se* *B1 p*)) *add Emp*) \wedge
Line-on (*Li* *B1 A1*) *p* \wedge \neg *Bet-Point* (*Se* *p A1*) *B1*
 \wedge *Eq* (*Geos* (*Seg* (*Se* *B C*)) *add Emp*) (*Geos* (*Seg* (*Se* *B1 q*)) *add Emp*) \wedge
Line-on (*Li* *B1 C1*) *q* \wedge \neg *Bet-Point* (*Se* *q C1*) *B1* \wedge *Def* (*Ang* (*An* *p B1 q*))
by *blast*
qed

Theorem11

theorem (in *Congruence-Rule*) *Tri-isosceles*:

assumes

Def (*Tri* (*Tr* *A B C*))

Eq (*Geos* (*Seg* (*Se* *A B*)) *add Emp*) (*Geos* (*Seg* (*Se* *A C*)) *add Emp*)

shows *Cong* (*Geos* (*Ang* (*An* *A B C*)) *add Emp*) (*Geos* (*Ang* (*An* *A C B*)) *add Emp*)

proof –

from *assms* **have** *P1* : *Eq* (*Geos* (*Seg* (*Se* *A C*)) *add Emp*) (*Geos* (*Seg* (*Se* *A B*)) *add Emp*) **by** (*simp add:Eq-rev*)

have *P2* : *Cong* (*Geos* (*Ang* (*An* *B A C*)) *add Emp*) (*Geos* (*Ang* (*An* *C A B*)) *add Emp*) **by** (*simp add:Ang-roll*)

from *assms* **have** *Def* (*Tri* (*Tr* *C B A*)) **by** (*simp add:Tri-def-rev*)

then **have** *P3* : *Def* (*Tri* (*Tr* *A C B*)) **by** (*simp add:Tri-def-trans*)

from *assms* *P1 P2 P3* **have** *P4* : *Cong* (*Geos* (*Ang* (*An* *C B A*)) *add Emp*) (*Geos* (*Ang* (*An* *B C A*)) *add Emp*) **by** (*simp add:Tri-week-SAS*)

have *P5* : *Eq* (*Geos* (*Ang* (*An* *C B A*)) *add Emp*) (*Geos* (*Ang* (*An* *A B C*)) *add Emp*) **by** (*simp add:Ang-roll*)

from *P4 P5* **have** *P6* : *Cong* (*Geos* (*Ang* (*An* *A B C*)) *add Emp*) (*Geos* (*Ang* (*An* *B C A*)) *add Emp*) **by** (*blast intro:Ang-weektrans Eq-rev*)

have *P7* : *Eq* (*Geos* (*Ang* (*An* *B C A*)) *add Emp*) (*Geos* (*Ang* (*An* *A C B*)) *add Emp*) **by** (*simp add:Ang-roll*)

from *P6 P7* **show** *Cong* (*Geos* (*Ang* (*An* *A B C*)) *add Emp*) (*Geos* (*Ang* (*An* *A C B*)) *add Emp*) **by** (*blast intro:Ang-weektrans Eq-rev Ang-rev*)

qed

lemma (in *Congruence-Rule*) *Tri-week-ASA* :

assumes *N* :

Def (*Tri* (*Tr* *A B C*))

Def (*Tri* (*Tr* *A1 B1 C1*))

Eq (*Geos* (*Seg* (*Se* *A B*)) *add Emp*) (*Geos* (*Seg* (*Se* *A1 B1*)) *add Emp*)

Cong (*Geos* (*Ang* (*An* *C A B*)) *add Emp*) (*Geos* (*Ang* (*An* *C1 A1 B1*)) *add Emp*)

Cong (*Geos* (*Ang* (*An* *C B A*)) *add Emp*) (*Geos* (*Ang* (*An* *C1 B1 A1*)) *add Emp*)

shows $\neg\neg$ *Eq* (*Geos* (*Seg* (*Se* *B C*)) *add Emp*) (*Geos* (*Seg* (*Se* *B1 C1*)) *add Emp*)

proof

assume *W* : \neg *Eq* (*Geos* (*Seg* (*Se* *B C*)) *add Emp*) (*Geos* (*Seg* (*Se* *B1 C1*)) *add Emp*)

have *P1* : *Line-on* (*Li* *B1 C1*) *B1* **by** (*simp add:Line-on-rule*)

have *P2* : *Line-on* (*Li* *B1 C1*) *C1* **by** (*simp add:Line-on-rule*)

from *assms* **have** $P3 : \neg \text{Eq} (\text{Geos} (\text{Poi } B1) \text{ add Emp}) (\text{Geos} (\text{Poi } C1) \text{ add Emp})$
by (*simp add:Tri-def*)
from *assms* **have** $P4 : \neg \text{Eq} (\text{Geos} (\text{Poi } B) \text{ add Emp}) (\text{Geos} (\text{Poi } C) \text{ add Emp})$
by (*simp add:Tri-def*)
from $P1 P2 P3 P4$ **have** $\exists p. \text{Eq} (\text{Geos} (\text{Seg} (\text{Se } B C)) \text{ add Emp}) (\text{Geos} (\text{Seg} (\text{Se } B1 p)) \text{ add Emp})$
 $\wedge \neg \text{Bet-Point} (\text{Se } p C1) B1 \wedge \text{Line-on} (\text{Li } B1 C1) p \wedge \neg \text{Eq} (\text{Geos} (\text{Poi } B1) \text{ add Emp}) (\text{Geos} (\text{Poi } p) \text{ add Emp})$ **by** (*simp add:Seg-move-sameside*)
then obtain $D1 :: \text{Point}$ **where** $P5 : \text{Eq} (\text{Geos} (\text{Seg} (\text{Se } B C)) \text{ add Emp}) (\text{Geos} (\text{Seg} (\text{Se } B1 D1)) \text{ add Emp})$
 $\wedge \neg \text{Bet-Point} (\text{Se } D1 C1) B1 \wedge \text{Line-on} (\text{Li } B1 C1) D1 \wedge \neg \text{Eq} (\text{Geos} (\text{Poi } B1) \text{ add Emp}) (\text{Geos} (\text{Poi } D1) \text{ add Emp})$ **by** *blast*
from W **have** $P6 : \neg \text{Eq} (\text{Geos} (\text{Seg} (\text{Se } B1 C1)) \text{ add Emp}) (\text{Geos} (\text{Seg} (\text{Se } B C)) \text{ add Emp})$ **by** (*blast intro:Eq-rev*)
from $P5 P6$ **have** $\neg \text{Eq} (\text{Geos} (\text{Seg} (\text{Se } B1 C1)) \text{ add Emp}) (\text{Geos} (\text{Seg} (\text{Se } B1 D1)) \text{ add Emp})$ **by** (*simp add:Eq-not-trans*)
then have $P7 : \neg \text{Eq} (\text{Geos} (\text{Poi } C1) \text{ add Emp}) (\text{Geos} (\text{Poi } D1) \text{ add Emp})$ **by** (*simp add:Seg-not-Eq-Point*)
from *assms* **have** $P8 : \neg \text{Line-on} (\text{Li } B1 C1) A1$ **by** (*simp add:Tri-def-Line*)
from $P5$ **have** $P9 : \text{Line-on} (\text{Li } B1 C1) D1$ **by** *simp*
then have $P10 : \text{Eq} (\text{Geos} (\text{Poi } D1) \text{ add Emp}) (\text{Geos} (\text{Poi } A1) \text{ add Emp}) \implies \text{Line-on} (\text{Li } B1 C1) A1$ **by** (*simp add:Point-Eq*)
from $P8 P10$ **have** $P11 : \neg \text{Eq} (\text{Geos} (\text{Poi } D1) \text{ add Emp}) (\text{Geos} (\text{Poi } A1) \text{ add Emp})$ **by** *blast*
from *assms* **have** $P12 : \neg \text{Eq} (\text{Geos} (\text{Poi } A1) \text{ add Emp}) (\text{Geos} (\text{Poi } B1) \text{ add Emp})$ **by** (*simp add:Tri-def*)
have $P13 : \text{Line-on} (\text{Li } A1 B1) B1$ **by** (*simp add:Line-on-rule*)
from $P5$ **have** $P14 : \neg \text{Eq} (\text{Geos} (\text{Poi } B1) \text{ add Emp}) (\text{Geos} (\text{Poi } D1) \text{ add Emp})$
by *simp*
from *assms* $P7 P9 P14$ **have** $\text{Def} (\text{Tri} (\text{Tr } A1 B1 D1))$ **by** (*blast intro:Tri-def-extension*)
then have $\text{Def} (\text{Tri} (\text{Tr } D1 B1 A1))$ **by** (*simp add:Tri-def-rev*)
then have $P15 : \text{Def} (\text{Tri} (\text{Tr } B1 A1 D1))$ **by** (*simp add:Tri-def-trans*)
from *assms* **have** $P16 : \text{Def} (\text{Tri} (\text{Tr } B A C))$ **by** (*blast intro:Tri-def-rev Tri-def-trans*)
from *assms* **have** $P17 : \text{Eq} (\text{Geos} (\text{Seg} (\text{Se } B A)) \text{ add Emp}) (\text{Geos} (\text{Seg} (\text{Se } B1 A1)) \text{ add Emp})$ **by** (*blast intro:Seg-rev Eq-trans*)
from $P5$ **have** $P18 : \text{Eq} (\text{Geos} (\text{Seg} (\text{Se } B C)) \text{ add Emp}) (\text{Geos} (\text{Seg} (\text{Se } B1 D1)) \text{ add Emp})$ **by** *simp*
have $P19 : \text{Line-on} (\text{Li } B1 A1) A1$ **by** (*simp add:Line-on-rule*)
from *assms* **have** $\text{Def} (\text{Tri} (\text{Tr } C1 B1 A1))$ **by** (*simp add:Tri-def-rev*)
then have $P20 : \text{Def} (\text{Ang} (\text{An } C1 B1 A1))$ **by** (*simp add:Tri-to-Ang*)
have $P21 : \text{Line-on} (\text{Li } B1 A1) A1$ **by** (*simp add:Line-on-rule*)
have $P22 : \neg \text{Bet-Point} (\text{Se } A1 A1) B1$ **by** (*simp add:Bet-end-Point*)
from $P5$ **have** $P23 : \neg \text{Bet-Point} (\text{Se } C1 D1) B1$ **by** (*blast intro:Bet-rev*)
from $P12$ **have** $P24 : \neg \text{Eq} (\text{Geos} (\text{Poi } B1) \text{ add Emp}) (\text{Geos} (\text{Poi } A1) \text{ add Emp})$
by (*blast intro:Eq-rev*)
from $P9 P14 P19 P20 P21 P22 P23 P24$ **have** $P25 : \text{Eq} (\text{Geos} (\text{Ang} (\text{An } C1 B1 A1)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } D1 B1 A1)) \text{ add Emp})$ **by** (*simp add:Ang-Point-swap*)
from *assms* $P25$ **have** $P26 : \text{Cong} (\text{Geos} (\text{Ang} (\text{An } C B A)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } C B A)) \text{ add Emp})$

(*Ang (An D1 B1 A1) add Emp*) **by** (*blast intro:Ang-weektrans Ang-rev Eq-rev*)
have *P27 : Eq (Geos (Ang (An C B A) add Emp) (Geos (Ang (An A B C)) add Emp)* **by** (*simp add:Ang-roll*)
from *P26 P27* **have** *P28 : Cong (Geos (Ang (An A B C)) add Emp) (Geos (Ang (An D1 B1 A1) add Emp)* **by** (*blast intro:Ang-weektrans Eq-rev*)
have *P29 : Eq (Geos (Ang (An D1 B1 A1) add Emp) (Geos (Ang (An A1 B1 D1) add Emp)* **by** (*simp add:Ang-roll*)
from *P28 P29* **have** *P30 : Cong (Geos (Ang (An A B C)) add Emp) (Geos (Ang (An A1 B1 D1) add Emp)* **by** (*blast intro:Ang-weektrans Eq-rev Ang-rev*)
from *P15 P16 P17 P18 P30* **have** *P31 : Cong (Geos (Ang (An C A B)) add Emp) (Geos (Ang (An D1 A1 B1) add Emp)* **by** (*simp add:Tri-week-SAS*)
from *P1 P2 P3 P5 P8 P23* **have** *P32 : Plane-sameside (Li A1 B1) C1 D1 \vee Eq (Geos (Poi C1) add Emp) (Geos (Poi D1) add Emp)* **by** (*simp add:Seg-Plane-sameside*)
from *assms P31* **have** *P33 : Plane-sameside (Li A1 B1) C1 D1 \implies Eq (Geos (Lin (Li C1 A1) add Emp) (Geos (Lin (Li D1 A1) add Emp)* **by** (*simp add:Ang-move-unique*)
from *assms* **have** \neg *Eq (Geos (Poi C1) add Emp) (Geos (Poi A1) add Emp)* **by** (*simp add:Tri-def*)
then **have** *P34 : Eq (Geos (Lin (Li C1 A1) add Emp) (Geos (Lin (Li A1 C1) add Emp)* **by** (*simp add:Line-rev*)
from *P11* **have** *P35 : Eq (Geos (Lin (Li D1 A1) add Emp) (Geos (Lin (Li A1 D1) add Emp)* **by** (*simp add:Line-rev*)
from *P33 P34 P35* **have** *P36 : Plane-sameside (Li A1 B1) C1 D1 \implies Eq (Geos (Lin (Li A1 C1) add Emp) (Geos (Lin (Li A1 D1) add Emp)* **by** (*blast intro:Eq-trans Eq-rev*)
have *P37 : Line-on (Li A1 C1) C1* **by** (*simp add:Line-on-rule*)
have *P38 : Line-on (Li A1 C1) A1* **by** (*simp add:Line-on-rule*)
have *P39 : Line-on (Li A1 D1) D1* **by** (*simp add:Line-on-rule*)
have *P40 : Line-on (Li A1 D1) A1* **by** (*simp add:Line-on-rule*)
from *P37* **have** *P41 : Eq (Geos (Poi C1) add Emp) (Geos (Poi D1) add Emp)* \implies *Line-on (Li A1 C1) D1* **by** (*simp add:Point-Eq*)
from *P11 P38 P39 P40 P41* **have** *P42 : Eq (Geos (Poi C1) add Emp) (Geos (Poi D1) add Emp)* \implies
Eq (Geos (Lin (Li A1 C1) add Emp) (Geos (Lin (Li A1 D1) add Emp) **by** (*simp add:Line-unique*)
from *P32 P36 P42* **have** *P43 : Eq (Geos (Lin (Li A1 C1) add Emp) (Geos (Lin (Li A1 D1) add Emp)* **by** *blast*
from *P2 P7 P9 P37* **have** *P44 : Line-on (Li A1 C1) D1 \implies Eq (Geos (Lin (Li B1 C1) add Emp) (Geos (Lin (Li A1 C1) add Emp)* **by** (*simp add:Line-unique*)
from *P1 P44* **have** *P45 : Line-on (Li A1 C1) D1 \implies Line-on (Li A1 C1) B1* **by** (*simp add:Line-on-trans*)
from *assms* **have** *Def (Tri (Tr C1 B1 A1))* **by** (*simp add:Tri-def-rev*)
then **have** *P46 : \neg Line-on (Li A1 C1) B1* **by** (*simp add:Tri-def-Line*)
from *P45 P46* **have** *P47 : \neg Line-on (Li A1 C1) D1* **by** *blast*
have *P48 : Line-on (Li A1 D1) D1* **by** (*simp add:Line-on-rule*)
from *P47 P48* **have** *P49 : \neg Eq (Geos (Lin (Li A1 C1) add Emp) (Geos (Lin (Li A1 D1) add Emp)* **by** (*simp add:Line-not-on-Eq*)
from *P43 P49* **show** *False* **by** *blast*
qed

Theorem12

theorem (in *Congruence-Rule*) *Tri-SAS*:

assumes

Def (*Tri* (*Tr* *A B C*))

Def (*Tri* (*Tr* *A1 B1 C1*))

Eq (*Geos* (*Seg* (*Se* *A B*)) *add Emp*) (*Geos* (*Seg* (*Se* *A1 B1*)) *add Emp*)

Eq (*Geos* (*Seg* (*Se* *A C*)) *add Emp*) (*Geos* (*Seg* (*Se* *A1 C1*)) *add Emp*)

Cong (*Geos* (*Ang* (*An* *B A C*)) *add Emp*) (*Geos* (*Ang* (*An* *B1 A1 C1*)) *add Emp*)

shows *Cong* (*Geos* (*Tri* (*Tr* *A B C*)) *add Emp*) (*Geos* (*Tri* (*Tr* *A1 B1 C1*)) *add Emp*)

proof –

from *assms* **have** *P1* : *Cong* (*Geos* (*Ang* (*An* *C B A*)) *add Emp*) (*Geos* (*Ang* (*An* *C1 B1 A1*)) *add Emp*) **by** (*simp add:Tri-week-SAS*)

have *P2* : *Eq* (*Geos* (*Ang* (*An* *B A C*)) *add Emp*) (*Geos* (*Ang* (*An* *C A B*)) *add Emp*) **by** (*simp add:Ang-roll*)

have *P3* : *Eq* (*Geos* (*Ang* (*An* *C1 A1 B1*)) *add Emp*) (*Geos* (*Ang* (*An* *B1 A1 C1*)) *add Emp*) **by** (*simp add:Ang-roll*)

from *assms* *P2* **have** *P4* : *Cong* (*Geos* (*Ang* (*An* *C A B*)) *add Emp*) (*Geos* (*Ang* (*An* *B1 A1 C1*)) *add Emp*) **by** (*blast intro:Ang-weektrans Eq-rev*)

from *P3* *P4* **have** *P5* : *Cong* (*Geos* (*Ang* (*An* *C A B*)) *add Emp*) (*Geos* (*Ang* (*An* *C1 A1 B1*)) *add Emp*) **by** (*blast intro:Ang-weektrans Ang-rev Eq-rev*)

from *assms* *P1* *P5* **have** *P6* : $\neg\neg$ *Eq* (*Geos* (*Seg* (*Se* *B C*)) *add Emp*) (*Geos* (*Seg* (*Se* *B1 C1*)) *add Emp*) **by** (*simp add:Tri-week-ASA*)

from *assms* **have** *P7* : *Eq* (*Geos* (*Seg* (*Se* *C A*)) *add Emp*) (*Geos* (*Seg* (*Se* *C1 A1*)) *add Emp*) **by** (*blast intro:Seg-rev Eq-rev Eq-trans*)

from *assms* **have** *P8* : *Def* (*Tri* (*Tr* *A C B*)) **by** (*blast intro:Tri-def-rev Tri-def-trans*)

from *assms* **have** *P9* : *Def* (*Tri* (*Tr* *A1 C1 B1*)) **by** (*blast intro:Tri-def-rev Tri-def-trans*)

from *assms* *P5* *P8* *P9* **have** *P10* : *Cong* (*Geos* (*Ang* (*An* *B C A*)) *add Emp*) (*Geos* (*Ang* (*An* *B1 C1 A1*)) *add Emp*) **by** (*simp add:Tri-week-SAS*)

have *P11* : *Eq* (*Geos* (*Ang* (*An* *B C A*)) *add Emp*) (*Geos* (*Ang* (*An* *A C B*)) *add Emp*) **by** (*simp add:Ang-roll*)

have *P12* : *Eq* (*Geos* (*Ang* (*An* *B1 C1 A1*)) *add Emp*) (*Geos* (*Ang* (*An* *A1 C1 B1*)) *add Emp*) **by** (*simp add:Ang-roll*)

from *P10* *P11* **have** *P13* : *Cong* (*Geos* (*Ang* (*An* *A C B*)) *add Emp*) (*Geos* (*Ang* (*An* *B1 C1 A1*)) *add Emp*) **by** (*blast intro:Ang-weektrans Eq-rev*)

from *P10* *P12* *P13* **have** *P14* : *Cong* (*Geos* (*Ang* (*An* *A C B*)) *add Emp*) (*Geos* (*Ang* (*An* *A1 C1 B1*)) *add Emp*) **by** (*blast intro:Ang-weektrans Ang-rev Eq-rev*)

from *assms* *P1* *P6* *P7* *P14* **show** *Cong* (*Geos* (*Tri* (*Tr* *A B C*)) *add Emp*) (*Geos* (*Tri* (*Tr* *A1 B1 C1*)) *add Emp*) **by** (*simp add:Tri-Cong-def*)

qed

Theorem13

theorem (in *Congruence-Rule*) *Tri-ASA*:

assumes

Def (*Tri* (*Tr* *A B C*))

Def (*Tri* (*Tr* *A1 B1 C1*))

Eq (*Geos* (*Seg* (*Se* *A B*)) *add Emp*) (*Geos* (*Seg* (*Se* *A1 B1*)) *add Emp*)

Cong (Geos (Ang (An C B A)) add Emp) (Geos (Ang (An C1 B1 A1)) add Emp)
Cong (Geos (Ang (An C A B)) add Emp) (Geos (Ang (An C1 A1 B1)) add Emp)
shows *Cong (Geos (Tri (Tr A B C)) add Emp) (Geos (Tri (Tr A1 B1 C1)) add Emp)*
proof –
from *assms* **have** $P1 : Eq (Geos (Seg (Se B A)) add Emp) (Geos (Seg (Se B1 A1)) add Emp)$ **by** (*blast intro:Seg-rev Eq-rev Eq-trans*)
from *assms* **have** $P2 : Def (Tri (Tr B A C))$ **by** (*blast intro:Tri-def-rev Tri-def-trans*)
from *assms* **have** $P3 : Def (Tri (Tr B1 A1 C1))$ **by** (*blast intro:Tri-def-rev Tri-def-trans*)
from *assms* $P1 P2 P3$ **have** $P4 : \neg\neg Eq (Geos (Seg (Se A C)) add Emp) (Geos (Seg (Se A1 C1)) add Emp)$ **by** (*simp add:Tri-week-ASA*)
have $P5 : Eq (Geos (Ang (An C A B)) add Emp) (Geos (Ang (An B A C)) add Emp)$ **by** (*simp add:Ang-roll*)
have $P6 : Eq (Geos (Ang (An C1 A1 B1)) add Emp) (Geos (Ang (An B1 A1 C1)) add Emp)$ **by** (*simp add:Ang-roll*)
from *assms* $P5$ **have** $P7 : Cong (Geos (Ang (An B A C)) add Emp) (Geos (Ang (An C1 A1 B1)) add Emp)$ **by** (*blast intro:Ang-weektrans Eq-rev*)
from $P6 P7$ **have** $P8 : Cong (Geos (Ang (An B A C)) add Emp) (Geos (Ang (An B1 A1 C1)) add Emp)$ **by** (*blast intro:Ang-weektrans Eq-rev Ang-rev*)
from *assms* $P1 P4 P8$ **show** *Cong (Geos (Tri (Tr A B C)) add Emp) (Geos (Tri (Tr A1 B1 C1)) add Emp)* **by** (*simp add:Tri-SAS*)
qed

Theorem14

theorem (in *Congruence-Rule*) *Ang-complementary* :

assumes

Def (Ang (An A B C))

Def (Ang (An A1 B1 C1))

Cong (Geos (Ang (An A B C)) add Emp) (Geos (Ang (An A1 B1 C1)) add Emp)

Bet-Point (Se A D) B

Bet-Point (Se A1 D1) B1

shows

Cong (Geos (Ang (An C B D)) add Emp) (Geos (Ang (An C1 B1 D1)) add Emp)

proof –

from *assms* **have** $\exists p. Cong (Geos (Ang (An A B C)) add Emp) (Geos (Ang (An p B1 C1)) add Emp)$

$\wedge Eq (Geos (Ang (An A1 B1 C1)) add Emp) (Geos (Ang (An p B1 C1)) add Emp)$

$\wedge Eq (Geos (Seg (Se B A)) add Emp) (Geos (Seg (Se B1 p)) add Emp)$

$\wedge Line-on (Li B1 A1) p \wedge \neg Bet-Point (Se p A1) B1 \wedge Def (Ang (An p B1 C1))$ **by** (*simp add:Ang-replace*)

then obtain $A2 :: Point$ **where** $P1 : Cong (Geos (Ang (An A B C)) add Emp) (Geos (Ang (An A2 B1 C1)) add Emp)$

$\wedge \text{Eq} (\text{Geos} (\text{Seg} (\text{Se} B A)) \text{ add Emp}) (\text{Geos} (\text{Seg} (\text{Se} B1 A2)) \text{ add Emp}) \wedge$
Line-on (*Li* B1 A1) A2
 $\wedge \neg \text{Bet-Point} (\text{Se} A2 A1) B1 \wedge \text{Def} (\text{Ang} (\text{An} A2 B1 C1))$ **by** *blast*
from *assms* P1 **have** $\exists p. \text{Cong} (\text{Geos} (\text{Ang} (\text{An} A B C)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An} A2 B1 p)) \text{ add Emp})$
 $\wedge \text{Eq} (\text{Geos} (\text{Ang} (\text{An} A2 B1 C1)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An} A2 B1 p)) \text{ add Emp})$
 $\wedge \text{Eq} (\text{Geos} (\text{Seg} (\text{Se} B C)) \text{ add Emp}) (\text{Geos} (\text{Seg} (\text{Se} B1 p)) \text{ add Emp})$
 $\wedge \text{Line-on} (\text{Li} B1 C1) p \wedge \neg \text{Bet-Point} (\text{Se} p C1) B1 \wedge \text{Def} (\text{Ang} (\text{An} A2 B1 p))$ **by** (*simp add:Ang-replace*)
then obtain C2 :: *Point* **where** P2 : $\text{Cong} (\text{Geos} (\text{Ang} (\text{An} A B C)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An} A2 B1 C2)) \text{ add Emp})$
 $\wedge \text{Eq} (\text{Geos} (\text{Seg} (\text{Se} B C)) \text{ add Emp}) (\text{Geos} (\text{Seg} (\text{Se} B1 C2)) \text{ add Emp}) \wedge$
Line-on (*Li* B1 C1) C2
 $\wedge \neg \text{Bet-Point} (\text{Se} C2 C1) B1 \wedge \text{Def} (\text{Ang} (\text{An} A2 B1 C2))$ **by** *blast*
from *assms* **have** $\text{Def} (\text{Tri} (\text{Tr} A B C))$ **by** (*simp add:Ang-to-Tri*)
then have P3 : $\text{Def} (\text{Tri} (\text{Tr} B A C))$ **by** (*blast intro:Tri-def-rev Tri-def-trans*)
from P2 **have** $\text{Def} (\text{Tri} (\text{Tr} A2 B1 C2))$ **by** (*simp add:Ang-to-Tri*)
then have P4 : $\text{Def} (\text{Tri} (\text{Tr} B1 A2 C2))$ **by** (*blast intro:Tri-def-rev Tri-def-trans*)
from P1 P2 P3 P4 **have** P5 : $\text{Cong} (\text{Geos} (\text{Tri} (\text{Tr} B A C)) \text{ add Emp}) (\text{Geos} (\text{Tri} (\text{Tr} B1 A2 C2)) \text{ add Emp})$ **by** (*simp add:Tri-SAS*)
then have P6 : $\text{Eq} (\text{Geos} (\text{Seg} (\text{Se} A C)) \text{ add Emp}) (\text{Geos} (\text{Seg} (\text{Se} A2 C2)) \text{ add Emp})$ **by** (*simp add:Tri-Cong-def*)
from P5 **have** P7 : $\text{Cong} (\text{Geos} (\text{Ang} (\text{An} C A B)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An} C2 A2 B1)) \text{ add Emp})$ **by** (*simp add:Tri-Cong-def*)
have P8 : *Line-on* (*Li* B1 D1) B1 **by** (*simp add:Line-on-rule*)
from *assms* **have** P9 : *Line-on* (*Li* A1 D1) B1 **by** (*simp add:Line-Bet-on*)
from *assms* **have** p10 : $\neg \text{Eq} (\text{Geos} (\text{Poi} B1) \text{ add Emp}) (\text{Geos} (\text{Poi} A1) \text{ add Emp})$ **by** (*simp add:Bet-Point-def*)
from *assms* **have** P11 : *Line-on* (*Li* B1 A1) D1 **by** (*simp add:Line-Bet-on*)
have P12 : *Line-on* (*Li* B1 D1) D1 **by** (*simp add:Line-on-rule*)
have P13 : *Line-on* (*Li* B1 A1) B1 **by** (*simp add:Line-on-rule*)
from *assms* **have** P14 : $\neg \text{Eq} (\text{Geos} (\text{Poi} D1) \text{ add Emp}) (\text{Geos} (\text{Poi} B1) \text{ add Emp})$ **by** (*simp add:Bet-Point-def*)
from P8 P11 P12 P13 P14 **have** P15 : $\text{Eq} (\text{Geos} (\text{Lin} (\text{Li} B1 A1)) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li} B1 D1)) \text{ add Emp})$ **by** (*simp add:Line-unique*)
from P1 P15 **have** P16 : *Line-on* (*Li* B1 D1) A2 **by** (*simp add:Line-on-trans*)
from P4 **have** P17 : $\neg \text{Eq} (\text{Geos} (\text{Poi} B1) \text{ add Emp}) (\text{Geos} (\text{Poi} A2) \text{ add Emp})$ **by** (*simp add:Tri-def*)
from *assms* **have** P18 : $\neg \text{Eq} (\text{Geos} (\text{Poi} D) \text{ add Emp}) (\text{Geos} (\text{Poi} B) \text{ add Emp})$ **by** (*simp add:Bet-Point-def*)
then have P19 : $\neg \text{Eq} (\text{Geos} (\text{Poi} B) \text{ add Emp}) (\text{Geos} (\text{Poi} D) \text{ add Emp})$ **by** (*blast intro:Eq-rev*)
from P8 P16 P17 P19 **have** $\exists p. \text{Eq} (\text{Geos} (\text{Seg} (\text{Se} B D)) \text{ add Emp}) (\text{Geos} (\text{Seg} (\text{Se} B1 p)) \text{ add Emp})$
 $\wedge \text{Bet-Point} (\text{Se} p A2) B1 \wedge \text{Line-on} (\text{Li} B1 D1) p \wedge \neg \text{Eq} (\text{Geos} (\text{Poi} B1) \text{ add Emp}) (\text{Geos} (\text{Poi} p) \text{ add Emp})$ **by** (*simp add:Seg-move-diffside*)
then obtain D2 :: *Point* **where** P20 : $\text{Eq} (\text{Geos} (\text{Seg} (\text{Se} B D)) \text{ add Emp}) (\text{Geos} (\text{Seg} (\text{Se} B1 D2)) \text{ add Emp})$

\wedge *Bet-Point* (*Se D2 A2*) *B1* \wedge *Line-on* (*Li B1 D1*) *D2* \wedge \neg *Eq* (*Geos (Poi B1)*
add Emp) (*Geos (Poi D2)* *add Emp*) **by** *blast*
from *P1* **have** *P21* : *Eq* (*Geos (Seg (Se A B))* *add Emp*) (*Geos (Seg (Se A2 B1))*
add Emp) **by** (*blast intro:Seg-rev Eq-trans*)
from *P20* **have** *P22* : *Bet-Point* (*Se A2 D2*) *B1* **by** (*blast intro:Bet-rev*)
from *assms P20 P21 P22* **have** *P23* : *Eq* (*Geos (Seg (Se A D))* *add Emp*) (*Geos*
(*Seg (Se A2 D2)*) *add Emp*) **by** (*blast intro:Seg-Bet-add*)
from *P3* **have** *P24* : *Def* (*Tri (Tr C A B)*) **by** (*blast intro:Tri-def-rev*)
from *assms* **have** *P25* : \neg *Eq* (*Geos (Poi A)* *add Emp*) (*Geos (Poi D)* *add Emp*)
by (*simp add:Bet-Point-def*)
from *assms* **have** *P26* : *Line-on* (*Li A B*) *D* **by** (*simp add:Line-Bet-on*)
from *P24 P25 P26* **have** *P27* : *Def* (*Tri (Tr C A D)*) **by** (*simp add:Tri-def-extension*)
from *P4* **have** *P28* : *Def* (*Tri (Tr C2 A2 B1)*) **by** (*blast intro:Tri-def-rev*)
from *P22* **have** *P29* : \neg *Eq* (*Geos (Poi A2)* *add Emp*) (*Geos (Poi D2)* *add Emp*)
by (*simp add:Bet-Point-def*)
from *P22* **have** *P30* : *Line-on* (*Li A2 B1*) *D2* **by** (*simp add:Line-Bet-on*)
from *P28 P29 P30* **have** *P31* : *Def* (*Tri (Tr C2 A2 D2)*) **by** (*simp add:Tri-def-extension*)
from *P24* **have** *P32* : *Def* (*Ang (An C A B)*) **by** (*simp add:Tri-to-Ang*)
from *P27* **have** *P33* : *Def* (*Ang (An C A D)*) **by** (*simp add:Tri-to-Ang*)
have *P34* : *Line-on* (*Li A C*) *C* **by** (*simp add:Line-on-rule*)
have *P35* : \neg *Bet-Point* (*Se C C*) *A* **by** (*simp add:Bet-end-Point*)
from *assms* **have** *Inv (Bet-Point (Se D B) A)* **by** (*simp add:Bet-iff*)
then **have** \neg *Bet-Point* (*Se D B*) *A* **by** (*simp add:Inv-def*)
then **have** *P36* : \neg *Bet-Point* (*Se B D*) *A* **by** (*blast intro:Bet-rev*)
from *P33* **have** \neg *Eq* (*Geos (Poi C)* *add Emp*) (*Geos (Poi A)* *add Emp*) **by**
(*simp add:Ang-def*)
then **have** *P37* : \neg *Eq* (*Geos (Poi A)* *add Emp*) (*Geos (Poi C)* *add Emp*) **by**
(*blast intro:Eq-rev*)
from *P25 P26 P32 P33 P34 P35 P36 P37* **have** *P38* : *Eq* (*Geos (Ang (An C A*
B)) *add Emp*) (*Geos (Ang (An C A D))* *add Emp*) **by** (*simp add:Ang-Point-swap*)
from *P28* **have** *P39* : *Def* (*Ang (An C2 A2 B1)*) **by** (*simp add:Tri-to-Ang*)
from *P31* **have** *P40* : *Def* (*Ang (An C2 A2 D2)*) **by** (*simp add:Tri-to-Ang*)
have *P41* : *Line-on* (*Li A2 C2*) *C2* **by** (*simp add:Line-on-rule*)
have *P42* : \neg *Bet-Point* (*Se C2 C2*) *A2* **by** (*simp add:Bet-end-Point*)
from *P20* **have** *Inv (Bet-Point (Se B1 D2) A2)* **by** (*simp add:Bet-iff*)
then **have** *P43* : \neg *Bet-Point* (*Se B1 D2*) *A2* **by** (*simp add:Inv-def*)
from *P40* **have** \neg *Eq* (*Geos (Poi C2)* *add Emp*) (*Geos (Poi A2)* *add Emp*) **by**
(*simp add:Ang-def*)
then **have** *P44* : \neg *Eq* (*Geos (Poi A2)* *add Emp*) (*Geos (Poi C2)* *add Emp*) **by**
(*blast intro:Eq-rev*)
from *P29 P30 P39 P40 P41 P42 P43 P44* **have** *P45* : *Eq* (*Geos (Ang (An C2 A2*
B1)) *add Emp*) (*Geos (Ang (An C2 A2 D2))* *add Emp*) **by** (*simp add:Ang-Point-swap*)
from *P7 P38* **have** *P46* : *Cong* (*Geos (Ang (An C A D))* *add Emp*) (*Geos (Ang*
(*An C2 A2 B1))* *add Emp*) **by** (*blast intro:Ang-weektrans Eq-rev*)
from *P45 P46* **have** *P47* : *Cong* (*Geos (Ang (An C A D))* *add Emp*) (*Geos (Ang*
(*An C2 A2 D2))* *add Emp*) **by** (*blast intro:Ang-weektrans Eq-rev Ang-rev*)
from *P27* **have** *P48* : *Def* (*Tri (Tr A C D)*) **by** (*blast intro:Tri-def-trans*
Tri-def-rev)
from *P31* **have** *P49* : *Def* (*Tri (Tr A2 C2 D2)*) **by** (*blast intro:Tri-def-trans*

Tri-def-rev)
from $P6\ P23\ P47\ P48\ P49$ **have** $P50 : Cong\ (Geos\ (Tri\ (Tr\ A\ C\ D))\ add\ Emp)$
 $(Geos\ (Tri\ (Tr\ A2\ C2\ D2))\ add\ Emp)$ **by** $(simp\ add:Tri-SAS)$
then have $P51 : Cong\ (Geos\ (Ang\ (An\ A\ D\ C))\ add\ Emp)$ $(Geos\ (Ang\ (An\ A2\ D2\ C2))\ add\ Emp)$ **by** $(simp\ add:Tri-Cong-def)$
from $P50$ **have** $Eq\ (Geos\ (Seg\ (Se\ C\ D))\ add\ Emp)$ $(Geos\ (Seg\ (Se\ C2\ D2))\ add\ Emp)$ **by** $(simp\ add:Tri-Cong-def)$
then have $P52 : Eq\ (Geos\ (Seg\ (Se\ D\ C))\ add\ Emp)$ $(Geos\ (Seg\ (Se\ D2\ C2))\ add\ Emp)$ **by** $(blast\ intro:Seg-rev\ Eq-trans)$
from *assms* **have** $P53 : Line-on\ (Li\ D\ A)\ B$ **by** $(simp\ add:Bet-rev\ Line-Bet-on)$
from *assms* **have** $Inv\ (Bet-Point\ (Se\ B\ A)\ D)$ **by** $(simp\ add:Bet-iff)$
then have $\neg\ Bet-Point\ (Se\ B\ A)\ D$ **by** $(simp\ add:Inv-def)$
then have $P54 : \neg\ Bet-Point\ (Se\ A\ B)\ D$ **by** $(blast\ intro:Bet-rev)$
have $P55 : Line-on\ (Li\ D\ C)\ C$ **by** $(simp\ add:Line-on-rule)$
have $P56 : \neg\ Bet-Point\ (Se\ C\ C)\ D$ **by** $(simp\ add:Bet-end-Point)$
from $P48$ **have** $P57 : Def\ (Ang\ (An\ A\ D\ C))$ **by** $(simp\ add:Tri-to-Ang\ Ang-def-inv)$
from $P57$ **have** $P58 : \neg\ Eq\ (Geos\ (Poi\ D)\ add\ Emp)$ $(Geos\ (Poi\ C)\ add\ Emp)$ **by** $(simp\ add:Ang-def)$
from $P18\ P53\ P54\ P55\ P56\ P57\ P58$ **have** $P59 : Eq\ (Geos\ (Ang\ (An\ A\ D\ C))\ add\ Emp)$
 $(Geos\ (Ang\ (An\ B\ D\ C))\ add\ Emp) \wedge Def\ (Ang\ (An\ B\ D\ C))$ **by** $(simp\ add:Ang-Point-swap)$
from $P22$ **have** $P60 : Line-on\ (Li\ D2\ A2)\ B1$ **by** $(simp\ add:Line-Bet-on)$
from $P20$ **have** $Inv\ (Bet-Point\ (Se\ A2\ B1)\ D2)$ **by** $(simp\ add:Bet-iff)$
then have $P61 : \neg\ Bet-Point\ (Se\ A2\ B1)\ D2$ **by** $(simp\ add:Inv-def)$
have $P62 : Line-on\ (Li\ D2\ C2)\ C2$ **by** $(simp\ add:Line-on-rule)$
have $P63 : \neg\ Bet-Point\ (Se\ C2\ C2)\ D2$ **by** $(simp\ add:Bet-end-Point)$
from $P49$ **have** $P64 : Def\ (Ang\ (An\ A2\ D2\ C2))$ **by** $(simp\ add:Tri-to-Ang\ Ang-def-inv)$
from $P20$ **have** $P65 : \neg\ Eq\ (Geos\ (Poi\ D2)\ add\ Emp)$ $(Geos\ (Poi\ B1)\ add\ Emp)$ **by** $(blast\ intro:Eq-rev)$
from $P64$ **have** $P66 : \neg\ Eq\ (Geos\ (Poi\ D2)\ add\ Emp)$ $(Geos\ (Poi\ C2)\ add\ Emp)$ **by** $(simp\ add:Ang-def)$
from $P60\ P61\ P62\ P63\ P64\ P65\ P66$ **have** $P67 : Eq\ (Geos\ (Ang\ (An\ A2\ D2\ C2))\ add\ Emp)$
 $(Geos\ (Ang\ (An\ B1\ D2\ C2))\ add\ Emp) \wedge Def\ (Ang\ (An\ B1\ D2\ C2))$ **by** $(simp\ add:Ang-Point-swap)$
from $P51\ P59$ **have** $P68 : Cong\ (Geos\ (Ang\ (An\ B\ D\ C))\ add\ Emp)$ $(Geos\ (Ang\ (An\ A2\ D2\ C2))\ add\ Emp)$ **by** $(blast\ intro:Ang-weektrans\ Eq-rev)$
from $P67\ P68$ **have** $P69 : Cong\ (Geos\ (Ang\ (An\ B\ D\ C))\ add\ Emp)$ $(Geos\ (Ang\ (An\ B1\ D2\ C2))\ add\ Emp)$ **by** $(blast\ intro:Ang-weektrans\ Eq-rev\ Ang-rev)$
from $P59$ **have** $Def\ (Tri\ (Tr\ B\ D\ C))$ **by** $(simp\ add:Ang-to-Tri)$
then have $P70 : Def\ (Tri\ (Tr\ D\ B\ C))$ **by** $(blast\ intro:Tri-def-trans\ Tri-def-rev)$
from $P67$ **have** $Def\ (Tri\ (Tr\ B1\ D2\ C2))$ **by** $(simp\ add:Ang-to-Tri)$
then have $P71 : Def\ (Tri\ (Tr\ D2\ B1\ C2))$ **by** $(blast\ intro:Tri-def-trans\ Tri-def-rev)$
from $P20$ **have** $Eq\ (Geos\ (Seg\ (Se\ B\ D))\ add\ Emp)$ $(Geos\ (Seg\ (Se\ B1\ D2))\ add\ Emp)$ **by** *simp*
then have $P72 : Eq\ (Geos\ (Seg\ (Se\ D\ B))\ add\ Emp)$ $(Geos\ (Seg\ (Se\ D2\ B1))\ add\ Emp)$ **by** $(blast\ intro:Seg-rev\ Eq-trans)$
from $P52\ P69\ P70\ P71\ P72$ **have** $Cong\ (Geos\ (Tri\ (Tr\ D\ B\ C))\ add\ Emp)$

(Geos (Tri (Tr D2 B1 C2)) add Emp) **by** (simp add:Tri-SAS)
then have P73 : Cong (Geos (Ang (An C B D)) add Emp) (Geos (Ang (An C2 B1 D2)) add Emp) **by** (simp add:Tri-Cong-def)
from P71 **have** Def (Tri (Tr C2 B1 D2)) **by** (blast intro:Tri-def-rev)
then have P74 : Def (Ang (An C2 B1 D2)) **by** (simp add:Tri-to-Ang)
from *assms* **have** P75 : \neg Eq (Geos (Poi B1) add Emp) (Geos (Poi C1) add Emp) **by** (simp add:Ang-def)
from P71 **have** P76 : \neg Eq (Geos (Poi B1) add Emp) (Geos (Poi C2) add Emp) **by** (simp add:Tri-def)
from P2 P75 P76 **have** P77 : Line-on (Li B1 C2) C1 **by** (simp add:Line-on-rev)
from P14 **have** P78 : \neg Eq (Geos (Poi B1) add Emp) (Geos (Poi D1) add Emp) **by** (blast intro:Eq-rev)
from P74 **have** P79 : \neg Eq (Geos (Poi B1) add Emp) (Geos (Poi D2) add Emp) **by** (simp add:Ang-def)
from P20 P78 P79 **have** P80 : Line-on (Li B1 D2) D1 **by** (simp add:Line-on-rev)
from *assms* **have** \neg Eq (Geos (Lin (Li B1 A1)) add Emp) (Geos (Lin (Li B1 C1)) add Emp) **by** (simp add:Ang-def)
then have P81 : \neg Eq (Geos (Lin (Li B1 C1)) add Emp) (Geos (Lin (Li B1 A1)) add Emp) **by** (blast intro:Eq-rev)
have P82 : Line-on (Li A1 D1) D1 **by** (simp add:Line-on-rule)
from P9 P11 P13 P14 P82 **have** P83 : Eq (Geos (Lin (Li B1 A1)) add Emp) (Geos (Lin (Li A1 D1)) add Emp) **by** (simp add:Line-unique)
from P81 P83 **have** P84 : \neg Eq (Geos (Lin (Li B1 C1)) add Emp) (Geos (Lin (Li A1 D1)) add Emp) **by** (simp add:Eq-not-trans)
then have P85 : \neg Eq (Geos (Lin (Li A1 D1)) add Emp) (Geos (Lin (Li B1 C1)) add Emp) **by** (blast intro:Eq-rev)
have P86 : Line-on (Li B1 C1) B1 **by** (simp add:Line-on-rule)
from *assms* P85 P86 **have** P87 : Plane-diffside (Li B1 C1) A1 D1 **by** (simp add:Plane-Bet-diffside)
then have P88 : Plane-diffside (Li B1 C1) D1 A1 **by** (simp add:Plane-diffside-rev)
have P89 : Bet-Point (Se D2 D1) B1 \implies Line-on (Li D2 D1) B1 **by** (simp add:Line-Bet-on)
have P90 : Line-on (Li D2 D1) D1 **by** (simp add:Line-on-rule)
from P9 P14 P82 P89 P90 **have** P91 : Bet-Point (Se D2 D1) B1 \implies Eq (Geos (Lin (Li A1 D1)) add Emp) (Geos (Lin (Li D2 D1)) add Emp) **by** (simp add:Line-unique)
from P84 P91 **have** Bet-Point (Se D2 D1) B1 \implies \neg Eq (Geos (Lin (Li B1 C1)) add Emp) (Geos (Lin (Li D2 D1)) add Emp) **by** (simp add:Eq-not-trans)
then have P92 : Bet-Point (Se D2 D1) B1 \implies \neg Eq (Geos (Lin (Li D2 D1)) add Emp) (Geos (Lin (Li B1 C1)) add Emp) **by** (blast intro:Eq-rev)
from P86 P92 **have** Bet-Point (Se D2 D1) B1 \implies Plane-diffside (Li B1 C1) D2 D1 **by** (simp add:Plane-Bet-diffside)
then have P93 : Bet-Point (Se D2 D1) B1 \implies Plane-diffside (Li B1 C1) D1 D2 **by** (simp add:Plane-diffside-rev)
from P20 **have** Eq (Geos (Poi D2) add Emp) (Geos (Poi A1) add Emp) \implies Bet-Point (Se A1 A2) B1 **by** (blast intro:Bet-Point-Eq)
then have P94 : Eq (Geos (Poi D2) add Emp) (Geos (Poi A1) add Emp) \implies Bet-Point (Se A2 A1) B1 **by** (simp add:Bet-rev)
from P1 P94 **have** P95 : \neg Eq (Geos (Poi A1) add Emp) (Geos (Poi D2) add

Emp) **by** (*blast intro:Eq-rev*)
from *P88 P93 P95* **have** *Bet-Point (Se D2 D1) B1 \implies Plane-sameside (Li B1 C1) A1 D2* **by** (*blast intro:Plane-trans-inv*)
then have *P96 : Bet-Point (Se D2 D1) B1 \implies Plane-sameside (Li B1 C1) D2 A1* **by** (*simp add:Plane-sameside-rev*)
from *P1* **have** *Def (Tri (Tr A2 B1 C1))* **by** (*simp add:Ang-to-Tri*)
then have *P97 : \neg Line-on (Li B1 C1) A2* **by** (*simp add:Tri-def-Line*)
have *Line-on (Li D2 A2) A2* **by** (*simp add:Line-on-rule*)
then have *P98 : Eq (Geos (Lin (Li D2 A2)) add Emp) (Geos (Lin (Li B1 C1)) add Emp) \implies Line-on (Li B1 C1) A2* **by** (*simp add:Line-on-trans*)
from *P97 P98* **have** *P99 : \neg Eq (Geos (Lin (Li D2 A2)) add Emp) (Geos (Lin (Li B1 C1)) add Emp)* **by** *blast*
from *P20* **have** *P100 : Bet-Point (Se D2 A2) B1* **by** *simp*
from *P86 P99 P100* **have** *P101 : Plane-diffside (Li B1 C1) D2 A2* **by** (*simp add:Plane-Bet-diffside*)
from *P96 P101* **have** \neg *Eq (Geos (Poi A1) add Emp) (Geos (Poi A2) add Emp) \implies Bet-Point (Se D2 D1) B1 \implies Plane-diffside (Li B1 C1) A1 A2* **by** (*simp add:Plane-trans*)
then have *P102 : \neg Eq (Geos (Poi A1) add Emp) (Geos (Poi A2) add Emp) \implies Bet-Point (Se D2 D1) B1 \implies \neg Plane-sameside (Li B1 C1) A1 A2* **by** (*simp add:Plane-diffside-not-sameside*)
have *P103 : Line-on (Li B1 A1) A1* **by** (*simp add:Line-on-rule*)
from *assms* **have** *Def (Tri (Tr C1 B1 A1))* **by** (*simp add:Ang-to-Tri Tri-def-rev*)
then have *P104 : \neg Line-on (Li B1 A1) C1* **by** (*simp add:Tri-def-Line*)
from *P4* **have** *P105 : \neg Eq (Geos (Poi B1) add Emp) (Geos (Poi A2) add Emp)* **by** (*simp add:Tri-def*)
from *P1 p10 P13 P103 P104 P105* **have** *P106 : Plane-sameside (Li C1 B1) A2 A1 \vee Eq (Geos (Poi A2) add Emp) (Geos (Poi A1) add Emp)* **by** (*simp add:Seg-Plane-sameside*)
from *assms* **have** \neg *Eq (Geos (Poi B1) add Emp) (Geos (Poi C1) add Emp)* **by** (*simp add:Ang-def*)
then have *P107 : Eq (Geos (Lin (Li C1 B1)) add Emp) (Geos (Lin (Li B1 C1)) add Emp)* **by** (*simp add:Line-rev Eq-rev*)
from *P106 P107* **have** *P108 : Plane-sameside (Li B1 C1) A1 A2 \vee Eq (Geos (Poi A1) add Emp) (Geos (Poi A2) add Emp)* **by** (*blast intro:Plane-sameside-rev Plane-Line-trans Eq-rev*)
from *P102 P108* **have** *P109 : \neg Eq (Geos (Poi A1) add Emp) (Geos (Poi A2) add Emp) \implies \neg Bet-Point (Se D2 D1) B1* **by** *blast*
from *P22* **have** *P110 : Eq (Geos (Poi A1) add Emp) (Geos (Poi A2) add Emp) \implies Bet-Point (Se A1 D2) B1* **by** (*blast intro:Bet-Point-Eq Eq-rev*)
then have *P111 : Eq (Geos (Poi A1) add Emp) (Geos (Poi A2) add Emp) \implies Line-on (Li A1 D2) B1* **by** (*simp add:Line-Bet-on*)
have *P112 : Line-on (Li A1 D2) A1* **by** (*simp add:Line-on-rule*)
have *P113 : Line-on (Li A1 D1) A1* **by** (*simp add:Line-on-rule*)
from *p10* **have** *P114 : \neg Eq (Geos (Poi A1) add Emp) (Geos (Poi B1) add Emp)* **by** (*blast intro:Eq-rev*)
from *P9 P111 P112 P113 P114* **have** *P115 : Eq (Geos (Poi A1) add Emp) (Geos (Poi A2) add Emp) \implies Eq (Geos (Lin (Li A1 D1)) add Emp) (Geos (Lin (Li A1 D2)) add Emp)* **by** (*simp add:Line-unique*)

from $P84$ $P115$ **have** Eq ($Geos$ (Poi $A1$) add Emp) ($Geos$ (Poi $A2$) add Emp)
 $\implies \neg Eq$ ($Geos$ (Lin (Li $B1$ $C1$)) add Emp) ($Geos$ (Lin (Li $A1$ $D2$)) add Emp)
by (*simp add:Eq-not-trans*)
then have $P116 : Eq$ ($Geos$ (Poi $A1$) add Emp) ($Geos$ (Poi $A2$) add Emp) \implies
 $\neg Eq$ ($Geos$ (Lin (Li $A1$ $D2$)) add Emp) ($Geos$ (Lin (Li $B1$ $C1$)) add Emp) **by**
(*blast intro:Eq-rev*)
from $P86$ $P110$ $P116$ **have** $P117 : Eq$ ($Geos$ (Poi $A1$) add Emp) ($Geos$ (Poi $A2$)
 add Emp) \implies $Plane\text{-}diffside$ (Li $B1$ $C1$) $A1$ $D2$ **by** (*simp add:Plane-Bet-diffside*)
have Eq ($Geos$ (Poi $A1$) add Emp) ($Geos$ (Poi $A2$) add Emp) \implies $Bet\text{-}Point$
(Se $D2$ $D1$) $B1 \implies \neg Eq$ ($Geos$ (Poi $D2$) add Emp) ($Geos$ (Poi $D1$) add Emp) **by**
(*simp add:Bet-Point-def*)
then have $P118 : Eq$ ($Geos$ (Poi $A1$) add Emp) ($Geos$ (Poi $A2$) add Emp) \implies
 $Bet\text{-}Point$ (Se $D2$ $D1$) $B1 \implies$
 $\neg Eq$ ($Geos$ (Poi $D1$) add Emp) ($Geos$ (Poi $D2$) add Emp) **by** (*blast intro:Eq-rev*)
from $P87$ $P117$ $P118$ **have** $P119 : Eq$ ($Geos$ (Poi $A1$) add Emp) ($Geos$ (Poi $A2$)
 add Emp) \implies $Bet\text{-}Point$ (Se $D2$ $D1$) $B1 \implies$
 $Plane\text{-}sameside$ (Li $B1$ $C1$) $D1$ $D2$ **by** (*blast intro:Plane-trans-inv*)
from $P93$ **have** $P120 : Bet\text{-}Point$ (Se $D2$ $D1$) $B1 \implies \neg Plane\text{-}sameside$ (Li $B1$
 $C1$) $D1$ $D2$ **by** (*simp add:Plane-diffside-not-sameside*)
from $P119$ $P120$ **have** $P121 : Eq$ ($Geos$ (Poi $A1$) add Emp) ($Geos$ (Poi $A2$) add
 Emp) $\implies \neg Bet\text{-}Point$ (Se $D2$ $D1$) $B1$ **by** *blast*
from $P109$ $P121$ **have** $P122 : \neg Bet\text{-}Point$ (Se $D2$ $D1$) $B1$ **by** *blast*
from $P2$ $P74$ $P75$ $P77$ $P78$ $P80$ $P122$ **have** $P123 : Eq$ ($Geos$ (Ang (An $C2$ $B1$ $D2$))
 add Emp) ($Geos$ (Ang (An $C1$ $B1$ $D1$)) add Emp) **by** (*simp add:Ang-Point-swap*)
from $P73$ $P123$ **show** $Cong$ ($Geos$ (Ang (An C B D)) add Emp) ($Geos$ (Ang (An
 $C1$ $B1$ $D1$)) add Emp) **by** (*blast intro:Ang-weektrans Ang-rev Eq-rev*)
qed

theorem (*in Congruence-Rule*) $Ang\text{-vertical}$:

assumes

Def (Ang (An A B C))

$Bet\text{-}Point$ (Se A D) B

$Bet\text{-}Point$ (Se C E) B

shows $Cong$ ($Geos$ (Ang (An A B C)) add Emp) ($Geos$ (Ang (An D B E)) add
 Emp)

and $Cong$ ($Geos$ (Ang (An C B D)) add Emp) ($Geos$ (Ang (An A B E)) add
 Emp)

proof –

have $P1 : Cong$ ($Geos$ (Ang (An A B C)) add Emp) ($Geos$ (Ang (An C B A))
 add Emp) **by** (*simp add:Ang-roll*)

from *assms* **have** $P2 : Def$ (Ang (An C B A)) **by** (*simp add:Ang-def-rev*)

from *assms* $P1$ $P2$ **show** $Cong$ ($Geos$ (Ang (An C B D)) add Emp) ($Geos$ (Ang (An
 A B E)) add Emp) **by** (*simp add:Ang-complementary*)

from *assms* **have** $P3 : Line\text{-}on$ (Li B A) D **by** (*simp add:Line-Bet-on*)

from *assms* **have** $\neg Eq$ ($Geos$ (Poi D) add Emp) ($Geos$ (Poi B) add Emp) **by**
(*simp add:Bet-Point-def*)

then have $P4 : \neg Eq$ ($Geos$ (Poi B) add Emp) ($Geos$ (Poi D) add Emp) **by**
(*blast intro:Eq-rev*)

from $P2$ $P3$ $P4$ **have** $P5 : Def$ (Ang (An C B D)) **by** (*simp add:Ang-def-extension*)

then have $P6 : \text{Def } (\text{Ang } (\text{An } D B C))$ **by** (*simp add:Ang-def-rev*)
have $P7 : \text{Cong } (\text{Geos } (\text{Ang } (\text{An } C B D)) \text{ add Emp})$ ($\text{Geos } (\text{Ang } (\text{An } D B C))$
add Emp) **by** (*simp add:Ang-roll*)
from *assms* **have** $P8 : \text{Bet-Point } (\text{Se } D A) B$ **by** (*simp add:Bet-rev*)
from *assms* $P5 P6 P7 P8$ **have** $P9 : \text{Cong } (\text{Geos } (\text{Ang } (\text{An } D B E)) \text{ add Emp})$
 $(\text{Geos } (\text{Ang } (\text{An } C B A)) \text{ add Emp})$ **by** (*simp add:Ang-complementary*)
from $P1$ **have** $P10 : \text{Eq } (\text{Geos } (\text{Ang } (\text{An } C B A)) \text{ add Emp})$ ($\text{Geos } (\text{Ang } (\text{An } A$
 $B C)) \text{ add Emp})$ **by** (*simp add:Ang-roll*)
from $P9 P10$ **show** $\text{Cong } (\text{Geos } (\text{Ang } (\text{An } A B C)) \text{ add Emp})$ ($\text{Geos } (\text{Ang } (\text{An } D$
 $B E)) \text{ add Emp})$ **by** (*blast intro:Ang-weektrans Ang-rev Eq-rev*)
qed

lemma (in *Congruence-Rule*) *Ang-inside-Planeside* :

assumes *Ang-inside* ($\text{An } A B C$) D

shows *Plane-diffside* ($\text{Li } B D$) $A C$

proof –

from *assms* **have** $P1 : \text{Plane-sameside } (\text{Li } B A) C D \wedge \text{Plane-sameside } (\text{Li } B$
 $C) A D$ **by** (*simp add:Ang-inside-def*)
then have $P2 : \neg \text{Line-on } (\text{Li } B A) C$ **by** (*simp add:Plane-sameside-def*)
have $P3 : \text{Line-on } (\text{Li } B A) B$ **by** (*simp add:Line-on-rule*)
then have $P4 : \text{Eq } (\text{Geos } (\text{Poi } B) \text{ add Emp})$ ($\text{Geos } (\text{Poi } C) \text{ add Emp}) \implies$
 $\text{Line-on } (\text{Li } B A) C$ **by** (*simp add:Point-Eq*)
from $P2 P4$ **have** $P5 : \neg \text{Eq } (\text{Geos } (\text{Poi } C) \text{ add Emp})$ ($\text{Geos } (\text{Poi } B) \text{ add Emp})$
by (*blast intro:Eq-rev*)
have $P6 : \text{Line-on } (\text{Li } B C) B$ **by** (*simp add:Line-on-rule*)
have $P7 : \text{Line-on } (\text{Li } B C) C$ **by** (*simp add:Line-on-rule*)
from $P5 P6 P7$ **have** $\exists p. \text{Bet-Point } (\text{Se } C p) B \wedge \text{Line-on } (\text{Li } B C) p$ **by** (*simp*
add:Bet-extension)
then obtain $E :: \text{Point where } P8 : \text{Bet-Point } (\text{Se } C E) B \wedge \text{Line-on } (\text{Li } B C)$
 E **by** *blast*
then have $P9 : \text{Line-on } (\text{Li } C E) B$ **by** (*simp add:Line-Bet-on*)
have $P10 : \text{Line-on } (\text{Li } C E) C$ **by** (*simp add:Line-on-rule*)
from $P5 P6 P7 P9 P10$ **have** $P11 : \text{Eq } (\text{Geos } (\text{Lin } (\text{Li } B C)) \text{ add Emp})$ (Geos
 $(\text{Lin } (\text{Li } C E)) \text{ add Emp})$ **by** (*simp add:Line-unique*)
from $P1$ **have** $P12 : \neg \text{Line-on } (\text{Li } B C) A$ **by** (*simp add:Plane-sameside-def*)
from $P11 P12$ **have** $P13 : \neg \text{Line-on } (\text{Li } C E) A$ **by** (*simp add:Line-not-on-trans*)
have $P14 : \text{Line-on } (\text{Li } B D) B$ **by** (*simp add:Line-on-rule*)
from $P1$ **have** $P15 : \neg \text{Line-on } (\text{Li } B A) D$ **by** (*simp add:Plane-sameside-def*)
from $P3$ **have** $P16 : \text{Eq } (\text{Geos } (\text{Poi } B) \text{ add Emp})$ ($\text{Geos } (\text{Poi } D) \text{ add Emp}) \implies$
 $\text{Line-on } (\text{Li } B A) D$ **by** (*simp add:Point-Eq*)
from $P15 P16$ **have** $P17 : \neg \text{Eq } (\text{Geos } (\text{Poi } B) \text{ add Emp})$ ($\text{Geos } (\text{Poi } D) \text{ add}$
 $\text{Emp})$ **by** *blast*
from $P6$ **have** $P18 : \text{Eq } (\text{Geos } (\text{Poi } B) \text{ add Emp})$ ($\text{Geos } (\text{Poi } A) \text{ add Emp}) \implies$
 $\text{Line-on } (\text{Li } B C) A$ **by** (*simp add:Point-Eq*)
from $P12 P18$ **have** $P19 : \neg \text{Eq } (\text{Geos } (\text{Poi } B) \text{ add Emp})$ ($\text{Geos } (\text{Poi } A) \text{ add}$
 $\text{Emp})$ **by** *blast*
from $P17 P19$ **have** $P20 : \text{Line-on } (\text{Li } B D) A \implies \text{Line-on } (\text{Li } B A) D$ **by**
(simp add:Line-on-rev)
from $P15 P20$ **have** $P21 : \neg \text{Line-on } (\text{Li } B D) A$ **by** *blast*

from $P1$ **have** $P22 : \neg \text{Line-on } (Li\ B\ C)\ D$ **by** (*simp add:Plane-sameside-def*)
from $P5$ **have** $P23 : \neg \text{Eq } (\text{Geos } (Poi\ B)\ \text{add Emp})\ (\text{Geos } (Poi\ C)\ \text{add Emp})$
by (*blast intro:Eq-rev*)
from $P17\ P23$ **have** $P24 : \text{Line-on } (Li\ B\ D)\ C \implies \text{Line-on } (Li\ B\ C)\ D$ **by**
(*simp add:Line-on-rev*)
from $P22\ P24$ **have** $P25 : \neg \text{Line-on } (Li\ B\ D)\ C$ **by** *blast*
from $P8$ **have** $P26 : \text{Bet-Point } (Se\ C\ E)\ B$ **by** *simp*
then **have** $P27 : \neg \text{Eq } (\text{Geos } (Poi\ E)\ \text{add Emp})\ (\text{Geos } (Poi\ B)\ \text{add Emp})$ **by**
(*simp add:Bet-Point-def*)
from $P8$ **have** $P28 : \text{Line-on } (Li\ B\ C)\ E$ **by** *simp*
from $P6\ P14\ P27\ P28$ **have** $P29 : \text{Line-on } (Li\ B\ D)\ E \implies \text{Eq } (\text{Geos } (Lin\ (Li\ B\ C))\ \text{add Emp})\ (\text{Geos } (Lin\ (Li\ B\ D))\ \text{add Emp})$ **by** (*simp add:Line-unique*)
from $P7\ P29$ **have** $P30 : \text{Line-on } (Li\ B\ D)\ E \implies \text{Line-on } (Li\ B\ D)\ C$ **by** (*simp add:Line-on-trans*)
from $P25\ P30$ **have** $P31 : \neg \text{Line-on } (Li\ B\ D)\ E$ **by** *blast*
from $P13\ P14\ P21\ P25\ P26\ P31$ **have** $P32 : \text{Line-on-Seg } (Li\ B\ D)\ (Se\ C\ A) \wedge \neg \text{Line-on-Seg } (Li\ B\ D)\ (Se\ E\ A) \vee \text{Line-on-Seg } (Li\ B\ D)\ (Se\ E\ A) \wedge \neg \text{Line-on-Seg } (Li\ B\ D)\ (Se\ C\ A)$ **by**
(*simp add:Pachets-axiom*)
have $\text{Line-on-Seg } (Li\ B\ D)\ (Se\ E\ A) \implies \exists p. \text{Line-on } (Li\ B\ D)\ p \wedge \text{Bet-Point } (Se\ E\ A)\ p$ **by** (*simp add:Line-on-Seg-rule*)
then **obtain** $F :: \text{Point}$ **where** $P33 : \text{Line-on-Seg } (Li\ B\ D)\ (Se\ E\ A) \implies \text{Line-on } (Li\ B\ D)\ F \wedge \text{Bet-Point } (Se\ E\ A)\ F$ **by** *blast*
then **have** $P34 : \text{Line-on-Seg } (Li\ B\ D)\ (Se\ E\ A) \implies \text{Bet-Point } (Se\ A\ E)\ F$ **by**
(*simp add:Bet-rev*)
from $P3$ **have** $P35 : \text{Eq } (\text{Geos } (Lin\ (Li\ B\ A))\ \text{add Emp})\ (\text{Geos } (Lin\ (Li\ A\ E))\ \text{add Emp}) \implies \text{Line-on } (Li\ A\ E)\ B$ **by** (*simp add:Line-on-trans*)
have $P36 : \text{Line-on } (Li\ A\ E)\ E$ **by** (*simp add:Line-on-rule*)
from $P6\ P27\ P28\ P35\ P36$ **have** $\text{Eq } (\text{Geos } (Lin\ (Li\ B\ A))\ \text{add Emp})\ (\text{Geos } (Lin\ (Li\ A\ E))\ \text{add Emp}) \implies \text{Eq } (\text{Geos } (Lin\ (Li\ A\ E))\ \text{add Emp})\ (\text{Geos } (Lin\ (Li\ B\ C))\ \text{add Emp})$ **by** (*simp add:Line-unique*)
then **have** $P37 : \text{Eq } (\text{Geos } (Lin\ (Li\ B\ A))\ \text{add Emp})\ (\text{Geos } (Lin\ (Li\ A\ E))\ \text{add Emp}) \implies \text{Eq } (\text{Geos } (Lin\ (Li\ B\ A))\ \text{add Emp})\ (\text{Geos } (Lin\ (Li\ B\ C))\ \text{add Emp})$ **by** (*blast intro:Eq-trans*)
have $P38 : \text{Line-on } (Li\ B\ A)\ A$ **by** (*simp add:Line-on-rule*)
from $P37\ P38$ **have** $P39 : \text{Eq } (\text{Geos } (Lin\ (Li\ B\ A))\ \text{add Emp})\ (\text{Geos } (Lin\ (Li\ B\ C))\ \text{add Emp}) \implies \text{Line-on } (Li\ B\ C)\ A$ **by** (*simp add:Line-on-trans*)
from $P12\ P39$ **have** $P40 : \neg \text{Eq } (\text{Geos } (Lin\ (Li\ B\ A))\ \text{add Emp})\ (\text{Geos } (Lin\ (Li\ B\ C))\ \text{add Emp})$ **by** *blast*
from $P37\ P40$ **have** $P41 : \neg \text{Eq } (\text{Geos } (Lin\ (Li\ A\ E))\ \text{add Emp})\ (\text{Geos } (Lin\ (Li\ B\ A))\ \text{add Emp})$ **by** (*blast intro:Eq-rev*)
from $P34\ P38\ P41$ **have** $P42 : \text{Line-on-Seg } (Li\ B\ D)\ (Se\ E\ A) \implies \text{Plane-sameside } (Li\ B\ A)\ E\ F$ **by** (*simp add:Plane-Bet-sameside Plane-sameside-rev*)
from $P11$ **have** $P43 : \text{Eq } (\text{Geos } (Lin\ (Li\ C\ E))\ \text{add Emp})\ (\text{Geos } (Lin\ (Li\ B\ A))\ \text{add Emp}) \implies \text{Eq } (\text{Geos } (Lin\ (Li\ B\ A))\ \text{add Emp})\ (\text{Geos } (Lin\ (Li\ B\ C))\ \text{add Emp})$ **by** (*blast intro:Eq-trans*)
from $P40\ P43$ **have** $P44 : \neg \text{Eq } (\text{Geos } (Lin\ (Li\ C\ E))\ \text{add Emp})\ (\text{Geos } (Lin\ (Li\ B\ A))\ \text{add Emp})$

$B A$) *add Emp*) **by** *blast*
from $P3 P26 P44$ **have** $P45 : \text{Plane-diffside } (Li B A) E C$ **by** (*simp add:Plane-Bet-diffside*
Plane-diffside-rev)
from $P34$ **have** $P46 : \text{Line-on-Seg } (Li B D) (Se E A) \implies \text{Line-on } (Li A E) F$
by (*simp add:Line-Bet-on*)
have $P47 : \text{Line-on } (Li C E) E$ **by** (*simp add:Line-on-rule*)
have $P48 : \text{Line-on } (Li A E) A$ **by** (*simp add:Line-on-rule*)
from $P42 P45$ **have** $P49 : \text{Line-on-Seg } (Li B D) (Se E A) \implies \text{Plane-diffside}$
 $(Li B A) F C$ **by** (*simp add:Plane-trans*)
from $P1$ **have** $P50 : \text{Plane-sameside } (Li B A) C D$ **by** (*simp add:Point-Eq*)
then have $Eq (\text{Geos } (Poi D) \text{ add Emp}) (\text{Geos } (Poi F) \text{ add Emp}) \implies \text{Plane-sameside}$
 $(Li B A) C F$ **by** (*simp add:Point-Eq*)
then have $Eq (\text{Geos } (Poi D) \text{ add Emp}) (\text{Geos } (Poi F) \text{ add Emp}) \implies \neg$
 $\text{Plane-diffside } (Li B A) C F$ **by** (*simp add:Plane-sameside-not-diffside*)
then have $P51 : Eq (\text{Geos } (Poi D) \text{ add Emp}) (\text{Geos } (Poi F) \text{ add Emp}) \implies \neg$
 $\text{Plane-diffside } (Li B A) F C$ **by** (*blast intro:Plane-diffside-rev*)
from $P49 P51$ **have** $P52 : \text{Line-on-Seg } (Li B D) (Se E A) \implies \neg Eq (\text{Geos } (Poi$
 $D) \text{ add Emp}) (\text{Geos } (Poi F) \text{ add Emp})$ **by** *blast*
from $P49 P50$ **have** $P53 : \text{Line-on-Seg } (Li B D) (Se E A) \implies \text{Plane-diffside}$
 $(Li B A) D F$ **by** (*simp add:Plane-trans Plane-diffside-rev*)
from $P46$ **have** $P54 : \text{Line-on-Seg } (Li B D) (Se E A) \implies Eq (\text{Geos } (Poi F) \text{ add}$
 $\text{Emp}) (\text{Geos } (Poi B) \text{ add Emp}) \implies$
 $\text{Line-on } (Li A E) B$ **by** (*simp add:Point-Eq*)
from $P26$ **have** $P55 : \text{Line-on } (Li C E) B$ **by** (*simp add:Line-Bet-on*)
from $P27 P36 P47 P54 P55$ **have** $P56 : \text{Line-on-Seg } (Li B D) (Se E A) \implies Eq$
 $(\text{Geos } (Poi F) \text{ add Emp}) (\text{Geos } (Poi B) \text{ add Emp}) \implies$
 $Eq (\text{Geos } (Lin (Li A E)) \text{ add Emp}) (\text{Geos } (Lin (Li C E)) \text{ add Emp})$ **by** (*simp*
add:Line-unique)
from $P48 P56$ **have** $P57 : \text{Line-on-Seg } (Li B D) (Se E A) \implies Eq (\text{Geos } (Poi$
 $F) \text{ add Emp}) (\text{Geos } (Poi B) \text{ add Emp}) \implies$
 $\text{Line-on } (Li C E) A$ **by** (*simp add:Line-on-trans*)
from $P13 P57$ **have** $P58 : \text{Line-on-Seg } (Li B D) (Se E A) \implies \neg Eq (\text{Geos } (Poi$
 $F) \text{ add Emp}) (\text{Geos } (Poi B) \text{ add Emp})$ **by** *blast*
have $P59 : \text{Line-on } (Li B D) D$ **by** (*simp add:Line-on-rule*)
from $P14 P17 P33 P52 P58 P59$ **have** $P60 : \text{Line-on-Seg } (Li B D) (Se E A)$
 $\implies \text{Bet-Point } (Se B F) D \vee \text{Bet-Point } (Se F D) B \vee \text{Bet-Point } (Se D B) F$ **by**
(simp add:Bet-case)
have $P61 : \text{Line-on } (Li B F) B$ **by** (*simp add:Line-on-rule*)
have $P62 : \text{Line-on } (Li B F) F$ **by** (*simp add:Line-on-rule*)
from $P33$ **have** $P63 : \text{Line-on-Seg } (Li B D) (Se E A) \implies \text{Line-on } (Li B D) F$
by *simp*
from $P14 P58 P61 P62 P63$ **have** $\text{Line-on-Seg } (Li B D) (Se E A) \implies Eq$
 $(\text{Geos } (Lin (Li B F)) \text{ add Emp}) (\text{Geos } (Lin (Li B D)) \text{ add Emp})$ **by** (*simp*
add:Line-unique)
then have $P64 : \text{Line-on-Seg } (Li B D) (Se E A) \implies Eq (\text{Geos } (Lin (Li B F))$
 $\text{add Emp}) (\text{Geos } (Lin (Li B A)) \text{ add Emp}) \implies$
 $Eq (\text{Geos } (Lin (Li B D)) \text{ add Emp}) (\text{Geos } (Lin (Li B A)) \text{ add Emp})$ **by** (*blast*
intro:Eq-trans)
from $P38$ **have** $P65 : Eq (\text{Geos } (Lin (Li B A)) \text{ add Emp}) (\text{Geos } (Lin (Li B D))$

$\text{add Emp} \implies \text{Line-on } (Li B D) A \text{ by } (\text{simp add:Line-on-trans})$
from $P21 P65$ **have** $P66 : \neg \text{Eq } (\text{Geos } (\text{Lin } (Li B D)) \text{ add Emp}) (\text{Geos } (\text{Lin } (Li B A)) \text{ add Emp})$ **by** $(\text{blast intro:Eq-rev})$
from $P64 P66$ **have** $P67 : \text{Line-on-Seg } (Li B D) (Se E A) \implies \neg \text{Eq } (\text{Geos } (\text{Lin } (Li B F)) \text{ add Emp}) (\text{Geos } (\text{Lin } (Li B A)) \text{ add Emp})$ **by** blast
from $P3 P67$ **have** $\text{Line-on-Seg } (Li B D) (Se E A) \implies \text{Bet-Point } (Se B F) D \implies \text{Plane-sameside } (Li B A) D F$ **by** $(\text{simp add:Plane-Bet-sameside Plane-sameside-rev})$
then have $P68 : \text{Line-on-Seg } (Li B D) (Se E A) \implies \text{Bet-Point } (Se B F) D \implies \neg \text{Plane-diffside } (Li B A) D F$ **by** $(\text{simp add:Plane-sameside-not-diffside})$
from $P53 P68$ **have** $P69 : \text{Line-on-Seg } (Li B D) (Se E A) \implies \neg \text{Bet-Point } (Se B F) D$ **by** blast
have $P70 : \text{Bet-Point } (Se D B) F \implies \text{Bet-Point } (Se B D) F$ **by** $(\text{simp add:Bet-rev})$
from $P3 P66 P70$ **have** $\text{Line-on-Seg } (Li B D) (Se E A) \implies \text{Bet-Point } (Se D B) F \implies \text{Plane-sameside } (Li B A) D F$ **by** $(\text{simp add:Plane-Bet-sameside Plane-sameside-rev})$
then have $P71 : \text{Line-on-Seg } (Li B D) (Se E A) \implies \text{Bet-Point } (Se D B) F \implies \neg \text{Plane-diffside } (Li B A) D F$ **by** $(\text{simp add:Plane-sameside-not-diffside})$
from $P53 P71$ **have** $P72 : \text{Line-on-Seg } (Li B D) (Se E A) \implies \neg \text{Bet-Point } (Se D B) F$ **by** blast
from $P60 P69 P72$ **have** $P73 : \text{Line-on-Seg } (Li B D) (Se E A) \implies \text{Bet-Point } (Se F D) B$ **by** blast
have $\text{Line-on } (Li F D) D$ **by** $(\text{simp add:Line-on-rule})$
then have $P74 : \text{Eq } (\text{Geos } (\text{Lin } (Li F D)) \text{ add Emp}) (\text{Geos } (\text{Lin } (Li B C)) \text{ add Emp}) \implies \text{Line-on } (Li B C) D$ **by** $(\text{simp add:Line-on-trans})$
from $P22 P74$ **have** $P75 : \neg \text{Eq } (\text{Geos } (\text{Lin } (Li F D)) \text{ add Emp}) (\text{Geos } (\text{Lin } (Li B C)) \text{ add Emp})$ **by** blast
from $P6 P73 P75$ **have** $P76 : \text{Line-on-Seg } (Li B D) (Se E A) \implies \text{Plane-diffside } (Li B C) F D$ **by** $(\text{simp add:Plane-Bet-diffside})$
from $P33$ **have** $P77 : \text{Line-on-Seg } (Li B D) (Se E A) \implies \text{Bet-Point } (Se E A) F$ **by** simp
have $\text{Line-on } (Li E A) A$ **by** $(\text{simp add:Line-on-rule})$
then have $P78 : \text{Eq } (\text{Geos } (\text{Lin } (Li E A)) \text{ add Emp}) (\text{Geos } (\text{Lin } (Li C E)) \text{ add Emp}) \implies \text{Line-on } (Li C E) A$ **by** $(\text{simp add:Line-on-trans})$
from $P13 P78$ **have** $P79 : \neg \text{Eq } (\text{Geos } (\text{Lin } (Li E A)) \text{ add Emp}) (\text{Geos } (\text{Lin } (Li C E)) \text{ add Emp})$ **by** blast
from $P47 P77 P79$ **have** $P80 : \text{Line-on-Seg } (Li B D) (Se E A) \implies \text{Plane-sameside } (Li C E) F A$ **by** $(\text{simp add:Plane-Bet-sameside})$
from $P11 P80$ **have** $P81 : \text{Line-on-Seg } (Li B D) (Se E A) \implies \text{Plane-sameside } (Li B C) F A$ **by** $(\text{blast intro:Eq-rev Plane-Line-trans})$
from $P76 P81$ **have** $\text{Line-on-Seg } (Li B D) (Se E A) \implies \text{Plane-diffside } (Li B C) A D$ **by** $(\text{simp add:Plane-trans})$
then have $P82 : \text{Line-on-Seg } (Li B D) (Se E A) \implies \neg \text{Plane-sameside } (Li B C) A D$ **by** $(\text{simp add:Plane-diffside-not-sameside})$
from $P1 P82$ **have** $P83 : \neg \text{Line-on-Seg } (Li B D) (Se E A)$ **by** blast
from $P32 P83$ **have** $\text{Line-on-Seg } (Li B D) (Se C A)$ **by** blast
then have $P84 : \exists p. \text{Line-on } (Li B D) p \wedge \text{Bet-Point } (Se C A) p$ **by** $(\text{simp add:Line-on-Seg-rule})$
from $P21 P25 P84$ **have** $\exists p. \text{Bet-Point } (Se C A) p \wedge \text{Line-on } (Li B D) p \wedge \neg$

Line-on (Li B D) C \wedge \neg *Line-on (Li B D) A* **by** *blast*
then have *Plane-diffside (Li B D) C A* **by** (*simp add:Plane-diffside-def*)
thus *Plane-diffside (Li B D) A C* **by** (*simp add:Plane-diffside-rev*)
qed

lemma (in *Congruence-Rule*) *Ang-inside-Bet-Point* :

assumes

Bet-Point (Se p1 p3) p2
 \neg *Eq (Geos (Lin (Li p4 p1)) add Emp) (Geos (Lin (Li p4 p3)) add Emp)*
 \neg *Eq (Geos (Poi p4) add Emp) (Geos (Poi p1) add Emp)*
 \neg *Eq (Geos (Poi p4) add Emp) (Geos (Poi p3) add Emp)*

shows *Ang-inside (An p1 p4 p3) p2*

proof –

have *P1 : Line-on (Li p1 p3) p1* **by** (*simp add:Line-on-rule*)
have *P2 : Line-on (Li p1 p3) p3* **by** (*simp add:Line-on-rule*)
have *P3 : Line-on (Li p4 p1) p4* **by** (*simp add:Line-on-rule*)
have *P4 : Line-on (Li p4 p1) p1* **by** (*simp add:Line-on-rule*)
have *P5 : Line-on (Li p4 p3) p4* **by** (*simp add:Line-on-rule*)
have *P6 : Line-on (Li p4 p3) p3* **by** (*simp add:Line-on-rule*)
from *assms* **have** *P7 : \neg Eq (Geos (Poi p1) add Emp) (Geos (Poi p3) add Emp)*
by (*simp add:Bet-Point-def*)
from *P2* **have** *P8 : Eq (Geos (Lin (Li p1 p3)) add Emp) (Geos (Lin (Li p4 p1)) add Emp) \implies Line-on (Li p4 p1) p3* **by** (*simp add:Line-on-trans*)
from *assms P3 P5 P6 P8* **have** *P9 : Eq (Geos (Lin (Li p1 p3)) add Emp) (Geos (Lin (Li p4 p1)) add Emp) \implies*
Eq (Geos (Lin (Li p4 p1)) add Emp) (Geos (Lin (Li p4 p3)) add Emp) **by** (*simp add:Line-unique*)
from *assms P9* **have** *P10 : \neg Eq (Geos (Lin (Li p1 p3)) add Emp) (Geos (Lin (Li p4 p1)) add Emp)* **by** *blast*
from *P1* **have** *P11 : Eq (Geos (Lin (Li p1 p3)) add Emp) (Geos (Lin (Li p4 p3)) add Emp) \implies Line-on (Li p4 p3) p1* **by** (*simp add:Line-on-trans*)
from *assms P3 P4 P5 P11* **have** *P12 : Eq (Geos (Lin (Li p1 p3)) add Emp) (Geos (Lin (Li p4 p3)) add Emp) \implies*
Eq (Geos (Lin (Li p4 p1)) add Emp) (Geos (Lin (Li p4 p3)) add Emp) **by** (*simp add:Line-unique*)
from *assms P12* **have** *P13 : \neg Eq (Geos (Lin (Li p1 p3)) add Emp) (Geos (Lin (Li p4 p3)) add Emp)* **by** *blast*
from *assms P4 P10* **have** *Plane-sameside (Li p4 p1) p2 p3* **by** (*simp add:Plane-Bet-sameside*)
then have *P14 : Plane-sameside (Li p4 p1) p3 p2* **by** (*simp add:Plane-sameside-rev*)
from *assms* **have** *P15 : Bet-Point (Se p3 p1) p2* **by** (*simp add:Bet-rev*)
from *P7* **have** *P16 : Eq (Geos (Lin (Li p1 p3)) add Emp) (Geos (Lin (Li p3 p1)) add Emp)* **by** (*simp add:Line-rev*)
from *P13 P16* **have** *P17 : \neg Eq (Geos (Lin (Li p3 p1)) add Emp) (Geos (Lin (Li p4 p3)) add Emp)* **by** (*blast intro:Eq-rev Eq-trans*)
from *assms P6 P15 P17* **have** *Plane-sameside (Li p4 p3) p2 p1* **by** (*simp add:Plane-Bet-sameside*)
then have *P18 : Plane-sameside (Li p4 p3) p1 p2* **by** (*simp add:Plane-sameside-rev*)
from *assms* **have** *P19 : \neg Eq (Geos (Poi p1) add Emp) (Geos (Poi p4) add Emp)*
by (*blast intro:Eq-rev*)

from $P7$ **have** $P20 : \neg Eq (Geos (Poi\ p3) add\ Emp) (Geos (Poi\ p1) add\ Emp)$
by (*blast intro:Eq-rev*)
from *assms* $P19\ P20$ **have** $P21 : Def (Ang (An\ p1\ p4\ p3))$ **by** (*simp add:Ang-def*)
from $P14\ P18\ P21$ **show** $Ang\ inside (An\ p1\ p4\ p3)\ p2$ **by** (*simp add:Ang-inside-def*)
qed

lemma (in *Congruence-Rule*) *Ang-inside-HalfLine* :

assumes

$Ang\ inside (An\ A\ B\ C)\ D$
 $\neg Eq (Geos (Poi\ B) add\ Emp) (Geos (Poi\ E) add\ Emp)$
 $Line\ on (Li\ B\ D)\ E$
 $\neg Bet\ Point (Se\ E\ D)\ B$

shows

$Ang\ inside (An\ A\ B\ C)\ E$

proof –

from *assms* **have** $P1 : Def (Ang (An\ A\ B\ C)) \wedge Plane\ sameside (Li\ B\ A)\ C\ D$
 $\wedge Plane\ sameside (Li\ B\ C)\ A\ D$ **by** (*simp add:Ang-inside-def*)
have $Plane\ diffside (Li\ B\ A)\ C\ E \implies \exists p. Bet\ Point (Se\ C\ E)\ p \wedge Line\ on (Li\ B\ A)\ p$
 $\wedge \neg Line\ on (Li\ B\ A)\ C \wedge \neg Line\ on (Li\ B\ A)\ E$ **by** (*simp add:Plane-diffside-def*)
then obtain $p1 :: Point$ **where** $Plane\ diffside (Li\ B\ A)\ C\ E \implies Bet\ Point (Se\ C\ E)\ p1$ **by** *blast*
from *assms* $P1$ **have** $Plane\ diffside (Li\ B\ A)\ C\ E \implies Plane\ diffside (Li\ B\ A)\ D\ E$ **by** (*blast intro:Plane-trans*)
then have $Plane\ diffside (Li\ B\ A)\ C\ E \implies \exists p. Bet\ Point (Se\ D\ E)\ p \wedge Line\ on (Li\ B\ A)\ p$
 $\wedge \neg Line\ on (Li\ B\ A)\ D \wedge \neg Line\ on (Li\ B\ A)\ E$ **by** (*simp add:Plane-diffside-def*)
then obtain $F :: Point$ **where** $P2 : Plane\ diffside (Li\ B\ A)\ C\ E \implies Bet\ Point (Se\ D\ E)\ F \wedge Line\ on (Li\ B\ A)\ F$ **by** *blast*
then have $Plane\ diffside (Li\ B\ A)\ C\ E \implies Bet\ Point (Se\ E\ D)\ F$ **by** (*simp add:Bet-rev*)
then have $P3 : Plane\ diffside (Li\ B\ A)\ C\ E \implies Eq (Geos (Poi\ F) add\ Emp) (Geos (Poi\ B) add\ Emp)$
 $\implies Bet\ Point (Se\ E\ D)\ B$ **by** (*simp add:Point-Eq*)
from *assms* $P3$ **have** $P4 : Plane\ diffside (Li\ B\ A)\ C\ E \implies \neg Eq (Geos (Poi\ F) add\ Emp) (Geos (Poi\ B) add\ Emp)$ **by** *blast*
have $P5 : Line\ on (Li\ B\ D)\ D$ **by** (*simp add:Line-on-rule*)
have $P6 : Line\ on (Li\ E\ D)\ E$ **by** (*simp add:Line-on-rule*)
have $P7 : Line\ on (Li\ E\ D)\ D$ **by** (*simp add:Line-on-rule*)
from *assms* $P5\ P6\ P7$ **have** $P8 : \neg Eq (Geos (Poi\ D) add\ Emp) (Geos (Poi\ E) add\ Emp) \implies$
 $Eq (Geos (Lin (Li\ E\ D)) add\ Emp) (Geos (Lin (Li\ B\ D)) add\ Emp)$ **by** (*simp add:Line-unique*)
from $P2$ **have** $P9 : Plane\ diffside (Li\ B\ A)\ C\ E \implies Line\ on (Li\ E\ D)\ F$ **by** (*simp add:Line-Bet-on*)
from $P8\ P9$ **have** $P10 : Plane\ diffside (Li\ B\ A)\ C\ E \implies$
 $\neg Eq (Geos (Poi\ D) add\ Emp) (Geos (Poi\ E) add\ Emp) \implies Line\ on (Li\ B\ D)\ F$ **by** (*simp add:Line-on-trans*)
have $P11 : Line\ on (Li\ B\ A)\ B$ **by** (*simp add:Line-on-rule*)

have $P12 : \text{Line-on } (Li\ B\ D)\ B$ **by** (*simp add:Line-on-rule*)
from $P2\ P4\ P10\ P11\ P12$ **have** $P13 : \text{Plane-diffside } (Li\ B\ A)\ C\ E \implies$
 $\neg \text{Eq } (\text{Geos } (Poi\ D)\ \text{add Emp})\ (\text{Geos } (Poi\ E)\ \text{add Emp}) \implies \text{Eq } (\text{Geos } (\text{Lin } (Li\ B\ D))\ \text{add Emp})\ (\text{Geos } (\text{Lin } (Li\ B\ A))\ \text{add Emp})$ **by** (*blast intro:Line-unique*)
from $P5\ P13$ **have** $P14 : \text{Plane-diffside } (Li\ B\ A)\ C\ E \implies$
 $\neg \text{Eq } (\text{Geos } (Poi\ D)\ \text{add Emp})\ (\text{Geos } (Poi\ E)\ \text{add Emp}) \implies \text{Line-on } (Li\ B\ A)$
 D **by** (*simp add:Line-on-trans*)
from $P1$ **have** $P15 : \neg \text{Line-on } (Li\ B\ A)\ D$ **by** (*simp add:Plane-sameside-def*)
from $P14\ P15$ **have** $P16 : \neg \text{Eq } (\text{Geos } (Poi\ D)\ \text{add Emp})\ (\text{Geos } (Poi\ E)\ \text{add Emp}) \implies \neg \text{Plane-diffside } (Li\ B\ A)\ C\ E$ **by** *blast*
from *assms* $P1$ **have** $\text{Plane-diffside } (Li\ B\ C)\ A\ E \implies \text{Plane-diffside } (Li\ B\ C)\ D\ E$ **by** (*blast intro:Plane-trans*)
then **have** $\text{Plane-diffside } (Li\ B\ C)\ A\ E \implies \exists p. \text{Bet-Point } (Se\ D\ E)\ p \wedge \text{Line-on } (Li\ B\ C)\ p$
 $\wedge \neg \text{Line-on } (Li\ B\ C)\ D \wedge \neg \text{Line-on } (Li\ B\ C)\ E$ **by** (*simp add:Plane-diffside-def*)
then **obtain** $G :: \text{Point}$ **where** $P17 : \text{Plane-diffside } (Li\ B\ C)\ A\ E \implies \text{Bet-Point } (Se\ D\ E)\ G \wedge \text{Line-on } (Li\ B\ C)\ G$ **by** *blast*
then **have** $\text{Plane-diffside } (Li\ B\ C)\ A\ E \implies \text{Bet-Point } (Se\ E\ D)\ G$ **by** (*simp add:Bet-rev*)
then **have** $P18 : \text{Plane-diffside } (Li\ B\ C)\ A\ E \implies \text{Eq } (\text{Geos } (Poi\ G)\ \text{add Emp})\ (\text{Geos } (Poi\ B)\ \text{add Emp})$
 $\implies \text{Bet-Point } (Se\ E\ D)\ B$ **by** (*simp add:Point-Eq*)
from *assms* $P18$ **have** $P19 : \text{Plane-diffside } (Li\ B\ C)\ A\ E \implies \neg \text{Eq } (\text{Geos } (Poi\ G)\ \text{add Emp})\ (\text{Geos } (Poi\ B)\ \text{add Emp})$ **by** *blast*
from $P17$ **have** $P20 : \text{Plane-diffside } (Li\ B\ C)\ A\ E \implies \text{Line-on } (Li\ E\ D)\ G$ **by** (*simp add:Line-Bet-on*)
from $P8\ P20$ **have** $P21 : \text{Plane-diffside } (Li\ B\ C)\ A\ E \implies$
 $\neg \text{Eq } (\text{Geos } (Poi\ D)\ \text{add Emp})\ (\text{Geos } (Poi\ E)\ \text{add Emp}) \implies \text{Line-on } (Li\ B\ D)$
 G **by** (*simp add:Line-on-trans*)
have $P22 : \text{Line-on } (Li\ B\ C)\ B$ **by** (*simp add:Line-on-rule*)
from $P12\ P17\ P19\ P21\ P22$ **have** $P23 : \text{Plane-diffside } (Li\ B\ C)\ A\ E \implies$
 $\neg \text{Eq } (\text{Geos } (Poi\ D)\ \text{add Emp})\ (\text{Geos } (Poi\ E)\ \text{add Emp})$
 $\implies \text{Eq } (\text{Geos } (\text{Lin } (Li\ B\ D))\ \text{add Emp})\ (\text{Geos } (\text{Lin } (Li\ B\ C))\ \text{add Emp})$ **by**
(*blast intro:Line-unique*)
from $P5\ P23$ **have** $P24 : \text{Plane-diffside } (Li\ B\ C)\ A\ E \implies$
 $\neg \text{Eq } (\text{Geos } (Poi\ D)\ \text{add Emp})\ (\text{Geos } (Poi\ E)\ \text{add Emp}) \implies \text{Line-on } (Li\ B\ C)$
 D **by** (*simp add:Line-on-trans*)
from $P1$ **have** $P25 : \neg \text{Line-on } (Li\ B\ C)\ D$ **by** (*simp add:Plane-sameside-def*)
from $P24\ P25$ **have** $P26 : \neg \text{Eq } (\text{Geos } (Poi\ D)\ \text{add Emp})\ (\text{Geos } (Poi\ E)\ \text{add Emp}) \implies \neg \text{Plane-diffside } (Li\ B\ C)\ A\ E$ **by** *blast*
from $P1$ **have** $P27 : \neg \text{Line-on } (Li\ B\ A)\ C$ **by** (*simp add:Plane-sameside-def*)
from $P8\ P12$ **have** $P28 : \neg \text{Eq } (\text{Geos } (Poi\ D)\ \text{add Emp})\ (\text{Geos } (Poi\ E)\ \text{add Emp}) \implies \text{Line-on } (Li\ E\ D)\ B$ **by** (*blast intro:Line-on-trans Eq-rev*)
from *assms* $P6\ P11\ P28$ **have** $P29 : \neg \text{Eq } (\text{Geos } (Poi\ D)\ \text{add Emp})\ (\text{Geos } (Poi\ E)\ \text{add Emp}) \implies \text{Line-on } (Li\ B\ A)\ E \implies$
 $\text{Eq } (\text{Geos } (\text{Lin } (Li\ E\ D))\ \text{add Emp})\ (\text{Geos } (\text{Lin } (Li\ B\ A))\ \text{add Emp})$ **by** (*simp add:Line-unique*)
from $P7\ P29$ **have** $P30 : \neg \text{Eq } (\text{Geos } (Poi\ D)\ \text{add Emp})\ (\text{Geos } (Poi\ E)\ \text{add Emp}) \implies$

$Line-on (Li B A) E \implies Line-on (Li B A) D$ **by** (*simp add:Line-on-trans*)
from $P15 P30$ **have** $P31 : \neg Eq (Geos (Poi D) add Emp) (Geos (Poi E) add Emp) \implies$
 $\neg Line-on (Li B A) E$ **by** *blast*
from $P1$ **have** $P32 : \neg Eq (Geos (Poi B) add Emp) (Geos (Poi C) add Emp)$
by (*simp add:Ang-def*)
have $P33 : Line-on (Li B C) C$ **by** (*simp add:Line-on-rule*)
from $P12 P22 P32 P33$ **have** $P34 : Line-on (Li B D) C \implies$
 $Eq (Geos (Lin (Li B D)) add Emp) (Geos (Lin (Li B C)) add Emp)$ **by** (*simp add:Line-unique*)
from $P5 P34$ **have** $P35 : Line-on (Li B D) C \implies Line-on (Li B C) D$ **by** (*simp add:Line-on-trans*)
from $P25 P35$ **have** $P36 : \neg Line-on (Li B D) C$ **by** *blast*
from *assms* **have** $P37 : Eq (Geos (Poi E) add Emp) (Geos (Poi C) add Emp)$
 $\implies Line-on (Li B D) C$ **by** (*simp add:Point-Eq*)
from $P36 P37$ **have** $P38 : \neg Eq (Geos (Poi C) add Emp) (Geos (Poi E) add Emp)$ **by** (*blast intro:Eq-rev*)
from $P16 P27 P31 P38$ **have** $P39 : \neg Eq (Geos (Poi D) add Emp) (Geos (Poi E) add Emp) \implies$
 $Plane-sameside (Li B A) C E$ **by** (*simp add:Plane-not-diffside-sameside*)
from $P1$ **have** $P40 : \neg Line-on (Li B C) A$ **by** (*simp add:Plane-sameside-def*)
from *assms* $P6 P22 P28$ **have** $P41 : \neg Eq (Geos (Poi D) add Emp) (Geos (Poi E) add Emp) \implies$
 $Line-on (Li B C) E \implies$
 $Eq (Geos (Lin (Li E D)) add Emp) (Geos (Lin (Li B C)) add Emp)$ **by** (*simp add:Line-unique*)
from $P7 P41$ **have** $P42 : \neg Eq (Geos (Poi D) add Emp) (Geos (Poi E) add Emp) \implies$
 $Line-on (Li B C) E \implies Line-on (Li B C) D$ **by** (*simp add:Line-on-trans*)
from $P25 P42$ **have** $P43 : \neg Eq (Geos (Poi D) add Emp) (Geos (Poi E) add Emp) \implies$
 $\neg Line-on (Li B C) E$ **by** *blast*
from $P39$ **have** $P44 : \neg Eq (Geos (Poi D) add Emp) (Geos (Poi E) add Emp) \implies$
 $\neg Line-on (Li B A) E$ **by** (*simp add:Plane-sameside-def*)
have $Line-on (Li B A) A$ **by** (*simp add:Line-on-rule*)
then **have** $P45 : \neg Eq (Geos (Poi D) add Emp) (Geos (Poi E) add Emp) \implies$
 $Eq (Geos (Poi A) add Emp) (Geos (Poi E) add Emp) \implies Line-on (Li B A) E$
by (*simp add:Point-Eq*)
from $P44 P45$ **have** $P46 : \neg Eq (Geos (Poi D) add Emp) (Geos (Poi E) add Emp) \implies$
 $\neg Eq (Geos (Poi A) add Emp) (Geos (Poi E) add Emp)$ **by** *blast*
from $P26 P40 P43 P46$ **have** $P47 : \neg Eq (Geos (Poi D) add Emp) (Geos (Poi E) add Emp) \implies$
 $Plane-sameside (Li B C) A E$ **by** (*simp add:Plane-not-diffside-sameside*)
from $P1 P39 P47$ **have** $P48 : \neg Eq (Geos (Poi D) add Emp) (Geos (Poi E) add Emp) \implies$
 $Ang-inside (An A B C) E$ **by** (*simp add:Ang-inside-def*)
from *assms* **have** $P49 : Eq (Geos (Poi D) add Emp) (Geos (Poi E) add Emp) \implies$
 $Ang-inside (An A B C) E$ **by** (*simp add:Point-Eq*)
from $P48 P49$ **show** $Ang-inside (An A B C) E$ **by** *blast*
qed

lemma (in Congruence-Rule) Ang-outside-Planeside :

assumes

Def (Ang (An A B C))

\neg *Ang-inside (An A B C) D*

shows \neg (*Plane-sameside (Li B A) C D* \wedge *Plane-sameside (Li B C) A D*)

and \neg *Plane-sameside (Li B A) C D* \wedge *Plane-sameside (Li B C) A D*

\vee *Plane-sameside (Li B A) C D* \wedge \neg *Plane-sameside (Li B C) A D*

\vee \neg *Plane-sameside (Li B A) C D* \wedge \neg *Plane-sameside (Li B C) A D*

proof –

from *assms* **have** *P1* : *Plane-sameside (Li B A) C D* \wedge *Plane-sameside (Li B C) A D* \implies *Ang-inside (An A B C) D* **by** (*simp add:Ang-inside-def*)

from *assms* *P1* **show** \neg (*Plane-sameside (Li B A) C D* \wedge *Plane-sameside (Li B C) A D*) **by** *blast*

thus \neg *Plane-sameside (Li B A) C D* \wedge *Plane-sameside (Li B C) A D*

\vee *Plane-sameside (Li B A) C D* \wedge \neg *Plane-sameside (Li B C) A D*

\vee \neg *Plane-sameside (Li B A) C D* \wedge \neg *Plane-sameside (Li B C) A D* **by** *blast*

qed

lemma (in Congruence-Rule) Ang-outside-exclusive :

assumes

Plane-sameside (Li B C) A D

\neg *Eq (Geos (Poi B) add Emp) (Geos (Poi C) add Emp)*

\neg *Eq (Geos (Lin (Li B A)) add Emp) (Geos (Lin (Li B D)) add Emp)*

shows

\neg (\neg *Ang-inside (An A B C) D* \wedge \neg *Ang-inside (An D B C) A*)

proof –

from *assms* **have** *P1* : \neg *Line-on (Li B C) A* **by** (*simp add:Plane-sameside-def*)

from *assms* *P1* **have** *Def (Ang (An B C A))* **by** (*simp add:Ang-simple-def*)

then **have** *P2* : *Def (Ang (An A B C))* **by** (*blast intro:Ang-def-rev Ang-def-inv*)

then **have** *P3* : \neg *Ang-inside (An A B C) D* \implies \neg *Plane-sameside (Li B A) C D* \wedge *Plane-sameside (Li B C) A D*

\vee *Plane-sameside (Li B A) C D* \wedge \neg *Plane-sameside (Li B C) A D*

\vee \neg *Plane-sameside (Li B A) C D* \wedge \neg *Plane-sameside (Li B C) A D* **by**

(*simp add:Ang-outside-Planeside*)

from *assms* **have** *P4* : \neg *Line-on (Li B C) D* **by** (*simp add:Plane-sameside-def*)

from *assms* *P4* **have** *Def (Ang (An B C D))* **by** (*simp add:Ang-simple-def*)

then **have** *P5* : *Def (Ang (An D B C))* **by** (*blast intro:Ang-def-rev Ang-def-inv*)

then **have** *P6* : \neg *Ang-inside (An D B C) A* \implies \neg *Plane-sameside (Li B D) C A* \wedge *Plane-sameside (Li B C) D A*

\vee *Plane-sameside (Li B D) C A* \wedge \neg *Plane-sameside (Li B C) D A*

\vee \neg *Plane-sameside (Li B D) C A* \wedge \neg *Plane-sameside (Li B C) D A* **by**

(*simp add:Ang-outside-Planeside*)

from *P3* *P6* **have** *P7* : \neg *Ang-inside (An A B C) D* \wedge \neg *Ang-inside (An D B C) A* \implies

\neg *Plane-sameside (Li B A) C D* \wedge *Plane-sameside (Li B C) A D* \wedge \neg

Plane-sameside (Li B D) C A \wedge *Plane-sameside (Li B C) D A*

\vee \neg *Plane-sameside (Li B A) C D* \wedge *Plane-sameside (Li B C) A D* \wedge

Plane-sameside (Li B D) C A \wedge \neg *Plane-sameside (Li B C) D A*

\vee \neg *Plane-sameside (Li B A) C D* \wedge *Plane-sameside (Li B C) A D* \wedge \neg

Plane-sameside (Li B D) C A \wedge \neg *Plane-sameside (Li B C) D A*
 \vee *Plane-sameside (Li B A) C D* \wedge \neg *Plane-sameside (Li B C) A D* \wedge \neg
Plane-sameside (Li B D) C A \wedge *Plane-sameside (Li B C) D A*
 \vee *Plane-sameside (Li B A) C D* \wedge \neg *Plane-sameside (Li B C) A D* \wedge
Plane-sameside (Li B D) C A \wedge \neg *Plane-sameside (Li B C) D A*
 \vee *Plane-sameside (Li B A) C D* \wedge \neg *Plane-sameside (Li B C) A D* \wedge \neg
Plane-sameside (Li B D) C A \wedge \neg *Plane-sameside (Li B C) D A*
 \vee \neg *Plane-sameside (Li B A) C D* \wedge \neg *Plane-sameside (Li B C) A D* \wedge \neg
Plane-sameside (Li B D) C A \wedge *Plane-sameside (Li B C) D A*
 \vee \neg *Plane-sameside (Li B A) C D* \wedge \neg *Plane-sameside (Li B C) A D* \wedge
Plane-sameside (Li B D) C A \wedge \neg *Plane-sameside (Li B C) D A*
 \vee \neg *Plane-sameside (Li B A) C D* \wedge \neg *Plane-sameside (Li B C) A D* \wedge \neg
Plane-sameside (Li B D) C A \wedge \neg *Plane-sameside (Li B C) D A* **by blast**
from *assms* **have** *P8* : *Plane-sameside (Li B A) C D* \wedge \neg *Plane-sameside (Li B C) A D* \wedge \neg *Plane-sameside (Li B D) C A* \wedge *Plane-sameside (Li B C) D A*
 \vee *Plane-sameside (Li B A) C D* \wedge \neg *Plane-sameside (Li B C) A D* \wedge
Plane-sameside (Li B D) C A \wedge \neg *Plane-sameside (Li B C) D A*
 \vee *Plane-sameside (Li B A) C D* \wedge \neg *Plane-sameside (Li B C) A D* \wedge \neg
Plane-sameside (Li B D) C A \wedge \neg *Plane-sameside (Li B C) D A*
 \vee \neg *Plane-sameside (Li B A) C D* \wedge \neg *Plane-sameside (Li B C) A D* \wedge \neg
Plane-sameside (Li B D) C A \wedge *Plane-sameside (Li B C) D A*
 \vee \neg *Plane-sameside (Li B A) C D* \wedge \neg *Plane-sameside (Li B C) A D* \wedge
Plane-sameside (Li B D) C A \wedge \neg *Plane-sameside (Li B C) D A*
 \vee \neg *Plane-sameside (Li B A) C D* \wedge \neg *Plane-sameside (Li B C) A D* \wedge \neg
Plane-sameside (Li B D) C A \wedge \neg *Plane-sameside (Li B C) D A* \implies **False** **by**
blast
from *assms* **have** *Plane-sameside (Li B C) D A* **by** (*simp add:Plane-sameside-rev*)
then **have** *P9* : \neg *Plane-sameside (Li B A) C D* \wedge *Plane-sameside (Li B C) A D* \wedge *Plane-sameside (Li B D) C A* \wedge \neg *Plane-sameside (Li B C) D A*
 \vee \neg *Plane-sameside (Li B A) C D* \wedge *Plane-sameside (Li B C) A D* \wedge \neg
Plane-sameside (Li B D) C A \wedge \neg *Plane-sameside (Li B C) D A* \implies **False** **by**
blast
have *P10* : *Line-on (Li C B) C* **by** (*simp add:Line-on-rule*)
have *P11* : *Line-on (Li C B) B* **by** (*simp add:Line-on-rule*)
from *assms* *P10 P11* **have** $\exists p.$ *Bet-Point (Se C p) B* \wedge *Line-on (Li C B) p* **by**
(*blast intro:Bet-extension Eq-rev*)
then **obtain** *E* :: *Point* **where** *P12* : *Bet-Point (Se C E) B* \wedge *Line-on (Li C B) E* **by** *blast*
then **have** *P13* : *Line-on (Li E C) B* **by** (*simp add:Line-Bet-on*)
have *P14* : *Line-on (Li B A) B* **by** (*simp add:Line-on-rule*)
have *P15* : *Line-on (Li B A) A* **by** (*simp add:Line-on-rule*)
have *P16* : *Line-on (Li B C) B* **by** (*simp add:Line-on-rule*)
have *P17* : *Line-on (Li B C) C* **by** (*simp add:Line-on-rule*)
from *P16* **have** *P18* : *Eq (Geos (Poi B) add Emp) (Geos (Poi A) add Emp)* \implies
Line-on (Li B C) A **by** (*simp add:Point-Eq*)
from *P1 P18* **have** *P19* : \neg *Eq (Geos (Poi B) add Emp) (Geos (Poi A) add Emp)* **by** *blast*
from *P13 P14 P15 P19* **have** *P20* : *Line-on (Li E C) A* \implies *Eq (Geos (Lin (Li B A)) add Emp) (Geos (Lin (Li E C)) add Emp)* **by** (*simp add:Line-unique*)

have $P21 : \text{Line-on } (Li\ E\ C)\ C$ **by** (*simp add:Line-on-rule*)
from *assms P13 P16 P17 P21* **have** $P22 : \text{Eq } (\text{Geos } (\text{Lin } (Li\ E\ C)))\ \text{add Emp}$
(*Geos (Lin (Li B C)) add Emp*) **by** (*simp add:Line-unique*)
from $P20\ P22$ **have** $P23 : \text{Line-on } (Li\ E\ C)\ A \implies \text{Eq } (\text{Geos } (\text{Lin } (Li\ B\ A)))$
add Emp (*Geos (Lin (Li B C)) add Emp*) **by** (*blast intro:Eq-trans*)
from $P15\ P23$ **have** $P24 : \text{Line-on } (Li\ E\ C)\ A \implies \text{Line-on } (Li\ B\ C)\ A$ **by**
(*simp add:Line-on-trans*)
from $P1\ P24$ **have** $P25 : \neg \text{Line-on } (Li\ E\ C)\ A$ **by** *blast*
have $P26 : \text{Line-on } (Li\ E\ C)\ E$ **by** (*simp add:Line-on-rule*)
from $P21$ **have** $P27 : \text{Eq } (\text{Geos } (\text{Poi } C))\ \text{add Emp} \implies (\text{Geos } (\text{Poi } A))\ \text{add Emp} \implies$
 $\text{Line-on } (Li\ E\ C)\ A$ **by** (*simp add:Point-Eq*)
from $P25\ P27$ **have** $P28 : \neg \text{Eq } (\text{Geos } (\text{Poi } C))\ \text{add Emp} \implies (\text{Geos } (\text{Poi } A))\ \text{add Emp}$
by *blast*
from $P12$ **have** $P29 : \text{Bet-Point } (Se\ C\ E)\ B$ **by** *simp*
then **have** $P30 : \neg \text{Eq } (\text{Geos } (\text{Poi } C))\ \text{add Emp} \implies (\text{Geos } (\text{Poi } E))\ \text{add Emp}$ **by**
(*simp add:Bet-Point-def Eq-rev*)
have $\text{Plane-diffside } (Li\ B\ D)\ C\ A \implies \exists p. \text{Bet-Point } (Se\ C\ A)\ p \wedge \text{Line-on}$
 $(Li\ B\ D)\ p \wedge \neg \text{Line-on } (Li\ B\ D)\ C \wedge \neg \text{Line-on } (Li\ B\ D)\ A$ **by** (*simp*
add:Plane-diffside-def)
then **have** $P31 : \text{Plane-diffside } (Li\ B\ D)\ C\ A \implies \neg \text{Line-on } (Li\ B\ D)\ C \wedge \neg$
 $\text{Line-on } (Li\ B\ D)\ A$ **by** *blast*
have $P32 : \text{Line-on } (Li\ B\ D)\ B$ **by** (*simp add:Line-on-rule*)
from $P29$ **have** $P33 : \neg \text{Eq } (\text{Geos } (\text{Poi } E))\ \text{add Emp} \implies (\text{Geos } (\text{Poi } B))\ \text{add Emp}$
by (*simp add:Bet-Point-def*)
from $P13\ P26\ P32\ P33$ **have** $P34 : \text{Line-on } (Li\ B\ D)\ E \implies \text{Eq } (\text{Geos } (\text{Lin } (Li\ E\ C)))$
add Emp (*Geos (Lin (Li B D)) add Emp*) **by** (*simp add:Line-unique*)
from $P21\ P34$ **have** $P35 : \text{Line-on } (Li\ B\ D)\ E \implies \text{Line-on } (Li\ B\ D)\ C$ **by**
(*simp add:Line-on-trans*)
from $P31\ P35$ **have** $P36 : \text{Plane-diffside } (Li\ B\ D)\ C\ A \implies \neg \text{Line-on } (Li\ B\ D)$
 E **by** *blast*
from $P29$ **have** $P37 : \text{Bet-Point } (Se\ E\ C)\ B$ **by** (*simp add:Bet-rev*)
from $P25\ P31\ P32\ P36\ P37$ **have** $P38 : \text{Plane-diffside } (Li\ B\ D)\ C\ A \implies$
 $\text{Line-on-Seg } (Li\ B\ D)\ (Se\ E\ A) \wedge \neg \text{Line-on-Seg } (Li\ B\ D)\ (Se\ C\ A)$
 $\vee \text{Line-on-Seg } (Li\ B\ D)\ (Se\ C\ A) \wedge \neg \text{Line-on-Seg } (Li\ B\ D)\ (Se\ E\ A)$ **by** (*simp*
add:Pachets-axiom)
have $P39 : \text{Line-on-Seg } (Li\ B\ D)\ (Se\ E\ A) \implies \exists p. \text{Line-on } (Li\ B\ D)\ p \wedge$
 $\text{Bet-Point } (Se\ E\ A)\ p$ **by** (*simp add:Line-on-Seg-rule*)
from $P31\ P36\ P39$ **have** $\text{Plane-diffside } (Li\ B\ D)\ C\ A \wedge \text{Line-on-Seg } (Li\ B\ D)$
 $(Se\ E\ A) \implies$
 $\exists p. \text{Bet-Point } (Se\ E\ A)\ p \wedge \text{Line-on } (Li\ B\ D)\ p \wedge \neg \text{Line-on } (Li\ B\ D)\ E \wedge \neg$
 $\text{Line-on } (Li\ B\ D)\ A$ **by** *blast*
then **have** $P40 : \text{Plane-diffside } (Li\ B\ D)\ C\ A \wedge \text{Line-on-Seg } (Li\ B\ D)\ (Se\ E\ A)$
 $\implies \text{Plane-diffside } (Li\ B\ D)\ E\ A$ **by** (*simp add:Plane-diffside-def*)
from $P26$ **have** $P41 : \text{Eq } (\text{Geos } (\text{Lin } (Li\ E\ C)))\ \text{add Emp} \implies (\text{Geos } (\text{Lin } (Li\ B\ D)))$
add Emp $\implies \text{Line-on } (Li\ B\ D)\ E$ **by** (*simp add:Line-on-trans*)
from $P36\ P41$ **have** $P42 : \text{Plane-diffside } (Li\ B\ D)\ C\ A \implies \neg \text{Eq } (\text{Geos } (\text{Lin}$
 $(Li\ E\ C)))\ \text{add Emp} \implies (\text{Geos } (\text{Lin } (Li\ B\ D)))\ \text{add Emp}$ **by** *blast*
from $P32\ P37\ P42$ **have** $P43 : \text{Plane-diffside } (Li\ B\ D)\ C\ A \implies \text{Plane-diffside}$
 $(Li\ B\ D)\ E\ C$ **by** (*simp add:Plane-Bet-diffside*)

from P_{28} **have** $P_{44} : \neg \text{Eq} (\text{Geos} (\text{Poi } A) \text{ add Emp}) (\text{Geos} (\text{Poi } C) \text{ add Emp})$
by (*blast intro:Eq-rev*)
from $P_{30} P_{40} P_{43} P_{44}$ **have** $\text{Plane-diffside} (Li\ B\ D)\ C\ A \wedge \text{Line-on-Seg} (Li\ B\ D)\ (Se\ E\ A) \implies \text{Plane-sameside} (Li\ B\ D)\ A\ C$ **by** (*blast intro:Plane-trans-inv*)
then have $P_{45} : \text{Plane-diffside} (Li\ B\ D)\ C\ A \wedge \text{Line-on-Seg} (Li\ B\ D)\ (Se\ E\ A) \implies \text{Plane-sameside} (Li\ B\ D)\ C\ A$ **by** (*blast intro:Plane-sameside-rev*)
have $P_{46} : \text{Plane-diffside} (Li\ B\ D)\ C\ A \implies \neg \text{Plane-sameside} (Li\ B\ D)\ C\ A$
by (*simp add:Plane-diffside-not-sameside*)
from $P_{45} P_{46}$ **have** $P_{47} : \text{Plane-diffside} (Li\ B\ D)\ C\ A \implies \neg \text{Line-on-Seg} (Li\ B\ D)\ (Se\ E\ A)$ **by** *blast*
from $P_{38} P_{47}$ **have** $\text{Plane-diffside} (Li\ B\ D)\ C\ A \implies \text{Line-on-Seg} (Li\ B\ D)\ (Se\ C\ A)$ **by** *blast*
then have $\text{Plane-diffside} (Li\ B\ D)\ C\ A \implies \exists p. \text{Line-on} (Li\ B\ D)\ p \wedge \text{Bet-Point} (Se\ C\ A)\ p$ **by** (*simp add:Line-on-Seg-rule*)
then obtain $F :: \text{Point}$ **where** $P_{48} : \text{Plane-diffside} (Li\ B\ D)\ C\ A \implies \text{Line-on} (Li\ B\ D)\ F \wedge \text{Bet-Point} (Se\ C\ A)\ F$ **by** *blast*
from P_{15} **have** $P_{49} : \text{Eq} (\text{Geos} (\text{Lin} (Li\ B\ A)) \text{ add Emp}) (\text{Geos} (\text{Lin} (Li\ B\ C)) \text{ add Emp}) \implies \text{Line-on} (Li\ B\ C)\ A$ **by** (*simp add:Line-on-trans*)
from $P_1 P_{49}$ **have** $P_{50} : \neg \text{Eq} (\text{Geos} (\text{Lin} (Li\ B\ A)) \text{ add Emp}) (\text{Geos} (\text{Lin} (Li\ B\ C)) \text{ add Emp})$ **by** *blast*
from P_{48} **have** $P_{51} : \text{Plane-diffside} (Li\ B\ D)\ C\ A \implies \text{Bet-Point} (Se\ A\ C)\ F$ **by** (*simp add:Bet-rev*)
from *assms* $P_{19} P_{50} P_{51}$ **have** $P_{52} : \text{Plane-diffside} (Li\ B\ D)\ C\ A \implies \text{Ang-inside} (An\ A\ B\ C)\ F$ **by** (*simp add:Ang-inside-Bet-Point*)
then have $\text{Plane-diffside} (Li\ B\ D)\ C\ A \implies \text{Eq} (\text{Geos} (\text{Poi } F) \text{ add Emp}) (\text{Geos} (\text{Poi } D) \text{ add Emp}) \implies \text{Ang-inside} (An\ A\ B\ C)\ D$ **by** (*simp add:Point-Eq*)
then have $P_{53} : \neg \text{Ang-inside} (An\ A\ B\ C)\ D \implies \text{Plane-diffside} (Li\ B\ D)\ C\ A \implies \neg \text{Eq} (\text{Geos} (\text{Poi } D) \text{ add Emp}) (\text{Geos} (\text{Poi } F) \text{ add Emp})$ **by** (*blast intro:Eq-rev*)
from P_{48} **have** $\text{Plane-diffside} (Li\ B\ D)\ C\ A \implies \text{Bet-Point} (Se\ C\ A)\ F$ **by** *simp*
then have $\text{Plane-diffside} (Li\ B\ D)\ C\ A \implies \text{Eq} (\text{Geos} (\text{Poi } F) \text{ add Emp}) (\text{Geos} (\text{Poi } B) \text{ add Emp}) \implies \text{Bet-Point} (Se\ C\ A)\ B$ **by** (*simp add:Point-Eq*)
then have $P_{54} : \text{Plane-diffside} (Li\ B\ D)\ C\ A \implies \text{Eq} (\text{Geos} (\text{Poi } F) \text{ add Emp}) (\text{Geos} (\text{Poi } B) \text{ add Emp}) \implies \text{Line-on} (Li\ B\ C)\ A$ **by** (*simp add:Line-Bet-on*)
from $P_1 P_{54}$ **have** $P_{55} : \text{Plane-diffside} (Li\ B\ D)\ C\ A \implies \neg \text{Eq} (\text{Geos} (\text{Poi } F) \text{ add Emp}) (\text{Geos} (\text{Poi } B) \text{ add Emp})$ **by** *blast*
have $P_{56} : \text{Line-on} (Li\ B\ D)\ D$ **by** (*simp add:Line-on-rule*)
from P_{16} **have** $P_{57} : \text{Eq} (\text{Geos} (\text{Poi } B) \text{ add Emp}) (\text{Geos} (\text{Poi } D) \text{ add Emp}) \implies \text{Line-on} (Li\ B\ C)\ D$ **by** (*simp add:Point-Eq*)
from $P_4 P_{57}$ **have** $P_{58} : \neg \text{Eq} (\text{Geos} (\text{Poi } B) \text{ add Emp}) (\text{Geos} (\text{Poi } D) \text{ add Emp})$ **by** *blast*
from $P_{32} P_{48} P_{53} P_{55} P_{56} P_{58}$ **have** $\neg \text{Ang-inside} (An\ A\ B\ C)\ D \implies \text{Plane-diffside} (Li\ B\ D)\ C\ A \implies$
 $\text{Bet-Point} (Se\ B\ F)\ D \vee \text{Bet-Point} (Se\ F\ D)\ B \vee \text{Bet-Point} (Se\ D\ B)\ F$ **by** (*simp add:Bet-case*)
then have $P_{59} : \neg \text{Ang-inside} (An\ A\ B\ C)\ D \implies \text{Plane-diffside} (Li\ B\ D)\ C\ A \implies$
 $\text{Bet-Point} (Se\ B\ F)\ D \wedge \neg \text{Bet-Point} (Se\ F\ D)\ B \wedge \neg \text{Bet-Point} (Se\ D\ B)\ F$
 $\vee \neg \text{Bet-Point} (Se\ B\ F)\ D \wedge \text{Bet-Point} (Se\ F\ D)\ B \wedge \neg \text{Bet-Point} (Se\ D\ B)\ F$
 $\vee \neg \text{Bet-Point} (Se\ B\ F)\ D \wedge \neg \text{Bet-Point} (Se\ F\ D)\ B \wedge \text{Bet-Point} (Se\ D\ B)$

F **by** (*simp add:Bet-case-fact*)
from $P26$ **have** $P60 : Eq (Geos (Lin (Li E C)) add Emp) (Geos (Lin (Li B D)) add Emp) \implies Line-on (Li B D) E$ **by** (*simp add:Line-on-trans*)
from $P36 P60$ **have** $P61 : Plane-diffside (Li B D) C A \implies \neg Eq (Geos (Lin (Li B D)) add Emp) (Geos (Lin (Li E C)) add Emp)$ **by** (*blast intro:Eq-rev*)
have $P62 : Bet-Point (Se F D) B \implies Line-on (Li F D) B$ **by** (*simp add:Line-Bet-on*)
have $P63 : Line-on (Li F D) D$ **by** (*simp add:Line-on-rule*)
from $P32 P56 P58 P62 P63$ **have** $P64 : Bet-Point (Se F D) B \implies Eq (Geos (Lin (Li F D)) add Emp) (Geos (Lin (Li B D)) add Emp)$ **by** (*simp add:Line-unique*)
from $P61 P64$ **have** $P65 : Plane-diffside (Li B D) C A \implies Bet-Point (Se F D) B \implies$
 $\neg Eq (Geos (Lin (Li F D)) add Emp) (Geos (Lin (Li E C)) add Emp)$ **by** (*blast intro:Eq-trans*)
from $P13 P65$ **have** $P66 : Plane-diffside (Li B D) C A \implies Bet-Point (Se F D) B \implies Plane-diffside (Li E C) F D$ **by** (*simp add:Plane-Bet-diffside*)
have $Line-on (Li C A) A$ **by** (*simp add:Line-on-rule*)
then have $P67 : Eq (Geos (Lin (Li C A)) add Emp) (Geos (Lin (Li E C)) add Emp) \implies Line-on (Li E C) A$ **by** (*simp add:Line-on-trans*)
from $P25 P67$ **have** $P68 : \neg Eq (Geos (Lin (Li C A)) add Emp) (Geos (Lin (Li E C)) add Emp)$ **by** *blast*
from $P48$ **have** $P69 : Plane-diffside (Li B D) C A \implies Bet-Point (Se C A) F$ **by** *simp*
from $P21 P68 P69$ **have** $P71 : Plane-diffside (Li B D) C A \implies Plane-sameside (Li E C) F A$ **by** (*simp add:Plane-Bet-sameside*)
from $P66 P71$ **have** $P72 : Plane-diffside (Li B D) C A \implies Bet-Point (Se F D) B \implies$
 $Plane-diffside (Li E C) A D$ **by** (*simp add:Plane-trans*)
from $P22 P72$ **have** $Plane-diffside (Li B D) C A \implies Bet-Point (Se F D) B \implies$
 $Plane-diffside (Li B C) A D$ **by** (*simp add:Plane-Line-diff-trans*)
then have $Plane-diffside (Li B D) C A \implies Bet-Point (Se F D) B \implies$
 $\neg Plane-sameside (Li B C) A D$ **by** (*simp add:Plane-diffside-not-sameside*)
then have $P73 : Plane-sameside (Li B C) A D \wedge Plane-diffside (Li B D) C A \implies \neg Bet-Point (Se F D) B$ **by** *blast*
have $P74 : Bet-Point (Se B F) D \implies Line-on (Li B F) D$ **by** (*simp add:Line-Bet-on*)
from $P59$ **have** $P75 : \neg Ang-inside (An A B C) D \implies Plane-diffside (Li B D) C A \implies$
 $Bet-Point (Se B F) D \implies \neg Bet-Point (Se D F) B$ **by** (*blast intro:Bet-rev*)
from $P52 P58 P74 P75$ **have** $\neg Ang-inside (An A B C) D \implies Plane-diffside (Li B D) C A \implies$
 $Bet-Point (Se B F) D \implies Ang-inside (An A B C) D$ **by** (*simp add:Ang-inside-HalfLine*)
then have $P76 : \neg Ang-inside (An A B C) D \implies Plane-diffside (Li B D) C A \implies \neg Bet-Point (Se B F) D$ **by** *blast*
have $P77 : Bet-Point (Se D B) F \implies Line-on (Li B F) D$ **by** (*simp add:Line-Bet-on*)
from $P59$ **have** $P78 : \neg Ang-inside (An A B C) D \implies Plane-diffside (Li B D) C A \implies$
 $Bet-Point (Se D B) F \implies \neg Bet-Point (Se D F) B$ **by** (*blast intro:Bet-rev*)
from $P52 P58 P77 P78$ **have** $\neg Ang-inside (An A B C) D \implies Plane-diffside (Li B D) C A \implies$

Bet-Point (Se D B) F \implies *Ang-inside (An A B C) D* **by** (*simp add:Ang-inside-HalfLine*)
then have *P79* : \neg *Ang-inside (An A B C) D* \implies *Plane-diffside (Li B D) C A*
 $\implies \neg$ *Bet-Point (Se D B) F* **by** *blast*
from *P59 P73 P76 P79* **have** \neg *Ang-inside (An A B C) D* \wedge *Plane-sameside (Li B C) A D* \wedge *Plane-diffside (Li B D) C A* \implies *False* **by** *blast*
then have *P80* : \neg *Ang-inside (An A B C) D* \wedge \neg *Plane-sameside (Li B A) C D*
 \wedge *Plane-sameside (Li B C) A D* \wedge *Plane-diffside (Li B D) C A* \wedge *Plane-sameside (Li B C) D A* \implies *False* **by** *simp*
from *P5* **have** *Def (Tri (Tr B D C))* **by** (*blast intro:Ang-to-Tri Tri-def-rev Tri-def-trans*)
then have *P81* : \neg *Line-on (Li B D) C* **by** (*simp add:Tri-def-Line*)
from *P14 P15 P19 P32* **have** *P82* : *Line-on (Li B D) A* \implies
Eq (Geos (Lin (Li B A)) add Emp) (Geos (Lin (Li B D)) add Emp) **by** (*simp add:Line-unique*)
from *assms P82* **have** *P83* : \neg *Line-on (Li B D) A* **by** *blast*
from *P28 P81 P83* **have** *P84* : \neg *Plane-sameside (Li B D) C A* \implies *Plane-diffside (Li B D) C A* **by** (*simp add:Plane-not-sameside-diffside*)
from *P80 P84* **have** *P85* : \neg *Ang-inside (An A B C) D* \wedge \neg *Plane-sameside (Li B A) C D*
 \wedge *Plane-sameside (Li B C) A D* \wedge \neg *Plane-sameside (Li B D) C A* \wedge *Plane-sameside (Li B C) D A* \implies *False* **by** *simp*
from *P7 P8 P9 P85* **show** \neg (\neg *Ang-inside (An A B C) D* \wedge \neg *Ang-inside (An D B C) A*) **by** *blast*
qed

lemma (in Congruence-Rule) Ang-inside-case :

assumes

Def (Ang (An A B C))

Def (Ang (An D B C))

Plane-sameside (Li B C) A D

\neg *Eq (Geos (Lin (Li B A)) add Emp) (Geos (Lin (Li B D)) add Emp)*

shows

Ang-inside (An A B C) D \wedge \neg *Ang-inside (An D B C) A*

$\vee \neg$ *Ang-inside (An A B C) D* \wedge *Ang-inside (An D B C) A*

proof –

have *P1* : *Ang-inside (An A B C) D* \implies *Plane-sameside (Li B A) C D* \wedge *Plane-sameside (Li B C) A D* **by** (*simp add:Ang-inside-def*)

have *P2* : *Ang-inside (An D B C) A* \implies *Plane-sameside (Li B D) C A* \wedge *Plane-sameside (Li B C) D A* **by** (*simp add:Ang-inside-def*)

have *P3* : *Ang-inside (An A B C) D* \implies *Plane-diffside (Li B D) A C* **by** (*simp add:Ang-inside-Planeside*)

have *Plane-diffside (Li B D) A C* \implies *Plane-diffside (Li B D) C A* **by** (*simp add:Plane-diffside-rev*)

then have *P4* : *Plane-diffside (Li B D) A C* \implies \neg *Plane-sameside (Li B D) C A* **by** (*simp add:Plane-diffside-not-sameside*)

from *P3 P4* **have** *P5* : *Ang-inside (An A B C) D* \implies \neg *Plane-sameside (Li B D) C A* **by** (*simp add:Plane-diffside-rev*)

from *P2 P5* **have** *P6* : *Ang-inside (An A B C) D* \wedge *Ang-inside (An D B C) A*

\implies *False* **by** *simp*
from *assms* **have** $P7 : \neg \text{Eq} (\text{Geos} (\text{Poi } B) \text{ add Emp}) (\text{Geos} (\text{Poi } C) \text{ add Emp})$
by (*simp add:Ang-def*)
from *assms* $P7$ **have** $P8 : \neg \text{Ang-inside} (\text{An } A \ B \ C) \ D \wedge \neg \text{Ang-inside} (\text{An } D \ B \ C) \ A \implies \text{False}$ **by** (*simp add:Ang-outside-exclusive*)
from $P6 \ P8$ **show** $\text{Ang-inside} (\text{An } A \ B \ C) \ D \wedge \neg \text{Ang-inside} (\text{An } D \ B \ C) \ A$
 $\vee \neg \text{Ang-inside} (\text{An } A \ B \ C) \ D \wedge \text{Ang-inside} (\text{An } D \ B \ C) \ A$ **by** *blast*
qed

lemma (in *Congruence-Rule*) *Plane-sameside-HalfLine* :

assumes

Plane-sameside $l1 \ p1 \ p2$

Line-on $l1 \ p3$

Line-on (*Li* $p3 \ p1$) $p4$

$\neg \text{Bet-Point} (\text{Se } p4 \ p1) \ p3$

$\neg \text{Eq} (\text{Geos} (\text{Poi } p1) \text{ add Emp}) (\text{Geos} (\text{Poi } p4) \text{ add Emp})$

$\neg \text{Eq} (\text{Geos} (\text{Poi } p3) \text{ add Emp}) (\text{Geos} (\text{Poi } p4) \text{ add Emp})$

shows *Plane-sameside* $l1 \ p1 \ p4$

proof –

from *assms* **have** $P1 : \neg \text{Line-on } l1 \ p1 \wedge \neg \text{Line-on } l1 \ p2$

$\wedge \neg \text{Eq} (\text{Geos} (\text{Poi } p1) \text{ add Emp}) (\text{Geos} (\text{Poi } p2) \text{ add Emp})$ **by** (*simp add:Plane-sameside-def*)

have *Plane-diffside* $l1 \ p1 \ p4 \implies$

$\exists p. \text{Bet-Point} (\text{Se } p1 \ p4) \ p \wedge \text{Line-on } l1 \ p \wedge \neg \text{Line-on } l1 \ p1 \wedge \neg \text{Line-on } l1 \ p4$ **by** (*simp add:Plane-diffside-def*)

then obtain $p5 :: \text{Point}$ **where** $P2 : \text{Plane-diffside } l1 \ p1 \ p4 \implies$

$\text{Bet-Point} (\text{Se } p1 \ p4) \ p5 \wedge \text{Line-on } l1 \ p5 \wedge \neg \text{Line-on } l1 \ p4$ **by** *blast*

from *assms* **have** $P3 : \text{Eq} (\text{Geos} (\text{Poi } p3) \text{ add Emp}) (\text{Geos} (\text{Poi } p1) \text{ add Emp})$
 $\implies \text{Line-on } l1 \ p1$ **by** (*simp add:Point-Eq*)

from $P1 \ P3$ **have** $P4 : \neg \text{Eq} (\text{Geos} (\text{Poi } p3) \text{ add Emp}) (\text{Geos} (\text{Poi } p1) \text{ add Emp})$
by *blast*

then have $P5 : \text{Eq} (\text{Geos} (\text{Lin} (\text{Li } p3 \ p1)) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li } p1 \ p3)) \text{ add Emp})$ **by** (*simp add:Line-rev*)

from *assms* $P5$ **have** $P6 : \text{Line-on} (\text{Li } p1 \ p3) \ p4$ **by** (*simp add:Line-rev Line-on-trans*)

from $P4$ **have** $P7 : \neg \text{Eq} (\text{Geos} (\text{Poi } p1) \text{ add Emp}) (\text{Geos} (\text{Poi } p3) \text{ add Emp})$
by (*blast intro:Eq-rev*)

from *assms* $P6 \ P7$ **have** $P8 : \text{Line-on} (\text{Li } p1 \ p4) \ p3$ **by** (*simp add:Line-on-rev*)

from $P2$ **have** $P9 : \text{Plane-diffside } l1 \ p1 \ p4 \implies \text{Line-on} (\text{Li } p1 \ p4) \ p5$ **by** (*simp add:Line-Bet-on*)

from *assms* $P2 \ P8 \ P9$ **have** $P10 : \text{Plane-diffside } l1 \ p1 \ p4 \implies \neg \text{Eq} (\text{Geos} (\text{Poi } p3) \text{ add Emp}) (\text{Geos} (\text{Poi } p5) \text{ add Emp}) \implies$

$\text{Eq} (\text{Geos} (\text{Lin} (\text{Li } p1 \ p4)) \text{ add Emp}) (\text{Geos} (\text{Lin } l1) \text{ add Emp})$ **by** (*simp add:Line-unique*)

have $P11 : \text{Line-on} (\text{Li } p1 \ p4) \ p1$ **by** (*simp add:Line-on-rule*)

from $P10 \ P11$ **have** $P12 : \text{Plane-diffside } l1 \ p1 \ p4 \implies \neg \text{Eq} (\text{Geos} (\text{Poi } p3) \text{ add Emp}) (\text{Geos} (\text{Poi } p5) \text{ add Emp}) \implies$

$\text{Line-on } l1 \ p1$ **by** (*simp add:Line-on-trans*)

from $P1 \ P12$ **have** $P13 : \text{Plane-diffside } l1 \ p1 \ p4 \implies \text{Eq} (\text{Geos} (\text{Poi } p5) \text{ add$

Emp) (*Geos* (*Poi* *p3*) *add Emp*) **by** (*blast intro:Eq-rev*)
from *P2* **have** *P14* : *Plane-diffside* *l1 p1 p4* \implies *Bet-Point* (*Se p1 p4*) *p5* **by**
simp
from *P13 P14* **have** *Plane-diffside* *l1 p1 p4* \implies *Bet-Point* (*Se p1 p4*) *p3* **by**
(*simp add:Point-Eq*)
then have *P15* : *Plane-diffside* *l1 p1 p4* \implies *Bet-Point* (*Se p4 p1*) *p3* **by** (*simp*
add:Bet-rev)
from *assms P15* **have** *P16* : \neg *Plane-diffside* *l1 p1 p4* **by** *blast*
have *P17* : *Line-on* (*Li p3 p1*) *p3* **by** (*simp add:Line-on-rule*)
from *assms P17* **have** *P18* : *Line-on* *l1 p4* \implies
Eq (*Geos* (*Lin* (*Li p3 p1*)) *add Emp*) (*Geos* (*Lin* *l1*) *add Emp*) **by** (*simp*
add:Line-unique)
have *P19* : *Line-on* (*Li p3 p1*) *p1* **by** (*simp add:Line-on-rule*)
from *P18 P19* **have** *P20* : *Line-on* *l1 p4* \implies *Line-on* *l1 p1* **by** (*simp add:Line-on-trans*)
from *P1 P20* **have** *P21* : \neg *Line-on* *l1 p4* **by** *blast*
from *assms P1 P16 P21* **show** *Plane-sameside* *l1 p1 p4* **by** (*simp add:Plane-not-diffside-sameside*)
qed

lemma (in *Congruence-Rule*) *Plane-Bet-sameside-rev* :

assumes

Plane-sameside *l1 p1 p3*

Line-on *l1 p2*

\neg *Eq* (*Geos* (*Poi* *p1*) *add Emp*) (*Geos* (*Poi* *p2*) *add Emp*)

\neg *Eq* (*Geos* (*Poi* *p2*) *add Emp*) (*Geos* (*Poi* *p3*) *add Emp*)

\neg *Eq* (*Geos* (*Poi* *p3*) *add Emp*) (*Geos* (*Poi* *p1*) *add Emp*)

Line-on *l2 p1* *Line-on* *l2 p2* *Line-on* *l2 p3*

\neg *Eq* (*Geos* (*Lin* *l1*) *add Emp*) (*Geos* (*Lin* *l2*) *add Emp*)

shows *Bet-Point* (*Se p3 p2*) *p1* \vee *Bet-Point* (*Se p2 p1*) *p3*

proof –

from *assms* **have** *P1* : *Bet-Point* (*Se p1 p3*) *p2* \vee *Bet-Point* (*Se p3 p2*) *p1* \vee
Bet-Point (*Se p2 p1*) *p3* **by** (*simp add:Bet-case*)

have *P2* : *Line-on* (*Li p1 p3*) *p1* **by** (*simp add:Line-on-rule*)

have *P3* : *Line-on* (*Li p1 p3*) *p3* **by** (*simp add:Line-on-rule*)

from *assms P2 P3* **have** *P4* : *Eq* (*Geos* (*Lin* *l2*) *add Emp*) (*Geos* (*Lin* (*Li p1*
p3)) *add Emp*) **by** (*simp add:Line-unique*)

from *assms P4* **have** \neg *Eq* (*Geos* (*Lin* *l1*) *add Emp*) (*Geos* (*Lin* (*Li p1 p3*)) *add*
Emp) **by** (*blast intro:Eq-trans*)

then have *P6* : \neg *Eq* (*Geos* (*Lin* (*Li p1 p3*)) *add Emp*) (*Geos* (*Lin* *l1*) *add Emp*)
by (*blast intro:Eq-rev*)

from *assms P6* **have** *Bet-Point* (*Se p1 p3*) *p2* \implies *Plane-diffside* *l1 p1 p3* **by**
(*simp add:Plane-Bet-diffside*)

then have *P7* : *Bet-Point* (*Se p1 p3*) *p2* \implies \neg *Plane-sameside* *l1 p1 p3* **by**
(*simp add:Plane-diffside-not-sameside*)

from *assms P7* **have** *P8* : \neg *Bet-Point* (*Se p1 p3*) *p2* **by** *blast*

from *P1 P8* **show** *Bet-Point* (*Se p3 p2*) *p1* \vee *Bet-Point* (*Se p2 p1*) *p3* **by** *blast*

qed

lemma (in *Congruence-Rule*) *Seg-Bet-relation* :

assumes *N* :

$Bet\text{-}Point (Se\ p1\ p2)\ p3$
shows $\neg Eq (Geos (Seg (Se\ p1\ p2))\ add\ Emp) (Geos (Seg (Se\ p1\ p3))\ add\ Emp)$
proof
assume $W : Eq (Geos (Seg (Se\ p1\ p2))\ add\ Emp) (Geos (Seg (Se\ p1\ p3))\ add\ Emp)$
from N **have** $Inv (Bet\text{-}Point (Se\ p2\ p3)\ p1) \wedge Inv (Bet\text{-}Point (Se\ p3\ p1)\ p2)$
by (*simp add:Bet-iff*)
then have $P1 : \neg Bet\text{-}Point (Se\ p2\ p3)\ p1$ **by** (*simp add:Inv-def*)
have $P2 : Line\text{-}on (Li\ p1\ p2)\ p1$ **by** (*simp add:Line-on-rule*)
have $P3 : Line\text{-}on (Li\ p1\ p2)\ p2$ **by** (*simp add:Line-on-rule*)
from N **have** $P4 : Line\text{-}on (Li\ p1\ p2)\ p3$ **by** (*simp add:Line-Bet-on*)
from N **have** $P5 : \neg Eq (Geos (Poi\ p1)\ add\ Emp) (Geos (Poi\ p2)\ add\ Emp)$ **by**
(*simp add:Bet-Point-def*)
from N **have** $\neg Eq (Geos (Poi\ p3)\ add\ Emp) (Geos (Poi\ p1)\ add\ Emp)$ **by** (*simp*
add:Bet-Point-def)
then have $P6 : \neg Eq (Geos (Poi\ p1)\ add\ Emp) (Geos (Poi\ p3)\ add\ Emp)$ **by**
(*blast intro:Eq-rev*)
from $W\ P1\ P2\ P3\ P4\ P5\ P6$ **have** $P7 : Eq (Geos (Poi\ p2)\ add\ Emp) (Geos$
($Poi\ p3$) $add\ Emp)$ **by** (*blast intro:Seg-move-unique*)
from N **have** $P8 : \neg Eq (Geos (Poi\ p2)\ add\ Emp) (Geos (Poi\ p3)\ add\ Emp)$ **by**
(*simp add:Bet-Point-def*)
from $P7\ P8$ **show** $False$ **by** *blast*
qed

lemma (in Congruence-Rule) Seg-Bet-move-lemma1 :

assumes
 $Bet\text{-}Point (Se\ p11\ p13)\ p12$
 $Line\text{-}on\ l1\ p21\ Line\text{-}on\ l1\ p22\ Line\text{-}on\ l1\ p23$
 $\neg Eq (Geos (Poi\ p21)\ add\ Emp) (Geos (Poi\ p22)\ add\ Emp)$
 $\neg Eq (Geos (Poi\ p21)\ add\ Emp) (Geos (Poi\ p23)\ add\ Emp)$
 $Eq (Geos (Seg (Se\ p11\ p12))\ add\ Emp) (Geos (Seg (Se\ p21\ p22))\ add\ Emp)$
 $Eq (Geos (Seg (Se\ p11\ p13))\ add\ Emp) (Geos (Seg (Se\ p21\ p23))\ add\ Emp)$
 $\neg Bet\text{-}Point (Se\ p22\ p23)\ p21$
shows $Bet\text{-}Point (Se\ p21\ p23)\ p22$
proof –
from *assms* **have** $P1 : \neg Eq (Geos (Poi\ p22)\ add\ Emp) (Geos (Poi\ p21)\ add$
 $Emp)$ **by** (*blast intro:Eq-rev*)
from *assms* **have** $\neg Eq (Geos (Poi\ p12)\ add\ Emp) (Geos (Poi\ p11)\ add\ Emp)$
by (*simp add:Bet-Point-def*)
then have $P2 : \neg Eq (Geos (Poi\ p11)\ add\ Emp) (Geos (Poi\ p12)\ add\ Emp)$ **by**
(*blast intro:Eq-rev*)
from *assms* $P1\ P2$ **have** $\exists p. Eq (Geos (Seg (Se\ p11\ p12))\ add\ Emp) (Geos (Seg$
($Se\ p22\ p$)) $add\ Emp)$
 $\wedge \neg Bet\text{-}Point (Se\ p\ p21)\ p22 \wedge Line\text{-}on\ l1\ p \wedge \neg Eq (Geos (Poi\ p22)\ add$
 $Emp) (Geos (Poi\ p)\ add\ Emp)$ **by** (*simp add:Seg-move-sameside*)
then obtain $p211 :: Point$ **where** $P3 : Eq (Geos (Seg (Se\ p11\ p12))\ add\ Emp)$
($Seg (Se\ p22\ p211))\ add\ Emp)$
 $\wedge \neg Bet\text{-}Point (Se\ p211\ p21)\ p22 \wedge Line\text{-}on\ l1\ p211 \wedge \neg Eq (Geos (Poi\ p22)$
 $add\ Emp) (Geos (Poi\ p211)\ add\ Emp)$ **by** *blast*

from *assms* **have** $\neg \text{Eq} (\text{Geos} (\text{Poi } p13) \text{ add Emp}) (\text{Geos} (\text{Poi } p12) \text{ add Emp})$
by (*simp add:Bet-Point-def*)
then **have** $P_4 : \neg \text{Eq} (\text{Geos} (\text{Poi } p12) \text{ add Emp}) (\text{Geos} (\text{Poi } p13) \text{ add Emp})$ **by**
(*blast intro:Eq-rev*)
from *assms* $P_3 P_4$ **have** $\exists p. \text{Eq} (\text{Geos} (\text{Seg} (\text{Se } p12 \text{ } p13)) \text{ add Emp}) (\text{Geos} (\text{Seg}$
(*Se p22 p*)) *add Emp*)
 $\wedge \text{Bet-Point} (\text{Se } p \text{ } p211) \text{ } p22 \wedge \text{Line-on } l1 \text{ } p \wedge \neg \text{Eq} (\text{Geos} (\text{Poi } p22) \text{ add Emp})$
(*Geos (Poi p) add Emp*) **by** (*simp add:Seg-move-diffside*)
then **obtain** $p231 :: \text{Point}$ **where** $P_5 : \text{Eq} (\text{Geos} (\text{Seg} (\text{Se } p12 \text{ } p13)) \text{ add Emp})$
(*Geos (Seg (Se p22 p231)) add Emp*)
 $\wedge \text{Bet-Point} (\text{Se } p231 \text{ } p211) \text{ } p22 \wedge \text{Line-on } l1 \text{ } p231 \wedge \neg \text{Eq} (\text{Geos} (\text{Poi } p22)$
add Emp) (*Geos (Poi p231) add Emp*) **by** *blast*
have $P_6 : \text{Eq} (\text{Geos} (\text{Seg} (\text{Se } p21 \text{ } p22)) \text{ add Emp}) (\text{Geos} (\text{Seg} (\text{Se } p22 \text{ } p21)) \text{ add}$
Emp) **by** (*simp add:Seg-rev*)
from *assms* **have** $P_7 : \text{Eq} (\text{Geos} (\text{Seg} (\text{Se } p21 \text{ } p22)) \text{ add Emp}) (\text{Geos} (\text{Seg} (\text{Se}$
p11 p12)) *add Emp*) **by** (*simp add:Eq-rev*)
from $P_3 P_7$ **have** $P_8 : \text{Eq} (\text{Geos} (\text{Seg} (\text{Se } p22 \text{ } p211)) \text{ add Emp}) (\text{Geos} (\text{Seg} (\text{Se}$
p21 p22)) *add Emp*) **by** (*blast intro:Eq-trans Eq-rev*)
from $P_6 P_8$ **have** $P_9 : \text{Eq} (\text{Geos} (\text{Seg} (\text{Se } p22 \text{ } p211)) \text{ add Emp}) (\text{Geos} (\text{Seg} (\text{Se}$
p22 p21)) *add Emp*) **by** (*blast intro:Eq-trans*)
from *assms* $P_1 P_3 P_9$ **have** $P_{10} : \text{Eq} (\text{Geos} (\text{Poi } p211) \text{ add Emp}) (\text{Geos} (\text{Poi}$
p21) *add Emp*) **by** (*blast intro:Seg-move-unique*)
from P_5 **have** $P_{11} : \text{Bet-Point} (\text{Se } p211 \text{ } p231) \text{ } p22$ **by** (*simp add:Bet-rev*)
from $P_{10} P_{11}$ **have** $P_{12} : \text{Bet-Point} (\text{Se } p21 \text{ } p231) \text{ } p22$ **by** (*simp add:Bet-Point-Eq*)
have $P_{13} : \text{Line-on} (\text{Li } p11 \text{ } p12) \text{ } p11$ **by** (*simp add:Line-on-rule*)
have $P_{14} : \text{Line-on} (\text{Li } p11 \text{ } p12) \text{ } p12$ **by** (*simp add:Line-on-rule*)
from *assms* **have** $P_{15} : \text{Line-on} (\text{Li } p11 \text{ } p12) \text{ } p13$ **by** (*simp add:Line-Bet-on*)
from *assms* **have** $P_{16} : \neg \text{Seg-on-Seg} (\text{Se } p11 \text{ } p12) (\text{Se } p12 \text{ } p13)$ **by** (*simp*
add:Seg-Bet-not-on)
from P_{12} **have** $P_{17} : \neg \text{Seg-on-Seg} (\text{Se } p21 \text{ } p22) (\text{Se } p22 \text{ } p231)$ **by** (*simp*
add:Seg-Bet-not-on)
from *assms* $P_5 P_{13} P_{14} P_{15} P_{16} P_{17}$ **have** $P_{18} : \text{Eq} (\text{Geos} (\text{Seg} (\text{Se } p11 \text{ } p13))$
add Emp) (*Geos (Seg (Se p21 p231)) add Emp*) **by** (*simp add:Seg-add*)
from *assms* P_{18} **have** $P_{19} : \text{Eq} (\text{Geos} (\text{Seg} (\text{Se } p21 \text{ } p231)) \text{ add Emp}) (\text{Geos} (\text{Seg}$
(*Se p21 p23*)) *add Emp*) **by** (*blast intro:Eq-trans Eq-rev*)
from P_{12} **have** $P_{20} : \neg \text{Eq} (\text{Geos} (\text{Poi } p21) \text{ add Emp}) (\text{Geos} (\text{Poi } p231) \text{ add}$
Emp) **by** (*simp add:Bet-Point-def*)
from P_{12} **have** $P_{21} : \text{Bet-Point} (\text{Se } p231 \text{ } p21) \text{ } p22$ **by** (*simp add:Bet-rev*)
from P_{21} **have** $P_{22} : \text{Bet-Point} (\text{Se } p231 \text{ } p23) \text{ } p21 \implies \text{Bet-Point} (\text{Se } p22 \text{ } p23)$
p21 **by** (*blast intro:Bet-swap-134-234*)
from *assms* P_{22} **have** $P_{23} : \neg \text{Bet-Point} (\text{Se } p231 \text{ } p23) \text{ } p21$ **by** *blast*
from *assms* $P_5 P_{19} P_{20} P_{23}$ **have** $P_{24} : \text{Eq} (\text{Geos} (\text{Poi } p231) \text{ add Emp}) (\text{Geos}$
(*Poi p23*) *add Emp*) **by** (*blast intro:Seg-move-unique*)
from $P_{21} P_{24}$ **have** $\text{Bet-Point} (\text{Se } p23 \text{ } p21) \text{ } p22$ **by** (*simp add:Bet-Point-Eq*)
thus $\text{Bet-Point} (\text{Se } p21 \text{ } p23) \text{ } p22$ **by** (*simp add:Bet-rev*)
qed

lemma (*in Congruence-Rule*) *Seg-Bet-move-sameside* :
assumes

Bet-Point (Se p11 p13) p12
Line-on l1 p21 Line-on l1 p4
 $\neg \text{Eq (Geos (Poi p21) add Emp) (Geos (Poi p4) add Emp)}$
shows $\exists p q. \text{Bet-Point (Se p21 q) p} \wedge \text{Line-on l1 p} \wedge \text{Line-on l1 q}$
 $\wedge \text{Eq (Geos (Seg (Se p11 p12)) add Emp) (Geos (Seg (Se p21 p)) add Emp)}$
 $\wedge \text{Eq (Geos (Seg (Se p11 p13)) add Emp) (Geos (Seg (Se p21 q)) add Emp)}$
 $\wedge \neg \text{Bet-Point (Se p p4) p21} \wedge \neg \text{Bet-Point (Se q p4) p21}$
proof –
from *assms* **have** $\neg \text{Eq (Geos (Poi p12) add Emp) (Geos (Poi p11) add Emp)}$
by (*simp add:Bet-Point-def*)
then **have** $P1 : \neg \text{Eq (Geos (Poi p11) add Emp) (Geos (Poi p12) add Emp)}$ **by**
(*blast intro:Eq-rev*)
from *assms* $P1$ **have** $\exists p. \text{Eq (Geos (Seg (Se p11 p12)) add Emp) (Geos (Seg (Se p21 p)) add Emp)}$
 $\wedge \neg \text{Bet-Point (Se p p4) p21} \wedge \text{Line-on l1 p} \wedge \neg \text{Eq (Geos (Poi p21) add Emp)}$
(*Geos (Poi p) add Emp*) **by** (*simp add:Seg-move-sameside*)
then **obtain** $p22 :: \text{Point}$ **where** $P2 : \text{Eq (Geos (Seg (Se p11 p12)) add Emp)}$
(*Geos (Seg (Se p21 p22)) add Emp*)
 $\wedge \neg \text{Bet-Point (Se p22 p4) p21} \wedge \text{Line-on l1 p22} \wedge \neg \text{Eq (Geos (Poi p21) add}$
Emp) (Geos (Poi p22) add Emp) **by** *blast*
from *assms* **have** $P3 : \neg \text{Eq (Geos (Poi p11) add Emp) (Geos (Poi p13) add}$
Emp) **by** (*simp add:Bet-Point-def*)
from *assms* $P2 P3$ **have** $\exists p. \text{Eq (Geos (Seg (Se p11 p13)) add Emp) (Geos (Seg$
(*Se p21 p)) add Emp)*
 $\wedge \neg \text{Bet-Point (Se p p22) p21} \wedge \text{Line-on l1 p} \wedge \neg \text{Eq (Geos (Poi p21) add Emp)}$
(*Geos (Poi p) add Emp*) **by** (*simp add:Seg-move-sameside*)
then **obtain** $p23 :: \text{Point}$ **where** $P4 : \text{Eq (Geos (Seg (Se p11 p13)) add Emp)}$
(*Geos (Seg (Se p21 p23)) add Emp*)
 $\wedge \neg \text{Bet-Point (Se p23 p22) p21} \wedge \text{Line-on l1 p23} \wedge \neg \text{Eq (Geos (Poi p21) add}$
Emp) (Geos (Poi p23) add Emp) **by** *blast*
from $P4$ **have** $\neg \text{Bet-Point (Se p23 p22) p21}$ **by** *simp*
then **have** $P5 : \neg \text{Bet-Point (Se p22 p23) p21}$ **by** (*blast intro:Bet-rev*)
from *assms* $P2 P4 P5$ **have** $P6 : \text{Bet-Point (Se p21 p23) p22}$ **by** (*blast intro:Seg-Bet-move-lemma1*)
then **have** $P7 : \text{Bet-Point (Se p23 p21) p22}$ **by** (*simp add:Bet-rev*)
from $P7$ **have** $P8 : \text{Bet-Point (Se p23 p4) p21} \implies \text{Bet-Point (Se p22 p4) p21}$
by (*blast intro:Bet-swap-134-234*)
from $P2 P8$ **have** $P9 : \neg \text{Bet-Point (Se p23 p4) p21}$ **by** *blast*
from $P2 P4 P6 P9$ **show** $\exists p q. \text{Bet-Point (Se p21 q) p} \wedge \text{Line-on l1 p} \wedge \text{Line-on}$
l1 q
 $\wedge \text{Eq (Geos (Seg (Se p11 p12)) add Emp) (Geos (Seg (Se p21 p)) add Emp)}$
 $\wedge \text{Eq (Geos (Seg (Se p11 p13)) add Emp) (Geos (Seg (Se p21 q)) add Emp)}$
 $\wedge \neg \text{Bet-Point (Se p p4) p21} \wedge \neg \text{Bet-Point (Se q p4) p21}$ **by** *blast*
qed

lemma (in Congruence-Rule) Seg-Bet-move-diffside :

assumes

Bet-Point (Se p11 p13) p12
Line-on l1 p21 Line-on l1 p4

$\neg \text{Eq} (\text{Geos} (\text{Poi } p21) \text{ add Emp}) (\text{Geos} (\text{Poi } p4) \text{ add Emp})$
shows $\exists p q. \text{Bet-Point} (\text{Se } p21 \text{ } q) p \wedge \text{Line-on } l1 \text{ } p \wedge \text{Line-on } l1 \text{ } q$
 $\wedge \text{Eq} (\text{Geos} (\text{Seg} (\text{Se } p11 \text{ } p12)) \text{ add Emp}) (\text{Geos} (\text{Seg} (\text{Se } p21 \text{ } p)) \text{ add Emp})$
 $\wedge \text{Eq} (\text{Geos} (\text{Seg} (\text{Se } p11 \text{ } p13)) \text{ add Emp}) (\text{Geos} (\text{Seg} (\text{Se } p21 \text{ } q)) \text{ add Emp})$
 $\wedge \text{Bet-Point} (\text{Se } p \text{ } p4) p21 \wedge \text{Bet-Point} (\text{Se } q \text{ } p4) p21$
proof –
from *assms* **have** $\neg \text{Eq} (\text{Geos} (\text{Poi } p12) \text{ add Emp}) (\text{Geos} (\text{Poi } p11) \text{ add Emp})$
by (*simp add:Bet-Point-def*)
then **have** $P1 : \neg \text{Eq} (\text{Geos} (\text{Poi } p11) \text{ add Emp}) (\text{Geos} (\text{Poi } p12) \text{ add Emp})$ **by**
(*blast intro:Eq-rev*)
from *assms* $P1$ **have** $\exists p. \text{Eq} (\text{Geos} (\text{Seg} (\text{Se } p11 \text{ } p12)) \text{ add Emp}) (\text{Geos} (\text{Seg} (\text{Se } p21 \text{ } p)) \text{ add Emp})$
 $\wedge \text{Bet-Point} (\text{Se } p \text{ } p4) p21 \wedge \text{Line-on } l1 \text{ } p \wedge \neg \text{Eq} (\text{Geos} (\text{Poi } p21) \text{ add Emp})$
(*Geos (Poi p) add Emp*) **by** (*simp add:Seg-move-diffside*)
then **obtain** $p22 :: \text{Point}$ **where** $P2 : \text{Eq} (\text{Geos} (\text{Seg} (\text{Se } p11 \text{ } p12)) \text{ add Emp})$
(*Geos (Seg (Se p21 p22)) add Emp*)
 $\wedge \text{Bet-Point} (\text{Se } p22 \text{ } p4) p21 \wedge \text{Line-on } l1 \text{ } p22 \wedge \neg \text{Eq} (\text{Geos} (\text{Poi } p21) \text{ add Emp})$
(*Geos (Poi p22) add Emp*) **by** *blast*
from *assms* **have** $P3 : \neg \text{Eq} (\text{Geos} (\text{Poi } p11) \text{ add Emp}) (\text{Geos} (\text{Poi } p13) \text{ add Emp})$ **by** (*simp add:Bet-Point-def*)
from *assms* $P2 \text{ } P3$ **have** $\exists p. \text{Eq} (\text{Geos} (\text{Seg} (\text{Se } p11 \text{ } p13)) \text{ add Emp}) (\text{Geos} (\text{Seg} (\text{Se } p21 \text{ } p)) \text{ add Emp})$
 $\wedge \neg \text{Bet-Point} (\text{Se } p \text{ } p22) p21 \wedge \text{Line-on } l1 \text{ } p \wedge \neg \text{Eq} (\text{Geos} (\text{Poi } p21) \text{ add Emp})$
(*Geos (Poi p) add Emp*) **by** (*simp add:Seg-move-sameside*)
then **obtain** $p23 :: \text{Point}$ **where** $P4 : \text{Eq} (\text{Geos} (\text{Seg} (\text{Se } p11 \text{ } p13)) \text{ add Emp})$
(*Geos (Seg (Se p21 p23)) add Emp*)
 $\wedge \neg \text{Bet-Point} (\text{Se } p23 \text{ } p22) p21 \wedge \text{Line-on } l1 \text{ } p23 \wedge \neg \text{Eq} (\text{Geos} (\text{Poi } p21) \text{ add Emp})$
(*Geos (Poi p23) add Emp*) **by** *blast*
from $P4$ **have** $\neg \text{Bet-Point} (\text{Se } p23 \text{ } p22) p21$ **by** *simp*
then **have** $P5 : \neg \text{Bet-Point} (\text{Se } p22 \text{ } p23) p21$ **by** (*blast intro:Bet-rev*)
from *assms* $P2 \text{ } P4 \text{ } P5$ **have** $P6 : \text{Bet-Point} (\text{Se } p21 \text{ } p23) p22$ **by** (*blast intro:Seg-Bet-move-lemma1*)
then **have** $P7 : \text{Bet-Point} (\text{Se } p23 \text{ } p21) p22$ **by** (*simp add:Bet-rev*)
from $P2 \text{ } P7$ **have** $P8 : \text{Bet-Point} (\text{Se } p23 \text{ } p4) p21$ **by** (*blast intro:Bet-swap-234-134*)
from $P2 \text{ } P4 \text{ } P6 \text{ } P8$ **show** $\exists p q. \text{Bet-Point} (\text{Se } p21 \text{ } q) p \wedge \text{Line-on } l1 \text{ } p \wedge \text{Line-on } l1 \text{ } q$
 $\wedge \text{Eq} (\text{Geos} (\text{Seg} (\text{Se } p11 \text{ } p12)) \text{ add Emp}) (\text{Geos} (\text{Seg} (\text{Se } p21 \text{ } p)) \text{ add Emp})$
 $\wedge \text{Eq} (\text{Geos} (\text{Seg} (\text{Se } p11 \text{ } p13)) \text{ add Emp}) (\text{Geos} (\text{Seg} (\text{Se } p21 \text{ } q)) \text{ add Emp})$
 $\wedge \text{Bet-Point} (\text{Se } p \text{ } p4) p21 \wedge \text{Bet-Point} (\text{Se } q \text{ } p4) p21$ **by** *blast*
qed

lemma (in Congruence-Rule) Seg-Bet-wrong-relation :

assumes

Bet-Point (Se p11 p13) p12

Bet-Point (Se p21 p22) p23

Eq (Geos (Seg (Se p11 p12)) add Emp) (Geos (Seg (Se p21 p22)) add Emp)

Eq (Geos (Seg (Se p11 p13)) add Emp) (Geos (Seg (Se p21 p23)) add Emp)

shows *False*

proof –

have $P1 : \text{Line-on } (Li\ p21\ p22)\ p21$ **by** (*simp add:Line-on-rule*)
have $P2 : \text{Line-on } (Li\ p21\ p22)\ p22$ **by** (*simp add:Line-on-rule*)
from *assms* **have** $P3 : \neg \text{Eq } (\text{Geos } (\text{Poi } p21)\ \text{add } \text{Emp})\ (\text{Geos } (\text{Poi } p22)\ \text{add } \text{Emp})$ **by** (*simp add:Bet-Point-def*)
from *assms* $P1\ P2\ P3$ **have** $\exists p\ q. \text{Bet-Point } (\text{Se } p21\ q)\ p \wedge \text{Line-on } (Li\ p21\ p22)\ p \wedge \text{Line-on } (Li\ p21\ p22)\ q$
 $\wedge \text{Eq } (\text{Geos } (\text{Seg } (\text{Se } p11\ p12))\ \text{add } \text{Emp})\ (\text{Geos } (\text{Seg } (\text{Se } p21\ p))\ \text{add } \text{Emp})$
 $\wedge \text{Eq } (\text{Geos } (\text{Seg } (\text{Se } p11\ p13))\ \text{add } \text{Emp})\ (\text{Geos } (\text{Seg } (\text{Se } p21\ q))\ \text{add } \text{Emp})$
 $\wedge \neg \text{Bet-Point } (\text{Se } p\ p22)\ p21 \wedge \neg \text{Bet-Point } (\text{Se } q\ p22)\ p21$ **by** (*simp add:Seg-Bet-move-sameside*)
then obtain $pn2\ pn3 :: \text{Point}$ **where** $P4 : \text{Bet-Point } (\text{Se } p21\ pn3)\ pn2 \wedge \text{Line-on } (Li\ p21\ p22)\ pn2 \wedge \text{Line-on } (Li\ p21\ p22)\ pn3$
 $\wedge \text{Eq } (\text{Geos } (\text{Seg } (\text{Se } p11\ p12))\ \text{add } \text{Emp})\ (\text{Geos } (\text{Seg } (\text{Se } p21\ pn2))\ \text{add } \text{Emp})$
 $\wedge \text{Eq } (\text{Geos } (\text{Seg } (\text{Se } p11\ p13))\ \text{add } \text{Emp})\ (\text{Geos } (\text{Seg } (\text{Se } p21\ pn3))\ \text{add } \text{Emp})$
 $\wedge \neg \text{Bet-Point } (\text{Se } pn2\ p22)\ p21 \wedge \neg \text{Bet-Point } (\text{Se } pn3\ p22)\ p21$ **by** *blast*
then have $P5 : \text{Bet-Point } (\text{Se } p21\ pn3)\ pn2$ **by** *simp*
then have $\neg \text{Eq } (\text{Geos } (\text{Poi } pn2)\ \text{add } \text{Emp})\ (\text{Geos } (\text{Poi } p21)\ \text{add } \text{Emp})$ **by** (*simp add:Bet-Point-def*)
then have $P6 : \neg \text{Eq } (\text{Geos } (\text{Poi } p21)\ \text{add } \text{Emp})\ (\text{Geos } (\text{Poi } pn2)\ \text{add } \text{Emp})$ **by** (*blast intro:Eq-rev*)
from *assms* $P4$ **have** $P7 : \text{Eq } (\text{Geos } (\text{Seg } (\text{Se } p21\ pn2))\ \text{add } \text{Emp})\ (\text{Geos } (\text{Seg } (\text{Se } p21\ p22))\ \text{add } \text{Emp})$ **by** (*blast intro:Eq-trans*)
from $P1\ P2\ P3\ P4\ P6\ P7$ **have** $P8 : \text{Eq } (\text{Geos } (\text{Poi } pn2)\ \text{add } \text{Emp})\ (\text{Geos } (\text{Poi } p22)\ \text{add } \text{Emp})$ **by** (*blast intro:Seg-move-unique*)
from $P5\ P8$ **have** $\text{Bet-Point } (\text{Se } p21\ pn3)\ p22$ **by** (*simp add:Point-Eq*)
then have $P9 : \text{Bet-Point } (\text{Se } pn3\ p21)\ p22$ **by** (*simp add:Bet-rev*)
from *assms* **have** $P10 : \text{Line-on } (Li\ p21\ p22)\ p23$ **by** (*simp add:Line-Bet-on*)
from *assms* **have** $\neg \text{Eq } (\text{Geos } (\text{Poi } p23)\ \text{add } \text{Emp})\ (\text{Geos } (\text{Poi } p21)\ \text{add } \text{Emp})$
by (*simp add:Bet-Point-def*)
then have $P11 : \neg \text{Eq } (\text{Geos } (\text{Poi } p21)\ \text{add } \text{Emp})\ (\text{Geos } (\text{Poi } p23)\ \text{add } \text{Emp})$ **by** (*blast intro:Eq-rev*)
from $P5$ **have** $P12 : \neg \text{Eq } (\text{Geos } (\text{Poi } p21)\ \text{add } \text{Emp})\ (\text{Geos } (\text{Poi } pn3)\ \text{add } \text{Emp})$
by (*simp add:Bet-Point-def*)
from *assms* $P4$ **have** $P13 : \text{Eq } (\text{Geos } (\text{Seg } (\text{Se } p21\ pn3))\ \text{add } \text{Emp})\ (\text{Geos } (\text{Seg } (\text{Se } p21\ p23))\ \text{add } \text{Emp})$ **by** (*blast intro:Eq-trans*)
from *assms* **have** $P14 : \text{Bet-Point } (\text{Se } p22\ p21)\ p23$ **by** (*simp add:Bet-rev*)
have $P15 : \text{Bet-Point } (\text{Se } pn3\ p23)\ p21 \implies \text{Bet-Point } (\text{Se } p23\ pn3)\ p21$ **by** (*simp add:Bet-rev*)
from $P14\ P15$ **have** $\text{Bet-Point } (\text{Se } pn3\ p23)\ p21 \implies \text{Bet-Point } (\text{Se } p22\ pn3)\ p21$ **by** (*blast intro:Bet-swap-234-134*)
then have $P16 : \text{Bet-Point } (\text{Se } pn3\ p23)\ p21 \implies \text{Bet-Point } (\text{Se } pn3\ p22)\ p21$ **by** (*simp add:Bet-rev*)
from $P4\ P16$ **have** $P17 : \neg \text{Bet-Point } (\text{Se } pn3\ p23)\ p21$ **by** *blast*
from $P1\ P4\ P10\ P11\ P12\ P13\ P17$ **have** $P18 : \text{Eq } (\text{Geos } (\text{Poi } pn3)\ \text{add } \text{Emp})\ (\text{Geos } (\text{Poi } p23)\ \text{add } \text{Emp})$ **by** (*blast intro:Seg-move-unique*)
from $P9\ P18$ **have** $P19 : \text{Bet-Point } (\text{Se } p23\ p21)\ p22$ **by** (*simp add:Bet-Point-Eq*)
from *assms* **have** $\text{Inv } (\text{Bet-Point } (\text{Se } p22\ p23)\ p21) \wedge \text{Inv } (\text{Bet-Point } (\text{Se } p23\ p21)\ p22)$ **by** (*simp add:Bet-iff*)
then have $P20 : \neg \text{Bet-Point } (\text{Se } p23\ p21)\ p22$ **by** (*simp add:Inv-def*)

from $P19$ $P20$ **show** $False$ **by** $blast$
qed

lemma (in *Congruence-Rule*) *Ang-inside-trans* :

assumes

$Ang-inside$ (An A B C) D Def (Ang (An A B C))
 $Line-on$ (Li B $A1$) A \neg $Bet-Point$ (Se A $A1$) B
 $Line-on$ (Li B $C1$) C \neg $Bet-Point$ (Se C $C1$) B
 \neg Eq ($Geos$ (Poi B) add Emp) ($Geos$ (Poi $A1$) add Emp)
 \neg Eq ($Geos$ (Poi B) add Emp) ($Geos$ (Poi $C1$) add Emp)

shows $Ang-inside$ (An $A1$ B $C1$) D

proof –

from $assms$ **have** $P1$: $Plane-sameside$ (Li B A) C D \wedge $Plane-sameside$ (Li B C) A D **by** ($simp$ $add:Ang-inside-def$)

have $P2$: $Line-on$ (Li B A) B **by** ($simp$ $add:Line-on-rule$)

have $P3$: $Line-on$ (Li B $A1$) B **by** ($simp$ $add:Line-on-rule$)

have $P4$: $Line-on$ (Li B A) A **by** ($simp$ $add:Line-on-rule$)

from $assms$ **have** $P5$: Def (Tri (Tr A B C)) **by** ($simp$ $add:Ang-to-Tri$)

from $P5$ **have** $P6$: \neg Eq ($Geos$ (Poi A) add Emp) ($Geos$ (Poi B) add Emp) **by** ($simp$ $add:Tri-def$)

from $assms$ $P2$ $P3$ $P4$ $P6$ **have** $P7$: Eq ($Geos$ (Lin (Li B A)) add Emp) ($Geos$ (Lin (Li B $A1$)) add Emp) **by** ($simp$ $add:Line-unique$)

have $P8$: $Line-on$ (Li B C) B **by** ($simp$ $add:Line-on-rule$)

have $P9$: $Line-on$ (Li B $C1$) B **by** ($simp$ $add:Line-on-rule$)

have $P10$: $Line-on$ (Li B C) C **by** ($simp$ $add:Line-on-rule$)

from $P5$ **have** $P11$: \neg Eq ($Geos$ (Poi B) add Emp) ($Geos$ (Poi C) add Emp) **by** ($simp$ $add:Tri-def$)

from $assms$ $P8$ $P9$ $P10$ $P11$ **have** $P12$: Eq ($Geos$ (Lin (Li B C)) add Emp) ($Geos$ (Lin (Li B $C1$)) add Emp) **by** ($simp$ $add:Line-unique$)

have $P13$: $Plane-diffside$ (Li B A) $C1$ D \implies $Plane-diffside$ (Li B A) D $C1$ **by** ($simp$ $add:Plane-diffside-rev$)

from $P1$ **have** $P14$: $Plane-sameside$ (Li B A) D C **by** ($simp$ $add:Plane-sameside-rev$)

from $P13$ $P14$ **have** $P15$: $Plane-diffside$ (Li B A) $C1$ D \implies $Plane-diffside$ (Li B A) C $C1$ **by** ($simp$ $add:Plane-trans$)

then **have** $Plane-diffside$ (Li B A) $C1$ D \implies $\exists p.$ $Bet-Point$ (Se C $C1$) p
 \wedge $Line-on$ (Li B A) p \wedge \neg $Line-on$ (Li B A) C \wedge \neg $Line-on$ (Li B A) $C1$ **by** ($simp$ $add:Plane-diffside-def$)

then **obtain** $B1$:: $Point$ **where** $P16$: $Plane-diffside$ (Li B A) $C1$ D \implies $Bet-Point$ (Se C $C1$) $B1$

\wedge $Line-on$ (Li B A) $B1$ \wedge \neg $Line-on$ (Li B A) C \wedge \neg $Line-on$ (Li B A) $C1$ **by** $blast$

from $P16$ **have** $P17$: $Plane-diffside$ (Li B A) $C1$ D \implies $Bet-Point$ (Se C $C1$) $B1$ **by** $simp$

then **have** $P18$: $Plane-diffside$ (Li B A) $C1$ D \implies $Line-on$ (Li C $C1$) $B1$ **by** ($simp$ $add:Line-Bet-on$)

have $P19$: $Line-on$ (Li B $C1$) $C1$ **by** ($simp$ $add:Line-on-rule$)

have $P20$: $Line-on$ (Li C $C1$) C **by** ($simp$ $add:Line-on-rule$)

have $P21$: $Line-on$ (Li C $C1$) $C1$ **by** ($simp$ $add:Line-on-rule$)

from $assms$ $P19$ $P20$ $P21$ **have** $P22$: \neg Eq ($Geos$ (Poi C) add Emp) ($Geos$ (Poi

$C1) \text{ add Emp} \implies$
Plane-diffside $(Li\ B\ A)\ C1\ D \implies \text{Eq}(\text{Geos}(\text{Lin}(Li\ B\ C1)) \text{ add Emp})(\text{Geos}(\text{Lin}(Li\ C\ C1)) \text{ add Emp})$ **by** *(simp add:Line-unique)*
from $P9\ P22$ **have** $P23 : \neg \text{Eq}(\text{Geos}(Poi\ C) \text{ add Emp})(\text{Geos}(Poi\ C1) \text{ add Emp}) \implies$
Plane-diffside $(Li\ B\ A)\ C1\ D \implies \text{Line-on}(Li\ C\ C1)\ B$ **by** *(simp add:Line-on-trans)*
from $P21$ **have** $P24 : \text{Eq}(\text{Geos}(\text{Lin}(Li\ C\ C1)) \text{ add Emp})(\text{Geos}(\text{Lin}(Li\ B\ A)) \text{ add Emp}) \implies \text{Line-on}(Li\ B\ A)\ C1$ **by** *(simp add:Line-on-trans)*
from $P16\ P24$ **have** $P25 : \text{Plane-diffside}(Li\ B\ A)\ C1\ D \implies$
 $\neg \text{Eq}(\text{Geos}(\text{Lin}(Li\ C\ C1)) \text{ add Emp})(\text{Geos}(\text{Lin}(Li\ B\ A)) \text{ add Emp})$ **by** *blast*
from $P2\ P16\ P18\ P23\ P25$ **have** $P26 : \neg \text{Eq}(\text{Geos}(Poi\ C) \text{ add Emp})(\text{Geos}(Poi\ C1) \text{ add Emp}) \implies$
Plane-diffside $(Li\ B\ A)\ C1\ D \implies \text{Eq}(\text{Geos}(Poi\ B1) \text{ add Emp})(\text{Geos}(Poi\ B) \text{ add Emp})$ **by** *(simp add:Line-unique-Point)*
from $P17\ P26$ **have** $P27 : \neg \text{Eq}(\text{Geos}(Poi\ C) \text{ add Emp})(\text{Geos}(Poi\ C1) \text{ add Emp}) \implies$
Plane-diffside $(Li\ B\ A)\ C1\ D \implies \text{Bet-Point}(Se\ C\ C1)\ B$ **by** *(simp add:Point-Eq)*
from *assms* $P27$ **have** $P28 : \neg \text{Eq}(\text{Geos}(Poi\ C) \text{ add Emp})(\text{Geos}(Poi\ C1) \text{ add Emp}) \implies$
 $\neg \text{Plane-diffside}(Li\ B\ A)\ C1\ D$ **by** *blast*
from *assms* $P2\ P9\ P19$ **have** $P29 : \text{Line-on}(Li\ B\ A)\ C1 \implies$
 $\text{Eq}(\text{Geos}(\text{Lin}(Li\ B\ C1)) \text{ add Emp})(\text{Geos}(\text{Lin}(Li\ B\ A)) \text{ add Emp})$ **by** *(simp add:Line-unique)*
from $P12\ P29$ **have** $P30 : \text{Line-on}(Li\ B\ A)\ C1 \implies$
 $\text{Eq}(\text{Geos}(\text{Lin}(Li\ B\ C)) \text{ add Emp})(\text{Geos}(\text{Lin}(Li\ B\ A)) \text{ add Emp})$ **by** *(blast intro:Eq-trans)*
from $P10\ P30$ **have** $P31 : \text{Line-on}(Li\ B\ A)\ C1 \implies \text{Line-on}(Li\ B\ A)\ C$ **by** *(simp add:Line-on-trans)*
from $P1$ **have** $P32 : \neg \text{Line-on}(Li\ B\ A)\ C$ **by** *(simp add:Plane-sameside-def)*
from $P31\ P32$ **have** $P33 : \neg \text{Line-on}(Li\ B\ A)\ C1$ **by** *blast*
from $P1$ **have** $P34 : \neg \text{Line-on}(Li\ B\ A)\ D$ **by** *(simp add:Plane-sameside-def)*
from $P12\ P19$ **have** $\text{Line-on}(Li\ B\ C)\ C1$ **by** *(blast intro:Line-on-trans Eq-rev)*
then **have** $P35 : \text{Eq}(\text{Geos}(Poi\ C1) \text{ add Emp})(\text{Geos}(Poi\ D) \text{ add Emp}) \implies$
 $\text{Line-on}(Li\ B\ C)\ D$ **by** *(simp add:Point-Eq)*
from $P1$ **have** $P36 : \neg \text{Line-on}(Li\ B\ C)\ D$ **by** *(simp add:Plane-sameside-def)*
from $P35\ P36$ **have** $P37 : \neg \text{Eq}(\text{Geos}(Poi\ C1) \text{ add Emp})(\text{Geos}(Poi\ D) \text{ add Emp})$ **by** *blast*
from $P28\ P33\ P34\ P37$ **have** $P38 : \neg \text{Eq}(\text{Geos}(Poi\ C) \text{ add Emp})(\text{Geos}(Poi\ C1) \text{ add Emp}) \implies$
Plane-sameside $(Li\ B\ A)\ C1\ D$ **by** *(simp add:Plane-not-diffside-sameside)*
from $P14$ **have** $\text{Eq}(\text{Geos}(Poi\ C) \text{ add Emp})(\text{Geos}(Poi\ C1) \text{ add Emp}) \implies$
Plane-sameside $(Li\ B\ A)\ D\ C1$ **by** *(simp add:Point-Eq)*
then **have** $P39 : \text{Eq}(\text{Geos}(Poi\ C) \text{ add Emp})(\text{Geos}(Poi\ C1) \text{ add Emp}) \implies$
Plane-sameside $(Li\ B\ A)\ C1\ D$ **by** *(simp add:Plane-sameside-rev)*
from $P38\ P39$ **have** $P40 : \text{Plane-sameside}(Li\ B\ A)\ C1\ D$ **by** *blast*
from $P7\ P40$ **have** $P41 : \text{Plane-sameside}(Li\ B\ A1)\ C1\ D$ **by** *(simp add:Plane-Line-trans)*
have $P42 : \text{Plane-diffside}(Li\ B\ C)\ A1\ D \implies \text{Plane-diffside}(Li\ B\ C)\ D\ A1$ **by** *(simp add:Plane-diffside-rev)*
from $P1$ **have** $P43 : \text{Plane-sameside}(Li\ B\ C)\ D\ A$ **by** *(simp add:Plane-sameside-rev)*

from $P_{42} P_{43}$ **have** $P_{44} : \text{Plane-diffside } (Li\ B\ C)\ A1\ D \implies \text{Plane-diffside } (Li\ B\ C)\ A\ A1$ **by** (*simp add:Plane-trans*)
then have $\text{Plane-diffside } (Li\ B\ C)\ A1\ D \implies \exists p. \text{Bet-Point } (Se\ A\ A1)\ p$
 $\wedge \text{Line-on } (Li\ B\ C)\ p \wedge \neg \text{Line-on } (Li\ B\ C)\ A \wedge \neg \text{Line-on } (Li\ B\ C)\ A1$ **by**
(*simp add:Plane-diffside-def*)
then obtain $B2 :: \text{Point}$ **where** $P_{45} : \text{Plane-diffside } (Li\ B\ C)\ A1\ D \implies$
 $\text{Bet-Point } (Se\ A\ A1)\ B2$
 $\wedge \text{Line-on } (Li\ B\ C)\ B2 \wedge \neg \text{Line-on } (Li\ B\ C)\ A \wedge \neg \text{Line-on } (Li\ B\ C)\ A1$ **by**
blast
from P_{45} **have** $P_{46} : \text{Plane-diffside } (Li\ B\ C)\ A1\ D \implies \text{Bet-Point } (Se\ A\ A1)$
 $B2$ **by** *simp*
then have $P_{47} : \text{Plane-diffside } (Li\ B\ C)\ A1\ D \implies \text{Line-on } (Li\ A\ A1)\ B2$ **by**
(*simp add:Line-Bet-on*)
have $P_{48} : \text{Line-on } (Li\ B\ A1)\ A1$ **by** (*simp add:Line-on-rule*)
have $P_{49} : \text{Line-on } (Li\ A\ A1)\ A$ **by** (*simp add:Line-on-rule*)
have $P_{50} : \text{Line-on } (Li\ A\ A1)\ A1$ **by** (*simp add:Line-on-rule*)
from *assms* $P_{48} P_{49} P_{50}$ **have** $P_{51} : \neg \text{Eq } (\text{Geos } (\text{Poi } A)\ \text{add } \text{Emp})\ (\text{Geos } (\text{Poi}$
 $A1)\ \text{add } \text{Emp}) \implies$
 $\text{Plane-diffside } (Li\ B\ C)\ A1\ D \implies \text{Eq } (\text{Geos } (\text{Lin } (Li\ B\ A1))\ \text{add } \text{Emp})\ (\text{Geos } (\text{Lin } (Li\ A\ A1))\ \text{add } \text{Emp})$ **by** (*simp add:Line-unique*)
from $P_3 P_{51}$ **have** $P_{52} : \neg \text{Eq } (\text{Geos } (\text{Poi } A)\ \text{add } \text{Emp})\ (\text{Geos } (\text{Poi } A1)\ \text{add } \text{Emp}) \implies$
 $\text{Plane-diffside } (Li\ B\ C)\ A1\ D \implies \text{Line-on } (Li\ A\ A1)\ B$ **by** (*simp add:Line-on-trans*)
from P_{50} **have** $P_{53} : \text{Eq } (\text{Geos } (\text{Lin } (Li\ A\ A1))\ \text{add } \text{Emp})\ (\text{Geos } (\text{Lin } (Li\ B\ C))\ \text{add } \text{Emp}) \implies$
 $\text{Line-on } (Li\ B\ C)\ A1$ **by** (*simp add:Line-on-trans*)
from $P_{45} P_{53}$ **have** $P_{54} : \text{Plane-diffside } (Li\ B\ C)\ A1\ D \implies$
 $\neg \text{Eq } (\text{Geos } (\text{Lin } (Li\ A\ A1))\ \text{add } \text{Emp})\ (\text{Geos } (\text{Lin } (Li\ B\ C))\ \text{add } \text{Emp})$ **by**
blast
from $P_8 P_{45} P_{47} P_{52} P_{54}$ **have** $P_{55} : \neg \text{Eq } (\text{Geos } (\text{Poi } A)\ \text{add } \text{Emp})\ (\text{Geos } (\text{Poi } A1)\ \text{add } \text{Emp}) \implies$
 $\text{Plane-diffside } (Li\ B\ C)\ A1\ D \implies \text{Eq } (\text{Geos } (\text{Poi } B2)\ \text{add } \text{Emp})\ (\text{Geos } (\text{Poi } B)\ \text{add } \text{Emp})$ **by** (*simp add:Line-unique-Point*)
from $P_{46} P_{55}$ **have** $P_{56} : \neg \text{Eq } (\text{Geos } (\text{Poi } A)\ \text{add } \text{Emp})\ (\text{Geos } (\text{Poi } A1)\ \text{add } \text{Emp}) \implies$
 $\text{Plane-diffside } (Li\ B\ C)\ A1\ D \implies \text{Bet-Point } (Se\ A\ A1)\ B$ **by** (*simp add:Point-Eq*)
from *assms* P_{56} **have** $P_{57} : \neg \text{Eq } (\text{Geos } (\text{Poi } A)\ \text{add } \text{Emp})\ (\text{Geos } (\text{Poi } A1)\ \text{add } \text{Emp}) \implies$
 $\neg \text{Plane-diffside } (Li\ B\ C)\ A1\ D$ **by** *blast*
from *assms* $P_3 P_8 P_{48}$ **have** $P_{58} : \text{Line-on } (Li\ B\ C)\ A1 \implies$
 $\text{Eq } (\text{Geos } (\text{Lin } (Li\ B\ A1))\ \text{add } \text{Emp})\ (\text{Geos } (\text{Lin } (Li\ B\ C))\ \text{add } \text{Emp})$ **by** (*simp add:Line-unique*)
from $P_7 P_{58}$ **have** $P_{59} : \text{Line-on } (Li\ B\ C)\ A1 \implies$
 $\text{Eq } (\text{Geos } (\text{Lin } (Li\ B\ C))\ \text{add } \text{Emp})\ (\text{Geos } (\text{Lin } (Li\ B\ A))\ \text{add } \text{Emp})$ **by** (*blast intro:Eq-trans*)
from $P_{10} P_{59}$ **have** $P_{60} : \text{Line-on } (Li\ B\ C)\ A1 \implies \text{Line-on } (Li\ B\ A)\ C$ **by**
(*simp add:Line-on-trans*)
from $P_{32} P_{60}$ **have** $P_{61} : \neg \text{Line-on } (Li\ B\ C)\ A1$ **by** *blast*
from $P_7 P_{48}$ **have** $\text{Line-on } (Li\ B\ A)\ A1$ **by** (*blast intro:Line-on-trans Eq-rev*)
then have $P_{62} : \text{Eq } (\text{Geos } (\text{Poi } A1)\ \text{add } \text{Emp})\ (\text{Geos } (\text{Poi } D)\ \text{add } \text{Emp}) \implies$

Line-on (*Li B A*) *D* **by** (*simp add:Point-Eq*)
from *P34 P62* **have** *P63* : $\neg \text{Eq} (\text{Geos} (\text{Poi } A1) \text{ add Emp}) (\text{Geos} (\text{Poi } D) \text{ add Emp})$ **by** *blast*
from *P36 P57 P61 P63* **have** *P64* : $\neg \text{Eq} (\text{Geos} (\text{Poi } A) \text{ add Emp}) (\text{Geos} (\text{Poi } A1) \text{ add Emp}) \implies$
Plane-sameside (*Li B C*) *A1 D* **by** (*simp add:Plane-not-diffside-sameside*)
from *P43* **have** $\text{Eq} (\text{Geos} (\text{Poi } A) \text{ add Emp}) (\text{Geos} (\text{Poi } A1) \text{ add Emp}) \implies$
Plane-sameside (*Li B C*) *D A1* **by** (*simp add:Point-Eq*)
then **have** *P65* : $\text{Eq} (\text{Geos} (\text{Poi } A) \text{ add Emp}) (\text{Geos} (\text{Poi } A1) \text{ add Emp}) \implies$
Plane-sameside (*Li B C*) *A1 D* **by** (*simp add:Plane-sameside-rev*)
from *P64 P65* **have** *P66* : *Plane-sameside* (*Li B C*) *A1 D* **by** *blast*
from *P12 P66* **have** *P67* : *Plane-sameside* (*Li B C1*) *A1 D* **by** (*simp add:Plane-Line-trans*)
from *P12* **have** *P68* : *Line-on* (*Li B C1*) *A1* \implies *Line-on* (*Li B C*) *A1* **by** (*blast intro:Line-on-trans Eq-rev*)
from *P61 P68* **have** *P69* : $\neg \text{Line-on} (\text{Li B C1}) \text{ A1}$ **by** *blast*
from *assms P69* **have** *Def* (*Ang* (*An B C1 A1*)) **by** (*simp add:Ang-simple-def*)
then **have** *P70* : *Def* (*Ang* (*An A1 B C1*)) **by** (*blast intro:Ang-def-rev Ang-def-inv*)
from *P41 P67 P70* **show** *Ang-inside* (*An A1 B C1*) *D* **by** (*simp add:Ang-inside-def*)
qed

lemma (**in** *Congruence-Rule*) *Ang-sub-lemma1* :

assumes

Plane-sameside (*Li o1 l1*) *h1 k1*
 $\neg \text{Eq} (\text{Geos} (\text{Poi } o1) \text{ add Emp}) (\text{Geos} (\text{Poi } l1) \text{ add Emp})$
Plane-sameside (*Li o2 l2*) *h2 k2*
 $\neg \text{Eq} (\text{Geos} (\text{Poi } o2) \text{ add Emp}) (\text{Geos} (\text{Poi } l2) \text{ add Emp})$
Cong (*Geos* (*Ang* (*An h1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 l2*)) *add Emp*)
Cong (*Geos* (*Ang* (*An k1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An k2 o2 l2*)) *add Emp*)
 $\neg \text{Eq} (\text{Geos} (\text{Lin} (\text{Li } o1 \text{ h1})) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li } o1 \text{ k1})) \text{ add Emp})$
 $\neg \text{Eq} (\text{Geos} (\text{Lin} (\text{Li } o2 \text{ h2})) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li } o2 \text{ k2})) \text{ add Emp})$
Ang-inside (*An k1 o1 l1*) *h1*
shows
Cong (*Geos* (*Ang* (*An h1 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 k2*)) *add Emp*)
Ang-inside (*An k2 o2 l2*) *h2*

proof –

from *assms* **have** *P1* : $\neg \text{Line-on} (\text{Li } o1 \text{ l1}) \text{ h1}$ **by** (*simp add:Plane-sameside-def*)
from *assms P1* **have** *Def* (*Ang* (*An o1 l1 h1*)) **by** (*simp add:Ang-simple-def*)
then **have** *P2* : *Def* (*Ang* (*An h1 o1 l1*)) **by** (*blast intro:Ang-def-rev Ang-def-inv*)
from *assms* **have** *P3* : $\neg \text{Line-on} (\text{Li } o1 \text{ l1}) \text{ k1}$ **by** (*simp add:Plane-sameside-def*)
from *assms P3* **have** *Def* (*Ang* (*An o1 l1 k1*)) **by** (*simp add:Ang-simple-def*)
then **have** *P4* : *Def* (*Ang* (*An k1 o1 l1*)) **by** (*blast intro:Ang-def-rev Ang-def-inv*)
from *assms* **have** *P5* : $\neg \text{Line-on} (\text{Li } o2 \text{ l2}) \text{ h2}$ **by** (*simp add:Plane-sameside-def*)
from *assms P5* **have** *Def* (*Ang* (*An o2 l2 h2*)) **by** (*simp add:Ang-simple-def*)
then **have** *P6* : *Def* (*Ang* (*An h2 o2 l2*)) **by** (*blast intro:Ang-def-rev Ang-def-inv*)
from *assms* **have** *P7* : $\neg \text{Line-on} (\text{Li } o2 \text{ l2}) \text{ k2}$ **by** (*simp add:Plane-sameside-def*)
from *assms P7* **have** *Def* (*Ang* (*An o2 l2 k2*)) **by** (*simp add:Ang-simple-def*)
then **have** *P8* : *Def* (*Ang* (*An k2 o2 l2*)) **by** (*blast intro:Ang-def-rev Ang-def-inv*)

from *assms P4 P8* **have** $\exists p q.$ *Cong* (*Geos* (*Ang* (*An* *k1* *o1* *l1*)) *add Emp*) (*Geos* (*Ang* (*An* *p* *o2* *q*)) *add Emp*)
 \wedge *Eq* (*Geos* (*Ang* (*An* *k2* *o2* *l2*)) *add Emp*) (*Geos* (*Ang* (*An* *p* *o2* *q*)) *add Emp*)
 \wedge *Eq* (*Geos* (*Seg* (*Se* *o1* *k1*)) *add Emp*) (*Geos* (*Seg* (*Se* *o2* *p*)) *add Emp*) \wedge
Line-on (*Li* *o2* *k2*) *p* \wedge \neg *Bet-Point* (*Se* *p* *k2*) *o2*
 \wedge *Eq* (*Geos* (*Seg* (*Se* *o1* *l1*)) *add Emp*) (*Geos* (*Seg* (*Se* *o2* *q*)) *add Emp*) \wedge
Line-on (*Li* *o2* *l2*) *q* \wedge \neg *Bet-Point* (*Se* *q* *l2*) *o2* \wedge *Def* (*Ang* (*An* *p* *o2* *q*)) **by** (*simp*
add:Ang-replace)
then obtain *k21 l21* :: *Point* **where** *P9* : *Cong* (*Geos* (*Ang* (*An* *k1* *o1* *l1*)) *add*
Emp) (*Geos* (*Ang* (*An* *k21* *o2* *l21*)) *add Emp*)
 \wedge *Eq* (*Geos* (*Ang* (*An* *k2* *o2* *l2*)) *add Emp*) (*Geos* (*Ang* (*An* *k21* *o2* *l21*)) *add*
Emp)
 \wedge *Eq* (*Geos* (*Seg* (*Se* *o1* *k1*)) *add Emp*) (*Geos* (*Seg* (*Se* *o2* *k21*)) *add Emp*) \wedge
Line-on (*Li* *o2* *k2*) *k21* \wedge \neg *Bet-Point* (*Se* *k21* *k2*) *o2*
 \wedge *Eq* (*Geos* (*Seg* (*Se* *o1* *l1*)) *add Emp*) (*Geos* (*Seg* (*Se* *o2* *l21*)) *add Emp*) \wedge
Line-on (*Li* *o2* *l2*) *l21* \wedge \neg *Bet-Point* (*Se* *l21* *l2*) *o2* \wedge *Def* (*Ang* (*An* *k21* *o2* *l21*))
by *blast*
from *assms* **have** *Plane-diffside* (*Li* *o1* *h1*) *k1* *l1* **by** (*simp* *add:Ang-inside-Planeside*)
then have $\exists p.$ *Bet-Point* (*Se* *k1* *l1*) *p* \wedge *Line-on* (*Li* *o1* *h1*) *p* \wedge \neg *Line-on* (*Li*
o1 *h1*) *k1* \wedge \neg *Line-on* (*Li* *o1* *h1*) *l1* **by** (*simp* *add:Plane-diffside-def*)
then obtain *h11* :: *Point* **where** *P10* : *Bet-Point* (*Se* *k1* *l1*) *h11* \wedge *Line-on* (*Li*
o1 *h1*) *h11* \wedge \neg *Line-on* (*Li* *o1* *h1*) *k1* \wedge \neg *Line-on* (*Li* *o1* *h1*) *l1* **by** *blast*
then have *Eq* (*Geos* (*Poi* *h11*) *add Emp*) (*Geos* (*Poi* *o1*) *add Emp*) \implies
Bet-Point (*Se* *k1* *l1*) *o1* **by** (*blast* *intro:Point-Eq*)
then have *P11* : *Eq* (*Geos* (*Poi* *h11*) *add Emp*) (*Geos* (*Poi* *o1*) *add Emp*) \implies
Line-on (*Li* *o1* *l1*) *k1* **by** (*simp* *add:Line-Bet-on*)
from *P3* *P11* **have** *P12* : \neg *Eq* (*Geos* (*Poi* *o1*) *add Emp*) (*Geos* (*Poi* *h11*) *add*
Emp) **by** (*blast* *intro:Eq-rev*)
have *P13* : *Line-on* (*Li* *o2* *h2*) *o2* **by** (*simp* *add:Line-on-rule*)
have *P14* : *Line-on* (*Li* *o2* *h2*) *h2* **by** (*simp* *add:Line-on-rule*)
from *P6* **have** \neg *Eq* (*Geos* (*Poi* *h2*) *add Emp*) (*Geos* (*Poi* *o2*) *add Emp*) **by**
(*simp* *add:Ang-def*)
then have *P15* : \neg *Eq* (*Geos* (*Poi* *o2*) *add Emp*) (*Geos* (*Poi* *h2*) *add Emp*) **by**
(*blast* *intro:Eq-rev*)
from *P12* *P13* *P14* *P15* **have** $\exists p.$ *Eq* (*Geos* (*Seg* (*Se* *o1* *h11*)) *add Emp*) (*Geos*
(*Seg* (*Se* *o2* *p*)) *add Emp*)
 \wedge \neg *Bet-Point* (*Se* *p* *h2*) *o2* \wedge *Line-on* (*Li* *o2* *h2*) *p* \wedge \neg *Eq* (*Geos* (*Poi* *o2*) *add*
Emp) (*Geos* (*Poi* *p*) *add Emp*) **by** (*simp* *add:Seg-move-sameside*)
then obtain *h21* :: *Point* **where** *P16* : *Eq* (*Geos* (*Seg* (*Se* *o1* *h11*)) *add Emp*)
(*Geos* (*Seg* (*Se* *o2* *h21*)) *add Emp*)
 \wedge \neg *Bet-Point* (*Se* *h21* *h2*) *o2* \wedge *Line-on* (*Li* *o2* *h2*) *h21* \wedge \neg *Eq* (*Geos* (*Poi*
o2) *add Emp*) (*Geos* (*Poi* *h21*) *add Emp*) **by** *blast*
have *P17* : *Line-on* (*Li* *o1* *l1*) *o1* **by** (*simp* *add:Line-on-rule*)
have *Line-on* (*Li* *h1* *h11*) *h1* **by** (*simp* *add:Line-on-rule*)
then have *P18* : *Eq* (*Geos* (*Lin* (*Li* *h1* *h11*)) *add Emp*) (*Geos* (*Lin* (*Li* *o1* *l1*))
add Emp) \implies *Line-on* (*Li* *o1* *l1*) *h1* **by** (*simp* *add:Line-on-trans*)
from *P1* *P18* **have** *P19* : \neg *Eq* (*Geos* (*Lin* (*Li* *h1* *h11*)) *add Emp*) (*Geos* (*Lin*
(*Li* *o1* *l1*)) *add Emp*) **by** *blast*
from *P17* *P19* **have** *Bet-Point* (*Se* *h1* *h11*) *o1* \implies *Plane-diffside* (*Li* *o1* *l1*) *h1*

$h11$ **by** (*simp add:Plane-Bet-diffside*)
then have $P20 : \text{Bet-Point (Se } h1 \ h11) \ o1 \implies \text{Plane-diffside (Li } o1 \ l1) \ h11 \ h1}$
by (*simp add:Plane-diffside-rev*)
from $P10$ **have** $P21 : \text{Bet-Point (Se } l1 \ k1) \ h11$ **by** (*simp add:Bet-rev*)
have $P22 : \text{Line-on (Li } o1 \ l1) \ l1$ **by** (*simp add:Line-on-rule*)
have $\text{Line-on (Li } l1 \ k1) \ k1$ **by** (*simp add:Line-on-rule*)
then have $P23 : \text{Eq (Geos (Lin (Li } l1 \ k1)) \ \text{add Emp) (Geos (Lin (Li } o1 \ l1)) \ \text{add Emp)} \implies \text{Line-on (Li } o1 \ l1) \ k1$ **by** (*simp add:Line-on-trans*)
from $P3 \ P23$ **have** $P24 : \neg \text{Eq (Geos (Lin (Li } l1 \ k1)) \ \text{add Emp) (Geos (Lin (Li } o1 \ l1)) \ \text{add Emp)}$ **by blast**
from $P21 \ P22 \ P24$ **have** $P25 : \text{Plane-sameside (Li } o1 \ l1) \ h11 \ k1$ **by** (*simp add:Plane-Bet-sameside*)
from $P20 \ P25$ **have** $\text{Bet-Point (Se } h1 \ h11) \ o1 \implies \text{Plane-diffside (Li } o1 \ l1) \ h1 \ k1$ **by** (*simp add:Plane-trans Plane-diffside-rev*)
then have $P26 : \text{Bet-Point (Se } h1 \ h11) \ o1 \implies \neg \text{Plane-sameside (Li } o1 \ l1) \ h1 \ k1$ **by** (*simp add:Plane-diffside-not-sameside*)
from *assms* $P26$ **have** $P27 : \neg \text{Bet-Point (Se } h1 \ h11) \ o1$ **by blast**
have $P28 : \neg \text{Bet-Point (Se } l1 \ l1) \ o1$ **by** (*simp add:Bet-end-Point*)
from *assms* $P2 \ P10 \ P22 \ P27 \ P28 \ P12$ **have** $P29 : \text{Eq (Geos (Ang (An } h1 \ o1 \ l1)) \ \text{add Emp) (Geos (Ang (An } h11 \ o1 \ l1)) \ \text{add Emp})} \wedge \text{Def (Ang (An } h11 \ o1 \ l1))$ **by** (*simp add:Ang-Point-swap*)
from $P9$ **have** $P30 : \neg \text{Eq (Geos (Poi } o2) \ \text{add Emp) (Geos (Poi } l21) \ \text{add Emp)}$ **by** (*simp add:Ang-def*)
from $P16$ **have** $P31 : \neg \text{Bet-Point (Se } h2 \ h21) \ o2$ **by** (*blast intro:Bet-rev*)
from $P9$ **have** $P32 : \neg \text{Bet-Point (Se } l2 \ l21) \ o2$ **by** (*blast intro:Bet-rev*)
from $P6 \ P9 \ P16 \ P30 \ P31 \ P32$ **have** $P33 : \text{Eq (Geos (Ang (An } h2 \ o2 \ l2)) \ \text{add Emp) (Geos (Ang (An } h21 \ o2 \ l21)) \ \text{add Emp})} \wedge \text{Def (Ang (An } h21 \ o2 \ l21))$ **by** (*simp add:Ang-Point-swap*)
from *assms* $P29$ **have** $P34 : \text{Cong (Geos (Ang (An } h11 \ o1 \ l1)) \ \text{add Emp) (Geos (Ang (An } h2 \ o2 \ l2)) \ \text{add Emp)}$ **by** (*blast intro:Ang-weektrans Ang-rev Eq-rev*)
from $P33 \ P34$ **have** $P35 : \text{Cong (Geos (Ang (An } h11 \ o1 \ l1)) \ \text{add Emp) (Geos (Ang (An } h21 \ o2 \ l21)) \ \text{add Emp)}$ **by** (*blast intro:Ang-weektrans Ang-rev Eq-rev*)
from $P29$ **have** $\text{Def (Tri (Tr } h11 \ o1 \ l1))$ **by** (*simp add:Ang-to-Tri*)
then have $P36 : \text{Def (Tri (Tr } o1 \ h11 \ l1))$ **by** (*blast intro:Tri-def-rev Tri-def-trans*)
from $P33$ **have** $\text{Def (Tri (Tr } h21 \ o2 \ l21))$ **by** (*simp add:Ang-to-Tri*)
then have $P37 : \text{Def (Tri (Tr } o2 \ h21 \ l21))$ **by** (*blast intro:Tri-def-rev Tri-def-trans*)
from $P9 \ P16 \ P35 \ P36 \ P37$ **have** $P38 : \text{Cong (Geos (Tri (Tr } o1 \ h11 \ l1)) \ \text{add Emp) (Geos (Tri (Tr } o2 \ h21 \ l21)) \ \text{add Emp)}$ **by** (*simp add:Tri-SAS*)
then have $P39 : \text{Cong (Geos (Ang (An } o1 \ l1 \ h11)) \ \text{add Emp) (Geos (Ang (An } o2 \ l21 \ h21)) \ \text{add Emp)}$ **by** (*simp add:Tri-Cong-def*)
from $P4$ **have** $P40 : \text{Def (Tri (Tr } o1 \ k1 \ l1))$ **by** (*simp add:Ang-to-Tri Tri-def-rev Tri-def-trans*)
from $P9$ **have** $P41 : \text{Def (Tri (Tr } o2 \ k21 \ l21))$ **by** (*simp add:Ang-to-Tri Tri-def-rev Tri-def-trans*)
from $P9 \ P40 \ P41$ **have** $P42 : \text{Cong (Geos (Tri (Tr } o1 \ k1 \ l1)) \ \text{add Emp) (Geos (Tri (Tr } o2 \ k21 \ l21)) \ \text{add Emp)}$ **by** (*simp add:Tri-SAS*)
then have $P43 : \text{Cong (Geos (Ang (An } o1 \ l1 \ k1)) \ \text{add Emp) (Geos (Ang (An } o2 \ l21 \ k21)) \ \text{add Emp)}$ **by** (*simp add:Tri-Cong-def*)
from $P4$ **have** $P44 : \text{Def (Ang (An } o1 \ l1 \ k1))$ **by** (*blast intro:Ang-def-rev*)

Ang-def-inv)
have $P_{45} : \text{Line-on } (Li\ l1\ o1)\ o1$ **by** (*simp add:Line-on-rule*)
have $P_{46} : \neg \text{Bet-Point } (Se\ o1\ o1)\ l1$ **by** (*simp add:Bet-end-Point*)
from P_{10} **have** $P_{47} : \text{Line-on } (Li\ l1\ k1)\ h11$ **by** (*simp add:Line-Bet-on*)
from P_{10} **have** $\text{Inv}(\text{Bet-Point } (Se\ h11\ k1)\ l1)$ **by** (*simp add:Bet-iff*)
then have $\neg \text{Bet-Point } (Se\ h11\ k1)\ l1$ **by** (*simp add:Inv-def*)
then have $P_{48} : \neg \text{Bet-Point } (Se\ k1\ h11)\ l1$ **by** (*blast intro:Bet-rev*)
from *assms* **have** $P_{49} : \neg \text{Eq } (\text{Geos } (Poi\ l1)\ \text{add } \text{Emp})\ (\text{Geos } (Poi\ o1)\ \text{add } \text{Emp})$
by (*blast intro:Eq-rev*)
from P_{10} **have** $\text{Bet-Point } (Se\ k1\ l1)\ h11$ **by** *simp*
then have $P_{50} : \neg \text{Eq } (\text{Geos } (Poi\ l1)\ \text{add } \text{Emp})\ (\text{Geos } (Poi\ h11)\ \text{add } \text{Emp})$ **by**
(*simp add:Bet-Point-def*)
from $P_{44}\ P_{45}\ P_{46}\ P_{47}\ P_{48}\ P_{49}\ P_{50}$ **have** $P_{51} : \text{Eq } (\text{Geos } (\text{Ang } (An\ o1\ l1\ k1))\ \text{add } \text{Emp})\ (\text{Geos } (\text{Ang } (An\ o1\ l1\ h11))\ \text{add } \text{Emp}) \wedge \text{Def } (\text{Ang } (An\ o1\ l1\ h11))$ **by**
(*simp add:Ang-Point-swap*)
have $P_{52} : \text{Eq } (\text{Geos } (\text{Ang } (An\ o2\ l21\ k21))\ \text{add } \text{Emp})\ (\text{Geos } (\text{Ang } (An\ k21\ l21\ o2))\ \text{add } \text{Emp})$ **by** (*simp add:Ang-roll*)
from $P_{43}\ P_{52}$ **have** $P_{53} : \text{Cong } (\text{Geos } (\text{Ang } (An\ o1\ l1\ k1))\ \text{add } \text{Emp})\ (\text{Geos } (\text{Ang } (An\ k21\ l21\ o2))\ \text{add } \text{Emp})$ **by** (*blast intro:Ang-weektrans Ang-rev Eq-rev*)
from $P_{51}\ P_{53}$ **have** $P_{54} : \text{Cong } (\text{Geos } (\text{Ang } (An\ o1\ l1\ h11))\ \text{add } \text{Emp})\ (\text{Geos } (\text{Ang } (An\ k21\ l21\ o2))\ \text{add } \text{Emp})$ **by** (*blast intro:Ang-weektrans Ang-rev Eq-rev*)
have $P_{55} : \text{Eq } (\text{Geos } (\text{Ang } (An\ o2\ l21\ h21))\ \text{add } \text{Emp})\ (\text{Geos } (\text{Ang } (An\ h21\ l21\ o2))\ \text{add } \text{Emp})$ **by** (*simp add:Ang-roll*)
from $P_{39}\ P_{55}$ **have** $P_{56} : \text{Cong } (\text{Geos } (\text{Ang } (An\ o1\ l1\ h11))\ \text{add } \text{Emp})\ (\text{Geos } (\text{Ang } (An\ h21\ l21\ o2))\ \text{add } \text{Emp})$ **by** (*blast intro:Ang-weektrans Ang-rev Eq-rev*)
have $P_{57} : \text{Line-on } (Li\ o2\ l2)\ o2$ **by** (*simp add:Line-on-rule*)
have $P_{58} : \text{Line-on } (Li\ l21\ o2)\ o2$ **by** (*simp add:Line-on-rule*)
have $P_{59} : \text{Line-on } (Li\ l21\ o2)\ l21$ **by** (*simp add:Line-on-rule*)
from P_9 **have** $P_{60} : \text{Line-on } (Li\ o2\ l2)\ l21$ **by** *simp*
from $P_{30}\ P_{57}\ P_{58}\ P_{59}\ P_{60}$ **have** $P_{61} : \text{Eq } (\text{Geos } (\text{Lin } (Li\ o2\ l2))\ \text{add } \text{Emp})\ (\text{Geos } (\text{Lin } (Li\ l21\ o2))\ \text{add } \text{Emp})$ **by** (*simp add:Line-unique*)
from *assms* $P_{16}\ P_{57}$ **have** $P_{62} : \neg \text{Eq } (\text{Geos } (Poi\ h2)\ \text{add } \text{Emp})\ (\text{Geos } (Poi\ h21)\ \text{add } \text{Emp}) \implies$
Plane-sameside $(Li\ o2\ l2)\ h21\ h2$ **by** (*blast intro:Plane-sameside-HalfLine Plane-sameside-rev*)
from *assms* **have** $P_{63} : \text{Plane-sameside } (Li\ o2\ l2)\ k2\ h2$ **by** (*simp add:Plane-sameside-rev*)
from P_{41} **have** $P_{64} : \neg \text{Eq } (\text{Geos } (Poi\ o2)\ \text{add } \text{Emp})\ (\text{Geos } (Poi\ k21)\ \text{add } \text{Emp})$
by (*simp add:Tri-def*)
from $P_9\ P_{57}\ P_{63}\ P_{64}$ **have** $P_{65} : \neg \text{Eq } (\text{Geos } (Poi\ k2)\ \text{add } \text{Emp})\ (\text{Geos } (Poi\ k21)\ \text{add } \text{Emp}) \implies$
Plane-sameside $(Li\ o2\ l2)\ k2\ k21$ **by** (*simp add:Plane-sameside-HalfLine*)
then have $P_{66} : \neg \text{Eq } (\text{Geos } (Poi\ k2)\ \text{add } \text{Emp})\ (\text{Geos } (Poi\ k21)\ \text{add } \text{Emp}) \implies$
Plane-sameside $(Li\ o2\ l2)\ k21\ k2$ **by** (*simp add:Plane-sameside-rev*)
from *assms* P_{62} **have** $P_{67} : \neg \text{Eq } (\text{Geos } (Poi\ h2)\ \text{add } \text{Emp})\ (\text{Geos } (Poi\ h21)\ \text{add } \text{Emp}) \implies$
 $\neg \text{Eq } (\text{Geos } (Poi\ k2)\ \text{add } \text{Emp})\ (\text{Geos } (Poi\ h21)\ \text{add } \text{Emp}) \implies$
Plane-sameside $(Li\ o2\ l2)\ h21\ k2$ **by** (*blast intro:Plane-sameside-trans*)
have $P_{68} : \text{Line-on } (Li\ o2\ k2)\ o2$ **by** (*simp add:Line-on-rule*)

from $P9$ **have** $P69 : Eq (Geos (Poi k21) add Emp) (Geos (Poi h21) add Emp)$
 \implies *Line-on* ($Li\ o2\ k2$) $h21$ **by** (*blast intro:Point-Eq*)
from $P9\ P13\ P16\ P68\ P69$ **have** $P70 : Eq (Geos (Poi k21) add Emp) (Geos (Poi h21) add Emp) \implies$
 $Eq (Geos (Lin (Li\ o2\ h2)) add Emp) (Geos (Lin (Li\ o2\ k2)) add Emp)$ **by** (*blast intro:Line-unique*)
from *assms* $P70$ **have** $P71 : \neg Eq (Geos (Poi k21) add Emp) (Geos (Poi h21) add Emp)$ **by** *blast*
from $P65\ P67\ P71$ **have** $P72 : \neg Eq (Geos (Poi h2) add Emp) (Geos (Poi h21) add Emp) \implies$
 $\neg Eq (Geos (Poi k2) add Emp) (Geos (Poi k21) add Emp) \implies$
 $\neg Eq (Geos (Poi k2) add Emp) (Geos (Poi h21) add Emp) \implies$
Plane-sameside ($Li\ o2\ l2$) $k21\ h21$ **by** (*blast intro:Plane-sameside-trans Plane-sameside-rev*)
from $P66$ **have** $P73 : \neg Eq (Geos (Poi k2) add Emp) (Geos (Poi k21) add Emp)$
 \implies
 $Eq (Geos (Poi k2) add Emp) (Geos (Poi h21) add Emp) \implies$
Plane-sameside ($Li\ o2\ l2$) $k21\ h21$ **by** (*simp add:Point-Eq*)
from $P71\ P72\ P73$ **have** $P74 : \neg Eq (Geos (Poi h2) add Emp) (Geos (Poi h21) add Emp) \implies$
 $\neg Eq (Geos (Poi k2) add Emp) (Geos (Poi k21) add Emp) \implies$
Plane-sameside ($Li\ o2\ l2$) $k21\ h21$ **by** *blast*
from $P63$ **have** $P75 : Eq (Geos (Poi h2) add Emp) (Geos (Poi h21) add Emp)$
 \implies
 $\neg Eq (Geos (Poi k2) add Emp) (Geos (Poi k21) add Emp) \implies$
Plane-sameside ($Li\ o2\ l2$) $k2\ h21$ **by** (*simp add:Point-Eq*)
from $P66\ P71\ P75$ **have** $P76 : Eq (Geos (Poi h2) add Emp) (Geos (Poi h21) add Emp) \implies$
 $\neg Eq (Geos (Poi k2) add Emp) (Geos (Poi k21) add Emp) \implies$
Plane-sameside ($Li\ o2\ l2$) $k21\ h21$ **by** (*blast intro:Plane-sameside-trans Eq-rev*)

from *assms* **have** $P77 : Eq (Geos (Poi k2) add Emp) (Geos (Poi k21) add Emp)$
 \implies
Plane-sameside ($Li\ o2\ l2$) $h2\ k21$ **by** (*simp add:Point-Eq*)
from $P62\ P71\ P77$ **have** $P78 : \neg Eq (Geos (Poi h2) add Emp) (Geos (Poi h21) add Emp) \implies$
 $Eq (Geos (Poi k2) add Emp) (Geos (Poi k21) add Emp) \implies$
Plane-sameside ($Li\ o2\ l2$) $k21\ h21$ **by** (*blast intro:Plane-sameside-trans Plane-sameside-rev*)

from $P77$ **have** $P79 : Eq (Geos (Poi h2) add Emp) (Geos (Poi h21) add Emp)$
 \implies
 $Eq (Geos (Poi k2) add Emp) (Geos (Poi k21) add Emp) \implies$
Plane-sameside ($Li\ o2\ l2$) $k21\ h21$ **by** (*blast intro:Point-Eq Plane-sameside-rev*)

from $P71\ P74\ P76\ P78\ P79$ **have** $P80 : Plane-sameside (Li\ o2\ l2) k21\ h21$ **by** *blast*
from $P61\ P74\ P80$ **have** $P81 : Plane-sameside (Li\ l21\ o2) k21\ h21$ **by** (*simp add:Plane-Line-trans*)
from $P54\ P56\ P74\ P81$ **have** $P82 : Eq (Geos (Lin (Li\ k21\ l21)) add Emp) (Geos (Lin (Li\ h21\ l21)) add Emp) \wedge \neg Bet-Point (Se\ k21\ h21) l21$ **by** (*simp*)

add:Ang-move-unique)
have $P83 : \text{Line-on } (Li\ h21\ l21)\ h21$ **by** (*simp add:Line-on-rule*)
from $P74\ P82\ P83$ **have** $P84 : \text{Line-on } (Li\ k21\ l21)\ h21$ **by** (*blast intro:Line-on-trans Eq-rev*)
have $P85 : \text{Line-on } (Li\ k21\ l21)\ k21$ **by** (*simp add:Line-on-rule*)
have $P86 : \text{Line-on } (Li\ k21\ l21)\ l21$ **by** (*simp add:Line-on-rule*)
from $P9$ **have** $\neg \text{Eq } (\text{Geos } (Poi\ l21)\ \text{add Emp})\ (\text{Geos } (Poi\ k21)\ \text{add Emp})$ **by** (*simp add:Ang-def*)
then have $P87 : \neg \text{Eq } (\text{Geos } (Poi\ k21)\ \text{add Emp})\ (\text{Geos } (Poi\ l21)\ \text{add Emp})$ **by** (*blast intro:Eq-rev*)
from $P33$ **have** $P88 : \neg \text{Eq } (\text{Geos } (Poi\ l21)\ \text{add Emp})\ (\text{Geos } (Poi\ h21)\ \text{add Emp})$ **by** (*simp add:Ang-def*)
from $P71$ **have** $P89 : \neg \text{Eq } (\text{Geos } (Poi\ h21)\ \text{add Emp})\ (\text{Geos } (Poi\ k21)\ \text{add Emp})$ **by** (*blast intro:Eq-rev*)
from $P81$ **have** $P90 : \neg \text{Line-on } (Li\ l21\ o2)\ k21$ **by** (*simp add:Plane-sameside-def*)
from $P85$ **have** $P91 : \text{Eq } (\text{Geos } (Lin\ (Li\ k21\ l21))\ \text{add Emp})\ (\text{Geos } (Lin\ (Li\ l21\ o2))\ \text{add Emp})$ \implies
Line-on $(Li\ l21\ o2)\ k21$ **by** (*simp add:Line-on-trans*)
from $P90\ P91$ **have** $P92 : \neg \text{Eq } (\text{Geos } (Lin\ (Li\ l21\ o2))\ \text{add Emp})\ (\text{Geos } (Lin\ (Li\ k21\ l21))\ \text{add Emp})$ **by** (*blast intro:Eq-rev*)
from *assms* $P59\ P81\ P84\ P85\ P86\ P87\ P88\ P89\ P92$ **have** $P93 : \text{Bet-Point } (Se\ h21\ l21)\ k21 \vee \text{Bet-Point } (Se\ l21\ k21)\ h21$ **by** (*simp add:Plane-Bet-sameside-rev*)
from $P10$ **have** $P94 : \text{Bet-Point } (Se\ k1\ l1)\ h11$ **by** *simp*
then have $P95 : \text{Bet-Point } (Se\ l1\ k1)\ h11$ **by** (*simp add:Bet-rev*)
from $P42$ **have** $P96 : \text{Eq } (\text{Geos } (Seg\ (Se\ k1\ l1))\ \text{add Emp})\ (\text{Geos } (Seg\ (Se\ k21\ l21))\ \text{add Emp})$ **by** (*simp add:Tri-Cong-def*)
then have $P97 : \text{Eq } (\text{Geos } (Seg\ (Se\ l1\ k1))\ \text{add Emp})\ (\text{Geos } (Seg\ (Se\ l21\ k21))\ \text{add Emp})$ **by** (*blast intro:Eq-rev Eq-trans Seg-rev*)
from $P38$ **have** $\text{Eq } (\text{Geos } (Seg\ (Se\ h11\ l1))\ \text{add Emp})\ (\text{Geos } (Seg\ (Se\ h21\ l21))\ \text{add Emp})$ **by** (*simp add:Tri-Cong-def*)
then have $P98 : \text{Eq } (\text{Geos } (Seg\ (Se\ l1\ h11))\ \text{add Emp})\ (\text{Geos } (Seg\ (Se\ l21\ h21))\ \text{add Emp})$ **by** (*blast intro:Eq-rev Eq-trans Seg-rev*)
from $P95\ P97\ P98$ **have** $P99 : \neg \text{Bet-Point } (Se\ l21\ h21)\ k21$ **by** (*blast intro:Seg-Bet-wrong-relation*)
then have $P100 : \neg \text{Bet-Point } (Se\ h21\ l21)\ k21$ **by** (*blast intro:Bet-rev*)
from $P93\ P100$ **have** $\text{Bet-Point } (Se\ l21\ k21)\ h21$ **by** *blast*
then have $P101 : \neg \text{Seg-on-Seg } (Se\ k21\ h21)\ (Se\ h21\ l21)$ **by** (*simp add:Bet-rev Seg-Bet-not-on*)
have $P102 : \text{Line-on } (Li\ k1\ l1)\ k1$ **by** (*simp add:Line-on-rule*)
have $P103 : \text{Line-on } (Li\ k1\ l1)\ l1$ **by** (*simp add:Line-on-rule*)
from $P94$ **have** $P104 : \text{Line-on } (Li\ k1\ l1)\ h11$ **by** (*simp add:Line-Bet-on*)
from $P94$ **have** $P105 : \neg \text{Seg-on-Seg } (Se\ k1\ h11)\ (Se\ h11\ l1)$ **by** (*simp add:Seg-Bet-not-on*)
from *assms* $P84\ P85\ P86\ P96\ P101\ P102\ P103\ P104\ P105$ **have** $P106 : \text{Eq } (\text{Geos } (Seg\ (Se\ k1\ h11))\ \text{add Emp})\ (\text{Geos } (Seg\ (Se\ k21\ h21))\ \text{add Emp})$ **by** (*simp add:Seg-sub*)
from $P42$ **have** $P107 : \text{Cong } (\text{Geos } (Ang\ (An\ l1\ k1\ o1))\ \text{add Emp})\ (\text{Geos } (Ang\ (An\ l21\ k21\ o2))\ \text{add Emp})$ **by** (*simp add:Tri-Cong-def*)
from $P4$ **have** $P108 : \text{Def } (Ang\ (An\ l1\ k1\ o1))$ **by** (*blast intro:Ang-def-rev Ang-def-inv*)

from $P94$ **have** Inv ($Bet\text{-}Point$ (Se $l1$ $h11$) $k1$) **by** ($simp$ add : $Bet\text{-}iff$)
then have $P109$: \neg $Bet\text{-}Point$ (Se $l1$ $h11$) $k1$ **by** ($simp$ add : $Inv\text{-}def$)
have $P110$: $Line\text{-}on$ (Li $k1$ $o1$) $o1$ **by** ($simp$ add : $Line\text{-}on\text{-}rule$)
have $P111$: \neg $Bet\text{-}Point$ (Se $o1$ $o1$) $k1$ **by** ($simp$ add : $Bet\text{-}end\text{-}Point$)
from $P94$ **have** \neg Eq ($Geos$ (Poi $h11$) add Emp) ($Geos$ (Poi $k1$) add Emp) **by**
($simp$ add : $Bet\text{-}Point\text{-}def$)
then have $P112$: \neg Eq ($Geos$ (Poi $k1$) add Emp) ($Geos$ (Poi $h11$) add Emp) **by**
($blast$ $intro$: $Eq\text{-}rev$)
from $P108$ **have** $P113$: \neg Eq ($Geos$ (Poi $k1$) add Emp) ($Geos$ (Poi $o1$) add
 Emp) **by** ($simp$ add : $Ang\text{-}def$)
from $P104$ $P108$ $P109$ $P110$ $P111$ $P112$ $P113$ **have** $P114$: Eq ($Geos$ (Ang (An
 $l1$ $k1$ $o1$)) add Emp) ($Geos$ (Ang (An $h11$ $k1$ $o1$)) add Emp) \wedge Def (Ang (An $h11$
 $k1$ $o1$)) **by** ($simp$ add : $Ang\text{-}Point\text{-}swap$)
from $P9$ **have** $P115$: Def (Ang (An $l21$ $k21$ $o2$)) **by** ($blast$ $intro$: $Ang\text{-}def\text{-}rev$
 $Ang\text{-}def\text{-}inv$)
have $P116$: $Line\text{-}on$ (Li $k21$ $o2$) $o2$ **by** ($simp$ add : $Line\text{-}on\text{-}rule$)
have $P117$: \neg $Bet\text{-}Point$ (Se $o2$ $o2$) $k21$ **by** ($simp$ add : $Bet\text{-}end\text{-}Point$)
from $P64$ **have** $P119$: \neg Eq ($Geos$ (Poi $k21$) add Emp) ($Geos$ (Poi $o2$) add
 Emp) **by** ($blast$ $intro$: $Eq\text{-}rev$)
from $assms$ $P71$ $P84$ $P99$ $P115$ $P116$ $P117$ $P119$ **have** $P120$: Eq ($Geos$ (Ang (An
 $l21$ $k21$ $o2$)) add Emp) ($Geos$ (Ang (An $h21$ $k21$ $o2$)) add Emp) \wedge Def (Ang (An
 $h21$ $k21$ $o2$)) **by** ($simp$ add : $Ang\text{-}Point\text{-}swap$)
from $P107$ $P114$ **have** $P121$: $Cong$ ($Geos$ (Ang (An $h11$ $k1$ $o1$)) add Emp) ($Geos$
(Ang (An $l21$ $k21$ $o2$)) add Emp) **by** ($blast$ $intro$: $Ang\text{-}weektrans$ $Ang\text{-}rev$ $Eq\text{-}rev$)
from $P120$ $P121$ **have** $P122$: $Cong$ ($Geos$ (Ang (An $h11$ $k1$ $o1$)) add Emp) ($Geos$
(Ang (An $h21$ $k21$ $o2$)) add Emp) **by** ($blast$ $intro$: $Ang\text{-}weektrans$ $Ang\text{-}rev$ $Eq\text{-}rev$)
from $P114$ **have** $P123$: Def (Tri (Tr $k1$ $h11$ $o1$)) **by** ($blast$ $intro$: $Ang\text{-}to\text{-}Tri$
 $Tri\text{-}def\text{-}rev$ $Tri\text{-}def\text{-}trans$)
from $P120$ **have** $P124$: Def (Tri (Tr $k21$ $h21$ $o2$)) **by** ($blast$ $intro$: $Ang\text{-}to\text{-}Tri$
 $Tri\text{-}def\text{-}rev$ $Tri\text{-}def\text{-}trans$)
from $P9$ **have** $P125$: Eq ($Geos$ (Seg (Se $k1$ $o1$)) add Emp) ($Geos$ (Seg (Se $k21$
 $o2$)) add Emp) **by** ($blast$ $intro$: $Seg\text{-}rev$ $Eq\text{-}trans$ $Eq\text{-}rev$)
from $P106$ $P122$ $P123$ $P124$ $P125$ **have** $Cong$ ($Geos$ (Tri (Tr $k1$ $h11$ $o1$)) add
 Emp) ($Geos$ (Tri (Tr $k21$ $h21$ $o2$)) add Emp) **by** ($simp$ add : $Tri\text{-}SAS$)
then have $P126$: $Cong$ ($Geos$ (Ang (An $k1$ $o1$ $h11$)) add Emp) ($Geos$ (Ang (An
 $k21$ $o2$ $h21$)) add Emp) **by** ($simp$ add : $Tri\text{-}Cong\text{-}def$)
have $P127$: Eq ($Geos$ (Ang (An $k1$ $o1$ $h11$)) add Emp) ($Geos$ (Ang (An $h11$ $o1$
 $k1$)) add Emp) **by** ($simp$ add : $Ang\text{-}roll$)
have $P128$: Eq ($Geos$ (Ang (An $k21$ $o2$ $h21$)) add Emp) ($Geos$ (Ang (An $h21$ $o2$
 $k21$)) add Emp) **by** ($simp$ add : $Ang\text{-}roll$)
from $P126$ $P127$ **have** $P129$: $Cong$ ($Geos$ (Ang (An $h11$ $o1$ $k1$)) add Emp) ($Geos$
(Ang (An $k21$ $o2$ $h21$)) add Emp) **by** ($blast$ $intro$: $Ang\text{-}weektrans$ $Ang\text{-}rev$ $Eq\text{-}rev$)
from $P128$ $P129$ **have** $P130$: $Cong$ ($Geos$ (Ang (An $h11$ $o1$ $k1$)) add Emp) ($Geos$
(Ang (An $h21$ $o2$ $k21$)) add Emp) **by** ($blast$ $intro$: $Ang\text{-}weektrans$ $Ang\text{-}rev$ $Eq\text{-}rev$)
from $P114$ **have** $P131$: Def (Ang (An $h11$ $o1$ $k1$)) **by** ($simp$ add : $Ang\text{-}def\text{-}inv$)
from $P2$ **have** \neg Eq ($Geos$ (Poi $h1$) add Emp) ($Geos$ (Poi $o1$) add Emp) **by**
($simp$ add : $Ang\text{-}def$)
then have $P132$: \neg Eq ($Geos$ (Poi $o1$) add Emp) ($Geos$ (Poi $h1$) add Emp) **by**
($blast$ $intro$: $Eq\text{-}rev$)

from $P10 P12 P132$ **have** $P133 : \text{Line-on } (Li\ o1\ h11)\ h1$ **by** (*blast intro:Line-on-rev*)
from $P27$ **have** $P134 : \neg \text{Bet-Point } (Se\ h11\ h1)\ o1$ **by** (*blast intro:Bet-rev*)
have $P135 : \text{Line-on } (Li\ o1\ k1)\ k1$ **by** (*simp add:Line-on-rule*)
have $P136 : \neg \text{Bet-Point } (Se\ k1\ k1)\ o1$ **by** (*simp add:Bet-end-Point*)
from $P4$ **have** $\neg \text{Eq } (\text{Geos } (Poi\ k1)\ \text{add Emp})\ (\text{Geos } (Poi\ o1)\ \text{add Emp})$ **by**
(*simp add:Ang-def*)
then have $P137 : \neg \text{Eq } (\text{Geos } (Poi\ o1)\ \text{add Emp})\ (\text{Geos } (Poi\ k1)\ \text{add Emp})$ **by**
(*blast intro:Eq-rev*)
from $P131 P132 P133 P134 P135 P136 P137$ **have** $P138 : \text{Eq } (\text{Geos } (\text{Ang } (An\ h11\ o1\ k1))\ \text{add Emp})\ (\text{Geos } (\text{Ang } (An\ h1\ o1\ k1))\ \text{add Emp}) \wedge \text{Def } (\text{Ang } (An\ h1\ o1\ k1))$ **by** (*simp add:Ang-Point-swap*)
from $P120$ **have** $P139 : \text{Def } (\text{Ang } (An\ h21\ o2\ k21))$ **by** (*simp add:Ang-def-inv*)
from $P15 P16$ **have** $P140 : \text{Line-on } (Li\ o2\ h21)\ h2$ **by** (*simp add:Line-on-rev*)
from $P16$ **have** $P141 : \neg \text{Bet-Point } (Se\ h21\ h2)\ o2$ **by** (*blast intro:Bet-rev*)
from $P8$ **have** $\neg \text{Eq } (\text{Geos } (Poi\ k2)\ \text{add Emp})\ (\text{Geos } (Poi\ o2)\ \text{add Emp})$ **by**
(*simp add:Ang-def*)
then have $P142 : \neg \text{Eq } (\text{Geos } (Poi\ o2)\ \text{add Emp})\ (\text{Geos } (Poi\ k2)\ \text{add Emp})$ **by**
(*blast intro:Eq-rev*)
from $P9 P64 P142$ **have** $P143 : \text{Line-on } (Li\ o2\ k21)\ k2$ **by** (*simp add:Line-on-rev*)
from $P9 P15 P139 P140 P141 P142 P143$ **have** $P143 : \text{Eq } (\text{Geos } (\text{Ang } (An\ h21\ o2\ k21))\ \text{add Emp})\ (\text{Geos } (\text{Ang } (An\ h2\ o2\ k2))\ \text{add Emp}) \wedge \text{Def } (\text{Ang } (An\ h2\ o2\ k2))$ **by** (*simp add:Ang-Point-swap*)
from $P130 P138$ **have** $P145 : \text{Cong } (\text{Geos } (\text{Ang } (An\ h1\ o1\ k1))\ \text{add Emp})\ (\text{Geos } (\text{Ang } (An\ h21\ o2\ k21))\ \text{add Emp})$ **by** (*blast intro:Ang-weektrans Ang-rev Eq-rev*)
from $P143 P145$ **show** $\text{Cong } (\text{Geos } (\text{Ang } (An\ h1\ o1\ k1))\ \text{add Emp})\ (\text{Geos } (\text{Ang } (An\ h2\ o2\ k2))\ \text{add Emp})$ **by** (*blast intro:Ang-weektrans Ang-rev Eq-rev*)
from $P93 P100$ **have** $P146 : \text{Bet-Point } (Se\ k21\ l21)\ h21$ **by** (*blast intro:Bet-rev*)
from $P9$ **have** $P147 : \neg \text{Eq } (\text{Geos } (Lin\ (Li\ o2\ k21))\ \text{add Emp})\ (\text{Geos } (Lin\ (Li\ o2\ l21))\ \text{add Emp})$ **by** (*simp add:Ang-def*)
from $P30 P64 P146 P147$ **have** $P148 : \text{Ang-inside } (An\ k21\ o2\ l21)\ h21$ **by** (*simp add:Ang-inside-Bet-Point*)
from $P8$ **have** $P149 : \neg \text{Eq } (\text{Geos } (Poi\ o2)\ \text{add Emp})\ (\text{Geos } (Poi\ l2)\ \text{add Emp})$ **by** (*simp add:Ang-def*)
from $P9 P142 P148 P149$ **have** $P150 : \text{Ang-inside } (An\ k2\ o2\ l2)\ h21$ **by** (*simp add:Ang-inside-trans*)
from $P15 P16$ **have** $P151 : \text{Line-on } (Li\ o2\ h21)\ h2$ **by** (*simp add:Line-on-rev*)
from $P16$ **have** $P152 : \neg \text{Bet-Point } (Se\ h2\ h21)\ o2$ **by** (*blast intro:Bet-rev*)
from $P15 P150 P151 P152$ **show** $\text{Ang-inside } (An\ k2\ o2\ l2)\ h2$ **by** (*simp add:Ang-inside-HalfLine*)
qed

Theorem15

theorem (*in Congruence-Rule*) *Ang-sub* :

assumes

Plane-sameside $(Li\ o1\ l1)\ h1\ k1$

$\neg \text{Eq } (\text{Geos } (Poi\ o1)\ \text{add Emp})\ (\text{Geos } (Poi\ l1)\ \text{add Emp})$

Plane-sameside $(Li\ o2\ l2)\ h2\ k2$

$\neg \text{Eq } (\text{Geos } (Poi\ o2)\ \text{add Emp})\ (\text{Geos } (Poi\ l2)\ \text{add Emp})$

Cong $(\text{Geos } (\text{Ang } (An\ h1\ o1\ l1))\ \text{add Emp})\ (\text{Geos } (\text{Ang } (An\ h2\ o2\ l2))\ \text{add$

Emp)
Cong (*Geos* (*Ang* (*An* *k1* *o1* *l1*)) *add Emp*) (*Geos* (*Ang* (*An* *k2* *o2* *l2*)) *add Emp*)
 \neg *Eq* (*Geos* (*Lin* (*Li* *o1* *h1*)) *add Emp*) (*Geos* (*Lin* (*Li* *o1* *k1*)) *add Emp*)
 \neg *Eq* (*Geos* (*Lin* (*Li* *o2* *h2*)) *add Emp*) (*Geos* (*Lin* (*Li* *o2* *k2*)) *add Emp*)
shows
Cong (*Geos* (*Ang* (*An* *h1* *o1* *k1*)) *add Emp*) (*Geos* (*Ang* (*An* *h2* *o2* *k2*)) *add Emp*)
proof –
from *assms* **have** *P1* : \neg *Line-on* (*Li* *o1* *l1*) *h1* **by** (*simp add:Plane-sameside-def*)
from *assms* *P1* **have** *Def* (*Ang* (*An* *o1* *l1* *h1*)) **by** (*simp add:Ang-simple-def*)
then **have** *P2* : *Def* (*Ang* (*An* *h1* *o1* *l1*)) **by** (*blast intro:Ang-def-rev Ang-def-inv*)
from *assms* **have** *P3* : \neg *Line-on* (*Li* *o1* *l1*) *k1* **by** (*simp add:Plane-sameside-def*)
from *assms* *P3* **have** *Def* (*Ang* (*An* *o1* *l1* *k1*)) **by** (*simp add:Ang-simple-def*)
then **have** *P4* : *Def* (*Ang* (*An* *k1* *o1* *l1*)) **by** (*blast intro:Ang-def-rev Ang-def-inv*)
from *assms* *P2* *P4* **have** *P5* : *Ang-inside* (*An* *h1* *o1* *l1*) *k1* \wedge \neg *Ang-inside* (*An* *k1* *o1* *l1*) *h1*
 \vee \neg *Ang-inside* (*An* *h1* *o1* *l1*) *k1* \wedge *Ang-inside* (*An* *k1* *o1* *l1*) *h1* **by** (*simp add:Ang-inside-case*)
from *assms* **have** *P6* : *Ang-inside* (*An* *k1* *o1* *l1*) *h1* \implies
Cong (*Geos* (*Ang* (*An* *h1* *o1* *k1*)) *add Emp*) (*Geos* (*Ang* (*An* *h2* *o2* *k2*)) *add Emp*) **by** (*simp add:Ang-sub-lemma1*)
from *assms* **have** *P7* : *Plane-sameside* (*Li* *o1* *l1*) *k1* *h1* **by** (*simp add:Plane-sameside-rev*)
from *assms* **have** *P8* : *Plane-sameside* (*Li* *o2* *l2*) *k2* *h2* **by** (*simp add:Plane-sameside-rev*)
from *assms* **have** *P9* : \neg *Eq* (*Geos* (*Lin* (*Li* *o1* *k1*)) *add Emp*) (*Geos* (*Lin* (*Li* *o1* *h1*)) *add Emp*) **by** (*blast intro:Eq-rev*)
from *assms* **have** *P10* : \neg *Eq* (*Geos* (*Lin* (*Li* *o2* *k2*)) *add Emp*) (*Geos* (*Lin* (*Li* *o2* *h2*)) *add Emp*) **by** (*blast intro:Eq-rev*)
from *assms* *P7* *P8* *P9* *P10* **have** *P11* : *Ang-inside* (*An* *h1* *o1* *l1*) *k1* \implies
Cong (*Geos* (*Ang* (*An* *k1* *o1* *h1*)) *add Emp*) (*Geos* (*Ang* (*An* *k2* *o2* *h2*)) *add Emp*) **by** (*simp add:Ang-sub-lemma1*)
have *P12* : *Eq* (*Geos* (*Ang* (*An* *k1* *o1* *h1*)) *add Emp*) (*Geos* (*Ang* (*An* *h1* *o1* *k1*)) *add Emp*) **by** (*simp add:Ang-roll*)
have *P13* : *Eq* (*Geos* (*Ang* (*An* *k2* *o2* *h2*)) *add Emp*) (*Geos* (*Ang* (*An* *h2* *o2* *k2*)) *add Emp*) **by** (*simp add:Ang-roll*)
from *P11* *P12* **have** *P14* : *Ang-inside* (*An* *h1* *o1* *l1*) *k1* \implies
Cong (*Geos* (*Ang* (*An* *h1* *o1* *k1*)) *add Emp*) (*Geos* (*Ang* (*An* *k2* *o2* *h2*)) *add Emp*) **by** (*blast intro:Ang-weektrans Ang-rev Eq-rev*)
from *P13* *P14* **have** *P15* : *Ang-inside* (*An* *h1* *o1* *l1*) *k1* \implies
Cong (*Geos* (*Ang* (*An* *h1* *o1* *k1*)) *add Emp*) (*Geos* (*Ang* (*An* *h2* *o2* *k2*)) *add Emp*) **by** (*blast intro:Ang-weektrans Ang-rev Eq-rev*)
from *P5* *P6* *P15* **show** *Cong* (*Geos* (*Ang* (*An* *h1* *o1* *k1*)) *add Emp*) (*Geos* (*Ang* (*An* *h2* *o2* *k2*)) *add Emp*) **by** *blast*
qed

theorem (in Congruence-Rule) Ang-add :

assumes

Plane-diffside (*Li* *o1* *l1*) *h1* *k1*

\neg *Eq* (*Geos* (*Poi* *o1*) *add Emp*) (*Geos* (*Poi* *l1*) *add Emp*)

Plane-diffside (*Li* *o2* *l2*) *h2* *k2*

$\neg \text{Eq} (\text{Geos} (\text{Poi } o2) \text{ add Emp}) (\text{Geos} (\text{Poi } l2) \text{ add Emp})$
 $\text{Cong} (\text{Geos} (\text{Ang} (\text{An } h1 \text{ } o1 \text{ } l1)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } h2 \text{ } o2 \text{ } l2)) \text{ add Emp})$
 $\text{Cong} (\text{Geos} (\text{Ang} (\text{An } k1 \text{ } o1 \text{ } l1)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } k2 \text{ } o2 \text{ } l2)) \text{ add Emp})$
 $\neg \text{Eq} (\text{Geos} (\text{Lin} (\text{Li } o1 \text{ } h1)) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li } o1 \text{ } k1)) \text{ add Emp})$
 $\neg \text{Eq} (\text{Geos} (\text{Lin} (\text{Li } o2 \text{ } h2)) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li } o2 \text{ } k2)) \text{ add Emp})$
shows
 $\text{Cong} (\text{Geos} (\text{Ang} (\text{An } h1 \text{ } o1 \text{ } k1)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } h2 \text{ } o2 \text{ } k2)) \text{ add Emp})$
proof –
from *assms* **have** $\exists p. \text{Bet-Point} (\text{Se } h1 \text{ } k1) p \wedge \text{Line-on} (\text{Li } o1 \text{ } l1) p$
 $\wedge \neg \text{Line-on} (\text{Li } o1 \text{ } l1) h1 \wedge \neg \text{Line-on} (\text{Li } o1 \text{ } l1) k1$ **by** (*simp add:Plane-diffside-def*)
then obtain $l11 :: \text{Point}$ **where** $P1 : \text{Bet-Point} (\text{Se } h1 \text{ } k1) l11 \wedge \text{Line-on} (\text{Li } o1 \text{ } l1) l11$
 $\wedge \neg \text{Line-on} (\text{Li } o1 \text{ } l1) h1 \wedge \neg \text{Line-on} (\text{Li } o1 \text{ } l1) k1$ **by** *blast*
have $P2 : \text{Line-on} (\text{Li } o1 \text{ } l1) o1$ **by** (*simp add:Line-on-rule*)
then have $P3 : \text{Eq} (\text{Geos} (\text{Poi } o1) \text{ add Emp}) (\text{Geos} (\text{Poi } k1) \text{ add Emp}) \implies$
 $\text{Line-on} (\text{Li } o1 \text{ } l1) k1$ **by** (*simp add:Point-Eq*)
from $P1 \ P3$ **have** $P4 : \neg \text{Eq} (\text{Geos} (\text{Poi } o1) \text{ add Emp}) (\text{Geos} (\text{Poi } k1) \text{ add Emp})$
by *blast*
have $P5 : \text{Line-on} (\text{Li } o1 \text{ } k1) o1$ **by** (*simp add:Line-on-rule*)
have $P6 : \text{Line-on} (\text{Li } o1 \text{ } k1) k1$ **by** (*simp add:Line-on-rule*)
from $P4 \ P5 \ P6$ **have** $\exists p. \text{Eq} (\text{Geos} (\text{Seg} (\text{Se } o1 \text{ } k1)) \text{ add Emp}) (\text{Geos} (\text{Seg} (\text{Se } o1 \text{ } p)) \text{ add Emp})$
 $\wedge \text{Bet-Point} (\text{Se } p \text{ } k1) o1 \wedge \text{Line-on} (\text{Li } o1 \text{ } k1) p \wedge \neg \text{Eq} (\text{Geos} (\text{Poi } o1) \text{ add Emp}) (\text{Geos} (\text{Poi } p) \text{ add Emp})$ **by** (*simp add:Seg-move-diffside*)
then obtain $k11 :: \text{Point}$ **where** $P7 : \text{Eq} (\text{Geos} (\text{Seg} (\text{Se } o1 \text{ } k1)) \text{ add Emp}) (\text{Geos} (\text{Seg} (\text{Se } o1 \text{ } k11)) \text{ add Emp})$
 $\wedge \text{Bet-Point} (\text{Se } k11 \text{ } k1) o1 \wedge \text{Line-on} (\text{Li } o1 \text{ } k1) k11 \wedge \neg \text{Eq} (\text{Geos} (\text{Poi } o1) \text{ add Emp}) (\text{Geos} (\text{Poi } k11) \text{ add Emp})$ **by** *blast*
from $P7$ **have** $P8 : \text{Bet-Point} (\text{Se } k11 \text{ } k1) o1$ **by** *blast*
have $\text{Line-on} (\text{Li } k11 \text{ } k1) k1$ **by** (*simp add:Line-on-rule*)
then have $P9 : \text{Eq} (\text{Geos} (\text{Lin} (\text{Li } k11 \text{ } k1)) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li } o1 \text{ } l1)) \text{ add Emp}) \implies$
 $\text{Line-on} (\text{Li } o1 \text{ } l1) k1$ **by** (*simp add:Line-on-trans*)
from $P1 \ P9$ **have** $P10 : \neg \text{Eq} (\text{Geos} (\text{Lin} (\text{Li } k11 \text{ } k1)) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li } o1 \text{ } l1)) \text{ add Emp})$ **by** *blast*
from $P2 \ P8 \ P10$ **have** $\text{Plane-diffside} (\text{Li } o1 \text{ } l1) k11 \text{ } k1$ **by** (*simp add:Plane-Bet-diffside*)
then have $P11 : \text{Plane-diffside} (\text{Li } o1 \text{ } l1) k1 \text{ } k11$ **by** (*simp add:Plane-diffside-rev*)
from *assms* **have** $P12 : \text{Plane-diffside} (\text{Li } o1 \text{ } l1) k1 \text{ } h1$ **by** (*simp add:Plane-diffside-rev*)
from $P2$ **have** $P13 : \text{Eq} (\text{Geos} (\text{Poi } o1) \text{ add Emp}) (\text{Geos} (\text{Poi } h1) \text{ add Emp}) \implies$
 $\text{Line-on} (\text{Li } o1 \text{ } l1) h1$ **by** (*simp add:Point-Eq*)
from $P1 \ P13$ **have** $P14 : \neg \text{Eq} (\text{Geos} (\text{Poi } o1) \text{ add Emp}) (\text{Geos} (\text{Poi } h1) \text{ add Emp})$ **by** *blast*
have $P15 : \text{Line-on} (\text{Li } o1 \text{ } h1) o1$ **by** (*simp add:Line-on-rule*)
have $P16 : \text{Line-on} (\text{Li } o1 \text{ } h1) h1$ **by** (*simp add:Line-on-rule*)
from $P5 \ P14 \ P15 \ P16$ **have** $P17 : \text{Line-on} (\text{Li } o1 \text{ } k1) h1$
 $\implies \text{Eq} (\text{Geos} (\text{Lin} (\text{Li } o1 \text{ } h1)) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li } o1 \text{ } k1)) \text{ add Emp})$ **by**
(*simp add:Line-unique*)
from *assms* $P17$ **have** $P18 : \neg \text{Line-on} (\text{Li } o1 \text{ } k1) h1$ **by** *blast*

from $P7$ **have** $P19 : Eq (Geos (Poi k11) add Emp) (Geos (Poi h1) add Emp)$
 $\implies Line-on (Li o1 k1) h1$ **by** (blast intro:Point-Eq)
from $P18 P19$ **have** $P20 : \neg Eq (Geos (Poi k11) add Emp) (Geos (Poi h1) add Emp)$ **by** blast
from $P11 P12 P20$ **have** $Plane-sameside (Li o1 l1) k11 h1$ **by** (blast intro:Plane-trans-inv)
then **have** $P21 : Plane-sameside (Li o1 l1) h1 k11$ **by** (simp add:Plane-sameside-rev)
from **assms** **have** $\exists p. Bet-Point (Se h2 k2) p \wedge Line-on (Li o2 l2) p$
 $\wedge \neg Line-on (Li o2 l2) h2 \wedge \neg Line-on (Li o2 l2) k2$ **by** (simp add:Plane-diffside-def)
then **obtain** $l21 :: Point$ **where** $P22 : Bet-Point (Se h2 k2) l21 \wedge Line-on (Li o2 l2) l21$
 $\wedge \neg Line-on (Li o2 l2) h2 \wedge \neg Line-on (Li o2 l2) k2$ **by** blast
have $P23 : Line-on (Li o2 l2) o2$ **by** (simp add:Line-on-rule)
then **have** $P24 : Eq (Geos (Poi o2) add Emp) (Geos (Poi k2) add Emp)$ \implies
 $Line-on (Li o2 l2) k2$ **by** (simp add:Point-Eq)
from $P22 P24$ **have** $P25 : \neg Eq (Geos (Poi o2) add Emp) (Geos (Poi k2) add Emp)$ **by** blast
have $P26 : Line-on (Li o2 k2) o2$ **by** (simp add:Line-on-rule)
have $P27 : Line-on (Li o2 k2) k2$ **by** (simp add:Line-on-rule)
from $P4 P25 P26 P27$ **have** $\exists p. Eq (Geos (Seg (Se o1 k1)) add Emp) (Geos (Seg (Se o2 p)) add Emp)$
 $\wedge Bet-Point (Se p k2) o2 \wedge Line-on (Li o2 k2) p \wedge \neg Eq (Geos (Poi o2) add Emp) (Geos (Poi p) add Emp)$ **by** (simp add:Seg-move-diffside)
then **obtain** $k21 :: Point$ **where** $P28 : Eq (Geos (Seg (Se o1 k1)) add Emp) (Geos (Seg (Se o2 k21)) add Emp)$
 $\wedge Bet-Point (Se k21 k2) o2 \wedge Line-on (Li o2 k2) k21 \wedge \neg Eq (Geos (Poi o2) add Emp) (Geos (Poi k21) add Emp)$ **by** blast
from $P28$ **have** $P29 : Bet-Point (Se k21 k2) o2$ **by** blast
have $Line-on (Li k21 k2) k2$ **by** (simp add:Line-on-rule)
then **have** $P30 : Eq (Geos (Lin (Li k21 k2)) add Emp) (Geos (Lin (Li o2 l2)) add Emp)$ \implies $Line-on (Li o2 l2) k2$ **by** (simp add:Line-on-trans)
from **assms** **have** $\exists p. Bet-Point (Se h2 k2) p \wedge Line-on (Li o2 l2) p$
 $\wedge \neg Line-on (Li o2 l2) h2 \wedge \neg Line-on (Li o2 l2) k2$ **by** (simp add:Plane-diffside-def)
from $P22 P30$ **have** $P31 : \neg Eq (Geos (Lin (Li k21 k2)) add Emp) (Geos (Lin (Li o2 l2)) add Emp)$ **by** blast
from $P23 P29 P31$ **have** $Plane-diffside (Li o2 l2) k21 k2$ **by** (simp add:Plane-Bet-diffside)
then **have** $P32 : Plane-diffside (Li o2 l2) k2 k21$ **by** (simp add:Plane-diffside-rev)
from **assms** **have** $P33 : Plane-diffside (Li o2 l2) k2 h2$ **by** (simp add:Plane-diffside-rev)
from $P23$ **have** $P34 : Eq (Geos (Poi o2) add Emp) (Geos (Poi h2) add Emp)$
 $\implies Line-on (Li o2 l2) h2$ **by** (simp add:Point-Eq)
from $P22 P34$ **have** $P35 : \neg Eq (Geos (Poi o2) add Emp) (Geos (Poi h2) add Emp)$ **by** blast
have $P36 : Line-on (Li o2 h2) o2$ **by** (simp add:Line-on-rule)
have $P37 : Line-on (Li o2 h2) h2$ **by** (simp add:Line-on-rule)
from $P26 P35 P36 P37$ **have** $P38 : Line-on (Li o2 k2) h2$
 $\implies Eq (Geos (Lin (Li o2 h2)) add Emp) (Geos (Lin (Li o2 k2)) add Emp)$ **by** (simp add:Line-unique)
from **assms** $P38$ **have** $P39 : \neg Line-on (Li o2 k2) h2$ **by** blast
from $P28$ **have** $P40 : Eq (Geos (Poi k21) add Emp) (Geos (Poi h2) add Emp)$

\implies *Line-on* (*Li o2 k2*) *h2* **by** (*blast intro:Point-Eq*)
from *P39 P40* **have** *P41* : \neg *Eq* (*Geos (Poi k21)*) *add Emp*) (*Geos (Poi h2)*) *add Emp*) **by** *blast*
from *P32 P33 P41* **have** *Plane-sameside* (*Li o2 l2*) *k21 h2* **by** (*blast intro:Plane-trans-inv*)
then **have** *P42* : *Plane-sameside* (*Li o2 l2*) *h2 k21* **by** (*simp add:Plane-sameside-rev*)
from *assms P1* **have** *Def* (*Ang (An o1 l1 k1)*) **by** (*simp add:Ang-simple-def*)
then **have** *P43* : *Def* (*Ang (An k1 o1 l1)*) **by** (*blast intro:Ang-def-rev Ang-def-inv*)
from *assms P22* **have** *Def* (*Ang (An o2 l2 k2)*) **by** (*simp add:Ang-simple-def*)
then **have** *P44* : *Def* (*Ang (An k2 o2 l2)*) **by** (*blast intro:Ang-def-rev Ang-def-inv*)
from *P8* **have** *P45* : *Bet-Point* (*Se k1 k11*) *o1* **by** (*simp add:Bet-rev*)
from *P28* **have** *P46* : *Bet-Point* (*Se k2 k21*) *o2* **by** (*simp add:Bet-rev*)
from *assms P43 P44 P45 P46* **have** *P47* : *Cong* (*Geos (Ang (An l1 o1 k11))*) *add Emp*) (*Geos (Ang (An l2 o2 k21))*) *add Emp*) **by** (*simp add:Ang-complementary*)
have *P48* : *Eq* (*Geos (Ang (An l1 o1 k11))*) *add Emp*) (*Geos (Ang (An k11 o1 l1))*) *add Emp*) **by** (*simp add:Ang-roll*)
have *P49* : *Eq* (*Geos (Ang (An l2 o2 k21))*) *add Emp*) (*Geos (Ang (An k21 o2 l2))*) *add Emp*) **by** (*simp add:Ang-roll*)
from *P47 P48* **have** *P50* : *Cong* (*Geos (Ang (An k11 o1 l1))*) *add Emp*) (*Geos (Ang (An l2 o2 k21))*) *add Emp*) **by** (*blast intro:Ang-weektrans Ang-rev Eq-rev*)
from *P49 P50* **have** *P51* : *Cong* (*Geos (Ang (An k11 o1 l1))*) *add Emp*) (*Geos (Ang (An k21 o2 l2))*) *add Emp*) **by** (*blast intro:Ang-weektrans Ang-rev Eq-rev*)
have *P52* : *Line-on* (*Li o1 k11*) *k11* **by** (*simp add:Line-on-rule*)
have *P53* : *Line-on* (*Li o1 k11*) *o1* **by** (*simp add:Line-on-rule*)
from *P5 P7 P52 P53* **have** *P54* : *Eq* (*Geos (Lin (Li o1 k1))*) *add Emp*) (*Geos (Lin (Li o1 k11))*) *add Emp*) **by** (*blast intro:Line-unique*)
from *P54* **have** *P55* : *Eq* (*Geos (Lin (Li o1 h1))*) *add Emp*) (*Geos (Lin (Li o1 k11))*) *add Emp*)
 \implies *Eq* (*Geos (Lin (Li o1 h1))*) *add Emp*) (*Geos (Lin (Li o1 k1))*) *add Emp*) **by** (*blast intro:Eq-trans Eq-rev*)
from *assms P55* **have** *P56* : \neg *Eq* (*Geos (Lin (Li o1 h1))*) *add Emp*) (*Geos (Lin (Li o1 k11))*) *add Emp*) **by** *blast*
have *P57* : *Line-on* (*Li o2 k21*) *k21* **by** (*simp add:Line-on-rule*)
have *P58* : *Line-on* (*Li o2 k21*) *o2* **by** (*simp add:Line-on-rule*)
from *P26 P28 P57 P58* **have** *P59* : *Eq* (*Geos (Lin (Li o2 k2))*) *add Emp*) (*Geos (Lin (Li o2 k21))*) *add Emp*) **by** (*blast intro:Line-unique*)
from *P59* **have** *P60* : *Eq* (*Geos (Lin (Li o2 h2))*) *add Emp*) (*Geos (Lin (Li o2 k21))*) *add Emp*)
 \implies *Eq* (*Geos (Lin (Li o2 h2))*) *add Emp*) (*Geos (Lin (Li o2 k2))*) *add Emp*) **by** (*blast intro:Eq-trans Eq-rev*)
from *assms P60* **have** *P61* : \neg *Eq* (*Geos (Lin (Li o2 h2))*) *add Emp*) (*Geos (Lin (Li o2 k21))*) *add Emp*) **by** *blast*
from *assms P21 P42 P51 P56 P61* **have** *P62* : *Cong* (*Geos (Ang (An h1 o1 k11))*) *add Emp*) (*Geos (Ang (An h2 o2 k21))*) *add Emp*) **by** (*simp add:Ang-sub*)
have *P63* : *Eq* (*Geos (Ang (An h1 o1 k11))*) *add Emp*) (*Geos (Ang (An k11 o1 h1))*) *add Emp*) **by** (*simp add:Ang-roll*)
have *P64* : *Eq* (*Geos (Ang (An h2 o2 k21))*) *add Emp*) (*Geos (Ang (An k21 o2 h2))*) *add Emp*) **by** (*simp add:Ang-roll*)
from *P62 P63* **have** *P65* : *Cong* (*Geos (Ang (An k11 o1 h1))*) *add Emp*) (*Geos*

(*Ang (An h2 o2 k21) add Emp*) **by** (*blast intro:Ang-weektrans Ang-rev Eq-rev*)
from *P64 P65* **have** *P66 : Cong (Geos (Ang (An k11 o1 h1)) add Emp) (Geos (Ang (An k21 o2 h2)) add Emp)* **by** (*blast intro:Ang-weektrans Ang-rev Eq-rev*)
from *P54* **have** *P67 : Line-on (Li o1 k11) h1 \implies Line-on (Li o1 k1) h1* **by**
 (*blast intro:Line-on-trans Eq-rev*)
from *P18 P67* **have** *P68 : \neg Line-on (Li o1 k11) h1* **by** *blast*
from *P7 P68* **have** *Def (Ang (An o1 k11 h1))* **by** (*simp add:Ang-simple-def*)
then have *P69 : Def (Ang (An k11 o1 h1))* **by** (*blast intro:Ang-def-rev Ang-def-inv*)
from *P59* **have** *P70 : Line-on (Li o2 k21) h2 \implies Line-on (Li o2 k2) h2* **by**
 (*blast intro:Line-on-trans Eq-rev*)
from *P39 P70* **have** *P71 : \neg Line-on (Li o2 k21) h2* **by** *blast*
from *P28 P71* **have** *Def (Ang (An o2 k21 h2))* **by** (*simp add:Ang-simple-def*)
then have *P72 : Def (Ang (An k21 o2 h2))* **by** (*blast intro:Ang-def-rev Ang-def-inv*)
from *P8 P29 P66 P69 P72* **show** *Cong (Geos (Ang (An h1 o1 k1)) add Emp)*
(Geos (Ang (An h2 o2 k2)) add Emp) **by** (*simp add:Ang-complementary*)
qed

lemma (in *Congruence-Rule*) *Ang-split-lemma1* :

assumes *N* :

Def (Ang (An h1 o1 k1)) Def (Ang (An h2 o2 k2))

Cong (Geos (Ang (An h1 o1 k1)) add Emp) (Geos (Ang (An h2 o2 k2)) add Emp)

Cong (Geos (Ang (An l1 o1 k1)) add Emp) (Geos (Ang (An l2 o2 k2)) add Emp)

Plane-sameside (Li o1 k1) h1 l1

Plane-sameside (Li o2 k2) h2 l2

\neg Eq (Geos (Lin (Li o1 l1)) add Emp) (Geos (Lin (Li o1 h1)) add Emp)

shows

\neg Eq (Geos (Lin (Li o2 l2)) add Emp) (Geos (Lin (Li o2 h2)) add Emp)

proof

assume *W : Eq (Geos (Lin (Li o2 l2)) add Emp) (Geos (Lin (Li o2 h2)) add Emp)*

have *P1 : Line-on (Li o2 k2) o2* **by** (*simp add:Line-on-rule*)

from *N* **have** *P2 : \neg Line-on (Li o2 k2) h2 \wedge \neg Line-on (Li o2 k2) l2*

\wedge \neg Eq (Geos (Poi h2) add Emp) (Geos (Poi l2) add Emp) **by** (*simp add:Plane-sameside-def*)

from *P1 P2* **have** *Bet-Point (Se h2 l2) o2 \implies ($\exists p$. Bet-Point (Se h2 l2) p*

\wedge Line-on (Li o2 k2) p \wedge \neg Line-on (Li o2 k2) h2 \wedge \neg Line-on (Li o2 k2) l2)

by *blast*

then have *Bet-Point (Se h2 l2) o2 \implies Plane-diffside (Li o2 k2) h2 l2* **by** (*simp add:Plane-diffside-def*)

then have *P3 : Bet-Point (Se h2 l2) o2 \implies \neg Plane-sameside (Li o2 k2) h2 l2* **by** (*simp add:Plane-diffside-not-sameside*)

from *N P3* **have** *P4 : \neg Bet-Point (Se h2 l2) o2* **by** *blast*

have *P5 : Line-on (Li o2 l2) l2* **by** (*simp add:Line-on-rule*)

from *W P5* **have** *P6 : Line-on (Li o2 h2) l2* **by** (*simp add:Line-on-trans*)

have *P7 : Line-on (Li o2 k2) k2* **by** (*simp add:Line-on-rule*)

have *P8 : \neg Bet-Point (Se k2 k2) o2* **by** (*simp add:Bet-end-Point*)

from *P1* **have** *P9 : Eq (Geos (Poi o2) add Emp) (Geos (Poi l2) add Emp) \implies Line-on (Li o2 k2) l2* **by** (*simp add:Point-Eq*)

from $P2\ P9$ **have** $P10 : \neg Eq\ (Geos\ (Poi\ o2)\ add\ Emp)\ (Geos\ (Poi\ l2)\ add\ Emp)$ **by** *blast*
from N **have** $Def\ (Tri\ (Tr\ h2\ o2\ k2))$ **by** *(simp add:Ang-to-Tri)*
then have $P11 : \neg Eq\ (Geos\ (Poi\ o2)\ add\ Emp)\ (Geos\ (Poi\ k2)\ add\ Emp)$ **by** *(simp add:Tri-def)*
from $N\ P4\ P6\ P7\ P8\ P10\ P11$ **have** $P12 :$
 $Eq\ (Geos\ (Ang\ (An\ h2\ o2\ k2))\ add\ Emp)\ (Geos\ (Ang\ (An\ l2\ o2\ k2))\ add\ Emp)$
 $\wedge\ Def\ (Ang\ (An\ l2\ o2\ k2))$ **by** *(simp add:Ang-Point-swap)*
from $N\ P12$ **have** $P13 : Cong\ (Geos\ (Ang\ (An\ l2\ o2\ k2))\ add\ Emp)\ (Geos\ (Ang\ (An\ h1\ o1\ k1))\ add\ Emp)$ **by** *(blast intro:Ang-weektrans Ang-rev Eq-rev)*
from N **have** $P14 : Cong\ (Geos\ (Ang\ (An\ l2\ o2\ k2))\ add\ Emp)\ (Geos\ (Ang\ (An\ l1\ o1\ k1))\ add\ Emp)$ **by** *(blast intro:Ang-rev)*
from $N\ P13\ P14$ **have** $P15 : Eq\ (Geos\ (Lin\ (Li\ h1\ o1))\ add\ Emp)\ (Geos\ (Lin\ (Li\ l1\ o1))\ add\ Emp) \wedge \neg Bet\text{-}Point\ (Se\ h1\ l1)\ o1$ **by** *(simp add:Ang-move-unique)*
from N **have** $Def\ (Tri\ (Tr\ h1\ o1\ k1))$ **by** *(simp add:Ang-to-Tri)*
then have $\neg Eq\ (Geos\ (Poi\ h1)\ add\ Emp)\ (Geos\ (Poi\ o1)\ add\ Emp)$ **by** *(simp add:Tri-def)*
then have $P16 : Eq\ (Geos\ (Lin\ (Li\ h1\ o1))\ add\ Emp)\ (Geos\ (Lin\ (Li\ o1\ h1))\ add\ Emp)$ **by** *(simp add:Line-rev)*
from N **have** $P17 : \neg Line\text{-}on\ (Li\ o1\ k1)\ h1 \wedge \neg Line\text{-}on\ (Li\ o1\ k1)\ l1$
 $\wedge \neg Eq\ (Geos\ (Poi\ h1)\ add\ Emp)\ (Geos\ (Poi\ l1)\ add\ Emp)$ **by** *(simp add:Plane-sameside-def)*
have $P18 : Line\text{-}on\ (Li\ o1\ k1)\ o1$ **by** *(simp add:Line-on-rule)*
then have $P19 : Eq\ (Geos\ (Poi\ o1)\ add\ Emp)\ (Geos\ (Poi\ l1)\ add\ Emp) \implies$
 $Line\text{-}on\ (Li\ o1\ k1)\ l1$ **by** *(simp add:Point-Eq)*
from $P17\ P19$ **have** $\neg Eq\ (Geos\ (Poi\ o1)\ add\ Emp)\ (Geos\ (Poi\ l1)\ add\ Emp)$
by *blast*
then have $P20 : Eq\ (Geos\ (Lin\ (Li\ o1\ l1))\ add\ Emp)\ (Geos\ (Lin\ (Li\ l1\ o1))\ add\ Emp)$ **by** *(simp add:Line-rev)*
from $P15\ P16$ **have** $P21 : Eq\ (Geos\ (Lin\ (Li\ o1\ h1))\ add\ Emp)\ (Geos\ (Lin\ (Li\ l1\ o1))\ add\ Emp)$ **by** *(blast intro:Eq-rev Eq-trans)*
from $P20\ P21$ **have** $P22 : Eq\ (Geos\ (Lin\ (Li\ o1\ l1))\ add\ Emp)\ (Geos\ (Lin\ (Li\ o1\ h1))\ add\ Emp)$ **by** *(blast intro:Eq-rev Eq-trans)*
from $N\ P22$ **show** $False$ **by** *blast*
qed

Theorem16

theorem (in *Congruence-Rule*) *Ang-split* :

assumes

$Def\ (Ang\ (An\ h1\ o1\ k1))\ Def\ (Ang\ (An\ h2\ o2\ k2))$

$Cong\ (Geos\ (Ang\ (An\ h1\ o1\ k1))\ add\ Emp)\ (Geos\ (Ang\ (An\ h2\ o2\ k2))\ add\ Emp)$

$Ang\text{-}inside\ (An\ h1\ o1\ k1)\ l1$

shows

$\exists p.\ Ang\text{-}inside\ (An\ h2\ o2\ k2)\ p$

$\wedge Cong\ (Geos\ (Ang\ (An\ h1\ o1\ l1))\ add\ Emp)\ (Geos\ (Ang\ (An\ h2\ o2\ p))\ add\ Emp)$

$\wedge Cong\ (Geos\ (Ang\ (An\ k1\ o1\ l1))\ add\ Emp)\ (Geos\ (Ang\ (An\ k2\ o2\ p))\ add\ Emp)$

proof –

from *assms* **have** $P1 : \text{Plane-sameside } (Li\ o1\ h1)\ k1\ l1 \wedge \text{Plane-sameside } (Li\ o1\ k1)\ h1\ l1$ **by** (*simp add:Ang-inside-def*)
from *assms* **have** $P2 : \neg \text{Line-on } (Li\ o2\ k2)\ h2$ **by** (*simp add:Ang-to-Tri Tri-def-Line*)
from $P1$ **have** $P3 : \neg \text{Line-on } (Li\ o1\ k1)\ l1$ **by** (*simp add:Plane-sameside-def*)
from *assms* **have** $P4 : \text{Def } (\text{Tri } (\text{Tr } h1\ o1\ k1))$ **by** (*simp add:Ang-to-Tri*)
then **have** $P5 : \neg \text{Eq } (\text{Geos } (\text{Poi } o1)\ \text{add } \text{Emp}) (\text{Geos } (\text{Poi } k1)\ \text{add } \text{Emp})$ **by** (*simp add:Tri-def*)
from $P3\ P5$ **have** $\text{Def } (\text{Ang } (\text{An } o1\ k1\ l1))$ **by** (*simp add:Ang-simple-def*)
then **have** $P6 : \text{Def } (\text{Ang } (\text{An } k1\ o1\ l1))$ **by** (*blast intro:Ang-def-rev Ang-def-inv*)
from $P2\ P6$ **have** $\exists p. \text{Cong } (\text{Geos } (\text{Ang } (\text{An } k1\ o1\ l1))\ \text{add } \text{Emp}) (\text{Geos } (\text{Ang } (\text{An } p\ o2\ k2))\ \text{add } \text{Emp})$
 $\wedge \text{Plane-sameside } (Li\ o2\ k2)\ p\ h2$ **by** (*simp add:Ang-move-sameside*)
then **obtain** $l2 :: \text{Point}$ **where** $P7 : \text{Cong } (\text{Geos } (\text{Ang } (\text{An } k1\ o1\ l1))\ \text{add } \text{Emp}) (\text{Geos } (\text{Ang } (\text{An } l2\ o2\ k2))\ \text{add } \text{Emp})$
 $\wedge \text{Plane-sameside } (Li\ o2\ k2)\ l2\ h2$ **by** *blast*
from *assms* **have** $\text{Def } (\text{Tri } (\text{Tr } h2\ o2\ k2))$ **by** (*simp add:Ang-to-Tri*)
then **have** $P8 : \neg \text{Eq } (\text{Geos } (\text{Poi } o2)\ \text{add } \text{Emp}) (\text{Geos } (\text{Poi } k2)\ \text{add } \text{Emp})$ **by** (*simp add:Tri-def*)
have $P9 : \text{Eq } (\text{Geos } (\text{Ang } (\text{An } k1\ o1\ l1))\ \text{add } \text{Emp}) (\text{Geos } (\text{Ang } (\text{An } l1\ o1\ k1))\ \text{add } \text{Emp})$ **by** (*simp add:Ang-roll*)
from $P7\ P9$ **have** $P10 : \text{Cong } (\text{Geos } (\text{Ang } (\text{An } l1\ o1\ k1))\ \text{add } \text{Emp}) (\text{Geos } (\text{Ang } (\text{An } l2\ o2\ k2))\ \text{add } \text{Emp})$ **by** (*blast intro:Ang-weektrans Ang-rev Eq-rev*)
from $P1$ **have** $P11 : \neg \text{Line-on-Seg } (Li\ o1\ h1)\ (\text{Se } k1\ l1) \wedge \neg \text{Line-on } (Li\ o1\ h1)\ k1$
 $\wedge \neg \text{Line-on } (Li\ o1\ h1)\ l1 \wedge \neg \text{Eq } (\text{Geos } (\text{Poi } k1)\ \text{add } \text{Emp}) (\text{Geos } (\text{Poi } l1)\ \text{add } \text{Emp})$ **by** (*simp add:Plane-sameside-def*)
have $\text{Line-on } (Li\ o1\ l1)\ l1$ **by** (*simp add:Line-on-rule*)
then **have** $P12 : \text{Eq } (\text{Geos } (\text{Lin } (Li\ o1\ l1))\ \text{add } \text{Emp}) (\text{Geos } (\text{Lin } (Li\ o1\ h1))\ \text{add } \text{Emp}) \implies \text{Line-on } (Li\ o1\ h1)\ l1$ **by** (*simp add:Line-on-trans*)
from $P11\ P12$ **have** $P13 : \neg \text{Eq } (\text{Geos } (\text{Lin } (Li\ o1\ l1))\ \text{add } \text{Emp}) (\text{Geos } (\text{Lin } (Li\ o1\ h1))\ \text{add } \text{Emp})$ **by** *blast*
from $P7$ **have** $P14 : \text{Plane-sameside } (Li\ o2\ k2)\ h2\ l2$ **by** (*simp add:Plane-sameside-rev*)
from *assms* $P1\ P10\ P13\ P14$ **have** $P15 : \neg \text{Eq } (\text{Geos } (\text{Lin } (Li\ o2\ l2))\ \text{add } \text{Emp}) (\text{Geos } (\text{Lin } (Li\ o2\ h2))\ \text{add } \text{Emp})$ **by** (*simp add:Ang-split-lemma1*)
from $P1$ **have** $P16 : \text{Plane-sameside } (Li\ o1\ k1)\ l1\ h1$ **by** (*simp add:Plane-sameside-rev*)
from $P7$ **have** $P17 : \text{Plane-sameside } (Li\ o2\ k2)\ l2\ h2$ **by** *simp*
from *assms* $P5\ P8\ P10\ P13\ P15\ P16\ P17$ **have** $P18 :$
 $\text{Cong } (\text{Geos } (\text{Ang } (\text{An } l1\ o1\ h1))\ \text{add } \text{Emp}) (\text{Geos } (\text{Ang } (\text{An } l2\ o2\ h2))\ \text{add } \text{Emp})$
 $\wedge \text{Ang-inside } (\text{An } h2\ o2\ k2)\ l2$ **by** (*simp add:Ang-sub-lemma1*)
have $P19 : \text{Eq } (\text{Geos } (\text{Ang } (\text{An } l2\ o2\ k2))\ \text{add } \text{Emp}) (\text{Geos } (\text{Ang } (\text{An } k2\ o2\ l2))\ \text{add } \text{Emp})$ **by** (*simp add:Ang-roll*)
from $P7\ P19$ **have** $P20 : \text{Cong } (\text{Geos } (\text{Ang } (\text{An } k1\ o1\ l1))\ \text{add } \text{Emp}) (\text{Geos } (\text{Ang } (\text{An } k2\ o2\ l2))\ \text{add } \text{Emp})$ **by** (*blast intro:Ang-weektrans Ang-rev Eq-rev*)
have $P21 : \text{Eq } (\text{Geos } (\text{Ang } (\text{An } l1\ o1\ h1))\ \text{add } \text{Emp}) (\text{Geos } (\text{Ang } (\text{An } h1\ o1\ l1))\ \text{add } \text{Emp})$ **by** (*simp add:Ang-roll*)
have $P22 : \text{Eq } (\text{Geos } (\text{Ang } (\text{An } l2\ o2\ h2))\ \text{add } \text{Emp}) (\text{Geos } (\text{Ang } (\text{An } h2\ o2\ l2))\ \text{add } \text{Emp})$ **by** (*simp add:Ang-roll*)

from $P18$ $P21$ **have** $P23 : Cong (Geos (Ang (An h1 o1 l1)) add Emp) (Geos (Ang (An l2 o2 h2)) add Emp)$ **by** (*blast intro:Ang-weektrans Ang-rev Eq-rev*)
from $P22$ $P23$ **have** $P24 : Cong (Geos (Ang (An h1 o1 l1)) add Emp) (Geos (Ang (An h2 o2 l2)) add Emp)$ **by** (*blast intro:Ang-weektrans Ang-rev Eq-rev*)
from $P18$ $P20$ $P24$ **show** $\exists p. Ang-inside (An h2 o2 k2) p$
 $\wedge Cong (Geos (Ang (An h1 o1 l1)) add Emp) (Geos (Ang (An h2 o2 p)) add Emp)$
 $\wedge Cong (Geos (Ang (An k1 o1 l1)) add Emp) (Geos (Ang (An k2 o2 p)) add Emp)$ **by** *blast*
qed

theorem (in Congruence-Rule) Ang-split-unique :

assumes

$Def (Ang (An h1 o1 k1)) Def (Ang (An h2 o2 k2))$

$Cong (Geos (Ang (An h1 o1 k1)) add Emp) (Geos (Ang (An h2 o2 k2)) add Emp)$

$Ang-inside (An h1 o1 k1) l1$

$Ang-inside (An h2 o2 k2) l21$

$Cong (Geos (Ang (An h1 o1 l1)) add Emp) (Geos (Ang (An h2 o2 l21)) add Emp)$

$Cong (Geos (Ang (An k1 o1 l1)) add Emp) (Geos (Ang (An k2 o2 l21)) add Emp)$

$Ang-inside (An h2 o2 k2) l22$

$Cong (Geos (Ang (An h1 o1 l1)) add Emp) (Geos (Ang (An h2 o2 l22)) add Emp)$

$Cong (Geos (Ang (An k1 o1 l1)) add Emp) (Geos (Ang (An k2 o2 l22)) add Emp)$

shows

$Eq (Geos (Lin (Li o2 l21)) add Emp) (Geos (Lin (Li o2 l22)) add Emp)$

proof –

from *assms* **have** $Plane-sameside (Li o2 h2) k2 l21 \wedge Plane-sameside (Li o2 k2) h2 l21$ **by** (*simp add:Ang-inside-def*)

then have $P1 : Plane-sameside (Li o2 k2) l21 h2$ **by** (*simp add:Plane-sameside-rev*)

from *assms* **have** $P2 : Plane-sameside (Li o2 h2) k2 l22 \wedge Plane-sameside (Li o2 k2) h2 l22$ **by** (*simp add:Ang-inside-def*)

from $P1$ $P2$ **have** $P3 : \neg Eq (Geos (Poi l22) add Emp) (Geos (Poi l21) add Emp) \implies$

$Plane-sameside (Li o2 k2) l21 l22$ **by** (*simp add:Plane-sameside-trans*)

have $P4 : Eq (Geos (Ang (An k2 o2 l21)) add Emp) (Geos (Ang (An l21 o2 k2)) add Emp)$ **by** (*simp add:Ang-roll*)

from *assms* $P4$ **have** $P5 : Cong (Geos (Ang (An k1 o1 l1)) add Emp) (Geos (Ang (An l21 o2 k2)) add Emp)$ **by** (*blast intro:Ang-weektrans Ang-rev Eq-rev*)

have $P6 : Eq (Geos (Ang (An k2 o2 l22)) add Emp) (Geos (Ang (An l22 o2 k2)) add Emp)$ **by** (*simp add:Ang-roll*)

from *assms* $P6$ **have** $P7 : Cong (Geos (Ang (An k1 o1 l1)) add Emp) (Geos (Ang (An l22 o2 k2)) add Emp)$ **by** (*blast intro:Ang-weektrans Ang-rev Eq-rev*)

from $P3$ $P5$ $P7$ **have** $P8 : \neg Eq (Geos (Poi l22) add Emp) (Geos (Poi l21) add Emp) \implies$

$Eq (Geos (Lin (Li l21 o2)) add Emp) (Geos (Lin (Li l22 o2)) add Emp)$ **by**

(simp add:Ang-move-unique)
have $P9 : \text{Line-on } (Li\ o2\ k2)\ o2$ **by** *(simp add:Line-on-rule)*
from $P1$ **have** $P10 : \neg \text{Line-on } (Li\ o2\ k2)\ l21$ **by** *(simp add:Plane-sameside-def)*
from $P9$ **have** $P11 : \text{Eq } (\text{Geos } (Poi\ o2)\ \text{add Emp}) (\text{Geos } (Poi\ l21)\ \text{add Emp})$
 $\implies \text{Line-on } (Li\ o2\ k2)\ l21$ **by** *(simp add:Point-Eq)*
from $P10\ P11$ **have** $P12 : \neg \text{Eq } (\text{Geos } (Poi\ o2)\ \text{add Emp}) (\text{Geos } (Poi\ l21)\ \text{add Emp})$ **by** *blast*
then **have** $P13 : \text{Eq } (\text{Geos } (Lin\ (Li\ o2\ l21))\ \text{add Emp}) (\text{Geos } (Lin\ (Li\ l21\ o2))\ \text{add Emp})$ **by** *(simp add:Line-rev)*
from $P2$ **have** $P14 : \neg \text{Line-on } (Li\ o2\ k2)\ l22$ **by** *(simp add:Plane-sameside-def)*
from $P9$ **have** $P15 : \text{Eq } (\text{Geos } (Poi\ o2)\ \text{add Emp}) (\text{Geos } (Poi\ l22)\ \text{add Emp})$
 $\implies \text{Line-on } (Li\ o2\ k2)\ l22$ **by** *(simp add:Point-Eq)*
from $P14\ P15$ **have** $\neg \text{Eq } (\text{Geos } (Poi\ o2)\ \text{add Emp}) (\text{Geos } (Poi\ l22)\ \text{add Emp})$ **by** *blast*
then **have** $P16 : \text{Eq } (\text{Geos } (Lin\ (Li\ o2\ l22))\ \text{add Emp}) (\text{Geos } (Lin\ (Li\ l22\ o2))\ \text{add Emp})$ **by** *(simp add:Line-rev)*
from $P8\ P13$ **have** $P17 : \neg \text{Eq } (\text{Geos } (Poi\ l22)\ \text{add Emp}) (\text{Geos } (Poi\ l21)\ \text{add Emp})$ \implies
 $\text{Eq } (\text{Geos } (Lin\ (Li\ o2\ l21))\ \text{add Emp}) (\text{Geos } (Lin\ (Li\ l22\ o2))\ \text{add Emp})$ **by**
(blast intro:Eq-rev Eq-trans)
from $P16\ P17$ **have** $P18 : \neg \text{Eq } (\text{Geos } (Poi\ l22)\ \text{add Emp}) (\text{Geos } (Poi\ l21)\ \text{add Emp})$ \implies
 $\text{Eq } (\text{Geos } (Lin\ (Li\ o2\ l21))\ \text{add Emp}) (\text{Geos } (Lin\ (Li\ o2\ l22))\ \text{add Emp})$ **by**
(blast intro:Eq-rev Eq-trans)
have $P19 : \text{Line-on } (Li\ o2\ l21)\ o2$ **by** *(simp add:Line-on-rule)*
have $P20 : \text{Line-on } (Li\ o2\ l21)\ l21$ **by** *(simp add:Line-on-rule)*
have $P21 : \text{Line-on } (Li\ o2\ l22)\ o2$ **by** *(simp add:Line-on-rule)*
have $\text{Line-on } (Li\ o2\ l22)\ l22$ **by** *(simp add:Line-on-rule)*
then **have** $P22 : \text{Eq } (\text{Geos } (Poi\ l22)\ \text{add Emp}) (\text{Geos } (Poi\ l21)\ \text{add Emp})$ \implies
 $\text{Line-on } (Li\ o2\ l22)\ l21$ **by** *(simp add:Point-Eq)*
from $P12\ P19\ P20\ P21\ P22$ **have** $P23 : \text{Eq } (\text{Geos } (Poi\ l22)\ \text{add Emp}) (\text{Geos } (Poi\ l21)\ \text{add Emp})$ \implies
 $\text{Eq } (\text{Geos } (Lin\ (Li\ o2\ l21))\ \text{add Emp}) (\text{Geos } (Lin\ (Li\ o2\ l22))\ \text{add Emp})$ **by**
(simp add:Line-unique)
from $P18\ P23$ **show** $\text{Eq } (\text{Geos } (Lin\ (Li\ o2\ l21))\ \text{add Emp}) (\text{Geos } (Lin\ (Li\ o2\ l22))\ \text{add Emp})$ **by** *blast*
qed

lemma (in Congruence-Rule) Tri-week-SSS-lemma1 :

assumes
 $\text{Plane-diffside } (Li\ x\ y)\ z1\ z2$
 $\neg \text{Eq } (\text{Geos } (Poi\ x)\ \text{add Emp}) (\text{Geos } (Poi\ y)\ \text{add Emp})$
 $\text{Eq } (\text{Geos } (Seg\ (Se\ x\ z1))\ \text{add Emp}) (\text{Geos } (Seg\ (Se\ x\ z2))\ \text{add Emp})$
 $\text{Eq } (\text{Geos } (Seg\ (Se\ y\ z1))\ \text{add Emp}) (\text{Geos } (Seg\ (Se\ y\ z2))\ \text{add Emp})$
 $\exists p. \text{Bet-Point } (Se\ z1\ z2)\ p \wedge \text{Line-on } (Li\ x\ y)\ p \wedge \text{Eq } (\text{Geos } (Poi\ x)\ \text{add Emp})$
 $(\text{Geos } (Poi\ p)\ \text{add Emp})$
shows $\text{Cong } (\text{Geos } (Ang\ (An\ x\ z1\ y))\ \text{add Emp}) (\text{Geos } (Ang\ (An\ x\ z2\ y))\ \text{add Emp})$
proof –

from *assms* **have** $P1 : \exists p. \text{Bet-Point } (Se\ z1\ z2)\ p \wedge \text{Line-on } (Li\ x\ y)\ p$
 $\wedge \neg \text{Line-on } (Li\ x\ y)\ z1 \wedge \neg \text{Line-on } (Li\ x\ y)\ z2$ **by** (*simp add:Plane-diffside-def*)
from *assms* **obtain** $pn :: \text{Point}$ **where** $P2 : \text{Bet-Point } (Se\ z1\ z2)\ pn \wedge \text{Line-on}$
 $(Li\ x\ y)\ pn \wedge \text{Eq } (\text{Geos } (\text{Poi } x)\ \text{add } \text{Emp})\ (\text{Geos } (\text{Poi } pn)\ \text{add } \text{Emp})$ **by** *blast*
from $P2$ **have** $P3 : \text{Bet-Point } (Se\ z1\ z2)\ pn$ **by** *simp*
then **have** $P4 : \text{Line-on } (Li\ z1\ z2)\ pn$ **by** (*simp add:Line-Bet-on*)
from $P2\ P4$ **have** $P5 : \text{Line-on } (Li\ z1\ z2)\ x$ **by** (*blast intro:Eq-rev Point-Eq*)
from *assms* $P3$ **have** $P6 : \text{Bet-Point } (Se\ z1\ z2)\ x$ **by** (*blast intro:Eq-rev Point-Eq*)
have $P7 : \text{Line-on } (Li\ x\ y)\ x$ **by** (*simp add:Line-on-rule*)
have $P8 : \text{Line-on } (Li\ x\ y)\ y$ **by** (*simp add:Line-on-rule*)
from *assms* $P5\ P7\ P8$ **have** $P9 : \text{Line-on } (Li\ z1\ z2)\ y \implies \text{Eq } (\text{Geos } (\text{Lin } (Li\ z1\ z2))\ \text{add } \text{Emp})\ (\text{Geos } (\text{Lin } (Li\ x\ y))\ \text{add } \text{Emp})$ **by** (*simp add:Line-unique*)
have $P10 : \text{Line-on } (Li\ z1\ z2)\ z1$ **by** (*simp add:Line-on-rule*)
from $P9\ P10$ **have** $P11 : \text{Line-on } (Li\ z1\ z2)\ y \implies \text{Line-on } (Li\ x\ y)\ z1$ **by** (*simp add:Line-on-trans*)
from $P1\ P11$ **have** $P12 : \neg \text{Line-on } (Li\ z1\ z2)\ y$ **by** *blast*
from $P3$ **have** $P13 : \neg \text{Eq } (\text{Geos } (\text{Poi } z1)\ \text{add } \text{Emp})\ (\text{Geos } (\text{Poi } z2)\ \text{add } \text{Emp})$
by (*simp add:Bet-Point-def*)
from $P8$ **have** $P14 : \text{Eq } (\text{Geos } (\text{Poi } z2)\ \text{add } \text{Emp})\ (\text{Geos } (\text{Poi } y)\ \text{add } \text{Emp}) \implies$
 $\text{Line-on } (Li\ x\ y)\ z2$ **by** (*blast intro:Eq-rev Point-Eq*)
from $P1\ P14$ **have** $P15 : \neg \text{Eq } (\text{Geos } (\text{Poi } z2)\ \text{add } \text{Emp})\ (\text{Geos } (\text{Poi } y)\ \text{add } \text{Emp})$ **by** *blast*
from $P8$ **have** $P16 : \text{Eq } (\text{Geos } (\text{Poi } y)\ \text{add } \text{Emp})\ (\text{Geos } (\text{Poi } z1)\ \text{add } \text{Emp}) \implies$
 $\text{Line-on } (Li\ x\ y)\ z1$ **by** (*blast intro:Eq-rev Point-Eq*)
from $P1\ P16$ **have** $P17 : \neg \text{Eq } (\text{Geos } (\text{Poi } y)\ \text{add } \text{Emp})\ (\text{Geos } (\text{Poi } z1)\ \text{add } \text{Emp})$ **by** *blast*
from $P12\ P13$ **have** $P18 : \text{Def } (\text{Tri } (\text{Tr } z1\ z2\ y))$ **by** (*simp add:Ang-simple-def Ang-to-Tri*)
then **have** $P19 : \text{Def } (\text{Tri } (\text{Tr } y\ z1\ z2))$ **by** (*blast intro:Tri-def-trans*)
from *assms* $P19$ **have** $P20 : \text{Cong } (\text{Geos } (\text{Ang } (\text{An } y\ z1\ z2))\ \text{add } \text{Emp})\ (\text{Geos } (\text{Ang } (\text{An } y\ z2\ z1))\ \text{add } \text{Emp})$ **by** (*simp add:Tri-isosceles*)
from $P18$ **have** $P21 : \text{Def } (\text{Ang } (\text{An } z1\ z2\ y))$ **by** (*simp add:Tri-to-Ang*)
from $P6$ **have** $P22 : \text{Line-on } (Li\ z2\ z1)\ x$ **by** (*simp add:Line-Bet-on*)
from $P6$ **have** $P23 : \text{Inv } (\text{Bet-Point } (Se\ z2\ x)\ z1) \wedge \text{Inv } (\text{Bet-Point } (Se\ x\ z1)\ z2)$ **by** (*simp add:Bet-iff*)
then **have** $\neg \text{Bet-Point } (Se\ x\ z1)\ z2$ **by** (*simp add:Inv-def*)
then **have** $P24 : \neg \text{Bet-Point } (Se\ z1\ x)\ z2$ **by** (*blast intro:Bet-rev*)
have $P25 : \text{Line-on } (Li\ z2\ y)\ y$ **by** (*simp add:Line-on-rule*)
have $P26 : \neg \text{Bet-Point } (Se\ y\ y)\ z2$ **by** (*simp add:Bet-end-Point*)
from $P6$ **have** $P27 : \neg \text{Eq } (\text{Geos } (\text{Poi } z2)\ \text{add } \text{Emp})\ (\text{Geos } (\text{Poi } x)\ \text{add } \text{Emp})$
by (*simp add:Bet-Point-def*)
from $P15\ P21\ P22\ P24\ P25\ P26\ P27$ **have** $P28 :$
 $\text{Eq } (\text{Geos } (\text{Ang } (\text{An } z1\ z2\ y))\ \text{add } \text{Emp})\ (\text{Geos } (\text{Ang } (\text{An } x\ z2\ y))\ \text{add } \text{Emp}) \wedge$
 $\text{Def } (\text{Ang } (\text{An } x\ z2\ y))$ **by** (*simp add:Ang-Point-swap*)
have $P29 : \text{Eq } (\text{Geos } (\text{Ang } (\text{An } y\ z2\ z1))\ \text{add } \text{Emp})\ (\text{Geos } (\text{Ang } (\text{An } z1\ z2\ y))\ \text{add } \text{Emp})$ **by** (*simp add:Ang-roll*)
from $P20\ P29$ **have** $P30 : \text{Cong } (\text{Geos } (\text{Ang } (\text{An } y\ z1\ z2))\ \text{add } \text{Emp})\ (\text{Geos } (\text{Ang } (\text{An } z1\ z2\ y))\ \text{add } \text{Emp})$ **by** (*blast intro:Ang-weektrans Ang-rev Eq-rev*)
from $P28\ P30$ **have** $P31 : \text{Cong } (\text{Geos } (\text{Ang } (\text{An } y\ z1\ z2))\ \text{add } \text{Emp})\ (\text{Geos } (\text{Ang } (\text{An } x\ z2\ y))\ \text{add } \text{Emp})$

$(An\ x\ z2\ y))\ add\ Emp)$ **by** (*blast intro:Ang-weektrans Ang-rev Eq-rev*)
from $P19$ **have** $P32 : Def\ (Ang\ (An\ z2\ z1\ y))$ **by** (*blast intro:Tri-to-Ang Tri-def-rev*)
from $P6$ **have** $P33 : Line-on\ (Li\ z1\ z2)\ x$ **by** (*simp add:Line-Bet-on*)
from $P23$ **have** $P34 : \neg\ Bet-Point\ (Se\ z2\ x)\ z1$ **by** (*simp add:Inv-def*)
have $P35 : Line-on\ (Li\ z1\ y)\ y$ **by** (*simp add:Line-on-rule*)
have $P36 : \neg\ Bet-Point\ (Se\ y\ y)\ z1$ **by** (*simp add:Bet-end-Point*)
from $P6$ **have** $\neg\ Eq\ (Geos\ (Poi\ x)\ add\ Emp)\ (Geos\ (Poi\ z1)\ add\ Emp)$ **by** (*simp add:Bet-Point-def*)
then **have** $P37 : \neg\ Eq\ (Geos\ (Poi\ z1)\ add\ Emp)\ (Geos\ (Poi\ x)\ add\ Emp)$ **by** (*blast intro:Eq-rev*)
from $P17$ **have** $P38 : \neg\ Eq\ (Geos\ (Poi\ z1)\ add\ Emp)\ (Geos\ (Poi\ y)\ add\ Emp)$ **by** (*blast intro:Eq-rev*)
from $P32\ P33\ P34\ P35\ P36\ P37\ P38$ **have** $P39 :$
 $Eq\ (Geos\ (Ang\ (An\ z2\ z1\ y))\ add\ Emp)\ (Geos\ (Ang\ (An\ x\ z1\ y))\ add\ Emp) \wedge$
 $Def\ (Ang\ (An\ x\ z1\ y))$ **by** (*simp add:Ang-Point-swap*)
have $P40 : Eq\ (Geos\ (Ang\ (An\ y\ z1\ z2))\ add\ Emp)\ (Geos\ (Ang\ (An\ z2\ z1\ y))\ add\ Emp)$ **by** (*simp add:Ang-roll*)
from $P31\ P40$ **have** $P41 : Cong\ (Geos\ (Ang\ (An\ z2\ z1\ y))\ add\ Emp)\ (Geos\ (Ang\ (An\ x\ z2\ y))\ add\ Emp)$ **by** (*blast intro:Ang-weektrans Ang-rev Eq-rev*)
from $P39\ P41$ **show** $Cong\ (Geos\ (Ang\ (An\ x\ z1\ y))\ add\ Emp)\ (Geos\ (Ang\ (An\ x\ z2\ y))\ add\ Emp)$ **by** (*blast intro:Ang-weektrans Ang-rev Eq-rev*)
qed

Theorem17

theorem (*in Congruence-Rule*) *Tri-week-SSS* :

assumes

$Plane-diffside\ (Li\ x\ y)\ z1\ z2$

$\neg\ Eq\ (Geos\ (Poi\ x)\ add\ Emp)\ (Geos\ (Poi\ y)\ add\ Emp)$

$Eq\ (Geos\ (Seg\ (Se\ x\ z1))\ add\ Emp)\ (Geos\ (Seg\ (Se\ x\ z2))\ add\ Emp)$

$Eq\ (Geos\ (Seg\ (Se\ y\ z1))\ add\ Emp)\ (Geos\ (Seg\ (Se\ y\ z2))\ add\ Emp)$

shows $Cong\ (Geos\ (Ang\ (An\ x\ y\ z1))\ add\ Emp)\ (Geos\ (Ang\ (An\ x\ y\ z2))\ add\ Emp)$

proof –

from *assms* **have** $\exists p.\ Bet-Point\ (Se\ z1\ z2)\ p \wedge Line-on\ (Li\ x\ y)\ p$

$\wedge \neg\ Line-on\ (Li\ x\ y)\ z1 \wedge \neg\ Line-on\ (Li\ x\ y)\ z2$ **by** (*simp add:Plane-diffside-def*)

then **obtain** $pn :: Point$ **where** $P1 : Bet-Point\ (Se\ z1\ z2)\ pn \wedge Line-on\ (Li\ x\ y)\ pn$

$\wedge \neg\ Line-on\ (Li\ x\ y)\ z1 \wedge \neg\ Line-on\ (Li\ x\ y)\ z2$ **by** *blast*

have $P2 : Eq\ (Geos\ (Poi\ x)\ add\ Emp)\ (Geos\ (Poi\ pn)\ add\ Emp) \wedge Eq\ (Geos\ (Poi\ y)\ add\ Emp)\ (Geos\ (Poi\ pn)\ add\ Emp) \implies$

$Eq\ (Geos\ (Poi\ x)\ add\ Emp)\ (Geos\ (Poi\ y)\ add\ Emp)$ **by** (*blast intro:Eq-trans*)

from *assms* $P2$ **have** $\neg\ (Eq\ (Geos\ (Poi\ x)\ add\ Emp)\ (Geos\ (Poi\ pn)\ add\ Emp) \wedge Eq\ (Geos\ (Poi\ y)\ add\ Emp)\ (Geos\ (Poi\ pn)\ add\ Emp))$ **by** *blast*

then **have** $P3 : Eq\ (Geos\ (Poi\ x)\ add\ Emp)\ (Geos\ (Poi\ pn)\ add\ Emp) \wedge \neg\ Eq\ (Geos\ (Poi\ y)\ add\ Emp)\ (Geos\ (Poi\ pn)\ add\ Emp)$

$\vee \neg\ Eq\ (Geos\ (Poi\ x)\ add\ Emp)\ (Geos\ (Poi\ pn)\ add\ Emp) \wedge Eq\ (Geos\ (Poi\ y)\ add\ Emp)\ (Geos\ (Poi\ pn)\ add\ Emp)$

$\vee \neg\ Eq\ (Geos\ (Poi\ x)\ add\ Emp)\ (Geos\ (Poi\ pn)\ add\ Emp) \wedge \neg\ Eq\ (Geos\ (Poi\ y)\ add\ Emp)\ (Geos\ (Poi\ pn)\ add\ Emp)$

y) *add Emp*) (Geos (Poi pn) *add Emp*) **by** *blast*
from $P1$ **have** $P4 : Eq$ (Geos (Poi x) *add Emp*) (Geos (Poi pn) *add Emp*) \implies
 $\exists p. Bet\text{-}Point$ (Se $z1$ $z2$) $p \wedge Line\text{-}on$ (Li x y) $p \wedge Eq$ (Geos (Poi x) *add Emp*)
(Geos (Poi p) *add Emp*) **by** *blast*
from *assms* $P4$ **have** $P5 : Eq$ (Geos (Poi x) *add Emp*) (Geos (Poi pn) *add Emp*)
 \implies
Cong (Geos (Ang (An x $z1$ y)) *add Emp*) (Geos (Ang (An x $z2$ y)) *add Emp*)
by (*simp add:Tri-week-SSS-lemma1*)
have $P6 : Line\text{-}on$ (Li x y) x **by** (*simp add:Line-on-rule*)
then **have** $P7 : Eq$ (Geos (Poi x) *add Emp*) (Geos (Poi $z1$) *add Emp*) \implies
Line-on (Li x y) $z1$ **by** (*simp add:Point-Eq*)
from $P1$ $P7$ **have** $P8 : \neg Eq$ (Geos (Poi $z1$) *add Emp*) (Geos (Poi x) *add Emp*)
by (*blast intro:Eq-rev*)
have $P9 : Line\text{-}on$ (Li x y) y **by** (*simp add:Line-on-rule*)
then **have** $P10 : Eq$ (Geos (Poi y) *add Emp*) (Geos (Poi $z1$) *add Emp*) \implies
Line-on (Li x y) $z1$ **by** (*simp add:Point-Eq*)
from $P1$ $P10$ **have** $P11 : \neg Eq$ (Geos (Poi y) *add Emp*) (Geos (Poi $z1$) *add*
Emp) **by** *blast*
from *assms* $P1$ **have** *Def* (Tri (Tr x y $z1$)) **by** (*simp add:Ang-simple-def Ang-to-Tri*)
then **have** $P12 : Def$ (Tri (Tr $z1$ x y)) **by** (*simp add:Tri-def-trans*)
from $P6$ **have** $P13 : Eq$ (Geos (Poi x) *add Emp*) (Geos (Poi $z2$) *add Emp*) \implies
Line-on (Li x y) $z2$ **by** (*simp add:Point-Eq*)
from $P1$ $P13$ **have** $P14 : \neg Eq$ (Geos (Poi $z2$) *add Emp*) (Geos (Poi x) *add*
Emp) **by** (*blast intro:Eq-rev*)
from $P9$ **have** $P15 : Eq$ (Geos (Poi y) *add Emp*) (Geos (Poi $z2$) *add Emp*) \implies
Line-on (Li x y) $z2$ **by** (*simp add:Point-Eq*)
from $P1$ $P15$ **have** $P16 : \neg Eq$ (Geos (Poi y) *add Emp*) (Geos (Poi $z2$) *add*
Emp) **by** (*blast intro:Eq-rev*)
from *assms* $P1$ **have** *Def* (Tri (Tr x y $z2$)) **by** (*simp add:Ang-simple-def Ang-to-Tri*)
then **have** $P17 : Def$ (Tri (Tr $z2$ x y)) **by** (*simp add:Tri-def-trans*)
have $P18 : Eq$ (Geos (Seg (Se x $z1$)) *add Emp*) (Geos (Seg (Se $z1$ x)) *add Emp*)
by (*blast intro:Seg-rev*)
have $P19 : Eq$ (Geos (Seg (Se x $z2$)) *add Emp*) (Geos (Seg (Se $z2$ x)) *add Emp*)
by (*blast intro:Seg-rev*)
from *assms* $P18$ **have** $P20 : Eq$ (Geos (Seg (Se $z1$ x)) *add Emp*) (Geos (Seg (Se
 x $z2$)) *add Emp*) **by** (*blast intro:Eq-trans Eq-rev*)
from $P19$ $P20$ **have** $P21 : Eq$ (Geos (Seg (Se $z1$ x)) *add Emp*) (Geos (Seg (Se
 $z2$ x)) *add Emp*) **by** (*blast intro:Eq-trans Eq-rev*)
have $P22 : Eq$ (Geos (Seg (Se y $z1$)) *add Emp*) (Geos (Seg (Se $z1$ y)) *add Emp*)
by (*blast intro:Seg-rev*)
have $P23 : Eq$ (Geos (Seg (Se y $z2$)) *add Emp*) (Geos (Seg (Se $z2$ y)) *add Emp*)
by (*blast intro:Seg-rev*)
from *assms* $P22$ **have** $P24 : Eq$ (Geos (Seg (Se $z1$ y)) *add Emp*) (Geos (Seg (Se
 y $z2$)) *add Emp*) **by** (*blast intro:Eq-trans Eq-rev*)
from $P23$ $P24$ **have** $P25 : Eq$ (Geos (Seg (Se $z1$ y)) *add Emp*) (Geos (Seg (Se
 $z2$ y)) *add Emp*) **by** (*blast intro:Eq-trans Eq-rev*)
from $P5$ $P12$ $P17$ $P21$ $P25$ **have** *Eq* (Geos (Poi x) *add Emp*) (Geos (Poi pn)
add Emp) \implies
Cong (Geos (Tri (Tr $z1$ x y)) *add Emp*) (Geos (Tri (Tr $z2$ x y)) *add Emp*) **by**

(simp add:Tri-SAS)
then have $P26 : Eq (Geos (Poi x) add Emp) (Geos (Poi pn) add Emp) \implies$
 $Cong (Geos (Ang (An z1 y x)) add Emp) (Geos (Ang (An z2 y x)) add Emp)$
by *(simp add:Tri-Cong-def)*
have $P27 : Eq (Geos (Ang (An z1 y x)) add Emp) (Geos (Ang (An x y z1)) add$
 $Emp)$ **by** *(simp add:Ang-roll)*
have $P28 : Eq (Geos (Ang (An z2 y x)) add Emp) (Geos (Ang (An x y z2)) add$
 $Emp)$ **by** *(simp add:Ang-roll)*
from $P26 P27$ **have** $P29 : Eq (Geos (Poi x) add Emp) (Geos (Poi pn) add Emp)$
 \implies
 $Cong (Geos (Ang (An x y z1)) add Emp) (Geos (Ang (An z2 y x)) add Emp)$
by *(blast intro:Ang-weektrans Ang-rev Eq-rev)*
from $P28 P29$ **have** $P30 : Eq (Geos (Poi x) add Emp) (Geos (Poi pn) add Emp)$
 \implies
 $Cong (Geos (Ang (An x y z1)) add Emp) (Geos (Ang (An x y z2)) add Emp)$
by *(blast intro:Ang-weektrans Ang-rev Eq-rev)*
from *assms* **have** $P31 : Eq (Geos (Lin (Li x y)) add Emp) (Geos (Lin (Li y x))$
 $add Emp)$ **by** *(simp add:Line-rev)*
from *assms* $P31$ **have** $P32 : Plane-diffside (Li y x) z1 z2$ **by** *(simp add:Plane-Line-diff-trans)*
from *assms* **have** $P33 : \neg Eq (Geos (Poi y) add Emp) (Geos (Poi x) add Emp)$
by *(blast intro:Eq-rev)*
from $P1 P31$ **have** $P34 : Line-on (Li y x) pn$ **by** *(blast intro:Line-on-trans)*
from $P1 P34$ **have** $P35 : Eq (Geos (Poi y) add Emp) (Geos (Poi pn) add Emp)$
 \implies
 $\exists p. Bet-Point (Se z1 z2) p \wedge Line-on (Li y x) p \wedge Eq (Geos (Poi y) add Emp)$
 $(Geos (Poi p) add Emp)$ **by** *blast*
from *assms* $P32 P33 P35$ **have** $P36 : Eq (Geos (Poi y) add Emp) (Geos (Poi$
 $pn) add Emp) \implies$
 $Cong (Geos (Ang (An y z1 x)) add Emp) (Geos (Ang (An y z2 x)) add Emp)$
by *(simp add:Tri-week-SSS-lemma1)*
from $P12$ **have** $P37 : Def (Tri (Tr z1 y x))$ **by** *(blast intro:Tri-def-trans*
 $Tri-def-rev)$
from $P17$ **have** $P38 : Def (Tri (Tr z2 y x))$ **by** *(blast intro:Tri-def-trans*
 $Tri-def-rev)$
from $P21 P25 P36 P37 P38$ **have** $Eq (Geos (Poi y) add Emp) (Geos (Poi pn)$
 $add Emp) \implies$
 $Cong (Geos (Tri (Tr z1 y x)) add Emp) (Geos (Tri (Tr z2 y x)) add Emp)$ **by**
(simp add:Tri-SAS)
then have $P39 : Eq (Geos (Poi y) add Emp) (Geos (Poi pn) add Emp) \implies$
 $Cong (Geos (Ang (An x y z1)) add Emp) (Geos (Ang (An x y z2)) add Emp)$
by *(simp add:Tri-Cong-def)*
from $P1$ **have** $P40 : Bet-Point (Se z1 z2) pn$ **by** *simp*
then have $P41 : Line-on (Li z1 z2) pn$ **by** *(simp add:Line-Bet-on)*
have $Line-on (Li z1 z2) z1$ **by** *(simp add:Line-on-rule)*
then have $P42 : Eq (Geos (Lin (Li z1 z2)) add Emp) (Geos (Lin (Li x y)) add$
 $Emp) \implies Line-on (Li x y) z1$ **by** *(simp add:Line-on-trans)*
from $P1 P42$ **have** $P43 : \neg Eq (Geos (Lin (Li z1 z2)) add Emp) (Geos (Lin (Li$
 $x y)) add Emp)$ **by** *blast*
from $P1 P6 P41$ **have** $P44 : \neg Eq (Geos (Poi x) add Emp) (Geos (Poi pn) add$

$Emp) \implies Line-on (Li\ z1\ z2)\ x \implies$
 $Eq (Geos (Lin (Li\ z1\ z2))\ add\ Emp) (Geos (Lin (Li\ x\ y))\ add\ Emp) \text{ by } (simp\ add:Line-unique)$
from $P_{43}\ P_{44}$ **have** $P_{45} : \neg Eq (Geos (Poi\ x)\ add\ Emp) (Geos (Poi\ pn)\ add\ Emp) \implies$
 $\neg Line-on (Li\ z1\ z2)\ x \text{ by } blast$
from P_{40} **have** $P_{46} : \neg Eq (Geos (Poi\ z1)\ add\ Emp) (Geos (Poi\ z2)\ add\ Emp)$
by $(simp\ add:Bet-Point-def)$
from $P_{45}\ P_{46}$ **have** $\neg Eq (Geos (Poi\ x)\ add\ Emp) (Geos (Poi\ pn)\ add\ Emp)$
 \implies
 $Def (Tri (Tr\ z1\ z2\ x)) \text{ by } (simp\ add:Ang-simple-def\ Ang-to-Tri)$
then have $P_{47} : \neg Eq (Geos (Poi\ x)\ add\ Emp) (Geos (Poi\ pn)\ add\ Emp) \implies$
 $Def (Tri (Tr\ x\ z1\ z2)) \text{ by } (simp\ add:Tri-def-trans)$
from $assms\ P_{47}$ **have** $P_{48} : \neg Eq (Geos (Poi\ x)\ add\ Emp) (Geos (Poi\ pn)\ add\ Emp) \implies$
 $Cong (Geos (Ang (An\ x\ z1\ z2))\ add\ Emp) (Geos (Ang (An\ x\ z2\ z1))\ add\ Emp)$
by $(simp\ add:Tri-isosceles)$
have $P_{49} : Line-on (Li\ z1\ x)\ x \text{ by } (simp\ add:Line-on-rule)$
have $P_{50} : \neg Bet-Point (Se\ x\ x)\ z1 \text{ by } (simp\ add:Bet-end-Point)$
from P_{40} **have** $P_{51} : Inv (Bet-Point (Se\ z2\ pn)\ z1) \wedge Inv (Bet-Point (Se\ pn\ z1)\ z2) \text{ by } (simp\ add:Bet-iff)$
then have $P_{52} : \neg Bet-Point (Se\ z2\ pn)\ z1 \text{ by } (simp\ add:Inv-def)$
from P_{40} **have** $\neg Eq (Geos (Poi\ pn)\ add\ Emp) (Geos (Poi\ z1)\ add\ Emp) \text{ by } (simp\ add:Bet-Point-def)$
then have $P_{53} : \neg Eq (Geos (Poi\ z1)\ add\ Emp) (Geos (Poi\ pn)\ add\ Emp) \text{ by } (blast\ intro:Eq-rev)$
from P_{47} **have** $P_{54} : \neg Eq (Geos (Poi\ x)\ add\ Emp) (Geos (Poi\ pn)\ add\ Emp) \implies$
 $Def (Ang (An\ x\ z1\ z2)) \text{ by } (simp\ add:Tri-to-Ang)$
from $P_8\ P_{41}\ P_{49}\ P_{50}\ P_{52}\ P_{53}\ P_{54}$ **have** $P_{55} : \neg Eq (Geos (Poi\ x)\ add\ Emp) (Geos (Poi\ pn)\ add\ Emp) \implies$
 $Eq (Geos (Ang (An\ x\ z1\ z2))\ add\ Emp) (Geos (Ang (An\ x\ z1\ pn))\ add\ Emp) \wedge$
 $Def (Ang (An\ x\ z1\ pn)) \text{ by } (simp\ add:Ang-Point-swap)$
from $P_{48}\ P_{55}$ **have** $P_{56} : \neg Eq (Geos (Poi\ x)\ add\ Emp) (Geos (Poi\ pn)\ add\ Emp) \implies$
 $Cong (Geos (Ang (An\ x\ z1\ pn))\ add\ Emp) (Geos (Ang (An\ x\ z2\ z1))\ add\ Emp)$
by $(blast\ intro:Ang-weektrans\ Ang-rev\ Eq-rev)$
from P_{47} **have** $P_{57} : \neg Eq (Geos (Poi\ x)\ add\ Emp) (Geos (Poi\ pn)\ add\ Emp) \implies$
 $Def (Ang (An\ x\ z2\ z1)) \text{ by } (blast\ intro:Tri-def-rev\ Tri-def-trans\ Tri-to-Ang)$
have $P_{58} : Line-on (Li\ z2\ x)\ x \text{ by } (simp\ add:Line-on-rule)$
have $P_{59} : \neg Bet-Point (Se\ x\ x)\ z2 \text{ by } (simp\ add:Bet-end-Point)$
from P_{40} **have** $P_{60} : Line-on (Li\ z2\ z1)\ pn \text{ by } (simp\ add:Line-Bet-on)$
from P_{51} **have** $\neg Bet-Point (Se\ pn\ z1)\ z2 \text{ by } (simp\ add:Inv-def)$
then have $P_{61} : \neg Bet-Point (Se\ z1\ pn)\ z2 \text{ by } (blast\ intro:Bet-rev)$
from P_{40} **have** $P_{62} : \neg Eq (Geos (Poi\ z2)\ add\ Emp) (Geos (Poi\ pn)\ add\ Emp)$
by $(simp\ add:Bet-Point-def)$
from $P_{14}\ P_{57}\ P_{58}\ P_{59}\ P_{60}\ P_{61}\ P_{62}$ **have** $P_{63} : \neg Eq (Geos (Poi\ x)\ add\ Emp) (Geos (Poi\ pn)\ add\ Emp) \implies$

$Eq (Geos (Ang (An x z2 z1)) add Emp) (Geos (Ang (An x z2 pn)) add Emp) \wedge$
Def (Ang (An x z2 pn)) **by** (*simp add:Ang-Point-swap*)
from *P56 P63* **have** $P64 : \neg Eq (Geos (Poi x) add Emp) (Geos (Poi pn) add Emp) \implies$
 $Cong (Geos (Ang (An x z1 pn)) add Emp) (Geos (Ang (An x z2 pn)) add Emp)$
by (*blast intro:Ang-weektrans Ang-rev Eq-rev*)
from *P1 P9 P41* **have** $P65 : \neg Eq (Geos (Poi y) add Emp) (Geos (Poi pn) add Emp) \implies$
 $Line-on (Li z1 z2) y \implies$
 $Eq (Geos (Lin (Li z1 z2)) add Emp) (Geos (Lin (Li x y)) add Emp)$ **by** (*simp add:Line-unique*)
from *P43 P65* **have** $P66 : \neg Eq (Geos (Poi y) add Emp) (Geos (Poi pn) add Emp) \implies$
 $\neg Line-on (Li z1 z2) y$ **by** *blast*
from *P46 P66* **have** $\neg Eq (Geos (Poi y) add Emp) (Geos (Poi pn) add Emp) \implies$
 $Def (Tri (Tr z1 z2 y))$ **by** (*simp add:Ang-simple-def Ang-to-Tri*)
then have $P67 : \neg Eq (Geos (Poi y) add Emp) (Geos (Poi pn) add Emp) \implies$
 $Def (Tri (Tr y z1 z2))$ **by** (*simp add:Tri-def-trans*)
from *assms P67* **have** $P68 : \neg Eq (Geos (Poi y) add Emp) (Geos (Poi pn) add Emp) \implies$
 $Cong (Geos (Ang (An y z1 z2)) add Emp) (Geos (Ang (An y z2 z1)) add Emp)$
by (*simp add:Tri-isosceles*)
from *P67* **have** $P69 : \neg Eq (Geos (Poi y) add Emp) (Geos (Poi pn) add Emp) \implies$
 $Def (Ang (An y z1 z2))$ **by** (*simp add:Tri-to-Ang*)
have $P70 : Line-on (Li z1 y) y$ **by** (*simp add:Line-on-rule*)
have $P71 : \neg Bet-Point (Se y y) z1$ **by** (*simp add:Bet-end-Point*)
from *P11* **have** $P72 : \neg Eq (Geos (Poi z1) add Emp) (Geos (Poi y) add Emp)$
by (*blast intro:Eq-rev*)
from *P41 P52 P53 P69 P70 P71 P72* **have** $P73 : \neg Eq (Geos (Poi y) add Emp) (Geos (Poi pn) add Emp) \implies$
 $Eq (Geos (Ang (An y z1 z2)) add Emp) (Geos (Ang (An y z1 pn)) add Emp) \wedge$
 $Def (Ang (An y z1 pn))$ **by** (*simp add:Ang-Point-swap*)
from *P68 P73* **have** $P74 : \neg Eq (Geos (Poi y) add Emp) (Geos (Poi pn) add Emp) \implies$
 $Cong (Geos (Ang (An y z1 pn)) add Emp) (Geos (Ang (An y z2 z1)) add Emp)$
by (*blast intro:Ang-weektrans Ang-rev Eq-rev*)
from *P69* **have** $P75 : \neg Eq (Geos (Poi y) add Emp) (Geos (Poi pn) add Emp) \implies$
 $Def (Ang (An y z2 z1))$ **by** (*simp add:Ang-def-inv*)
have $P76 : Line-on (Li z2 y) y$ **by** (*simp add:Line-on-rule*)
have $P77 : \neg Bet-Point (Se y y) z2$ **by** (*simp add:Bet-end-Point*)
from *P16* **have** $P78 : \neg Eq (Geos (Poi z2) add Emp) (Geos (Poi y) add Emp)$
by (*blast intro:Eq-rev*)
from *P60 P61 P62 P75 P76 P77 P78* **have** $P79 : \neg Eq (Geos (Poi y) add Emp) (Geos (Poi pn) add Emp) \implies$
 $Eq (Geos (Ang (An y z2 z1)) add Emp) (Geos (Ang (An y z2 pn)) add Emp) \wedge$
 $Def (Ang (An y z2 pn))$ **by** (*simp add:Ang-Point-swap*)
from *P74 P79* **have** $P80 : \neg Eq (Geos (Poi y) add Emp) (Geos (Poi pn) add Emp)$

$Emp) \implies$
Cong ($Geos (Ang (An y z1 pn)) add Emp$) ($Geos (Ang (An y z2 pn)) add Emp$)
by (*blast intro:Ang-weektrans Ang-rev Eq-rev*)
have $P81 : \neg Eq (Geos (Poi x) add Emp) (Geos (Poi pn) add Emp) \implies$
 $\neg Eq (Geos (Poi pn) add Emp) (Geos (Poi x) add Emp)$ **by** (*blast intro:Eq-rev*)
from *assms P1 P6 P9 P81* **have**
 $\neg Eq (Geos (Poi x) add Emp) (Geos (Poi pn) add Emp) \wedge \neg Eq (Geos (Poi y)$
 $add Emp) (Geos (Poi pn) add Emp) \implies$
 $Bet-Point (Se x pn) y \vee Bet-Point (Se pn y) x \vee Bet-Point (Se y x) pn$ **by**
(*simp add:Bet-case*)
then have $P82 :$
 $\neg Eq (Geos (Poi x) add Emp) (Geos (Poi pn) add Emp) \wedge \neg Eq (Geos (Poi y)$
 $add Emp) (Geos (Poi pn) add Emp) \implies$
 $Bet-Point (Se x pn) y \wedge \neg Bet-Point (Se pn y) x \wedge \neg Bet-Point (Se y x) pn$
 $\vee \neg Bet-Point (Se x pn) y \wedge Bet-Point (Se pn y) x \wedge \neg Bet-Point (Se y x) pn$
 $\vee \neg Bet-Point (Se x pn) y \wedge \neg Bet-Point (Se pn y) x \wedge Bet-Point (Se y x) pn$
by (*simp add:Bet-case-fact*)
have $P83 : Bet-Point (Se x pn) y \implies Bet-Point (Se pn x) y$ **by** (*simp add:Bet-rev*)
have $P84 : Line-on (Li z1 pn) pn$ **by** (*simp add:Line-on-rule*)
have $P85 : Line-on (Li pn x) x$ **by** (*simp add:Line-on-rule*)
from $P83$ **have** $P86 : Bet-Point (Se x pn) y \implies Line-on (Li pn x) y$ **by** (*simp*
add:Line-Bet-on)
from *assms P6 P9 P85 P86* **have** $P87 : Bet-Point (Se x pn) y \implies$
 $Eq (Geos (Lin (Li pn x)) add Emp) (Geos (Lin (Li x y)) add Emp)$ **by** (*simp*
add:Line-unique)
have $P88 : Line-on (Li z1 pn) z1$ **by** (*simp add:Line-on-rule*)
from $P87 P88$ **have** $P89 : Bet-Point (Se x pn) y \implies$
 $Eq (Geos (Lin (Li pn x)) add Emp) (Geos (Lin (Li z1 pn)) add Emp) \implies$
 $Line-on (Li x y) z1$ **by** (*blast intro:Line-on-trans Eq-rev*)
from $P1 P89$ **have** $P90 : Bet-Point (Se x pn) y \implies$
 $\neg Eq (Geos (Lin (Li pn x)) add Emp) (Geos (Lin (Li z1 pn)) add Emp)$ **by**
blast
from $P83 P84 P90$ **have** $P91 : Bet-Point (Se x pn) y \implies Plane-sameside (Li$
 $z1 pn) y x$ **by** (*simp add:Plane-Bet-sameside*)
have $P92 : Line-on (Li z2 pn) pn$ **by** (*simp add:Line-on-rule*)
have $P93 : Line-on (Li z2 pn) z2$ **by** (*simp add:Line-on-rule*)
from $P87 P93$ **have** $P94 : Bet-Point (Se x pn) y \implies$
 $Eq (Geos (Lin (Li pn x)) add Emp) (Geos (Lin (Li z2 pn)) add Emp) \implies$
 $Line-on (Li x y) z2$ **by** (*blast intro:Line-on-trans Eq-rev*)
from $P1 P94$ **have** $P95 : Bet-Point (Se x pn) y \implies$
 $\neg Eq (Geos (Lin (Li pn x)) add Emp) (Geos (Lin (Li z2 pn)) add Emp)$ **by**
blast
from $P83 P92 P95$ **have** $P96 : Bet-Point (Se x pn) y \implies Plane-sameside (Li$
 $z2 pn) y x$ **by** (*simp add:Plane-Bet-sameside*)
from $P37$ **have** $Def (Ang (An y z1 x))$ **by** (*blast intro:Tri-to-Ang Ang-def-rev*
Ang-def-inv)
then have $P97 : \neg Eq (Geos (Lin (Li z1 y)) add Emp) (Geos (Lin (Li z1 x))$
 $add Emp)$ **by** (*simp add:Ang-def*)
from $P38$ **have** $Def (Ang (An y z2 x))$ **by** (*blast intro:Tri-to-Ang Ang-def-rev*

Ang-def-inv
then have $P98 : \neg \text{Eq} (\text{Geos} (\text{Lin} (\text{Li} \ z2 \ y)) \ \text{add} \ \text{Emp}) (\text{Geos} (\text{Lin} (\text{Li} \ z2 \ x)) \ \text{add} \ \text{Emp})$ **by** (*simp add:Ang-def*)
from $P53 \ P62 \ P64 \ P80 \ P91 \ P96 \ P97 \ P98$ **have** $P99 : \neg \text{Eq} (\text{Geos} (\text{Poi} \ x) \ \text{add} \ \text{Emp}) (\text{Geos} (\text{Poi} \ pn) \ \text{add} \ \text{Emp})$
 $\wedge \neg \text{Eq} (\text{Geos} (\text{Poi} \ y) \ \text{add} \ \text{Emp}) (\text{Geos} (\text{Poi} \ pn) \ \text{add} \ \text{Emp}) \implies \text{Bet-Point} (\text{Se} \ x \ pn) \ y \implies$
 $\text{Cong} (\text{Geos} (\text{Ang} (\text{An} \ y \ z1 \ x)) \ \text{add} \ \text{Emp}) (\text{Geos} (\text{Ang} (\text{An} \ y \ z2 \ x)) \ \text{add} \ \text{Emp})$
by (*simp add:Ang-sub*)
have $P100 : \text{Line-on} (\text{Li} \ pn \ y) \ y$ **by** (*simp add:Line-on-rule*)
from $P85$ **have** $P101 : \text{Bet-Point} (\text{Se} \ pn \ y) \ x \implies \text{Line-on} (\text{Li} \ pn \ y) \ x$ **by** (*simp add:Line-Bet-on*)
from *assms* $P6 \ P9 \ P100 \ P101$ **have** $P102 : \text{Bet-Point} (\text{Se} \ pn \ y) \ x \implies$
 $\text{Eq} (\text{Geos} (\text{Lin} (\text{Li} \ pn \ y)) \ \text{add} \ \text{Emp}) (\text{Geos} (\text{Lin} (\text{Li} \ x \ y)) \ \text{add} \ \text{Emp})$ **by** (*simp add:Line-unique*)
from $P88 \ P102$ **have** $P103 : \text{Bet-Point} (\text{Se} \ pn \ y) \ x \implies$
 $\text{Eq} (\text{Geos} (\text{Lin} (\text{Li} \ pn \ y)) \ \text{add} \ \text{Emp}) (\text{Geos} (\text{Lin} (\text{Li} \ z1 \ pn)) \ \text{add} \ \text{Emp}) \implies$
 $\text{Line-on} (\text{Li} \ x \ y) \ z1$ **by** (*blast intro:Line-on-trans Eq-rev*)
from $P1 \ P103$ **have** $P104 : \text{Bet-Point} (\text{Se} \ pn \ y) \ x \implies$
 $\neg \text{Eq} (\text{Geos} (\text{Lin} (\text{Li} \ pn \ y)) \ \text{add} \ \text{Emp}) (\text{Geos} (\text{Lin} (\text{Li} \ z1 \ pn)) \ \text{add} \ \text{Emp})$ **by**
blast
from $P84 \ P104$ **have** $\text{Bet-Point} (\text{Se} \ pn \ y) \ x \implies \text{Plane-sameside} (\text{Li} \ z1 \ pn) \ x \ y$
by (*simp add:Plane-Bet-sameside*)
then have $P105 : \text{Bet-Point} (\text{Se} \ pn \ y) \ x \implies \text{Plane-sameside} (\text{Li} \ z1 \ pn) \ y \ x$ **by**
(*simp add:Plane-sameside-rev*)
from $P93 \ P102$ **have** $P106 : \text{Bet-Point} (\text{Se} \ pn \ y) \ x \implies$
 $\text{Eq} (\text{Geos} (\text{Lin} (\text{Li} \ pn \ y)) \ \text{add} \ \text{Emp}) (\text{Geos} (\text{Lin} (\text{Li} \ z2 \ pn)) \ \text{add} \ \text{Emp}) \implies$
 $\text{Line-on} (\text{Li} \ x \ y) \ z2$ **by** (*blast intro:Line-on-trans Eq-rev*)
from $P1 \ P106$ **have** $P107 : \text{Bet-Point} (\text{Se} \ pn \ y) \ x \implies$
 $\neg \text{Eq} (\text{Geos} (\text{Lin} (\text{Li} \ pn \ y)) \ \text{add} \ \text{Emp}) (\text{Geos} (\text{Lin} (\text{Li} \ z2 \ pn)) \ \text{add} \ \text{Emp})$ **by**
blast
from $P92 \ P107$ **have** $\text{Bet-Point} (\text{Se} \ pn \ y) \ x \implies \text{Plane-sameside} (\text{Li} \ z2 \ pn) \ x \ y$
by (*simp add:Plane-Bet-sameside*)
then have $P108 : \text{Bet-Point} (\text{Se} \ pn \ y) \ x \implies \text{Plane-sameside} (\text{Li} \ z2 \ pn) \ y \ x$ **by**
(*simp add:Plane-sameside-rev*)
from $P53 \ P62 \ P64 \ P80 \ P97 \ P98 \ P105 \ P108$ **have** $P109 : \neg \text{Eq} (\text{Geos} (\text{Poi} \ x) \ \text{add} \ \text{Emp}) (\text{Geos} (\text{Poi} \ pn) \ \text{add} \ \text{Emp})$
 $\wedge \neg \text{Eq} (\text{Geos} (\text{Poi} \ y) \ \text{add} \ \text{Emp}) (\text{Geos} (\text{Poi} \ pn) \ \text{add} \ \text{Emp}) \implies \text{Bet-Point} (\text{Se} \ pn \ y) \ x \implies$
 $\text{Cong} (\text{Geos} (\text{Ang} (\text{An} \ y \ z1 \ x)) \ \text{add} \ \text{Emp}) (\text{Geos} (\text{Ang} (\text{An} \ y \ z2 \ x)) \ \text{add} \ \text{Emp})$
by (*simp add:Ang-sub*)
have $P110 : \text{Bet-Point} (\text{Se} \ y \ x) \ pn \implies \text{Bet-Point} (\text{Se} \ x \ y) \ pn$ **by** (*simp add:Bet-rev*)
from $P88$ **have** $P111 : \text{Eq} (\text{Geos} (\text{Lin} (\text{Li} \ x \ y)) \ \text{add} \ \text{Emp}) (\text{Geos} (\text{Lin} (\text{Li} \ z1 \ pn)) \ \text{add} \ \text{Emp})$
 $\implies \text{Line-on} (\text{Li} \ x \ y) \ z1$ **by** (*blast intro:Line-on-trans Eq-rev*)
from $P1 \ P111$ **have** $P112 : \neg \text{Eq} (\text{Geos} (\text{Lin} (\text{Li} \ x \ y)) \ \text{add} \ \text{Emp}) (\text{Geos} (\text{Lin} (\text{Li} \ z1 \ pn)) \ \text{add} \ \text{Emp})$ **by** *blast*
from $P84 \ P110 \ P112$ **have** $\text{Bet-Point} (\text{Se} \ y \ x) \ pn \implies \text{Plane-diffside} (\text{Li} \ z1 \ pn)$

$x y$ **by** (*simp add:Plane-Bet-diffside*)
then have $P113 : \text{Bet-Point } (Se\ y\ x)\ pn \implies \text{Plane-diffside } (Li\ z1\ pn)\ y\ x$ **by**
(*simp add:Plane-diffside-rev*)
from $P93$ **have** $P114 : \text{Eq } (Geos\ (Lin\ (Li\ x\ y))\ \text{add}\ Emp)\ (Geos\ (Lin\ (Li\ z2\ pn))\ \text{add}\ Emp)$
 $\implies \text{Line-on } (Li\ x\ y)\ z2$ **by** (*blast intro:Line-on-trans Eq-rev*)
from $P1\ P114$ **have** $P115 : \neg \text{Eq } (Geos\ (Lin\ (Li\ x\ y))\ \text{add}\ Emp)\ (Geos\ (Lin\ (Li\ z2\ pn))\ \text{add}\ Emp)$ **by** *blast*
from $P92\ P110\ P115$ **have** $\text{Bet-Point } (Se\ y\ x)\ pn \implies \text{Plane-diffside } (Li\ z2\ pn)\ x\ y$ **by**
(*simp add:Plane-Bet-diffside*)
then have $P116 : \text{Bet-Point } (Se\ y\ x)\ pn \implies \text{Plane-diffside } (Li\ z2\ pn)\ y\ x$ **by**
(*simp add:Plane-diffside-rev*)
from $P53\ P62\ P64\ P80\ P97\ P98\ P113\ P116$ **have** $P117 : \neg \text{Eq } (Geos\ (Poi\ x)\ \text{add}\ Emp)\ (Geos\ (Poi\ pn)\ \text{add}\ Emp)$
 $\wedge \neg \text{Eq } (Geos\ (Poi\ y)\ \text{add}\ Emp)\ (Geos\ (Poi\ pn)\ \text{add}\ Emp) \implies \text{Bet-Point } (Se\ y\ x)\ pn \implies$
 $\text{Cong } (Geos\ (Ang\ (An\ y\ z1\ x))\ \text{add}\ Emp)\ (Geos\ (Ang\ (An\ y\ z2\ x))\ \text{add}\ Emp)$
by (*simp add:Ang-add*)
from $P82\ P99\ P109\ P117$ **have** $P118 :$
 $\neg \text{Eq } (Geos\ (Poi\ x)\ \text{add}\ Emp)\ (Geos\ (Poi\ pn)\ \text{add}\ Emp) \wedge \neg \text{Eq } (Geos\ (Poi\ y)\ \text{add}\ Emp)\ (Geos\ (Poi\ pn)\ \text{add}\ Emp) \implies$
 $\text{Cong } (Geos\ (Ang\ (An\ y\ z1\ x))\ \text{add}\ Emp)\ (Geos\ (Ang\ (An\ y\ z2\ x))\ \text{add}\ Emp)$
by *blast*
from $P21\ P25\ P37\ P38\ P118$ **have**
 $\neg \text{Eq } (Geos\ (Poi\ x)\ \text{add}\ Emp)\ (Geos\ (Poi\ pn)\ \text{add}\ Emp) \wedge \neg \text{Eq } (Geos\ (Poi\ y)\ \text{add}\ Emp)\ (Geos\ (Poi\ pn)\ \text{add}\ Emp) \implies$
 $\text{Cong } (Geos\ (Tri\ (Tr\ z1\ y\ x))\ \text{add}\ Emp)\ (Geos\ (Tri\ (Tr\ z2\ y\ x))\ \text{add}\ Emp)$ **by**
(*simp add:Tri-SAS*)
then have $P119 :$
 $\neg \text{Eq } (Geos\ (Poi\ x)\ \text{add}\ Emp)\ (Geos\ (Poi\ pn)\ \text{add}\ Emp) \wedge \neg \text{Eq } (Geos\ (Poi\ y)\ \text{add}\ Emp)\ (Geos\ (Poi\ pn)\ \text{add}\ Emp) \implies$
 $\text{Cong } (Geos\ (Ang\ (An\ x\ y\ z1))\ \text{add}\ Emp)\ (Geos\ (Ang\ (An\ x\ y\ z2))\ \text{add}\ Emp)$
by (*simp add:Tri-Cong-def*)
from $P3\ P30\ P39\ P119$ **show** $\text{Cong } (Geos\ (Ang\ (An\ x\ y\ z1))\ \text{add}\ Emp)\ (Geos\ (Ang\ (An\ x\ y\ z2))\ \text{add}\ Emp)$ **by** *blast*
qed

Theorem18

theorem (*in Congruence-Rule*) *Tri-SSS* :

assumes

$\text{Def } (Tri\ (Tr\ A1\ B1\ C1))\ \text{Def } (Tri\ (Tr\ A2\ B2\ C2))$

$\text{Eq } (Geos\ (Seg\ (Se\ A1\ B1))\ \text{add}\ Emp)\ (Geos\ (Seg\ (Se\ A2\ B2))\ \text{add}\ Emp)$

$\text{Eq } (Geos\ (Seg\ (Se\ B1\ C1))\ \text{add}\ Emp)\ (Geos\ (Seg\ (Se\ B2\ C2))\ \text{add}\ Emp)$

$\text{Eq } (Geos\ (Seg\ (Se\ C1\ A1))\ \text{add}\ Emp)\ (Geos\ (Seg\ (Se\ C2\ A2))\ \text{add}\ Emp)$

shows $\text{Cong } (Geos\ (Tri\ (Tr\ A1\ B1\ C1))\ \text{add}\ Emp)\ (Geos\ (Tri\ (Tr\ A2\ B2\ C2))\ \text{add}\ Emp)$

proof –

from *assms* **have** $\text{Def } (Tri\ (Tr\ C2\ B2\ A2))$ **by** (*simp add:Tri-def-rev*)

then have $P1 : \neg \text{Line-on } (Li\ A2\ C2)\ B2$ **by** (*simp add:Tri-def-Line*)

from *assms* **have** $P2 : \text{Def } (\text{Ang } (\text{An } B1 \ A1 \ C1))$ **by** (*blast intro:Tri-def-rev*
Tri-def-trans Tri-to-Ang)
from $P1 \ P2$ **have** $\exists p. \text{Cong } (\text{Geos } (\text{Ang } (\text{An } B1 \ A1 \ C1)) \ \text{add } \text{Emp})$ ($\text{Geos } (\text{Ang } (\text{An } p \ A2 \ C2)) \ \text{add } \text{Emp}) \wedge \text{Plane-sameside } (\text{Li } A2 \ C2) \ p \ B2$ **by** (*simp*
add:Ang-move-sameside)
then obtain $B21 :: \text{Point}$ **where** $P3 : \text{Cong } (\text{Geos } (\text{Ang } (\text{An } B1 \ A1 \ C1)) \ \text{add } \text{Emp})$ ($\text{Geos } (\text{Ang } (\text{An } B21 \ A2 \ C2)) \ \text{add } \text{Emp})$
 $\wedge \text{Plane-sameside } (\text{Li } A2 \ C2) \ B21 \ B2$ **by** *blast*
then have $P4 : \neg \text{Line-on } (\text{Li } A2 \ C2) \ B21$ **by** (*simp add:Plane-sameside-def*)
from $P2 \ P4$ **have** $\exists p. \text{Cong } (\text{Geos } (\text{Ang } (\text{An } B1 \ A1 \ C1)) \ \text{add } \text{Emp})$ ($\text{Geos } (\text{Ang } (\text{An } p \ A2 \ C2)) \ \text{add } \text{Emp}) \wedge \text{Plane-diffside } (\text{Li } A2 \ C2) \ p \ B21$ **by** (*simp*
add:Ang-move-diffside)
then obtain $B22 :: \text{Point}$ **where** $P5 : \text{Cong } (\text{Geos } (\text{Ang } (\text{An } B1 \ A1 \ C1)) \ \text{add } \text{Emp})$ ($\text{Geos } (\text{Ang } (\text{An } B22 \ A2 \ C2)) \ \text{add } \text{Emp})$
 $\wedge \text{Plane-diffside } (\text{Li } A2 \ C2) \ B22 \ B21$ **by** *blast*
have $P6 : \text{Line-on } (\text{Li } A2 \ B21) \ A2$ **by** (*simp add:Line-on-rule*)
have $P7 : \text{Line-on } (\text{Li } A2 \ B21) \ B21$ **by** (*simp add:Line-on-rule*)
have $P8 : \text{Line-on } (\text{Li } A2 \ C2) \ A2$ **by** (*simp add:Line-on-rule*)
then have $P9 : \text{Eq } (\text{Geos } (\text{Poi } A2) \ \text{add } \text{Emp})$ ($\text{Geos } (\text{Poi } B21) \ \text{add } \text{Emp}) \implies$
 $\text{Line-on } (\text{Li } A2 \ C2) \ B21$ **by** (*simp add:Point-Eq*)
from $P4 \ P9$ **have** $P10 : \neg \text{Eq } (\text{Geos } (\text{Poi } A2) \ \text{add } \text{Emp})$ ($\text{Geos } (\text{Poi } B21) \ \text{add } \text{Emp})$ **by** *blast*
from *assms* **have** $P11 : \neg \text{Eq } (\text{Geos } (\text{Poi } A1) \ \text{add } \text{Emp})$ ($\text{Geos } (\text{Poi } B1) \ \text{add } \text{Emp})$ **by** (*simp add:Tri-def*)
from $P6 \ P7 \ P10 \ P11$ **have** $\exists p. \text{Eq } (\text{Geos } (\text{Seg } (\text{Se } A1 \ B1)) \ \text{add } \text{Emp})$ ($\text{Geos } (\text{Seg } (\text{Se } A2 \ p)) \ \text{add } \text{Emp})$
 $\wedge \neg \text{Bet-Point } (\text{Se } p \ B21) \ A2 \wedge \text{Line-on } (\text{Li } A2 \ B21) \ p \wedge \neg \text{Eq } (\text{Geos } (\text{Poi } A2) \ \text{add } \text{Emp})$ ($\text{Geos } (\text{Poi } p) \ \text{add } \text{Emp})$ **by** (*simp add:Seg-move-sameside*)
then obtain $B211 :: \text{Point}$ **where** $P12 : \text{Eq } (\text{Geos } (\text{Seg } (\text{Se } A1 \ B1)) \ \text{add } \text{Emp})$ ($\text{Geos } (\text{Seg } (\text{Se } A2 \ B211)) \ \text{add } \text{Emp})$
 $\wedge \neg \text{Bet-Point } (\text{Se } B211 \ B21) \ A2 \wedge \text{Line-on } (\text{Li } A2 \ B21) \ B211 \wedge \neg \text{Eq } (\text{Geos } (\text{Poi } A2) \ \text{add } \text{Emp})$ ($\text{Geos } (\text{Poi } B211) \ \text{add } \text{Emp})$ **by** *blast*
have $P13 : \text{Line-on } (\text{Li } A2 \ B22) \ A2$ **by** (*simp add:Line-on-rule*)
have $P14 : \text{Line-on } (\text{Li } A2 \ B22) \ B22$ **by** (*simp add:Line-on-rule*)
from $P8$ **have** $P15 : \text{Eq } (\text{Geos } (\text{Poi } A2) \ \text{add } \text{Emp})$ ($\text{Geos } (\text{Poi } B22) \ \text{add } \text{Emp})$
 $\implies \text{Line-on } (\text{Li } A2 \ C2) \ B22$ **by** (*simp add:Point-Eq*)
from $P5$ **have** $P16 : \exists p. \text{Bet-Point } (\text{Se } B22 \ B21) \ p \wedge \text{Line-on } (\text{Li } A2 \ C2) \ p$
 $\wedge \neg \text{Line-on } (\text{Li } A2 \ C2) \ B22 \wedge \neg \text{Line-on } (\text{Li } A2 \ C2) \ B21$ **by** (*simp*
add:Plane-diffside-def)
from $P15 \ P16$ **have** $P17 : \neg \text{Eq } (\text{Geos } (\text{Poi } A2) \ \text{add } \text{Emp})$ ($\text{Geos } (\text{Poi } B22) \ \text{add } \text{Emp})$ **by** *blast*
from $P11 \ P13 \ P14 \ P17$ **have** $\exists p. \text{Eq } (\text{Geos } (\text{Seg } (\text{Se } A1 \ B1)) \ \text{add } \text{Emp})$ ($\text{Geos } (\text{Seg } (\text{Se } A2 \ p)) \ \text{add } \text{Emp})$
 $\wedge \neg \text{Bet-Point } (\text{Se } p \ B22) \ A2 \wedge \text{Line-on } (\text{Li } A2 \ B22) \ p \wedge \neg \text{Eq } (\text{Geos } (\text{Poi } A2) \ \text{add } \text{Emp})$ ($\text{Geos } (\text{Poi } p) \ \text{add } \text{Emp})$ **by** (*simp add:Seg-move-sameside*)
then obtain $B221 :: \text{Point}$ **where** $P18 : \text{Eq } (\text{Geos } (\text{Seg } (\text{Se } A1 \ B1)) \ \text{add } \text{Emp})$ ($\text{Geos } (\text{Seg } (\text{Se } A2 \ B221)) \ \text{add } \text{Emp})$
 $\wedge \neg \text{Bet-Point } (\text{Se } B221 \ B22) \ A2 \wedge \text{Line-on } (\text{Li } A2 \ B22) \ B221 \wedge \neg \text{Eq } (\text{Geos } (\text{Poi } A2) \ \text{add } \text{Emp})$ ($\text{Geos } (\text{Poi } B221) \ \text{add } \text{Emp})$ **by** *blast*

from *assms* **have** $P19 : \neg Eq (Geos (Poi C2) add Emp) (Geos (Poi A2) add Emp)$ **by** (*simp add:Tri-def*)
then have $P20 : \neg Eq (Geos (Poi A2) add Emp) (Geos (Poi C2) add Emp)$ **by** (*blast intro:Eq-rev*)
from $P4 P20$ **have** $P21 : Def (Ang (An B21 A2 C2))$ **by** (*blast intro:Ang-simple-def Ang-def-rev Ang-def-inv*)
from $P12$ **have** $\neg Bet-Point (Se B211 B21) A2$ **by** *blast*
then have $P22 : \neg Bet-Point (Se B21 B211) A2$ **by** (*blast intro:Bet-rev*)
have $P23 : Line-on (Li A2 C2) C2$ **by** (*simp add:Line-on-rule*)
have $P24 : \neg Bet-Point (Se C2 C2) A2$ **by** (*simp add:Bet-end-Point*)
from $P12 P20 P21 P22 P23 P24$ **have** $P25 :$
 $Eq (Geos (Ang (An B21 A2 C2)) add Emp) (Geos (Ang (An B211 A2 C2)) add Emp) \wedge Def (Ang (An B211 A2 C2))$ **by** (*simp add:Ang-Point-swap*)
from $P3 P25$ **have** $P26 : Cong (Geos (Ang (An B1 A1 C1)) add Emp) (Geos (Ang (An B211 A2 C2)) add Emp)$ **by** (*blast intro:Ang-weektrans Ang-rev Eq-rev*)
from *assms* **have** $P27 : Def (Tri (Tr A1 B1 C1))$ **by** (*simp add:Ang-to-Tri*)
from $P25$ **have** $P28 : Def (Tri (Tr A2 B211 C2))$ **by** (*blast intro:Ang-to-Tri Tri-def-trans Tri-def-rev*)
from *assms* **have** $P29 : Eq (Geos (Seg (Se A1 C1)) add Emp) (Geos (Seg (Se A2 C2)) add Emp)$ **by** (*blast intro:Seg-rev Eq-rev Eq-trans*)
from $P12 P26 P27 P28 P29$ **have** $Cong (Geos (Tri (Tr A1 B1 C1)) add Emp) (Geos (Tri (Tr A2 B211 C2)) add Emp)$ **by** (*simp add:Tri-SAS*)
then have $P30 : Eq (Geos (Seg (Se B1 C1)) add Emp) (Geos (Seg (Se B211 C2)) add Emp)$ **by** (*simp add:Tri-Cong-def*)
from $P16 P20$ **have** $P31 : Def (Ang (An B22 A2 C2))$ **by** (*blast intro:Ang-simple-def Ang-def-rev Ang-def-inv*)
from $P18$ **have** $\neg Bet-Point (Se B221 B22) A2$ **by** *blast*
then have $P32 : \neg Bet-Point (Se B22 B221) A2$ **by** (*blast intro:Bet-rev*)
from $P18 P20 P23 P24 P31 P32$ **have** $P33 :$
 $Eq (Geos (Ang (An B22 A2 C2)) add Emp) (Geos (Ang (An B221 A2 C2)) add Emp) \wedge Def (Ang (An B221 A2 C2))$ **by** (*simp add:Ang-Point-swap*)
from $P5 P33$ **have** $P34 : Cong (Geos (Ang (An B1 A1 C1)) add Emp) (Geos (Ang (An B221 A2 C2)) add Emp)$ **by** (*blast intro:Ang-weektrans Ang-rev Eq-rev*)
from $P33$ **have** $P35 : Def (Tri (Tr A2 B221 C2))$ **by** (*blast intro:Ang-to-Tri Tri-def-trans Tri-def-rev*)
from $P18 P27 P29 P34 P35$ **have** $Cong (Geos (Tri (Tr A1 B1 C1)) add Emp) (Geos (Tri (Tr A2 B221 C2)) add Emp)$ **by** (*simp add:Tri-SAS*)
then have $P36 : Eq (Geos (Seg (Se B1 C1)) add Emp) (Geos (Seg (Se B221 C2)) add Emp)$ **by** (*simp add:Tri-Cong-def*)
from $P12$ **have** $P37 : Eq (Geos (Seg (Se A1 B1)) add Emp) (Geos (Seg (Se A2 B211)) add Emp)$ **by** *simp*
from $P18$ **have** $P38 : Eq (Geos (Seg (Se A1 B1)) add Emp) (Geos (Seg (Se A2 B221)) add Emp)$ **by** *simp*
from $P37 P38$ **have** $P39 : Eq (Geos (Seg (Se A2 B221)) add Emp) (Geos (Seg (Se A2 B211)) add Emp)$ **by** (*blast intro:Eq-trans Eq-rev*)
from *assms* $P38$ **have** $P40 : Eq (Geos (Seg (Se A2 B221)) add Emp) (Geos (Seg (Se A2 B2)) add Emp)$ **by** (*blast intro:Eq-trans Eq-rev*)
from $P30 P36$ **have** $P41 : Eq (Geos (Seg (Se B221 C2)) add Emp) (Geos (Seg (Se B211 C2)) add Emp)$ **by** (*blast intro:Eq-trans Eq-rev*)

from *assms P36* **have** $P_{42} : Eq (Geos (Seg (Se B221 C2)) add Emp) (Geos (Seg (Se B2 C2)) add Emp)$ **by** (*blast intro:Eq-trans Eq-rev*)
from *P5* **have** $P_{43} : Plane-diffside (Li A2 C2) B22 B21$ **by** *simp*
then **have** $P_{44} : Eq (Geos (Poi B21) add Emp) (Geos (Poi B211) add Emp) \implies$
 $Plane-diffside (Li A2 C2) B22 B211$ **by** (*blast intro:Point-Eq Eq-rev*)
from *P6 P8 P20 P23* **have** $P_{45} : Line-on (Li A2 B21) C2 \implies$
 $Eq (Geos (Lin (Li A2 B21)) add Emp) (Geos (Lin (Li A2 C2)) add Emp)$ **by**
(*simp add:Line-unique*)
from *P7 P45* **have** $P_{46} : Line-on (Li A2 B21) C2 \implies Line-on (Li A2 C2) B21$
by (*simp add:Line-on-trans*)
from *P4 P46* **have** $P_{47} : \neg Line-on (Li A2 B21) C2$ **by** *blast*
from *P6 P7 P10 P12 P22 P47* **have** $P_{48} : Plane-sameside (Li C2 A2) B21$
 $B211 \vee Eq (Geos (Poi B21) add Emp) (Geos (Poi B211) add Emp)$ **by** (*simp*
add:Seg-Plane-sameside)
from *P20* **have** $P_{49} : Plane-sameside (Li C2 A2) B21 B211 \implies$
 $Plane-sameside (Li A2 C2) B21 B211$ **by** (*blast intro:Line-rev Plane-Line-trans*
 $Eq-rev$)
from *P43* **have** $P_{50} : Plane-diffside (Li A2 C2) B21 B22$ **by** (*simp add:Plane-diffside-rev*)
from *P49 P50* **have** $P_{51} : Plane-sameside (Li C2 A2) B21 B211 \implies$
 $Plane-diffside (Li A2 C2) B22 B211$ **by** (*simp add:Plane-trans Plane-diffside-rev*)
from *P44 P48 P51* **have** $P_{52} : Plane-diffside (Li A2 C2) B22 B211$ **by** *blast*
then **have** $Plane-diffside (Li A2 C2) B211 B22$ **by** (*blast intro:Plane-diffside-rev*)
then **have** $P_{53} : Eq (Geos (Poi B22) add Emp) (Geos (Poi B221) add Emp)$
 \implies
 $Plane-diffside (Li A2 C2) B211 B221$ **by** (*blast intro:Point-Eq Eq-rev*)
from *P8 P13 P20 P23* **have** $P_{54} : Line-on (Li A2 B22) C2 \implies$
 $Eq (Geos (Lin (Li A2 B22)) add Emp) (Geos (Lin (Li A2 C2)) add Emp)$ **by**
(*simp add:Line-unique*)
from *P14 P54* **have** $P_{55} : Line-on (Li A2 B22) C2 \implies Line-on (Li A2 C2)$
 $B22$ **by** (*simp add:Line-on-trans*)
from *P16 P55* **have** $P_{56} : \neg Line-on (Li A2 B22) C2$ **by** *blast*
from *P31* **have** $\neg Eq (Geos (Poi B22) add Emp) (Geos (Poi A2) add Emp)$ **by**
(*simp add:Ang-def*)
then **have** $P_{57} : \neg Eq (Geos (Poi A2) add Emp) (Geos (Poi B22) add Emp)$
by (*blast intro:Eq-rev*)
from *P13 P14 P18 P32 P56 P57* **have** $P_{58} : Plane-sameside (Li C2 A2) B22$
 $B221 \vee Eq (Geos (Poi B22) add Emp) (Geos (Poi B221) add Emp)$ **by** (*simp*
add:Seg-Plane-sameside)
from *P20* **have** $P_{59} : Plane-sameside (Li C2 A2) B22 B221 \implies Plane-sameside$
 $(Li A2 C2) B22 B221$ **by** (*blast intro:Line-rev Plane-Line-trans Eq-rev*)
from *P52 P59* **have** $P_{60} : Plane-sameside (Li C2 A2) B22 B221 \implies$
 $Plane-diffside (Li A2 C2) B211 B221$ **by** (*simp add:Plane-trans Plane-diffside-rev*)
from *P53 P58 P60* **have** $Plane-diffside (Li A2 C2) B211 B221$ **by** *blast*
then **have** $P_{61} : Plane-diffside (Li A2 C2) B221 B211$ **by** (*simp add:Plane-diffside-rev*)
from *P20 P61* **have** $P_{62} : Plane-diffside (Li C2 A2) B221 B211$ **by** (*blast*
intro:Line-rev Plane-Line-diff-trans)
have $P_{63} : Eq (Geos (Seg (Se B221 C2)) add Emp) (Geos (Seg (Se C2 B221))$
 $add Emp)$ **by** (*simp add:Seg-rev*)
have $P_{64} : Eq (Geos (Seg (Se B211 C2)) add Emp) (Geos (Seg (Se C2 B211))$

add Emp) **by** (*simp add:Seg-rev*)
from $P_{41} P_{63} P_{64}$ **have** $P_{65} : Eq$ (*Geos (Seg (Se C2 B221)) add Emp*) (*Geos (Seg (Se C2 B211)) add Emp*) **by** (*blast intro:Eq-rev Eq-trans*)
from $P_{19} P_{39} P_{62} P_{65}$ **have** $P_{66} : Cong$ (*Geos (Ang (An C2 A2 B221)) add Emp*) (*Geos (Ang (An C2 A2 B211)) add Emp*) **by** (*simp add:Tri-week-SSS*)
have $P_{67} : Eq$ (*Geos (Ang (An C2 A2 B211)) add Emp*) (*Geos (Ang (An B211 A2 C2)) add Emp*) **by** (*simp add:Ang-roll*)
from $P_{66} P_{67}$ **have** $P_{68} : Cong$ (*Geos (Ang (An C2 A2 B221)) add Emp*) (*Geos (Ang (An B211 A2 C2)) add Emp*) **by** (*blast intro:Ang-weektrans Ang-rev Eq-rev*)
from P_3 **have** $P_{69} : Plane-sameside$ (*Li A2 C2*) $B_{21} B_2$ **by** *simp*
from $P_{50} P_{69}$ **have** $P_{70} : Plane-diffside$ (*Li A2 C2*) $B_2 B_{22}$ **by** (*simp add:Plane-trans*)
then have $P_{71} : Plane-diffside$ (*Li A2 C2*) $B_{22} B_2$ **by** (*simp add:Plane-diffside-rev*)
from P_{70} **have** $P_{72} : Eq$ (*Geos (Poi B22) add Emp*) (*Geos (Poi B221) add Emp*) \implies
Plane-diffside (Li A2 C2) B221 B2 **by** (*blast intro:Point-Eq Plane-diffside-rev*)
from $P_{59} P_{71}$ **have** $P_{73} : Plane-sameside$ (*Li C2 A2*) $B_{22} B_{221}$ \implies
Plane-diffside (Li A2 C2) B221 B2 **by** (*simp add:Plane-trans*)
from $P_{58} P_{72} P_{73}$ **have** $P_{74} : Plane-diffside$ (*Li A2 C2*) $B_{221} B_2$ **by** *blast*
from $P_{20} P_{74}$ **have** $P_{75} : Plane-diffside$ (*Li C2 A2*) $B_{221} B_2$ **by** (*blast intro:Line-rev Plane-Line-diff-trans*)
have $P_{76} : Eq$ (*Geos (Seg (Se B221 C2)) add Emp*) (*Geos (Seg (Se C2 B221)) add Emp*) **by** (*simp add:Seg-rev*)
have $P_{77} : Eq$ (*Geos (Seg (Se B2 C2)) add Emp*) (*Geos (Seg (Se C2 B2)) add Emp*) **by** (*simp add:Seg-rev*)
from $P_{42} P_{76} P_{77}$ **have** $P_{78} : Eq$ (*Geos (Seg (Se C2 B221)) add Emp*) (*Geos (Seg (Se C2 B2)) add Emp*) **by** (*blast intro:Eq-rev Eq-trans*)
from $P_{19} P_{40} P_{75} P_{78}$ **have** $P_{79} : Cong$ (*Geos (Ang (An C2 A2 B221)) add Emp*) (*Geos (Ang (An C2 A2 B2)) add Emp*) **by** (*simp add:Tri-week-SSS*)
have $P_{80} : Eq$ (*Geos (Ang (An C2 A2 B2)) add Emp*) (*Geos (Ang (An B2 A2 C2)) add Emp*) **by** (*simp add:Ang-roll*)
from $P_{79} P_{80}$ **have** $P_{81} : Cong$ (*Geos (Ang (An C2 A2 B221)) add Emp*) (*Geos (Ang (An B2 A2 C2)) add Emp*) **by** (*blast intro:Ang-weektrans Ang-rev Eq-rev*)
from $P_{52} P_{71}$ **have** $P_{82} : \neg Eq$ (*Geos (Poi B2) add Emp*) (*Geos (Poi B211) add Emp*) \implies
Plane-sameside (Li A2 C2) B2 B211 **by** (*blast intro:Plane-trans-inv*)
from $P_{68} P_{81} P_{82}$ **have** $P_{83} : \neg Eq$ (*Geos (Poi B2) add Emp*) (*Geos (Poi B211) add Emp*) \implies
Eq (Geos (Lin (Li B2 A2)) add Emp) (Geos (Lin (Li B211 A2)) add Emp) \wedge \neg Bet-Point (Se B2 B211) A2 **by** (*simp add:Ang-move-unique*)
from *assms* **have** $P_{84} : Def$ (*Ang (An B2 A2 C2)*) **by** (*blast intro:Tri-to-Ang Ang-def-rev Ang-def-inv*)
then have $\neg Eq$ (*Geos (Poi B2) add Emp*) (*Geos (Poi A2) add Emp*) **by** (*simp add:Ang-def*)
then have $P_{85} : Eq$ (*Geos (Lin (Li B2 A2)) add Emp*) (*Geos (Lin (Li A2 B2)) add Emp*) **by** (*simp add:Line-rev*)
have $P_{86} : Line-on$ (*Li B211 A2*) B_{211} **by** (*simp add:Line-on-rule*)
from $P_{83} P_{85} P_{86}$ **have** $P_{87} : \neg Eq$ (*Geos (Poi B2) add Emp*) (*Geos (Poi B211) add Emp*) \implies
Line-on (Li A2 B2) B211 **by** (*blast intro:Eq-rev Line-on-trans*)

have *Line-on* (*Li A2 B2*) *B2* **by** (*simp add:Line-on-rule*)
then have $P88 : Eq (Geos (Poi B2) add Emp) (Geos (Poi B211) add Emp) \implies$
Line-on (*Li A2 B2*) *B211* **by** (*simp add:Point-Eq*)
from $P87 P88$ **have** $P89 : Line-on (Li A2 B2) B211$ **by** *blast*
have $P90 : \neg Bet-Point (Se B2 B2) A2$ **by** (*simp add:Bet-end-Point*)
have $P91 : Eq (Geos (Poi B2) add Emp) (Geos (Poi B211) add Emp) \implies$
Bet-Point (*Se B211 B2*) *A2* \implies *Bet-Point* (*Se B2 B2*) *A2* **by** (*blast intro:Eq-rev*
Bet-Point-Eq)
from $P90 P91$ **have** $P92 : Eq (Geos (Poi B2) add Emp) (Geos (Poi B211) add$
Emp) $\implies \neg Bet-Point (Se B2 B211) A2$ **by** (*blast intro:Bet-rev*)
from $P83 P92$ **have** $P93 : \neg Bet-Point (Se B2 B211) A2$ **by** *blast*
from $P12 P20 P23 P24 P84 P89 P93$ **have** $P94 :$
 $Eq (Geos (Ang (An B2 A2 C2)) add Emp) (Geos (Ang (An B211 A2 C2)) add$
Emp) \wedge *Def* (*Ang (An B211 A2 C2)*) **by** (*simp add:Ang-Point-swap*)
from $P25 P94$ **have** $P95 : Eq (Geos (Ang (An B21 A2 C2)) add Emp) (Geos$
(*Ang (An B2 A2 C2)*) *add Emp*) **by** (*blast intro:Eq-rev Eq-trans*)
from $P3 P95$ **have** $P96 : Cong (Geos (Ang (An B1 A1 C1)) add Emp) (Geos$
(*Ang (An B2 A2 C2)*) *add Emp*) **by** (*blast intro:Ang-weektrans Ang-rev Eq-rev*)
have $P97 : Eq (Geos (Seg (Se C1 A1)) add Emp) (Geos (Seg (Se A1 C1)) add$
Emp) **by** (*simp add:Seg-rev*)
have $P98 : Eq (Geos (Seg (Se C2 A2)) add Emp) (Geos (Seg (Se A2 C2)) add$
Emp) **by** (*simp add:Seg-rev*)
from *assms* $P97 P98$ **have** $P99 : Eq (Geos (Seg (Se A1 C1)) add Emp) (Geos$
(*Seg (Se A2 C2)*) *add Emp*) **by** (*blast intro:Eq-rev Eq-trans*)
from *assms* **have** $P100 : Def (Tri (Tr A1 B1 C1))$ **by** (*simp add:Ang-to-Tri*)
from *assms* **have** $P101 : Def (Tri (Tr A2 B2 C2))$ **by** (*simp add:Ang-to-Tri*)
from *assms* $P96 P99 P100 P101$ **show** $Cong (Geos (Tri (Tr A1 B1 C1)) add$
Emp) (*Geos (Tri (Tr A2 B2 C2)) add Emp*) **by** (*simp add:Tri-SAS*)
qed

Theorem19

theorem (*in Congruence-Rule*) *Ang-trans* :

assumes

Def (Ang (An A1 B1 C1)) Def (Ang (An A2 B2 C2)) Def (Ang (An A3 B3 C3))

Cong (Geos (Ang (An A2 B2 C2)) add Emp) (Geos (Ang (An A1 B1 C1)) add Emp)

Cong (Geos (Ang (An A3 B3 C3)) add Emp) (Geos (Ang (An A1 B1 C1)) add Emp)

shows *Cong (Geos (Ang (An A2 B2 C2)) add Emp) (Geos (Ang (An A3 B3 C3)) add Emp)*

proof –

from *assms* **have** $P1 : Cong (Geos (Ang (An A1 B1 C1)) add Emp) (Geos (Ang$
(*An A2 B2 C2*) *add Emp*) **by** (*simp add:Ang-rev*)

from *assms* $P1$ **have** $\exists p q. Cong (Geos (Ang (An A1 B1 C1)) add Emp) (Geos$
(*Ang (An p B2 q)*) *add Emp*)

$\wedge Eq (Geos (Ang (An A2 B2 C2)) add Emp) (Geos (Ang (An p B2 q)) add$
Emp)

$\wedge Eq (Geos (Seg (Se B1 A1)) add Emp) (Geos (Seg (Se B2 p)) add Emp)$

\wedge *Line-on* (*Li B2 A2*) $p \wedge \neg$ *Bet-Point* (*Se p A2*) *B2*
 \wedge *Eq* (*Geos* (*Seg* (*Se B1 C1*)) *add Emp*) (*Geos* (*Seg* (*Se B2 q*)) *add Emp*)
 \wedge *Line-on* (*Li B2 C2*) $q \wedge \neg$ *Bet-Point* (*Se q C2*) *B2* \wedge *Def* (*Ang* (*An p B2*
q)) **by** (*simp add:Ang-replace*)
then obtain *A21 C21* :: *Point where P2* : *Cong* (*Geos* (*Ang* (*An A1 B1 C1*))
add Emp) (*Geos* (*Ang* (*An A21 B2 C21*)) *add Emp*)
 \wedge *Eq* (*Geos* (*Ang* (*An A2 B2 C2*)) *add Emp*) (*Geos* (*Ang* (*An A21 B2 C21*))
add Emp)
 \wedge *Eq* (*Geos* (*Seg* (*Se B1 A1*)) *add Emp*) (*Geos* (*Seg* (*Se B2 A21*)) *add Emp*)
 \wedge *Line-on* (*Li B2 A2*) *A21* $\wedge \neg$ *Bet-Point* (*Se A21 A2*) *B2*
 \wedge *Eq* (*Geos* (*Seg* (*Se B1 C1*)) *add Emp*) (*Geos* (*Seg* (*Se B2 C21*)) *add Emp*)
 \wedge *Line-on* (*Li B2 C2*) *C21* $\wedge \neg$ *Bet-Point* (*Se C21 C2*) *B2* \wedge *Def* (*Ang* (*An*
A21 B2 C21)) **by blast**
from *assms* **have** *P3* : *Cong* (*Geos* (*Ang* (*An A1 B1 C1*)) *add Emp*) (*Geos* (*Ang*
(*An A3 B3 C3*)) *add Emp*) **by** (*simp add:Ang-rev*)
from *assms P3* **have** $\exists p q.$ *Cong* (*Geos* (*Ang* (*An A1 B1 C1*)) *add Emp*) (*Geos*
(*Ang* (*An p B3 q*)) *add Emp*)
 \wedge *Eq* (*Geos* (*Ang* (*An A3 B3 C3*)) *add Emp*) (*Geos* (*Ang* (*An p B3 q*)) *add*
Emp)
 \wedge *Eq* (*Geos* (*Seg* (*Se B1 A1*)) *add Emp*) (*Geos* (*Seg* (*Se B3 p*)) *add Emp*)
 \wedge *Line-on* (*Li B3 A3*) $p \wedge \neg$ *Bet-Point* (*Se p A3*) *B3*
 \wedge *Eq* (*Geos* (*Seg* (*Se B1 C1*)) *add Emp*) (*Geos* (*Seg* (*Se B3 q*)) *add Emp*)
 \wedge *Line-on* (*Li B3 C3*) $q \wedge \neg$ *Bet-Point* (*Se q C3*) *B3* \wedge *Def* (*Ang* (*An p B3*
q)) **by** (*simp add:Ang-replace*)
then obtain *A31 C31* :: *Point where P4* : *Cong* (*Geos* (*Ang* (*An A1 B1 C1*))
add Emp) (*Geos* (*Ang* (*An A31 B3 C31*)) *add Emp*)
 \wedge *Eq* (*Geos* (*Ang* (*An A3 B3 C3*)) *add Emp*) (*Geos* (*Ang* (*An A31 B3 C31*))
add Emp)
 \wedge *Eq* (*Geos* (*Seg* (*Se B1 A1*)) *add Emp*) (*Geos* (*Seg* (*Se B3 A31*)) *add Emp*)
 \wedge *Line-on* (*Li B3 A3*) *A31* $\wedge \neg$ *Bet-Point* (*Se A31 A3*) *B3*
 \wedge *Eq* (*Geos* (*Seg* (*Se B1 C1*)) *add Emp*) (*Geos* (*Seg* (*Se B3 C31*)) *add Emp*)
 \wedge *Line-on* (*Li B3 C3*) *C31* $\wedge \neg$ *Bet-Point* (*Se C31 C3*) *B3* \wedge *Def* (*Ang* (*An*
A31 B3 C31)) **by blast**
from *assms* **have** *P5* : *Def* (*Tri* (*Tr B1 A1 C1*)) **by** (*blast intro:Tri-def-rev*
Tri-def-trans Ang-to-Tri)
from *P2* **have** *P6* : *Def* (*Tri* (*Tr B2 A21 C21*)) **by** (*blast intro:Tri-def-rev*
Tri-def-trans Ang-to-Tri)
from *P2 P5 P6* **have** *Cong* (*Geos* (*Tri* (*Tr B1 A1 C1*)) *add Emp*) (*Geos* (*Tri*
(*Tr B2 A21 C21*)) *add Emp*) **by** (*simp add:Tri-SAS*)
then have *P7* : *Eq* (*Geos* (*Seg* (*Se A1 C1*)) *add Emp*) (*Geos* (*Seg* (*Se A21 C21*))
add Emp) **by** (*simp add:Tri-Cong-def*)
from *P4* **have** *P8* : *Def* (*Tri* (*Tr B3 A31 C31*)) **by** (*blast intro:Tri-def-rev*
Tri-def-trans Ang-to-Tri)
from *P4 P5 P8* **have** *Cong* (*Geos* (*Tri* (*Tr B1 A1 C1*)) *add Emp*) (*Geos* (*Tri*
(*Tr B3 A31 C31*)) *add Emp*) **by** (*simp add:Tri-SAS*)
then have *P9* : *Eq* (*Geos* (*Seg* (*Se A1 C1*)) *add Emp*) (*Geos* (*Seg* (*Se A31 C31*))
add Emp) **by** (*simp add:Tri-Cong-def*)
from *P6* **have** *P10* : *Def* (*Tri* (*Tr A21 C21 B2*)) **by** (*blast intro:Tri-def-trans*)
from *P8* **have** *P11* : *Def* (*Tri* (*Tr A31 C31 B3*)) **by** (*blast intro:Tri-def-trans*)

from $P7\ P9$ **have** $P12 : Eq\ (Geos\ (Seg\ (Se\ A21\ C21))\ add\ Emp)\ (Geos\ (Seg\ (Se\ A31\ C31))\ add\ Emp)$ **by** $(blast\ intro:Eq-trans)$
from $P2\ P4$ **have** $P13 : Eq\ (Geos\ (Seg\ (Se\ B2\ A21))\ add\ Emp)\ (Geos\ (Seg\ (Se\ B3\ A31))\ add\ Emp)$ **by** $(blast\ intro:Eq-trans)$
from $P2\ P4$ **have** $P14 : Eq\ (Geos\ (Seg\ (Se\ B2\ C21))\ add\ Emp)\ (Geos\ (Seg\ (Se\ B3\ C31))\ add\ Emp)$ **by** $(blast\ intro:Eq-trans)$
have $P15 : Eq\ (Geos\ (Seg\ (Se\ B2\ C21))\ add\ Emp)\ (Geos\ (Seg\ (Se\ C21\ B2))\ add\ Emp)$ **by** $(simp\ add:Seg-rev)$
have $P16 : Eq\ (Geos\ (Seg\ (Se\ B3\ C31))\ add\ Emp)\ (Geos\ (Seg\ (Se\ C31\ B3))\ add\ Emp)$ **by** $(simp\ add:Seg-rev)$
from $P14\ P15\ P16$ **have** $P17 : Eq\ (Geos\ (Seg\ (Se\ C21\ B2))\ add\ Emp)\ (Geos\ (Seg\ (Se\ C31\ B3))\ add\ Emp)$ **by** $(blast\ intro:Eq-trans\ Eq-rev)$
from $P10\ P11\ P12\ P13\ P17$ **have** $Cong\ (Geos\ (Tri\ (Tr\ A21\ C21\ B2))\ add\ Emp)\ (Geos\ (Tri\ (Tr\ A31\ C31\ B3))\ add\ Emp)$ **by** $(simp\ add:Tri-SSS)$
then **have** $P18 : Cong\ (Geos\ (Ang\ (An\ A21\ B2\ C21))\ add\ Emp)\ (Geos\ (Ang\ (An\ A31\ B3\ C31))\ add\ Emp)$ **by** $(simp\ add:Tri-Cong-def)$
from $P2\ P18$ **have** $P19 : Cong\ (Geos\ (Ang\ (An\ A2\ B2\ C2))\ add\ Emp)\ (Geos\ (Ang\ (An\ A31\ B3\ C31))\ add\ Emp)$ **by** $(blast\ intro:Ang-weektrans\ Ang-rev)$
from $P4\ P19$ **show** $Cong\ (Geos\ (Ang\ (An\ A2\ B2\ C2))\ add\ Emp)\ (Geos\ (Ang\ (An\ A3\ B3\ C3))\ add\ Emp)$ **by** $(blast\ intro:Ang-weektrans\ Ang-rev)$
qed

lemma (in *Congruence-Rule*) *Ang-move-unique-inv* :

assumes

$Def\ (Ang\ (An\ p1\ p2\ p3))\ Def\ (Ang\ (An\ p4\ p2\ p3))$

$Plane-sameside\ (Li\ p2\ p3)\ p1\ p4$

$Eq\ (Geos\ (Lin\ (Li\ p2\ p1))\ add\ Emp)\ (Geos\ (Lin\ (Li\ p2\ p4))\ add\ Emp)$

shows

$Cong\ (Geos\ (Ang\ (An\ p1\ p2\ p3))\ add\ Emp)\ (Geos\ (Ang\ (An\ p4\ p2\ p3))\ add\ Emp)$

proof –

have $P1 : Line-on\ (Li\ p2\ p4)\ p4$ **by** $(simp\ add:Line-on-rule)$

from *assms* $P1$ **have** $P2 : Line-on\ (Li\ p2\ p1)\ p4$ **by** $(blast\ intro:Line-on-trans\ Eq-rev)$

have $P3 : Line-on\ (Li\ p2\ p3)\ p2$ **by** $(simp\ add:Line-on-rule)$

from *assms* **have** $P4 : \neg\ Line-on\ (Li\ p2\ p3)\ p1$ **by** $(simp\ add:Plane-sameside-def)$

from *assms* **have** $P5 : \neg\ Line-on\ (Li\ p2\ p3)\ p4$ **by** $(simp\ add:Plane-sameside-def)$

from $P3\ P4\ P5$ **have** $Bet-Point\ (Se\ p1\ p4)\ p2 \implies \exists p. Bet-Point\ (Se\ p1\ p4)\ p \wedge Line-on\ (Li\ p2\ p3)\ p$

$\neg\ Line-on\ (Li\ p2\ p3)\ p1 \wedge \neg\ Line-on\ (Li\ p2\ p3)\ p4$ **by** *blast*

then **have** $Bet-Point\ (Se\ p1\ p4)\ p2 \implies Plane-diffside\ (Li\ p2\ p3)\ p1\ p4$ **by** $(simp\ add:Plane-diffside-def)$

then **have** $P6 : Bet-Point\ (Se\ p1\ p4)\ p2 \implies \neg\ Plane-sameside\ (Li\ p2\ p3)\ p1\ p4$ **by** $(simp\ add:Plane-diffside-not-sameside)$

from *assms* $P6$ **have** $P7 : \neg\ Bet-Point\ (Se\ p1\ p4)\ p2$ **by** *blast*

have $P8 : Line-on\ (Li\ p2\ p3)\ p3$ **by** $(simp\ add:Line-on-rule)$

have $P9 : \neg\ Bet-Point\ (Se\ p3\ p3)\ p2$ **by** $(simp\ add:Bet-end-Point)$

from *assms* **have** $\neg\ Eq\ (Geos\ (Poi\ p4)\ add\ Emp)\ (Geos\ (Poi\ p2)\ add\ Emp)$ **by** $(simp\ add:Ang-def)$

then have $P10 : \neg Eq (Geos (Poi\ p2)\ add\ Emp) (Geos (Poi\ p4)\ add\ Emp)$ **by**
(blast intro:Eq-rev)
from *assms* **have** $P11 : \neg Eq (Geos (Poi\ p2)\ add\ Emp) (Geos (Poi\ p3)\ add\ Emp)$ **by** *(simp add:Ang-def)*
from *assms P2 P7 P8 P9 P10 P11* **have** $Eq (Geos (Ang (An\ p1\ p2\ p3))\ add\ Emp) (Geos (Ang (An\ p4\ p2\ p3))\ add\ Emp)$
 $\wedge Def (Ang (An\ p4\ p2\ p3))$ **by** *(simp add:Ang-Point-swap)*
thus $Cong (Geos (Ang (An\ p1\ p2\ p3))\ add\ Emp) (Geos (Ang (An\ p4\ p2\ p3))\ add\ Emp)$ **by** *(blast intro:Ang-weektrans)*
qed

Theorem20

theorem (in *Congruence-Rule*) *Ang-move-Greater* :

assumes

$Def (Ang (An\ h1\ o1\ k1))\ Def (Ang (An\ h2\ o2\ l2))$

$Cong (Geos (Ang (An\ h1\ o1\ k1))\ add\ Emp) (Geos (Ang (An\ h2\ o2\ k2))\ add\ Emp)$

$Plane-sameside (Li\ o2\ h2)\ k2\ l2$

$Cong (Geos (Ang (An\ h2\ o2\ l2))\ add\ Emp) (Geos (Ang (An\ h1\ o1\ l1))\ add\ Emp)$

$Plane-sameside (Li\ o1\ h1)\ k1\ l1$

$Ang-inside (An\ h2\ o2\ l2)\ k2$

shows

$\neg Ang-inside (An\ h1\ o1\ k1)\ l1$

$\neg Eq (Geos (Lin (Li\ o1\ k1))\ add\ Emp) (Geos (Lin (Li\ o1\ l1))\ add\ Emp)$

proof –

from *assms* **have** $P1 : \neg Line-on (Li\ o2\ h2)\ k2$ **by** *(simp add:Plane-sameside-def)*

from *assms* **have** $\neg Eq (Geos (Poi\ h2)\ add\ Emp) (Geos (Poi\ o2)\ add\ Emp)$ **by**
(simp add:Ang-def)

then have $P2 : \neg Eq (Geos (Poi\ o2)\ add\ Emp) (Geos (Poi\ h2)\ add\ Emp)$ **by**
(blast intro:Eq-rev)

from $P1\ P2$ **have** $Def (Ang (An\ o2\ h2\ k2))$ **by** *(simp add:Ang-simple-def)*

then have $P3 : Def (Ang (An\ h2\ o2\ k2))$ **by** *(blast intro:Ang-def-rev Ang-def-inv)*

from *assms P3* **have** $Ang-inside (An\ h1\ o1\ k1)\ l1 \implies \exists p. Ang-inside (An\ h2\ o2\ k2)\ p$

$\wedge Cong (Geos (Ang (An\ h1\ o1\ l1))\ add\ Emp) (Geos (Ang (An\ h2\ o2\ p))\ add\ Emp)$

$\wedge Cong (Geos (Ang (An\ k1\ o1\ l1))\ add\ Emp) (Geos (Ang (An\ k2\ o2\ p))\ add\ Emp)$ **by** *(simp add:Ang-split)*

then obtain $l21 :: Point$ **where** $P4 : Ang-inside (An\ h1\ o1\ k1)\ l1 \implies Ang-inside (An\ h2\ o2\ k2)\ l21$

$\wedge Cong (Geos (Ang (An\ h1\ o1\ l1))\ add\ Emp) (Geos (Ang (An\ h2\ o2\ l21))\ add\ Emp)$

$\wedge Cong (Geos (Ang (An\ k1\ o1\ l1))\ add\ Emp) (Geos (Ang (An\ k2\ o2\ l21))\ add\ Emp)$ **by** *blast*

then have $P5 : Ang-inside (An\ h1\ o1\ k1)\ l1 \implies$

$Plane-sameside (Li\ o2\ k2)\ h2\ l21 \wedge Plane-sameside (Li\ o2\ h2)\ k2\ l21$ **by** *(simp add:Ang-inside-def)*

from *assms* **have** $P6 : Plane-diffside (Li\ o2\ k2)\ h2\ l2$ **by** *(simp add:Ang-inside-Planeside)*

from $P5\ P6$ **have** $Ang\text{-}inside\ (An\ h1\ o1\ k1)\ l1 \implies$
 $Plane\text{-}diffside\ (Li\ o2\ k2)\ l21\ l2$ **by** $(blast\ intro:Plane\text{-}trans)$
then have $Ang\text{-}inside\ (An\ h1\ o1\ k1)\ l1 \implies \exists p. Bet\text{-}Point\ (Se\ l21\ l2)\ p \wedge Line\text{-}on$
 $(Li\ o2\ k2)\ p$
 $\wedge \neg Line\text{-}on\ (Li\ o2\ k2)\ l21 \wedge \neg Line\text{-}on\ (Li\ o2\ k2)\ l2$ **by** $(simp\ add:Plane\text{-}diffside\text{-}def)$
then obtain $pn :: Point$ **where** $P7 : Ang\text{-}inside\ (An\ h1\ o1\ k1)\ l1 \implies$
 $Bet\text{-}Point\ (Se\ l21\ l2)\ pn \wedge Line\text{-}on\ (Li\ o2\ k2)\ pn$
 $\wedge \neg Line\text{-}on\ (Li\ o2\ k2)\ l21 \wedge \neg Line\text{-}on\ (Li\ o2\ k2)\ l2$ **by** $blast$
then have $P8 : Ang\text{-}inside\ (An\ h1\ o1\ k1)\ l1 \implies Bet\text{-}Point\ (Se\ l21\ l2)\ pn$ **by**
 $simp$
then have $P9 : Ang\text{-}inside\ (An\ h1\ o1\ k1)\ l1 \implies$
 $\neg Eq\ (Geos\ (Poi\ l21)\ add\ Emp)\ (Geos\ (Poi\ l2)\ add\ Emp)$ **by** $(simp\ add:Bet\text{-}Point\text{-}def)$
from $assms$ **have** $P10 : Plane\text{-}sameside\ (Li\ o2\ h2)\ l2\ k2$ **by** $(simp\ add:Plane\text{-}sameside\text{-}rev)$
from $P5\ P9\ P10$ **have** $P11 : Ang\text{-}inside\ (An\ h1\ o1\ k1)\ l1 \implies Plane\text{-}sameside$
 $(Li\ o2\ h2)\ l2\ l21$ **by** $(blast\ intro:Plane\text{-}sameside\text{-}trans)$
from $P8$ **have** $P12 : Ang\text{-}inside\ (An\ h1\ o1\ k1)\ l1 \implies$
 $Eq\ (Geos\ (Poi\ pn)\ add\ Emp)\ (Geos\ (Poi\ o2)\ add\ Emp) \implies Bet\text{-}Point\ (Se\ l21$
 $l2)\ o2$ **by** $(simp\ add:Point\text{-}Eq)$
have $P13 : Line\text{-}on\ (Li\ o2\ h2)\ o2$ **by** $(simp\ add:Line\text{-}on\text{-}rule)$
have $P14 : Line\text{-}on\ (Li\ l21\ l2)\ l2$ **by** $(simp\ add:Line\text{-}on\text{-}rule)$
from $P14$ **have** $P15 : Eq\ (Geos\ (Lin\ (Li\ l21\ l2))\ add\ Emp)\ (Geos\ (Lin\ (Li\ o2$
 $h2))\ add\ Emp) \implies$
 $Line\text{-}on\ (Li\ o2\ h2)\ l2$ **by** $(simp\ add:Line\text{-}on\text{-}trans)$
from $assms$ **have** $Def\ (Tri\ (Tr\ o2\ h2\ l2))$ **by** $(blast\ intro:Ang\text{-}to\text{-}Tri\ Tri\text{-}def\text{-}rev$
 $Tri\text{-}def\text{-}trans)$
then have $P16 : \neg Line\text{-}on\ (Li\ o2\ h2)\ l2$ **by** $(simp\ add:Tri\text{-}def\text{-}Line)$
from $P15\ P16$ **have** $P17 : \neg Eq\ (Geos\ (Lin\ (Li\ l21\ l2))\ add\ Emp)\ (Geos\ (Lin$
 $(Li\ o2\ h2))\ add\ Emp)$ **by** $blast$
from $P12\ P13\ P17$ **have** $Ang\text{-}inside\ (An\ h1\ o1\ k1)\ l1 \implies$
 $Eq\ (Geos\ (Poi\ pn)\ add\ Emp)\ (Geos\ (Poi\ o2)\ add\ Emp) \implies$
 $Plane\text{-}diffside\ (Li\ o2\ h2)\ l2\ l21$ **by** $(simp\ add:Plane\text{-}Bet\text{-}diffside\ Plane\text{-}diffside\text{-}rev)$
then have $P18 : Ang\text{-}inside\ (An\ h1\ o1\ k1)\ l1 \implies$
 $Eq\ (Geos\ (Poi\ pn)\ add\ Emp)\ (Geos\ (Poi\ o2)\ add\ Emp) \implies$
 $\neg Plane\text{-}sameside\ (Li\ o2\ h2)\ l2\ l21$ **by** $(simp\ add:Plane\text{-}diffside\text{-}not\text{-}sameside)$
from $P11\ P18$ **have** $P19 : Ang\text{-}inside\ (An\ h1\ o1\ k1)\ l1 \implies$
 $\neg Eq\ (Geos\ (Poi\ pn)\ add\ Emp)\ (Geos\ (Poi\ o2)\ add\ Emp)$ **by** $blast$
have $P20 : Line\text{-}on\ (Li\ o2\ k2)\ o2$ **by** $(simp\ add:Line\text{-}on\text{-}rule)$
from $P8$ **have** $P21 : Ang\text{-}inside\ (An\ h1\ o1\ k1)\ l1 \implies Line\text{-}on\ (Li\ l21\ l2)\ pn$ **by**
 $(simp\ add:Line\text{-}Bet\text{-}on)$
from $P7\ P19\ P20\ P21$ **have** $P22 : Ang\text{-}inside\ (An\ h1\ o1\ k1)\ l1 \implies Line\text{-}on\ (Li$
 $l21\ l2)\ o2 \implies$
 $Eq\ (Geos\ (Lin\ (Li\ l21\ l2))\ add\ Emp)\ (Geos\ (Lin\ (Li\ o2\ k2))\ add\ Emp)$ **by** $(simp$
 $add:Line\text{-}unique)$
from $P14\ P22$ **have** $P23 : Ang\text{-}inside\ (An\ h1\ o1\ k1)\ l1 \implies$
 $Line\text{-}on\ (Li\ l21\ l2)\ o2 \implies Line\text{-}on\ (Li\ o2\ k2)\ l2$ **by** $(simp\ add:Line\text{-}on\text{-}trans)$
from $P7\ P23$ **have** $P24 : Ang\text{-}inside\ (An\ h1\ o1\ k1)\ l1 \implies \neg Line\text{-}on\ (Li\ l21\ l2)$
 $o2$ **by** $blast$
from $P9\ P24$ **have** $Ang\text{-}inside\ (An\ h1\ o1\ k1)\ l1 \implies Def\ (Ang\ (An\ l21\ l2\ o2))$
by $(simp\ add:Ang\text{-}simple\text{-}def)$

then have $P25 : \text{Ang-inside } (An\ h1\ o1\ k1)\ l1 \implies \text{Def } (\text{Ang } (An\ l21\ o2\ l2))$ **by**
(blast intro:Ang-def-rev Ang-def-inv)
then have $P26 : \text{Ang-inside } (An\ h1\ o1\ k1)\ l1 \implies$
 $\neg \text{Eq } (\text{Geos } (\text{Lin } (Li\ o2\ l21))\ \text{add Emp})\ (\text{Geos } (\text{Lin } (Li\ o2\ l2))\ \text{add Emp})$ **by**
(simp add:Ang-def)
have $P27 : \text{Eq } (\text{Geos } (\text{Ang } (An\ h2\ o2\ l21))\ \text{add Emp})\ (\text{Geos } (\text{Ang } (An\ l21\ o2\ h2))\ \text{add Emp})$ **by** *(simp add:Ang-roll)*
from $P4\ P27$ **have** $P28 : \text{Ang-inside } (An\ h1\ o1\ k1)\ l1 \implies$
 $\text{Cong } (\text{Geos } (\text{Ang } (An\ h1\ o1\ l1))\ \text{add Emp})\ (\text{Geos } (\text{Ang } (An\ l21\ o2\ h2))\ \text{add Emp})$ **by** *(blast intro:Ang-weektrans Ang-rev Eq-rev)*
have $P29 : \text{Eq } (\text{Geos } (\text{Ang } (An\ h2\ o2\ l2))\ \text{add Emp})\ (\text{Geos } (\text{Ang } (An\ l2\ o2\ h2))\ \text{add Emp})$ **by** *(simp add:Ang-roll)*
from *assms* $P29$ **have** $P30 : \text{Cong } (\text{Geos } (\text{Ang } (An\ h1\ o1\ l1))\ \text{add Emp})\ (\text{Geos } (\text{Ang } (An\ l2\ o2\ h2))\ \text{add Emp})$ **by** *(blast intro:Ang-weektrans Ang-rev Eq-rev)*
from $P11\ P28\ P30$ **have** $P31 : \text{Ang-inside } (An\ h1\ o1\ k1)\ l1 \implies$
 $\text{Eq } (\text{Geos } (\text{Lin } (Li\ l2\ o2))\ \text{add Emp})\ (\text{Geos } (\text{Lin } (Li\ l21\ o2))\ \text{add Emp})$ **by**
(simp add:Ang-move-unique)
from $P25$ **have** $P32 : \text{Ang-inside } (An\ h1\ o1\ k1)\ l1 \implies \text{Def } (\text{Tri } (Tr\ l21\ o2\ l2))$
by *(simp add:Ang-to-Tri)*
then have $\text{Ang-inside } (An\ h1\ o1\ k1)\ l1 \implies \neg \text{Eq } (\text{Geos } (Poi\ o2)\ \text{add Emp})\ (\text{Geos } (Poi\ l2)\ \text{add Emp})$ **by** *(simp add:Tri-def)*
then have $P33 : \text{Ang-inside } (An\ h1\ o1\ k1)\ l1 \implies$
 $\text{Eq } (\text{Geos } (\text{Lin } (Li\ o2\ l2))\ \text{add Emp})\ (\text{Geos } (\text{Lin } (Li\ l2\ o2))\ \text{add Emp})$ **by** *(simp add:Line-rev)*
from $P32$ **have** $\text{Ang-inside } (An\ h1\ o1\ k1)\ l1 \implies$
 $\neg \text{Eq } (\text{Geos } (Poi\ l21)\ \text{add Emp})\ (\text{Geos } (Poi\ o2)\ \text{add Emp})$ **by** *(simp add:Tri-def)*
then have $P34 : \text{Ang-inside } (An\ h1\ o1\ k1)\ l1 \implies$
 $\text{Eq } (\text{Geos } (\text{Lin } (Li\ l21\ o2))\ \text{add Emp})\ (\text{Geos } (\text{Lin } (Li\ o2\ l21))\ \text{add Emp})$ **by**
(simp add:Line-rev)
from $P31\ P33$ **have** $P35 : \text{Ang-inside } (An\ h1\ o1\ k1)\ l1 \implies$
 $\text{Eq } (\text{Geos } (\text{Lin } (Li\ o2\ l2))\ \text{add Emp})\ (\text{Geos } (\text{Lin } (Li\ l21\ o2))\ \text{add Emp})$ **by**
(blast intro:Eq-rev Eq-trans)
from $P34\ P35$ **have** $P36 : \text{Ang-inside } (An\ h1\ o1\ k1)\ l1 \implies$
 $\text{Eq } (\text{Geos } (\text{Lin } (Li\ o2\ l21))\ \text{add Emp})\ (\text{Geos } (\text{Lin } (Li\ o2\ l2))\ \text{add Emp})$ **by** *(blast intro:Eq-rev Eq-trans)*
from $P26\ P36$ **show** $\neg \text{Ang-inside } (An\ h1\ o1\ k1)\ l1$ **by** *blast*
from *assms* **have** $P37 : \text{Def } (\text{Ang } (An\ k1\ o1\ h1))$ **by** *(blast intro:Ang-def-rev)*
from *assms* **have** $P38 : \neg \text{Line-on } (Li\ o1\ h1)\ l1$ **by** *(simp add:Plane-sameside-def)*
from $P37$ **have** $P39 : \neg \text{Eq } (\text{Geos } (Poi\ o1)\ \text{add Emp})\ (\text{Geos } (Poi\ h1)\ \text{add Emp})$
by *(simp add:Ang-def)*
from $P38\ P39$ **have** $\text{Def } (\text{Ang } (An\ o1\ h1\ l1))$ **by** *(simp add:Ang-simple-def)*
then have $P40 : \text{Def } (\text{Ang } (An\ l1\ o1\ h1))$ **by** *(blast intro:Ang-def-rev Ang-def-inv)*
from *assms* $P37\ P40$ **have** $P41 : \text{Eq } (\text{Geos } (\text{Lin } (Li\ o1\ k1))\ \text{add Emp})\ (\text{Geos } (\text{Lin } (Li\ o1\ l1))\ \text{add Emp}) \implies$
 $\text{Cong } (\text{Geos } (\text{Ang } (An\ k1\ o1\ h1))\ \text{add Emp})\ (\text{Geos } (\text{Ang } (An\ l1\ o1\ h1))\ \text{add Emp})$ **by** *(simp add:Ang-move-unique-inv)*
have $P42 : \text{Cong } (\text{Geos } (\text{Ang } (An\ k1\ o1\ h1))\ \text{add Emp})\ (\text{Geos } (\text{Ang } (An\ h1\ o1\ k1))\ \text{add Emp})$ **by** *(simp add:Ang-roll)*
from *assms* $P37\ P40\ P41\ P42$ **have** $P43 : \text{Eq } (\text{Geos } (\text{Lin } (Li\ o1\ k1))\ \text{add Emp})$

(Geos (Lin (Li o1 l1)) add Emp) \implies
 Cong (Geos (Ang (An h1 o1 k1)) add Emp) (Geos (Ang (An l1 o1 h1)) add Emp) **by** (blast intro:Ang-trans Ang-rev)
 have P44 : Cong (Geos (Ang (An l1 o1 h1)) add Emp) (Geos (Ang (An h1 o1 l1)) add Emp) **by** (simp add:Ang-roll)
 from P40 have P45 : Def (Ang (An h1 o1 l1)) **by** (blast intro:Ang-def-rev)
 from assms P40 P43 P44 P45 have P46 : Eq (Geos (Lin (Li o1 k1)) add Emp) (Geos (Lin (Li o1 l1)) add Emp) \implies
 Cong (Geos (Ang (An h1 o1 k1)) add Emp) (Geos (Ang (An h1 o1 l1)) add Emp) **by** (blast intro:Ang-trans Ang-rev)
 from assms P45 P46 have P47 : Eq (Geos (Lin (Li o1 k1)) add Emp) (Geos (Lin (Li o1 l1)) add Emp) \implies
 Cong (Geos (Ang (An h1 o1 k1)) add Emp) (Geos (Ang (An h2 o2 l2)) add Emp) **by** (blast intro:Ang-trans Ang-rev)
 from assms P3 P47 have P48 : Eq (Geos (Lin (Li o1 k1)) add Emp) (Geos (Lin (Li o1 l1)) add Emp) \implies
 Cong (Geos (Ang (An h2 o2 k2)) add Emp) (Geos (Ang (An h2 o2 l2)) add Emp) **by** (blast intro:Ang-trans Ang-rev)
 from P29 P48 have P49 : Eq (Geos (Lin (Li o1 k1)) add Emp) (Geos (Lin (Li o1 l1)) add Emp) \implies
 Cong (Geos (Ang (An h2 o2 k2)) add Emp) (Geos (Ang (An l2 o2 h2)) add Emp) **by** (blast intro:Ang-weektrans Ang-rev Eq-rev)
 have P50 : Cong (Geos (Ang (An h2 o2 k2)) add Emp) (Geos (Ang (An k2 o2 h2)) add Emp) **by** (simp add:Ang-roll)
 from assms P49 P50 have P51 : Eq (Geos (Lin (Li o1 k1)) add Emp) (Geos (Lin (Li o1 l1)) add Emp) \implies
 Eq (Geos (Lin (Li k2 o2)) add Emp) (Geos (Lin (Li l2 o2)) add Emp)
 $\wedge \neg$ Bet-Point (Se k2 l2) o2 **by** (simp add:Ang-move-unique)
 from assms have Def (Ang (An h2 o2 l2)) \wedge Plane-sameside (Li o2 h2) l2 k2
 \wedge Plane-sameside (Li o2 l2) h2 k2 **by** (simp add:Ang-inside-def)
 then have P52 : \neg Line-on (Li o2 l2) k2 **by** (simp add:Plane-sameside-def)
 from assms have \neg Eq (Geos (Poi o2) add Emp) (Geos (Poi l2) add Emp) **by** (simp add:Ang-def)
 then have P53 : Eq (Geos (Lin (Li o2 l2)) add Emp) (Geos (Lin (Li l2 o2)) add Emp) **by** (simp add:Line-rev)
 from P51 P53 have P54 : Eq (Geos (Lin (Li o1 k1)) add Emp) (Geos (Lin (Li o1 l1)) add Emp) \implies
 Eq (Geos (Lin (Li k2 o2)) add Emp) (Geos (Lin (Li o2 l2)) add Emp) **by** (blast intro:Eq-trans)
 have P55 : Line-on (Li k2 o2) k2 **by** (simp add:Line-on-rule)
 from P54 P55 have P56 : Eq (Geos (Lin (Li o1 k1)) add Emp) (Geos (Lin (Li o1 l1)) add Emp) \implies
 Line-on (Li o2 l2) k2 **by** (simp add:Line-on-trans)
 from P52 P56 show \neg Eq (Geos (Lin (Li o1 k1)) add Emp) (Geos (Lin (Li o1 l1)) add Emp) **by** blast
qed

theorem (in Congruence-Rule) Ang-move-Smaller :
 assumes

$Def (Ang (An\ h1\ o1\ k1))\ Def (Ang (An\ h2\ o2\ l2))$
 $Cong (Geos (Ang (An\ h1\ o1\ k1))\ add\ Emp)\ (Geos (Ang (An\ h2\ o2\ k2))\ add\ Emp)$
 $Plane-sameside (Li\ o2\ h2)\ k2\ l2$
 $Cong (Geos (Ang (An\ h2\ o2\ l2))\ add\ Emp)\ (Geos (Ang (An\ h1\ o1\ l1))\ add\ Emp)$
 $Plane-sameside (Li\ o1\ h1)\ k1\ l1$
 $\neg Ang-inside (An\ h2\ o2\ l2)\ k2$
 $\neg Eq (Geos (Lin (Li\ o2\ k2))\ add\ Emp)\ (Geos (Lin (Li\ o2\ l2))\ add\ Emp)$
shows $Ang-inside (An\ h1\ o1\ k1)\ l1$
proof –
have $P1 : Ang-inside (An\ l2\ o2\ h2)\ k2 \implies Ang-inside (An\ h2\ o2\ l2)\ k2$ **by** (*simp add:Ang-inside-def Ang-def-rev*)
from *assms* $P1$ **have** $P2 : \neg Ang-inside (An\ l2\ o2\ h2)\ k2$ **by** *blast*
from *assms* **have** $P3 : \neg Line-on (Li\ o2\ h2)\ k2$ **by** (*simp add:Plane-sameside-def*)
from *assms* **have** $\neg Eq (Geos (Poi\ h2))\ add\ Emp)\ (Geos (Poi\ o2))\ add\ Emp$ **by** (*simp add:Ang-def*)
then **have** $P4 : \neg Eq (Geos (Poi\ o2))\ add\ Emp)\ (Geos (Poi\ h2))\ add\ Emp$ **by** (*blast intro:Eq-rev*)
from $P3\ P4$ **have** $Def (Ang (An\ o2\ h2\ k2))$ **by** (*simp add:Ang-simple-def*)
then **have** $P5 : Def (Ang (An\ k2\ o2\ h2))$ **by** (*blast intro:Ang-def-rev Ang-def-inv*)
from *assms* **have** $P6 : Def (Ang (An\ l2\ o2\ h2))$ **by** (*blast intro:Ang-def-rev*)
from *assms* $P5\ P6$ **have** $P7 : Ang-inside (An\ k2\ o2\ h2)\ l2 \wedge \neg Ang-inside (An\ l2\ o2\ h2)\ k2$
 $\vee \neg Ang-inside (An\ k2\ o2\ h2)\ l2 \wedge Ang-inside (An\ l2\ o2\ h2)\ k2$ **by** (*simp add:Ang-inside-case*)
from $P2\ P7$ **have** $Ang-inside (An\ k2\ o2\ h2)\ l2$ **by** *blast*
then **have** $P8 : Ang-inside (An\ h2\ o2\ k2)\ l2$ **by** (*simp add:Ang-inside-def Ang-def-rev*)
from *assms* **have** $P9 : \neg Line-on (Li\ o1\ h1)\ l1$ **by** (*simp add:Plane-sameside-def*)
from *assms* **have** $\neg Eq (Geos (Poi\ h1))\ add\ Emp)\ (Geos (Poi\ o1))\ add\ Emp$ **by** (*simp add:Ang-def*)
then **have** $P10 : \neg Eq (Geos (Poi\ o1))\ add\ Emp)\ (Geos (Poi\ h1))\ add\ Emp$ **by** (*blast intro:Eq-rev*)
from $P9\ P10$ **have** $Def (Ang (An\ o1\ h1\ l1))$ **by** (*simp add:Ang-simple-def*)
then **have** $P11 : Def (Ang (An\ h1\ o1\ l1))$ **by** (*blast intro:Ang-def-rev Ang-def-inv*)
from $P5\ P11$ **have** $P12 : Def (Ang (An\ h2\ o2\ k2))$ **by** (*blast intro:Ang-def-rev Ang-def-inv*)
from *assms* **have** $P13 : Cong (Geos (Ang (An\ h1\ o1\ l1))\ add\ Emp)\ (Geos (Ang (An\ h2\ o2\ l2))\ add\ Emp)$ **by** (*blast intro:Ang-rev*)
from *assms* **have** $P14 : Plane-sameside (Li\ o2\ h2)\ l2\ k2$ **by** (*simp add:Plane-sameside-rev*)
from *assms* **have** $P15 : Cong (Geos (Ang (An\ h2\ o2\ k2))\ add\ Emp)\ (Geos (Ang (An\ h1\ o1\ k1))\ add\ Emp)$ **by** (*blast intro:Ang-rev*)
from *assms* **have** $P16 : Plane-sameside (Li\ o1\ h1)\ l1\ k1$ **by** (*simp add:Plane-sameside-rev*)
from $P8\ P11\ P12\ P13\ P14\ P15\ P16$ **have** $P17 : \neg Ang-inside (An\ h1\ o1\ l1)\ k1$ **by** (*simp add:Ang-move-Greater*)
have $P18 : Ang-inside (An\ l1\ o1\ h1)\ k1 \implies Ang-inside (An\ h1\ o1\ l1)\ k1$ **by** (*simp add:Ang-inside-def Ang-def-rev*)
from $P17\ P18$ **have** $P19 : \neg Ang-inside (An\ l1\ o1\ h1)\ k1$ **by** *blast*

from *assms* **have** $P20 : \text{Def } (\text{Ang } (\text{An } k1 \ o1 \ h1))$ **by** (*blast intro:Ang-def-rev*)
from $P11$ **have** $P21 : \text{Def } (\text{Ang } (\text{An } l1 \ o1 \ h1))$ **by** (*blast intro:Ang-def-rev*)
have $\text{Line-on } (\text{Li } o1 \ k1) \ k1$ **by** (*simp add:Line-on-rule*)
then have $P22 : \text{Eq } (\text{Geos } (\text{Lin } (\text{Li } o1 \ k1)) \ \text{add Emp})$ ($\text{Geos } (\text{Lin } (\text{Li } o1 \ l1))$
 add Emp) \implies
 $\text{Line-on } (\text{Li } o1 \ l1) \ k1$ **by** (*simp add:Line-on-trans*)
have $P23 : \text{Line-on } (\text{Li } o1 \ h1) \ o1$ **by** (*simp add:Line-on-rule*)
have $\text{Line-on } (\text{Li } l1 \ k1) \ l1$ **by** (*simp add:Line-on-rule*)
then have $P24 : \text{Eq } (\text{Geos } (\text{Lin } (\text{Li } l1 \ k1)) \ \text{add Emp})$ ($\text{Geos } (\text{Lin } (\text{Li } o1 \ h1))$
 add Emp) $\implies \text{Line-on } (\text{Li } o1 \ h1) \ l1$ **by** (*simp add:Line-on-trans*)
from $P9 \ P24$ **have** $P25 : \neg \text{Eq } (\text{Geos } (\text{Lin } (\text{Li } l1 \ k1)) \ \text{add Emp})$ ($\text{Geos } (\text{Lin } (\text{Li } o1 \ h1))$
 add Emp) **by** *blast*
from $P23 \ P25$ **have** $\text{Bet-Point } (\text{Se } l1 \ k1) \ o1 \implies \text{Plane-diffside } (\text{Li } o1 \ h1) \ l1 \ k1$
by (*simp add:Plane-Bet-diffside*)
then have $P26 : \text{Bet-Point } (\text{Se } l1 \ k1) \ o1 \implies \neg \text{Plane-sameside } (\text{Li } o1 \ h1) \ l1 \ k1$
by (*simp add:Plane-diffside-not-sameside*)
from $P16 \ P26$ **have** $P27 : \neg \text{Bet-Point } (\text{Se } l1 \ k1) \ o1$ **by** *blast*
have $P28 : \text{Line-on } (\text{Li } o1 \ h1) \ h1$ **by** (*simp add:Line-on-rule*)
have $P29 : \neg \text{Bet-Point } (\text{Se } h1 \ h1) \ o1$ **by** (*simp add:Bet-end-Point*)
from *assms* **have** $P30 : \neg \text{Eq } (\text{Geos } (\text{Poi } o1) \ \text{add Emp})$ ($\text{Geos } (\text{Poi } k1) \ \text{add Emp}$)
by (*simp add:Ang-def*)
from $P10 \ P21 \ P22 \ P27 \ P28 \ P29 \ P30$ **have** $P31 : \text{Eq } (\text{Geos } (\text{Lin } (\text{Li } o1 \ k1)) \ \text{add}$
 $\text{Emp})$ ($\text{Geos } (\text{Lin } (\text{Li } o1 \ l1)) \ \text{add Emp}$) \implies
 $\text{Eq } (\text{Geos } (\text{Ang } (\text{An } l1 \ o1 \ h1)) \ \text{add Emp})$ ($\text{Geos } (\text{Ang } (\text{An } k1 \ o1 \ h1)) \ \text{add Emp}$)
 $\wedge \text{Def } (\text{Ang } (\text{An } k1 \ o1 \ h1))$ **by** (*simp add:Ang-Point-swap*)
have $P32 : \text{Eq } (\text{Geos } (\text{Ang } (\text{An } l1 \ o1 \ h1)) \ \text{add Emp})$ ($\text{Geos } (\text{Ang } (\text{An } h1 \ o1 \ l1))$
 add Emp) **by** (*simp add:Ang-roll*)
from $P31 \ P32$ **have** $P33 : \text{Eq } (\text{Geos } (\text{Lin } (\text{Li } o1 \ k1)) \ \text{add Emp})$ ($\text{Geos } (\text{Lin } (\text{Li } o1 \ l1))$
 add Emp) \implies
 $\text{Eq } (\text{Geos } (\text{Ang } (\text{An } h1 \ o1 \ l1)) \ \text{add Emp})$ ($\text{Geos } (\text{Ang } (\text{An } k1 \ o1 \ h1)) \ \text{add Emp}$)
by (*blast intro:Eq-trans Eq-rev*)
have $P34 : \text{Eq } (\text{Geos } (\text{Ang } (\text{An } k1 \ o1 \ h1)) \ \text{add Emp})$ ($\text{Geos } (\text{Ang } (\text{An } h1 \ o1 \ k1))$
 add Emp) **by** (*simp add:Ang-roll*)
from $P33 \ P34$ **have** $P35 : \text{Eq } (\text{Geos } (\text{Lin } (\text{Li } o1 \ k1)) \ \text{add Emp})$ ($\text{Geos } (\text{Lin } (\text{Li } o1 \ l1))$
 add Emp) \implies
 $\text{Eq } (\text{Geos } (\text{Ang } (\text{An } h1 \ o1 \ l1)) \ \text{add Emp})$ ($\text{Geos } (\text{Ang } (\text{An } h1 \ o1 \ k1)) \ \text{add Emp}$)
by (*blast intro:Eq-trans Eq-rev*)
from *assms* $P35$ **have** $P36 : \text{Eq } (\text{Geos } (\text{Lin } (\text{Li } o1 \ k1)) \ \text{add Emp})$ ($\text{Geos } (\text{Lin } (\text{Li } o1 \ l1))$
 add Emp) \implies
 $\text{Cong } (\text{Geos } (\text{Ang } (\text{An } h1 \ o1 \ k1)) \ \text{add Emp})$ ($\text{Geos } (\text{Ang } (\text{An } h2 \ o2 \ l2)) \ \text{add}$
 Emp) **by** (*blast intro:Ang-weektrans Ang-rev Eq-rev*)
have $P37 : \text{Eq } (\text{Geos } (\text{Ang } (\text{An } h2 \ o2 \ k2)) \ \text{add Emp})$ ($\text{Geos } (\text{Ang } (\text{An } k2 \ o2 \ h2))$
 add Emp) **by** (*simp add:Ang-roll*)
from *assms* $P37$ **have** $P38 : \text{Cong } (\text{Geos } (\text{Ang } (\text{An } h1 \ o1 \ k1)) \ \text{add Emp})$ ($\text{Geos } (\text{Ang } (\text{An } k2 \ o2 \ h2))$
 add Emp) **by** (*blast intro:Ang-weektrans Ang-rev Eq-rev*)
have $P39 : \text{Eq } (\text{Geos } (\text{Ang } (\text{An } h2 \ o2 \ l2)) \ \text{add Emp})$ ($\text{Geos } (\text{Ang } (\text{An } l2 \ o2 \ h2))$
 add Emp) **by** (*simp add:Ang-roll*)
from $P36 \ P39$ **have** $P40 : \text{Eq } (\text{Geos } (\text{Lin } (\text{Li } o1 \ k1)) \ \text{add Emp})$ ($\text{Geos } (\text{Lin } (\text{Li } o1 \ l1))$
 add Emp) \implies

Cong (*Geos* (*Ang* (*An* *h1* *o1* *k1*)) *add Emp*) (*Geos* (*Ang* (*An* *l2* *o2* *h2*)) *add Emp*) **by** (*blast intro:Ang-weektrans Ang-rev Eq-rev*)
from *assms P38 P40* **have** $P_{41} : Eq$ (*Geos* (*Lin* (*Li* *o1* *k1*)) *add Emp*) (*Geos* (*Lin* (*Li* *o1* *l1*)) *add Emp*) \implies
 Eq (*Geos* (*Lin* (*Li* *k2* *o2*)) *add Emp*) (*Geos* (*Lin* (*Li* *l2* *o2*)) *add Emp*) $\wedge \neg$
Bet-Point (*Se* *k2* *l2*) *o2* **by** (*simp add:Ang-move-unique*)
from *P12* **have** $\neg Eq$ (*Geos* (*Poi* *o2*) *add Emp*) (*Geos* (*Poi* *k2*) *add Emp*) **by**
(*simp add:Ang-def*)
then **have** $P_{42} : Eq$ (*Geos* (*Lin* (*Li* *o2* *k2*)) *add Emp*) (*Geos* (*Lin* (*Li* *k2* *o2*)) *add Emp*) **by** (*simp add:Line-rev*)
from P_{41} P_{42} **have** $P_{43} : Eq$ (*Geos* (*Lin* (*Li* *o1* *k1*)) *add Emp*) (*Geos* (*Lin* (*Li* *o1* *l1*)) *add Emp*) \implies
 Eq (*Geos* (*Lin* (*Li* *o2* *k2*)) *add Emp*) (*Geos* (*Lin* (*Li* *l2* *o2*)) *add Emp*) **by** (*blast intro:Eq-rev Eq-trans*)
from *assms* **have** $\neg Eq$ (*Geos* (*Poi* *o2*) *add Emp*) (*Geos* (*Poi* *l2*) *add Emp*) **by**
(*simp add:Ang-def*)
then **have** $P_{44} : Eq$ (*Geos* (*Lin* (*Li* *o2* *l2*)) *add Emp*) (*Geos* (*Lin* (*Li* *l2* *o2*)) *add Emp*) **by** (*simp add:Line-rev*)
from P_{43} P_{44} **have** $P_{45} : Eq$ (*Geos* (*Lin* (*Li* *o1* *k1*)) *add Emp*) (*Geos* (*Lin* (*Li* *o1* *l1*)) *add Emp*) \implies
 Eq (*Geos* (*Lin* (*Li* *o2* *k2*)) *add Emp*) (*Geos* (*Lin* (*Li* *o2* *l2*)) *add Emp*) **by** (*blast intro:Eq-rev Eq-trans*)
from *assms* P_{45} **have** $P_{46} : \neg Eq$ (*Geos* (*Lin* (*Li* *o1* *k1*)) *add Emp*) (*Geos* (*Lin* (*Li* *o1* *l1*)) *add Emp*) **by** *blast*
from *assms P20 P21 P46* **have** $P_{47} : Ang\text{-inside}$ (*An* *k1* *o1* *h1*) *l1* $\wedge \neg Ang\text{-inside}$
(*An* *l1* *o1* *h1*) *k1*
 $\vee \neg Ang\text{-inside}$ (*An* *k1* *o1* *h1*) *l1* $\wedge Ang\text{-inside}$ (*An* *l1* *o1* *h1*) *k1* **by** (*simp add:Ang-inside-case*)
from *P19 P47* **have** *Ang-inside* (*An* *k1* *o1* *h1*) *l1* **by** *blast*
thus *Ang-inside* (*An* *h1* *o1* *k1*) *l1* **by** (*simp add:Ang-inside-def Ang-def-rev*)
qed

lemma (in *Congruence-Rule*) *Ang-not-Gr-Eq-rev* :

assumes

Def (*Ang* (*An* *p11* *p12* *p13*)) *Def* (*Ang* (*An* *p21* *p22* *p23*))

$\neg Gr$ (*Geos* (*Ang* (*An* *p21* *p22* *p23*)) *add Emp*) (*Geos* (*Ang* (*An* *p11* *p12* *p13*)) *add Emp*)

shows

Cong (*Geos* (*Ang* (*An* *p11* *p12* *p13*)) *add Emp*) (*Geos* (*Ang* (*An* *p21* *p22* *p23*)) *add Emp*)

$\vee Gr$ (*Geos* (*Ang* (*An* *p11* *p12* *p13*)) *add Emp*) (*Geos* (*Ang* (*An* *p21* *p22* *p23*)) *add Emp*)

proof –

from *assms* **have** $\neg Line\text{-on}$ (*Li* *p12* *p13*) *p11* **by** (*simp add:Ang-to-Tri Tri-def-Line*)

then **have** $\exists p.$ *Cong* (*Geos* (*Ang* (*An* *p21* *p22* *p23*)) *add Emp*) (*Geos* (*Ang* (*An* *p* *p12* *p13*)) *add Emp*)

$\wedge Plane\text{-sameside}$ (*Li* *p12* *p13*) *p* *p11* **using** *assms* **by** (*simp add:Ang-move-sameside*)

then **obtain** $p_4 :: Point$ **where** $P1 : Cong$ (*Geos* (*Ang* (*An* *p21* *p22* *p23*)) *add Emp*) (*Geos* (*Ang* (*An* *p4* *p12* *p13*)) *add Emp*)

\wedge *Plane-sameside* (*Li* *p12* *p13*) *p4* *p11* **by** *blast*
from *assms* *P1* **have** *P2* : \neg *Ang-inside* (*An* *p11* *p12* *p13*) *p4* \wedge \neg *Eq* (*Geos* (*Lin* (*Li* *p12* *p11*)) *add Emp*) (*Geos* (*Lin* (*Li* *p12* *p4*)) *add Emp*) \implies
Gr (*Geos* (*Ang* (*An* *p21* *p22* *p23*)) *add Emp*) (*Geos* (*Ang* (*An* *p11* *p12* *p13*)) *add Emp*) **by** (*blast intro:Ang-less-def*)
from *assms* *P2* **have** *P3* : *Ang-inside* (*An* *p11* *p12* *p13*) *p4* \vee *Eq* (*Geos* (*Lin* (*Li* *p12* *p11*)) *add Emp*) (*Geos* (*Lin* (*Li* *p12* *p4*)) *add Emp*) **by** *blast*
from *P1* **have** *P4* : *Plane-sameside* (*Li* *p12* *p13*) *p11* *p4* **by** (*simp add:Plane-sameside-rev*)
then **have** *P5* : \neg *Line-on* (*Li* *p12* *p13*) *p4* **by** (*simp add:Plane-sameside-def*)
from *assms* **have** *P6* : \neg *Eq* (*Geos* (*Poi* *p12*) *add Emp*) (*Geos* (*Poi* *p13*) *add Emp*) **by** (*simp add:Ang-def*)
from *P5* *P6* **have** *Def* (*Ang* (*An* *p12* *p13* *p4*)) **by** (*simp add:Ang-simple-def*)
then **have** *P7* : *Def* (*Ang* (*An* *p4* *p12* *p13*)) **by** (*blast intro:Ang-def-inv Ang-def-rev*)
from *assms* *P4* *P7* **have** *P8* : *Eq* (*Geos* (*Lin* (*Li* *p12* *p11*)) *add Emp*) (*Geos* (*Lin* (*Li* *p12* *p4*)) *add Emp*) \implies
Cong (*Geos* (*Ang* (*An* *p11* *p12* *p13*)) *add Emp*) (*Geos* (*Ang* (*An* *p4* *p12* *p13*)) *add Emp*) **by** (*simp add:Ang-move-unique-inv*)
from *assms* *P1* *P7* *P8* **have** *P9* : *Eq* (*Geos* (*Lin* (*Li* *p12* *p11*)) *add Emp*) (*Geos* (*Lin* (*Li* *p12* *p4*)) *add Emp*) \implies
Cong (*Geos* (*Ang* (*An* *p11* *p12* *p13*)) *add Emp*) (*Geos* (*Ang* (*An* *p21* *p22* *p23*)) *add Emp*) **by** (*blast intro:Ang-trans Ang-rev*)
from *P1* **have** *P10* : *Ang-inside* (*An* *p11* *p12* *p13*) *p4* \longleftrightarrow
Gr (*Geos* (*Ang* (*An* *p11* *p12* *p13*)) *add Emp*) (*Geos* (*Ang* (*An* *p21* *p22* *p23*)) *add Emp*) **by** (*simp add:Ang-greater-def*)
from *P3* *P9* *P10* **show** *Cong* (*Geos* (*Ang* (*An* *p11* *p12* *p13*)) *add Emp*) (*Geos* (*Ang* (*An* *p21* *p22* *p23*)) *add Emp*)
 \vee *Gr* (*Geos* (*Ang* (*An* *p11* *p12* *p13*)) *add Emp*) (*Geos* (*Ang* (*An* *p21* *p22* *p23*)) *add Emp*) **by** *blast*
qed

lemma (in Congruence-Rule) Ang-not-Eq-Gr :

assumes

Def (*Ang* (*An* *p11* *p12* *p13*)) *Def* (*Ang* (*An* *p21* *p22* *p23*))

\neg *Cong* (*Geos* (*Ang* (*An* *p11* *p12* *p13*)) *add Emp*) (*Geos* (*Ang* (*An* *p21* *p22* *p23*)) *add Emp*)

shows

Gr (*Geos* (*Ang* (*An* *p11* *p12* *p13*)) *add Emp*) (*Geos* (*Ang* (*An* *p21* *p22* *p23*)) *add Emp*)

\vee *Gr* (*Geos* (*Ang* (*An* *p21* *p22* *p23*)) *add Emp*) (*Geos* (*Ang* (*An* *p11* *p12* *p13*)) *add Emp*)

proof –

from *assms* **have** \neg *Line-on* (*Li* *p12* *p13*) *p11* **by** (*simp add:Ang-to-Tri Tri-def-Line*)

then **have** $\exists p.$ *Cong* (*Geos* (*Ang* (*An* *p21* *p22* *p23*)) *add Emp*) (*Geos* (*Ang* (*An* *p* *p12* *p13*)) *add Emp*)

\wedge *Plane-sameside* (*Li* *p12* *p13*) *p* *p11* **using** *assms* **by** (*simp add:Ang-move-sameside*)

then **obtain** *p4* :: *Point* **where** *P1* : *Cong* (*Geos* (*Ang* (*An* *p21* *p22* *p23*)) *add Emp*) (*Geos* (*Ang* (*An* *p4* *p12* *p13*)) *add Emp*)

\wedge *Plane-sameside* (*Li* *p12* *p13*) *p4* *p11* **by** *blast*

from *P1* **have** *P2* : *Plane-sameside* (*Li* *p12* *p13*) *p11* *p4* **by** (*simp add:Plane-sameside-rev*)

then have $P3 : \neg \text{Line-on } (Li\ p12\ p13)\ p4$ **by** (*simp add:Plane-sameside-def*)
from *assms* **have** $P4 : \neg \text{Eq } (Geos\ (Poi\ p12)\ \text{add}\ Emp)\ (Geos\ (Poi\ p13)\ \text{add}\ Emp)$ **by** (*simp add:Ang-def*)
from $P3\ P4$ **have** $Def\ (Ang\ (An\ p12\ p13\ p4))$ **by** (*simp add:Ang-simple-def*)
then have $P5 : Def\ (Ang\ (An\ p4\ p12\ p13))$ **by** (*blast intro:Ang-def-inv Ang-def-rev*)
from *assms* $P2\ P5$ **have** $P6 : \text{Eq } (Geos\ (Lin\ (Li\ p12\ p11))\ \text{add}\ Emp)\ (Geos\ (Lin\ (Li\ p12\ p4))\ \text{add}\ Emp)$ \implies
Cong (*Geos* (*Ang* (*An* $p11\ p12\ p13$)) *add* *Emp*) (*Geos* (*Ang* (*An* $p4\ p12\ p13$)) *add* *Emp*) **by** (*simp add:Ang-move-unique-inv*)
from *assms* $P1\ P5\ P6$ **have** $P7 : \text{Eq } (Geos\ (Lin\ (Li\ p12\ p11))\ \text{add}\ Emp)\ (Geos\ (Lin\ (Li\ p12\ p4))\ \text{add}\ Emp)$ \implies
Cong (*Geos* (*Ang* (*An* $p11\ p12\ p13$)) *add* *Emp*) (*Geos* (*Ang* (*An* $p21\ p22\ p23$)) *add* *Emp*) **by** (*blast intro:Ang-trans Ang-rev*)
from *assms* $P7$ **have** $P8 : \neg \text{Eq } (Geos\ (Lin\ (Li\ p12\ p11))\ \text{add}\ Emp)\ (Geos\ (Lin\ (Li\ p12\ p4))\ \text{add}\ Emp)$ **by** *blast*
from *assms* $P2\ P5\ P8$ **have** $P9 : \text{Ang-inside } (An\ p11\ p12\ p13)\ p4 \wedge \neg \text{Ang-inside } (An\ p4\ p12\ p13)\ p11$
 $\vee \neg \text{Ang-inside } (An\ p11\ p12\ p13)\ p4 \wedge \text{Ang-inside } (An\ p4\ p12\ p13)\ p11$ **by** (*simp add:Ang-inside-case*)
from $P1$ **have** $P10 : \text{Ang-inside } (An\ p11\ p12\ p13)\ p4 \longleftrightarrow$
Gr (*Geos* (*Ang* (*An* $p11\ p12\ p13$)) *add* *Emp*) (*Geos* (*Ang* (*An* $p21\ p22\ p23$)) *add* *Emp*) **by** (*simp add:Ang-greater-def*)
from $P1\ P8$ **have** $P11 : \neg \text{Ang-inside } (An\ p11\ p12\ p13)\ p4 \implies$
Gr (*Geos* (*Ang* (*An* $p21\ p22\ p23$)) *add* *Emp*) (*Geos* (*Ang* (*An* $p11\ p12\ p13$)) *add* *Emp*) **by** (*blast intro:Ang-less-def*)
from $P9\ P10\ P11$ **show** *Gr* (*Geos* (*Ang* (*An* $p11\ p12\ p13$)) *add* *Emp*) (*Geos* (*Ang* (*An* $p21\ p22\ p23$)) *add* *Emp*)
 \vee *Gr* (*Geos* (*Ang* (*An* $p21\ p22\ p23$)) *add* *Emp*) (*Geos* (*Ang* (*An* $p11\ p12\ p13$)) *add* *Emp*) **by** *blast*
qed

lemma (in Congruence-Rule) Ang-relation-case :

assumes

Def (*Ang* (*An* $p11\ p12\ p13$)) *Def* (*Ang* (*An* $p21\ p22\ p23$))

shows

Cong (*Geos* (*Ang* (*An* $p11\ p12\ p13$)) *add* *Emp*) (*Geos* (*Ang* (*An* $p21\ p22\ p23$)) *add* *Emp*)

\vee *Gr* (*Geos* (*Ang* (*An* $p11\ p12\ p13$)) *add* *Emp*) (*Geos* (*Ang* (*An* $p21\ p22\ p23$)) *add* *Emp*)

\vee *Gr* (*Geos* (*Ang* (*An* $p21\ p22\ p23$)) *add* *Emp*) (*Geos* (*Ang* (*An* $p11\ p12\ p13$)) *add* *Emp*)

proof –

from *assms* **have** $P1 : \neg \text{Cong } (Geos\ (Ang\ (An\ p11\ p12\ p13))\ \text{add}\ Emp)\ (Geos\ (Ang\ (An\ p21\ p22\ p23))\ \text{add}\ Emp)$ \implies

Gr (*Geos* (*Ang* (*An* $p11\ p12\ p13$)) *add* *Emp*) (*Geos* (*Ang* (*An* $p21\ p22\ p23$)) *add* *Emp*)

\vee *Gr* (*Geos* (*Ang* (*An* $p21\ p22\ p23$)) *add* *Emp*) (*Geos* (*Ang* (*An* $p11\ p12\ p13$)) *add* *Emp*) **by** (*simp add:Ang-not-Eq-Gr*)

then have $P2 : \neg \text{Cong } (Geos\ (Ang\ (An\ p11\ p12\ p13))\ \text{add}\ Emp)\ (Geos\ (Ang\ (An\ p21\ p22\ p23))\ \text{add}\ Emp)$

$(An\ p21\ p22\ p23))\ add\ Emp) \implies$
 $\neg\ Gr\ (Geos\ (Ang\ (An\ p11\ p12\ p13))\ add\ Emp)\ (Geos\ (Ang\ (An\ p21\ p22\ p23))\ add\ Emp) \implies$
 $Gr\ (Geos\ (Ang\ (An\ p21\ p22\ p23))\ add\ Emp)\ (Geos\ (Ang\ (An\ p11\ p12\ p13))\ add\ Emp) \textbf{ by blast}$
from $P1$ **have** $P3 : \neg\ Cong\ (Geos\ (Ang\ (An\ p11\ p12\ p13))\ add\ Emp)\ (Geos\ (Ang\ (An\ p21\ p22\ p23))\ add\ Emp) \implies$
 $\neg\ Gr\ (Geos\ (Ang\ (An\ p21\ p22\ p23))\ add\ Emp)\ (Geos\ (Ang\ (An\ p11\ p12\ p13))\ add\ Emp) \implies$
 $Gr\ (Geos\ (Ang\ (An\ p11\ p12\ p13))\ add\ Emp)\ (Geos\ (Ang\ (An\ p21\ p22\ p23))\ add\ Emp) \textbf{ by blast}$
from *assms* **have** $\neg\ Gr\ (Geos\ (Ang\ (An\ p21\ p22\ p23))\ add\ Emp)\ (Geos\ (Ang\ (An\ p11\ p12\ p13))\ add\ Emp) \implies$
 $Cong\ (Geos\ (Ang\ (An\ p11\ p12\ p13))\ add\ Emp)\ (Geos\ (Ang\ (An\ p21\ p22\ p23))\ add\ Emp)$
 $\vee\ Gr\ (Geos\ (Ang\ (An\ p11\ p12\ p13))\ add\ Emp)\ (Geos\ (Ang\ (An\ p21\ p22\ p23))\ add\ Emp) \textbf{ by (simp add:Ang-not-Gr-Eq-rev)}$
then **have** $P4 : \neg\ Gr\ (Geos\ (Ang\ (An\ p21\ p22\ p23))\ add\ Emp)\ (Geos\ (Ang\ (An\ p11\ p12\ p13))\ add\ Emp) \implies$
 $\neg\ Gr\ (Geos\ (Ang\ (An\ p11\ p12\ p13))\ add\ Emp)\ (Geos\ (Ang\ (An\ p21\ p22\ p23))\ add\ Emp) \implies$
 $Cong\ (Geos\ (Ang\ (An\ p11\ p12\ p13))\ add\ Emp)\ (Geos\ (Ang\ (An\ p21\ p22\ p23))\ add\ Emp) \textbf{ by blast}$
from $P2\ P3\ P4$ **show** $Cong\ (Geos\ (Ang\ (An\ p11\ p12\ p13))\ add\ Emp)\ (Geos\ (Ang\ (An\ p21\ p22\ p23))\ add\ Emp)$
 $\vee\ Gr\ (Geos\ (Ang\ (An\ p11\ p12\ p13))\ add\ Emp)\ (Geos\ (Ang\ (An\ p21\ p22\ p23))\ add\ Emp)$
 $\vee\ Gr\ (Geos\ (Ang\ (An\ p21\ p22\ p23))\ add\ Emp)\ (Geos\ (Ang\ (An\ p11\ p12\ p13))\ add\ Emp) \textbf{ by blast}$
qed

lemma (in Congruence-Rule) Ang-not-Gr-lemma1 :

assumes

$Def\ (Ang\ (An\ p11\ p12\ p13))\ Def\ (Ang\ (An\ p21\ p22\ p23))$

$Cong\ (Geos\ (Ang\ (An\ p11\ p12\ p13))\ add\ Emp)\ (Geos\ (Ang\ (An\ p21\ p22\ p23))\ add\ Emp)$

shows

$\neg\ Gr\ (Geos\ (Ang\ (An\ p11\ p12\ p13))\ add\ Emp)\ (Geos\ (Ang\ (An\ p21\ p22\ p23))\ add\ Emp)$

proof –

from *assms* **have** $\neg\ Line-on\ (Li\ p12\ p13)\ p11$ **by** *(simp add:Ang-to-Tri Tri-def-Line)*

then **have** $\exists p. Cong\ (Geos\ (Ang\ (An\ p21\ p22\ p23))\ add\ Emp)\ (Geos\ (Ang\ (An\ p\ p12\ p13))\ add\ Emp)$

$\wedge\ Plane-sameside\ (Li\ p12\ p13)\ p\ p11$ **using** *assms* **by** *(simp add:Ang-move-sameside)*

then **obtain** $p14 :: Point$ **where** $P1 : Cong\ (Geos\ (Ang\ (An\ p21\ p22\ p23))\ add\ Emp)\ (Geos\ (Ang\ (An\ p14\ p12\ p13))\ add\ Emp)$

$\wedge\ Plane-sameside\ (Li\ p12\ p13)\ p14\ p11$ **by** *blast*

from $P1$ **have** $P2 : Plane-sameside\ (Li\ p12\ p13)\ p11\ p14$ **by** *(simp add:Plane-sameside-rev)*

then **have** $P3 : \neg\ Line-on\ (Li\ p12\ p13)\ p14$ **by** *(simp add:Plane-sameside-def)*

from *assms* **have** $P_4 : \neg \text{Eq} (\text{Geos} (\text{Poi } p12) \text{ add Emp}) (\text{Geos} (\text{Poi } p13) \text{ add Emp})$ **by** (*simp add:Ang-def*)
from $P_3 P_4$ **have** $\text{Def} (\text{Ang} (\text{An } p12 p13 p14))$ **by** (*simp add:Ang-simple-def*)
then **have** $P_5 : \text{Def} (\text{Ang} (\text{An } p14 p12 p13))$ **by** (*blast intro:Ang-def-inv Ang-def-rev*)
from *assms* $P_1 P_5$ **have** $P_6 : \text{Cong} (\text{Geos} (\text{Ang} (\text{An } p11 p12 p13)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } p14 p12 p13)) \text{ add Emp})$ **by** (*blast intro:Ang-rev Ang-trans*)
have $P_7 : \text{Cong} (\text{Geos} (\text{Ang} (\text{An } p13 p12 p11)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } p11 p12 p13)) \text{ add Emp})$ **by** (*simp add:Ang-roll*)
from *assms* **have** $P_8 : \text{Def} (\text{Ang} (\text{An } p13 p12 p11))$ **by** (*simp add:Ang-def-rev*)
from *assms* $P_5 P_6 P_7 P_8$ **have** $P_9 : \text{Cong} (\text{Geos} (\text{Ang} (\text{An } p13 p12 p11)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } p14 p12 p13)) \text{ add Emp})$ **by** (*blast intro:Ang-trans Ang-rev*)
from $P_2 P_7 P_9$ **have** $P_{10} : \text{Eq} (\text{Geos} (\text{Lin} (\text{Li } p11 p12)) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li } p14 p12)) \text{ add Emp}) \wedge \neg \text{Bet-Point} (\text{Se } p11 p14) p12$ **by** (*simp add:Ang-move-unique*)
from P_1 **have** $\text{Ang-inside} (\text{An } p11 p12 p13) p14 \longleftrightarrow$
 $\text{Gr} (\text{Geos} (\text{Ang} (\text{An } p11 p12 p13)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } p21 p22 p23)) \text{ add Emp})$ **by** (*simp add:Ang-greater-def*)
then **have** $\text{Gr} (\text{Geos} (\text{Ang} (\text{An } p11 p12 p13)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } p21 p22 p23)) \text{ add Emp}) \implies$
 $\text{Ang-inside} (\text{An } p11 p12 p13) p14$ **by** *blast*
then **have** $\text{Gr} (\text{Geos} (\text{Ang} (\text{An } p11 p12 p13)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } p21 p22 p23)) \text{ add Emp}) \implies$
 $\text{Plane-sameside} (\text{Li } p12 p11) p13 p14$ **by** (*simp add:Ang-inside-def*)
then **have** $P_{11} : \text{Gr} (\text{Geos} (\text{Ang} (\text{An } p11 p12 p13)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } p21 p22 p23)) \text{ add Emp}) \implies$
 $\neg \text{Line-on} (\text{Li } p12 p11) p14$ **by** (*simp add:Plane-sameside-def*)
from *assms* **have** $\neg \text{Eq} (\text{Geos} (\text{Poi } p11) \text{ add Emp}) (\text{Geos} (\text{Poi } p12) \text{ add Emp})$ **by** (*simp add:Ang-def*)
then **have** $P_{12} : \text{Eq} (\text{Geos} (\text{Lin} (\text{Li } p11 p12)) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li } p12 p11)) \text{ add Emp})$ **by** (*simp add:Line-rev*)
from $P_{10} P_{12}$ **have** $P_{13} : \text{Eq} (\text{Geos} (\text{Lin} (\text{Li } p14 p12)) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li } p12 p11)) \text{ add Emp})$ **by** (*blast intro:Eq-rev Eq-trans*)
have $P_{14} : \text{Line-on} (\text{Li } p14 p12) p14$ **by** (*simp add:Line-on-rule*)
from $P_{13} P_{14}$ **have** $P_{15} : \text{Line-on} (\text{Li } p12 p11) p14$ **by** (*simp add:Line-on-trans*)
from $P_{11} P_{15}$ **show** $\neg \text{Gr} (\text{Geos} (\text{Ang} (\text{An } p11 p12 p13)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } p21 p22 p23)) \text{ add Emp})$ **by** *blast*
qed

lemma (in Congruence-Rule) Ang-not-Gr :

assumes
 $\text{Def} (\text{Ang} (\text{An } p11 p12 p13)) \text{Def} (\text{Ang} (\text{An } p21 p22 p23))$
 $\text{Cong} (\text{Geos} (\text{Ang} (\text{An } p11 p12 p13)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } p21 p22 p23)) \text{ add Emp})$
shows
 $\neg \text{Gr} (\text{Geos} (\text{Ang} (\text{An } p11 p12 p13)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } p21 p22 p23)) \text{ add Emp})$
 $\neg \text{Gr} (\text{Geos} (\text{Ang} (\text{An } p21 p22 p23)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } p11 p12 p13)) \text{ add Emp})$
proof –

from *assms* **show** $P1 : \neg Gr (Geos (Ang (An p11 p12 p13)) add Emp) (Geos (Ang (An p21 p22 p23)) add Emp)$ **by** (*simp add:Ang-not-Gr-lemma1*)
from *assms* **have** $P2 : Cong (Geos (Ang (An p21 p22 p23)) add Emp) (Geos (Ang (An p11 p12 p13)) add Emp)$ **by** (*simp add:Ang-rev*)
from *assms* $P2$ **show** $\neg Gr (Geos (Ang (An p21 p22 p23)) add Emp) (Geos (Ang (An p11 p12 p13)) add Emp)$ **by** (*simp add:Ang-not-Gr-lemma1*)
qed

lemma (in *Congruence-Rule*) *Ang-Gr-not-Eq-rev* :

assumes
 $Def (Ang (An p11 p12 p13)) Def (Ang (An p21 p22 p23))$
 $Gr (Geos (Ang (An p21 p22 p23)) add Emp) (Geos (Ang (An p11 p12 p13)) add Emp)$
shows
 $\neg Cong (Geos (Ang (An p11 p12 p13)) add Emp) (Geos (Ang (An p21 p22 p23)) add Emp)$
 $\neg Gr (Geos (Ang (An p11 p12 p13)) add Emp) (Geos (Ang (An p21 p22 p23)) add Emp)$
proof –
from *assms* **have** $\neg Line-on (Li p12 p13) p11$ **by** (*simp add:Ang-to-Tri Tri-def-Line*)
then **have** $\exists p. Cong (Geos (Ang (An p21 p22 p23)) add Emp) (Geos (Ang (An p p12 p13)) add Emp)$
 $\wedge Plane-sameside (Li p12 p13) p p11$ **using** *assms* **by** (*simp add:Ang-move-sameside*)
then **obtain** $p14 :: Point$ **where** $P1 : Cong (Geos (Ang (An p21 p22 p23)) add Emp) (Geos (Ang (An p14 p12 p13)) add Emp)$
 $\wedge Plane-sameside (Li p12 p13) p14 p11$ **by** *blast*
from $P1$ **have** $P2 : Plane-sameside (Li p12 p13) p11 p14$ **by** (*simp add:Plane-sameside-rev*)
then **have** $P3 : \neg Line-on (Li p12 p13) p14$ **by** (*simp add:Plane-sameside-def*)
from *assms* **have** $P4 : \neg Eq (Geos (Poi p12) add Emp) (Geos (Poi p13) add Emp)$ **by** (*simp add:Ang-def*)
from $P3 P4$ **have** $Def (Ang (An p12 p13 p14))$ **by** (*simp add:Ang-simple-def*)
then **have** $P5 : Def (Ang (An p14 p12 p13))$ **by** (*blast intro:Ang-def-inv Ang-def-rev*)
from *assms* **have** $\neg Line-on (Li p22 p23) p21$ **by** (*simp add:Ang-to-Tri Tri-def-Line*)
then **have** $\exists p. Cong (Geos (Ang (An p11 p12 p13)) add Emp) (Geos (Ang (An p p22 p23)) add Emp)$
 $\wedge Plane-sameside (Li p22 p23) p p21$ **using** *assms* **by** (*simp add:Ang-move-sameside*)
then **obtain** $p24 :: Point$ **where** $P6 : Cong (Geos (Ang (An p11 p12 p13)) add Emp) (Geos (Ang (An p24 p22 p23)) add Emp)$
 $\wedge Plane-sameside (Li p22 p23) p24 p21$ **by** *blast*
then **have** $P7 : \neg Line-on (Li p22 p23) p24$ **by** (*simp add:Plane-sameside-def*)
from *assms* **have** $P8 : \neg Eq (Geos (Poi p22) add Emp) (Geos (Poi p23) add Emp)$ **by** (*simp add:Ang-def*)
from $P7 P8$ **have** $Def (Ang (An p22 p23 p24))$ **by** (*simp add:Ang-simple-def*)
then **have** $P9 : Def (Ang (An p24 p22 p23))$ **by** (*blast intro:Ang-def-inv Ang-def-rev*)
from $P6$ **have** $P10 : Plane-sameside (Li p22 p23) p21 p24$ **by** (*simp add:Plane-sameside-rev*)
from *assms* **have** $P11 : Def (Ang (An p13 p12 p11))$ **by** (*simp add:Ang-def-rev*)
from *assms* **have** $P12 : Def (Ang (An p23 p22 p21))$ **by** (*simp add:Ang-def-rev*)

from $P5$ **have** $P13 : \text{Def} (\text{Ang} (\text{An } p13 \ p12 \ p14))$ **by** (*simp add:Ang-def-rev*)
from $P9$ **have** $P14 : \text{Def} (\text{Ang} (\text{An } p23 \ p22 \ p24))$ **by** (*simp add:Ang-def-rev*)
have $P15 : \text{Cong} (\text{Geos} (\text{Ang} (\text{An } p11 \ p12 \ p13)) \text{add Emp}) (\text{Geos} (\text{Ang} (\text{An } p13 \ p12 \ p11)) \text{add Emp})$ **by** (*simp add:Ang-roll*)
have $P16 : \text{Cong} (\text{Geos} (\text{Ang} (\text{An } p21 \ p22 \ p23)) \text{add Emp}) (\text{Geos} (\text{Ang} (\text{An } p23 \ p22 \ p21)) \text{add Emp})$ **by** (*simp add:Ang-roll*)
have $P17 : \text{Cong} (\text{Geos} (\text{Ang} (\text{An } p14 \ p12 \ p13)) \text{add Emp}) (\text{Geos} (\text{Ang} (\text{An } p13 \ p12 \ p14)) \text{add Emp})$ **by** (*simp add:Ang-roll*)
have $P18 : \text{Cong} (\text{Geos} (\text{Ang} (\text{An } p24 \ p22 \ p23)) \text{add Emp}) (\text{Geos} (\text{Ang} (\text{An } p23 \ p22 \ p24)) \text{add Emp})$ **by** (*simp add:Ang-roll*)
from *assms* $P6 \ P9 \ P11 \ P15$ **have** $P19 :$
 $\text{Cong} (\text{Geos} (\text{Ang} (\text{An } p13 \ p12 \ p11)) \text{add Emp}) (\text{Geos} (\text{Ang} (\text{An } p24 \ p22 \ p23)) \text{add Emp})$ **by** (*blast intro:Ang-rev Ang-trans*)
from $P9 \ P11 \ P14 \ P18 \ P19$ **have** $P20 :$
 $\text{Cong} (\text{Geos} (\text{Ang} (\text{An } p13 \ p12 \ p11)) \text{add Emp}) (\text{Geos} (\text{Ang} (\text{An } p23 \ p22 \ p24)) \text{add Emp})$ **by** (*blast intro:Ang-rev Ang-trans*)
from *assms* $P1 \ P5 \ P12 \ P16$ **have** $P21 :$
 $\text{Cong} (\text{Geos} (\text{Ang} (\text{An } p23 \ p22 \ p21)) \text{add Emp}) (\text{Geos} (\text{Ang} (\text{An } p14 \ p12 \ p13)) \text{add Emp})$ **by** (*blast intro:Ang-rev Ang-trans*)
from $P5 \ P12 \ P13 \ P17 \ P21$ **have** $P22 :$
 $\text{Cong} (\text{Geos} (\text{Ang} (\text{An } p23 \ p22 \ p21)) \text{add Emp}) (\text{Geos} (\text{Ang} (\text{An } p13 \ p12 \ p14)) \text{add Emp})$ **by** (*blast intro:Ang-rev Ang-trans*)
from $P6$ **have** $P23 : \text{Ang-inside} (\text{An } p21 \ p22 \ p23) \ p24 \longleftrightarrow$
 $\text{Gr} (\text{Geos} (\text{Ang} (\text{An } p21 \ p22 \ p23)) \text{add Emp}) (\text{Geos} (\text{Ang} (\text{An } p11 \ p12 \ p13)) \text{add Emp})$ **by** (*simp add:Ang-greater-def*)
from *assms* $P23$ **have** $P24 : \text{Ang-inside} (\text{An } p21 \ p22 \ p23) \ p24$ **by** *blast*
from $P12 \ P24$ **have** $P25 : \text{Ang-inside} (\text{An } p23 \ p22 \ p21) \ p24$ **by** (*simp add:Ang-inside-def*)
from $P2 \ P6 \ P11 \ P12 \ P20 \ P22 \ P25$ **have** $P26 : \neg \text{Ang-inside} (\text{An } p13 \ p12 \ p11) \ p14$
 $\wedge \neg \text{Eq} (\text{Geos} (\text{Lin} (\text{Li } p12 \ p11)) \text{add Emp}) (\text{Geos} (\text{Lin} (\text{Li } p12 \ p14)) \text{add Emp})$ **by** (*simp add:Ang-move-Greater*)
from $P1$ **have** $\text{Ang-inside} (\text{An } p11 \ p12 \ p13) \ p14 \longleftrightarrow$
 $\text{Gr} (\text{Geos} (\text{Ang} (\text{An } p11 \ p12 \ p13)) \text{add Emp}) (\text{Geos} (\text{Ang} (\text{An } p21 \ p22 \ p23)) \text{add Emp})$ **by** (*simp add:Ang-greater-def*)
then **have** $P27 : \text{Gr} (\text{Geos} (\text{Ang} (\text{An } p11 \ p12 \ p13)) \text{add Emp}) (\text{Geos} (\text{Ang} (\text{An } p21 \ p22 \ p23)) \text{add Emp}) \implies$
 $\text{Ang-inside} (\text{An } p11 \ p12 \ p13) \ p14$ **by** *blast*
from $P11 \ P27$ **have** $P28 : \text{Gr} (\text{Geos} (\text{Ang} (\text{An } p11 \ p12 \ p13)) \text{add Emp}) (\text{Geos} (\text{Ang} (\text{An } p21 \ p22 \ p23)) \text{add Emp}) \implies$
 $\text{Ang-inside} (\text{An } p13 \ p12 \ p11) \ p14$ **by** (*simp add:Ang-inside-def*)
from $P26 \ P28$ **show** $\neg \text{Gr} (\text{Geos} (\text{Ang} (\text{An } p11 \ p12 \ p13)) \text{add Emp}) (\text{Geos} (\text{Ang} (\text{An } p21 \ p22 \ p23)) \text{add Emp})$ **by** *blast*
from *assms* $P1 \ P5$ **have** $P29 : \text{Cong} (\text{Geos} (\text{Ang} (\text{An } p11 \ p12 \ p13)) \text{add Emp}) (\text{Geos} (\text{Ang} (\text{An } p21 \ p22 \ p23)) \text{add Emp}) \implies$
 $\text{Cong} (\text{Geos} (\text{Ang} (\text{An } p11 \ p12 \ p13)) \text{add Emp}) (\text{Geos} (\text{Ang} (\text{An } p14 \ p12 \ p13)) \text{add Emp})$ **by** (*blast intro:Ang-rev Ang-trans*)
have $P30 : \text{Cong} (\text{Geos} (\text{Ang} (\text{An } p13 \ p12 \ p11)) \text{add Emp}) (\text{Geos} (\text{Ang} (\text{An } p11 \ p12 \ p13)) \text{add Emp})$ **by** (*simp add:Ang-roll*)
from *assms* $P5 \ P11 \ P29 \ P30$ **have** $P31 : \text{Cong} (\text{Geos} (\text{Ang} (\text{An } p11 \ p12 \ p13))$

$add\ Emp) (Geos (Ang (An\ p21\ p22\ p23))\ add\ Emp) \implies$
 $Cong (Geos (Ang (An\ p13\ p12\ p11))\ add\ Emp) (Geos (Ang (An\ p14\ p12\ p13))$
 $add\ Emp) \text{ by } (blast\ intro:Ang\ rev\ Ang\ trans)$
from $P2\ P30\ P31$ **have** $P32 : Cong (Geos (Ang (An\ p11\ p12\ p13))\ add\ Emp)$
 $(Geos (Ang (An\ p21\ p22\ p23))\ add\ Emp) \implies$
 $Eq (Geos (Lin (Li\ p11\ p12))\ add\ Emp) (Geos (Lin (Li\ p14\ p12))\ add\ Emp) \wedge$
 $\neg\ Bet\ Point (Se\ p11\ p14)\ p12 \text{ by } (simp\ add:Ang\ move\ unique)$
from *assms* **have** $\neg\ Eq (Geos (Poi\ p11)\ add\ Emp) (Geos (Poi\ p12)\ add\ Emp)$
by $(simp\ add:Ang\ def)$
then **have** $P33 : Eq (Geos (Lin (Li\ p11\ p12))\ add\ Emp) (Geos (Lin (Li\ p12$
 $p11))\ add\ Emp) \text{ by } (simp\ add:Line\ rev)$
from $P32\ P33$ **have** $P34 : Cong (Geos (Ang (An\ p11\ p12\ p13))\ add\ Emp) (Geos$
 $(Ang (An\ p21\ p22\ p23))\ add\ Emp) \implies$
 $Eq (Geos (Lin (Li\ p12\ p11))\ add\ Emp) (Geos (Lin (Li\ p14\ p12))\ add\ Emp) \text{ by}$
 $(blast\ intro:Eq\ trans\ Eq\ rev)$
from $P5$ **have** $\neg\ Eq (Geos (Poi\ p14)\ add\ Emp) (Geos (Poi\ p12)\ add\ Emp) \text{ by}$
 $(simp\ add:Ang\ def)$
then **have** $P35 : Eq (Geos (Lin (Li\ p14\ p12))\ add\ Emp) (Geos (Lin (Li\ p12$
 $p14))\ add\ Emp) \text{ by } (simp\ add:Line\ rev)$
from $P34\ P35$ **have** $P36 : Cong (Geos (Ang (An\ p11\ p12\ p13))\ add\ Emp) (Geos$
 $(Ang (An\ p21\ p22\ p23))\ add\ Emp) \implies$
 $Eq (Geos (Lin (Li\ p12\ p11))\ add\ Emp) (Geos (Lin (Li\ p12\ p14))\ add\ Emp) \text{ by}$
 $(blast\ intro:Eq\ trans\ Eq\ rev)$
from $P26\ P36$ **show** $\neg\ Cong (Geos (Ang (An\ p11\ p12\ p13))\ add\ Emp) (Geos$
 $(Ang (An\ p21\ p22\ p23))\ add\ Emp) \text{ by } blast$
qed

lemma (in Congruence-Rule) Ang-relation-case-fact :

assumes

$Def (Ang (An\ p11\ p12\ p13))\ Def (Ang (An\ p21\ p22\ p23))$

shows

$Cong (Geos (Ang (An\ p11\ p12\ p13))\ add\ Emp) (Geos (Ang (An\ p21\ p22\ p23))$
 $add\ Emp)$
 $\wedge \neg\ Gr (Geos (Ang (An\ p11\ p12\ p13))\ add\ Emp) (Geos (Ang (An\ p21\ p22$
 $p23))\ add\ Emp)$
 $\wedge \neg\ Gr (Geos (Ang (An\ p21\ p22\ p23))\ add\ Emp) (Geos (Ang (An\ p11\ p12$
 $p13))\ add\ Emp)$
 $\vee \neg\ Cong (Geos (Ang (An\ p11\ p12\ p13))\ add\ Emp) (Geos (Ang (An\ p21\ p22$
 $p23))\ add\ Emp)$
 $\wedge Gr (Geos (Ang (An\ p11\ p12\ p13))\ add\ Emp) (Geos (Ang (An\ p21\ p22\ p23))$
 $add\ Emp)$
 $\wedge \neg\ Gr (Geos (Ang (An\ p21\ p22\ p23))\ add\ Emp) (Geos (Ang (An\ p11\ p12$
 $p13))\ add\ Emp)$
 $\vee \neg\ Cong (Geos (Ang (An\ p11\ p12\ p13))\ add\ Emp) (Geos (Ang (An\ p21\ p22$
 $p23))\ add\ Emp)$
 $\wedge \neg\ Gr (Geos (Ang (An\ p11\ p12\ p13))\ add\ Emp) (Geos (Ang (An\ p21\ p22$
 $p23))\ add\ Emp)$
 $\wedge Gr (Geos (Ang (An\ p21\ p22\ p23))\ add\ Emp) (Geos (Ang (An\ p11\ p12\ p13))$
 $add\ Emp)$

proof –

from *assms* **have** $P1 : Cong (Geos (Ang (An p11 p12 p13)) add Emp) (Geos (Ang (An p21 p22 p23)) add Emp)$
 $\vee Gr (Geos (Ang (An p11 p12 p13)) add Emp) (Geos (Ang (An p21 p22 p23)) add Emp)$
 $\vee Gr (Geos (Ang (An p21 p22 p23)) add Emp) (Geos (Ang (An p11 p12 p13)) add Emp)$ **by** (*simp add:Ang-relation-case*)
from *assms* **have** $P2 : Cong (Geos (Ang (An p11 p12 p13)) add Emp) (Geos (Ang (An p21 p22 p23)) add Emp) ==>$
 $\neg Gr (Geos (Ang (An p11 p12 p13)) add Emp) (Geos (Ang (An p21 p22 p23)) add Emp)$
 $\wedge \neg Gr (Geos (Ang (An p21 p22 p23)) add Emp) (Geos (Ang (An p11 p12 p13)) add Emp)$ **by** (*simp add:Ang-not-Gr*)
from *assms* **have** $P3 : Gr (Geos (Ang (An p21 p22 p23)) add Emp) (Geos (Ang (An p11 p12 p13)) add Emp) ==>$
 $\neg Cong (Geos (Ang (An p11 p12 p13)) add Emp) (Geos (Ang (An p21 p22 p23)) add Emp)$
 $\wedge \neg Gr (Geos (Ang (An p11 p12 p13)) add Emp) (Geos (Ang (An p21 p22 p23)) add Emp)$ **by** (*simp add:Ang-Gr-not-Eq-rev*)
from *assms* **have** $Gr (Geos (Ang (An p11 p12 p13)) add Emp) (Geos (Ang (An p21 p22 p23)) add Emp) ==>$
 $\neg Cong (Geos (Ang (An p21 p22 p23)) add Emp) (Geos (Ang (An p11 p12 p13)) add Emp)$
 $\wedge \neg Gr (Geos (Ang (An p21 p22 p23)) add Emp) (Geos (Ang (An p11 p12 p13)) add Emp)$ **by** (*simp add:Ang-Gr-not-Eq-rev*)
then **have** $P4 : Gr (Geos (Ang (An p11 p12 p13)) add Emp) (Geos (Ang (An p21 p22 p23)) add Emp) ==>$
 $\neg Cong (Geos (Ang (An p11 p12 p13)) add Emp) (Geos (Ang (An p21 p22 p23)) add Emp)$
 $\wedge \neg Gr (Geos (Ang (An p21 p22 p23)) add Emp) (Geos (Ang (An p11 p12 p13)) add Emp)$ **by** (*blast intro:Ang-rev*)
from $P1 P2 P3 P4$ **show** $Cong (Geos (Ang (An p11 p12 p13)) add Emp) (Geos (Ang (An p21 p22 p23)) add Emp)$
 $\wedge \neg Gr (Geos (Ang (An p11 p12 p13)) add Emp) (Geos (Ang (An p21 p22 p23)) add Emp)$
 $\wedge \neg Gr (Geos (Ang (An p21 p22 p23)) add Emp) (Geos (Ang (An p11 p12 p13)) add Emp)$
 $\vee \neg Cong (Geos (Ang (An p11 p12 p13)) add Emp) (Geos (Ang (An p21 p22 p23)) add Emp)$
 $\wedge Gr (Geos (Ang (An p11 p12 p13)) add Emp) (Geos (Ang (An p21 p22 p23)) add Emp)$
 $\wedge \neg Gr (Geos (Ang (An p21 p22 p23)) add Emp) (Geos (Ang (An p11 p12 p13)) add Emp)$
 $\vee \neg Cong (Geos (Ang (An p11 p12 p13)) add Emp) (Geos (Ang (An p21 p22 p23)) add Emp)$
 $\wedge \neg Gr (Geos (Ang (An p11 p12 p13)) add Emp) (Geos (Ang (An p21 p22 p23)) add Emp)$
 $\wedge Gr (Geos (Ang (An p21 p22 p23)) add Emp) (Geos (Ang (An p11 p12 p13)) add Emp)$ **by** *blast*

qed

lemma (in *Congruence-Rule*) *Ang-Gr-trans-Eq-Gr* :

assumes

Def (*Ang* (*An* *p11* *p12* *p13*)) *Def* (*Ang* (*An* *p21* *p22* *p23*)) *Def* (*Ang* (*An* *p31* *p32* *p33*))

Cong (*Geos* (*Ang* (*An* *p11* *p12* *p13*)) *add Emp*) (*Geos* (*Ang* (*An* *p21* *p22* *p23*)) *add Emp*)

Gr (*Geos* (*Ang* (*An* *p21* *p22* *p23*)) *add Emp*) (*Geos* (*Ang* (*An* *p31* *p32* *p33*)) *add Emp*)

shows

Gr (*Geos* (*Ang* (*An* *p11* *p12* *p13*)) *add Emp*) (*Geos* (*Ang* (*An* *p31* *p32* *p33*)) *add Emp*)

proof –

from *assms* **have** *P1* : \neg *Line-on* (*Li* *p22* *p23*) *p21* **by** (*simp add:Ang-to-Tri Tri-def-Line*)

from *assms* *P1* **have** $\exists p$. *Cong* (*Geos* (*Ang* (*An* *p31* *p32* *p33*)) *add Emp*) (*Geos* (*Ang* (*An* *p* *p22* *p23*)) *add Emp*)

\wedge *Plane-sameside* (*Li* *p22* *p23*) *p* *p21* **by** (*simp add:Ang-move-sameside*)

then obtain *p24* :: *Point* **where** *P2* : *Cong* (*Geos* (*Ang* (*An* *p31* *p32* *p33*)) *add Emp*) (*Geos* (*Ang* (*An* *p24* *p22* *p23*)) *add Emp*)

\wedge *Plane-sameside* (*Li* *p22* *p23*) *p24* *p21* **by** *blast*

then have *P3* : *Cong* (*Geos* (*Ang* (*An* *p31* *p32* *p33*)) *add Emp*) (*Geos* (*Ang* (*An* *p24* *p22* *p23*)) *add Emp*) **by** *simp*

from *P2* **have** *P4* : *Plane-sameside* (*Li* *p22* *p23*) *p24* *p21* **by** *simp*

from *assms* *P3* *P4* **have** *Ang-inside* (*An* *p21* *p22* *p23*) *p24* **by** (*simp add:Ang-greater-def*)

then have *P5* : *Ang-inside* (*An* *p23* *p22* *p21*) *p24* **by** (*simp add:Ang-inside-def Ang-def-rev*)

from *assms* **have** *P6* : \neg *Line-on* (*Li* *p32* *p33*) *p31* **by** (*simp add:Ang-to-Tri Tri-def-Line*)

from *assms* *P6* **have** $\exists p$. *Cong* (*Geos* (*Ang* (*An* *p21* *p22* *p23*)) *add Emp*) (*Geos* (*Ang* (*An* *p* *p32* *p33*)) *add Emp*)

\wedge *Plane-sameside* (*Li* *p32* *p33*) *p* *p31* **by** (*simp add:Ang-move-sameside*)

then obtain *p34* :: *Point* **where** *P7* : *Cong* (*Geos* (*Ang* (*An* *p21* *p22* *p23*)) *add Emp*) (*Geos* (*Ang* (*An* *p34* *p32* *p33*)) *add Emp*)

\wedge *Plane-sameside* (*Li* *p32* *p33*) *p34* *p31* **by** *blast*

then have *P8* : *Cong* (*Geos* (*Ang* (*An* *p21* *p22* *p23*)) *add Emp*) (*Geos* (*Ang* (*An* *p34* *p32* *p33*)) *add Emp*) **by** *simp*

from *P7* **have** *P9* : *Plane-sameside* (*Li* *p32* *p33*) *p34* *p31* **by** *simp*

from *assms* **have** *P10* : *Def* (*Ang* (*An* *p33* *p32* *p31*)) **by** (*blast intro:Ang-def-rev*)

from *assms* **have** *P11* : *Def* (*Ang* (*An* *p23* *p22* *p21*)) **by** (*blast intro:Ang-def-rev*)

from *P4* **have** *P12* : \neg *Line-on* (*Li* *p22* *p23*) *p24* **by** (*simp add:Plane-sameside-def*)

from *assms* **have** *P13* : \neg *Eq* (*Geos* (*Poi* *p22*) *add Emp*) (*Geos* (*Poi* *p23*) *add Emp*) **by** (*simp add:Ang-def*)

from *P12* *P13* **have** *Def* (*Ang* (*An* *p22* *p23* *p24*)) **by** (*simp add:Ang-simple-def*)

then have *P14* : *Def* (*Ang* (*An* *p24* *p22* *p23*)) **by** (*blast intro:Ang-def-rev Ang-def-inv*)

then have *P15* : *Def* (*Ang* (*An* *p23* *p22* *p24*)) **by** (*blast intro:Ang-def-rev*)

have *P16* : *Cong* (*Geos* (*Ang* (*An* *p24* *p22* *p23*)) *add Emp*) (*Geos* (*Ang* (*An* *p23*

$p22\ p24$) *add Emp*) **by** (*simp add:Ang-roll*)
from *assms P3 P14 P15 P16* **have** $P17$:
Cong (Geos (Ang (An p31 p32 p33)) add Emp) (Geos (Ang (An p23 p22 p24))
add Emp) **by** (*blast intro:Ang-trans Ang-rev*)
have $P18$: *Cong (Geos (Ang (An p31 p32 p33)) add Emp) (Geos (Ang (An p33*
p32 p31)) add Emp) **by** (*simp add:Ang-roll*)
from *assms P10 P15 P17 P18* **have** $P19$:
Cong (Geos (Ang (An p33 p32 p31)) add Emp) (Geos (Ang (An p23 p22 p24))
add Emp) **by** (*blast intro:Ang-trans Ang-rev*)
from $P9$ **have** $P20$: \neg *Line-on (Li p32 p33) p34* **by** (*simp add:Plane-sameside-def*)
from *assms* **have** $P21$: \neg *Eq (Geos (Poi p32) add Emp) (Geos (Poi p33) add*
Emp) **by** (*simp add:Ang-def*)
from $P20\ P21$ **have** *Def (Ang (An p32 p33 p34))* **by** (*simp add:Ang-simple-def*)
then **have** $P22$: *Def (Ang (An p34 p32 p33))* **by** (*blast intro:Ang-def-rev*
Ang-def-inv)
then **have** $P23$: *Def (Ang (An p33 p32 p34))* **by** (*blast intro:Ang-def-rev*)
have $P24$: *Cong (Geos (Ang (An p34 p32 p33)) add Emp) (Geos (Ang (An p33*
p32 p34)) add Emp) **by** (*simp add:Ang-roll*)
from *assms P8 P22 P23 P24* **have** $P25$:
Cong (Geos (Ang (An p21 p22 p23)) add Emp) (Geos (Ang (An p33 p32 p34))
add Emp) **by** (*blast intro:Ang-trans Ang-rev*)
have $P26$: *Cong (Geos (Ang (An p21 p22 p23)) add Emp) (Geos (Ang (An p23*
p22 p21)) add Emp) **by** (*simp add:Ang-roll*)
from *assms P11 P23 P25 P26* **have** $P27$:
Cong (Geos (Ang (An p23 p22 p21)) add Emp) (Geos (Ang (An p33 p32 p34))
add Emp) **by** (*blast intro:Ang-trans Ang-rev*)
from $P9$ **have** $P28$: *Plane-sameside (Li p32 p33) p31 p34* **by** (*simp add:Plane-sameside-rev*)
from $P4\ P5\ P10\ P11\ P19\ P27\ P28$ **have** $P29$: \neg *Ang-inside (An p33 p32 p31)*
p34
 \wedge \neg *Eq (Geos (Lin (Li p32 p31)) add Emp) (Geos (Lin (Li p32 p34)) add Emp)*
by (*simp add:Ang-move-Greater*)
have $P30$: *Ang-inside (An p31 p32 p33) p34 \implies Ang-inside (An p33 p32 p31)*
p34 **by** (*simp add:Ang-inside-def Ang-def-rev*)
from $P29\ P30$ **have** $P31$: \neg *Ang-inside (An p31 p32 p33) p34* **by** *blast*
from *assms P8 P22* **have** $P32$: *Cong (Geos (Ang (An p11 p12 p13)) add Emp)*
(Geos (Ang (An p34 p32 p33)) add Emp) **by** (*blast intro:Ang-trans Ang-rev*)
from $P9\ P29\ P31\ P32$ **show** *Gr (Geos (Ang (An p11 p12 p13)) add Emp) (Geos*
(Ang (An p31 p32 p33)) add Emp) **by** (*simp add:Ang-less-def*)
qed

lemma (in Congruence-Rule) Ang-Gr-trans-Gr-Eq :

assumes
Def (Ang (An p11 p12 p13)) Def (Ang (An p21 p22 p23)) Def (Ang (An p31
p32 p33))
Gr (Geos (Ang (An p11 p12 p13)) add Emp) (Geos (Ang (An p21 p22 p23))
add Emp)
Cong (Geos (Ang (An p21 p22 p23)) add Emp) (Geos (Ang (An p31 p32 p33))
add Emp)
shows

$Gr (Geos (Ang (An p11 p12 p13)) add Emp) (Geos (Ang (An p31 p32 p33)) add Emp)$
proof –
from *assms* **have** $P1 : \neg Line-on (Li p12 p13) p11$ **by** (*simp add:Ang-to-Tri Tri-def-Line*)
from *assms* $P1$ **have** $\exists p. Cong (Geos (Ang (An p21 p22 p23)) add Emp) (Geos (Ang (An p p12 p13)) add Emp)$
 $\wedge Plane-sameside (Li p12 p13) p p11$ **by** (*simp add:Ang-move-sameside*)
then obtain $p14 :: Point$ **where** $P2 : Cong (Geos (Ang (An p21 p22 p23)) add Emp) (Geos (Ang (An p14 p12 p13)) add Emp)$
 $\wedge Plane-sameside (Li p12 p13) p14 p11$ **by** *blast*
then have $P3 : Cong (Geos (Ang (An p21 p22 p23)) add Emp) (Geos (Ang (An p14 p12 p13)) add Emp)$ **by** *simp*
from $P2$ **have** $P4 : Plane-sameside (Li p12 p13) p14 p11$ **by** *simp*
from $P3 P4$ **have** $P5 : Ang-inside (An p11 p12 p13) p14 \longleftrightarrow$
 $Gr (Geos (Ang (An p11 p12 p13)) add Emp) (Geos (Ang (An p21 p22 p23)) add Emp)$ **by** (*simp add:Ang-greater-def*)
from *assms* $P5$ **have** $P6 : Ang-inside (An p11 p12 p13) p14$ **by** *simp*
from $P4$ **have** $P7 : \neg Line-on (Li p12 p13) p14$ **by** (*simp add:Plane-sameside-def*)
from *assms* **have** $P8 : \neg Eq (Geos (Poi p12) add Emp) (Geos (Poi p13) add Emp)$ **by** (*simp add:Ang-def*)
from $P7 P8$ **have** $Def (Ang (An p12 p13 p14))$ **by** (*simp add:Ang-simple-def*)
then have $P9 : Def (Ang (An p14 p12 p13))$ **by** (*blast intro:Ang-def-rev Ang-def-inv*)
from *assms* $P3 P9$ **have** $P10 : Cong (Geos (Ang (An p31 p32 p33)) add Emp) (Geos (Ang (An p14 p12 p13)) add Emp)$ **by** (*blast intro:Ang-trans Ang-rev*)
from $P4 P10$ **have** $P11 : Ang-inside (An p11 p12 p13) p14 \longleftrightarrow$
 $Gr (Geos (Ang (An p11 p12 p13)) add Emp) (Geos (Ang (An p31 p32 p33)) add Emp)$ **by** (*simp add:Ang-greater-def*)
from $P6 P11$ **show** $Gr (Geos (Ang (An p11 p12 p13)) add Emp) (Geos (Ang (An p31 p32 p33)) add Emp)$ **by** *simp*
qed

lemma (in Congruence-Rule) Ang-Eq-Point :

assumes

$Def (Ang (An p1 p2 p3))$

$Eq (Geos (Poi p1) add Emp) (Geos (Poi p4) add Emp)$

shows

$Eq (Geos (Ang (An p1 p2 p3)) add Emp) (Geos (Ang (An p4 p2 p3)) add Emp)$

$\wedge Def (Ang (An p4 p2 p3))$

proof –

have $Line-on (Li p2 p1) p1$ **by** (*simp add:Line-on-rule*)

then have $P1 : Line-on (Li p2 p1) p4$ **using** *assms* **by** (*simp add:Point-Eq*)

from *assms* **have** $P2 : Bet-Point (Se p1 p4) p2 \implies Bet-Point (Se p4 p4) p2$ **by** (*simp add:Bet-Point-Eq*)

have $P3 : \neg Bet-Point (Se p4 p4) p2$ **by** (*simp add:Bet-end-Point*)

from $P2 P3$ **have** $P4 : \neg Bet-Point (Se p1 p4) p2$ **by** *blast*

have $P5 : Line-on (Li p2 p3) p3$ **by** (*simp add:Line-on-rule*)

have $P6 : \neg Bet-Point (Se p3 p3) p2$ **by** (*simp add:Bet-end-Point*)

from *assms* **have** $P7 : \neg \text{Eq} (\text{Geos} (\text{Poi } p1) \text{ add Emp}) (\text{Geos} (\text{Poi } p2) \text{ add Emp})$
by (*simp add:Ang-def*)
from *assms* **have** $P8 : \text{Eq} (\text{Geos} (\text{Poi } p2) \text{ add Emp}) (\text{Geos} (\text{Poi } p4) \text{ add Emp})$
 \implies
 $\text{Eq} (\text{Geos} (\text{Poi } p1) \text{ add Emp}) (\text{Geos} (\text{Poi } p2) \text{ add Emp})$ **by** (*blast intro:Eq-trans Eq-rev*)
from $P7 P8$ **have** $P9 : \neg \text{Eq} (\text{Geos} (\text{Poi } p2) \text{ add Emp}) (\text{Geos} (\text{Poi } p4) \text{ add Emp})$
by *blast*
from *assms* **have** $P10 : \neg \text{Eq} (\text{Geos} (\text{Poi } p2) \text{ add Emp}) (\text{Geos} (\text{Poi } p3) \text{ add Emp})$ **by** (*simp add:Ang-def*)
from *assms* $P1 P4 P5 P6 P9 P10$ **show**
 $\text{Eq} (\text{Geos} (\text{Ang} (\text{An } p1 p2 p3)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } p4 p2 p3)) \text{ add Emp})$
 $\wedge \text{Def} (\text{Ang} (\text{An } p4 p2 p3))$ **by** (*simp add:Ang-Point-swap*)
qed

lemma (in *Congruence-Rule*) *Planeside-wrong-relation* :

assumes

$\text{Plane-diffside} (\text{Li } p1 p2) p3 p4$
 $\text{Plane-diffside} (\text{Li } p1 p3) p2 p4$
 $\text{Plane-sameside} (\text{Li } p1 p5) p3 p2$
 $\text{Plane-sameside} (\text{Li } p1 p5) p4 p2$

shows *False*

proof –

from *assms* **have** $\exists p. \text{Bet-Point} (\text{Se } p3 p4) p \wedge \text{Line-on} (\text{Li } p1 p2) p$
 $\wedge \neg \text{Line-on} (\text{Li } p1 p2) p3 \wedge \neg \text{Line-on} (\text{Li } p1 p2) p4$ **by** (*simp add:Plane-diffside-def*)
then obtain $p6 :: \text{Point}$ **where** $P1 : \text{Bet-Point} (\text{Se } p3 p4) p6 \wedge \text{Line-on} (\text{Li } p1$
 $p2) p6$
 $\wedge \neg \text{Line-on} (\text{Li } p1 p2) p3 \wedge \neg \text{Line-on} (\text{Li } p1 p2) p4$ **by** *blast*
from *assms* **have** $\exists p. \text{Bet-Point} (\text{Se } p2 p4) p \wedge \text{Line-on} (\text{Li } p1 p3) p$
 $\wedge \neg \text{Line-on} (\text{Li } p1 p3) p2 \wedge \neg \text{Line-on} (\text{Li } p1 p3) p4$ **by** (*simp add:Plane-diffside-def*)
then obtain $p7 :: \text{Point}$ **where** $P2 : \text{Bet-Point} (\text{Se } p2 p4) p7 \wedge \text{Line-on} (\text{Li } p1$
 $p3) p7$
 $\wedge \neg \text{Line-on} (\text{Li } p1 p3) p2 \wedge \neg \text{Line-on} (\text{Li } p1 p3) p4$ **by** *blast*
then have $P3 : \text{Bet-Point} (\text{Se } p2 p4) p7$ **by** *simp*
then have $P4 : \text{Line-on} (\text{Li } p2 p4) p7$ **by** (*blast intro:Line-Bet-on*)
from $P3$ **have** $P5 : \neg \text{Eq} (\text{Geos} (\text{Poi } p4) \text{ add Emp}) (\text{Geos} (\text{Poi } p7) \text{ add Emp})$
by (*simp add:Bet-Point-def*)
have $P6 : \text{Line-on} (\text{Li } p2 p4) p4$ **by** (*simp add:Line-on-rule*)
have $P7 : \text{Line-on} (\text{Li } p3 p4) p4$ **by** (*simp add:Line-on-rule*)
from $P4 P5 P6 P7$ **have** $P8 : \text{Line-on} (\text{Li } p3 p4) p7 \implies$
 $\text{Eq} (\text{Geos} (\text{Lin} (\text{Li } p2 p4)) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li } p3 p4)) \text{ add Emp})$ **by** (*simp*
add:Line-unique)
have $P9 : \text{Line-on} (\text{Li } p2 p4) p2$ **by** (*simp add:Line-on-rule*)
from $P8 P9$ **have** $P10 : \text{Line-on} (\text{Li } p3 p4) p7 \implies \text{Line-on} (\text{Li } p3 p4) p2$ **by**
(*simp add:Line-on-trans*)
from $P7$ **have** $P11 : \text{Eq} (\text{Geos} (\text{Lin} (\text{Li } p3 p4)) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li } p1$
 $p2)) \text{ add Emp}) \implies$
 $\text{Line-on} (\text{Li } p1 p2) p4$ **by** (*simp add:Line-on-trans*)
from $P1 P11$ **have** $P12 : \neg \text{Eq} (\text{Geos} (\text{Lin} (\text{Li } p3 p4)) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li}$

$p1\ p2$) *add Emp*) **by** *blast*
from $P1$ **have** $P13 : \text{Bet-Point } (Se\ p3\ p4)\ p6$ **by** *simp*
then have $P14 : \text{Line-on } (Li\ p3\ p4)\ p6$ **by** (*blast intro:Line-Bet-on*)
have $P15 : \text{Line-on } (Li\ p1\ p2)\ p2$ **by** (*simp add:Line-on-rule*)
from $P1\ P10\ P12\ P14\ P15$ **have** $P16 : \text{Line-on } (Li\ p3\ p4)\ p7 \implies$
 $Eq\ (Geos\ (Poi\ p6)\ \text{add Emp})\ (Geos\ (Poi\ p2)\ \text{add Emp})$ **by** (*simp add:Line-unique-Point*)
from $P13\ P16$ **have** $P17 : \text{Line-on } (Li\ p3\ p4)\ p7 \implies \text{Bet-Point } (Se\ p3\ p4)\ p2$
by (*simp add:Point-Eq*)
from $P7$ **have** $P18 : Eq\ (Geos\ (Lin\ (Li\ p3\ p4))\ \text{add Emp})\ (Geos\ (Lin\ (Li\ p1\ p3))\ \text{add Emp}) \implies$
 $\text{Line-on } (Li\ p1\ p3)\ p4$ **by** (*simp add:Line-on-trans*)
from $P2\ P18$ **have** $P19 : \neg Eq\ (Geos\ (Lin\ (Li\ p3\ p4))\ \text{add Emp})\ (Geos\ (Lin\ (Li\ p1\ p3))\ \text{add Emp})$ **by** *blast*
have $P20 : \text{Line-on } (Li\ p1\ p3)\ p3$ **by** (*simp add:Line-on-rule*)
from $P17\ P19\ P20$ **have** $\text{Line-on } (Li\ p3\ p4)\ p7 \implies \text{Plane-sameside } (Li\ p1\ p3)\ p2\ p4$ **by** (*simp add:Plane-Bet-sameside*)
then have $\text{Line-on } (Li\ p3\ p4)\ p7 \implies \neg \text{Plane-diffside } (Li\ p1\ p3)\ p2\ p4$ **by** (*simp add:Plane-sameside-not-diffside*)
then have $P21 : \neg \text{Line-on } (Li\ p3\ p4)\ p7$ **using** *assms* **by** *blast*
from $P3$ **have** $P22 : \neg Eq\ (Geos\ (Poi\ p7)\ \text{add Emp})\ (Geos\ (Poi\ p2)\ \text{add Emp})$ **by** (*simp add:Bet-Point-def*)
from $P4\ P9\ P15\ P22$ **have** $P23 : \text{Line-on } (Li\ p1\ p2)\ p7 \implies$
 $Eq\ (Geos\ (Lin\ (Li\ p2\ p4))\ \text{add Emp})\ (Geos\ (Lin\ (Li\ p1\ p2))\ \text{add Emp})$ **by** (*simp add:Line-unique*)
from $P6\ P23$ **have** $P24 : \text{Line-on } (Li\ p1\ p2)\ p7 \implies \text{Line-on } (Li\ p1\ p2)\ p4$ **by** (*simp add:Line-on-trans*)
from $P1\ P24$ **have** $P25 : \neg \text{Line-on } (Li\ p1\ p2)\ p7$ **by** *blast*
from $P1\ P13\ P21\ P25$ **have** $P26 : \text{Line-on-Seg } (Li\ p1\ p2)\ (Se\ p3\ p7) \wedge \neg \text{Line-on-Seg } (Li\ p1\ p2)\ (Se\ p4\ p7)$
 $\vee \text{Line-on-Seg } (Li\ p1\ p2)\ (Se\ p4\ p7) \wedge \neg \text{Line-on-Seg } (Li\ p1\ p2)\ (Se\ p3\ p7)$
by (*simp add:Pachets-axiom*)
have $\text{Line-on-Seg } (Li\ p1\ p2)\ (Se\ p4\ p7) \implies \exists p. \text{Line-on } (Li\ p1\ p2)\ p \wedge \text{Bet-Point } (Se\ p4\ p7)\ p$ **by** (*simp add:Line-on-Seg-rule*)
then obtain $p8 :: \text{Point}$ **where** $P27 : \text{Line-on-Seg } (Li\ p1\ p2)\ (Se\ p4\ p7) \implies$
 $\text{Line-on } (Li\ p1\ p2)\ p8 \wedge \text{Bet-Point } (Se\ p4\ p7)\ p8$ **by** *blast*
then have $P28 : \text{Line-on-Seg } (Li\ p1\ p2)\ (Se\ p4\ p7) \implies \text{Line-on } (Li\ p4\ p7)\ p8$ **by** (*blast intro:Line-Bet-on*)
from $P3$ **have** $P29 : \text{Line-on } (Li\ p4\ p7)\ p2$ **by** (*blast intro:Line-Bet-on*)
from $P27$ **have** $\text{Line-on-Seg } (Li\ p1\ p2)\ (Se\ p4\ p7) \implies \text{Bet-Point } (Se\ p4\ p7)\ p8$ **by** *simp*
then have $P30 : \text{Line-on-Seg } (Li\ p1\ p2)\ (Se\ p4\ p7) \implies$
 $Eq\ (Geos\ (Poi\ p8)\ \text{add Emp})\ (Geos\ (Poi\ p2)\ \text{add Emp}) \implies \text{Bet-Point } (Se\ p4\ p7)\ p2$ **by** (*simp add:Point-Eq*)
from $P3$ **have** $Inv\ (\text{Bet-Point } (Se\ p4\ p7)\ p2)$ **by** (*simp add:Bet-iff*)
then have $P31 : \neg \text{Bet-Point } (Se\ p4\ p7)\ p2$ **by** (*simp add:Inv-def*)
from $P30\ P31$ **have** $P32 : \text{Line-on-Seg } (Li\ p1\ p2)\ (Se\ p4\ p7) \implies$
 $\neg Eq\ (Geos\ (Poi\ p8)\ \text{add Emp})\ (Geos\ (Poi\ p2)\ \text{add Emp})$ **by** *blast*
from $P15\ P27\ P28\ P29\ P32$ **have** $P33 : \text{Line-on-Seg } (Li\ p1\ p2)\ (Se\ p4\ p7) \implies$
 $Eq\ (Geos\ (Lin\ (Li\ p4\ p7))\ \text{add Emp})\ (Geos\ (Lin\ (Li\ p1\ p2))\ \text{add Emp})$ **by**

(simp add:Line-unique)
have $P34 : \text{Line-on } (Li\ p4\ p7)\ p4$ **by** *(simp add:Line-on-rule)*
from $P33\ P34$ **have** $P35 : \text{Line-on-Seg } (Li\ p1\ p2)\ (Se\ p4\ p7) \implies \text{Line-on } (Li\ p1\ p2)\ p4$ **by** *(simp add:Line-on-trans)*
from $P1\ P35$ **have** $P36 : \neg \text{Line-on-Seg } (Li\ p1\ p2)\ (Se\ p4\ p7)$ **by** *blast*
from $P26\ P36$ **have** $\text{Line-on-Seg } (Li\ p1\ p2)\ (Se\ p3\ p7)$ **by** *blast*
then **have** $\exists p. \text{Line-on } (Li\ p1\ p2)\ p \wedge \text{Bet-Point } (Se\ p3\ p7)\ p$ **by** *(simp add:Line-on-Seg-rule)*
then **obtain** $p8 :: \text{Point}$ **where** $P37 : \text{Line-on } (Li\ p1\ p2)\ p8 \wedge \text{Bet-Point } (Se\ p3\ p7)\ p8$ **by** *blast*
have $\text{Line-on } (Li\ p3\ p4)\ p3$ **by** *(simp add:Line-on-rule)*
then **have** $P38 : \text{Eq } (\text{Geos } (\text{Poi } p3)\ \text{add } \text{Emp})\ (\text{Geos } (\text{Poi } p7)\ \text{add } \text{Emp}) \implies \text{Line-on } (Li\ p3\ p4)\ p7$ **by** *(simp add:Point-Eq)*
from $P21\ P38$ **have** $P39 : \neg \text{Eq } (\text{Geos } (\text{Poi } p3)\ \text{add } \text{Emp})\ (\text{Geos } (\text{Poi } p7)\ \text{add } \text{Emp})$ **by** *blast*
have $P40 : \text{Line-on } (Li\ p3\ p7)\ p3$ **by** *(simp add:Line-on-rule)*
have $P41 : \text{Line-on } (Li\ p3\ p7)\ p7$ **by** *(simp add:Line-on-rule)*
from $P2$ **have** $P42 : \text{Line-on } (Li\ p1\ p3)\ p7$ **by** *simp*
from $P20\ P39\ P40\ P41\ P42$ **have** $P43 :$
 $\text{Eq } (\text{Geos } (\text{Lin } (Li\ p1\ p3))\ \text{add } \text{Emp})\ (\text{Geos } (\text{Lin } (Li\ p3\ p7))\ \text{add } \text{Emp})$ **by**
(simp add:Line-unique)
have $P44 : \text{Line-on } (Li\ p1\ p3)\ p1$ **by** *(simp add:Line-on-rule)*
from $P43\ P44$ **have** $P45 : \text{Line-on } (Li\ p3\ p7)\ p1$ **by** *(simp add:Line-on-trans)*
from $P37$ **have** $P46 : \text{Line-on } (Li\ p3\ p7)\ p8$ **by** *(blast intro:Line-Bet-on)*
have $P47 : \text{Line-on } (Li\ p1\ p2)\ p1$ **by** *(simp add:Line-on-rule)*
from $P40$ **have** $P48 : \text{Eq } (\text{Geos } (\text{Lin } (Li\ p3\ p7))\ \text{add } \text{Emp})\ (\text{Geos } (\text{Lin } (Li\ p1\ p2))\ \text{add } \text{Emp}) \implies$
 $\text{Line-on } (Li\ p1\ p2)\ p3$ **by** *(simp add:Line-on-trans)*
from $P1\ P48$ **have** $P49 : \neg \text{Eq } (\text{Geos } (\text{Lin } (Li\ p3\ p7))\ \text{add } \text{Emp})\ (\text{Geos } (\text{Lin } (Li\ p1\ p2))\ \text{add } \text{Emp})$ **by** *blast*
from $P37\ P45\ P46\ P48\ P47\ P49$ **have** $P50 : \text{Eq } (\text{Geos } (\text{Poi } p8)\ \text{add } \text{Emp})\ (\text{Geos } (\text{Poi } p1)\ \text{add } \text{Emp})$ **by** *(simp add:Line-unique-Point)*
from $P37$ **have** $P51 : \text{Bet-Point } (Se\ p3\ p7)\ p8$ **by** *simp*
from $P50\ P51$ **have** $P52 : \text{Bet-Point } (Se\ p3\ p7)\ p1$ **by** *(simp add:Point-Eq)*
from $P44$ **have** $P53 : \text{Eq } (\text{Geos } (\text{Poi } p1)\ \text{add } \text{Emp})\ (\text{Geos } (\text{Poi } p2)\ \text{add } \text{Emp}) \implies \text{Line-on } (Li\ p1\ p3)\ p2$ **by** *(simp add:Point-Eq)*
from $P2\ P53$ **have** $P54 : \neg \text{Eq } (\text{Geos } (\text{Poi } p1)\ \text{add } \text{Emp})\ (\text{Geos } (\text{Poi } p2)\ \text{add } \text{Emp})$ **by** *blast*
have $P55 : \text{Line-on } (Li\ p1\ p2)\ p2$ **by** *(simp add:Line-on-rule)*
from $P45\ P47\ P54\ P55$ **have** $P56 : \text{Line-on } (Li\ p3\ p7)\ p2 \implies$
 $\text{Eq } (\text{Geos } (\text{Lin } (Li\ p3\ p7))\ \text{add } \text{Emp})\ (\text{Geos } (\text{Lin } (Li\ p1\ p2))\ \text{add } \text{Emp})$ **by**
(simp add:Line-unique)
from $P49\ P56$ **have** $P57 : \neg \text{Line-on } (Li\ p3\ p7)\ p2$ **by** *blast*
from *assms* **have** $P58 : \neg \text{Line-on-Seg } (Li\ p1\ p5)\ (Se\ p3\ p2) \wedge \neg \text{Line-on } (Li\ p1\ p5)\ p3$
 $\wedge \neg \text{Line-on } (Li\ p1\ p5)\ p2 \wedge \neg \text{Eq } (\text{Geos } (\text{Poi } p3)\ \text{add } \text{Emp})\ (\text{Geos } (\text{Poi } p2)\ \text{add } \text{Emp})$ **by** *(simp add:Plane-sameside-def)*
from $P52$ **have** $P59 : \neg \text{Eq } (\text{Geos } (\text{Poi } p7)\ \text{add } \text{Emp})\ (\text{Geos } (\text{Poi } p1)\ \text{add } \text{Emp})$ **by** *(simp add:Bet-Point-def)*

have $P60 : \text{Line-on } (Li\ p1\ p5)\ p1$ **by** (*simp add:Line-on-rule*)
from $P41\ P45\ P59\ P60$ **have** $P61 : \text{Line-on } (Li\ p1\ p5)\ p7 \implies$
 $\text{Eq } (\text{Geos } (\text{Lin } (Li\ p3\ p7)))\ \text{add Emp}) (\text{Geos } (\text{Lin } (Li\ p1\ p5)))\ \text{add Emp})$ **by**
(*simp add:Line-unique*)
from $P40\ P61$ **have** $P62 : \text{Line-on } (Li\ p1\ p5)\ p7 \implies \text{Line-on } (Li\ p1\ p5)\ p3$ **by**
(*simp add:Line-on-trans*)
from $P58\ P62$ **have** $P63 : \neg \text{Line-on } (Li\ p1\ p5)\ p7$ **by** *blast*
from $P58$ **have** $P64 : \neg \text{Line-on } (Li\ p1\ p5)\ p3$ **by** *simp*
from $P58$ **have** $P65 : \neg \text{Line-on } (Li\ p1\ p5)\ p2$ **by** *simp*
from $P52\ P57\ P60\ P63\ P64\ P65$ **have** $P66 : \text{Line-on-Seg } (Li\ p1\ p5)\ (Se\ p3\ p2)$
 $\wedge \neg \text{Line-on-Seg } (Li\ p1\ p5)\ (Se\ p7\ p2)$
 $\vee \text{Line-on-Seg } (Li\ p1\ p5)\ (Se\ p7\ p2) \wedge \neg \text{Line-on-Seg } (Li\ p1\ p5)\ (Se\ p3\ p2)$
by (*simp add:Pachets-axiom*)
from $P58\ P66$ **have** $\text{Line-on-Seg } (Li\ p1\ p5)\ (Se\ p7\ p2)$ **by** *blast*
then **have** $\exists p. \text{Line-on } (Li\ p1\ p5)\ p \wedge \text{Bet-Point } (Se\ p7\ p2)\ p$ **by** (*simp*
add:Line-on-Seg-rule)
then **obtain** $p9 :: \text{Point}$ **where** $P67 : \text{Line-on } (Li\ p1\ p5)\ p9 \wedge \text{Bet-Point } (Se$
 $p7\ p2)\ p9$ **by** *blast*
then **have** $P68 : \text{Bet-Point } (Se\ p2\ p7)\ p9$ **by** (*simp add:Bet-rev*)
from $P3\ P68$ **have** $\text{Bet-Point } (Se\ p2\ p4)\ p9$ **by** (*blast intro:Bet-swap-134-124*)
then **have** $P69 : \text{Bet-Point } (Se\ p4\ p2)\ p9$ **by** (*simp add:Bet-rev*)
from $P67\ P69$ **have** $\exists p. \text{Line-on } (Li\ p1\ p5)\ p \wedge \text{Bet-Point } (Se\ p4\ p2)\ p$ **by** *blast*
then **have** $P70 : \text{Line-on-Seg } (Li\ p1\ p5)\ (Se\ p4\ p2)$ **by** (*simp add:Line-on-Seg-rule*)
from *assms* **have** $P71 : \neg \text{Line-on-Seg } (Li\ p1\ p5)\ (Se\ p4\ p2)$ **by** (*simp add:Plane-sameside-def*)
from $P70\ P71$ **show** *False* **by** *blast*
qed

lemma (*in Congruence-Rule*) *Ang-Gr-trans-Gr-Gr* :

assumes

$\text{Def } (\text{Ang } (\text{An } p11\ p12\ p13))\ \text{Def } (\text{Ang } (\text{An } p21\ p22\ p23))\ \text{Def } (\text{Ang } (\text{An } p31$
 $p32\ p33))$

$\text{Gr } (\text{Geos } (\text{Ang } (\text{An } p11\ p12\ p13)))\ \text{add Emp}) (\text{Geos } (\text{Ang } (\text{An } p21\ p22\ p23)))$
 $\text{add Emp})$

$\text{Gr } (\text{Geos } (\text{Ang } (\text{An } p21\ p22\ p23)))\ \text{add Emp}) (\text{Geos } (\text{Ang } (\text{An } p31\ p32\ p33)))$
 $\text{add Emp})$

shows

$\text{Gr } (\text{Geos } (\text{Ang } (\text{An } p11\ p12\ p13)))\ \text{add Emp}) (\text{Geos } (\text{Ang } (\text{An } p31\ p32\ p33)))$
 $\text{add Emp})$

proof –

from *assms* **have** $P1 : \neg \text{Line-on } (Li\ p12\ p13)\ p11$ **by** (*simp add:Ang-to-Tri*
Tri-def-Line)

from *assms* $P1$ **have** $\exists p. \text{Cong } (\text{Geos } (\text{Ang } (\text{An } p21\ p22\ p23)))\ \text{add Emp}) (\text{Geos}$
 $(\text{Ang } (\text{An } p\ p12\ p13)))\ \text{add Emp})$

$\wedge \text{Plane-sameside } (Li\ p12\ p13)\ p\ p11$ **by** (*simp add:Ang-move-sameside*)

then **obtain** $p14 :: \text{Point}$ **where** $P2 : \text{Cong } (\text{Geos } (\text{Ang } (\text{An } p21\ p22\ p23)))\ \text{add}$
 $\text{Emp}) (\text{Geos } (\text{Ang } (\text{An } p14\ p12\ p13)))\ \text{add Emp})$

$\wedge \text{Plane-sameside } (Li\ p12\ p13)\ p14\ p11$ **by** *blast*

from *assms* $P1$ **have** $\exists p. \text{Cong } (\text{Geos } (\text{Ang } (\text{An } p31\ p32\ p33)))\ \text{add Emp}) (\text{Geos}$
 $(\text{Ang } (\text{An } p\ p12\ p13)))\ \text{add Emp})$

\wedge *Plane-sameside* (*Li* *p12 p13*) *p p11* **by** (*simp add:Ang-move-sameside*)
then obtain *p15* :: *Point* **where** *P3* : *Cong* (*Geos* (*Ang* (*An* *p31 p32 p33*))) *add Emp*) (*Geos* (*Ang* (*An* *p15 p12 p13*))) *add Emp*)
 \wedge *Plane-sameside* (*Li* *p12 p13*) *p15 p11* **by** *blast*
from *P2* **have** *P4* : *Cong* (*Geos* (*Ang* (*An* *p21 p22 p23*))) *add Emp*) (*Geos* (*Ang* (*An* *p14 p12 p13*))) *add Emp*) **by** *simp*
from *P2* **have** *P5* : *Plane-sameside* (*Li* *p12 p13*) *p14 p11* **by** *simp*
from *P3* **have** *P6* : *Cong* (*Geos* (*Ang* (*An* *p31 p32 p33*))) *add Emp*) (*Geos* (*Ang* (*An* *p15 p12 p13*))) *add Emp*) **by** *simp*
from *P3* **have** *P7* : *Plane-sameside* (*Li* *p12 p13*) *p15 p11* **by** *simp*
from *P4 P5* **have** *Ang-inside* (*An* *p11 p12 p13*) *p14* \longleftrightarrow
Gr (*Geos* (*Ang* (*An* *p11 p12 p13*))) *add Emp*) (*Geos* (*Ang* (*An* *p21 p22 p23*))) *add Emp*) **by** (*simp add:Ang-greater-def*)
then have *P8* : *Ang-inside* (*An* *p11 p12 p13*) *p14* **using** *assms* **by** *blast*
from *P5* **have** *P9* : \neg *Line-on* (*Li* *p12 p13*) *p14* **by** (*simp add:Plane-sameside-def*)
from *assms* **have** *P10* : \neg *Eq* (*Geos* (*Poi* *p12*)) *add Emp*) (*Geos* (*Poi* *p13*)) *add Emp*) **by** (*simp add:Ang-def*)
from *P9 P10* **have** *Def* (*Ang* (*An* *p12 p13 p14*))) **by** (*simp add:Ang-simple-def*)
then have *P11* : *Def* (*Ang* (*An* *p14 p12 p13*))) **by** (*blast intro:Ang-def-rev Ang-def-inv*)
from *assms P4 P11* **have** *P12* : *Gr* (*Geos* (*Ang* (*An* *p14 p12 p13*))) *add Emp*) (*Geos* (*Ang* (*An* *p31 p32 p33*))) *add Emp*) **by** (*blast intro:Ang-Gr-trans-Eq-Gr Ang-rev*)
from *P11* **have** *P13* : *Eq* (*Geos* (*Poi* *p14*)) *add Emp*) (*Geos* (*Poi* *p15*)) *add Emp*)
 \implies
Eq (*Geos* (*Ang* (*An* *p14 p12 p13*))) *add Emp*) (*Geos* (*Ang* (*An* *p15 p12 p13*))) *add Emp*) \wedge *Def* (*Ang* (*An* *p15 p12 p13*))) **by** (*simp add:Ang-Eq-Point*)
then have *P14* : *Eq* (*Geos* (*Poi* *p14*)) *add Emp*) (*Geos* (*Poi* *p15*)) *add Emp*) \implies
Cong (*Geos* (*Ang* (*An* *p14 p12 p13*))) *add Emp*) (*Geos* (*Ang* (*An* *p15 p12 p13*))) *add Emp*) **by** (*blast intro:Ang-weektrans*)
from *assms P4 P11 P13 P14* **have** *P15* : *Eq* (*Geos* (*Poi* *p14*)) *add Emp*) (*Geos* (*Poi* *p15*)) *add Emp*) \implies
Cong (*Geos* (*Ang* (*An* *p21 p22 p23*))) *add Emp*) (*Geos* (*Ang* (*An* *p15 p12 p13*))) *add Emp*) **by** (*blast intro:Ang-trans Ang-rev*)
from *assms P6 P13 P15* **have** *P16* : *Eq* (*Geos* (*Poi* *p14*)) *add Emp*) (*Geos* (*Poi* *p15*)) *add Emp*) \implies
Cong (*Geos* (*Ang* (*An* *p31 p32 p33*))) *add Emp*) (*Geos* (*Ang* (*An* *p21 p22 p23*))) *add Emp*) **by** (*blast intro:Ang-trans Ang-rev*)
from *assms* **have** *P17* : \neg *Cong* (*Geos* (*Ang* (*An* *p31 p32 p33*))) *add Emp*) (*Geos* (*Ang* (*An* *p21 p22 p23*))) *add Emp*) **by** (*simp add:Ang-Gr-not-Eq-rev*)
from *P16 P17* **have** *P18* : \neg *Eq* (*Geos* (*Poi* *p15*)) *add Emp*) (*Geos* (*Poi* *p14*)) *add Emp*) **by** (*blast intro:Eq-rev*)
from *P7* **have** *P19* : *Plane-sameside* (*Li* *p12 p13*) *p11 p15* **by** (*simp add:Plane-sameside-rev*)
from *P5 P18 P19* **have** *P20* : *Plane-sameside* (*Li* *p12 p13*) *p15 p14* **by** (*blast intro:Plane-sameside-trans Plane-sameside-rev*)
from *P6 P20* **have** *P21* : *Ang-inside* (*An* *p14 p12 p13*) *p15* \longleftrightarrow
Gr (*Geos* (*Ang* (*An* *p14 p12 p13*))) *add Emp*) (*Geos* (*Ang* (*An* *p31 p32 p33*))) *add Emp*) **by** (*simp add:Ang-greater-def*)
from *P12 P21* **have** *Ang-inside* (*An* *p14 p12 p13*) *p15* **by** *simp*
then have *P22* : *Plane-sameside* (*Li* *p12 p14*) *p13 p15* \wedge *Plane-sameside* (*Li*

$p12\ p13) p14\ p15$ **by** (*simp add:Ang-inside-def*)
from $P8$ **have** $P23 : \text{Plane-sameside } (Li\ p12\ p11)\ p13\ p14 \wedge \text{Plane-sameside } (Li\ p12\ p13)\ p11\ p14$ **by** (*simp add:Ang-inside-def*)
then have $P24 : \text{Plane-diffside } (Li\ p12\ p11)\ p13\ p15 \implies \text{Plane-diffside } (Li\ p12\ p11)\ p14\ p15$ **by** (*blast intro:Plane-trans*)
from $P8$ **have** $P25 : \text{Plane-diffside } (Li\ p12\ p14)\ p11\ p13$ **by** (*simp add:Ang-inside-Planeside*)
from $P22\ P25$ **have** $P26 : \text{Plane-diffside } (Li\ p12\ p14)\ p11\ p15$ **by** (*blast intro:Plane-trans Plane-diffside-rev*)
from $P5\ P7\ P24\ P26$ **have** $\text{Plane-diffside } (Li\ p12\ p11)\ p13\ p15 \implies \text{False}$ **by** (*blast intro:Planeside-wrong-relation*)
then have $P27 : \neg \text{Plane-diffside } (Li\ p12\ p11)\ p13\ p15$ **by** *blast*
from $P23$ **have** $P28 : \neg \text{Line-on } (Li\ p12\ p11)\ p13$ **by** (*simp add:Plane-sameside-def*)
have $P29 : \text{Line-on } (Li\ p12\ p13)\ p12$ **by** (*simp add:Line-on-rule*)
from $P19$ **have** $P30 : \neg \text{Line-on } (Li\ p12\ p13)\ p11$ **by** (*simp add:Plane-sameside-def*)
from $P19$ **have** $P31 : \neg \text{Line-on } (Li\ p12\ p13)\ p15$ **by** (*simp add:Plane-sameside-def*)
from $P29\ P30\ P31$ **have** $\text{Bet-Point } (Se\ p11\ p15)\ p12 \implies \exists p. \text{Bet-Point } (Se\ p11\ p15)\ p \wedge \text{Line-on } (Li\ p12\ p13)\ p \wedge \neg \text{Line-on } (Li\ p12\ p13)\ p11 \wedge \neg \text{Line-on } (Li\ p12\ p13)\ p15$ **by** *blast*
then have $\text{Bet-Point } (Se\ p11\ p15)\ p12 \implies \text{Plane-diffside } (Li\ p12\ p13)\ p11\ p15$ **by** (*simp add:Plane-diffside-def*)
then have $P32 : \text{Bet-Point } (Se\ p11\ p15)\ p12 \implies \neg \text{Plane-sameside } (Li\ p12\ p13)\ p15\ p11$ **by** (*simp add:Plane-diffside-rev Plane-diffside-not-sameside*)
from $P7\ P32$ **have** $P33 : \neg \text{Bet-Point } (Se\ p11\ p15)\ p12$ **by** *blast*
have $P34 : \text{Line-on } (Li\ p12\ p13)\ p13$ **by** (*simp add:Line-on-rule*)
have $P35 : \neg \text{Bet-Point } (Se\ p13\ p13)\ p12$ **by** (*simp add:Bet-end-Point*)
from $P29$ **have** $P36 : \text{Eq } (\text{Geos } (Poi\ p12)\ \text{add } \text{Emp})\ (\text{Geos } (Poi\ p15)\ \text{add } \text{Emp})$
 \implies
 $\text{Line-on } (Li\ p12\ p13)\ p15$ **by** (*simp add:Point-Eq*)
from $P31\ P36$ **have** $P37 : \neg \text{Eq } (\text{Geos } (Poi\ p12)\ \text{add } \text{Emp})\ (\text{Geos } (Poi\ p15)\ \text{add } \text{Emp})$ **by** *blast*
from *assms* **have** $P38 : \neg \text{Eq } (\text{Geos } (Poi\ p12)\ \text{add } \text{Emp})\ (\text{Geos } (Poi\ p13)\ \text{add } \text{Emp})$ **by** (*simp add:Ang-def*)
from *assms* $P33\ P34\ P35\ P37\ P38$ **have** $P39 : \text{Line-on } (Li\ p12\ p11)\ p15 \implies \text{Eq } (\text{Geos } (\text{Ang } (\text{An } p11\ p12\ p13))\ \text{add } \text{Emp})\ (\text{Geos } (\text{Ang } (\text{An } p15\ p12\ p13))\ \text{add } \text{Emp}) \wedge \text{Def } (\text{Ang } (\text{An } p15\ p12\ p13))$ **by** (*simp add:Ang-Point-swap*)
then have $P40 : \text{Line-on } (Li\ p12\ p11)\ p15 \implies \text{Cong } (\text{Geos } (\text{Ang } (\text{An } p11\ p12\ p13))\ \text{add } \text{Emp})\ (\text{Geos } (\text{Ang } (\text{An } p15\ p12\ p13))\ \text{add } \text{Emp})$ **by** (*blast intro:Ang-weektrans*)
from *assms* $P6\ P39\ P40$ **have** $P41 : \text{Line-on } (Li\ p12\ p11)\ p15 \implies \text{Cong } (\text{Geos } (\text{Ang } (\text{An } p31\ p32\ p33))\ \text{add } \text{Emp})\ (\text{Geos } (\text{Ang } (\text{An } p11\ p12\ p13))\ \text{add } \text{Emp})$ **by** (*blast intro:Ang-trans Ang-rev*)
from *assms* $P41$ **have** $P42 : \text{Line-on } (Li\ p12\ p11)\ p15 \implies \text{Gr } (\text{Geos } (\text{Ang } (\text{An } p21\ p22\ p23))\ \text{add } \text{Emp})\ (\text{Geos } (\text{Ang } (\text{An } p11\ p12\ p13))\ \text{add } \text{Emp})$ **by** (*blast intro:Ang-Gr-trans-Gr-Eq*)
from *assms* **have** $P43 : \neg \text{Gr } (\text{Geos } (\text{Ang } (\text{An } p21\ p22\ p23))\ \text{add } \text{Emp})\ (\text{Geos } (\text{Ang } (\text{An } p11\ p12\ p13))\ \text{add } \text{Emp})$ **by** (*simp add:Ang-Gr-not-Eq-rev*)
from $P42\ P43$ **have** $P44 : \neg \text{Line-on } (Li\ p12\ p11)\ p15$ **by** *blast*
from $P34$ **have** $P45 : \text{Eq } (\text{Geos } (Poi\ p13)\ \text{add } \text{Emp})\ (\text{Geos } (Poi\ p15)\ \text{add } \text{Emp})$

\implies
Line-on (*Li* *p12* *p13*) *p15* **by** (*simp add:Point-Eq*)
from *P31 P45* **have** *P46* : $\neg \text{Eq} (\text{Geos} (\text{Poi } p13) \text{ add Emp}) (\text{Geos} (\text{Poi } p15) \text{ add Emp})$ **by** *blast*
from *P27 P28 P44 P46* **have** *P47* : *Plane-sameside* (*Li* *p12* *p11*) *p13* *p15* **by** (*simp add:Plane-not-diffside-sameside*)
from *assms P19 P47* **have** *P48* : *Ang-inside* (*An* *p11* *p12* *p13*) *p15* **by** (*simp add:Ang-inside-def*)
from *P6 P7* **have** *P49* : *Ang-inside* (*An* *p11* *p12* *p13*) *p15* \longleftrightarrow
Gr (*Geos* (*Ang* (*An* *p11* *p12* *p13*)) *add Emp*) (*Geos* (*Ang* (*An* *p31* *p32* *p33*)) *add Emp*) **by** (*simp add:Ang-greater-def*)
from *P48 P49* **show** *Gr* (*Geos* (*Ang* (*An* *p11* *p12* *p13*)) *add Emp*) (*Geos* (*Ang* (*An* *p31* *p32* *p33*)) *add Emp*) **by** *simp*
qed

lemma (in *Congruence-Rule*) *Ang-complementary-inside* :

assumes

Def (*Ang* (*An* *p1* *p2* *p3*))

Bet-Point (*Se* *p3* *p4*) *p2*

Ang-inside (*An* *p5* *p2* *p3*) *p1*

shows

Ang-inside (*An* *p1* *p2* *p4*) *p5*

proof –

from *assms* **have** *P1* : *Plane-sameside* (*Li* *p2* *p5*) *p3* *p1* \wedge *Plane-sameside* (*Li* *p2* *p3*) *p5* *p1* **by** (*simp add:Ang-inside-def*)

from *assms* **have** *P2* : *Line-on* (*Li* *p2* *p3*) *p4* **by** (*simp add:Line-Bet-on*)

have *P3* : *Line-on* (*Li* *p2* *p3*) *p2* **by** (*simp add:Line-on-rule*)

have *P4* : *Line-on* (*Li* *p2* *p4*) *p2* **by** (*simp add:Line-on-rule*)

have *P5* : *Line-on* (*Li* *p2* *p4*) *p4* **by** (*simp add:Line-on-rule*)

from *assms* **have** *P6* : $\neg \text{Eq} (\text{Geos} (\text{Poi } p4) \text{ add Emp}) (\text{Geos} (\text{Poi } p2) \text{ add Emp})$ **by** (*simp add:Bet-Point-def*)

from *P2 P3 P4 P5 P6* **have** *P7* : $\text{Eq} (\text{Geos} (\text{Lin} (\text{Li } p2 \text{ } p3)) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li } p2 \text{ } p4)) \text{ add Emp})$ **by** (*simp add:Line-unique*)

from *P1 P7* **have** *P8* : *Plane-sameside* (*Li* *p2* *p4*) *p5* *p1* **by** (*blast intro:Plane-Line-trans*)

have *P9* : *Line-on* (*Li* *p2* *p1*) *p2* **by** (*simp add:Line-on-rule*)

have *Line-on* (*Li* *p2* *p1*) *p1* **by** (*simp add:Line-on-rule*)

then **have** *P10* : $\text{Eq} (\text{Geos} (\text{Lin} (\text{Li } p3 \text{ } p4)) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li } p2 \text{ } p1)) \text{ add Emp}) \implies$

Line-on (*Li* *p3* *p4*) *p1* **by** (*blast intro:Eq-rev Line-on-trans*)

from *P1* **have** *P11* : $\neg \text{Line-on} (\text{Li } p2 \text{ } p3) \text{ } p1$ **by** (*simp add:Plane-sameside-def*)

from *assms* **have** *P12* : *Line-on* (*Li* *p3* *p4*) *p2* **by** (*simp add:Line-Bet-on*)

have *P13* : *Line-on* (*Li* *p3* *p4*) *p3* **by** (*simp add:Line-on-rule*)

have *P14* : *Line-on* (*Li* *p2* *p3*) *p3* **by** (*simp add:Line-on-rule*)

from *assms* **have** *P15* : $\neg \text{Eq} (\text{Geos} (\text{Poi } p2) \text{ add Emp}) (\text{Geos} (\text{Poi } p3) \text{ add Emp})$ **by** (*simp add:Bet-Point-def*)

from *P3 P12 P13 P14 P15* **have** *P16* : $\text{Eq} (\text{Geos} (\text{Lin} (\text{Li } p3 \text{ } p4)) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li } p2 \text{ } p3)) \text{ add Emp})$ **by** (*simp add:Line-unique*)

from *P10 P16* **have** *P17* : $\text{Eq} (\text{Geos} (\text{Lin} (\text{Li } p3 \text{ } p4)) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li } p2 \text{ } p1)) \text{ add Emp}) \implies$

Line-on (Li p2 p3) p1 **by** (*simp add:Line-on-trans*)
from *P11 P17* **have** *P18* : $\neg \text{Eq (Geos (Lin (Li p3 p4)) add Emp) (Geos (Lin (Li p2 p1)) add Emp)}$ **by** *blast*
from *assms P9 P18* **have** *P19* : *Plane-diffside (Li p2 p1) p3 p4* **by** (*simp add:Plane-Bet-diffside*)
from *assms* **have** *P20* : *Plane-diffside (Li p2 p1) p3 p5* **by** (*simp add:Ang-inside-Planeside Plane-diffside-rev*)
from *P5* **have** *P21* : *Eq (Geos (Poi p4) add Emp) (Geos (Poi p5) add Emp)* \implies *Line-on (Li p2 p4) p5* **by** (*simp add:Point-Eq*)
from *P8* **have** *P22* : $\neg \text{Line-on (Li p2 p4) p5}$ **by** (*simp add:Plane-sameside-def*)
from *P21 P22* **have** *P23* : $\neg \text{Eq (Geos (Poi p4) add Emp) (Geos (Poi p5) add Emp)}$ **by** *blast*
from *P19 P20 P23* **have** *P24* : *Plane-sameside (Li p2 p1) p4 p5* **by** (*simp add:Plane-trans-inv*)
then **have** *P25* : $\neg \text{Line-on (Li p2 p1) p4}$ **by** (*simp add:Plane-sameside-def*)
from *assms* **have** $\neg \text{Eq (Geos (Poi p1) add Emp) (Geos (Poi p2) add Emp)}$ **by** (*simp add:Ang-def*)
then **have** *P26* : $\neg \text{Eq (Geos (Poi p2) add Emp) (Geos (Poi p1) add Emp)}$ **by** (*blast intro:Eq-rev*)
from *P25 P26* **have** *Def (Ang (An p2 p1 p4))* **by** (*simp add:Ang-simple-def*)
then **have** *P27* : *Def (Ang (An p1 p2 p4))* **by** (*blast intro:Ang-def-rev Ang-def-inv*)
from *P8* **have** *P28* : *Plane-sameside (Li p2 p4) p1 p5* **by** (*simp add:Plane-sameside-rev*)
from *P24 P27 P28* **show** *Ang-inside (An p1 p2 p4) p5* **by** (*simp add:Ang-inside-def*)
qed

Theorem21

theorem (*in Congruence-Rule*) *Ang-Right-angle-Cong* :

assumes

Right-angle (An l1 o1 h1) Right-angle (An l2 o2 h2)

shows

Cong (Geos (Ang (An l1 o1 h1)) add Emp) (Geos (Ang (An l2 o2 h2)) add Emp)

proof –

from *assms* **have** $\exists p. \text{Cong (Geos (Ang (An l1 o1 h1)) add Emp) (Geos (Ang (An l1 o1 p)) add Emp)}$

$\wedge \text{Bet-Point (Se h1 p) o1} \wedge \text{Def (Ang (An l1 o1 h1))} \wedge \text{Def (Ang (An l1 o1 p))}$ **by** (*simp add:Ang-Right-angle-def*)

then obtain *k1* :: *Point* **where** *P1* : *Cong (Geos (Ang (An l1 o1 h1)) add Emp) (Geos (Ang (An l1 o1 k1)) add Emp)*

$\wedge \text{Bet-Point (Se h1 k1) o1} \wedge \text{Def (Ang (An l1 o1 h1))} \wedge \text{Def (Ang (An l1 o1 k1))}$ **by** *blast*

from *assms* **have** $\exists p. \text{Cong (Geos (Ang (An l2 o2 h2)) add Emp) (Geos (Ang (An l2 o2 p)) add Emp)}$

$\wedge \text{Bet-Point (Se h2 p) o2} \wedge \text{Def (Ang (An l2 o2 h2))} \wedge \text{Def (Ang (An l2 o2 p))}$ **by** (*simp add:Ang-Right-angle-def*)

then obtain *k2* :: *Point* **where** *P2* : *Cong (Geos (Ang (An l2 o2 h2)) add Emp) (Geos (Ang (An l2 o2 k2)) add Emp)*

$\wedge \text{Bet-Point (Se h2 k2) o2} \wedge \text{Def (Ang (An l2 o2 h2))} \wedge \text{Def (Ang (An l2 o2 k2))}$ **by** *blast*

from $P1$ **have** $P3 : \neg \text{Line-on } (Li\ o1\ h1)\ l1$ **by** (*simp add:Ang-to-Tri Tri-def-Line*)
from $P1\ P2\ P3$ **have** $\exists p. \text{Cong } (\text{Geos } (\text{Ang } (\text{An } l2\ o2\ h2))) \text{ add Emp}$ (*Geos*
(Ang (An p o1 h1)) add Emp)
 $\wedge \text{Plane-sameside } (Li\ o1\ h1)\ p\ l1$ **by** (*simp add:Ang-move-sameside*)
then obtain $l11 :: \text{Point}$ **where** $P4 : \text{Cong } (\text{Geos } (\text{Ang } (\text{An } l2\ o2\ h2))) \text{ add Emp}$
(Geos (Ang (An l11 o1 h1)) add Emp)
 $\wedge \text{Plane-sameside } (Li\ o1\ h1)\ l11\ l1$ **by** *blast*
then have $P5 : \neg \text{Line-on } (Li\ o1\ h1)\ l11$ **by** (*simp add:Plane-sameside-def*)
from $P1$ **have** $P6 : \neg \text{Eq } (\text{Geos } (\text{Poi } o1)) \text{ add Emp}$ (*Geos (Poi h1) add Emp*)
by (*simp add:Ang-def*)
from $P5\ P6$ **have** $\text{Def } (\text{Ang } (\text{An } o1\ h1)\ l11)$ **by** (*simp add:Ang-simple-def*)
then have $P7 : \text{Def } (\text{Ang } (\text{An } l11\ o1\ h1))$ **by** (*blast intro:Ang-def-rev Ang-def-inv*)
from $P2\ P4\ P7$ **have** $P8 : \text{Cong } (\text{Geos } (\text{Ang } (\text{An } l2\ o2\ k2))) \text{ add Emp}$ (*Geos*
(Ang (An l11 o1 h1)) add Emp) **by** (*blast intro:Ang-rev Ang-trans*)
from $P2$ **have** $P9 : \text{Def } (\text{Ang } (\text{An } h2\ o2\ l2))$ **by** (*simp add:Ang-def-rev*)
from $P7$ **have** $P10 : \text{Def } (\text{Ang } (\text{An } h1\ o1\ l11))$ **by** (*simp add:Ang-def-rev*)
have $P11 : \text{Cong } (\text{Geos } (\text{Ang } (\text{An } l2\ o2\ h2))) \text{ add Emp}$ (*Geos (Ang (An h2 o2*
l2)) add Emp) **by** (*simp add:Ang-roll*)
have $P12 : \text{Cong } (\text{Geos } (\text{Ang } (\text{An } l11\ o1\ h1))) \text{ add Emp}$ (*Geos (Ang (An h1 o1*
l11)) add Emp) **by** (*simp add:Ang-roll*)
from $P2\ P4\ P7\ P9\ P11$ **have** $P13 : \text{Cong } (\text{Geos } (\text{Ang } (\text{An } h2\ o2\ l2))) \text{ add Emp}$
(Geos (Ang (An l11 o1 h1)) add Emp) **by** (*blast intro:Ang-rev Ang-trans*)
from $P7\ P9\ P10\ P12\ P13$ **have** $P14 : \text{Cong } (\text{Geos } (\text{Ang } (\text{An } h2\ o2\ l2))) \text{ add$
Emp) (*Geos (Ang (An h1 o1 l11)) add Emp*) **by** (*blast intro:Ang-rev Ang-trans*)
from $P1\ P2\ P9\ P10\ P14$ **have** $P15 : \text{Cong } (\text{Geos } (\text{Ang } (\text{An } l2\ o2\ k2))) \text{ add Emp}$
(Geos (Ang (An l11 o1 k1)) add Emp) **by** (*simp add:Ang-complementary*)
from $P1\ P4\ P7$ **have** $P16 : \text{Eq } (\text{Geos } (\text{Lin } (\text{Li } o1\ l11))) \text{ add Emp}$ (*Geos (Lin*
(Li o1 l1)) add Emp) \implies
 $\text{Cong } (\text{Geos } (\text{Ang } (\text{An } l11\ o1\ h1))) \text{ add Emp}$ (*Geos (Ang (An l1 o1 h1)) add*
Emp) **by** (*simp add:Ang-move-unique-inv*)
from $P1\ P2\ P4\ P7\ P16$ **have** $\text{Eq } (\text{Geos } (\text{Lin } (\text{Li } o1\ l11))) \text{ add Emp}$ (*Geos (Lin*
(Li o1 l1)) add Emp) \implies
 $\text{Cong } (\text{Geos } (\text{Ang } (\text{An } l1\ o1\ h1))) \text{ add Emp}$ (*Geos (Ang (An l2 o2 h2)) add*
Emp) **by** (*blast intro:Ang-trans Ang-rev*)
then have $P17 : \neg \text{Cong } (\text{Geos } (\text{Ang } (\text{An } l1\ o1\ h1))) \text{ add Emp}$ (*Geos (Ang (An*
l2 o2 h2)) add Emp) \implies
 $\neg \text{Eq } (\text{Geos } (\text{Lin } (\text{Li } o1\ l11))) \text{ add Emp}$ (*Geos (Lin (Li o1 l1)) add Emp*) **by**
blast
from $P1\ P4\ P7\ P17$ **have** $P18 : \neg \text{Cong } (\text{Geos } (\text{Ang } (\text{An } l1\ o1\ h1))) \text{ add Emp}$
(Geos (Ang (An l2 o2 h2)) add Emp) \implies
 $\text{Ang-inside } (\text{An } l11\ o1\ h1)\ l1 \wedge \neg \text{Ang-inside } (\text{An } l1\ o1\ h1)\ l11$
 $\vee \neg \text{Ang-inside } (\text{An } l11\ o1\ h1)\ l1 \wedge \text{Ang-inside } (\text{An } l1\ o1\ h1)\ l11$ **by** (*simp*
add:Ang-inside-case)
from $P4$ **have** $P19 : \text{Plane-sameside } (Li\ o1\ h1)\ l1\ l11$ **by** (*simp add:Plane-sameside-rev*)
have $P20 : \text{Cong } (\text{Geos } (\text{Ang } (\text{An } l1\ o1\ h1))) \text{ add Emp}$ (*Geos (Ang (An l1 o1*
h1)) add Emp) **by** *simp*
from $P19\ P20$ **have** $P21 : \text{Ang-inside } (\text{An } l11\ o1\ h1)\ l1 \implies$
 $\text{Gr } (\text{Geos } (\text{Ang } (\text{An } l11\ o1\ h1))) \text{ add Emp}$ (*Geos (Ang (An l1 o1 h1)) add Emp*)
by (*simp add:Ang-greater-def*)

from $P1$ **have** $P22 : \text{Line-on } (Li\ o1\ h1)\ k1$ **by** (*simp add:Line-Bet-on*)
have $P23 : \text{Line-on } (Li\ o1\ h1)\ o1$ **by** (*simp add:Line-on-rule*)
have $P24 : \text{Line-on } (Li\ o1\ k1)\ k1$ **by** (*simp add:Line-on-rule*)
have $P25 : \text{Line-on } (Li\ o1\ k1)\ o1$ **by** (*simp add:Line-on-rule*)
from $P1$ **have** $P26 : \text{Bet-Point } (Se\ h1\ k1)\ o1$ **by** *simp*
then have $P27 : \neg \text{Eq } (\text{Geos } (\text{Poi } k1)\ \text{add Emp})\ (\text{Geos } (\text{Poi } o1)\ \text{add Emp})$ **by**
(*simp add:Bet-Point-def*)
from $P22\ P23\ P24\ P25\ P27$ **have** $P28 : \text{Eq } (\text{Geos } (\text{Lin } (Li\ o1\ h1))\ \text{add Emp})\ (\text{Geos } (\text{Lin } (Li\ o1\ k1))\ \text{add Emp})$ **by** (*simp add:Line-unique*)
from $P19\ P28$ **have** $P29 : \text{Plane-sameside } (Li\ o1\ k1)\ l1\ l11$ **by** (*simp add:Plane-Line-trans*)
then have $P30 : \text{Plane-sameside } (Li\ o1\ k1)\ l11\ l1$ **by** (*simp add:Plane-sameside-rev*)
then have $P31 : \neg \text{Line-on } (Li\ o1\ k1)\ l11$ **by** (*simp add:Plane-sameside-def*)
from $P1$ **have** $P32 : \neg \text{Eq } (\text{Geos } (\text{Poi } o1)\ \text{add Emp})\ (\text{Geos } (\text{Poi } k1)\ \text{add Emp})$
by (*simp add:Ang-def*)
from $P31\ P32$ **have** $\text{Def } (\text{Ang } (An\ o1\ k1\ l11))$ **by** (*simp add:Ang-simple-def*)
then have $P33 : \text{Def } (\text{Ang } (An\ l11\ o1\ k1))$ **by** (*blast intro:Ang-def-rev Ang-def-inv*)
from $P2\ P7\ P8\ P15\ P33$ **have** $P34 : \text{Cong } (\text{Geos } (\text{Ang } (An\ l11\ o1\ h1))\ \text{add Emp})\ (\text{Geos } (\text{Ang } (An\ l11\ o1\ k1))\ \text{add Emp})$ **by** (*blast intro:Ang-rev Ang-trans*)
from $P1$ **have** $P35 : \text{Def } (\text{Ang } (An\ l1\ o1\ h1))$ **by** *simp*
from $P26\ P35$ **have** $P36 : \text{Ang-inside } (An\ l11\ o1\ h1)\ l1 \implies \text{Ang-inside } (An\ l1\ o1\ k1)\ l11$ **by** (*simp add:Ang-complementary-inside*)
have $P37 : \text{Cong } (\text{Geos } (\text{Ang } (An\ l11\ o1\ k1))\ \text{add Emp})\ (\text{Geos } (\text{Ang } (An\ l11\ o1\ k1))\ \text{add Emp})$ **by** *simp*
from $P30\ P36\ P37$ **have** $P38 : \text{Ang-inside } (An\ l11\ o1\ h1)\ l1 \implies \text{Gr } (\text{Geos } (\text{Ang } (An\ l1\ o1\ k1))\ \text{add Emp})\ (\text{Geos } (\text{Ang } (An\ l11\ o1\ k1))\ \text{add Emp})$
by (*simp add:Ang-greater-def*)
from $P1\ P7\ P21$ **have** $P39 : \text{Ang-inside } (An\ l11\ o1\ h1)\ l1 \implies \text{Gr } (\text{Geos } (\text{Ang } (An\ l11\ o1\ h1))\ \text{add Emp})\ (\text{Geos } (\text{Ang } (An\ l1\ o1\ k1))\ \text{add Emp})$
by (*blast intro:Ang-Gr-trans-Gr-Eq*)
from $P1\ P7\ P33\ P38\ P39$ **have** $P40 : \text{Ang-inside } (An\ l11\ o1\ h1)\ l1 \implies \text{Gr } (\text{Geos } (\text{Ang } (An\ l11\ o1\ h1))\ \text{add Emp})\ (\text{Geos } (\text{Ang } (An\ l11\ o1\ k1))\ \text{add Emp})$
by (*blast intro:Ang-Gr-trans-Gr-Gr*)
from $P7\ P33$ **have** $P41 :$
 $\text{Cong } (\text{Geos } (\text{Ang } (An\ l11\ o1\ h1))\ \text{add Emp})\ (\text{Geos } (\text{Ang } (An\ l11\ o1\ k1))\ \text{add Emp})$
 $\wedge \neg \text{Gr } (\text{Geos } (\text{Ang } (An\ l11\ o1\ h1))\ \text{add Emp})\ (\text{Geos } (\text{Ang } (An\ l11\ o1\ k1))\ \text{add Emp})$
 $\wedge \neg \text{Gr } (\text{Geos } (\text{Ang } (An\ l11\ o1\ k1))\ \text{add Emp})\ (\text{Geos } (\text{Ang } (An\ l11\ o1\ h1))\ \text{add Emp})$
 $\vee \neg \text{Cong } (\text{Geos } (\text{Ang } (An\ l11\ o1\ h1))\ \text{add Emp})\ (\text{Geos } (\text{Ang } (An\ l11\ o1\ k1))\ \text{add Emp})$
 $\wedge \text{Gr } (\text{Geos } (\text{Ang } (An\ l11\ o1\ h1))\ \text{add Emp})\ (\text{Geos } (\text{Ang } (An\ l11\ o1\ k1))\ \text{add Emp})$
 $\wedge \neg \text{Gr } (\text{Geos } (\text{Ang } (An\ l11\ o1\ k1))\ \text{add Emp})\ (\text{Geos } (\text{Ang } (An\ l11\ o1\ h1))\ \text{add Emp})$
 $\vee \neg \text{Cong } (\text{Geos } (\text{Ang } (An\ l11\ o1\ h1))\ \text{add Emp})\ (\text{Geos } (\text{Ang } (An\ l11\ o1\ k1))\ \text{add Emp})$
 $\wedge \neg \text{Gr } (\text{Geos } (\text{Ang } (An\ l11\ o1\ h1))\ \text{add Emp})\ (\text{Geos } (\text{Ang } (An\ l11\ o1\ k1))\ \text{add Emp})$

\wedge *Gr* (*Geos* (*Ang* (*An l11 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An l11 o1 h1*)) *add Emp*) **by** (*simp add:Ang-relation-case-fact*)
from *P40 P41* **have** *P42* : *Ang-inside* (*An l11 o1 h1*) *l1* \implies
 \neg *Cong* (*Geos* (*Ang* (*An l11 o1 h1*)) *add Emp*) (*Geos* (*Ang* (*An l11 o1 k1*)) *add Emp*) **by** *blast*
from *P34 P42* **have** *P43* : \neg *Ang-inside* (*An l11 o1 h1*) *l1* **by** *blast*
have *P44* : *Cong* (*Geos* (*Ang* (*An l11 o1 h1*)) *add Emp*) (*Geos* (*Ang* (*An l11 o1 h1*)) *add Emp*) **by** *simp*
from *P4 P44* **have** *P45* : *Ang-inside* (*An l1 o1 h1*) *l11* \implies
Gr (*Geos* (*Ang* (*An l1 o1 h1*)) *add Emp*) (*Geos* (*Ang* (*An l11 o1 h1*)) *add Emp*) **by** (*simp add:Ang-greater-def*)
from *P1 P7* **have** *P46* : *Ang-inside* (*An l1 o1 h1*) *l11* \implies *Ang-inside* (*An l11 o1 k1*) *l1* **by** (*simp add:Ang-complementary-inside*)
have *P47* : *Cong* (*Geos* (*Ang* (*An l1 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An l1 o1 k1*)) *add Emp*) **by** *simp*
from *P29 P46 P47* **have** *P48* : *Ang-inside* (*An l1 o1 h1*) *l11* \implies
Gr (*Geos* (*Ang* (*An l11 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An l1 o1 k1*)) *add Emp*) **by** (*simp add:Ang-greater-def*)
from *P1 P33 P48* **have** *P49* : *Ang-inside* (*An l1 o1 h1*) *l11* \implies
Gr (*Geos* (*Ang* (*An l11 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An l1 o1 h1*)) *add Emp*) **by** (*blast intro:Ang-Gr-trans-Gr-Eq Ang-rev*)
from *P1 P7 P33 P45 P49* **have** *P50* : *Ang-inside* (*An l1 o1 h1*) *l11* \implies
Gr (*Geos* (*Ang* (*An l11 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An l11 o1 h1*)) *add Emp*) **by** (*blast intro:Ang-Gr-trans-Gr-Gr*)
from *P41 P50* **have** *P51* : *Ang-inside* (*An l1 o1 h1*) *l11* \implies
 \neg *Cong* (*Geos* (*Ang* (*An l11 o1 h1*)) *add Emp*) (*Geos* (*Ang* (*An l11 o1 k1*)) *add Emp*) **by** *blast*
from *P34 P51* **have** *P52* : \neg *Ang-inside* (*An l1 o1 h1*) *l11* **by** *blast*
from *P18 P43 P52* **show** *Cong* (*Geos* (*Ang* (*An l1 o1 h1*)) *add Emp*) (*Geos* (*Ang* (*An l2 o2 h2*)) *add Emp*) **by** *blast*
qed

lemma (in *Congruence-Rule*) *Ang-external-Gr-lemma1* :

assumes *N* :

Def (*Tri* (*Tr A B C*))

Bet-Point (*Se B D*) *A*

shows \neg *Cong* (*Geos* (*Ang* (*An C A D*)) *add Emp*) (*Geos* (*Ang* (*An A C B*)) *add Emp*)

proof

assume *W* : *Cong* (*Geos* (*Ang* (*An C A D*)) *add Emp*) (*Geos* (*Ang* (*An A C B*)) *add Emp*)

from *N* **have** *P1* : *Line-on* (*Li B D*) *A* **by** (*simp add:Line-Bet-on*)

have *P2* : *Line-on* (*Li B D*) *D* **by** (*simp add:Line-on-rule*)

from *N* **have** \neg *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) **by** (*simp add:Bet-Point-def*)

then **have** *P3* : \neg *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) **by** (*blast intro:Eq-rev*)

from *N* **have** *P4* : \neg *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) **by** (*simp add:Tri-def*)

then have $P5 : \neg \text{Eq} (\text{Geos} (\text{Poi } C) \text{ add Emp}) (\text{Geos} (\text{Poi } B) \text{ add Emp})$ **by**
(blast intro:Eq-rev)
from $P1 P2 P3 P5$ **have** $\exists p. \text{Eq} (\text{Geos} (\text{Seg} (\text{Se } C B)) \text{ add Emp}) (\text{Geos} (\text{Seg} (\text{Se } A p)) \text{ add Emp})$
 $\wedge \neg \text{Bet-Point} (\text{Se } p D) A \wedge \text{Line-on} (\text{Li } B D) p \wedge \neg \text{Eq} (\text{Geos} (\text{Poi } A) \text{ add Emp}) (\text{Geos} (\text{Poi } p) \text{ add Emp})$ **by** *(simp add:Seg-move-sameside)*
then obtain $D1 :: \text{Point}$ **where** $P6 : \text{Eq} (\text{Geos} (\text{Seg} (\text{Se } C B)) \text{ add Emp}) (\text{Geos} (\text{Seg} (\text{Se } A D1)) \text{ add Emp})$
 $\wedge \neg \text{Bet-Point} (\text{Se } D1 D) A \wedge \text{Line-on} (\text{Li } B D) D1 \wedge \neg \text{Eq} (\text{Geos} (\text{Poi } A) \text{ add Emp}) (\text{Geos} (\text{Poi } D1) \text{ add Emp})$ **by** *blast*
have $P7 : \text{Line-on} (\text{Li } A D) A$ **by** *(simp add:Line-on-rule)*
from N **have** $P8 : \text{Line-on} (\text{Li } A D) B$ **by** *(simp add:Line-Bet-on)*
have $P9 : \text{Line-on} (\text{Li } A B) A$ **by** *(simp add:Line-on-rule)*
have $P10 : \text{Line-on} (\text{Li } A B) B$ **by** *(simp add:Line-on-rule)*
from N **have** $P11 : \neg \text{Eq} (\text{Geos} (\text{Poi } A) \text{ add Emp}) (\text{Geos} (\text{Poi } B) \text{ add Emp})$ **by**
(simp add:Tri-def)
from $P7 P8 P9 P10 P11$ **have** $\text{Eq} (\text{Geos} (\text{Lin} (\text{Li } A D)) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li } A B)) \text{ add Emp})$ **by** *(simp add:Line-unique)*
then have $P12 : \text{Line-on} (\text{Li } A D) C \implies \text{Line-on} (\text{Li } A B) C$ **by** *(simp add:Line-on-trans)*
from N **have** $P13 : \neg \text{Line-on} (\text{Li } A B) C$ **by** *(simp add:Tri-def-Line)*
from $P12 P13$ **have** $P14 : \neg \text{Line-on} (\text{Li } A D) C$ **by** *blast*
from $P3 P14$ **have** $\text{Def} (\text{Ang} (\text{An } A D C))$ **by** *(simp add:Ang-simple-def)*
then have $P15 : \text{Def} (\text{Ang} (\text{An } C A D))$ **by** *(blast intro:Ang-def-rev Ang-def-inv)*
have $P16 : \text{Line-on} (\text{Li } A C) C$ **by** *(simp add:Line-on-rule)*
have $P17 : \neg \text{Bet-Point} (\text{Se } C C) A$ **by** *(simp add:Bet-end-Point)*
have $P18 : \text{Line-on} (\text{Li } A D) D$ **by** *(simp add:Line-on-rule)*
from $P1 P2 P3 P7 P18$ **have** $P19 : \text{Eq} (\text{Geos} (\text{Lin} (\text{Li } B D)) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li } A D)) \text{ add Emp})$ **by** *(simp add:Line-unique)*
from $P6 P19$ **have** $P20 : \text{Line-on} (\text{Li } A D) D1$ **by** *(simp add:Line-on-trans)*
from $P6$ **have** $P21 : \neg \text{Bet-Point} (\text{Se } D D1) A$ **by** *(blast intro:Bet-rev)*
from N **have** $\neg \text{Eq} (\text{Geos} (\text{Poi } C) \text{ add Emp}) (\text{Geos} (\text{Poi } A) \text{ add Emp})$ **by** *(simp add:Tri-def)*
then have $P22 : \neg \text{Eq} (\text{Geos} (\text{Poi } A) \text{ add Emp}) (\text{Geos} (\text{Poi } C) \text{ add Emp})$ **by**
(blast intro:Eq-rev)
from $P6 P15 P16 P17 P20 P21 P22$ **have** $P23 :$
 $\text{Eq} (\text{Geos} (\text{Ang} (\text{An } C A D)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } C A D1)) \text{ add Emp})$
 $\wedge \text{Def} (\text{Ang} (\text{An } C A D1))$ **by** *(simp add:Ang-Point-swap)*
have $P24 : \text{Cong} (\text{Geos} (\text{Ang} (\text{An } C A D1)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } C A D1)) \text{ add Emp})$ **by** *simp*
from $P23 P24$ **have** $P25 : \text{Cong} (\text{Geos} (\text{Ang} (\text{An } C A D)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } C A D1)) \text{ add Emp})$ **by** *(blast intro:Ang-weektrans)*
from $P23$ **have** $P26 : \text{Def} (\text{Tri} (\text{Tr } A C D1))$ **by** *(blast intro:Ang-to-Tri Tri-def-rev Tri-def-trans)*
from N **have** $P27 : \text{Def} (\text{Tri} (\text{Tr } C A B))$ **by** *(blast intro:Ang-to-Tri Tri-def-rev Tri-def-trans)*
have $P28 : \text{Eq} (\text{Geos} (\text{Seg} (\text{Se } A C)) \text{ add Emp}) (\text{Geos} (\text{Seg} (\text{Se } C A)) \text{ add Emp})$ **by** *(simp add:Seg-rev)*
from $P6$ **have** $P29 : \text{Eq} (\text{Geos} (\text{Seg} (\text{Se } A D1)) \text{ add Emp}) (\text{Geos} (\text{Seg} (\text{Se } C B))$

add Emp) **by** (*simp add:Eq-rev*)
from *P29 P26 P27 P28* **have**
Cong (Geos (Ang (An C A D1)) add Emp) (Geos (Ang (An A C B)) add Emp)
 \implies
Cong (Geos (Tri (Tr A C D1)) add Emp) (Geos (Tri (Tr C A B)) add Emp)
by (*simp add:Tri-SAS*)
then have *P29 : Cong (Geos (Ang (An C A D1)) add Emp) (Geos (Ang (An A C B)) add Emp) \implies*
Cong (Geos (Ang (An D1 C A)) add Emp) (Geos (Ang (An B A C)) add Emp)
by (*simp add:Tri-Cong-def*)
from *N* **have** *P30 : Def (Ang (An A C B))* **by** (*blast intro:Tri-to-Ang Ang-def-inv*)
from *W P15 P23 P25 P30* **have** *P31 : Cong (Geos (Ang (An C A D1)) add Emp) (Geos (Ang (An A C B)) add Emp)* **by** (*blast intro:Ang-trans Ang-rev*)
from *P29 P31* **have** *P32 : Cong (Geos (Ang (An D1 C A)) add Emp) (Geos (Ang (An B A C)) add Emp)* **by** *simp*
have *P33 : Line-on (Li B C) B* **by** (*simp add:Line-on-rule*)
have *P34 : Line-on (Li B C) C* **by** (*simp add:Line-on-rule*)
from *P4 P33 P34* **have** $\exists p. \text{Bet-Point (Se B p) C} \wedge \text{Line-on (Li B C) p}$ **by**
(*simp add:Bet-extension*)
then obtain *E :: Point* **where** *P35 : Bet-Point (Se B E) C* \wedge *Line-on (Li B C) E* **by** *blast*
have *P36 : Cong (Geos (Ang (An C A D)) add Emp) (Geos (Ang (An D A C)) add Emp)* **by** (*simp add:Ang-roll*)
from *P15* **have** *P37 : Def (Ang (An D A C))* **by** (*simp add:Ang-def-rev*)
have *P38 : Cong (Geos (Ang (An A C B)) add Emp) (Geos (Ang (An B C A)) add Emp)* **by** (*simp add:Ang-roll*)
from *P30* **have** *P39 : Def (Ang (An B C A))* **by** (*simp add:Ang-def-rev*)
from *W P15 P30 P36 P37 P38* **have** *P40 : Cong (Geos (Ang (An D A C)) add Emp) (Geos (Ang (An A C B)) add Emp)* **by** (*blast intro:Ang-trans Ang-rev*)
from *P30 P37 P38 P39 P40* **have** *P41 : Cong (Geos (Ang (An D A C)) add Emp) (Geos (Ang (An B C A)) add Emp)* **by** (*blast intro:Ang-trans Ang-rev*)
from *N* **have** *P42 : Bet-Point (Se D B) A* **by** (*simp add:Bet-rev*)
from *P35 P37 P39 P41 P42* **have** *P43 : Cong (Geos (Ang (An C A B)) add Emp) (Geos (Ang (An A C E)) add Emp)* **by** (*simp add:Ang-complementary*)
have *P44 : Cong (Geos (Ang (An B A C)) add Emp) (Geos (Ang (An C A B)) add Emp)* **by** (*simp add:Ang-roll*)
from *P39* **have** *P45 : Def (Ang (An B A C))* **by** (*simp add:Ang-def-inv*)
then have *P46 : Def (Ang (An C A B))* **by** (*simp add:Ang-def-rev*)
from *P23* **have** *P47 : Def (Ang (An D1 C A))* **by** (*blast intro:Ang-def-rev Ang-def-inv*)
from *P32 P44 P45 P46 P47* **have** *P48 : Cong (Geos (Ang (An D1 C A)) add Emp) (Geos (Ang (An C A B)) add Emp)* **by** (*blast intro:Ang-trans Ang-rev*)
from *P35* **have** *P49 : Bet-Point (Se B E) C* **by** *simp*
then have *P50 : $\neg \text{Eq (Geos (Poi E) add Emp) (Geos (Poi C) add Emp)}$* **by**
(*simp add:Bet-Point-def*)
from *P35* **have** *P51 : Line-on (Li B C) E* **by** *simp*
from *P16 P34 P50 P51* **have** *P52 : Line-on (Li A C) E \implies*
Eq (Geos (Lin (Li B C)) add Emp) (Geos (Lin (Li A C)) add Emp) **by** (*simp add:Line-unique*)

from $P33$ $P52$ **have** $P53 : \text{Line-on } (Li\ A\ C)\ E \implies \text{Line-on } (Li\ A\ C)\ B$ **by**
(simp add:Line-on-trans)
from N **have** $Def\ (Tri\ (Tr\ A\ C\ B))$ **by** *(blast intro:Tri-def-rev Tri-def-trans)*
then have $P54 : \neg \text{Line-on } (Li\ A\ C)\ B$ **by** *(simp add:Tri-def-Line)*
from $P53$ $P54$ **have** $P55 : \neg \text{Line-on } (Li\ A\ C)\ E$ **by** *blast*
from $P22$ $P55$ **have** $P56 : Def\ (Ang\ (An\ A\ C\ E))$ **by** *(simp add:Ang-simple-def)*
from $P43$ $P46$ $P47$ $P48$ $P56$ **have** $P57 : Cong\ (Geos\ (Ang\ (An\ D1\ C\ A))\ add\ Emp)\ (Geos\ (Ang\ (An\ A\ C\ E))\ add\ Emp)$ **by** *(blast intro:Ang-trans Ang-rev)*
then have $P58 : Cong\ (Geos\ (Ang\ (An\ A\ C\ E))\ add\ Emp)\ (Geos\ (Ang\ (An\ D1\ C\ A))\ add\ Emp)$ **by** *(simp add:Ang-rev)*
have $P59 : Cong\ (Geos\ (Ang\ (An\ A\ C\ E))\ add\ Emp)\ (Geos\ (Ang\ (An\ E\ C\ A))\ add\ Emp)$ **by** *(simp add:Ang-roll)*
have $P60 : \text{Line-on } (Li\ C\ A)\ A$ **by** *(simp add:Line-on-rule)*
have $\text{Line-on } (Li\ B\ D)\ B$ **by** *(simp add:Line-on-rule)*
then have $P61 : Eq\ (Geos\ (Lin\ (Li\ B\ D))\ add\ Emp)\ (Geos\ (Lin\ (Li\ C\ A))\ add\ Emp) \implies \text{Line-on } (Li\ C\ A)\ B$ **by** *(simp add:Line-on-trans)*
from N **have** $P62 : \neg \text{Line-on } (Li\ C\ A)\ B$ **by** *(simp add:Tri-def-Line)*
from $P61$ $P62$ **have** $P63 : \neg Eq\ (Geos\ (Lin\ (Li\ B\ D))\ add\ Emp)\ (Geos\ (Lin\ (Li\ C\ A))\ add\ Emp)$ **by** *blast*
from N $P60$ $P63$ **have** $P64 : \text{Plane-diffside } (Li\ C\ A)\ B\ D$ **by** *(simp add:Plane-Bet-diffside)*
then have $P65 : Eq\ (Geos\ (Poi\ D)\ add\ Emp)\ (Geos\ (Poi\ D1)\ add\ Emp) \implies$
 $\text{Plane-diffside } (Li\ C\ A)\ B\ D1$ **by** *(simp add:Point-Eq)*
from $P6$ **have** $P66 : \text{Line-on } (Li\ B\ D)\ D1$ **by** *simp*
from $P6$ **have** $P67 : \neg Eq\ (Geos\ (Poi\ D1)\ add\ Emp)\ (Geos\ (Poi\ A)\ add\ Emp)$
by *(blast intro:Eq-rev)*
from $P1$ $P2$ $P3$ $P66$ $P67$ **have** $P68 : \neg Eq\ (Geos\ (Poi\ D)\ add\ Emp)\ (Geos\ (Poi\ D1)\ add\ Emp) \implies$
 $\text{Bet-Point } (Se\ D\ A)\ D1 \vee \text{Bet-Point } (Se\ A\ D1)\ D \vee \text{Bet-Point } (Se\ D1\ D)\ A$
by *(simp add:Bet-case)*
from $P19$ **have** $P69 : Eq\ (Geos\ (Lin\ (Li\ A\ D))\ add\ Emp)\ (Geos\ (Lin\ (Li\ C\ A))\ add\ Emp) \implies$
 $Eq\ (Geos\ (Lin\ (Li\ B\ D))\ add\ Emp)\ (Geos\ (Lin\ (Li\ C\ A))\ add\ Emp)$ **by** *(blast intro:Eq-trans)*
from $P63$ $P69$ **have** $P70 : \neg Eq\ (Geos\ (Lin\ (Li\ A\ D))\ add\ Emp)\ (Geos\ (Lin\ (Li\ C\ A))\ add\ Emp)$ **by** *blast*
from $P60$ $P70$ **have** $P71 : \text{Bet-Point } (Se\ A\ D)\ D1 \implies \text{Plane-sameside } (Li\ C\ A)\ D\ D1$ **by** *(simp add:Plane-Bet-sameside Plane-sameside-rev)*
have $\text{Line-on } (Li\ A\ D1)\ D1$ **by** *(simp add:Line-on-rule)*
then have $P72 : Eq\ (Geos\ (Lin\ (Li\ A\ D1))\ add\ Emp)\ (Geos\ (Lin\ (Li\ C\ A))\ add\ Emp) \implies$
 $\text{Line-on } (Li\ C\ A)\ D1$ **by** *(simp add:Line-on-trans)*
from $P23$ **have** $Def\ (Tri\ (Tr\ C\ A\ D1))$ **by** *(simp add:Ang-to-Tri)*
then have $P73 : \neg \text{Line-on } (Li\ C\ A)\ D1$ **by** *(simp add:Tri-def-Line)*
from $P72$ $P73$ **have** $P74 : \neg Eq\ (Geos\ (Lin\ (Li\ A\ D1))\ add\ Emp)\ (Geos\ (Lin\ (Li\ C\ A))\ add\ Emp)$ **by** *blast*
from $P60$ $P74$ **have** $P75 : \text{Bet-Point } (Se\ A\ D1)\ D \implies \text{Plane-sameside } (Li\ C\ A)\ D\ D1$ **by** *(simp add:Plane-Bet-sameside)*
from $P6$ $P64$ $P68$ $P71$ $P75$ **have** $P76 : \neg Eq\ (Geos\ (Poi\ D)\ add\ Emp)\ (Geos\ (Poi\ D1)\ add\ Emp) \implies$

Plane-diffside (Li C A) B D1 **by** (*blast intro:Plane-trans Plane-diffside-rev Bet-rev*)
from *P65 P76* **have** *P77 : Plane-diffside (Li C A) B D1* **by** *blast*
have *P78 : Line-on (Li C A) C* **by** (*simp add:Line-on-rule*)
have *Line-on (Li B E) B* **by** (*simp add:Line-on-rule*)
then have *P79 : Eq (Geos (Lin (Li B E)) add Emp) (Geos (Lin (Li C A)) add Emp)* \implies *Line-on (Li C A) B* **by** (*simp add:Line-on-trans*)
from *P62 P79* **have** *P80 : \neg Eq (Geos (Lin (Li B E)) add Emp) (Geos (Lin (Li C A)) add Emp)* **by** *blast*
from *P49 P78 P80* **have** *P81 : Plane-diffside (Li C A) B E* **by** (*simp add:Plane-Bet-diffside*)
from *P51* **have** *P82 : Eq (Geos (Poi D1) add Emp) (Geos (Poi E) add Emp)*
 \implies *Line-on (Li B C) D1* **by** (*blast intro:Eq-rev Point-Eq*)
from *P77* **have** $\exists p. \text{Bet-Point (Se B D1) } p \wedge \text{Line-on (Li C A) } p$
 $\wedge \neg \text{Line-on (Li C A) B} \wedge \neg \text{Line-on (Li C A) D1}$ **by** (*simp add:Plane-diffside-def*)
then obtain *F :: Point* **where** *Bet-Point (Se B D1) F* **by** *blast*
then have *P83 : \neg Eq (Geos (Poi B) add Emp) (Geos (Poi D1) add Emp)* **by**
(*simp add:Bet-Point-def*)
from *P8 P20 P33 P82 P83* **have** *P84 : Eq (Geos (Poi D1) add Emp) (Geos (Poi E) add Emp)*
 \implies *Eq (Geos (Lin (Li A D)) add Emp) (Geos (Lin (Li B C)) add Emp)* **by** (*simp add:Line-unique*)
from *P7 P84* **have** *P85 : Eq (Geos (Poi D1) add Emp) (Geos (Poi E) add Emp)*
 \implies *Line-on (Li B C) A* **by** (*simp add:Line-on-trans*)
from *N* **have** *P86 : \neg Line-on (Li B C) A* **by** (*simp add:Tri-def-Line*)
from *P85 P86* **have** *P87 : \neg Eq (Geos (Poi D1) add Emp) (Geos (Poi E) add Emp)* **by** *blast*
from *P77 P81 P87* **have** *P88 : Plane-sameside (Li C A) D1 E* **by** (*simp add:Plane-trans-inv*)
from *P58 P59 P88* **have** *P89 : Eq (Geos (Lin (Li D1 C)) add Emp) (Geos (Lin (Li E C)) add Emp) \wedge \neg Bet-Point (Se D1 E) C* **by** (*simp add:Ang-move-unique*)
from *P49* **have** *P90 : Line-on (Li E C) B* **by** (*simp add:Line-Bet-on*)
from *P89 P90* **have** *P91 : Line-on (Li D1 C) B* **by** (*blast intro:Eq-rev Line-on-trans*)
have *P92 : Line-on (Li D1 C) D1* **by** (*simp add:Line-on-rule*)
from *P8 P20 P83 P91 P92* **have** *P93 : Eq (Geos (Lin (Li A D)) add Emp) (Geos (Lin (Li D1 C)) add Emp)* **by** (*simp add:Line-unique*)
from *P89 P93* **have** *P94 : Eq (Geos (Lin (Li A D)) add Emp) (Geos (Lin (Li E C)) add Emp)* **by** (*blast intro:Eq-trans*)
from *P7 P94* **have** *P95 : Line-on (Li E C) A* **by** (*simp add:Line-on-trans*)
from *P56* **have** *Def (Tri (Tr E C A))* **by** (*blast intro:Ang-to-Tri Tri-def-rev Tri-def-trans*)
then have *P96 : \neg Line-on (Li E C) A* **by** (*simp add:Tri-def-Line*)
from *P95 P96* **show** *False* **by** *blast*
qed

lemma (*in Congruence-Rule*) *Ang-external-Gr-lemma2* :

assumes *N* :

Def (Tri (Tr A B C))

Bet-Point (Se B D) A

shows $\neg \text{Gr (Geos (Ang (An A C B)) add Emp) (Geos (Ang (An C A D)) add$

Emp)
proof
assume $W : Gr (Geos (Ang (An A C B)) add Emp) (Geos (Ang (An C A D)) add Emp)$
from N **have** $\neg Eq (Geos (Poi D) add Emp) (Geos (Poi A) add Emp)$ **by** (*simp add:Bet-Point-def*)
then have $P1 : \neg Eq (Geos (Poi A) add Emp) (Geos (Poi D) add Emp)$ **by** (*blast intro:Eq-rev*)
have $P2 : Line-on (Li A D) A$ **by** (*simp add:Line-on-rule*)
from N **have** $P3 : Line-on (Li A D) B$ **by** (*simp add:Line-Bet-on*)
have $P4 : Line-on (Li A B) A$ **by** (*simp add:Line-on-rule*)
have $P5 : Line-on (Li A B) B$ **by** (*simp add:Line-on-rule*)
from N **have** $P6 : \neg Eq (Geos (Poi A) add Emp) (Geos (Poi B) add Emp)$ **by** (*simp add:Tri-def*)
from $P2 P3 P4 P5 P6$ **have** $Eq (Geos (Lin (Li A D)) add Emp) (Geos (Lin (Li A B)) add Emp)$ **by** (*simp add:Line-unique*)
then have $P7 : Line-on (Li A D) C \implies Line-on (Li A B) C$ **by** (*simp add:Line-on-trans*)
from N **have** $P8 : \neg Line-on (Li A B) C$ **by** (*simp add:Tri-def-Line*)
from $P7 P8$ **have** $P9 : \neg Line-on (Li A D) C$ **by** *blast*
from $P1 P9$ **have** $Def (Ang (An A D C))$ **by** (*simp add:Ang-simple-def*)
then have $P10 : Def (Ang (An C A D))$ **by** (*blast intro:Ang-def-rev Ang-def-inv*)
have $P11 : Cong (Geos (Ang (An A C B)) add Emp) (Geos (Ang (An B C A)) add Emp)$ **by** (*simp add:Ang-roll*)
from N **have** $P12 : Def (Ang (An A C B))$ **by** (*blast intro:Tri-to-Ang Ang-def-inv*)
then have $P13 : Def (Ang (An B C A))$ **by** (*simp add:Ang-def-rev*)
from $W P10 P11 P12 P13$ **have** $P14 : Gr (Geos (Ang (An B C A)) add Emp) (Geos (Ang (An C A D)) add Emp)$ **by** (*blast intro:Ang-Gr-trans-Eq-Gr Ang-rev*)
from N **have** $P15 : \neg Line-on (Li C A) B$ **by** (*simp add:Tri-def-Line*)
from N **have** $P16 : \neg Eq (Geos (Poi C) add Emp) (Geos (Poi A) add Emp)$ **by** (*simp add:Tri-def*)
from $P10 P15$ **have** $\exists p. Cong (Geos (Ang (An C A D)) add Emp) (Geos (Ang (An p C A)) add Emp)$
 $\wedge Plane-sameside (Li C A) p B$ **by** (*simp add:Ang-move-sameside*)
then obtain $B1 :: Point$ **where** $P17 : Cong (Geos (Ang (An C A D)) add Emp) (Geos (Ang (An B1 C A)) add Emp)$
 $\wedge Plane-sameside (Li C A) B1 B$ **by** *blast*
then have $P18 : \neg Line-on (Li C A) B1$ **by** (*simp add:Plane-sameside-def*)
from $P16 P18$ **have** $Def (Ang (An C A B1))$ **by** (*blast intro:Ang-simple-def Eq-rev*)
then have $P19 : Def (Ang (An B1 C A))$ **by** (*blast intro:Ang-def-rev Ang-def-inv*)
from N **have** $P20 : Def (Ang (An A C B))$ **by** (*blast intro:Tri-def-rev Tri-def-trans Tri-to-Ang*)
from $P17$ **have** $P21 : Cong (Geos (Ang (An C A D)) add Emp) (Geos (Ang (An B1 C A)) add Emp)$ **by** *simp*
from $P17$ **have** $P22 : Plane-sameside (Li C A) B1 B$ **by** *simp*
from $P10 P14 P19 P20 P21 P22$ **have** $Ang-inside (An B C A) B1$ **by** (*simp add:Ang-greater-def*)
then have $Plane-diffside (Li C B1) B A$ **by** (*simp add:Ang-inside-Planeside*)

then have $\exists p. \text{Bet-Point } (Se\ B\ A)\ p \wedge \text{Line-on } (Li\ C\ B1)\ p \wedge \neg \text{Line-on } (Li\ C\ B1)\ B \wedge \neg \text{Line-on } (Li\ C\ B1)\ A$ **by** (*simp add:Plane-diffside-def*)
then obtain $B2 :: \text{Point}$ **where** $P23 : \text{Bet-Point } (Se\ B\ A)\ B2 \wedge \text{Line-on } (Li\ C\ B1)\ B2$
 $\wedge \neg \text{Line-on } (Li\ C\ B1)\ B \wedge \neg \text{Line-on } (Li\ C\ B1)\ A$ **by** *blast*
then have $\text{Line-on } (Li\ B\ A)\ B2$ **by** (*simp add:Line-Bet-on*)
then have $P24 : \text{Eq } (Geos\ (Poi\ B2)\ \text{add}\ Emp)\ (Geos\ (Poi\ C)\ \text{add}\ Emp) \implies \text{Line-on } (Li\ B\ A)\ C$ **by** (*simp add:Point-Eq*)
from N **have** $Def\ (Tri\ (Tr\ B\ A\ C))$ **by** (*blast intro:Tri-def-rev Tri-def-trans*)
then have $P25 : \neg \text{Line-on } (Li\ B\ A)\ C$ **by** (*simp add:Tri-def-Line*)
from $P24\ P25$ **have** $P26 : \neg \text{Eq } (Geos\ (Poi\ C)\ \text{add}\ Emp)\ (Geos\ (Poi\ B2)\ \text{add}\ Emp)$ **by** (*blast intro:Eq-rev*)
from $P23$ **have** $P27 : \text{Bet-Point } (Se\ A\ B)\ B2$ **by** (*simp add:Bet-rev*)
have $P28 : \text{Line-on } (Li\ C\ A)\ A$ **by** (*simp add:Line-on-rule*)
from $P5$ **have** $P29 : \text{Eq } (Geos\ (Lin\ (Li\ A\ B))\ \text{add}\ Emp)\ (Geos\ (Lin\ (Li\ C\ A))\ \text{add}\ Emp) \implies \text{Line-on } (Li\ C\ A)\ B$ **by** (*simp add:Line-on-trans*)
from $P15\ P29$ **have** $P30 : \neg \text{Eq } (Geos\ (Lin\ (Li\ A\ B))\ \text{add}\ Emp)\ (Geos\ (Lin\ (Li\ C\ A))\ \text{add}\ Emp)$ **by** *blast*
from $P27\ P28\ P30$ **have** $P31 : \text{Plane-sameside } (Li\ C\ A)\ B\ B2$ **by** (*simp add:Plane-Bet-sameside Plane-sameside-rev*)
have $P32 : \text{Line-on } (Li\ C\ A)\ C$ **by** (*simp add:Line-on-rule*)
have $\text{Line-on } (Li\ B1\ B2)\ B1$ **by** (*simp add:Line-on-rule*)
then have $P33 : \text{Eq } (Geos\ (Lin\ (Li\ B1\ B2))\ \text{add}\ Emp)\ (Geos\ (Lin\ (Li\ C\ A))\ \text{add}\ Emp) \implies \text{Line-on } (Li\ C\ A)\ B1$ **by** (*simp add:Line-on-trans*)
from $P18\ P33$ **have** $P34 : \neg \text{Eq } (Geos\ (Lin\ (Li\ B1\ B2))\ \text{add}\ Emp)\ (Geos\ (Lin\ (Li\ C\ A))\ \text{add}\ Emp)$ **by** *blast*
from $P32\ P34$ **have** $\text{Bet-Point } (Se\ B1\ B2)\ C \implies \text{Plane-diffside } (Li\ C\ A)\ B1\ B2$ **by** (*simp add:Plane-Bet-diffside*)
then have $P35 : \text{Bet-Point } (Se\ B1\ B2)\ C \implies \neg \text{Plane-sameside } (Li\ C\ A)\ B1\ B2$ **by** (*simp add:Plane-diffside-not-sameside*)
from $P22\ P31$ **have** $P36 : \neg \text{Eq } (Geos\ (Poi\ B1)\ \text{add}\ Emp)\ (Geos\ (Poi\ B2)\ \text{add}\ Emp) \implies \text{Plane-sameside } (Li\ C\ A)\ B1\ B2$ **by** (*blast intro:Plane-sameside-trans Eq-rev*)
from $P35\ P36$ **have** $P37 : \neg \text{Eq } (Geos\ (Poi\ B1)\ \text{add}\ Emp)\ (Geos\ (Poi\ B2)\ \text{add}\ Emp) \implies \neg \text{Bet-Point } (Se\ B1\ B2)\ C$ **by** *blast*
have $P38 : \text{Eq } (Geos\ (Poi\ B1)\ \text{add}\ Emp)\ (Geos\ (Poi\ B2)\ \text{add}\ Emp) \implies \text{Bet-Point } (Se\ B1\ B2)\ C \implies \text{Bet-Point } (Se\ B2\ B2)\ C$ **by** (*simp add:Bet-Point-Eq*)
have $P39 : \neg \text{Bet-Point } (Se\ B2\ B2)\ C$ **by** (*simp add:Bet-end-Point*)
from $P38\ P39$ **have** $P40 : \text{Eq } (Geos\ (Poi\ B1)\ \text{add}\ Emp)\ (Geos\ (Poi\ B2)\ \text{add}\ Emp) \implies \neg \text{Bet-Point } (Se\ B1\ B2)\ C$ **by** *blast*
from $P37\ P40$ **have** $P41 : \neg \text{Bet-Point } (Se\ B1\ B2)\ C$ **by** *blast*
have $P42 : \neg \text{Bet-Point } (Se\ A\ A)\ C$ **by** (*simp add:Bet-end-Point*)
from $P16\ P19\ P23\ P26\ P28\ P41\ P42$ **have** $P43 : \text{Eq } (Geos\ (Ang\ (An\ B1\ C\ A))\ \text{add}\ Emp)\ (Geos\ (Ang\ (An\ B2\ C\ A))\ \text{add}\ Emp) \wedge \text{Def } (Ang\ (An\ B2\ C\ A))$ **by** (*simp add:Ang-Point-swap*)

from $P_{21} P_{43}$ **have** $P_{44} : \text{Cong} (\text{Geos} (\text{Ang} (\text{An } C A D)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } B_2 C A)) \text{ add Emp})$ **by** (*blast intro:Ang-weektrans Ang-rev Eq-rev*)
from P_{43} **have** $P_{45} : \text{Def} (\text{Tri} (\text{Tr } A B_2 C))$ **by** (*blast intro:Ang-to-Tri Tri-def-rev Tri-def-trans*)
from P_{27} **have** $P_{46} : \text{Bet-Point} (\text{Se } B A) B_2$ **by** (*simp add:Bet-rev*)
from $N P_{46}$ **have** $P_{47} : \text{Bet-Point} (\text{Se } B_2 D) A$ **by** (*blast intro:Bet-swap-134-234*)
from $P_{45} P_{47}$ **have** $P_{48} : \neg \text{Cong} (\text{Geos} (\text{Ang} (\text{An } C A D)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } A C B_2)) \text{ add Emp})$ **by** (*simp add:Ang-external-Gr-lemma1*)
have $P_{49} : \text{Cong} (\text{Geos} (\text{Ang} (\text{An } B_2 C A)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } A C B_2)) \text{ add Emp})$ **by** (*simp add:Ang-roll*)
from P_{43} **have** $P_{50} : \text{Def} (\text{Ang} (\text{An } A C B_2))$ **by** (*simp add:Ang-def-rev*)
from $P_{10} P_{43} P_{44} P_{49} P_{50}$ **have** $P_{51} : \text{Cong} (\text{Geos} (\text{Ang} (\text{An } C A D)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } A C B_2)) \text{ add Emp})$ **by** (*blast intro:Ang-trans Ang-rev*)
from $P_{48} P_{51}$ **show** *False* **by** *blast*
qed

Theorem22

theorem (*in Congruence-Rule*) *Ang-external-Gr* :

assumes

Def (*Tri* (*Tr* $A B C$))

Bet-Point (*Se* $B D$) A

shows

Gr (*Geos* (*Ang* (*An* $C A D$)) *add Emp*) (*Geos* (*Ang* (*An* $A C B$)) *add Emp*)

Gr (*Geos* (*Ang* (*An* $C A D$)) *add Emp*) (*Geos* (*Ang* (*An* $A B C$)) *add Emp*)

proof –

from *assms* **have** $\neg \text{Eq} (\text{Geos} (\text{Poi } D) \text{ add Emp}) (\text{Geos} (\text{Poi } A) \text{ add Emp})$ **by** (*simp add:Bet-Point-def*)

then **have** $P_1 : \neg \text{Eq} (\text{Geos} (\text{Poi } A) \text{ add Emp}) (\text{Geos} (\text{Poi } D) \text{ add Emp})$ **by** (*blast intro:Eq-rev*)

have $P_2 : \text{Line-on} (\text{Li } A D) A$ **by** (*simp add:Line-on-rule*)

from *assms* **have** $P_3 : \text{Line-on} (\text{Li } A D) B$ **by** (*simp add:Line-Bet-on*)

have $P_4 : \text{Line-on} (\text{Li } A B) A$ **by** (*simp add:Line-on-rule*)

have $P_5 : \text{Line-on} (\text{Li } A B) B$ **by** (*simp add:Line-on-rule*)

from *assms* **have** $P_6 : \neg \text{Eq} (\text{Geos} (\text{Poi } A) \text{ add Emp}) (\text{Geos} (\text{Poi } B) \text{ add Emp})$ **by** (*simp add:Tri-def*)

from $P_2 P_3 P_4 P_5 P_6$ **have** $\text{Eq} (\text{Geos} (\text{Lin} (\text{Li } A D)) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li } A B)) \text{ add Emp})$ **by** (*simp add:Line-unique*)

then **have** $P_7 : \text{Line-on} (\text{Li } A D) C \implies \text{Line-on} (\text{Li } A B) C$ **by** (*simp add:Line-on-trans*)

from *assms* **have** $P_8 : \neg \text{Line-on} (\text{Li } A B) C$ **by** (*simp add:Tri-def-Line*)

from $P_7 P_8$ **have** $P_9 : \neg \text{Line-on} (\text{Li } A D) C$ **by** *blast*

from $P_1 P_9$ **have** *Def* (*Ang* (*An* $A D C$)) **by** (*simp add:Ang-simple-def*)

then **have** $P_{10} : \text{Def} (\text{Ang} (\text{An } C A D))$ **by** (*blast intro:Ang-def-rev Ang-def-inv*)

from *assms* **have** $P_{11} : \text{Def} (\text{Ang} (\text{An } A C B))$ **by** (*simp add:Tri-to-Ang Ang-def-inv*)

from $P_{10} P_{11}$ **have** $P_{12} :$

Cong (*Geos* (*Ang* (*An* $C A D$)) *add Emp*) (*Geos* (*Ang* (*An* $A C B$)) *add Emp*)

\vee *Gr* (*Geos* (*Ang* (*An* $C A D$)) *add Emp*) (*Geos* (*Ang* (*An* $A C B$)) *add Emp*)

\vee *Gr* (*Geos* (*Ang* (*An* $A C B$)) *add Emp*) (*Geos* (*Ang* (*An* $C A D$)) *add Emp*)

by (*simp add:Ang-relation-case*)
from *assms* **have** $P13 : \neg \text{Cong} (\text{Geos} (\text{Ang} (\text{An } C \ A \ D)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } A \ C \ B)) \text{ add Emp})$ **by** (*simp add:Ang-external-Gr-lemma1*)
from *assms* **have** $P14 : \neg \text{Gr} (\text{Geos} (\text{Ang} (\text{An } A \ C \ B)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } C \ A \ D)) \text{ add Emp})$ **by** (*simp add:Ang-external-Gr-lemma2*)
from $P12 \ P13 \ P14$ **show** $P15 : \text{Gr} (\text{Geos} (\text{Ang} (\text{An } C \ A \ D)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } A \ C \ B)) \text{ add Emp})$ **by** *blast*
have $P16 : \text{Line-on} (\text{Li } C \ A) \ C$ **by** (*simp add:Line-on-rule*)
have $P17 : \text{Line-on} (\text{Li } C \ A) \ A$ **by** (*simp add:Line-on-rule*)
from *assms* **have** $P18 : \neg \text{Eq} (\text{Geos} (\text{Poi } C) \text{ add Emp}) (\text{Geos} (\text{Poi } A) \text{ add Emp})$
by (*simp add:Tri-def*)
from $P16 \ P17 \ P18$ **have** $\exists p. \text{Bet-Point} (\text{Se } C \ p) \ A \wedge \text{Line-on} (\text{Li } C \ A) \ p$ **by** (*simp add:Bet-extension*)
then obtain $E :: \text{Point}$ **where** $P19 : \text{Bet-Point} (\text{Se } C \ E) \ A$ **by** *blast*
from *assms* **have** $P20 : \text{Bet-Point} (\text{Se } D \ B) \ A$ **by** (*simp add:Bet-rev*)
from $P10 \ P19 \ P20$ **have** $P21 : \text{Cong} (\text{Geos} (\text{Ang} (\text{An } C \ A \ D)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } E \ A \ B)) \text{ add Emp})$ **by** (*simp add:Ang-vertical*)
have $P22 : \text{Eq} (\text{Geos} (\text{Ang} (\text{An } E \ A \ B)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } B \ A \ E)) \text{ add Emp})$ **by** (*simp add:Ang-roll*)
from $P21 \ P22$ **have** $P23 : \text{Cong} (\text{Geos} (\text{Ang} (\text{An } C \ A \ D)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } B \ A \ E)) \text{ add Emp})$ **by** (*blast intro:Ang-weektrans Ang-rev Eq-rev*)
from $P19$ **have** $P24 : \text{Line-on} (\text{Li } C \ E) \ A$ **by** (*simp add:Line-Bet-on*)
have $P25 : \text{Line-on} (\text{Li } C \ E) \ E$ **by** (*simp add:Line-on-rule*)
from $P19$ **have** $P26 : \neg \text{Eq} (\text{Geos} (\text{Poi } E) \text{ add Emp}) (\text{Geos} (\text{Poi } A) \text{ add Emp})$
by (*simp add:Bet-Point-def*)
from $P4 \ P24 \ P25 \ P26$ **have** $P27 : \text{Line-on} (\text{Li } A \ B) \ E \implies \text{Eq} (\text{Geos} (\text{Lin} (\text{Li } C \ E)) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li } A \ B)) \text{ add Emp})$ **by** (*simp add:Line-unique*)
have $P28 : \text{Line-on} (\text{Li } C \ E) \ C$ **by** (*simp add:Line-on-rule*)
from $P27 \ P28$ **have** $P29 : \text{Line-on} (\text{Li } A \ B) \ E \implies \text{Line-on} (\text{Li } A \ B) \ C$ **by** (*simp add:Line-on-trans*)
from $P8 \ P29$ **have** $P30 : \neg \text{Line-on} (\text{Li } A \ B) \ E$ **by** *blast*
from $P6 \ P30$ **have** $\text{Def} (\text{Ang} (\text{An } A \ B \ E))$ **by** (*simp add:Ang-simple-def*)
then have $P31 : \text{Def} (\text{Ang} (\text{An } B \ A \ E))$ **by** (*blast intro:Ang-def-rev Ang-def-inv*)
from $P11$ **have** $P32 : \text{Def} (\text{Tri} (\text{Tr } A \ C \ B))$ **by** (*simp add:Ang-to-Tri*)
from $P19 \ P32$ **have** $P33 : \neg \text{Cong} (\text{Geos} (\text{Ang} (\text{An } B \ A \ E)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } A \ B \ C)) \text{ add Emp})$ **by** (*simp add:Ang-external-Gr-lemma1*)
from $P19 \ P32$ **have** $P34 : \neg \text{Gr} (\text{Geos} (\text{Ang} (\text{An } A \ B \ C)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } B \ A \ E)) \text{ add Emp})$ **by** (*simp add:Ang-external-Gr-lemma2*)
from *assms* **have** $P35 : \text{Def} (\text{Ang} (\text{An } A \ B \ C))$ **by** (*simp add:Tri-to-Ang*)
from $P31 \ P35$ **have** $P36 :$
 $\text{Cong} (\text{Geos} (\text{Ang} (\text{An } B \ A \ E)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } A \ B \ C)) \text{ add Emp})$
 $\vee \text{Gr} (\text{Geos} (\text{Ang} (\text{An } B \ A \ E)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } A \ B \ C)) \text{ add Emp})$
 $\vee \text{Gr} (\text{Geos} (\text{Ang} (\text{An } A \ B \ C)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } B \ A \ E)) \text{ add Emp})$
by (*simp add:Ang-relation-case*)
from $P33 \ P34 \ P36$ **have** $P37 : \text{Gr} (\text{Geos} (\text{Ang} (\text{An } B \ A \ E)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } A \ B \ C)) \text{ add Emp})$ **by** *blast*
from $P10 \ P23 \ P31 \ P35 \ P37$ **show** $\text{Gr} (\text{Geos} (\text{Ang} (\text{An } C \ A \ D)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } A \ B \ C)) \text{ add Emp})$ **by** (*blast intro:Ang-Gr-trans-Eq-Gr*)

qed

lemma (in Congruence-Rule) Seg-not-Eq-move :

assumes

- $\neg Eq (Geos (Poi A1) add Emp) (Geos (Poi B1) add Emp)$
- $\neg Eq (Geos (Poi A2) add Emp) (Geos (Poi B2) add Emp)$
- $\neg Eq (Geos (Poi A2) add Emp) (Geos (Poi B3) add Emp)$
- $Line-on\ l1\ A2\ Line-on\ l1\ B2\ Line-on\ l1\ B3$
- $\neg Bet-Point (Se B3 B2) A2$
- $Eq (Geos (Seg (Se A1 B1)) add Emp) (Geos (Seg (Se A2 B3)) add Emp)$
- $\neg Eq (Geos (Seg (Se A1 B1)) add Emp) (Geos (Seg (Se A2 B2)) add Emp)$

shows

- $Bet-Point (Se B2 A2) B3 \wedge \neg Bet-Point (Se A2 B3) B2$
- $\vee \neg Bet-Point (Se B2 A2) B3 \wedge Bet-Point (Se A2 B3) B2$

proof –

from *assms* **have** $P1 : Eq (Geos (Seg (Se A2 B2)) add Emp) (Geos (Seg (Se A2 B3)) add Emp) \implies$

$Eq (Geos (Seg (Se A1 B1)) add Emp) (Geos (Seg (Se A2 B2)) add Emp)$ **by**
(*blast intro:Eq-trans Eq-rev*)

from *assms* $P1$ **have** $\neg Eq (Geos (Seg (Se A2 B2)) add Emp) (Geos (Seg (Se A2 B3)) add Emp)$ **by** *blast*

then **have** $P2 : \neg Eq (Geos (Poi B2) add Emp) (Geos (Poi B3) add Emp)$ **by**
(*simp add:Seg-not-Eq-Point*)

from *assms* **have** $P3 : \neg Eq (Geos (Poi B3) add Emp) (Geos (Poi A2) add Emp)$
by (*blast intro:Eq-rev*)

from *assms* **have** $P4 : Line-on\ l1\ A2$ **by** *simp*

from *assms* **have** $P5 : Line-on\ l1\ B2$ **by** *simp*

from *assms* **have** $P6 : Line-on\ l1\ B3$ **by** *simp*

from *assms* **have** $P7 : \neg Eq (Geos (Poi A2) add Emp) (Geos (Poi B2) add Emp)$ **by** *simp*

from $P2\ P3\ P4\ P5\ P6\ P7$ **have** $Bet-Point (Se A2 B3) B2 \vee Bet-Point (Se B3 B2) A2 \vee Bet-Point (Se B2 A2) B3$ **by** (*simp add:Bet-case*)

then **have** $P8 :$

$Bet-Point (Se A2 B3) B2 \wedge \neg Bet-Point (Se B3 B2) A2 \wedge \neg Bet-Point (Se B2 A2) B3$

$\vee \neg Bet-Point (Se A2 B3) B2 \wedge Bet-Point (Se B3 B2) A2 \wedge \neg Bet-Point (Se B2 A2) B3$

$\vee \neg Bet-Point (Se A2 B3) B2 \wedge \neg Bet-Point (Se B3 B2) A2 \wedge Bet-Point (Se B2 A2) B3$ **by** (*simp add:Bet-case-fact*)

from *assms* $P8$ **show** $Bet-Point (Se B2 A2) B3 \wedge \neg Bet-Point (Se A2 B3) B2$

$\vee \neg Bet-Point (Se B2 A2) B3 \wedge Bet-Point (Se A2 B3) B2$ **by** *blast*

qed

lemma (in Congruence-Rule) Tri-Seg-diagonal :

assumes

$Def (Tri (Tr A B C))$

$Bet-Point (Se B C) D$

$Eq (Geos (Seg (Se A C)) add Emp) (Geos (Seg (Se C D)) add Emp)$

shows

$Gr (Geos (Ang (An B A C)) add Emp) (Geos (Ang (An A B C)) add Emp)$
proof –
from *assms* **have** $P1 : \neg Line-on (Li A B) C$ **by** (*simp add:Tri-def-Line*)
have $Line-on (Li A C) C$ **by** (*simp add:Line-on-rule*)
then have $P2 : Eq (Geos (Lin (Li A C)) add Emp) (Geos (Lin (Li A B)) add Emp) \implies$
 $Line-on (Li A B) C$ **by** (*simp add:Line-on-trans*)
from $P1 P2$ **have** $P3 : \neg Eq (Geos (Lin (Li A B)) add Emp) (Geos (Lin (Li A C)) add Emp)$ **by** (*blast intro:Eq-rev*)
from *assms* **have** $P4 : \neg Eq (Geos (Poi A) add Emp) (Geos (Poi B) add Emp)$ **by** (*simp add:Tri-def*)
from *assms* **have** $\neg Eq (Geos (Poi C) add Emp) (Geos (Poi A) add Emp)$ **by** (*simp add:Tri-def*)
then have $P5 : \neg Eq (Geos (Poi A) add Emp) (Geos (Poi C) add Emp)$ **by** (*blast intro:Eq-rev*)
from *assms* $P3 P4 P5$ **have** $P6 : Ang-inside (An B A C) D$ **by** (*simp add:Ang-inside-Bet-Point*)
then have $Plane-sameside (Li A C) B D$ **by** (*simp add:Ang-inside-def*)
then have $P7 : Plane-sameside (Li A C) D B$ **by** (*simp add:Plane-sameside-rev*)
have $P8 : Cong (Geos (Ang (An D A C)) add Emp) (Geos (Ang (An D A C)) add Emp)$ **by** *simp*
from $P6 P7 P8$ **have** $P9 : Gr (Geos (Ang (An B A C)) add Emp) (Geos (Ang (An D A C)) add Emp)$ **by** (*simp add:Ang-greater-def*)
from *assms* **have** $P10 : \neg Eq (Geos (Poi C) add Emp) (Geos (Poi D) add Emp)$ **by** (*simp add:Bet-Point-def*)
from *assms* **have** $P11 : Line-on (Li C B) D$ **by** (*simp add:Line-Bet-on*)
from *assms* **have** $P12 : Def (Tri (Tr A C B))$ **by** (*blast intro:Tri-def-rev Tri-def-trans*)
from $P10 P11 P12$ **have** $Def (Tri (Tr A C D))$ **by** (*simp add:Tri-def-extension*)
then have $P13 : Def (Tri (Tr C A D))$ **by** (*blast intro:Tri-def-rev Tri-def-trans*)
have $P14 : Eq (Geos (Seg (Se A C)) add Emp) (Geos (Seg (Se C A)) add Emp)$ **by** (*simp add:Seg-rev*)
from *assms* $P14$ **have** $P15 : Eq (Geos (Seg (Se C A)) add Emp) (Geos (Seg (Se C D)) add Emp)$ **by** (*blast intro:Eq-rev Eq-trans*)
from $P13 P15$ **have** $P16 : Cong (Geos (Ang (An C A D)) add Emp) (Geos (Ang (An C D A)) add Emp)$ **by** (*simp add:Tri-isosceles*)
from *assms* **have** $\neg Eq (Geos (Poi D) add Emp) (Geos (Poi B) add Emp)$ **by** (*simp add:Bet-Point-def*)
then have $P17 : \neg Eq (Geos (Poi B) add Emp) (Geos (Poi D) add Emp)$ **by** (*blast intro:Eq-rev*)
from *assms* **have** $P18 : Line-on (Li B C) D$ **by** (*simp add:Line-Bet-on*)
from *assms* $P17 P18$ **have** $Def (Tri (Tr A B D))$ **by** (*simp add:Tri-def-extension*)
then have $P19 : Def (Tri (Tr D B A))$ **by** (*simp add:Tri-def-rev*)
from *assms* $P19$ **have** $P20 : Gr (Geos (Ang (An A D C)) add Emp) (Geos (Ang (An D B A)) add Emp)$ **by** (*simp add:Ang-external-Gr*)
from *assms* **have** $P21 : Def (Ang (An A B C))$ **by** (*simp add:Tri-to-Ang*)
have $P22 : Line-on (Li B A) A$ **by** (*simp add:Line-on-rule*)
have $P23 : \neg Bet-Point (Se A A) B$ **by** (*simp add:Bet-end-Point*)
from *assms* **have** $P24 : Line-on (Li B C) D$ **by** (*simp add:Line-Bet-on*)
from *assms* **have** $Inv (Bet-Point (Se C D) B)$ **by** (*simp add:Bet-iff*)

then have $P25 : \neg \text{Bet-Point } (Se\ C\ D)\ B$ **by** (*simp add:Inv-def*)
from $P4$ **have** $P26 : \neg \text{Eq } (Geos\ (Poi\ B)\ \text{add}\ Emp)\ (Geos\ (Poi\ A)\ \text{add}\ Emp)$
by (*blast intro:Eq-rev*)
from $P17\ P21\ P22\ P23\ P24\ P25\ P26$ **have** $P27 :$
 $\text{Eq } (Geos\ (Ang\ (An\ A\ B\ C))\ \text{add}\ Emp)\ (Geos\ (Ang\ (An\ A\ B\ D))\ \text{add}\ Emp)$
 $\wedge \text{Def } (Ang\ (An\ A\ B\ D))$ **by** (*simp add:Ang-Point-swap*)
have $P28 : \text{Cong } (Geos\ (Ang\ (An\ D\ B\ A))\ \text{add}\ Emp)\ (Geos\ (Ang\ (An\ A\ B\ D))\ \text{add}\ Emp)$ **by** (*simp add:Ang-roll*)
from $P27\ P28$ **have** $P29 : \text{Cong } (Geos\ (Ang\ (An\ A\ B\ C))\ \text{add}\ Emp)\ (Geos\ (Ang\ (An\ D\ B\ A))\ \text{add}\ Emp)$ **by** (*blast intro:Ang-weektrans Ang-rev*)
have $P30 : \text{Eq } (Geos\ (Ang\ (An\ C\ D\ A))\ \text{add}\ Emp)\ (Geos\ (Ang\ (An\ A\ D\ C))\ \text{add}\ Emp)$ **by** (*simp add:Ang-roll*)
from $P16\ P30$ **have** $P31 : \text{Cong } (Geos\ (Ang\ (An\ C\ A\ D))\ \text{add}\ Emp)\ (Geos\ (Ang\ (An\ A\ D\ C))\ \text{add}\ Emp)$ **by** (*blast intro:Ang-weektrans Ang-rev Eq-rev*)
from $P13$ **have** $P32 : \text{Def } (Ang\ (An\ C\ A\ D))$ **by** (*simp add:Tri-to-Ang*)
then have $P33 : \text{Def } (Ang\ (An\ A\ D\ C))$ **by** (*blast intro:Ang-def-rev Ang-def-inv*)
from $P27$ **have** $P34 : \text{Def } (Ang\ (An\ D\ B\ A))$ **by** (*blast intro:Ang-def-rev*)
from $P20\ P31\ P32\ P33\ P34$ **have** $P35 : \text{Gr } (Geos\ (Ang\ (An\ C\ A\ D))\ \text{add}\ Emp)\ (Geos\ (Ang\ (An\ D\ B\ A))\ \text{add}\ Emp)$ **by** (*blast intro:Ang-Gr-trans-Eq-Gr*)
from $P21\ P29\ P32\ P34\ P35$ **have** $P36 : \text{Gr } (Geos\ (Ang\ (An\ C\ A\ D))\ \text{add}\ Emp)\ (Geos\ (Ang\ (An\ A\ B\ C))\ \text{add}\ Emp)$ **by** (*blast intro:Ang-Gr-trans-Gr-Eq Ang-rev*)
have $P37 : \text{Cong } (Geos\ (Ang\ (An\ D\ A\ C))\ \text{add}\ Emp)\ (Geos\ (Ang\ (An\ C\ A\ D))\ \text{add}\ Emp)$ **by** (*simp add:Ang-roll*)
from $P32$ **have** $P38 : \text{Def } (Ang\ (An\ D\ A\ C))$ **by** (*simp add:Ang-def-rev*)
from *assms* **have** $P39 : \text{Def } (Ang\ (An\ B\ A\ C))$ **by** (*blast intro:Tri-to-Ang Ang-def-rev Ang-def-inv*)
from $P9\ P32\ P37\ P38\ P39$ **have** $P40 : \text{Gr } (Geos\ (Ang\ (An\ B\ A\ C))\ \text{add}\ Emp)\ (Geos\ (Ang\ (An\ C\ A\ D))\ \text{add}\ Emp)$ **by** (*blast intro:Ang-Gr-trans-Gr-Eq*)
from $P21\ P32\ P36\ P39\ P40$ **show** $\text{Gr } (Geos\ (Ang\ (An\ B\ A\ C))\ \text{add}\ Emp)\ (Geos\ (Ang\ (An\ A\ B\ C))\ \text{add}\ Emp)$ **by** (*blast intro:Ang-Gr-trans-Gr-Gr*)
qed

lemma (*in Congruence-Rule*) *Tri-Bet-Ang-Gr* :

assumes

$\text{Def } (Tri\ (Tr\ A\ B\ C))$

$\text{Bet-Point } (Se\ A\ C)\ D$

$\text{Eq } (Geos\ (Seg\ (Se\ A\ B))\ \text{add}\ Emp)\ (Geos\ (Seg\ (Se\ A\ C))\ \text{add}\ Emp)$

shows

$\text{Gr } (Geos\ (Ang\ (An\ A\ D\ B))\ \text{add}\ Emp)\ (Geos\ (Ang\ (An\ A\ B\ D))\ \text{add}\ Emp)$

proof –

from *assms* **have** $P1 : \text{Cong } (Geos\ (Ang\ (An\ A\ B\ C))\ \text{add}\ Emp)\ (Geos\ (Ang\ (An\ A\ C\ B))\ \text{add}\ Emp)$ **by** (*simp add:Tri-isosceles*)

from *assms* **have** $P2 : \text{Def } (Tri\ (Tr\ B\ C\ A))$ **by** (*simp add:Tri-def-trans*)

from *assms* **have** $P3 : \text{Line-on } (Li\ C\ A)\ D$ **by** (*simp add:Line-Bet-on*)

from *assms* **have** $P4 : \neg \text{Eq } (Geos\ (Poi\ C)\ \text{add}\ Emp)\ (Geos\ (Poi\ D)\ \text{add}\ Emp)$ **by** (*simp add:Bet-Point-def*)

from $P2\ P3\ P4$ **have** $\text{Def } (Tri\ (Tr\ B\ C\ D))$ **by** (*simp add:Tri-def-extension*)

then have $P5 : \text{Def } (Tri\ (Tr\ D\ C\ B))$ **by** (*blast intro:Tri-def-rev Tri-def-trans*)

from *assms* **have** $P6 : \text{Bet-Point } (Se\ C\ A)\ D$ **by** (*simp add:Bet-rev*)

from $P5\ P6$ **have** $P7 : Gr (Geos (Ang (An\ B\ D\ A))\ add\ Emp) (Geos (Ang (An\ D\ C\ B))\ add\ Emp)$ **by** (*simp add:Ang-external-Gr*)
from $P5$ **have** $P8 : Def (Ang (An\ D\ C\ B))$ **by** (*simp add:Tri-to-Ang*)
from *assms* **have** $P9 : Line-on (Li\ C\ D)\ A$ **by** (*simp add:Line-Bet-on*)
from *assms* **have** $Inv (Bet-Point (Se\ D\ A)\ C)$ **by** (*simp add:Bet-iff*)
then **have** $P10 : \neg Bet-Point (Se\ D\ A)\ C$ **by** (*simp add:Inv-def*)
have $P11 : Line-on (Li\ C\ B)\ B$ **by** (*simp add:Line-on-rule*)
have $P12 : \neg Bet-Point (Se\ B\ B)\ C$ **by** (*simp add:Bet-end-Point*)
from $P6$ **have** $P13 : \neg Eq (Geos (Poi\ C)\ add\ Emp) (Geos (Poi\ A)\ add\ Emp)$
by (*simp add:Bet-Point-def*)
from $P5$ **have** $P14 : \neg Eq (Geos (Poi\ C)\ add\ Emp) (Geos (Poi\ B)\ add\ Emp)$
by (*simp add:Tri-def*)
from $P8\ P9\ P10\ P11\ P12\ P13\ P14$ **have** $P15 : Eq (Geos (Ang (An\ D\ C\ B))\ add\ Emp) (Geos (Ang (An\ A\ C\ B))\ add\ Emp)$
 $\wedge Def (Ang (An\ A\ C\ B))$ **by** (*simp add:Ang-Point-swap*)
from $P1\ P15$ **have** $P16 : Cong (Geos (Ang (An\ A\ B\ C))\ add\ Emp) (Geos (Ang (An\ D\ C\ B))\ add\ Emp)$ **by** (*blast intro:Ang-weektrans Ang-rev*)
have $P17 : Cong (Geos (Ang (An\ B\ D\ A))\ add\ Emp) (Geos (Ang (An\ A\ D\ B))\ add\ Emp)$ **by** (*simp add:Ang-roll*)
from $P2$ **have** $P18 : Def (Tri (Tr\ B\ A\ C))$ **by** (*blast intro:Tri-def-rev Tri-def-trans*)
from *assms* **have** $P19 : Line-on (Li\ A\ C)\ D$ **by** (*simp add:Line-Bet-on*)
from $P6$ **have** $P20 : \neg Eq (Geos (Poi\ A)\ add\ Emp) (Geos (Poi\ D)\ add\ Emp)$
by (*simp add:Bet-Point-def*)
from $P18\ P19\ P20$ **have** $Def (Tri (Tr\ B\ A\ D))$ **by** (*simp add:Tri-def-extension*)
then **have** $P21 : Def (Ang (An\ B\ D\ A))$ **by** (*blast intro:Tri-to-Ang Ang-def-inv*)
then **have** $P22 : Def (Ang (An\ A\ D\ B))$ **by** (*simp add:Ang-def-rev*)
from $P7\ P8\ P17\ P21\ P22$ **have** $P23 : Gr (Geos (Ang (An\ A\ D\ B))\ add\ Emp) (Geos (Ang (An\ D\ C\ B))\ add\ Emp)$ **by** (*blast intro:Ang-Gr-trans-Eq-Gr Ang-rev*)
from *assms* **have** $P24 : Def (Ang (An\ A\ B\ C))$ **by** (*simp add:Tri-to-Ang*)
from $P8\ P16\ P22\ P23\ P24$ **have** $P25 : Gr (Geos (Ang (An\ A\ D\ B))\ add\ Emp) (Geos (Ang (An\ A\ B\ C))\ add\ Emp)$ **by** (*blast intro:Ang-Gr-trans-Gr-Eq Ang-rev*)
from $P24$ **have** $\neg Eq (Geos (Lin (Li\ B\ A))\ add\ Emp) (Geos (Lin (Li\ B\ C))\ add\ Emp)$ **by** (*simp add:Ang-def*)
then **have** $P26 : \neg Eq (Geos (Lin (Li\ B\ C))\ add\ Emp) (Geos (Lin (Li\ B\ A))\ add\ Emp)$ **by** (*blast intro:Eq-rev*)
from *assms* **have** $\neg Eq (Geos (Poi\ A)\ add\ Emp) (Geos (Poi\ B)\ add\ Emp)$ **by** (*simp add:Tri-def*)
then **have** $P27 : \neg Eq (Geos (Poi\ B)\ add\ Emp) (Geos (Poi\ A)\ add\ Emp)$ **by** (*blast intro:Eq-rev*)
from *assms* **have** $P28 : \neg Eq (Geos (Poi\ B)\ add\ Emp) (Geos (Poi\ C)\ add\ Emp)$
by (*simp add:Tri-def*)
from $P6\ P26\ P27\ P28$ **have** $P29 : Ang-inside (An\ C\ B\ A)\ D$ **by** (*simp add:Ang-inside-Bet-Point*)
then **have** $P30 : Plane-sameside (Li\ B\ A)\ D\ C$ **by** (*simp add:Ang-inside-def Plane-sameside-rev*)
have $P31 : Cong (Geos (Ang (An\ D\ B\ A))\ add\ Emp) (Geos (Ang (An\ D\ B\ A))\ add\ Emp)$ **by** *simp*
from $P29\ P30\ P31$ **have** $P32 : Gr (Geos (Ang (An\ C\ B\ A))\ add\ Emp) (Geos (Ang (An\ D\ B\ A))\ add\ Emp)$ **by** (*simp add:Ang-greater-def*)

have $P33 : Cong (Geos (Ang (An C B A)) add Emp) (Geos (Ang (An A B C)) add Emp)$ **by** (*simp add:Ang-roll*)
from $P24$ **have** $P34 : Def (Ang (An C B A))$ **by** (*simp add:Ang-def-rev*)
from $P22$ **have** $P35 : Def (Ang (An D B A))$ **by** (*blast intro:Ang-def-rev Ang-def-inv*)
from $P24 P32 P33 P34 P35$ **have** $P36 : Gr (Geos (Ang (An A B C)) add Emp) (Geos (Ang (An D B A)) add Emp)$ **by** (*blast intro:Ang-Gr-trans-Eq-Gr Ang-rev*)
have $P37 : Cong (Geos (Ang (An D B A)) add Emp) (Geos (Ang (An A B D)) add Emp)$ **by** (*simp add:Ang-roll*)
from $P35$ **have** $P38 : Def (Ang (An A B D))$ **by** (*simp add:Ang-def-rev*)
from $P24 P35 P36 P37 P38$ **have** $P39 : Gr (Geos (Ang (An A B C)) add Emp) (Geos (Ang (An A B D)) add Emp)$ **by** (*blast intro:Ang-Gr-trans-Gr-Eq Ang-rev*)
from $P22 P24 P25 P38 P39$ **show** $Gr (Geos (Ang (An A D B)) add Emp) (Geos (Ang (An A B D)) add Emp)$ **by** (*blast intro:Ang-Gr-trans-Gr-Gr Ang-rev*)
qed

Theorem24

theorem (*in Congruence-Rule*) *Tri-isosceles-inv* :

assumes N :

$Def (Tri (Tr A B C))$

$Cong (Geos (Ang (An A B C)) add Emp) (Geos (Ang (An A C B)) add Emp)$

shows

$\neg Eq (Geos (Seg (Se A B)) add Emp) (Geos (Seg (Se A C)) add Emp)$

proof

assume $W : \neg Eq (Geos (Seg (Se A B)) add Emp) (Geos (Seg (Se A C)) add Emp)$

have $P1 : Line-on (Li A C) A$ **by** (*simp add:Line-on-rule*)

have $P2 : Line-on (Li A C) C$ **by** (*simp add:Line-on-rule*)

from N **have** $P3 : \neg Eq (Geos (Poi A) add Emp) (Geos (Poi B) add Emp)$ **by** (*simp add:Tri-def*)

from N **have** $\neg Eq (Geos (Poi C) add Emp) (Geos (Poi A) add Emp)$ **by** (*simp add:Tri-def*)

then have $P4 : \neg Eq (Geos (Poi A) add Emp) (Geos (Poi C) add Emp)$ **by** (*blast intro:Eq-rev*)

from $P1 P2 P3 P4$ **have** $\exists p. Eq (Geos (Seg (Se A B)) add Emp) (Geos (Seg (Se A p)) add Emp)$

$\wedge \neg Bet-Point (Se p C) A \wedge Line-on (Li A C) p \wedge \neg Eq (Geos (Poi A) add Emp) (Geos (Poi p) add Emp)$ **by** (*simp add:Seg-move-sameside*)

then obtain $D :: Point$ **where** $P5 : Eq (Geos (Seg (Se A B)) add Emp) (Geos (Seg (Se A D)) add Emp)$

$\wedge \neg Bet-Point (Se D C) A \wedge Line-on (Li A C) D \wedge \neg Eq (Geos (Poi A) add Emp) (Geos (Poi D) add Emp)$ **by** *blast*

from $W P1 P2 P3 P4 P5$ **have** $P6 : Bet-Point (Se C A) D \wedge \neg Bet-Point (Se A D) C$

$\vee \neg Bet-Point (Se C A) D \wedge Bet-Point (Se A D) C$ **by** (*simp add:Seg-not-Eq-move*)

from N **have** $P7 : Def (Tri (Tr B C A))$ **by** (*blast intro:Tri-def-rev Tri-def-trans*)

have $P8 : Eq (Geos (Seg (Se A B)) add Emp) (Geos (Seg (Se B A)) add Emp)$ **by** (*simp add:Seg-rev*)

from $P5 P8$ **have** $P9 : Eq (Geos (Seg (Se B A)) add Emp) (Geos (Seg (Se A$

$D))$ *add Emp* **by** (*blast intro:Eq-trans Eq-rev*)
from $P7 P9$ **have** $P10 : \text{Bet-Point } (Se\ C\ A)\ D \implies$
 $Gr\ (Geos\ (Ang\ (An\ C\ B\ A))\ \text{add}\ Emp)\ (Geos\ (Ang\ (An\ B\ C\ A))\ \text{add}\ Emp)$ **by**
(*simp add:Tri-Seg-diagonal*)
have $P11 : \text{Cong}\ (Geos\ (Ang\ (An\ C\ B\ A))\ \text{add}\ Emp)\ (Geos\ (Ang\ (An\ A\ B\ C))\ \text{add}\ Emp)$ **by** (*simp add:Ang-roll*)
from *assms* **have** $P12 : \text{Def}\ (Ang\ (An\ A\ B\ C))$ **by** (*simp add:Tri-to-Ang*)
then **have** $P13 : \text{Def}\ (Ang\ (An\ C\ B\ A))$ **by** (*simp add:Ang-def-rev*)
then **have** $P14 : \text{Def}\ (Ang\ (An\ B\ C\ A))$ **by** (*blast intro:Ang-def-rev Ang-def-inv*)
from $P10 P11 P12 P13 P14$ **have** $P15 : \text{Bet-Point } (Se\ C\ A)\ D \implies$
 $Gr\ (Geos\ (Ang\ (An\ A\ B\ C))\ \text{add}\ Emp)\ (Geos\ (Ang\ (An\ B\ C\ A))\ \text{add}\ Emp)$ **by**
(*blast intro:Ang-Gr-trans-Eq-Gr Ang-rev*)
from $P14$ **have** $P16 : \text{Def}\ (Ang\ (An\ A\ C\ B))$ **by** (*simp add:Ang-def-rev*)
have $P17 : \text{Cong}\ (Geos\ (Ang\ (An\ B\ C\ A))\ \text{add}\ Emp)\ (Geos\ (Ang\ (An\ A\ C\ B))\ \text{add}\ Emp)$ **by** (*simp add:Ang-roll*)
from $P12 P14 P15 P16 P17$ **have** $P18 : \text{Bet-Point } (Se\ C\ A)\ D \implies$
 $Gr\ (Geos\ (Ang\ (An\ A\ B\ C))\ \text{add}\ Emp)\ (Geos\ (Ang\ (An\ A\ C\ B))\ \text{add}\ Emp)$ **by**
(*blast intro:Ang-Gr-trans-Gr-Eq Ang-rev*)
from $P12 P16$ **have** $P19 : \text{Cong}\ (Geos\ (Ang\ (An\ A\ B\ C))\ \text{add}\ Emp)\ (Geos\ (Ang\ (An\ A\ C\ B))\ \text{add}\ Emp)$
 $\wedge \neg Gr\ (Geos\ (Ang\ (An\ A\ B\ C))\ \text{add}\ Emp)\ (Geos\ (Ang\ (An\ A\ C\ B))\ \text{add}\ Emp)$
 $\wedge \neg Gr\ (Geos\ (Ang\ (An\ A\ C\ B))\ \text{add}\ Emp)\ (Geos\ (Ang\ (An\ A\ B\ C))\ \text{add}\ Emp)$
 $\vee \neg \text{Cong}\ (Geos\ (Ang\ (An\ A\ B\ C))\ \text{add}\ Emp)\ (Geos\ (Ang\ (An\ A\ C\ B))\ \text{add}\ Emp)$
 $\wedge Gr\ (Geos\ (Ang\ (An\ A\ B\ C))\ \text{add}\ Emp)\ (Geos\ (Ang\ (An\ A\ C\ B))\ \text{add}\ Emp)$
 $\wedge \neg Gr\ (Geos\ (Ang\ (An\ A\ C\ B))\ \text{add}\ Emp)\ (Geos\ (Ang\ (An\ A\ B\ C))\ \text{add}\ Emp)$
 $\vee \neg \text{Cong}\ (Geos\ (Ang\ (An\ A\ B\ C))\ \text{add}\ Emp)\ (Geos\ (Ang\ (An\ A\ C\ B))\ \text{add}\ Emp)$
 $\wedge \neg Gr\ (Geos\ (Ang\ (An\ A\ B\ C))\ \text{add}\ Emp)\ (Geos\ (Ang\ (An\ A\ C\ B))\ \text{add}\ Emp)$
 $\wedge Gr\ (Geos\ (Ang\ (An\ A\ C\ B))\ \text{add}\ Emp)\ (Geos\ (Ang\ (An\ A\ B\ C))\ \text{add}\ Emp)$
by (*simp add:Ang-relation-case-fact*)
from $P18 P19$ **have** $P20 : \text{Bet-Point } (Se\ C\ A)\ D \implies$
 $\neg \text{Cong}\ (Geos\ (Ang\ (An\ A\ B\ C))\ \text{add}\ Emp)\ (Geos\ (Ang\ (An\ A\ C\ B))\ \text{add}\ Emp)$
by *blast*
from *assms* **have** $P21 : \text{Def}\ (Tri\ (Tr\ B\ A\ C))$ **by** (*blast intro:Tri-def-rev Tri-def-trans*)
from $P5 P21$ **have** $\text{Def}\ (Tri\ (Tr\ B\ A\ D))$ **by** (*simp add:Tri-def-extension*)
then **have** $P22 : \text{Def}\ (Tri\ (Tr\ A\ B\ D))$ **by** (*blast intro:Tri-def-rev Tri-def-trans*)
from $P5 P22$ **have** $P23 : \text{Bet-Point } (Se\ A\ D)\ C \implies$
 $Gr\ (Geos\ (Ang\ (An\ A\ C\ B))\ \text{add}\ Emp)\ (Geos\ (Ang\ (An\ A\ B\ C))\ \text{add}\ Emp)$ **by**
(*simp add:Tri-Bet-Ang-Gr*)
from $P19 P23$ **have** $P24 : \text{Bet-Point } (Se\ A\ D)\ C \implies$
 $\neg \text{Cong}\ (Geos\ (Ang\ (An\ A\ B\ C))\ \text{add}\ Emp)\ (Geos\ (Ang\ (An\ A\ C\ B))\ \text{add}\ Emp)$
by *blast*
from *assms* $P6 P20 P24$ **show** *False* **by** *blast*
qed

lemma (*in Congruence-Rule*) *Tri-AAS-lemma1* :
assumes

$Def (Tri (Tr A1 B1 C1)) Def (Tri (Tr A2 B2 C2))$
 $Eq (Geos (Seg (Se A1 B1)) add Emp) (Geos (Seg (Se A2 B2)) add Emp)$
 $Cong (Geos (Ang (An A1 C1 B1)) add Emp) (Geos (Ang (An A2 C2 B2)) add Emp)$
 $Cong (Geos (Ang (An B1 A1 C1)) add Emp) (Geos (Ang (An B2 A2 C2)) add Emp)$
shows
 $Cong (Geos (Tri (Tr A1 B1 C1)) add Emp) (Geos (Tri (Tr A2 B2 C2)) add Emp)$
proof –
have $P1 : Line-on (Li A2 C2) A2$ **by** $(simp add:Line-on-rule)$
have $P2 : Line-on (Li A2 C2) C2$ **by** $(simp add:Line-on-rule)$
from *assms* **have** $\neg Eq (Geos (Poi C1) add Emp) (Geos (Poi A1) add Emp)$ **by**
 $(simp add:Tri-def)$
then have $P3 : \neg Eq (Geos (Poi A1) add Emp) (Geos (Poi C1) add Emp)$ **by**
 $(blast intro:Eq-rev)$
from *assms* **have** $P4 : \neg Eq (Geos (Poi C2) add Emp) (Geos (Poi A2) add Emp)$
by $(simp add:Tri-def)$
then have $P5 : \neg Eq (Geos (Poi A2) add Emp) (Geos (Poi C2) add Emp)$ **by**
 $(blast intro:Eq-rev)$
from $P1 P2 P3 P5$ **have** $\exists p. Eq (Geos (Seg (Se A1 C1)) add Emp) (Geos (Seg (Se A2 p)) add Emp)$
 $\wedge \neg Bet-Point (Se p C2) A2 \wedge Line-on (Li A2 C2) p \wedge \neg Eq (Geos (Poi A2) add Emp) (Geos (Poi p) add Emp)$ **by** $(simp add:Seg-move-sameside)$
then obtain $C3 :: Point$ **where** $P6 : Eq (Geos (Seg (Se A1 C1)) add Emp) (Geos (Seg (Se A2 C3)) add Emp)$
 $\wedge \neg Bet-Point (Se C3 C2) A2 \wedge Line-on (Li A2 C2) C3 \wedge \neg Eq (Geos (Poi A2) add Emp) (Geos (Poi C3) add Emp)$ **by** *blast*
from $P6$ **have** $P7 : Eq (Geos (Seg (Se A2 C2)) add Emp) (Geos (Seg (Se A2 C3)) add Emp) \implies$
 $Eq (Geos (Seg (Se A1 C1)) add Emp) (Geos (Seg (Se A2 C2)) add Emp)$ **by**
 $(blast intro:Eq-rev Eq-trans)$
from *assms* $P7$ **have** $P8 : Eq (Geos (Seg (Se A2 C2)) add Emp) (Geos (Seg (Se A2 C3)) add Emp) \implies$
 $Cong (Geos (Tri (Tr A1 B1 C1)) add Emp) (Geos (Tri (Tr A2 B2 C2)) add Emp)$ **by** $(simp add:Tri-SAS)$
from $P6$ **have** $Eq (Geos (Seg (Se A1 C1)) add Emp) (Geos (Seg (Se A2 C2)) add Emp) \implies$
 $Eq (Geos (Seg (Se A2 C2)) add Emp) (Geos (Seg (Se A2 C3)) add Emp)$ **by**
 $(blast intro:Eq-rev Eq-trans)$
then have $P9 : \neg Eq (Geos (Seg (Se A2 C2)) add Emp) (Geos (Seg (Se A2 C3)) add Emp) \implies$
 $\neg Eq (Geos (Seg (Se A1 C1)) add Emp) (Geos (Seg (Se A2 C2)) add Emp)$ **by**
blast
from $P1 P2 P3 P5 P6 P9$ **have** $P10 : \neg Eq (Geos (Seg (Se A2 C2)) add Emp) (Geos (Seg (Se A2 C3)) add Emp) \implies$
 $Bet-Point (Se C2 A2) C3 \wedge \neg Bet-Point (Se A2 C3) C2$
 $\vee \neg Bet-Point (Se C2 A2) C3 \wedge Bet-Point (Se A2 C3) C2$ **by** $(simp add:Seg-not-Eq-move)$

from *assms* **have** $P11 : \text{Def } (\text{Tri } (\text{Tr } B2 \ A2 \ C2))$ **by** (*blast intro:Tri-def-rev Tri-def-trans*)
from $P6 \ P11$ **have** $\text{Def } (\text{Tri } (\text{Tr } B2 \ A2 \ C3))$ **by** (*simp add:Tri-def-extension*)
then have $P12 : \text{Def } (\text{Tri } (\text{Tr } A2 \ B2 \ C3))$ **by** (*blast intro:Tri-def-rev Tri-def-trans*)
from $P11$ **have** $P13 : \text{Def } (\text{Ang } (\text{An } B2 \ A2 \ C2))$ **by** (*simp add:Tri-to-Ang*)
have $P14 : \text{Line-on } (\text{Li } A2 \ B2) \ B2$ **by** (*simp add:Line-on-rule*)
have $P15 : \neg \text{Bet-Point } (\text{Se } B2 \ B2) \ A2$ **by** (*simp add:Bet-end-Point*)
from $P6$ **have** $P16 : \neg \text{Bet-Point } (\text{Se } C2 \ C3) \ A2$ **by** (*blast intro:Bet-rev*)
from *assms* **have** $P17 : \neg \text{Eq } (\text{Geos } (\text{Poi } A2) \ \text{add Emp}) \ (\text{Geos } (\text{Poi } B2) \ \text{add Emp})$ **by** (*simp add:Tri-def*)
from $P6 \ P13 \ P14 \ P15 \ P16 \ P17$ **have** $P18 :$
 $\text{Eq } (\text{Geos } (\text{Ang } (\text{An } B2 \ A2 \ C2)) \ \text{add Emp}) \ (\text{Geos } (\text{Ang } (\text{An } B2 \ A2 \ C3)) \ \text{add Emp}) \wedge \text{Def } (\text{Ang } (\text{An } B2 \ A2 \ C3))$ **by** (*simp add:Ang-Point-swap*)
from *assms* $P18$ **have** $P19 : \text{Cong } (\text{Geos } (\text{Ang } (\text{An } B1 \ A1 \ C1)) \ \text{add Emp}) \ (\text{Geos } (\text{Ang } (\text{An } B2 \ A2 \ C3)) \ \text{add Emp})$ **by** (*blast intro:Ang-weektrans Ang-rev Eq-rev*)
from *assms* $P6 \ P12 \ P19$ **have** $\text{Cong } (\text{Geos } (\text{Tri } (\text{Tr } A1 \ B1 \ C1)) \ \text{add Emp}) \ (\text{Geos } (\text{Tri } (\text{Tr } A2 \ B2 \ C3)) \ \text{add Emp})$ **by** (*simp add:Tri-SAS*)
then have $P20 : \text{Cong } (\text{Geos } (\text{Ang } (\text{An } A1 \ C1 \ B1)) \ \text{add Emp}) \ (\text{Geos } (\text{Ang } (\text{An } A2 \ C3 \ B2)) \ \text{add Emp})$ **by** (*simp add:Tri-Cong-def*)
from *assms* **have** $P21 : \text{Def } (\text{Ang } (\text{An } A1 \ C1 \ B1))$ **by** (*blast intro:Tri-to-Ang Ang-def-inv*)
from *assms* **have** $P22 : \text{Def } (\text{Ang } (\text{An } A2 \ C2 \ B2))$ **by** (*blast intro:Tri-to-Ang Ang-def-inv*)
from $P18$ **have** $P23 : \text{Def } (\text{Ang } (\text{An } A2 \ C3 \ B2))$ **by** (*blast intro:Ang-def-rev Ang-def-inv*)
from *assms* $P20 \ P21 \ P22 \ P23$ **have** $P24 : \text{Cong } (\text{Geos } (\text{Ang } (\text{An } A2 \ C2 \ B2)) \ \text{add Emp}) \ (\text{Geos } (\text{Ang } (\text{An } A2 \ C3 \ B2)) \ \text{add Emp})$ **by** (*blast intro:Ang-trans Ang-rev*)
from $P22 \ P23 \ P24$ **have** $P25 : \neg \text{Gr } (\text{Geos } (\text{Ang } (\text{An } A2 \ C2 \ B2)) \ \text{add Emp}) \ (\text{Geos } (\text{Ang } (\text{An } A2 \ C3 \ B2)) \ \text{add Emp})$
 $\wedge \neg \text{Gr } (\text{Geos } (\text{Ang } (\text{An } A2 \ C3 \ B2)) \ \text{add Emp}) \ (\text{Geos } (\text{Ang } (\text{An } A2 \ C2 \ B2)) \ \text{add Emp})$ **by** (*simp add:Ang-not-Gr*)
from *assms* **have** $P26 : \text{Def } (\text{Tri } (\text{Tr } B2 \ C2 \ A2))$ **by** (*blast intro:Tri-def-trans*)
have $P27 : \text{Bet-Point } (\text{Se } C2 \ A2) \ C3 \implies \neg \text{Eq } (\text{Geos } (\text{Poi } C3) \ \text{add Emp}) \ (\text{Geos } (\text{Poi } C2) \ \text{add Emp})$ **by** (*simp add:Bet-Point-def*)
have $P28 : \text{Bet-Point } (\text{Se } A2 \ C3) \ C2 \implies \neg \text{Eq } (\text{Geos } (\text{Poi } C3) \ \text{add Emp}) \ (\text{Geos } (\text{Poi } C2) \ \text{add Emp})$ **by** (*simp add:Bet-Point-def*)
from $P10 \ P27 \ P28$ **have** $P29 : \neg \text{Eq } (\text{Geos } (\text{Seg } (\text{Se } A2 \ C2)) \ \text{add Emp}) \ (\text{Geos } (\text{Seg } (\text{Se } A2 \ C3)) \ \text{add Emp}) \implies$
 $\neg \text{Eq } (\text{Geos } (\text{Poi } C2) \ \text{add Emp}) \ (\text{Geos } (\text{Poi } C3) \ \text{add Emp})$ **by** (*blast intro:Eq-rev*)
from $P5 \ P6$ **have** $P30 : \text{Line-on } (\text{Li } C2 \ A2) \ C3$ **by** (*blast intro:Line-rev Line-on-trans*)
from $P26 \ P29 \ P30$ **have** $\neg \text{Eq } (\text{Geos } (\text{Seg } (\text{Se } A2 \ C2)) \ \text{add Emp}) \ (\text{Geos } (\text{Seg } (\text{Se } A2 \ C3)) \ \text{add Emp}) \implies$
 $\text{Def } (\text{Tri } (\text{Tr } B2 \ C2 \ C3))$ **by** (*simp add:Tri-def-extension*)
then have $P31 : \neg \text{Eq } (\text{Geos } (\text{Seg } (\text{Se } A2 \ C2)) \ \text{add Emp}) \ (\text{Geos } (\text{Seg } (\text{Se } A2 \ C3)) \ \text{add Emp}) \implies$
 $\text{Def } (\text{Tri } (\text{Tr } C3 \ C2 \ B2))$ **by** (*blast intro:Tri-def-rev Tri-def-trans*)
then have $P32 : \neg \text{Eq } (\text{Geos } (\text{Seg } (\text{Se } A2 \ C2)) \ \text{add Emp}) \ (\text{Geos } (\text{Seg } (\text{Se } A2 \ C3)) \ \text{add Emp}) \implies$

Bet-Point (Se C2 A2) C3 \implies Gr (Geos (Ang (An B2 C3 A2)) add Emp) (Geos (Ang (An C3 C2 B2)) add Emp) **by** (simp add:Ang-external-Gr)
have P33 : Cong (Geos (Ang (An B2 C3 A2)) add Emp) (Geos (Ang (An A2 C3 B2)) add Emp) **by** (simp add:Ang-roll)
from P31 **have** P34 : \neg Eq (Geos (Seg (Se A2 C2)) add Emp) (Geos (Seg (Se A2 C3)) add Emp) \implies
Def (Ang (An C3 C2 B2)) **by** (simp add:Tri-to-Ang)
from P23 **have** P35 : Def (Ang (An B2 C3 A2)) **by** (blast intro:Ang-def-rev Ang-def-inv)
from P23 P32 P33 P34 P35 **have** P36 : \neg Eq (Geos (Seg (Se A2 C2)) add Emp) (Geos (Seg (Se A2 C3)) add Emp) \implies
Bet-Point (Se C2 A2) C3 \implies Gr (Geos (Ang (An A2 C3 B2)) add Emp) (Geos (Ang (An C3 C2 B2)) add Emp) **by** (blast intro:Ang-Gr-trans-Eq-Gr Ang-rev)
have P37 : *Bet-Point* (Se C2 A2) C3 \implies Line-on (Li C2 C3) A2 **by** (simp add:Line-Bet-on)
from P10 **have** P38 : \neg Eq (Geos (Seg (Se A2 C2)) add Emp) (Geos (Seg (Se A2 C3)) add Emp) \implies
Bet-Point (Se C2 A2) C3 \implies \neg *Bet-Point* (Se C3 A2) C2 **by** (blast intro:Bet-rev)
have P39 : Line-on (Li C2 B2) B2 **by** (simp add:Line-on-rule)
have P40 : \neg *Bet-Point* (Se B2 B2) C2 **by** (simp add:Bet-end-Point)
from P13 **have** P41 : \neg Eq (Geos (Poi C2) add Emp) (Geos (Poi B2) add Emp) **by** (simp add:Ang-def)
from P4 P34 P37 P38 P39 P40 P41 **have** \neg Eq (Geos (Seg (Se A2 C2)) add Emp) (Geos (Seg (Se A2 C3)) add Emp) \implies
Bet-Point (Se C2 A2) C3 \implies Eq (Geos (Ang (An C3 C2 B2)) add Emp) (Geos (Ang (An A2 C2 B2)) add Emp)
 \wedge Def (Ang (An A2 C2 B2)) **by** (simp add:Ang-Point-swap)
then **have** P42 : \neg Eq (Geos (Seg (Se A2 C2)) add Emp) (Geos (Seg (Se A2 C3)) add Emp) \implies
Bet-Point (Se C2 A2) C3 \implies Cong (Geos (Ang (An C3 C2 B2)) add Emp) (Geos (Ang (An A2 C2 B2)) add Emp) **by** (blast intro:Ang-weektrans)
from P22 P23 P34 P36 P42 **have** P43 : \neg Eq (Geos (Seg (Se A2 C2)) add Emp) (Geos (Seg (Se A2 C3)) add Emp) \implies
Bet-Point (Se C2 A2) C3 \implies Gr (Geos (Ang (An A2 C3 B2)) add Emp) (Geos (Ang (An A2 C2 B2)) add Emp) **by** (blast intro:Ang-Gr-trans-Gr-Eq Ang-rev)
from P31 **have** \neg Eq (Geos (Seg (Se A2 C2)) add Emp) (Geos (Seg (Se A2 C3)) add Emp) \implies
Def (Tri (Tr C2 C3 B2)) **by** (blast intro:Tri-def-rev Tri-def-trans)
then **have** P44 : \neg Eq (Geos (Seg (Se A2 C2)) add Emp) (Geos (Seg (Se A2 C3)) add Emp) \implies
Bet-Point (Se A2 C3) C2 \implies Gr (Geos (Ang (An B2 C2 A2)) add Emp) (Geos (Ang (An C2 C3 B2)) add Emp) **by** (simp add:Ang-external-Gr Bet-rev)
have P45 : Cong (Geos (Ang (An B2 C2 A2)) add Emp) (Geos (Ang (An A2 C2 B2)) add Emp) **by** (simp add:Ang-roll)
from P26 **have** P46 : Def (Ang (An B2 C2 A2)) **by** (simp add:Tri-to-Ang)
from P34 **have** P47 : \neg Eq (Geos (Seg (Se A2 C2)) add Emp) (Geos (Seg (Se A2 C3)) add Emp) \implies
Def (Ang (An C2 C3 B2)) **by** (blast intro:Ang-def-rev Ang-def-inv)
from P22 P44 P45 P46 P47 **have** P48 : \neg Eq (Geos (Seg (Se A2 C2)) add Emp)

(Geos (Seg (Se A2 C3)) add Emp) \implies
 Bet-Point (Se A2 C3) C2 \implies Gr (Geos (Ang (An A2 C2 B2)) add Emp) (Geos
 (Ang (An C2 C3 B2)) add Emp) **by** (blast intro:Ang-Gr-trans-Eq-Gr Ang-rev)
have P49 : Bet-Point (Se A2 C3) C2 \implies Line-on (Li C3 C2) A2 **by** (simp
 add:Line-Bet-on)
from P10 **have** P50 : \neg Eq (Geos (Seg (Se A2 C2)) add Emp) (Geos (Seg (Se
 A2 C3)) add Emp) \implies
 Bet-Point (Se A2 C3) C2 \implies \neg Bet-Point (Se C2 A2) C3 **by** (blast intro:Bet-rev)
have P51 : Line-on (Li C3 B2) B2 **by** (simp add:Line-on-rule)
have P52 : \neg Bet-Point (Se B2 B2) C3 **by** (simp add:Bet-end-Point)
from P6 **have** P53 : \neg Eq (Geos (Poi C3) add Emp) (Geos (Poi A2) add Emp)
by (blast intro:Eq-rev)
from P47 **have** P54 : \neg Eq (Geos (Seg (Se A2 C2)) add Emp) (Geos (Seg (Se
 A2 C3)) add Emp) \implies
 \neg Eq (Geos (Poi C3) add Emp) (Geos (Poi B2) add Emp) **by** (simp add:Ang-def)
from P47 P49 P50 P51 P52 P53 P54 **have** \neg Eq (Geos (Seg (Se A2 C2)) add
 Emp) (Geos (Seg (Se A2 C3)) add Emp) \implies
 Bet-Point (Se A2 C3) C2 \implies Eq (Geos (Ang (An C2 C3 B2)) add Emp) (Geos
 (Ang (An A2 C3 B2)) add Emp)
 \wedge Def (Ang (An A2 C3 B2)) **by** (simp add:Ang-Point-swap)
then have P55 : \neg Eq (Geos (Seg (Se A2 C2)) add Emp) (Geos (Seg (Se A2
 C3)) add Emp) \implies
 Bet-Point (Se A2 C3) C2 \implies Cong (Geos (Ang (An C2 C3 B2)) add Emp)
 (Geos (Ang (An A2 C3 B2)) add Emp) **by** (blast intro:Ang-weektrans)
from P22 P23 P47 P48 P55 **have** P56 : \neg Eq (Geos (Seg (Se A2 C2)) add Emp)
 (Geos (Seg (Se A2 C3)) add Emp) \implies
 Bet-Point (Se A2 C3) C2 \implies Gr (Geos (Ang (An A2 C2 B2)) add Emp) (Geos
 (Ang (An A2 C3 B2)) add Emp) **by** (blast intro:Ang-Gr-trans-Gr-Eq Ang-rev)
from P10 P25 P43 P56 **have** P57 : Eq (Geos (Seg (Se A2 C2)) add Emp) (Geos
 (Seg (Se A2 C3)) add Emp) **by** blast
from P8 P57 **show** Cong (Geos (Tri (Tr A1 B1 C1)) add Emp) (Geos (Tri (Tr
 A2 B2 C2)) add Emp) **by** blast
qed

Theorem25

theorem (in Congruence-Rule) Tri-AAS :

assumes

Def (Tri (Tr A1 B1 C1)) Def (Tri (Tr A2 B2 C2))

Eq (Geos (Seg (Se A1 B1)) add Emp) (Geos (Seg (Se A2 B2)) add Emp)

Cong (Geos (Ang (An A1 C1 B1)) add Emp) (Geos (Ang (An A2 C2 B2)) add
 Emp)

Cong (Geos (Ang (An A1 B1 C1)) add Emp) (Geos (Ang (An A2 B2 C2)) add
 Emp)

\vee Cong (Geos (Ang (An B1 A1 C1)) add Emp) (Geos (Ang (An B2 A2 C2))
 add Emp)

shows

Cong (Geos (Tri (Tr A1 B1 C1)) add Emp) (Geos (Tri (Tr A2 B2 C2)) add
 Emp)

proof –

from *assms* **have** $P1 : \text{Def } (\text{Tri } (\text{Tr } B1 \ A1 \ C1))$ **by** (*blast intro:Tri-def-rev Tri-def-trans*)
from *assms* **have** $P2 : \text{Def } (\text{Tri } (\text{Tr } B2 \ A2 \ C2))$ **by** (*blast intro:Tri-def-rev Tri-def-trans*)
have $P3 : \text{Eq } (\text{Geos } (\text{Seg } (\text{Se } A1 \ B1)) \ \text{add } \text{Emp})$ ($\text{Geos } (\text{Seg } (\text{Se } B1 \ A1)) \ \text{add } \text{Emp}$) **by** (*simp add:Seg-rev*)
have $P4 : \text{Eq } (\text{Geos } (\text{Seg } (\text{Se } A2 \ B2)) \ \text{add } \text{Emp})$ ($\text{Geos } (\text{Seg } (\text{Se } B2 \ A2)) \ \text{add } \text{Emp}$) **by** (*simp add:Seg-rev*)
from *assms* $P3 \ P4$ **have** $P5 : \text{Eq } (\text{Geos } (\text{Seg } (\text{Se } B1 \ A1)) \ \text{add } \text{Emp})$ ($\text{Geos } (\text{Seg } (\text{Se } B2 \ A2)) \ \text{add } \text{Emp}$) **by** (*blast intro:Eq-trans Eq-rev*)
have $P6 : \text{Eq } (\text{Geos } (\text{Ang } (\text{An } A1 \ C1 \ B1)) \ \text{add } \text{Emp})$ ($\text{Geos } (\text{Ang } (\text{An } B1 \ C1 \ A1)) \ \text{add } \text{Emp}$) **by** (*simp add:Ang-roll*)
from *assms* $P6$ **have** $P7 : \text{Cong } (\text{Geos } (\text{Ang } (\text{An } B1 \ C1 \ A1)) \ \text{add } \text{Emp})$ ($\text{Geos } (\text{Ang } (\text{An } A2 \ C2 \ B2)) \ \text{add } \text{Emp}$) **by** (*blast intro:Ang-weektrans Ang-rev Eq-rev*)
have $P8 : \text{Eq } (\text{Geos } (\text{Ang } (\text{An } A2 \ C2 \ B2)) \ \text{add } \text{Emp})$ ($\text{Geos } (\text{Ang } (\text{An } B2 \ C2 \ A2)) \ \text{add } \text{Emp}$) **by** (*simp add:Ang-roll*)
from $P7 \ P8$ **have** $P9 : \text{Cong } (\text{Geos } (\text{Ang } (\text{An } B1 \ C1 \ A1)) \ \text{add } \text{Emp})$ ($\text{Geos } (\text{Ang } (\text{An } B2 \ C2 \ A2)) \ \text{add } \text{Emp}$) **by** (*blast intro:Ang-weektrans Ang-rev Eq-rev*)
from $P1 \ P2 \ P5 \ P9$ **have** $\text{Cong } (\text{Geos } (\text{Ang } (\text{An } A1 \ B1 \ C1)) \ \text{add } \text{Emp})$ ($\text{Geos } (\text{Ang } (\text{An } A2 \ B2 \ C2)) \ \text{add } \text{Emp}$) \implies
 $\text{Cong } (\text{Geos } (\text{Tri } (\text{Tr } B1 \ A1 \ C1)) \ \text{add } \text{Emp})$ ($\text{Geos } (\text{Tri } (\text{Tr } B2 \ A2 \ C2)) \ \text{add } \text{Emp}$) **by** (*simp add:Tri-AAS-lemma1*)
then **have** $P10 : \text{Cong } (\text{Geos } (\text{Ang } (\text{An } A1 \ B1 \ C1)) \ \text{add } \text{Emp})$ ($\text{Geos } (\text{Ang } (\text{An } A2 \ B2 \ C2)) \ \text{add } \text{Emp}$) \implies
 $\text{Cong } (\text{Geos } (\text{Ang } (\text{An } C1 \ A1 \ B1)) \ \text{add } \text{Emp})$ ($\text{Geos } (\text{Ang } (\text{An } C2 \ A2 \ B2)) \ \text{add } \text{Emp}$) **by** (*simp add:Tri-Cong-def*)
have $P11 : \text{Eq } (\text{Geos } (\text{Ang } (\text{An } C1 \ A1 \ B1)) \ \text{add } \text{Emp})$ ($\text{Geos } (\text{Ang } (\text{An } B1 \ A1 \ C1)) \ \text{add } \text{Emp}$) **by** (*simp add:Ang-roll*)
from $P10 \ P11$ **have** $P12 : \text{Cong } (\text{Geos } (\text{Ang } (\text{An } A1 \ B1 \ C1)) \ \text{add } \text{Emp})$ ($\text{Geos } (\text{Ang } (\text{An } A2 \ B2 \ C2)) \ \text{add } \text{Emp}$) \implies
 $\text{Cong } (\text{Geos } (\text{Ang } (\text{An } B1 \ A1 \ C1)) \ \text{add } \text{Emp})$ ($\text{Geos } (\text{Ang } (\text{An } C2 \ A2 \ B2)) \ \text{add } \text{Emp}$) **by** (*blast intro:Ang-weektrans Ang-rev Eq-rev*)
have $P13 : \text{Eq } (\text{Geos } (\text{Ang } (\text{An } C2 \ A2 \ B2)) \ \text{add } \text{Emp})$ ($\text{Geos } (\text{Ang } (\text{An } B2 \ A2 \ C2)) \ \text{add } \text{Emp}$) **by** (*simp add:Ang-roll*)
from $P12 \ P13$ **have** $\text{Cong } (\text{Geos } (\text{Ang } (\text{An } A1 \ B1 \ C1)) \ \text{add } \text{Emp})$ ($\text{Geos } (\text{Ang } (\text{An } A2 \ B2 \ C2)) \ \text{add } \text{Emp}$) \implies
 $\text{Cong } (\text{Geos } (\text{Ang } (\text{An } B1 \ A1 \ C1)) \ \text{add } \text{Emp})$ ($\text{Geos } (\text{Ang } (\text{An } B2 \ A2 \ C2)) \ \text{add } \text{Emp}$) **by** (*blast intro:Ang-weektrans Ang-rev Eq-rev*)
then **have** $P14 : \text{Cong } (\text{Geos } (\text{Ang } (\text{An } A1 \ B1 \ C1)) \ \text{add } \text{Emp})$ ($\text{Geos } (\text{Ang } (\text{An } A2 \ B2 \ C2)) \ \text{add } \text{Emp}$)
 $\vee \text{Cong } (\text{Geos } (\text{Ang } (\text{An } B1 \ A1 \ C1)) \ \text{add } \text{Emp})$ ($\text{Geos } (\text{Ang } (\text{An } B2 \ A2 \ C2)) \ \text{add } \text{Emp}$) \implies
 $\text{Cong } (\text{Geos } (\text{Ang } (\text{An } B1 \ A1 \ C1)) \ \text{add } \text{Emp})$ ($\text{Geos } (\text{Ang } (\text{An } B2 \ A2 \ C2)) \ \text{add } \text{Emp}$) **by** *blast*
from *assms* **have** $P15 : \text{Cong } (\text{Geos } (\text{Ang } (\text{An } B1 \ A1 \ C1)) \ \text{add } \text{Emp})$ ($\text{Geos } (\text{Ang } (\text{An } B2 \ A2 \ C2)) \ \text{add } \text{Emp}$) \implies
 $\text{Cong } (\text{Geos } (\text{Tri } (\text{Tr } A1 \ B1 \ C1)) \ \text{add } \text{Emp})$ ($\text{Geos } (\text{Tri } (\text{Tr } A2 \ B2 \ C2)) \ \text{add } \text{Emp}$) **by** (*simp add:Tri-AAS-lemma1*)
from *assms* $P14 \ P15$ **show** $\text{Cong } (\text{Geos } (\text{Tri } (\text{Tr } A1 \ B1 \ C1)) \ \text{add } \text{Emp})$ ($\text{Geos } (\text{Tri } (\text{Tr } A2 \ B2 \ C2)) \ \text{add } \text{Emp}$)

(*Tri (Tr A2 B2 C2)*) *add Emp* **by** *blast*
qed

Theorem26

theorem (in *Congruence-Rule*) *Seg-bisection* :

assumes

$\neg \text{Eq} (\text{Geos} (\text{Poi } A) \text{ add Emp}) (\text{Geos} (\text{Poi } B) \text{ add Emp})$

shows

$\exists p. \text{Eq} (\text{Geos} (\text{Seg} (\text{Se } A \ p)) \text{ add Emp}) (\text{Geos} (\text{Seg} (\text{Se } p \ B)) \text{ add Emp}) \wedge$
Bet-Point (*Se A B*) *p*

proof –

have $\exists p \ q \ r. \neg \text{Line-on} (\text{Li } A \ B) \ p \wedge \neg \text{Line-on} (\text{Li } A \ B) \ q \wedge \neg \text{Line-on} (\text{Li } A \ B) \ r$

$\wedge \neg \text{Eq} (\text{Geos} (\text{Poi } p) \text{ add Emp}) (\text{Geos} (\text{Poi } q) \text{ add Emp}) \wedge \neg \text{Eq} (\text{Geos} (\text{Poi } q) \text{ add Emp}) (\text{Geos} (\text{Poi } r) \text{ add Emp})$

$\wedge \neg \text{Eq} (\text{Geos} (\text{Poi } r) \text{ add Emp}) (\text{Geos} (\text{Poi } p) \text{ add Emp})$ **by** (*blast intro:Line-not-on-exist*)

then obtain *C* :: *Point* **where** *P1* : $\neg \text{Line-on} (\text{Li } A \ B) \ C$ **by** *blast*

from *assms P1* **have** *P2* : *Def* (*Ang (An A B C)*) **by** (*simp add:Ang-simple-def*)

then have *P2* : *Def* (*Ang (An C A B)*) **by** (*blast intro:Ang-def-rev Ang-def-inv*)

from *assms* **have** *P3* : $\text{Eq} (\text{Geos} (\text{Lin} (\text{Li } A \ B)) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li } B \ A)) \text{ add Emp})$ **by** (*simp add:Line-rev*)

from *P1 P3* **have** *P4* : $\neg \text{Line-on} (\text{Li } B \ A) \ C$ **by** (*simp add:Line-not-on-trans*)

from *P2 P4* **have** $\exists p. \text{Cong} (\text{Geos} (\text{Ang} (\text{An } C \ A \ B)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } p \ B \ A)) \text{ add Emp})$

$\wedge \text{Plane-diffside} (\text{Li } B \ A) \ p \ C$ **by** (*simp add:Ang-move-diffside*)

then obtain *D1* :: *Point* **where** *P5* : $\text{Cong} (\text{Geos} (\text{Ang} (\text{An } C \ A \ B)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } D1 \ B \ A)) \text{ add Emp})$

$\wedge \text{Plane-diffside} (\text{Li } B \ A) \ D1 \ C$ **by** *blast*

then have $\exists p. \text{Bet-Point} (\text{Se } D1 \ C) \ p \wedge \text{Line-on} (\text{Li } B \ A) \ p \wedge \neg \text{Line-on} (\text{Li } B \ A) \ D1 \wedge \neg \text{Line-on} (\text{Li } B \ A) \ C$ **by** (*simp add:Plane-diffside-def*)

then have *P6* : $\neg \text{Line-on} (\text{Li } B \ A) \ D1$ **by** *blast*

from *assms* **have** *P7* : $\neg \text{Eq} (\text{Geos} (\text{Poi } B) \text{ add Emp}) (\text{Geos} (\text{Poi } A) \text{ add Emp})$ **by** (*blast intro:Eq-rev*)

from *P6 P7* **have** *Def* (*Ang (An B A D1)*) **by** (*simp add:Ang-simple-def*)

then have *P8* : *Def* (*Ang (An D1 B A)*) **by** (*blast intro:Ang-def-rev Ang-def-inv*)

from *P2 P5 P8* **have** $\exists p. \text{Cong} (\text{Geos} (\text{Ang} (\text{An } C \ A \ B)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } p \ B \ A)) \text{ add Emp})$

$\wedge \text{Eq} (\text{Geos} (\text{Ang} (\text{An } D1 \ B \ A)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } p \ B \ A)) \text{ add Emp})$

$\wedge \text{Eq} (\text{Geos} (\text{Seg} (\text{Se } A \ C)) \text{ add Emp}) (\text{Geos} (\text{Seg} (\text{Se } B \ p)) \text{ add Emp}) \wedge \text{Line-on} (\text{Li } B \ D1) \ p$

$\wedge \neg \text{Bet-Point} (\text{Se } p \ D1) \ B \wedge \text{Def} (\text{Ang} (\text{An } p \ B \ A))$ **by** (*simp add:Ang-replace*)

then obtain *D* :: *Point* **where** *P9* : $\text{Cong} (\text{Geos} (\text{Ang} (\text{An } C \ A \ B)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } D \ B \ A)) \text{ add Emp})$

$\wedge \text{Eq} (\text{Geos} (\text{Ang} (\text{An } D1 \ B \ A)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } D \ B \ A)) \text{ add Emp})$

$\wedge \text{Eq} (\text{Geos} (\text{Seg} (\text{Se } A \ C)) \text{ add Emp}) (\text{Geos} (\text{Seg} (\text{Se } B \ D)) \text{ add Emp}) \wedge \text{Line-on} (\text{Li } B \ D1) \ D$

$\wedge \neg \text{Bet-Point} (\text{Se } D \ D1) \ B \wedge \text{Def} (\text{Ang} (\text{An } D \ B \ A))$ **by** *blast*

have $\text{Plane-diffside} (\text{Li } B \ A) \ D \ D1 \implies$

$\exists p. \text{Bet-Point (Se D D1) } p \wedge \text{Line-on (Li B A) } p \wedge \neg \text{Line-on (Li B A) } D \wedge$
 $\neg \text{Line-on (Li B A) } D1$ **by** (simp add:Plane-diffside-def)
then obtain $B1 :: \text{Point}$ **where** $P10 : \text{Plane-diffside (Li B A) } D D1 \implies$
 $\text{Bet-Point (Se D D1) } B1 \wedge \text{Line-on (Li B A) } B1 \wedge \neg \text{Line-on (Li B A) } D \wedge \neg$
 $\text{Line-on (Li B A) } D1$ **by** blast
then have $P11 : \text{Plane-diffside (Li B A) } D D1 \implies \text{Line-on (Li D D1) } B1$ **by**
(simp add:Line-Bet-on)
from $P10$ **have** $\text{Plane-diffside (Li B A) } D D1 \implies \text{Bet-Point (Se D D1) } B1$ **by**
simp
then have $P12 : \text{Plane-diffside (Li B A) } D D1 \implies$
 $\neg \text{Eq (Geos (Poi D) add Emp) (Geos (Poi D1) add Emp)}$ **by** (simp add:Bet-Point-def)
have $P13 : \text{Line-on (Li B D1) } D1$ **by** (simp add:Line-on-rule)
have $P14 : \text{Line-on (Li D D1) } D$ **by** (simp add:Line-on-rule)
have $P15 : \text{Line-on (Li D D1) } D1$ **by** (simp add:Line-on-rule)
from $P9$ **have** $P16 : \text{Line-on (Li B D1) } D$ **by** simp
from $P12 P13 P14 P15 P16$ **have** $P17 : \text{Plane-diffside (Li B A) } D D1 \implies$
 $\text{Eq (Geos (Lin (Li B D1)) add Emp) (Geos (Lin (Li D D1)) add Emp)}$ **by** (simp
add:Line-unique)
have $P18 : \text{Line-on (Li B D1) } B$ **by** (simp add:Line-on-rule)
from $P17 P18$ **have** $P19 : \text{Plane-diffside (Li B A) } D D1 \implies \text{Line-on (Li D D1)}$
 B **by** (simp add:Line-on-trans)
from $P14$ **have** $P20 : \text{Eq (Geos (Lin (Li D D1)) add Emp) (Geos (Lin (Li B$
 $A)) add Emp) \implies$
 $\text{Line-on (Li B A) } D$ **by** (simp add:Line-on-trans)
from $P10 P20$ **have** $P21 : \text{Plane-diffside (Li B A) } D D1 \implies$
 $\neg \text{Eq (Geos (Lin (Li D D1)) add Emp) (Geos (Lin (Li B A)) add Emp)}$ **by** blast
have $P22 : \text{Line-on (Li B A) } B$ **by** (simp add:Line-on-rule)
from $P10 P11 P19 P21 P22$ **have** $P23 : \text{Plane-diffside (Li B A) } D D1 \implies$
 $\text{Eq (Geos (Poi B1) add Emp) (Geos (Poi B) add Emp)}$ **by** (simp add:Line-unique-Point)
from $P10 P23$ **have** $P24 : \text{Plane-diffside (Li B A) } D D1 \implies \text{Bet-Point (Se D}$
 $D1) B$ **by** (blast intro:Point-Eq)
from $P9 P24$ **have** $P25 : \neg \text{Plane-diffside (Li B A) } D D1$ **by** blast
from $P5$ **have** $P26 : \text{Plane-sameside (Li B A) } C D \implies \text{Plane-diffside (Li B A)}$
 $D D1$ **by** (simp add:Plane-diffside-rev Plane-trans)
from $P25 P26$ **have** $P27 : \neg \text{Plane-sameside (Li B A) } C D$ **by** blast
from $P5$ **have** $P28 : \text{Eq (Geos (Poi C) add Emp) (Geos (Poi D) add Emp) \implies}$
 $\text{Plane-diffside (Li B A) } D1 D$ **by** (blast intro:Point-Eq)
from $P25 P28$ **have** $P29 : \neg \text{Eq (Geos (Poi C) add Emp) (Geos (Poi D) add}$
 Emp) **by** (blast intro:Plane-diffside-rev)
from $P9$ **have** $\text{Def (Tri (Tr B A D))}$ **by** (blast intro:Ang-to-Tri Tri-def-rev
Tri-def-trans)
then have $P30 : \neg \text{Line-on (Li B A) } D$ **by** (simp add:Tri-def-Line)
from $P4 P27 P29 P30$ **have** $\text{Plane-diffside (Li B A) } C D$ **by** (simp add:Plane-not-sameside-diffside)
then have $\exists p. \text{Bet-Point (Se C D) } p \wedge \text{Line-on (Li B A) } p \wedge \neg \text{Line-on (Li B}$
 $A) C \wedge \neg \text{Line-on (Li B A) } D$ **by** (simp add:Plane-diffside-def)
then obtain $E :: \text{Point}$ **where** $P31 : \text{Bet-Point (Se C D) } E \wedge \text{Line-on (Li B A)}$
 E
 $\wedge \neg \text{Line-on (Li B A) } C \wedge \neg \text{Line-on (Li B A) } D$ **by** blast
then have $P32 : \text{Bet-Point (Se C D) } E$ **by** simp

then have $P33 : Eq (Geos (Poi E) add Emp) (Geos (Poi A) add Emp) \implies$
Bet-Point (Se D C) A **by** (*blast intro:Point-Eq Bet-rev*)
from $P9$ **have** $P34 : Def (Tri (Tr A D B))$ **by** (*blast intro:Ang-to-Tri Tri-def-rev*
Tri-def-trans)
from $P33 P34$ **have** $P35 : Eq (Geos (Poi E) add Emp) (Geos (Poi A) add Emp)$
 \implies
 $Gr (Geos (Ang (An B A C)) add Emp) (Geos (Ang (An A B D)) add Emp)$ **by**
(*simp add:Ang-external-Gr*)
from $P32$ **have** $P36 : Eq (Geos (Poi E) add Emp) (Geos (Poi B) add Emp) \implies$
Bet-Point (Se C D) B **by** (*simp add:Point-Eq*)
from $P2$ **have** $P37 : Def (Tri (Tr B C A))$ **by** (*blast intro:Ang-to-Tri Tri-def-rev*
Tri-def-trans)
from $P36 P37$ **have** $P38 : Eq (Geos (Poi E) add Emp) (Geos (Poi B) add Emp)$
 \implies
 $Gr (Geos (Ang (An A B D)) add Emp) (Geos (Ang (An B A C)) add Emp)$ **by**
(*simp add:Ang-external-Gr*)
have $P39 : Eq (Geos (Ang (An C A B)) add Emp) (Geos (Ang (An B A C))$
add Emp) **by** (*simp add:Ang-roll*)
from $P9 P39$ **have** $P40 : Cong (Geos (Ang (An B A C)) add Emp) (Geos (Ang$
(An D B A)) add Emp) **by** (*blast intro:Ang-weektrans Ang-rev Eq-rev*)
have $P41 : Eq (Geos (Ang (An D B A)) add Emp) (Geos (Ang (An A B D)) add$
Emp) **by** (*simp add:Ang-roll*)
from $P40 P41$ **have** $P42 : Cong (Geos (Ang (An B A C)) add Emp) (Geos (Ang$
(An A B D)) add Emp) **by** (*blast intro:Ang-weektrans Ang-rev Eq-rev*)
from $P2$ **have** $P43 : Def (Ang (An B A C))$ **by** (*blast intro:Ang-def-rev*
Ang-def-inv)
from $P9$ **have** $P44 : Def (Ang (An A B D))$ **by** (*simp add:Ang-def-rev*)
from $P42 P43 P44$ **have** $P45 : \neg Gr (Geos (Ang (An B A C)) add Emp) (Geos$
(Ang (An A B D)) add Emp)
 $\wedge \neg Gr (Geos (Ang (An A B D)) add Emp) (Geos (Ang (An B A C)) add Emp)$
by (*simp add:Ang-not-Gr*)
from $P35 P45$ **have** $P46 : \neg Eq (Geos (Poi E) add Emp) (Geos (Poi A) add$
Emp) **by** *blast*
from $P38 P45$ **have** $P47 : \neg Eq (Geos (Poi B) add Emp) (Geos (Poi E) add$
Emp) **by** (*blast intro:Eq-rev*)
have $P48 : Line-on (Li B A) B$ **by** (*simp add:Line-on-rule*)
have $P49 : Line-on (Li B A) A$ **by** (*simp add:Line-on-rule*)
from *assms* $P31 P46 P47 P48 P49$ **have** $P50 : Bet-Point (Se A E) B \vee Bet-Point$
(Se E B) A
 $\vee Bet-Point (Se B A) E$ **by** (*simp add:Bet-case*)
have $P51 : Bet-Point (Se A E) B \implies Line-on (Li B E) A$ **by** (*simp add:Line-Bet-on*)
have $P52 : Line-on (Li B E) B$ **by** (*simp add:Line-on-rule*)
from *assms* $P48 P49 P51 P52$ **have** $Bet-Point (Se A E) B \implies$
 $Eq (Geos (Lin (Li B E)) add Emp) (Geos (Lin (Li B A)) add Emp)$ **by** (*simp*
add:Line-unique)
then have $P53 : Bet-Point (Se A E) B \implies Line-on (Li B E) D \implies Line-on$
(Li B A) D **by** (*simp add:Line-on-trans*)
from $P30 P53$ **have** $P54 : Bet-Point (Se A E) B \implies \neg Line-on (Li B E) D$ **by**
blast

from $P47\ P54$ **have** $P55 : \text{Bet-Point } (Se\ A\ E)\ B \implies \text{Def } (Tri\ (Tr\ B\ E\ D))$ **by**
(simp add:Ang-simple-def Ang-to-Tri)
have $P56 : \text{Bet-Point } (Se\ A\ E)\ B \implies \text{Bet-Point } (Se\ E\ A)\ B$ **by** *(simp add:Bet-rev)*
from $P55\ P56$ **have** $P57 : \text{Bet-Point } (Se\ A\ E)\ B \implies$
 $Gr\ (Geos\ (Ang\ (An\ D\ B\ A))\ \text{add}\ Emp)\ (Geos\ (Ang\ (An\ B\ E\ D))\ \text{add}\ Emp)$ **by**
(simp add:Ang-external-Gr)
have $P58 : \text{Bet-Point } (Se\ A\ E)\ B \implies \text{Line-on } (Li\ E\ A)\ B$ **by** *(simp add:Line-Bet-on)*
have $P59 : \text{Line-on } (Li\ E\ A)\ A$ **by** *(simp add:Line-on-rule)*
from *assms* $P48\ P49\ P58\ P59$ **have** $P60 : \text{Bet-Point } (Se\ A\ E)\ B \implies$
 $Eq\ (Geos\ (Lin\ (Li\ E\ A))\ \text{add}\ Emp)\ (Geos\ (Lin\ (Li\ B\ A))\ \text{add}\ Emp)$ **by** *(simp*
add:Line-unique)
then **have** $P61 : \text{Bet-Point } (Se\ A\ E)\ B \implies \text{Line-on } (Li\ E\ A)\ C \implies \text{Line-on}$
 $(Li\ B\ A)\ C$ **by** *(simp add:Line-on-trans)*
from $P31\ P61$ **have** $P62 : \text{Bet-Point } (Se\ A\ E)\ B \implies \neg \text{Line-on } (Li\ E\ A)\ C$ **by**
blast
from $P46\ P62$ **have** $\text{Bet-Point } (Se\ A\ E)\ B \implies \text{Def } (Tri\ (Tr\ E\ A\ C))$ **by** *(simp*
add:Ang-simple-def Ang-to-Tri)
then **have** $P63 : \text{Bet-Point } (Se\ A\ E)\ B \implies \text{Def } (Tri\ (Tr\ E\ C\ A))$ **by** *(blast*
intro:Tri-def-rev Tri-def-trans)
from $P31\ P63$ **have** $P64 : \text{Bet-Point } (Se\ A\ E)\ B \implies$
 $Gr\ (Geos\ (Ang\ (An\ A\ E\ D))\ \text{add}\ Emp)\ (Geos\ (Ang\ (An\ E\ A\ C))\ \text{add}\ Emp)$ **by**
(simp add:Ang-external-Gr)
from $P63$ **have** $P65 : \text{Bet-Point } (Se\ A\ E)\ B \implies \text{Def } (Ang\ (An\ E\ A\ C))$ **by**
(blast intro:Tri-to-Ang Ang-def-inv)
have $P66 : \text{Bet-Point } (Se\ A\ E)\ B \implies \text{Line-on } (Li\ A\ E)\ B$ **by** *(simp add:Line-Bet-on)*
have $\text{Bet-Point } (Se\ A\ E)\ B \implies \text{Inv } (\text{Bet-Point } (Se\ E\ B)\ A)$ **by** *(simp add:Bet-iff)*
then **have** $P67 : \text{Bet-Point } (Se\ A\ E)\ B \implies \neg \text{Bet-Point } (Se\ E\ B)\ A$ **by** *(simp*
add:Inv-def)
have $P68 : \text{Line-on } (Li\ A\ C)\ C$ **by** *(simp add:Line-on-rule)*
have $P69 : \neg \text{Bet-Point } (Se\ C\ C)\ A$ **by** *(simp add:Bet-end-Point)*
from $P2$ **have** $\neg Eq\ (Geos\ (Poi\ C)\ \text{add}\ Emp)\ (Geos\ (Poi\ A)\ \text{add}\ Emp)$ **by** *(simp*
add:Ang-def)
then **have** $P70 : \neg Eq\ (Geos\ (Poi\ A)\ \text{add}\ Emp)\ (Geos\ (Poi\ C)\ \text{add}\ Emp)$ **by**
(blast intro:Eq-rev)
from *assms* $P65\ P66\ P67\ P68\ P69\ P70$ **have** $P71 : \text{Bet-Point } (Se\ A\ E)\ B \implies$
 $Eq\ (Geos\ (Ang\ (An\ E\ A\ C))\ \text{add}\ Emp)\ (Geos\ (Ang\ (An\ B\ A\ C))\ \text{add}\ Emp) \wedge$
 $\text{Def } (Ang\ (An\ B\ A\ C))$ **by** *(simp add:Ang-Point-swap)*
then **have** $P72 : \text{Bet-Point } (Se\ A\ E)\ B \implies$
 $Cong\ (Geos\ (Ang\ (An\ E\ A\ C))\ \text{add}\ Emp)\ (Geos\ (Ang\ (An\ B\ A\ C))\ \text{add}\ Emp)$
by *(blast intro:Ang-weektrans Ang-rev)*
from $P60$ **have** $P73 : \text{Bet-Point } (Se\ A\ E)\ B \implies \text{Line-on } (Li\ E\ A)\ D \implies$
 $\text{Line-on } (Li\ B\ A)\ D$ **by** *(simp add:Line-on-trans)*
from $P31\ P73$ **have** $P74 : \text{Bet-Point } (Se\ A\ E)\ B \implies \neg \text{Line-on } (Li\ E\ A)\ D$ **by**
blast
from $P46\ P74$ **have** $\text{Bet-Point } (Se\ A\ E)\ B \implies \text{Def } (Ang\ (An\ E\ A\ D))$ **by** *(simp*
add:Ang-simple-def)
then **have** $P75 : \text{Bet-Point } (Se\ A\ E)\ B \implies \text{Def } (Ang\ (An\ A\ E\ D))$ **by** *(blast*
intro:Ang-def-rev Ang-def-inv)
from $P64\ P65\ P71\ P72\ P75$ **have** $P76 : \text{Bet-Point } (Se\ A\ E)\ B \implies$

$Gr (Geos (Ang (An A E D)) add Emp) (Geos (Ang (An B A C)) add Emp)$ **by**
(blast intro:Ang-Gr-trans-Gr-Eq Ang-rev)
have $Bet-Point (Se A E) B \implies Inv (Bet-Point (Se B A) E)$ **by** *(simp add:Bet-iff)*
then have $Bet-Point (Se A E) B \implies \neg Bet-Point (Se B A) E$ **by** *(simp add:Inv-def)*
then have $P77 : Bet-Point (Se A E) B \implies \neg Bet-Point (Se A B) E$ **by** *(blast intro:Bet-rev)*
have $P78 : Line-on (Li E D) D$ **by** *(simp add:Line-on-rule)*
have $P79 : \neg Bet-Point (Se D D) E$ **by** *(simp add:Bet-end-Point)*
from $P47$ **have** $P80 : \neg Eq (Geos (Poi E) add Emp) (Geos (Poi B) add Emp)$
by *(blast intro:Eq-rev)*
from $P32$ **have** $\neg Eq (Geos (Poi D) add Emp) (Geos (Poi E) add Emp)$ **by**
(simp add:Bet-Point-def)
then have $P81 : \neg Eq (Geos (Poi E) add Emp) (Geos (Poi D) add Emp)$ **by**
(blast intro:Eq-rev)
from $P58 P75 P77 P78 P79 P80 P81$ **have** $P82 : Bet-Point (Se A E) B \implies$
 $Eq (Geos (Ang (An A E D)) add Emp) (Geos (Ang (An B E D)) add Emp) \wedge$
 $Def (Ang (An B E D))$ **by** *(simp add:Ang-Point-swap)*
then have $P83 : Bet-Point (Se A E) B \implies$
 $Cong (Geos (Ang (An A E D)) add Emp) (Geos (Ang (An B E D)) add Emp)$
by *(blast intro:Ang-weektrans)*
from $P9 P57 P75 P82 P83$ **have** $P84 : Bet-Point (Se A E) B \implies$
 $Gr (Geos (Ang (An D B A)) add Emp) (Geos (Ang (An A E D)) add Emp)$ **by**
(blast intro:Ang-Gr-trans-Gr-Eq Ang-rev)
from $P9 P71 P75 P76 P84$ **have** $P85 : Bet-Point (Se A E) B \implies$
 $Gr (Geos (Ang (An D B A)) add Emp) (Geos (Ang (An B A C)) add Emp)$ **by**
(blast intro:Ang-Gr-trans-Gr-Gr)
from $P9 P40 P71$ **have** $P86 : Bet-Point (Se A E) B \implies$
 $\neg Gr (Geos (Ang (An D B A)) add Emp) (Geos (Ang (An B A C)) add Emp)$
by *(simp add:Ang-not-Gr)*
from $P85 P86$ **have** $P87 : \neg Bet-Point (Se A E) B$ **by** *blast*
have $P88 : Bet-Point (Se E B) A \implies Line-on (Li E A) B$ **by** *(simp add:Line-Bet-on)*
from *assms* $P48 P49 P59 P88$ **have** $P89 : Bet-Point (Se E B) A \implies$
 $Eq (Geos (Lin (Li E A)) add Emp) (Geos (Lin (Li B A)) add Emp)$ **by** *(simp add:Line-unique)*
then have $P90 : Bet-Point (Se E B) A \implies Line-on (Li E A) C \implies Line-on$
 $(Li B A) C$ **by** *(simp add:Line-on-trans)*
from $P31 P90$ **have** $P91 : Bet-Point (Se E B) A \implies \neg Line-on (Li E A) C$ **by**
blast
from $P46 P91$ **have** $Bet-Point (Se E B) A \implies Def (Ang (An E A C))$ **by** *(simp add:Ang-simple-def)*
then have $P92 : Bet-Point (Se E B) A \implies Def (Tri (Tr A E C))$ **by** *(blast intro:Ang-to-Tri Tri-def-rev Tri-def-trans)*
then have $P93 : Bet-Point (Se E B) A \implies Gr (Geos (Ang (An C A B)) add$
 $Emp) (Geos (Ang (An A E C)) add Emp)$ **by** *(simp add:Ang-external-Gr)*
have $P94 : Bet-Point (Se E B) A \implies Line-on (Li B E) A$ **by** *(simp add:Line-Bet-on)*
from *assms* $P48 P49 P52 P94$ **have** $Bet-Point (Se E B) A \implies$
 $Eq (Geos (Lin (Li B E)) add Emp) (Geos (Lin (Li B A)) add Emp)$ **by** *(simp add:Line-unique)*

then have $P95 : \text{Bet-Point } (Se\ E\ B)\ A \implies \text{Line-on } (Li\ B\ E)\ D \implies \text{Line-on } (Li\ B\ A)\ D$ **by** (*simp add:Line-on-trans*)
from $P31\ P95$ **have** $P96 : \text{Bet-Point } (Se\ E\ B)\ A \implies \neg \text{Line-on } (Li\ B\ E)\ D$ **by** *blast*
from $P47\ P96$ **have** $\text{Bet-Point } (Se\ E\ B)\ A \implies \text{Def } (\text{Ang } (An\ B\ E\ D))$ **by** (*simp add:Ang-simple-def*)
then have $P97 : \text{Bet-Point } (Se\ E\ B)\ A \implies \text{Def } (\text{Tri } (Tr\ E\ D\ B))$ **by** (*blast intro:Ang-to-Tri Tri-def-rev Tri-def-trans*)
from $P32$ **have** $P98 : \text{Bet-Point } (Se\ D\ C)\ E$ **by** (*simp add:Bet-rev*)
from $P97\ P98$ **have** $P99 : \text{Bet-Point } (Se\ E\ B)\ A \implies$
 $\text{Gr } (\text{Geos } (\text{Ang } (An\ B\ E\ C))\ \text{add } \text{Emp})\ (\text{Geos } (\text{Ang } (An\ E\ B\ D))\ \text{add } \text{Emp})$ **by** (*simp add:Ang-external-Gr*)
from $P92$ **have** $P100 : \text{Bet-Point } (Se\ E\ B)\ A \implies \text{Def } (\text{Ang } (An\ A\ E\ C))$ **by** (*simp add:Tri-to-Ang*)
have $\text{Bet-Point } (Se\ E\ B)\ A \implies \text{Inv } (\text{Bet-Point } (Se\ B\ A)\ E)$ **by** (*simp add:Bet-iff*)
then have $\text{Bet-Point } (Se\ E\ B)\ A \implies \neg \text{Bet-Point } (Se\ B\ A)\ E$ **by** (*simp add:Inv-def*)
then have $P101 : \text{Bet-Point } (Se\ E\ B)\ A \implies \neg \text{Bet-Point } (Se\ A\ B)\ E$ **by** (*blast intro:Bet-rev*)
have $P102 : \text{Line-on } (Li\ E\ C)\ C$ **by** (*simp add:Line-on-rule*)
have $P103 : \neg \text{Bet-Point } (Se\ C\ C)\ E$ **by** (*simp add:Bet-end-Point*)
from $P32$ **have** $P104 : \neg \text{Eq } (\text{Geos } (Poi\ E)\ \text{add } \text{Emp})\ (\text{Geos } (Poi\ C)\ \text{add } \text{Emp})$ **by** (*simp add:Bet-Point-def*)
from $P80\ P88\ P100\ P101\ P102\ P103\ P104$ **have** $P105 : \text{Bet-Point } (Se\ E\ B)\ A \implies$
 $\text{Eq } (\text{Geos } (\text{Ang } (An\ A\ E\ C))\ \text{add } \text{Emp})\ (\text{Geos } (\text{Ang } (An\ B\ E\ C))\ \text{add } \text{Emp}) \wedge$
 $\text{Def } (\text{Ang } (An\ B\ E\ C))$ **by** (*simp add:Ang-Point-swap*)
then have $P106 : \text{Bet-Point } (Se\ E\ B)\ A \implies$
 $\text{Cong } (\text{Geos } (\text{Ang } (An\ A\ E\ C))\ \text{add } \text{Emp})\ (\text{Geos } (\text{Ang } (An\ B\ E\ C))\ \text{add } \text{Emp})$ **by** (*blast intro:Ang-weektrans*)
have $P107 : \text{Bet-Point } (Se\ E\ B)\ A \implies \text{Line-on } (Li\ B\ A)\ E$ **by** (*simp add:Line-Bet-on*)
have $\text{Bet-Point } (Se\ E\ B)\ A \implies \text{Inv } (\text{Bet-Point } (Se\ A\ E)\ B)$ **by** (*simp add:Bet-iff*)
then have $P108 : \text{Bet-Point } (Se\ E\ B)\ A \implies \neg \text{Bet-Point } (Se\ A\ E)\ B$ **by** (*simp add:Inv-def*)
have $P109 : \text{Line-on } (Li\ B\ D)\ D$ **by** (*simp add:Line-on-rule*)
have $P110 : \neg \text{Bet-Point } (Se\ D\ D)\ B$ **by** (*simp add:Bet-end-Point*)
from $P44$ **have** $P111 : \neg \text{Eq } (\text{Geos } (Poi\ B)\ \text{add } \text{Emp})\ (\text{Geos } (Poi\ D)\ \text{add } \text{Emp})$ **by** (*simp add:Ang-def*)
from $P44\ P47\ P107\ P108\ P109\ P110\ P111$ **have** $P112 : \text{Bet-Point } (Se\ E\ B)\ A \implies$
 $\text{Eq } (\text{Geos } (\text{Ang } (An\ A\ B\ D))\ \text{add } \text{Emp})\ (\text{Geos } (\text{Ang } (An\ E\ B\ D))\ \text{add } \text{Emp}) \wedge$
 $\text{Def } (\text{Ang } (An\ E\ B\ D))$ **by** (*simp add:Ang-Point-swap*)
then have $P113 : \text{Bet-Point } (Se\ E\ B)\ A \implies$
 $\text{Cong } (\text{Geos } (\text{Ang } (An\ A\ B\ D))\ \text{add } \text{Emp})\ (\text{Geos } (\text{Ang } (An\ E\ B\ D))\ \text{add } \text{Emp})$ **by** (*blast intro:Ang-weektrans*)
from $P2\ P93\ P100\ P105\ P106$ **have** $P114 : \text{Bet-Point } (Se\ E\ B)\ A \implies$
 $\text{Gr } (\text{Geos } (\text{Ang } (An\ C\ A\ B))\ \text{add } \text{Emp})\ (\text{Geos } (\text{Ang } (An\ B\ E\ C))\ \text{add } \text{Emp})$ **by** (*blast intro:Ang-Gr-trans-Gr-Eq*)
from $P44\ P99\ P105\ P112\ P113$ **have** $P115 : \text{Bet-Point } (Se\ E\ B)\ A \implies$

$Gr (Geos (Ang (An B E C)) add Emp) (Geos (Ang (An A B D)) add Emp)$ **by**
(blast intro:Ang-Gr-trans-Gr-Eq Ang-rev)
from $P2 P44 P105 P114 P115$ **have** $P116 : Bet-Point (Se E B) A \implies$
 $Gr (Geos (Ang (An C A B)) add Emp) (Geos (Ang (An A B D)) add Emp)$ **by**
(blast intro:Ang-Gr-trans-Gr-Gr)
from $P9 P41$ **have** $P117 : Cong (Geos (Ang (An C A B)) add Emp) (Geos (Ang$
 $(An A B D)) add Emp)$ **by** *(blast intro:Ang-weektrans Ang-rev Eq-rev)*
from $P2 P44 P117$ **have** $P118 : \neg Gr (Geos (Ang (An C A B)) add Emp) (Geos$
 $(Ang (An A B D)) add Emp)$
 $\wedge \neg Gr (Geos (Ang (An A B D)) add Emp) (Geos (Ang (An C A B)) add Emp)$
by *(simp add:Ang-not-Gr)*
from $P116 P118$ **have** $P119 : \neg Bet-Point (Se E B) A$ **by** *blast*
from $P50 P87 P119$ **have** $P120 : Bet-Point (Se B A) E$ **by** *blast*
then have $P121 : Line-on (Li B E) A$ **by** *(simp add:Line-Bet-on)*
from *assms* $P48 P49 P52 P121$ **have** $Eq (Geos (Lin (Li B E)) add Emp) (Geos$
 $(Lin (Li B A)) add Emp)$ **by** *(simp add:Line-unique)*
then have $P122 : Line-on (Li B E) C \implies Line-on (Li B A) C$ **by** *(simp*
add:Line-on-trans)
from $P31 P122$ **have** $P123 : \neg Line-on (Li B E) C$ **by** *blast*
from $P47 P123$ **have** $P124 : Def (Ang (An B E C))$ **by** *(simp add:Ang-simple-def)*
from $P32 P120 P124$ **have** $P125 : Cong (Geos (Ang (An C E A)) add Emp)$
 $(Geos (Ang (An B E D)) add Emp)$ **by** *(simp add:Ang-vertical)*
have $P126 : Eq (Geos (Seg (Se A C)) add Emp) (Geos (Seg (Se C A)) add Emp)$
by *(simp add:Seg-rev)*
have $P127 : Eq (Geos (Seg (Se B D)) add Emp) (Geos (Seg (Se D B)) add Emp)$
by *(simp add:Seg-rev)*
from $P9 P126 P127$ **have** $P128 : Eq (Geos (Seg (Se C A)) add Emp) (Geos (Seg$
 $(Se D B)) add Emp)$ **by** *(blast intro:Eq-trans Eq-rev)*
have $P129 : Eq (Geos (Ang (An B E D)) add Emp) (Geos (Ang (An D E B))$
 $add Emp)$ **by** *(simp add:Ang-roll)*
from $P125 P129$ **have** $P130 : Cong (Geos (Ang (An C E A)) add Emp) (Geos$
 $(Ang (An D E B)) add Emp)$ **by** *(blast intro:Ang-weektrans Ang-rev Eq-rev)*
from $P120$ **have** $P131 : Line-on (Li A B) E$ **by** *(simp add:Line-Bet-on)*
from $P119$ **have** $P132 : \neg Bet-Point (Se B E) A$ **by** *(blast intro:Bet-rev)*
from $P46$ **have** $P133 : \neg Eq (Geos (Poi A) add Emp) (Geos (Poi E) add Emp)$
by *(blast intro:Eq-rev)*
from $P2 P68 P69 P70 P131 P132 P133$ **have** $P134 :$
 $Eq (Geos (Ang (An C A B)) add Emp) (Geos (Ang (An C A E)) add Emp) \wedge$
 $Def (Ang (An C A E))$ **by** *(simp add:Ang-Point-swap)*
then have $P135 : Def (Tri (Tr C A E))$ **by** *(simp add:Ang-to-Tri)*
from $P9 P134$ **have** $P136 : Cong (Geos (Ang (An C A E)) add Emp) (Geos$
 $(Ang (An D B A)) add Emp)$ **by** *(blast intro:Ang-weektrans Ang-rev Eq-rev)*
from $P9$ **have** $P137 : Def (Ang (An D B A))$ **by** *simp*
from $P31 P47 P87 P109 P110 P111 P137$ **have** $P138 :$
 $Eq (Geos (Ang (An D B A)) add Emp) (Geos (Ang (An D B E)) add Emp) \wedge$
 $Def (Ang (An D B E))$ **by** *(simp add:Ang-Point-swap)*
then have $P139 : Def (Tri (Tr D B E))$ **by** *(simp add:Ang-to-Tri)*
from $P136 P138$ **have** $P140 : Cong (Geos (Ang (An C A E)) add Emp) (Geos$
 $(Ang (An D B E)) add Emp)$ **by** *(blast intro:Ang-weektrans Ang-rev Eq-rev)*

from $P128 P130 P135 P139 P140$ **have** $Cong (Geos (Tri (Tr C A E)) add Emp)$
 $(Geos (Tri (Tr D B E)) add Emp)$ **by** $(simp add:Tri-AAS)$
then have $P141 : Eq (Geos (Seg (Se A E)) add Emp) (Geos (Seg (Se B E)) add Emp)$ **by** $(simp add:Tri-Cong-def)$
have $P142 : Eq (Geos (Seg (Se B E)) add Emp) (Geos (Seg (Se E B)) add Emp)$
by $(simp add:Seg-rev)$
from $P141 P142$ **have** $P143 : Eq (Geos (Seg (Se A E)) add Emp) (Geos (Seg (Se E B)) add Emp)$ **by** $(blast intro:Eq-trans)$
from $P120$ **have** $P144 : Bet-Point (Se A B) E$ **by** $(simp add:Bet-rev)$
from $P143 P144$ **show** $\exists p. Eq (Geos (Seg (Se A p)) add Emp) (Geos (Seg (Se p B)) add Emp) \wedge Bet-Point (Se A B) p$ **by** $blast$
qed

theorem (in Congruence-Rule) Ang-bisection :

assumes

$Def (Ang (An A B C))$

shows

$\exists p. Cong (Geos (Ang (An A B p)) add Emp) (Geos (Ang (An p B C)) add Emp)$

$\wedge Ang-inside (An A B C) p \wedge Def (Ang (An A B p)) \wedge Def (Ang (An p B C))$

proof –

have $P1 : Line-on (Li B C) B$ **by** $(simp add:Line-on-rule)$

have $P2 : Line-on (Li B C) C$ **by** $(simp add:Line-on-rule)$

from $assms$ **have** $P3 : \neg Eq (Geos (Poi B) add Emp) (Geos (Poi C) add Emp)$ **by** $(simp add:Ang-def)$

from $assms$ **have** $\neg Eq (Geos (Poi A) add Emp) (Geos (Poi B) add Emp)$ **by** $(simp add:Ang-def)$

then have $P4 : \neg Eq (Geos (Poi B) add Emp) (Geos (Poi A) add Emp)$ **by** $(blast intro:Eq-rev)$

from $P1 P2 P3 P4$ **have** $\exists p. Eq (Geos (Seg (Se B A)) add Emp) (Geos (Seg (Se B p)) add Emp)$

$\wedge \neg Bet-Point (Se p C) B \wedge Line-on (Li B C) p \wedge \neg Eq (Geos (Poi B) add Emp) (Geos (Poi p) add Emp)$ **by** $(simp add:Seg-move-sameside)$

then obtain $C2 :: Point$ **where** $P5 : Eq (Geos (Seg (Se B A)) add Emp) (Geos (Seg (Se B C2)) add Emp)$

$\wedge \neg Bet-Point (Se C2 C) B \wedge Line-on (Li B C) C2 \wedge \neg Eq (Geos (Poi B) add Emp) (Geos (Poi C2) add Emp)$ **by** $blast$

then have $P6 : Line-on (Li B C) C2$ **by** $simp$

then have $P7 : Eq (Geos (Poi C2) add Emp) (Geos (Poi A) add Emp) \implies Line-on (Li B C) A$ **by** $(simp add:Point-Eq)$

from $assms$ **have** $P8 : Def (Tri (Tr A B C))$ **by** $(simp add:Ang-to-Tri)$

then have $P9 : \neg Line-on (Li B C) A$ **by** $(simp add:Tri-def-Line)$

from $P7 P9$ **have** $P10 : \neg Eq (Geos (Poi C2) add Emp) (Geos (Poi A) add Emp)$ **by** $blast$

then have $\exists p. Eq (Geos (Seg (Se C2 p)) add Emp) (Geos (Seg (Se p A)) add Emp) \wedge Bet-Point (Se C2 A) p$ **by** $(simp add:Seg-bisection)$

then obtain $D :: Point$ **where** $P11 : Eq (Geos (Seg (Se C2 D)) add Emp) (Geos (Seg (Se D A)) add Emp)$

$\wedge Bet-Point (Se C2 A) D$ **by** $blast$

have $P12 : \text{Line-on } (Li\ B\ A)\ B$ **by** (*simp add:Line-on-rule*)
from $P1\ P5\ P6\ P12$ **have** $P13 : \text{Line-on } (Li\ B\ A)\ C2 \implies$
 $\text{Eq } (\text{Geos } (\text{Lin } (Li\ B\ A))\ \text{add Emp})\ (\text{Geos } (\text{Lin } (Li\ B\ C))\ \text{add Emp})$ **by** (*simp*
add:Line-unique)
from *assms* **have** $P14 : \neg \text{Eq } (\text{Geos } (\text{Lin } (Li\ B\ A))\ \text{add Emp})\ (\text{Geos } (\text{Lin } (Li\ B\ C))\ \text{add Emp})$ **by** (*simp add:Ang-def*)
from $P13\ P14$ **have** $P15 : \neg \text{Line-on } (Li\ B\ A)\ C2$ **by** *blast*
from $P11$ **have** $P16 : \text{Bet-Point } (\text{Se } C2\ A)\ D$ **by** *simp*
then **have** $P17 : \text{Line-on } (Li\ C2\ A)\ D$ **by** (*simp add:Line-Bet-on*)
have $P18 : \text{Line-on } (Li\ C2\ A)\ A$ **by** (*simp add:Line-on-rule*)
have $P19 : \text{Line-on } (Li\ B\ A)\ A$ **by** (*simp add:Line-on-rule*)
from $P16$ **have** $P20 : \neg \text{Eq } (\text{Geos } (\text{Poi } A)\ \text{add Emp})\ (\text{Geos } (\text{Poi } D)\ \text{add Emp})$
by (*simp add:Bet-Point-def*)
from $P17\ P18\ P19\ P20$ **have** $P21 : \text{Line-on } (Li\ B\ A)\ D \implies$
 $\text{Eq } (\text{Geos } (\text{Lin } (Li\ C2\ A))\ \text{add Emp})\ (\text{Geos } (\text{Lin } (Li\ B\ A))\ \text{add Emp})$ **by** (*simp*
add:Line-unique)
have $P22 : \text{Line-on } (Li\ C2\ A)\ C2$ **by** (*simp add:Line-on-rule*)
from $P21\ P22$ **have** $P23 : \text{Line-on } (Li\ B\ A)\ D \implies \text{Line-on } (Li\ B\ A)\ C2$ **by**
(*simp add:Line-on-trans*)
from $P15\ P23$ **have** $P24 : \neg \text{Line-on } (Li\ B\ A)\ D$ **by** *blast*
from $P4\ P24$ **have** $\text{Def } (\text{Ang } (An\ B\ A\ D))$ **by** (*simp add:Ang-simple-def*)
then **have** $P25 : \text{Def } (\text{Tri } (Tr\ A\ B\ D))$ **by** (*blast intro:Ang-to-Tri Tri-def-rev*
Tri-def-trans)
from $P4\ P15$ **have** $P26 : \text{Def } (\text{Ang } (An\ B\ A\ C2))$ **by** (*simp add:Ang-simple-def*)
then **have** $\text{Def } (\text{Tri } (Tr\ C2\ B\ A))$ **by** (*blast intro:Ang-to-Tri Tri-def-rev Tri-def-trans*)
then **have** $P27 : \neg \text{Line-on } (Li\ C2\ B)\ A$ **by** (*simp add:Tri-def-Line*)
have $P28 : \text{Line-on } (Li\ C2\ B)\ C2$ **by** (*simp add:Line-on-rule*)
from $P16$ **have** $P29 : \neg \text{Eq } (\text{Geos } (\text{Poi } D)\ \text{add Emp})\ (\text{Geos } (\text{Poi } C2)\ \text{add Emp})$
by (*simp add:Bet-Point-def*)
from $P17\ P22\ P28\ P29$ **have** $P30 : \text{Line-on } (Li\ C2\ B)\ D \implies$
 $\text{Eq } (\text{Geos } (\text{Lin } (Li\ C2\ A))\ \text{add Emp})\ (\text{Geos } (\text{Lin } (Li\ C2\ B))\ \text{add Emp})$ **by** (*simp*
add:Line-unique)
from $P18\ P30$ **have** $P31 : \text{Line-on } (Li\ C2\ B)\ D \implies \text{Line-on } (Li\ C2\ B)\ A$ **by**
(*simp add:Line-on-trans*)
from $P27\ P31$ **have** $P32 : \neg \text{Line-on } (Li\ C2\ B)\ D$ **by** *blast*
from $P5$ **have** $P33 : \neg \text{Eq } (\text{Geos } (\text{Poi } C2)\ \text{add Emp})\ (\text{Geos } (\text{Poi } B)\ \text{add Emp})$
by (*blast intro:Eq-rev*)
from $P32\ P33$ **have** $\text{Def } (\text{Ang } (An\ C2\ B\ D))$ **by** (*simp add:Ang-simple-def*)
then **have** $P34 : \text{Def } (\text{Tri } (Tr\ C2\ B\ D))$ **by** (*simp add:Ang-to-Tri*)
have $P35 : \text{Eq } (\text{Geos } (\text{Seg } (\text{Se } B\ A))\ \text{add Emp})\ (\text{Geos } (\text{Seg } (\text{Se } A\ B))\ \text{add Emp})$
by (*simp add:Seg-rev*)
have $P36 : \text{Eq } (\text{Geos } (\text{Seg } (\text{Se } B\ C2))\ \text{add Emp})\ (\text{Geos } (\text{Seg } (\text{Se } C2\ B))\ \text{add
Emp})$ **by** (*simp add:Seg-rev*)
from $P5$ **have** $P37 : \text{Eq } (\text{Geos } (\text{Seg } (\text{Se } B\ A))\ \text{add Emp})\ (\text{Geos } (\text{Seg } (\text{Se } B\ C2))\ \text{add
Emp})$ **by** *simp*
from $P35\ P36\ P37$ **have** $P38 : \text{Eq } (\text{Geos } (\text{Seg } (\text{Se } A\ B))\ \text{add Emp})\ (\text{Geos } (\text{Seg } (\text{Se } C2\ B))\ \text{add
Emp})$ **by** (*blast intro:Eq-rev Eq-trans*)
have $P39 : \text{Eq } (\text{Geos } (\text{Seg } (\text{Se } C2\ D))\ \text{add Emp})\ (\text{Geos } (\text{Seg } (\text{Se } D\ C2))\ \text{add
Emp})$ **by** (*simp add:Seg-rev*)

from $P11\ P39$ **have** $P40 : Eq\ (Geos\ (Seg\ (Se\ D\ A))\ add\ Emp)\ (Geos\ (Seg\ (Se\ D\ C2))\ add\ Emp)$ **by** $(blast\ intro:Eq\ rev\ Eq\ trans)$
from $P25\ P34\ P38\ P40$ **have** $Cong\ (Geos\ (Tri\ (Tr\ A\ B\ D))\ add\ Emp)\ (Geos\ (Tri\ (Tr\ C2\ B\ D))\ add\ Emp)$ **by** $(simp\ add:Tri\ SSS)$
then **have** $P41 : Cong\ (Geos\ (Ang\ (An\ D\ B\ A))\ add\ Emp)\ (Geos\ (Ang\ (An\ D\ B\ C2))\ add\ Emp)$ **by** $(simp\ add:Tri\ Cong\ def)$
have $P42 : Eq\ (Geos\ (Ang\ (An\ D\ B\ A))\ add\ Emp)\ (Geos\ (Ang\ (An\ A\ B\ D))\ add\ Emp)$ **by** $(simp\ add:Ang\ roll)$
from $P41\ P42$ **have** $P43 : Cong\ (Geos\ (Ang\ (An\ A\ B\ D))\ add\ Emp)\ (Geos\ (Ang\ (An\ D\ B\ C2))\ add\ Emp)$ **by** $(blast\ intro:Ang\ weektrans\ Ang\ rev\ Eq\ rev)$
from $P34$ **have** $P44 : Def\ (Ang\ (An\ D\ B\ C2))$ **by** $(blast\ intro:Tri\ to\ Ang\ Ang\ def\ rev\ Ang\ def\ inv)$
have $P45 : Line\ on\ (Li\ B\ D)\ D$ **by** $(simp\ add:Line\ on\ rule)$
have $P46 : \neg\ Bet\ Point\ (Se\ D\ D)\ B$ **by** $(simp\ add:Bet\ end\ Point)$
from $P33$ **have** $P47 : \neg\ Eq\ (Geos\ (Poi\ B)\ add\ Emp)\ (Geos\ (Poi\ C2)\ add\ Emp)$ **by** $(blast\ intro:Eq\ rev)$
from $P3\ P6\ P47$ **have** $P48 : Line\ on\ (Li\ B\ C2)\ C$ **by** $(simp\ add:Line\ on\ rev)$
from $P34$ **have** $P49 : \neg\ Eq\ (Geos\ (Poi\ B)\ add\ Emp)\ (Geos\ (Poi\ D)\ add\ Emp)$ **by** $(simp\ add:Tri\ def)$
from $P3\ P5\ P44\ P45\ P46\ P48\ P49$ **have** $P50 : Eq\ (Geos\ (Ang\ (An\ D\ B\ C2))\ add\ Emp)\ (Geos\ (Ang\ (An\ D\ B\ C))\ add\ Emp)$
 $\wedge\ Def\ (Ang\ (An\ D\ B\ C))$ **by** $(simp\ add:Ang\ Point\ swap)$
from $P43\ P50$ **have** $P51 : Cong\ (Geos\ (Ang\ (An\ A\ B\ D))\ add\ Emp)\ (Geos\ (Ang\ (An\ D\ B\ C))\ add\ Emp)$ **by** $(blast\ intro:Ang\ weektrans\ Ang\ rev\ Eq\ rev)$
from $P25$ **have** $P52 : Def\ (Ang\ (An\ A\ B\ D))$ **by** $(simp\ add:Tri\ to\ Ang)$
from $P26$ **have** $P53 : Def\ (Ang\ (An\ A\ B\ C2))$ **by** $(blast\ intro:Ang\ def\ rev\ Ang\ def\ inv)$
then **have** $P54 : \neg\ Eq\ (Geos\ (Lin\ (Li\ B\ A))\ add\ Emp)\ (Geos\ (Lin\ (Li\ B\ C2))\ add\ Emp)$ **by** $(simp\ add:Ang\ def)$
from $P16$ **have** $P55 : Bet\ Point\ (Se\ A\ C2)\ D$ **by** $(simp\ add:Bet\ rev)$
from $P4\ P5\ P54\ P55$ **have** $P56 : Ang\ inside\ (An\ A\ B\ C2)\ D$ **by** $(simp\ add:Ang\ inside\ Bet\ Point)$
have $P57 : \neg\ Bet\ Point\ (Se\ A\ A)\ B$ **by** $(simp\ add:Bet\ end\ Point)$
from $P3\ P4\ P5\ P6\ P19\ P53\ P56\ P57$ **have** $P58 : Ang\ inside\ (An\ A\ B\ C)\ D$ **by** $(simp\ add:Ang\ inside\ trans)$
from $P50\ P51\ P52\ P58$ **show** $\exists p.\ Cong\ (Geos\ (Ang\ (An\ A\ B\ p))\ add\ Emp)\ (Geos\ (Ang\ (An\ p\ B\ C))\ add\ Emp)$
 $\wedge\ Ang\ inside\ (An\ A\ B\ C)\ p \wedge Def\ (Ang\ (An\ A\ B\ p)) \wedge Def\ (Ang\ (An\ p\ B\ C))$
by $blast$
qed

end

References

- [1] D. Hilbert. *The Foundations of Geometry*. <https://math.berkeley.edu/wodzicki/160/Hilbert.pdf>.