Verified Construction of Static Single Assignment Form

Sebastian Ullrich Denis Lohner

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Abstract

We define a functional variant of the static single assignment (SSA) form construction algorithm described by Braun et al. [2], which combines simplicity and efficiency. The definition is based on a general, abstract control flow graph representation using Isabelle locales. We prove that the algorithm's output is semantically equivalent to the input according to a small-step semantics, and that it is in minimal SSA form for the common special case of reducible inputs. We then show the satisfiability of the locale assumptions by giving instantiations for a simple While language. Furthermore, we use a generic instantiation based on typedefs in order to extract ML code and replace the unverified SSA construction algorithm of the CompCertSSA project [1] with it.

Contents

1 Prelude

1.1 Miscellaneous Lemmata

theory *FormalSSA-Misc*
imports *Main HOL−Library.Sublist*
begin

lemma *length-1-last-hd*: $length\ ns = 1 \Longrightarrow last\ ns = hd\ ns$
⟨*proof*⟩

lemma *not-in-butlast*[*simp*]: $[\![x \in set\ ys;\ x \notin set\ (butlast\ ys)]\!] \Longrightarrow x = last\ ys$
⟨*proof*⟩

lemma *in-set-butlastI*: $x \in set\ xs \Longrightarrow x \neq last\ xs \Longrightarrow x \in set\ (butlast\ xs)$
⟨*proof*⟩

lemma *butlast-strict-prefix*: $xs \neq [] \Longrightarrow strict\text{-}prefix\ (butlast\ xs)\ xs$
⟨*proof*⟩

lemma *set-tl*: $set\ (tl\ xs) \subseteq set\ xs$
⟨*proof*⟩

lemma *in-set-tlD*[*elim*]: $x \in set\ (tl\ xs) \Longrightarrow x \in set\ xs$
⟨*proof*⟩

lemma *suffix-unsnoc*:
assumes *suffix xs ys xs* ≠ []
obtains *x* **where** *xs* = *butlast xs*@[*x*] *ys* = *butlast ys*@[*x*]
⟨*proof*⟩

lemma *prefix-split-first*:
assumes $x \in set\ xs$
obtains *as* **where** *prefix* (*as*@[*x*]) *xs* **and** $x \notin set\ as$
⟨*proof*⟩

lemma *in-prefix*[*elim*]:
assumes *prefix xs ys* **and** $x \in set\ xs$

shows $x \in set\ ys$
⟨*proof*⟩

lemma *strict-prefix-butlast*:
assumes *prefix xs* (*butlast ys*) $ys \neq []$
shows *strict-prefix xs ys*
⟨*proof*⟩

lemma *prefix-tl-subset*: *prefix xs ys* $\Longrightarrow$ *set* (*tl xs*) $\subseteq$ *set* (*tl ys*)
⟨*proof*⟩

lemma *suffix-tl-subset*: *suffix xs ys* $\Longrightarrow$ *set* (*tl xs*) $\subseteq$ *set* (*tl ys*)
⟨*proof*⟩

lemma *set-tl-append'*: *set* (*tl* (*xs* @ *ys*)) $\subseteq$ *set* (*tl xs*) $\cup$ *set ys*
⟨*proof*⟩

lemma *last-in-tl*: *length xs* $>$ *1* $\Longrightarrow$ *last xs* $\in$ *set* (*tl xs*)
⟨*proof*⟩

lemma *concat-join*: $xs \neq [] \Longrightarrow ys \neq [] \Longrightarrow$ *last xs* $=$ *hd ys* $\Longrightarrow$ *butlast xs*@*ys* $=$ *xs*@*tl ys*
⟨*proof*⟩

lemma *fold-induct*[*case-names Nil Cons*]: *P s* $\Longrightarrow$ ($\bigwedge$*x s*. *x* $\in$ *set xs* $\Longrightarrow$ *P s* $\Longrightarrow$ *P* (*f x s*)) $\Longrightarrow$ *P* (*fold f xs s*)
⟨*proof*⟩

lemma *fold-union-elem*:
assumes $x \in$ *fold* ($\cup$) *xss xs*
obtains *ys* **where** $x \in ys$ $ys \in$ *set xss* $\cup$ $\{xs\}$
⟨*proof*⟩

lemma *fold-union-elemI*:
assumes $x \in ys$ $ys \in$ *set xss* $\cup$ $\{xs\}$
shows $x \in$ *fold* ($\cup$) *xss xs*
⟨*proof*⟩

lemma *fold-union-elemI'*:
assumes $x \in xs \vee (\exists\, xs \in set\ xss.\ x \in xs)$
shows $x \in$ *fold* ($\cup$) *xss xs*
⟨*proof*⟩

lemma *fold-union-finite*[*intro*!]:
assumes *finite xs* $\forall\, xs \in set\ xss.$ *finite xs*
shows *finite* (*fold* ($\cup$) *xss xs*)
⟨*proof*⟩

lemma *in-set-zip-map*:

assumes $(x,y) \in set\ (zip\ xs\ (map\ f\ ys))$
obtains y' **where** $(x,y') \in set\ (zip\ xs\ ys)\ f\ y' = y$
⟨*proof*⟩

lemma *dom-comp-subset*: $g\ `\ dom\ (f \circ g) \subseteq dom\ f$
⟨*proof*⟩

lemma *finite-dom-comp*:
assumes *finite* $(dom\ f)$ *inj-on* $g\ (dom\ (f \circ g))$
shows *finite* $(dom\ (f \circ g))$
⟨*proof*⟩

lemma *the1-list*: $\exists!x \in set\ xs.\ P\ x \Longrightarrow (THE\ x.\ x \in set\ xs \wedge P\ x) = hd\ (filter\ P\ xs)$
⟨*proof*⟩

lemma *set-zip-leftI*:
assumes *length xs* = *length ys*
assumes $y \in set\ ys$
obtains x **where** $(x,y) \in set\ (zip\ xs\ ys)$
⟨*proof*⟩

lemma *butlast-idx*:
assumes $y \in set\ (butlast\ xs)$
obtains i **where** $xs\ !\ i = y\ i < length\ xs - 1$
⟨*proof*⟩

lemma *butlast-idx'*:
assumes $xs\ !\ i = y\ i < length\ xs - 1\ length\ xs > 1$
shows $y \in set\ (butlast\ xs)$
⟨*proof*⟩

lemma *card-eq-1-singleton*:
assumes *card A* = *1*
obtains x **where** $A = \{x\}$
⟨*proof*⟩

lemma *set-take-two*:
assumes *card* $A \geq 2$
obtains $x\ y$ **where** $x \in A\ y \in A\ x \neq y$
⟨*proof*⟩

lemma *singleton-list-hd-last*: *length xs* = *1* $\Longrightarrow$ *hd xs* = *last xs*
⟨*proof*⟩

lemma *distinct-hd-tl*: *distinct xs* $\Longrightarrow$ *hd xs* $\notin$ *set* (*tl xs*)
⟨*proof*⟩

lemma *set-mono-strict-prefix*: *strict-prefix xs ys* $\Longrightarrow$ *set xs* $\subseteq$ *set* (*butlast ys*)

⟨*proof*⟩

lemma *set-butlast-distinct*: *distinct xs* ⟹ *set* (*butlast xs*) ∩ {*last xs*} = {}
⟨*proof*⟩

lemma *disjoint-elem*[*elim*]: *A* ∩ *B* = {} ⟹ *x* ∈ *A* ⟹ *x* ∉ *B* ⟨*proof*⟩

lemma *prefix-butlastD*[*elim*]: *prefix xs* (*butlast ys*) ⟹ *prefix xs ys*
⟨*proof*⟩

lemma *butlast-prefix*: *prefix xs ys* ⟹ *prefix* (*butlast xs*) (*butlast ys*)
⟨*proof*⟩

lemma *hd-in-butlast*: *length xs* > *1* ⟹ *hd xs* ∈ *set* (*butlast xs*)
⟨*proof*⟩

lemma *nonsimple-length-gt-1*: *xs* ≠ [] ⟹ *hd xs* ≠ *last xs* ⟹ *length xs* > *1*
⟨*proof*⟩

lemma *set-hd-tl*: *xs* ≠ [] ⟹ *set* [*hd xs*] ∪ *set* (*tl xs*) = *set xs*
⟨*proof*⟩

lemma *fold-update-conv*:
fold (λ*n m*. *m*(*n* ↦ *g n*)) *xs m x* =
(*if* (*x* ∈ *set xs*) *then Some* (*g x*) *else m x*)
⟨*proof*⟩

lemmas *removeAll-le* = *length-removeAll-less-eq*

lemmas *removeAll-less* [*intro*] = *length-removeAll-less*

lemma *removeAll-induct*:
assumes ⋀*xs*. (⋀*x*. *x* ∈ *set xs* ⟹ *P* (*removeAll x xs*)) ⟹ *P xs*
shows *P xs*
⟨*proof*⟩

lemma *The-Min*: *Ex1 P* ⟹ *The P* = *Min* {*x*. *P x*}
⟨*proof*⟩

lemma *The-Max*: *Ex1 P* ⟹ *The P* = *Max* {*x*. *P x*}
⟨*proof*⟩

lemma *set-sorted-list-of-set-remove* [*simp*]:
set (*sorted-list-of-set* (*Set.remove x A*)) = *Set.remove x* (*set* (*sorted-list-of-set A*))
⟨*proof*⟩

lemma *set-minus-one*: ⟦*v* ≠ *v*′; *v*′ ∈ *set vs*⟧ ⟹ *set vs* − {*v*′} ⊆ {*v*} ⟷ *set vs*
= {*v*′} ∨ *set vs* = {*v*,*v*′}
⟨*proof*⟩

lemma *set-single-hd*: *set vs* = {*v*} ⟹ *hd vs* = *v*
⟨*proof*⟩

lemma *set-double-filter-hd*: ⟦ *set vs* = {*v*,*v'*}; *v* ≠ *v'* ⟧ ⟹ *hd* [*v'*←*vs* . *v'* ≠ *v*] = *v'*
⟨*proof*⟩

lemma *map-option-the*: *x* = *map-option f y* ⟹ *x* ≠ *None* ⟹ *the x* = *f* (*the y*)
⟨*proof*⟩

end

1.2 Serial Relations

A serial relation on a finite carrier induces a cycle.

theory *Serial-Rel*
imports *Main*
begin

definition *serial-on A r* ⟷ (∀ *x* ∈ *A*. ∃ *y* ∈ *A*. (*x*,*y*) ∈ *r*)
lemmas *serial-onI* = *serial-on-def*[*THEN iffD2*, *rule-format*]
lemmas *serial-onE* = *serial-on-def*[*THEN iffD1*, *rule-format*, *THEN bexE*]

fun *iterated-serial-on* :: *'a set* ⇒ *'a rel* ⇒ *'a* ⇒ *nat* ⇒ *'a* **where**
iterated-serial-on A r x 0 = *x*
| *iterated-serial-on A r x* (*Suc n*) = (*SOME y*. *y* ∈ *A* ∧ (*iterated-serial-on A r x n*,*y*) ∈ *r*)

lemma *iterated-serial-on-linear*: *iterated-serial-on A r x* (*n*+*m*) = *iterated-serial-on A r* (*iterated-serial-on A r x n*) *m*
⟨*proof*⟩

lemma *iterated-serial-on-in-A*:
assumes *serial-on A r a* ∈ *A*
shows *iterated-serial-on A r a n* ∈ *A*
⟨*proof*⟩

lemma *iterated-serial-on-in-power*:
assumes *serial-on A r a* ∈ *A*
shows (*a*,*iterated-serial-on A r a n*) ∈ *r* ^^ *n*
⟨*proof*⟩

lemma *trancl-powerI*: *a* ∈ *R* ^^ *n* ⟹ *n* > *0* ⟹ *a* ∈ R^+
⟨*proof*⟩

theorem *serial-on-finite-cycle*:
assumes *serial-on A r A* ≠ {} *finite A*
obtains *a* **where** *a* ∈ *A* (*a*,*a*) ∈ r^+

⟨*proof*⟩

end

1.3 Mapping Extensions

Some lifted definition on mapping and efficient implementations.

theory *Mapping-Exts*
imports *HOL−Library.Mapping FormalSSA-Misc*
begin

lift-definition *mapping-delete-all* :: ('*a* ⇒ *bool*) ⇒ ('*a*,'*b*) *mapping* ⇒ ('*a*,'*b*) *mapping*
  is λ*P m x*. *if* (*P x*) *then None else m x* ⟨*proof*⟩
lift-definition *map-keys* :: ('*a* ⇒ '*b*) ⇒ ('*a*,'*c*) *mapping* ⇒ ('*b*,'*c*) *mapping*
  is λ*f m x*. *if f* −‘ {*x*} ≠ {} *then m* (*THE k*. *f* −‘ {*x*} = {*k*}) *else None* ⟨*proof*⟩
lift-definition *map-values* :: ('*a* ⇒ '*b* ⇒ '*c option*) ⇒ ('*a*,'*b*) *mapping* ⇒ ('*a*,'*c*) *mapping*
  is λ*f m x*. *Option.bind* (*m x*) (*f x*) ⟨*proof*⟩
lift-definition *restrict-mapping* :: ('*a* ⇒ '*b*) ⇒ '*a set* ⇒ ('*a*, '*b*) *mapping*
  is λ*f*. *restrict-map* (*Some* ∘ *f*) ⟨*proof*⟩
lift-definition *mapping-add* :: ('*a*, '*b*) *mapping* ⇒ ('*a*, '*b*) *mapping* ⇒ ('*a*, '*b*) *mapping*
  is (++) ⟨*proof*⟩

definition *mmap* = *Mapping.map id*

lemma *lookup-map-keys*: *Mapping.lookup* (*map-keys f m*) *x* = (*if f* −‘ {*x*} ≠ {} *then Mapping.lookup m* (*THE k*. *f* −‘ {*x*} = {*k*}) *else None*)
⟨*proof*⟩

lemma *Mapping-Mapping-lookup* [*simp*, *code-unfold*]: *Mapping.Mapping* (*Mapping.lookup m*) = *m* ⟨*proof*⟩
declare *Mapping.lookup.abs-eq*[*simp*]

lemma *Mapping-eq-lookup*: *m* = *m*′ ⟷ *Mapping.lookup m* = *Mapping.lookup m*′
⟨*proof*⟩

lemma *map-of-map-if-conv*:
  map-of (*map* (λ*k*. (*k*, *f k*)) *xs*) *x* = (*if* (*x* ∈ *set xs*) *then Some* (*f x*) *else None*)
  ⟨*proof*⟩

lemma *Mapping-lookup-map*: *Mapping.lookup* (*Mapping.map f g m*) *a* = *map-option g* (*Mapping.lookup m* (*f a*))
  ⟨*proof*⟩

lemma *Mapping-lookup-map-default*: *Mapping.lookup* (*Mapping.map-default k d f m*) *k*′ = (*if k* = *k*′
  then (*Some* ∘ *f*) (*case Mapping.lookup m k of None* ⇒ *d* | *Some x* ⇒ *x*)

else Mapping.lookup m k′)
⟨*proof*⟩

lemma *Mapping-lookup-mapping-add*: *Mapping.lookup* (*mapping-add m1 m2*) *k* =
case-option (*Mapping.lookup m1 k*) *Some* (*Mapping.lookup m2 k*)
⟨*proof*⟩

lemma *Mapping-lookup-map-values*: *Mapping.lookup* (*map-values f m*) *k* =
Option.bind (*Mapping.lookup m k*) (*f k*)
⟨*proof*⟩

lemma *lookup-fold-update* [*simp*]: *Mapping.lookup* (*fold* (λ*n*. *Mapping.update n* (*g n*)) *xs m*) *x*
= (*if* (*x* ∈ *set xs*) *then Some* (*g x*) *else Mapping.lookup m x*)
⟨*proof*⟩

lemma *mapping-eq-iff*: *m1* = *m2* ⟷ (∀ *k*. *Mapping.lookup m1 k* = *Mapping.lookup m2 k*)
⟨*proof*⟩

lemma *lookup-delete*: *Mapping.lookup* (*Mapping.delete k m*) *k′* = (*if k* = *k′ then None else Mapping.lookup m k′*)
⟨*proof*⟩

lemma *keys-map-values*: *Mapping.keys* (*map-values f m*) = *Mapping.keys m* − {*k*∈*Mapping.keys m*. *f k* (*the* (*Mapping.lookup m k*)) = *None*}
⟨*proof*⟩

lemma *map-default-eq*: *Mapping.map-default k v f m* = *m* ⟷ (∃ *v*. *Mapping.lookup m k* = *Some v* ∧ *f v* = *v*)
⟨*proof*⟩

lemma *lookup-update-cases*: *Mapping.lookup* (*Mapping.update k v m*) *k′* = (*if k*=*k′ then Some v else Mapping.lookup m k′*)
⟨*proof*⟩

end

theory *RBT-Mapping-Exts*
imports
Mapping-Exts
HOL−*Library.RBT-Mapping*
HOL−*Library.RBT-Set*
begin

lemma *restrict-mapping-code* [*code*]:
restrict-mapping f (*RBT-Set.Set r*) = *RBT-Mapping.Mapping* (*RBT.map* (λ*a* -. *f a*) *r*)

⟨*proof*⟩

lemma *map-keys-code*:
assumes *inj f*
shows *map-keys f* (*RBT-Mapping.Mapping t*) = *RBT.fold* (λ*x v m*. *Mapping.update* (*f x*) *v m*) *t Mapping.empty*
⟨*proof*⟩

lemma *map-values-code* [*code*]:
map-values f (*RBT-Mapping.Mapping t*) = *RBT.fold* (λ*x v m*. *case* (*f x v*) *of None* ⇒ *m* | *Some v'* ⇒ *Mapping.update x v' m*) *t Mapping.empty*
⟨*proof*⟩

lemma [*code-unfold*]: *set* (*RBT.keys t*) = *RBT-Set.Set* (*RBT.map* (λ- -. ()) *t*)
⟨*proof*⟩

lemma *mmap-rbt-code* [*code*]: *mmap f* (*RBT-Mapping.Mapping t*) = *RBT-Mapping.Mapping* (*RBT.map* (λ-. *f*) *t*)
⟨*proof*⟩

lemma *mapping-add-code* [*code*]: *mapping-add* (*RBT-Mapping.Mapping t1*) (*RBT-Mapping.Mapping t2*) = *RBT-Mapping.Mapping* (*RBT.union t1 t2*)
⟨*proof*⟩

end

2 SSA Representation

2.1 Inductive Graph Paths

We extend the Graph framework with inductively defined paths. We adopt the convention of separating locale definitions into assumption-less base locales.

theory *Graph-path* **imports**
FormalSSA-Misc
Dijkstra-Shortest-Path.GraphSpec
CAVA-Automata.Digraph-Basic
begin

hide-const *Omega-Words-Fun.prefix Omega-Words-Fun.suffix*

type-synonym ('*n*, '*ed*) *edge* = ('*n* × '*ed* × '*n*)

definition *getFrom* :: ('*n*, '*ed*) *edge* ⇒ '*n* **where**
getFrom ≡ *fst*
definition *getData* :: ('*n*, '*ed*) *edge* ⇒ '*ed* **where**
getData ≡ *fst* ∘ *snd*
definition *getTo* :: ('*n*, '*ed*) *edge* ⇒ '*n* **where**

getTo ≡ *snd* ∘ *snd*

lemma *get-edge-simps* [*simp*]:
getFrom (*f*,*d*,*t*) = *f*
getData (*f*,*d*,*t*) = *d*
getTo (*f*,*d*,*t*) = *t*
⟨*proof*⟩

Predecessors of a node.

definition *pred* :: ('*v*,'*w*) *graph* ⇒ '*v* ⇒ ('*v*×'*w*) *set*
where *pred G v* ≡ {(*v'*,*w*). (*v'*,*w*,*v*)∈*edges G*}

lemma *pred-finite*[*simp*, *intro*]: *finite* (*edges G*) ⟹ *finite* (*pred G v*)
⟨*proof*⟩

lemma *pred-empty*[*simp*]: *pred empty v* = {} ⟨*proof*⟩

lemma (**in** *valid-graph*) *pred-subset*: *pred G v* ⊆ *V*×*UNIV*
⟨*proof*⟩

type-synonym ('*V*,'*W*,'*σ*,'*G*) *graph-pred-it* =
'*G* ⇒ '*V* ⇒ ('*V*×'*W*,'*σ*) *set-iterator*

locale *graph-pred-it-defs* =
fixes *pred-list-it* :: '*G* ⇒ '*V* ⇒ ('*V*×'*W*,('*V*×'*W*) *list*) *set-iterator*
begin
definition *pred-it g v* ≡ *it-to-it* (*pred-list-it g v*)
end

locale *graph-pred-it* = *graph* *α* *invar* + *graph-pred-it-defs pred-list-it*
for *α* :: '*G* ⇒ ('*V*,'*W*) *graph* **and** *invar* **and**
pred-list-it :: '*G* ⇒ '*V* ⇒ ('*V*×'*W*,('*V*×'*W*) *list*) *set-iterator* +
assumes *pred-list-it-correct*:
invar g ⟹ *set-iterator* (*pred-list-it g v*) (*pred* (*α g*) *v*)
begin
lemma *pred-it-correct*:
invar g ⟹ *set-iterator* (*pred-it g v*) (*pred* (*α g*) *v*)
⟨*proof*⟩

lemma *pi-pred-it*[*icf-proper-iteratorI*]:
proper-it (*pred-it S v*) (*pred-it S v*)
⟨*proof*⟩

lemma *pred-it-proper*[*proper-it*]:
proper-it' (λ*S*. *pred-it S v*) (λ*S*. *pred-it S v*)
⟨*proof*⟩
end

record ('*V*,'*W*,'*G*) *graph-ops* = ('*V*,'*W*,'*G*) *GraphSpec.graph-ops* +

gop-pred-list-it :: $'G \Rightarrow 'V \Rightarrow ('V \times 'W, ('V \times 'W)\ list)\ set\text{-}iterator$

lemma (**in** *graph-pred-it*) *pred-it-is-iterator*[*refine-transfer*]:
invar g $\Longrightarrow$ *set-iterator* (*pred-it g v*) (*pred* (α *g*) *v*)
⟨*proof*⟩

locale *StdGraphDefs* = *GraphSpec.StdGraphDefs ops*
+ *graph-pred-it-defs gop-pred-list-it ops*
for *ops* :: $('V,'W,'G,'m)$ *graph-ops-scheme*
begin
abbreviation *pred-list-it* **where** *pred-list-it* ≡ *gop-pred-list-it ops*
end

locale *StdGraph* = *StdGraphDefs* + *org:StdGraph* +
graph-pred-it α *invar pred-list-it*

locale *graph-path-base* =
graph-nodes-it-defs λg. *foldri* (αn *g*) +
graph-pred-it-defs λg *n*. *foldri* (*inEdges'* *g n*)
for
αe :: $'g \Rightarrow ('node \times 'edgeD \times 'node)\ set$ **and**
αn :: $'g \Rightarrow 'node\ list$ **and**
invar :: $'g \Rightarrow bool$ **and**
inEdges' :: $'g \Rightarrow 'node \Rightarrow ('node \times 'edgeD)\ list$
begin

definition *inEdges* :: $'g \Rightarrow 'node \Rightarrow ('node \times 'edgeD \times 'node)\ list$
where *inEdges g n* ≡ *map* ($\lambda(f,d)$. (f,d,n)) (*inEdges' g n*)

definition *predecessors* :: $'g \Rightarrow 'node \Rightarrow 'node\ list$ **where**
predecessors g n ≡ *map getFrom* (*inEdges g n*)

definition *successors* :: $'g \Rightarrow 'node \Rightarrow 'node\ list$ **where**
successors g m ≡ $[n\ .\ n \leftarrow \alpha n\ g,\ m \in set\ (predecessors\ g\ n)]$

declare *predecessors-def* [*code*]

declare [[*inductive-internals*]]
inductive *path* :: $'g \Rightarrow 'node\ list \Rightarrow bool$
for *g* :: $'g$
where
empty-path[*intro*]: $[\![n \in set\ (\alpha n\ g);\ invar\ g]\!] \Longrightarrow path\ g\ [n]$
| *Cons-path*[*intro*]: $[\![path\ g\ ns;\ n' \in set\ (predecessors\ g\ (hd\ ns))]\!] \Longrightarrow path\ g\ (n'\#ns)$

definition *path2* :: $'g \Rightarrow 'node \Rightarrow 'node\ list \Rightarrow 'node \Rightarrow bool$ (- ⊢ -–-→-

[51,0,0,51] 80) **where**
path2 g n ns m ≡ *path g ns* ∧ *n* = *hd ns* ∧ *m* = *last ns*

abbreviation *α g* ≡ (|*nodes* = *set* (*αn g*), *edges* = *αe g*|)
end

locale *graph-path* =
graph-path-base αe αn invar inEdges′ +
graph α invar +
ni: *graph-nodes-it α invar* λ*g*. *foldri* (*αn g*) +
pi: *graph-pred-it α invar* λ*g n*. *foldri* (*inEdges′ g n*)
for
αe :: *′g* ⇒ (*′node* × *′edgeD* × *′node*) *set* **and**
αn :: *′g* ⇒ *′node list* **and**
invar :: *′g* ⇒ *bool* **and**
inEdges′ :: *′g* ⇒ *′node* ⇒ (*′node* × *′edgeD*) *list*
begin
lemma *αn-correct*: *invar g* ⟹ *set* (*αn g*) ⊇ *getFrom* ‘ *αe g* ∪ *getTo* ‘ *αe g*
⟨*proof*⟩

lemma *αn-distinct*: *invar g* ⟹ *distinct* (*αn g*)
⟨*proof*⟩

lemma *inEdges-correct′*:
assumes *invar g*
shows *set* (*inEdges g n*) = (λ(*f*,*d*). (*f*,*d*,*n*)) ‘ (*pred* (*α g*) *n*)
⟨*proof*⟩

lemma *inEdges-correct* [*intro*!, *simp*]:
invar g ⟹ *set* (*inEdges g n*) = {(-, -, *t*). *t* = *n*} ∩ *αe g*
⟨*proof*⟩

lemma *in-set-αnI1* [*intro*]: ⟦*invar g*; *x* ∈ *getFrom* ‘ *αe g*⟧ ⟹ *x* ∈ *set* (*αn g*)
⟨*proof*⟩
lemma *in-set-αnI2* [*intro*]: ⟦*invar g*; *x* ∈ *getTo* ‘ *αe g*⟧ ⟹ *x* ∈ *set* (*αn g*)
⟨*proof*⟩

lemma *edge-to-node*:
assumes *invar g* **and** *e* ∈ *αe g*
obtains *getFrom e* ∈ *set* (*αn g*) **and** *getTo e* ∈ *set* (*αn g*)
⟨*proof*⟩

lemma *inEdge-to-edge*:
assumes *e* ∈ *set* (*inEdges g n*) **and** *invar g*
obtains *eD n′* **where** (*n′*,*eD*,*n*) ∈ *αe g*

⟨*proof*⟩

lemma *edge-to-inEdge*:
assumes $(n,eD,m) \in \alpha e\ g\ invar\ g$
obtains $(n,eD,m) \in set\ (inEdges\ g\ m)$
⟨*proof*⟩

lemma *edge-to-predecessors*:
assumes $(n,eD,m) \in \alpha e\ g\ invar\ g$
obtains $n \in set\ (predecessors\ g\ m)$
⟨*proof*⟩

lemma *predecessor-is-node*[*elim*]: $[\![n \in set\ (predecessors\ g\ n'); invar\ g]\!] \Longrightarrow n \in set\ (\alpha n\ g)$
⟨*proof*⟩

lemma *successor-is-node*[*elim*]: $[\![n \in set\ (predecessors\ g\ n'); n \in set\ (\alpha n\ g); invar\ g]\!] \Longrightarrow n' \in set\ (\alpha n\ g)$
⟨*proof*⟩

lemma *successors-predecessors*[*simp*]: $n \in set\ (\alpha n\ g) \Longrightarrow n \in set\ (successors\ g\ m) \longleftrightarrow m \in set\ (predecessors\ g\ n)$
⟨*proof*⟩

lemma *path-not-Nil*[*simp*, *dest*]: $path\ g\ ns \Longrightarrow ns \neq []$
⟨*proof*⟩

lemma *path2-not-Nil*[*simp*]: $g \vdash n - ns \rightarrow m \Longrightarrow ns \neq []$
⟨*proof*⟩

lemma *path2-not-Nil2*[*simp*]: $\neg\ g \vdash n - [] \rightarrow m$
⟨*proof*⟩

lemma *path2-not-Nil3*[*simp*]: $g \vdash n - ns \rightarrow m \Longrightarrow length\ ns \geq 1$
⟨*proof*⟩

lemma *empty-path2*[*intro*]: $[\![n \in set\ (\alpha n\ g); invar\ g]\!] \Longrightarrow g \vdash n - [n] \rightarrow n$
⟨*proof*⟩

lemma *Cons-path2*[*intro*]: $[\![g \vdash n - ns \rightarrow m; n' \in set\ (predecessors\ g\ n)]\!] \Longrightarrow g \vdash n' - n' \# ns \rightarrow m$
⟨*proof*⟩

lemma *path2-cases*:
assumes $g \vdash n - ns \rightarrow m$
obtains (*empty-path*) $ns = [n]\ m = n$
| (*Cons-path*) $g \vdash hd\ (tl\ ns) - tl\ ns \rightarrow m\ n \in set\ (predecessors\ g\ (hd\ (tl\ ns)))$
⟨*proof*⟩

lemma *path2-induct*[*consumes 1*, *case-names empty-path Cons-path*]:
assumes $g \vdash n{-}ns{\rightarrow}m$
assumes *empty*: *invar* $g \Longrightarrow P\ m\ [m]\ m$
assumes *Cons*: $\bigwedge ns\ n'\ n.\ g \vdash n{-}ns{\rightarrow}m \Longrightarrow P\ n\ ns\ m \Longrightarrow n' \in set\ (predecessors\ g\ n) \Longrightarrow P\ n'\ (n'\ \#\ ns)\ m$
shows $P\ n\ ns\ m$
⟨*proof*⟩

lemma *path-invar*[*intro*]: *path* $g\ ns \Longrightarrow$ *invar* g
⟨*proof*⟩

lemma *path-in-αn*[*intro*]: $[\![path\ g\ ns;\ n \in set\ ns]\!] \Longrightarrow n \in set\ (\alpha n\ g)$
⟨*proof*⟩

lemma *path2-in-αn*[*elim*]: $[\![g \vdash n{-}ns{\rightarrow}m;\ l \in set\ ns]\!] \Longrightarrow l \in set\ (\alpha n\ g)$
⟨*proof*⟩

lemma *path2-hd-in-αn*[*elim*]: $g \vdash n{-}ns{\rightarrow}m \Longrightarrow n \in set\ (\alpha n\ g)$
⟨*proof*⟩

lemma *path2-tl-in-αn*[*elim*]: $g \vdash n{-}ns{\rightarrow}m \Longrightarrow m \in set\ (\alpha n\ g)$
⟨*proof*⟩

lemma *path2-forget-hd*[*simp*]: $g \vdash n{-}ns{\rightarrow}m \Longrightarrow g \vdash hd\ ns{-}ns{\rightarrow}m$
⟨*proof*⟩

lemma *path2-forget-last*[*simp*]: $g \vdash n{-}ns{\rightarrow}m \Longrightarrow g \vdash n{-}ns{\rightarrow}last\ ns$
⟨*proof*⟩

lemma *path-hd*[*dest*]: *path* $g\ (n\#ns) \Longrightarrow$ *path* $g\ [n]$
⟨*proof*⟩

lemma *path-by-tail*[*intro*]: $[\![path\ g\ (n\#n'\#ns);\ path\ g\ (n'\#ns) \Longrightarrow path\ g\ (n'\#ms)]\!] \Longrightarrow path\ g\ (n\#n'\#ms)$
⟨*proof*⟩

lemma *αn-in-αnE* [*elim*]:
assumes $(n,e,m) \in \alpha e\ g$ **and** *invar* g
obtains $n \in set\ (\alpha n\ g)$ **and** $m \in set\ (\alpha n\ g)$
⟨*proof*⟩

lemma *path-split*:
assumes *path* $g\ (ns@m\#ns')$
shows *path* $g\ (ns@[m])$ *path* $g(m\#ns')$
⟨*proof*⟩

lemma *path2-split*:
assumes $g \vdash n{-}ns@n'\#ns'{\rightarrow}m$

shows $g \vdash n{-}ns@[n']{\rightarrow}n'$ $g \vdash n'{-}n'\#ns'{\rightarrow}m$
⟨*proof*⟩

lemma *elem-set-implies-elem-tl-app-cons*[*simp*]: $x \in set\ xs \Longrightarrow x \in set\ (tl\ (ys@y\#xs))$
⟨*proof*⟩

lemma *path2-split-ex*:
assumes $g \vdash n{-}ns{\rightarrow}m$ $x \in set\ ns$
obtains ns_1 ns_2 **where** $g \vdash n{-}ns_1{\rightarrow}x$ $g \vdash x{-}ns_2{\rightarrow}m$ $ns = ns_1@tl\ ns_2$ $ns = butlast\ ns_1@ns_2$
⟨*proof*⟩

lemma *path2-split-ex'*:
assumes $g \vdash n{-}ns{\rightarrow}m$ $x \in set\ ns$
obtains ns_1 ns_2 **where** $g \vdash n{-}ns_1{\rightarrow}x$ $g \vdash x{-}ns_2{\rightarrow}m$ $ns = butlast\ ns_1@ns_2$
⟨*proof*⟩

lemma *path-snoc*:
assumes *path* g $(ns@[n])$ $n \in set\ (predecessors\ g\ m)$
shows *path* g $(ns@[n,m])$
⟨*proof*⟩

lemma *path2-snoc*[*elim*]:
assumes $g \vdash n{-}ns{\rightarrow}m$ $m \in set\ (predecessors\ g\ m')$
shows $g \vdash n{-}ns@[m']{\rightarrow}m'$
⟨*proof*⟩

lemma *path-unsnoc*:
assumes *path* g *ns length ns* $\geq$ *2*
obtains *path* g $(butlast\ ns) \wedge last\ (butlast\ ns) \in set\ (predecessors\ g\ (last\ ns))$
⟨*proof*⟩

lemma *path2-unsnoc*:
assumes $g \vdash n{-}ns{\rightarrow}m$ *length ns* $\geq$ *2*
obtains $g \vdash n{-}butlast\ ns{\rightarrow}last\ (butlast\ ns)$ $last\ (butlast\ ns) \in set\ (predecessors\ g\ m)$
⟨*proof*⟩

lemma *path2-rev-induct*[*consumes 1*, *case-names empty snoc*]:
assumes $g \vdash n{-}ns{\rightarrow}m$
assumes *empty*: $n \in set\ (\alpha n\ g) \Longrightarrow P\ n\ [n]\ n$
assumes *snoc*: $\bigwedge ns\ m'\ m.\ g \vdash n{-}ns{\rightarrow}m' \Longrightarrow P\ n\ ns\ m' \Longrightarrow m' \in set\ (predecessors\ g\ m) \Longrightarrow P\ n\ (ns@[m])\ m$
shows $P\ n\ ns\ m$
⟨*proof*⟩

lemma *path2-hd*[*elim*, *dest?*]: $g \vdash n{-}ns{\rightarrow}m \Longrightarrow n = hd\ ns$
⟨*proof*⟩

lemma *path2-hd-in-ns*[*elim*]: $g \vdash n-ns\rightarrow m \Longrightarrow n \in set\ ns$
⟨*proof*⟩

lemma *path2-last*[*elim*, *dest?*]: $g \vdash n-ns\rightarrow m \Longrightarrow m = last\ ns$
⟨*proof*⟩

lemma *path2-last-in-ns*[*elim*]: $g \vdash n-ns\rightarrow m \Longrightarrow m \in set\ ns$
⟨*proof*⟩

lemma *path-app*[*elim*]:
assumes *path g ns path g ms last ns = hd ms*
shows *path g (ns@tl ms)*
⟨*proof*⟩

lemma *path2-app*[*elim*]:
assumes $g \vdash n-ns\rightarrow m$ $g \vdash m-ms\rightarrow l$
shows $g \vdash n-ns@tl\ ms\rightarrow l$
⟨*proof*⟩

lemma *butlast-tl*:
assumes *last xs = hd ys xs ≠ [] ys ≠ []*
shows *butlast xs@ys = xs@tl ys*
⟨*proof*⟩

lemma *path2-app'*[*elim*]:
assumes $g \vdash n-ns\rightarrow m$ $g \vdash m-ms\rightarrow l$
shows $g \vdash n-butlast\ ns@ms\rightarrow l$
⟨*proof*⟩

lemma *path2-nontrivial*[*elim*]:
assumes $g \vdash n-ns\rightarrow m$ $n \neq m$
shows *length ns ≥ 2*
⟨*proof*⟩

lemma *simple-path2-aux*:
assumes $g \vdash n-ns\rightarrow m$
obtains *ns'* **where** $g \vdash n-ns'\rightarrow m$ *distinct ns' set ns' ⊆ set ns length ns' ≤ length ns*
⟨*proof*⟩

lemma *simple-path2*:
assumes $g \vdash n-ns\rightarrow m$
obtains *ns'* **where** $g \vdash n-ns'\rightarrow m$ *distinct ns' set ns' ⊆ set ns length ns' ≤ length ns n ∉ set (tl ns') m ∉ set (butlast ns')*
⟨*proof*⟩

lemma *simple-path2-unsnoc*:
assumes $g \vdash n-ns\rightarrow m$ $n \neq m$
obtains *ns'* **where** $g \vdash n-ns'\rightarrow last\ ns'$ *last ns' ∈ set (predecessors g m) distinct*

ns' *set* $ns' \subseteq$ *set* ns $m \notin$ *set* ns'
⟨*proof*⟩

lemma *path2-split-first-last*:
assumes $g \vdash n - ns \rightarrow m$ $x \in$ *set* ns
obtains ns_1 ns_3 ns_2 **where** $ns = ns_1 @ ns_3 @ ns_2$ *prefix* $(ns_1 @ [x])$ ns *suffix* $(x \# ns_2)$ ns
and $g \vdash n - ns_1 @ [x] \rightarrow x$ $x \notin$ *set* ns_1
and $g \vdash x - ns_3 \rightarrow x$
and $g \vdash x - x \# ns_2 \rightarrow m$ $x \notin$ *set* ns_2
⟨*proof*⟩

lemma *path2-simple-loop*:
assumes $g \vdash n - ns \rightarrow n$ $n' \in$ *set* ns
obtains ns' **where** $g \vdash n - ns' \rightarrow n$ $n' \in$ *set* ns' $n \notin$ *set* $(tl\ (butlast\ ns'))$ *set* $ns' \subseteq$ *set* ns
⟨*proof*⟩

lemma *path2-split-first-prop*:
assumes $g \vdash n - ns \rightarrow m$ $\exists x \in set\ ns.\ P\ x$
obtains m' ns' **where** $g \vdash n - ns' \rightarrow m'$ $P\ m'$ $\forall x \in set\ (butlast\ ns').\ \neg P\ x$ *prefix* ns' ns
⟨*proof*⟩

lemma *path2-split-last-prop*:
assumes $g \vdash n - ns \rightarrow m$ $\exists x \in set\ ns.\ P\ x$
obtains n' ns' **where** $g \vdash n' - ns' \rightarrow m$ $P\ n'$ $\forall x \in set\ (tl\ ns').\ \neg P\ x$ *suffix* ns' ns
⟨*proof*⟩

lemma *path2-prefix*[*elim*]:
assumes *1*: $g \vdash n - ns \rightarrow m$
assumes *2*: *prefix* $(ns' @ [m'])$ ns
shows $g \vdash n - ns' @ [m'] \rightarrow m'$
⟨*proof*⟩

lemma *path2-prefix-ex*:
assumes $g \vdash n - ns \rightarrow m$ $m' \in$ *set* ns
obtains ns' **where** $g \vdash n - ns' \rightarrow m'$ *prefix* ns' ns $m' \notin$ *set* $(butlast\ ns')$
⟨*proof*⟩

lemma *path2-strict-prefix-ex*:
assumes $g \vdash n - ns \rightarrow m$ $m' \in$ *set* $(butlast\ ns)$
obtains ns' **where** $g \vdash n - ns' \rightarrow m'$ *strict-prefix* ns' ns $m' \notin$ *set* $(butlast\ ns')$
⟨*proof*⟩

lemma *path2-nontriv*[*elim*]: $[\![g \vdash n - ns \rightarrow m;\ n \neq m]\!] \Longrightarrow$ *length* $ns > 1$
⟨*proof*⟩

declare *path-not-Nil* [*simp del*]

declare *path2-not-Nil* [*simp del*]
declare *path2-not-Nil3* [*simp del*]
end

2.2 Domination

We fix an entry node per graph and use it to define node domination.

locale *graph-Entry-base* = *graph-path-base* αe αn *invar inEdges'*
for
$\alpha e :: 'g \Rightarrow ('node \times 'edgeD \times 'node)\ set$ **and**
$\alpha n :: 'g \Rightarrow 'node\ list$ **and**
invar $:: 'g \Rightarrow bool$ **and**
inEdges' $:: 'g \Rightarrow 'node \Rightarrow ('node \times 'edgeD)\ list$
+
fixes *Entry* $:: 'g \Rightarrow 'node$
begin
definition *dominates* $:: 'g \Rightarrow 'node \Rightarrow 'node \Rightarrow bool$ **where**
dominates g n m $\equiv m \in set\ (\alpha n\ g) \wedge (\forall ns.\ g \vdash Entry\ g{-}ns{\rightarrow}m \longrightarrow n \in set\ ns)$

abbreviation *strict-dom g n m* $\equiv n \neq m \wedge$ *dominates g n m*
end

locale *graph-Entry* = *graph-Entry-base* αe αn *invar inEdges' Entry*
+ *graph-path* αe αn *invar inEdges'*
for
$\alpha e :: 'g \Rightarrow ('node \times 'edgeD \times 'node)\ set$ **and**
$\alpha n :: 'g \Rightarrow 'node\ list$ **and**
invar $:: 'g \Rightarrow bool$ **and**
inEdges' $:: 'g \Rightarrow 'node \Rightarrow ('node \times 'edgeD)\ list$ **and**
Entry $:: 'g \Rightarrow 'node$
+
assumes *Entry-in-graph*[*simp*]: *Entry g* $\in set\ (\alpha n\ g)$
assumes *Entry-unreachable*: *invar g* $\Longrightarrow$ *inEdges g* (*Entry g*) $= []$
assumes *Entry-reaches*[*intro*]:
$[\![n \in set\ (\alpha n\ g);\ invar\ g]\!] \Longrightarrow \exists ns.\ g \vdash Entry\ g{-}ns{\rightarrow}n$
begin
lemma *Entry-dominates*[*simp*,*intro*]: $[\![invar\ g;\ n \in set\ (\alpha n\ g)]\!] \Longrightarrow$ *dominates g* (*Entry g*) *n*
⟨*proof*⟩

lemma *Entry-iff-unreachable*[*simp*]:
assumes *invar g n* $\in set\ (\alpha n\ g)$
shows *predecessors g n* $= [] \longleftrightarrow n =$ *Entry g*
⟨*proof*⟩

lemma *Entry-loop*:
assumes *invar g g* $\vdash$ *Entry g*$-ns\rightarrow$*Entry g*
shows *ns*$=[$*Entry g*$]$

⟨*proof*⟩

lemma *simple-Entry-path*:
assumes *invar g n* ∈ *set* (*αn g*)
obtains *ns* **where** *g* ⊢ *Entry g*−*ns*→*n* **and** *n* ∉ *set* (*butlast ns*)
⟨*proof*⟩

lemma *dominatesI* [*intro*]:
⟦*m* ∈ *set* (*αn g*); ⋀*ns*. ⟦*g* ⊢ *Entry g*−*ns*→*m*⟧ ⟹ *n* ∈ *set ns*⟧ ⟹ *dominates g n m*
⟨*proof*⟩

lemma *dominatesE*:
assumes *dominates g n m*
obtains *m* ∈ *set* (*αn g*) **and** ⋀*ns*. *g* ⊢ *Entry g*−*ns*→*m* ⟹ *n* ∈ *set ns*
⟨*proof*⟩

lemma[*simp*]: *dominates g n m* ⟹ *m* ∈ *set* (*αn g*) ⟨*proof*⟩

lemma[*simp*]:
assumes *dominates g n m* **and**[*simp*]: *invar g*
shows *n* ∈ *set* (*αn g*)
⟨*proof*⟩

lemma *strict-domE*[*elim*]:
assumes *strict-dom g n m*
obtains *m* ∈ *set* (*αn g*) **and** ⋀*ns*. *g* ⊢ *Entry g*−*ns*→*m* ⟹ *n* ∈ *set* (*butlast ns*)
⟨*proof*⟩

lemma *dominates-refl*[*intro*!]: ⟦*invar g*; *n* ∈ *set* (*αn g*)⟧ ⟹ *dominates g n n*
⟨*proof*⟩

lemma *dominates-trans*:
assumes *invar g*
assumes *part1*: *dominates g n n′*
assumes *part2*: *dominates g n′ n″*
shows *dominates g n n″*
⟨*proof*⟩

lemma *dominates-antisymm*:
assumes *invar g*
assumes *dom1*: *dominates g n n′*
assumes *dom2*: *dominates g n′ n*
shows *n* = *n′*
⟨*proof*⟩

lemma *dominates-unsnoc*:
assumes [*simp*]: *invar g* **and** *dominates g n m m′* ∈ *set* (*predecessors g m*) *n*

$\neq m$
shows *dominates g n m′*
⟨*proof*⟩

lemma *dominates-unsnoc′*:
assumes [*simp*]: *invar g* **and** *dominates g n m g* ⊢ *m′*−*ms*→*m* ∀ *x* ∈ *set* (*tl ms*). *x* ≠ *n*
shows *dominates g n m′*
⟨*proof*⟩

lemma *dominates-path*:
assumes *dominates g n m* **and**[*simp*]: *invar g*
obtains *ns* **where** *g* ⊢ *n*−*ns*→*m*
⟨*proof*⟩

lemma *dominates-antitrans*:
assumes[*simp*]: *invar g* **and** *dominates g n*$_1$ *m dominates g n*$_2$ *m*
obtains (*1*) *dominates g n*$_1$ *n*$_2$
| (*2*) *dominates g n*$_2$ *n*$_1$
⟨*proof*⟩

lemma *dominates-extend*:
assumes *dominates g n m*
assumes *g* ⊢ *m′*−*ms*→*m n* ∉ *set* (*tl ms*)
shows *dominates g n m′*
⟨*proof*⟩

definition *dominators* :: *′g* ⇒ *′node* ⇒ *′node set* **where**
dominators g n ≡ {*m* ∈ *set* (*αn g*). *dominates g m n*}

definition *isIdom g n m* ⟷ *strict-dom g m n* ∧ (∀ *m′* ∈ *set* (*αn g*). *strict-dom g m′ n* ⟶ *dominates g m′ m*)
definition *idom* :: *′g* ⇒ *′node* ⇒ *′node* **where**
idom g n ≡ *THE m*. *isIdom g n m*

lemma *idom-ex*:
assumes[*simp*]: *invar g n* ∈ *set* (*αn g*) *n* ≠ *Entry g*
shows ∃!*m*. *isIdom g n m*
⟨*proof*⟩

lemma *idom*: ⟦*invar g*; *n* ∈ *set* (*αn g*) − {*Entry g*}⟧ ⟹ *isIdom g n* (*idom g n*)
⟨*proof*⟩

lemma *dominates-mid*:
assumes *dominates g n x dominates g x m g* ⊢ *n*−*ns*→*m* **and**[*simp*]: *invar g*
shows *x* ∈ *set ns*
⟨*proof*⟩

definition *shortestPath* :: *′g* ⇒ *′node* ⇒ *nat* **where**

shortestPath g n ≡ (*LEAST l*. ∃ *ns*. *length ns* = *l* ∧ *g* ⊢ *Entry g*−*ns*→*n*)

lemma *shortestPath-ex*:
assumes *n* ∈ *set* (*αn g*) *invar g*
obtains *ns* **where** *g* ⊢ *Entry g*−*ns*→*n distinct ns length ns* = *shortestPath g n*
⟨*proof*⟩

lemma[*simp*]: ⟦*n* ∈ *set* (*αn g*); *invar g*⟧ ⟹ *shortestPath g n* ≠ *0*
⟨*proof*⟩

lemma *shortestPath-upper-bound*:
assumes *n* ∈ *set* (*αn g*) *invar g*
shows *shortestPath g n* ≤ *length* (*αn g*)
⟨*proof*⟩

lemma *shortestPath-predecessor*:
assumes *n* ∈ *set* (*αn g*) − {*Entry g*} **and**[*simp*]: *invar g*
obtains *n′* **where** *Suc* (*shortestPath g n′*) = *shortestPath g n n′* ∈ *set* (*predecessors g n*)
⟨*proof*⟩

lemma *successor-in-αn*[*simp*]:
assumes *predecessors g n* ≠ [] **and**[*simp*]: *invar g*
shows *n* ∈ *set* (*αn g*)
⟨*proof*⟩

lemma *shortestPath-single-predecessor*:
assumes *predecessors g n* = [*m*] **and**[*simp*]: *invar g*
shows *shortestPath g m* < *shortestPath g n*
⟨*proof*⟩

lemma *strict-dom-shortestPath-order*:
assumes *strict-dom g n m m* ∈ *set* (*αn g*) *invar g*
shows *shortestPath g n* < *shortestPath g m*
⟨*proof*⟩

lemma *dominates-shortestPath-order*:
assumes *dominates g n m m* ∈ *set* (*αn g*) *invar g*
shows *shortestPath g n* ≤ *shortestPath g m*
⟨*proof*⟩

lemma *strict-dom-trans*:
assumes[*simp*]: *invar g*
assumes *strict-dom g n m strict-dom g m m′*
shows *strict-dom g n m′*
⟨*proof*⟩

inductive *EntryPath* :: *′g* ⇒ *′node list* ⇒ *bool* **where**
EntryPath-triv[*simp*]: *EntryPath g* [*n*]

| *EntryPath-snoc*[*intro*]: *EntryPath g ns* $\Longrightarrow$ *shortestPath g m* = *Suc* (*shortestPath g* (*last ns*)) $\Longrightarrow$ *EntryPath g* (*ns*@[*m*])

lemma[*simp*]:
assumes *EntryPath g ns prefix ns' ns ns'* $\neq$ []
shows *EntryPath g ns'*
⟨*proof*⟩

lemma *EntryPath-suffix*:
assumes *EntryPath g ns suffix ns' ns ns'* $\neq$ []
shows *EntryPath g ns'*
⟨*proof*⟩

lemma *EntryPath-butlast-less-last*:
assumes *EntryPath g ns z* $\in$ *set* (*butlast ns*)
shows *shortestPath g z* < *shortestPath g* (*last ns*)
⟨*proof*⟩

lemma *EntryPath-distinct*:
assumes *EntryPath g ns*
shows *distinct ns*
⟨*proof*⟩

lemma *Entry-reachesE*:
assumes *n* $\in$ *set* (*αn g*) **and**[*simp*]: *invar g*
obtains *ns* **where** *g* $\vdash$ *Entry g*−*ns*→*n EntryPath g ns*
⟨*proof*⟩
end

end

theory *SSA-CFG*
imports *Graph-path HOL*−*Library.Sublist*
begin

2.3 CFG

locale *CFG-base* = *graph-Entry-base αe αn invar inEdges' Entry*
for
αe :: *'g* ⇒ (*'node*::*linorder* × *'edgeD* × *'node*) *set* **and**
αn :: *'g* ⇒ *'node list* **and**
invar :: *'g* ⇒ *bool* **and**
inEdges' :: *'g* ⇒ *'node* ⇒ (*'node* × *'edgeD*) *list* **and**
Entry :: *'g* ⇒ *'node* +
fixes *defs* :: *'g* ⇒ *'node* ⇒ *'var*::*linorder set*
fixes *uses* :: *'g* ⇒ *'node* ⇒ *'var set*
begin
definition *vars g* ≡ *fold* (∪) (*map* (*uses g*) (*αn g*)) {}

definition *defAss'* :: *'g* ⇒ *'node* ⇒ *'var* ⇒ *bool* **where**
defAss' g m v ⟷ (∀ *ns*. *g* ⊢ *Entry g*−*ns*→*m* ⟶ (∃ *n* ∈ *set ns*. *v* ∈ *defs g n*))

definition *defAss'Uses* :: *'g* ⇒ *bool* **where**
defAss'Uses g ≡ ∀ *m* ∈ *set* (*αn g*). ∀ *v* ∈ *uses g m*. *defAss' g m v*
end

locale *CFG* = *CFG-base αe αn invar inEdges' Entry defs uses*
+ *graph-Entry αe αn invar inEdges' Entry*
for
αe :: *'g* ⇒ (*'node*::*linorder* × *'edgeD* × *'node*) *set* **and**
αn :: *'g* ⇒ *'node list* **and**
invar :: *'g* ⇒ *bool* **and**
inEdges' :: *'g* ⇒ *'node* ⇒ (*'node* × *'edgeD*) *list* **and**
Entry :: *'g* ⇒ *'node* **and**
defs :: *'g* ⇒ *'node* ⇒ *'var*::*linorder set* **and**
uses :: *'g* ⇒ *'node* ⇒ *'var set* +
assumes *defs-uses-disjoint*: *n* ∈ *set* (*αn g*) ⟹ *defs g n* ∩ *uses g n* = {}
assumes *defs-finite*[*simp*]: *finite* (*defs g n*)
assumes *uses-in-αn*: *v* ∈ *uses g n* ⟹ *n* ∈ *set* (*αn g*)
assumes *uses-finite*[*simp*, *intro*!]: *finite* (*uses g n*)
assumes *invar*[*intro*!]: *invar g*
begin
lemma *vars-finite*[*simp*]: *finite* (*vars g*)
⟨*proof*⟩

lemma *Entry-no-predecessor*[*simp*]: *predecessors g* (*Entry g*) = []
⟨*proof*⟩

lemma *uses-in-vars*[*elim*, *simp*]: *v* ∈ *uses g n* ⟹ *v* ∈ *vars g*
⟨*proof*⟩

lemma *varsE*:
assumes *v* ∈ *vars g*
obtains *n* **where** *n* ∈ *set* (*αn g*) *v* ∈ *uses g n*
⟨*proof*⟩

lemma *defs-uses-disjoint'*[*simp*]: *n* ∈ *set* (*αn g*) ⟹ *v* ∈ *defs g n* ⟹ *v* ∈ *uses g n* ⟹ *False*
⟨*proof*⟩
end

context *CFG*
begin
lemma *defAss'E*:
assumes *defAss' g m v g* ⊢ *Entry g*−*ns*→*m*
obtains *n* **where** *n* ∈ *set ns v* ∈ *defs g n*
⟨*proof*⟩

lemmas *defAss'I* = *defAss'-def*[*THEN iffD2*, *rule-format*]

lemma *defAss'-extend*:
assumes *defAss' g m v*
assumes $g \vdash n{-}ns{\rightarrow}m \; \forall\, n \in set\ (tl\ ns).\ v \notin defs\ g\ n$
shows *defAss' g n v*
⟨*proof*⟩
end

A CFG is well-formed if it satisfies definite assignment.

locale *CFG-wf* = *CFG* $\alpha e\ \alpha n$ *invar inEdges' Entry defs uses*
for
$\alpha e :: 'g \Rightarrow ('node{::}linorder \times 'edgeD \times 'node)\ set$ **and**
$\alpha n :: 'g \Rightarrow 'node\ list$ **and**
invar :: $'g \Rightarrow bool$ **and**
inEdges' :: $'g \Rightarrow 'node \Rightarrow ('node \times 'edgeD)\ list$ **and**
Entry::$'g \Rightarrow 'node$ **and**
defs :: $'g \Rightarrow 'node \Rightarrow 'var{::}linorder\ set$ **and**
uses :: $'g \Rightarrow 'node \Rightarrow 'var\ set$ +
assumes *def-ass-uses*: $\forall\, m \in set\ (\alpha n\ g).\ \forall\, v \in uses\ g\ m.\ defAss'\ g\ m\ v$

2.4 SSA CFG

type-synonym ($'node$, $'val$) *phis* = $'node \times 'val \rightharpoonup 'val\ list$

declare *in-set-zipE*[*elim*]
declare *zip-same*[*simp*]

locale *CFG-SSA-base* = *CFG-base* $\alpha e\ \alpha n$ *invar inEdges' Entry defs uses*
for
$\alpha e :: 'g \Rightarrow ('node{::}linorder \times 'edgeD \times 'node)\ set$ **and**
$\alpha n :: 'g \Rightarrow 'node\ list$ **and**
invar :: $'g \Rightarrow bool$ **and**
inEdges' :: $'g \Rightarrow 'node \Rightarrow ('node \times 'edgeD)\ list$ **and**
Entry::$'g \Rightarrow 'node$ **and**
defs :: $'g \Rightarrow 'node \Rightarrow 'val{::}linorder\ set$ **and**
uses :: $'g \Rightarrow 'node \Rightarrow 'val\ set$ +
fixes *phis* :: $'g \Rightarrow ('node, 'val)\ phis$
begin
definition *phiDefs g n* ≡ {*v*. (*n*,*v*) ∈ *dom* (*phis g*)}
definition[*code*]: *allDefs g n* ≡ *defs g n* ∪ *phiDefs g n*

definition[*code*]: *phiUses g n* ≡
$\bigcup n' \in set\ (successors\ g\ n).\ \bigcup v' \in phiDefs\ g\ n'.\ snd\ `\ Set.filter\ (\lambda(n'',v).\ n'' = n)\ (set\ (zip\ (predecessors\ g\ n')\ (the\ (phis\ g\ (n',v')))))$
definition[*code*]: *allUses g n* ≡ *uses g n* ∪ *phiUses g n*
definition[*code*]: *allVars g* ≡ $\bigcup n \in set\ (\alpha n\ g).\ allDefs\ g\ n \cup allUses\ g\ n$

definition *defAss* :: $'g \Rightarrow 'node \Rightarrow 'val \Rightarrow bool$ **where**

defAss g m v $\longleftrightarrow$ ($\forall$ *ns. g* $\vdash$ *Entry g*$-$*ns*$\rightarrow$*m* $\longrightarrow$ ($\exists$ *n* $\in$ *set ns. v* $\in$ *allDefs g n*))

lemmas *CFG-SSA-defs = phiDefs-def allDefs-def phiUses-def allUses-def allVars-def defAss-def*
end

locale *CFG-SSA = CFG* αe αn *invar inEdges' Entry defs uses + CFG-SSA-base* αe αn *invar inEdges' Entry defs uses phis*
for
αe :: *'g* $\Rightarrow$ (*'node*::*linorder* $\times$ *'edgeD* $\times$ *'node*) *set* **and**
αn :: *'g* $\Rightarrow$ *'node list* **and**
invar :: *'g* $\Rightarrow$ *bool* **and**
inEdges' :: *'g* $\Rightarrow$ *'node* $\Rightarrow$ (*'node* $\times$ *'edgeD*) *list* **and**
Entry::*'g* $\Rightarrow$ *'node* **and**
defs :: *'g* $\Rightarrow$ *'node* $\Rightarrow$ *'val*::*linorder set* **and**
uses :: *'g* $\Rightarrow$ *'node* $\Rightarrow$ *'val set* **and**
phis :: *'g* $\Rightarrow$ (*'node*, *'val*) *phis* +
assumes *phis-finite*: *finite* (*dom* (*phis g*))
assumes *phis-in-*αn: *phis g* (*n*,*v*) = *Some vs* $\Longrightarrow$ *n* $\in$ *set* (αn *g*)
assumes *phis-wf*:
phis g (*n*,*v*) = *Some args* $\Longrightarrow$ *length* (*predecessors g n*) = *length args*
assumes *simpleDefs-phiDefs-disjoint*:
n $\in$ *set* (αn *g*) $\Longrightarrow$ *defs g n* $\cap$ *phiDefs g n* = {}
assumes *allDefs-disjoint*:
⟦*n* $\in$ *set* (αn *g*); *m* $\in$ *set* (αn *g*); *n* $\neq$ *m*⟧ $\Longrightarrow$ *allDefs g n* $\cap$ *allDefs g m* = {}
begin
lemma *phis-disj*:
assumes *phis g* (*n*,*v*) = *Some vs*
and *phis g* (*n'*,*v*) = *Some vs'*
shows *n* = *n'* **and** *vs* = *vs'*
⟨*proof*⟩

lemma *allDefs-disjoint'*: ⟦*n* $\in$ *set* (αn *g*); *m* $\in$ *set* (αn *g*); *v* $\in$ *allDefs g n*; *v* $\in$ *allDefs g m*⟧ $\Longrightarrow$ *n* = *m*
⟨*proof*⟩

lemma *phiUsesI*:
assumes *n'* $\in$ *set* (αn *g*) *phis g* (*n'*,*v'*) = *Some vs* (*n*,*v*) $\in$ *set* (*zip* (*predecessors g n'*) *vs*)
shows *v* $\in$ *phiUses g n*
⟨*proof*⟩

lemma *phiUsesE*:
assumes *v* $\in$ *phiUses g n*
obtains *n' v' vs* **where** *n'* $\in$ *set* (*successors g n*) (*n*,*v*) $\in$ *set* (*zip* (*predecessors g n'*) *vs*) *phis g* (*n'*, *v'*) = *Some vs*
⟨*proof*⟩

lemma *defs-in-allDefs*[*simp*]: $v \in defs\ g\ n \Longrightarrow v \in allDefs\ g\ n$ ⟨*proof*⟩
lemma *phiDefs-in-allDefs*[*simp*, *elim*]: $v \in phiDefs\ g\ n \Longrightarrow v \in allDefs\ g\ n$ ⟨*proof*⟩
lemma *uses-in-allUses*[*simp*]: $v \in uses\ g\ n \Longrightarrow v \in allUses\ g\ n$ ⟨*proof*⟩
lemma *phiUses-in-allUses*[*simp*]: $v \in phiUses\ g\ n \Longrightarrow v \in allUses\ g\ n$ ⟨*proof*⟩
lemma *allDefs-in-allVars*[*simp*, *intro*]: $[\![v \in allDefs\ g\ n;\ n \in set\ (\alpha n\ g)]\!] \Longrightarrow v \in allVars\ g$ ⟨*proof*⟩
lemma *allUses-in-allVars*[*simp*, *intro*]: $[\![v \in allUses\ g\ n;\ n \in set\ (\alpha n\ g)]\!] \Longrightarrow v \in allVars\ g$ ⟨*proof*⟩

lemma *phiDefs-finite*[*simp*]: *finite* (*phiDefs g n*)
⟨*proof*⟩

lemma *phiUses-finite*[*simp*]:
assumes $n \in set\ (\alpha n\ g)$
shows *finite* (*phiUses g n*)
⟨*proof*⟩

lemma *allDefs-finite*[*simp*]: $n \in set\ (\alpha n\ g) \Longrightarrow$ *finite* (*allDefs g n*) ⟨*proof*⟩
lemma *allUses-finite*[*simp*]: $n \in set\ (\alpha n\ g) \Longrightarrow$ *finite* (*allUses g n*) ⟨*proof*⟩
lemma *allVars-finite*[*simp*]: *finite* (*allVars g*) ⟨*proof*⟩

lemmas *defAssI* = *defAss-def*[*THEN iffD2*, *rule-format*]
lemmas *defAssD* = *defAss-def*[*THEN iffD1*, *rule-format*]

lemma *defAss-extend*:
assumes *defAss g m v*
assumes $g \vdash n - ns \rightarrow m\ \forall n \in set\ (tl\ ns).\ v \notin allDefs\ g\ n$
shows *defAss g n v*
⟨*proof*⟩

lemma *defAss-dominating*:
assumes[*simp*]: $n \in set\ (\alpha n\ g)$
shows $defAss\ g\ n\ v \longleftrightarrow (\exists m \in set\ (\alpha n\ g).\ dominates\ g\ m\ n \wedge v \in allDefs\ g\ m)$
⟨*proof*⟩
end

locale *CFG-SSA-wf-base* = *CFG-SSA-base* $\alpha e\ \alpha n$ *invar inEdges' Entry defs uses phis*
for
$\alpha e :: 'g \Rightarrow ('node::linorder \times 'edgeD \times 'node)\ set$ **and**
$\alpha n :: 'g \Rightarrow 'node\ list$ **and**
$invar :: 'g \Rightarrow bool$ **and**
$inEdges' :: 'g \Rightarrow 'node \Rightarrow ('node \times 'edgeD)\ list$ **and**
$Entry::'g \Rightarrow 'node$ **and**
$defs :: 'g \Rightarrow 'node \Rightarrow 'val::linorder\ set$ **and**
$uses :: 'g \Rightarrow 'node \Rightarrow 'val\ set$ **and**
$phis :: 'g \Rightarrow ('node, 'val)\ phis$

begin

Using the SSA properties, we can map every value to its unique defining node and remove the *'node* parameter of the *phis* map.

definition *defNode* :: *'g* ⇒ *'val* ⇒ *'node* **where**
defNode-code [*code*]: *defNode g v* ≡ *hd* [*n* ← *αn g*. *v* ∈ *allDefs g n*]

abbreviation *def-dominates g v' v* ≡ *dominates g* (*defNode g v'*) (*defNode g v*)
abbreviation *strict-def-dom g v' v* ≡ *defNode g v'* ≠ *defNode g v* ∧ *def-dominates g v' v*

definition *phi g v* = *phis g* (*defNode g v*,*v*)
definition[*simp*]: *phiArg g v v'* ≡ ∃ *vs*. *phi g v* = *Some vs* ∧ *v'* ∈ *set vs*

definition[*code*]: *isTrivialPhi g v v'* ⟷ *v'* ≠ *v* ∧
(*case phi g v of*
Some vs ⇒ *set vs* = {*v*,*v'*} ∨ *set vs* = {*v'*}
| *None* ⇒ *False*)
definition[*code*]: *trivial g v* ≡ ∃ *v'* ∈ *allVars g*. *isTrivialPhi g v v'*
definition[*code*]: *redundant g* ≡ ∃ *v* ∈ *allVars g*. *trivial g v*

definition *defAssUses g* ≡ ∀ *n* ∈ *set* (*αn g*). ∀ *v* ∈ *allUses g n*. *defAss g n v*

'liveness' of an SSA value is defined inductively starting from simple uses so that a circle of ϕ functions is not considered live.

declare [[*inductive-internals*]]
inductive *liveVal* :: *'g* ⇒ *'val* ⇒ *bool*
for *g* :: *'g*
where
liveSimple: ⟦*n* ∈ *set* (*αn g*); *val* ∈ *uses g n*⟧ ⟹ *liveVal g val*
| *livePhi*: ⟦*liveVal g v*; *phiArg g v v'*⟧ ⟹ *liveVal g v'*

definition *pruned g* = (∀ *n* ∈ *set* (*αn g*). ∀ *val*. *val* ∈ *phiDefs g n* ⟶ *liveVal g val*)

lemmas *CFG-SSA-wf-defs* = *CFG-SSA-defs defNode-code phi-def isTrivialPhi-def trivial-def redundant-def liveVal-def pruned-def*
end

locale *CFG-SSA-wf* = *CFG-SSA αe αn invar inEdges' Entry defs uses phis* + *CFG-SSA-wf-base αe αn invar inEdges' Entry defs uses phis*
for
αe :: *'g* ⇒ (*'node*::*linorder* × *'edgeD* × *'node*) *set* **and**
αn :: *'g* ⇒ *'node list* **and**
invar :: *'g* ⇒ *bool* **and**
inEdges' :: *'g* ⇒ *'node* ⇒ (*'node* × *'edgeD*) *list* **and**
Entry::*'g* ⇒ *'node* **and**
defs :: *'g* ⇒ *'node* ⇒ *'val*::*linorder set* **and**
uses :: *'g* ⇒ *'node* ⇒ *'val set* **and**

phis :: *'g* ⇒ (*'node*, *'val*) *phis* +
assumes *allUses-def-ass*: ⟦*v* ∈ *allUses g n*; *n* ∈ *set* (*αn g*)⟧ ⟹ *defAss g n v*
assumes *Entry-no-phis*[*simp*]: *phis g* (*Entry g*,*v*) = *None*
begin
lemma *allVars-in-allDefs*: *v* ∈ *allVars g* ⟹ ∃ *n* ∈ *set* (*αn g*). *v* ∈ *allDefs g n*
⟨*proof*⟩

lemma *phiDefs-Entry-empty*[*simp*]: *phiDefs g* (*Entry g*) = {}
⟨*proof*⟩

lemma *phi-Entry-empty*[*simp*]: *defNode g v* = *Entry g* ⟹ *phi g v* = *None*
⟨*proof*⟩

lemma *defNode-ex1*:
assumes *v* ∈ *allVars g*
shows ∃!*n*. *n* ∈ *set* (*αn g*) ∧ *v* ∈ *allDefs g n*
⟨*proof*⟩

lemma *defNode-def*: *v* ∈ *allVars g* ⟹ *defNode g v* = (*THE n*. *n* ∈ *set* (*αn g*)
∧ *v* ∈ *allDefs g n*)
⟨*proof*⟩

lemma *defNode*[*simp*]:
assumes *v* ∈ *allVars g*
shows (*defNode g v*) ∈ *set* (*αn g*) *v* ∈ *allDefs g* (*defNode g v*)
⟨*proof*⟩

lemma *defNode-eq*[*intro*]:
assumes *n* ∈ *set* (*αn g*) *v* ∈ *allDefs g n*
shows *defNode g v* = *n*
⟨*proof*⟩

lemma *defNode-cases*[*consumes 1*]:
assumes *v* ∈ *allVars g*
obtains (*simpleDef*) *v* ∈ *defs g* (*defNode g v*)
| (*phi*) *phi g v* ≠ *None*
⟨*proof*⟩

lemma *phi-phiDefs*[*simp*]: *phi g v* = *Some vs* ⟹ *v* ∈ *phiDefs g* (*defNode g v*)
⟨*proof*⟩

lemma *simpleDef-not-phi*:
assumes *n* ∈ *set* (*αn g*) *v* ∈ *defs g n*
shows *phi g v* = *None*
⟨*proof*⟩

lemma *phi-wf*: *phi g v* = *Some vs* ⟹ *length* (*predecessors g* (*defNode g v*)) =
length vs
⟨*proof*⟩

lemma *phi-finite*: *finite* (*dom* (*phi g*))
⟨*proof*⟩

lemma *phiUses-exI*:
assumes $m \in$ *set* (*predecessors g n*) *phis g* (*n*,*v*) = *Some vs n* ∈ *set* (*αn g*)
obtains v' **where** $v' \in$ *phiUses g m v'* ∈ *set vs*
⟨*proof*⟩

lemma *phiArg-exI*:
assumes $m \in$ *set* (*predecessors g* (*defNode g v*)) *phi g v* ≠ *None* **and**[*simp*]: *v* ∈ *allVars g*
obtains v' **where** $v' \in$ *phiUses g m phiArg g v v'*
⟨*proof*⟩

lemma *phiUses-exI′*:
assumes *phiArg g p q* **and**[*simp*]: *p* ∈ *allVars g*
obtains *m* **where** *q* ∈ *phiUses g m m* ∈ *set* (*predecessors g* (*defNode g p*))
⟨*proof*⟩

lemma *phiArg-in-allVars*[*simp*]:
assumes *phiArg g v v′*
shows $v' \in$ *allVars g*
⟨*proof*⟩

lemma *defAss-defNode*:
assumes *defAss g m v v* ∈ *allVars g g* ⊢ *Entry g*−*ns*→*m*
shows *defNode g v* ∈ *set ns*
⟨*proof*⟩

lemma *defUse-path-ex*:
assumes *v* ∈ *allUses g m m* ∈ *set* (*αn g*)
obtains *ns* **where** *g* ⊢ *defNode g v*−*ns*→*m EntryPath g ns*
⟨*proof*⟩

lemma *defUse-path-dominated*:
assumes *g* ⊢ *defNode g v*−*ns*→*n defNode g v* ∉ *set* (*tl ns*) *v* ∈ *allUses g n n′* ∈ *set ns*
shows *dominates g* (*defNode g v*) *n′*
⟨*proof*⟩

lemma *allUses-dominated*:
assumes *v* ∈ *allUses g n n* ∈ *set* (*αn g*)
shows *dominates g* (*defNode g v*) *n*
⟨*proof*⟩

lemma *phiArg-path-ex′*:
assumes *phiArg g p q* **and**[*simp*]: *p* ∈ *allVars g*
obtains *ns m* **where** *g* ⊢ *defNode g q*−*ns*→*m EntryPath g ns q* ∈ *phiUses g m*

$m \in set\ (predecessors\ g\ (defNode\ g\ p))$
⟨*proof*⟩

lemma *phiArg-path-ex*:
assumes *phiArg g p q* **and**[*simp*]: $p \in allVars\ g$
obtains *ns* **where** $g \vdash defNode\ g\ q{-}ns{\rightarrow}defNode\ g\ p\ length\ ns > 1$
⟨*proof*⟩

lemma *phiArg-tranclp-path-ex*:
assumes $r^{++}\ p\ q\ p \in allVars\ g$ **and**[*simp*]: $\bigwedge p\ q.\ r\ p\ q \Longrightarrow phiArg\ g\ p\ q$
obtains *ns* **where** $g \vdash defNode\ g\ q{-}ns{\rightarrow}defNode\ g\ p\ length\ ns > 1$
$\forall n \in set\ (butlast\ ns).\ \exists p\ q\ m\ ns'.\ r\ p\ q \wedge g \vdash defNode\ g\ q{-}ns'{\rightarrow}m \wedge (defNode\ g\ q) \notin set\ (tl\ ns') \wedge q \in phiUses\ g\ m \wedge m \in set\ (predecessors\ g\ (defNode\ g\ p)) \wedge n \in set\ ns' \wedge set\ ns' \subseteq set\ ns \wedge defNode\ g\ p \in set\ ns$
⟨*proof*⟩

lemma *non-dominated-predecessor*:
assumes $n \in set\ (\alpha n\ g)\ n \neq Entry\ g$
obtains *m* **where** $m \in set\ (predecessors\ g\ n)\ \neg dominates\ g\ n\ m$
⟨*proof*⟩

lemmas *dominates-trans'*[*trans*, *elim*] = *dominates-trans*[*OF invar*]
lemmas *strict-dom-trans'*[*trans*, *elim*] = *strict-dom-trans*[*OF invar*]
lemmas *dominates-refl'*[*simp*] = *dominates-refl*[*OF invar*]
lemmas *dominates-antisymm'*[*dest*] = *dominates-antisymm*[*OF invar*]

lemma *liveVal-in-allVars*[*simp*]: $liveVal\ g\ v \Longrightarrow v \in allVars\ g$
⟨*proof*⟩

lemma *phi-no-closed-loop*:
assumes[*simp*]: $p \in allVars\ g$ **and** $phi\ g\ p = Some\ vs$
shows $set\ vs \neq \{p\}$
⟨*proof*⟩

lemma *phis-phi*: $phis\ g\ (n, v) = Some\ vs \Longrightarrow phi\ g\ v = Some\ vs$
⟨*proof*⟩

lemma *trivial-phi*: $trivial\ g\ v \Longrightarrow phi\ g\ v \neq None$
⟨*proof*⟩

lemma *trivial-finite*: *finite* $\{v.\ trivial\ g\ v\}$
⟨*proof*⟩

lemma *trivial-in-allVars*: $trivial\ g\ v \Longrightarrow v \in allVars\ g$
⟨*proof*⟩

declare *phiArg-def* [*simp del*]
end

2.5 Bundling of CFG and Equivalent SSA CFG

locale *CFG-SSA-Transformed-base = old*: *CFG-base* αe αn *invar inEdges' Entry oldDefs oldUses + CFG-SSA-wf-base* αe αn *invar inEdges' Entry defs uses phis*
for
$\alpha e :: 'g \Rightarrow ('node::linorder \times 'edgeD \times 'node)\ set$ **and**
$\alpha n :: 'g \Rightarrow 'node\ list$ **and**
$invar :: 'g \Rightarrow bool$ **and**
$inEdges' :: 'g \Rightarrow 'node \Rightarrow ('node \times 'edgeD)\ list$ **and**
$Entry::'g \Rightarrow 'node$ **and**
$oldDefs :: 'g \Rightarrow 'node \Rightarrow 'var::linorder\ set$ **and**
$oldUses :: 'g \Rightarrow 'node \Rightarrow 'var\ set$ **and**
$defs :: 'g \Rightarrow 'node \Rightarrow 'val::linorder\ set$ **and**
$uses :: 'g \Rightarrow 'node \Rightarrow 'val\ set$ **and**
$phis :: 'g \Rightarrow ('node, 'val)\ phis$ +
fixes $var :: 'g \Rightarrow 'val \Rightarrow 'var$

locale *CFG-SSA-Transformed = CFG-SSA-Transformed-base* αe αn *invar inEdges' Entry oldDefs oldUses defs uses phis var*
+ *old*: *CFG-wf* αe αn *invar inEdges' Entry oldDefs oldUses + CFG-SSA-wf* αe αn *invar inEdges' Entry defs uses phis*
for
$\alpha e :: 'g \Rightarrow ('node::linorder \times 'edgeD \times 'node)\ set$ **and**
$\alpha n :: 'g \Rightarrow 'node\ list$ **and**
$invar :: 'g \Rightarrow bool$ **and**
$inEdges' :: 'g \Rightarrow 'node \Rightarrow ('node \times 'edgeD)\ list$ **and**
$Entry::'g \Rightarrow 'node$ **and**
$oldDefs :: 'g \Rightarrow 'node \Rightarrow 'var::linorder\ set$ **and**
$oldUses :: 'g \Rightarrow 'node \Rightarrow 'var\ set$ **and**
$defs :: 'g \Rightarrow 'node \Rightarrow 'val::linorder\ set$ **and**
$uses :: 'g \Rightarrow 'node \Rightarrow 'val\ set$ **and**
$phis :: 'g \Rightarrow ('node, 'val)\ phis$ **and**
$var :: 'g \Rightarrow 'val \Rightarrow 'var$ +
assumes *oldDefs-def*: $oldDefs\ g\ n = var\ g\ `\ defs\ g\ n$
assumes *oldUses-def*: $n \in set\ (\alpha n\ g) \Longrightarrow oldUses\ g\ n = var\ g\ `\ uses\ g\ n$
assumes *conventional*:
$[\![g \vdash n -ns\rightarrow m;\ n \notin set\ (tl\ ns);\ v \in allDefs\ g\ n;\ v \in allUses\ g\ m;\ x \in set\ (tl\ ns);\ v' \in allDefs\ g\ x]\!] \Longrightarrow var\ g\ v' \neq var\ g\ v$
assumes *phis-same-var*[*elim*]: $phis\ g\ (n,v) = Some\ vs \Longrightarrow v' \in set\ vs \Longrightarrow var\ g\ v' = var\ g\ v$
assumes *allDefs-var-disjoint*: $[\![n \in set\ (\alpha n\ g);\ v \in allDefs\ g\ n;\ v' \in allDefs\ g\ n;\ v \neq v']\!] \Longrightarrow var\ g\ v' \neq var\ g\ v$
begin
lemma *conventional'*: $[\![g \vdash n -ns\rightarrow m;\ n \notin set\ (tl\ ns);\ v \in allDefs\ g\ n;\ v \in allUses\ g\ m;\ v' \in allDefs\ g\ x;\ var\ g\ v' = var\ g\ v]\!] \Longrightarrow x \notin set\ (tl\ ns)$
⟨*proof*⟩

lemma *conventional''*: $[\![g \vdash defNode\ g\ v -ns\rightarrow m;\ defNode\ g\ v \notin set\ (tl\ ns);\ v \in allUses\ g\ m;\ var\ g\ v' = var\ g\ v;\ v \in allVars\ g;\ v' \in allVars\ g]\!] \Longrightarrow defNode\ g\ v' \notin set\ (tl\ ns)$

⟨*proof*⟩

lemma *phiArg-same-var*: *phiArg g p q* ⟹ *var g q* = *var g p*
⟨*proof*⟩

lemma *oldDef-defAss*:
assumes *v* ∈ *allUses g n g* ⊢ *Entry g*−*ns*→*n*
obtains *m* **where** *m* ∈ *set ns var g v* ∈ *oldDefs g m*
⟨*proof*⟩

lemma *allDef-path-from-simpleDef*:
assumes[*simp*]: *v* ∈ *allVars g*
obtains *n ns* **where** *g* ⊢ *n*−*ns*→*defNode g v old.EntryPath g ns var g v* ∈ *oldDefs g n*
⟨*proof*⟩

lemma *defNode-var-disjoint*:
assumes *p* ∈ *allVars g q* ∈ *allVars g p* ≠ *q defNode g p* = *defNode g q*
shows *var g p* ≠ *var g q*
⟨*proof*⟩

lemma *phiArg-distinct-nodes*:
assumes *phiArg g p q p* ≠ *q* **and**[*simp*]: *p* ∈ *allVars g*
shows *defNode g p* ≠ *defNode g q*
⟨*proof*⟩

lemma *phiArgs-def-distinct*:
assumes *phiArg g p q phiArg g p r q* ≠ *r p* ∈ *allVars g*
shows *defNode g q* ≠ *defNode g r*
⟨*proof*⟩

lemma *defNode-not-on-defUse-path*:
assumes *p*: *g* ⊢ *defNode g p*−*ns*→*n defNode g p* ∉ *set* (*tl ns*) *p* ∈ *allUses g n*
assumes[*simp*]: *q* ∈ *allVars g p* ≠ *q var g p* = *var g q*
shows *defNode g q* ∉ *set ns*
⟨*proof*⟩

lemma *defUse-paths-disjoint*:
assumes *p*: *g* ⊢ *defNode g p*−*ns*→*n defNode g p* ∉ *set* (*tl ns*) *p* ∈ *allUses g n*
assumes *q*: *g* ⊢ *defNode g q*−*ms*→*m defNode g q* ∉ *set* (*tl ms*) *q* ∈ *allUses g m*
assumes[*simp*]: *p* ≠ *q var g p* = *var g q*
shows *set ns* ∩ *set ms* = {}
⟨*proof*⟩

lemma *oldDefsI*: *v* ∈ *defs g n* ⟹ *var g v* ∈ *oldDefs g n* ⟨*proof*⟩

lemma *simpleDefs-phiDefs-var-disjoint*:
assumes *v* ∈ *phiDefs g n n* ∈ *set* (*α n g*)
shows *var g v* ∉ *oldDefs g n*

⟨*proof*⟩

lemma *liveVal-use-path*:
assumes *liveVal g v*
obtains *ns m* **where** $g \vdash defNode\ g\ v{-}ns{\rightarrow}m\ var\ g\ v \in oldUses\ g\ m$
$\bigwedge x.\ x \in set\ (tl\ ns) \Longrightarrow var\ g\ v \notin oldDefs\ g\ x$
⟨*proof*⟩
end

end

3 Minimality

We show that every reducible CFG without trivial ϕ functions is minimal, recreating the proof in [2]. The original proof is inlined as prose text.

theory *Minimality*
imports *SSA-CFG Serial-Rel*
begin

context *graph-path*
begin

Cytron's definition of path convergence

definition $pathsConverge\ g\ x\ xs\ y\ ys\ z \equiv g \vdash x{-}xs{\rightarrow}z \wedge g \vdash y{-}ys{\rightarrow}z \wedge length\ xs > 1 \wedge length\ ys > 1 \wedge x \neq y \wedge$
$(\forall j \in \{0..<length\ xs\}.\ \forall k \in \{0..<length\ ys\}.\ xs\ !\ j = ys\ !\ k \longrightarrow j = length\ xs - 1 \vee k = length\ ys - 1)$

Simplified definition

definition $pathsConverge'\ g\ x\ xs\ y\ ys\ z \equiv g \vdash x{-}xs{\rightarrow}z \wedge g \vdash y{-}ys{\rightarrow}z \wedge length\ xs > 1 \wedge length\ ys > 1 \wedge x \neq y \wedge$
$set\ (butlast\ xs) \cap set\ (butlast\ ys) = \{\}$

lemma *pathsConverge'*[*simp*]: $pathsConverge\ g\ x\ xs\ y\ ys\ z \longleftrightarrow pathsConverge'\ g\ x\ xs\ y\ ys\ z$
⟨*proof*⟩

lemma *pathsConvergeI*:
assumes $g \vdash x{-}xs{\rightarrow}z\ g \vdash y{-}ys{\rightarrow}z\ length\ xs > 1\ length\ ys > 1\ set\ (butlast\ xs) \cap set\ (butlast\ ys) = \{\}$
shows *pathsConverge g x xs y ys z*
⟨*proof*⟩
end

A (control) flow graph G is reducible iff for each cycle C of G there is a node of C that dominates all other nodes in C.

definition (**in** *graph-Entry*) $reducible\ g \equiv \forall n\ ns.\ g \vdash n{-}ns{\rightarrow}n \longrightarrow (\exists m \in set\ ns.\ \forall n \in set\ ns.\ dominates\ g\ m\ n)$

context *CFG-SSA-Transformed*
begin

A ϕ function for variable v is necessary in block Z iff two non-null paths $X \rightarrow^+ Z$ and $Y \rightarrow^+ Z$ converge at a block Z, such that the blocks X and Y contain assignments to v.

definition *necessaryPhi g v z* ≡ ∃ *n ns m ms. old.pathsConverge g n ns m ms z* ∧ *v* ∈ *oldDefs g n* ∧ *v* ∈ *oldDefs g m*
abbreviation *necessaryPhi′ g val* ≡ *necessaryPhi g* (*var g val*) (*defNode g val*)
definition *unnecessaryPhi g val* ≡ *phi g val* ≠ *None* ∧ ¬*necessaryPhi′ g val*

lemma *necessaryPhiI*: *old.pathsConverge g n ns m ms z* ⟹ *v* ∈ *oldDefs g n* ⟹ *v* ∈ *oldDefs g m* ⟹ *necessaryPhi g v z*
⟨*proof*⟩

A program with only necessary ϕ functions is in minimal SSA form.

definition *cytronMinimal g* ≡ ∀ *v* ∈ *allVars g. phi g v* ≠ *None* ⟶ *necessaryPhi′ g v*

Let p be a ϕ function in a block P. Furthermore, let q in a block Q and r in a block R be two operands of p, such that p, q and r are pairwise distinct. Then at least one of Q and R does not dominate P.

lemma *2*:
assumes *phiArg g p q phiArg g p r distinct* [*p*, *q*, *r*] **and**[*simp*]: *p* ∈ *allVars g*
shows ¬(*def-dominates g q p* ∧ *def-dominates g r p*)
⟨*proof*⟩

lemma *convergence-prop*:
assumes *necessaryPhi g* (*var g v*) *n g* ⊢ *n*−*ns*→*m v* ∈ *allUses g m* ⋀*x*. *x* ∈ *set* (*tl ns*) ⟹ *v* ∉ *allDefs g x v* ∉ *defs g n*
shows *phis g* (*n*,*v*) ≠ *None*
⟨*proof*⟩

lemma *convergence-prop′*:
assumes *necessaryPhi g v n g* ⊢ *n*−*ns*→*m v* ∈ *var g* ‘ *allUses g m* ⋀*x*. *x* ∈ *set ns* ⟹ *v* ∉ *oldDefs g x*
obtains *val* **where** *var g val* = *v phis g* (*n*,*val*) ≠ *None*
⟨*proof*⟩

lemma *nontrivialE*:
assumes ¬*trivial g p phi g p* ≠ *None* **and**[*simp*]: *p* ∈ *allVars g*
obtains *r s* **where** *phiArg g p r phiArg g p s distinct* [*p*, *r*, *s*]
⟨*proof*⟩

lemma *paths-converge-prefix*:
assumes *g* ⊢ *x*−*xs*→*z g* ⊢ *y*−*ys*→*z x* ≠ *y length xs* > *1 length ys* > *1 x* ∉ *set* (*butlast ys*) *y* ∉ *set* (*butlast xs*)

obtains *xs′ ys′ z′* **where** *old.pathsConverge g x xs′ y ys′ z′ prefix xs′ xs prefix ys′ ys*
⟨*proof*⟩

lemma *ununnecessaryPhis-disjoint-paths-aux*:
assumes ¬*unnecessaryPhi g r* **and**[*simp*]: *r* ∈ *allVars g*
obtains n_1 ns_1 n_2 ns_2 **where**
var g r ∈ *oldDefs g* n_1 *g* ⊢ n_1−ns_1→*defNode g r* **and**
var g r ∈ *oldDefs g* n_2 *g* ⊢ n_2−ns_2→*defNode g r* **and**
set (*butlast* ns_1) ∩ *set* (*butlast* ns_2) = {}
⟨*proof*⟩

lemma *ununnecessaryPhis-disjoint-paths*:
assumes ¬*unnecessaryPhi g r* ¬*unnecessaryPhi g s*

and *rs*: *defNode g r* ≠ *defNode g s*
and[*simp*]: *r* ∈ *allVars g s* ∈ *allVars g var g r* = *V var g s* = *V*
obtains *n ns m ms* **where** *V* ∈ *oldDefs g n g* ⊢ *n*−*ns*→*defNode g r* **and** *V* ∈ *oldDefs g m g* ⊢ *m*−*ms*→*defNode g s*
and *set ns* ∩ *set ms* = {}
⟨*proof*⟩

Lemma 3. If a ϕ function p in a block P for a variable v is unnecessary, but non-trivial, then it has an operand q in a block Q, such that q is an unnecessary ϕ function and Q does not dominate P.

lemma *3*:
assumes *unnecessaryPhi g p* ¬*trivial g p* **and**[*simp*]: *p* ∈ *allVars g*
obtains *q* **where** *phiArg g p q unnecessaryPhi g q* ¬*def-dominates g q p*
⟨*proof*⟩

Theorem 1. A program in SSA form with a reducible CFG G without any trivial ϕ functions is in minimal SSA form.

theorem *reducible-nonredundant-imp-minimal*:
assumes *old.reducible g* ¬*redundant g*
shows *cytronMinimal g*
⟨*proof*⟩
end

context *CFG-SSA-Transformed*
begin
definition *phiCount g* = *card* ((λ(*n*,*v*). (*n*, *var g v*)) ‘ *dom* (*phis g*))

lemma *phiCount*: *phiCount g* = *card* (*dom* (*phis g*))
⟨*proof*⟩

theorem *phi-count-minimal*:
assumes *cytronMinimal g pruned g*
assumes *CFG-SSA-Transformed αe αn invar inEdges′ Entry oldDefs oldUses defs′ uses′ phis′ var′*

shows *card* (*dom* (*phis g*)) ≤ *card* (*dom* (*phis′ g*))
⟨*proof*⟩
end

end

4 SSA Construction

4.1 CFG to SSA CFG

theory *Construct-SSA* **imports** *SSA-CFG*
HOL−Library.While-Combinator
HOL−Library.Product-Lexorder
begin

datatype *Def* = *SimpleDef* | *PhiDef*
type-synonym (*'node*, *'var*) *ssaVal* = *'var* × *'node* × *Def*

instantiation *Def* :: *linorder*
begin
definition *x* < *y* ⟷ *x* = *SimpleDef* ∧ *y* = *PhiDef*
definition *less-eq-Def* (*x* :: *Def*) *y* ⟷ *x* = *y* ∨ *x* < *y*
instance ⟨*proof*⟩
end

locale *CFG-Construct* = *CFG* α*e* α*n invar inEdges′ Entry defs uses*
for
α*e* :: *'g* ⇒ (*'node*::*linorder* × *'edgeD* × *'node*) *set* **and**
α*n* :: *'g* ⇒ *'node list* **and**
invar :: *'g* ⇒ *bool* **and**
inEdges′ :: *'g* ⇒ *'node* ⇒ (*'node* × *'edgeD*) *list* **and**
Entry::*'g* ⇒ *'node* **and**
defs :: *'g* ⇒ *'node* ⇒ *'var*::*linorder set* **and**
uses :: *'g* ⇒ *'node* ⇒ *'var set*
begin
fun *phiDefNodes-aux* :: *'g* ⇒ *'var* ⇒ *'node list* ⇒ *'node* ⇒ *'node set* **where**
phiDefNodes-aux g v unvisited n =(
if n ∉ *set unvisited* ∨ *v* ∈ *defs g n then* {}
else fold (∪)
[*phiDefNodes-aux g v* (*removeAll n unvisited*) *m* . *m* ← *predecessors g n*]
(*if length* (*predecessors g n*) ≠ *1 then* {*n*} *else* {})
)

definition *phiDefNodes* :: *'g* ⇒ *'var* ⇒ *'node set* **where**
phiDefNodes g v ≡ *fold* (∪)
[*phiDefNodes-aux g v* (α*n g*) *n* . *n* ← α*n g*, *v* ∈ *uses g n*]
{}

definition *var* :: *'g* ⇒ (*'node*, *'var*) *ssaVal* ⇒ *'var* **where** *var g* ≡ *fst*

abbreviation *defNode* :: (*'node*, *'var*) *ssaVal* ⇒ *'node* **where** *defNode v* ≡ *fst* (*snd v*)
abbreviation *defKind* :: (*'node*, *'var*) *ssaVal* ⇒ *Def* **where** *defKind v* ≡ *snd* (*snd v*)

declare *var-def*[*simp*]

function *lookupDef* :: *'g* ⇒ *'node* ⇒ *'var* ⇒ (*'node*, *'var*) *ssaVal* **where**
lookupDef g n v =(
if n ∉ *set* (*αn g*) *then undefined*
else if v ∈ *defs g n then* (*v*,*n*,*SimpleDef*)
else case predecessors g n of
[*m*] ⇒ *lookupDef g m v*
| - ⇒ (*v*,*n*,*PhiDef*)
)

⟨*proof*⟩
termination ⟨*proof*⟩
declare *lookupDef.simps* [*code*]

definition *defs'* :: *'g* ⇒ *'node* ⇒ (*'node*, *'var*) *ssaVal set* **where**
defs' g n ≡ (*λv*. (*v*,*n*,*SimpleDef*)) ‘ *defs g n*
definition *uses'* :: *'g* ⇒ *'node* ⇒ (*'node*, *'var*) *ssaVal set* **where**
uses' g n ≡ *lookupDef g n* ‘ *uses g n*
definition *phis'* :: *'g* ⇒ (*'node*, (*'node*, *'var*) *ssaVal*) *phis* **where**
phis' ≡ *λg* (*n*,(*v*,*m*,*def*)).
if m = *n* ∧ *n* ∈ *phiDefNodes g v* ∧ *v* ∈ *vars g* ∧ *def* = *PhiDef then*
Some [*lookupDef g m v* . *m* ← *predecessors g n*]
else None
declare *uses'-def* [*code*] *defs'-def* [*code*] *phis'-def* [*code*]

abbreviation *lookupDefNode g n v* ≡ *defNode* (*lookupDef g n v*)
declare *lookupDef.simps* [*simp del*]
declare *phiDefNodes-aux.simps* [*simp del*]

lemma *phiDefNodes-aux-cases*:
obtains (*nonrec*) *phiDefNodes-aux g v unvisited n* = {} (*n* ∉ *set unvisited* ∨ *v* ∈ *defs g n*)
| (*rec*) *phiDefNodes-aux g v unvisited n* = *fold union* (*map* (*phiDefNodes-aux g v* (*removeAll n unvisited*)) (*predecessors g n*))
(*if length* (*predecessors g n*) = *1 then* {} *else* {*n*})
n ∈ *set unvisited v* ∉ *defs g n*
⟨*proof*⟩

lemma *phiDefNode-aux-is-join-node*:
assumes *n* ∈ *phiDefNodes-aux g v un m*
shows *length* (*predecessors g n*) ≠ *1*
⟨*proof*⟩

lemma *phiDefNode-is-join-node*:
assumes $n \in phiDefNodes\ g\ v$
shows $length\ (predecessors\ g\ n) \neq 1$
⟨*proof*⟩

abbreviation *unvisitedPath* :: $'node\ list \Rightarrow 'node\ list \Rightarrow bool$ **where**
$unvisitedPath\ un\ ns \equiv distinct\ ns \wedge set\ ns \subseteq set\ un$

lemma *unvisitedPath-removeLast*:
assumes $unvisitedPath\ un\ ns\ length\ ns \geq 2$
shows $unvisitedPath\ (removeAll\ (last\ ns)\ un)\ (butlast\ ns)$
⟨*proof*⟩

lemma *phiDefNodes-auxI*:
assumes $g \vdash n{-}ns{\rightarrow}m\ unvisitedPath\ un\ ns\ \forall n \in set\ ns.\ v \notin defs\ g\ n\ length\ (predecessors\ g\ n) \neq 1$
shows $n \in phiDefNodes\text{-}aux\ g\ v\ un\ m$
⟨*proof*⟩

lemma *phiDefNodes-auxE*:
assumes $n \in phiDefNodes\text{-}aux\ g\ v\ un\ m\ m \in set\ (\alpha n\ g)$
obtains *ns* **where** $g \vdash n{-}ns{\rightarrow}m\ \forall n \in set\ ns.\ v \notin defs\ g\ n\ length\ (predecessors\ g\ n) \neq 1\ unvisitedPath\ un\ ns$
⟨*proof*⟩

lemma *phiDefNodesE*:
assumes $n \in phiDefNodes\ g\ v$
obtains *ns m* **where** $g \vdash n{-}ns{\rightarrow}m\ \forall n \in set\ ns.\ v \notin defs\ g\ n\ v \in uses\ g\ m$
⟨*proof*⟩

lemma *phiDefNodes-αn*[*simp*]: $n \in phiDefNodes\ g\ v \Longrightarrow n \in set\ (\alpha n\ g)$
⟨*proof*⟩

lemma *phiDefNodesI*:
assumes $g \vdash n{-}ns{\rightarrow}m\ v \in uses\ g\ m\ \forall n \in set\ ns.\ v \notin defs\ g\ n\ length\ (predecessors\ g\ n) \neq 1$
shows $n \in phiDefNodes\ g\ v$
⟨*proof*⟩

lemma *lookupDef-cases*[*consumes 1*]:
assumes $n \in set\ (\alpha n\ g)$
obtains $(SimpleDef)\ v \in defs\ g\ n\ lookupDef\ g\ n\ v = (v,n,SimpleDef)$
| $(PhiDef)\ \ v \notin defs\ g\ n\ length\ (predecessors\ g\ n) \neq 1\ lookupDef\ g\ n\ v = (v,n,PhiDef)$
| $(rec)\ m$ **where** $v \notin defs\ g\ n\ predecessors\ g\ n = [m]\ m \in set\ (\alpha n\ g)\ lookupDef\ g\ n\ v = lookupDef\ g\ m\ v$
⟨*proof*⟩

lemma *lookupDef-cases'*[*consumes 1*]:

assumes $n \in set\ (\alpha n\ g)$
obtains (*SimpleDef*) $v \in defs\ g\ n\ defNode\ (lookupDef\ g\ n\ v) = n\ defKind\ (lookupDef\ g\ n\ v) = SimpleDef$
| (*PhiDef*) $v \notin defs\ g\ n\ length\ (predecessors\ g\ n) \neq 1\ lookupDefNode\ g\ n\ v = n\ defKind\ (lookupDef\ g\ n\ v) = PhiDef$
| (*rec*) m **where** $v \notin defs\ g\ n\ predecessors\ g\ n = [m]\ m \in set\ (\alpha n\ g)\ lookupDef\ g\ n\ v = lookupDef\ g\ m\ v$
⟨*proof*⟩

lemma *lookupDefE*:
assumes $lookupDef\ g\ n\ v = v'\ n \in set\ (\alpha n\ g)$
obtains (*SimpleDef*) $v \in defs\ g\ n\ v' = (v,n,SimpleDef)$
| (*PhiDef*) $v \notin defs\ g\ n\ length\ (predecessors\ g\ n) \neq 1\ v' = (v,n,PhiDef)$
| (*rec*) m **where** $v \notin defs\ g\ n\ predecessors\ g\ n = [m]\ m \in set\ (\alpha n\ g)\ v' = lookupDef\ g\ m\ v$
⟨*proof*⟩

lemma *lookupDef-induct*[*consumes 1*, *case-names SimpleDef PhiDef rec*]:
assumes $n \in set\ (\alpha n\ g)$
$\bigwedge n.\ [\![n \in set\ (\alpha n\ g);\ v \in defs\ g\ n;\ lookupDef\ g\ n\ v = (v,n,SimpleDef)]\!] \Longrightarrow P\ n$
$\bigwedge n.\ [\![n \in set\ (\alpha n\ g);\ v \notin defs\ g\ n;\ length\ (predecessors\ g\ n) \neq 1;\ lookupDef\ g\ n\ v = (v,n,PhiDef)]\!] \Longrightarrow P\ n$
$\bigwedge n\ m.\ [\![v \notin defs\ g\ n;\ predecessors\ g\ n = [m];\ m \in set\ (\alpha n\ g);\ lookupDef\ g\ n\ v = lookupDef\ g\ m\ v;\ P\ m]\!] \Longrightarrow P\ n$
shows $P\ n$
⟨*proof*⟩

lemma *lookupDef-induct′*[*consumes 2*, *case-names SimpleDef PhiDef rec*]:
assumes $n \in set\ (\alpha n\ g)\ lookupDef\ g\ n\ v = (v,n',def)$
$[\![v \in defs\ g\ n';\ def = SimpleDef]\!] \Longrightarrow P\ n'$
$[\![v \notin defs\ g\ n';\ length\ (predecessors\ g\ n') \neq 1;\ def = PhiDef]\!] \Longrightarrow P\ n'$
$\bigwedge n\ m.\ [\![v \notin defs\ g\ n;\ predecessors\ g\ n = [m];\ m \in set\ (\alpha n\ g);\ lookupDef\ g\ n\ v = lookupDef\ g\ m\ v;\ P\ m]\!] \Longrightarrow P\ n$
shows $P\ n$
⟨*proof*⟩

lemma *lookupDef-looksup*[*simp*]:
assumes $lookupDef\ g\ n\ v = (v',n',def)\ n \in set\ (\alpha n\ g)$
shows $v' = v$
⟨*proof*⟩

lemma *lookupDef-looksup′*:
assumes $(v',n',def) = lookupDef\ g\ n\ v\ n \in set\ (\alpha n\ g)$
shows $v' = v$
⟨*proof*⟩

lemma *lookupDef-looksup″*:
assumes $n \in set\ (\alpha n\ g)$

obtains n' *def* **where** *lookupDef g n v* = (*v*,*n'*,*def*)
⟨*proof*⟩

lemma *lookupDef-fst*[*simp*]: $n \in set\ (\alpha n\ g) \Longrightarrow fst\ (lookupDef\ g\ n\ v) = v$
⟨*proof*⟩

lemma *lookupDef-to-αn*:
assumes *lookupDef g n v* = (*v'*,*n'*,*def*) $n \in set\ (\alpha n\ g)$
shows $n' \in set\ (\alpha n\ g)$
⟨*proof*⟩

lemma *lookupDef-to-αn'*[*simp*]:
assumes *lookupDef g n v* = *val* $n \in set\ (\alpha n\ g)$
shows *defNode val* $\in set\ (\alpha n\ g)$
⟨*proof*⟩

lemma *lookupDef-induct''*[*consumes 2*, *case-names SimpleDef PhiDef rec*]:
assumes *lookupDef g n v* = *val* $n \in set\ (\alpha n\ g)$
$[\![v \in defs\ g\ (defNode\ val);\ defKind\ val = SimpleDef]\!] \Longrightarrow P\ (defNode\ val)$
$[\![v \notin defs\ g\ (defNode\ val);\ length\ (predecessors\ g\ (defNode\ val)) \neq 1;\ defKind\ val = PhiDef]\!] \Longrightarrow P\ (defNode\ val)$
$\bigwedge n\ m.\ [\![v \notin defs\ g\ n;\ predecessors\ g\ n = [m];\ m \in set\ (\alpha n\ g);\ lookupDef\ g\ n\ v = lookupDef\ g\ m\ v;\ P\ m]\!] \Longrightarrow P\ n$
shows *P n*
⟨*proof*⟩

lemma *defs'-finite*: *finite* (*defs'* *g n*)
⟨*proof*⟩

lemma *uses'-finite*: *finite* (*uses'* *g n*)
⟨*proof*⟩

lemma *defs'-uses'-disjoint*: $n \in set\ (\alpha n\ g) \Longrightarrow defs'\ g\ n \cap uses'\ g\ n = \{\}$
⟨*proof*⟩

lemma *allDefs'-disjoint*: $n \in set\ (\alpha n\ g) \Longrightarrow m \in set\ (\alpha n\ g) \Longrightarrow n \neq m \Longrightarrow (defs'\ g\ n \cup \{v.\ (n,\ v) \in dom\ (phis'\ g)\}) \cap (defs'\ g\ m \cup \{v.\ (m,\ v) \in dom\ (phis'\ g)\}) = \{\}$
⟨*proof*⟩

lemma *phiDefNodes-aux-finite*: *finite* (*phiDefNodes-aux g v un m*)
⟨*proof*⟩

lemma *phis'-finite*: *finite* (*dom* (*phis'* *g*))
⟨*proof*⟩

lemma *phis'-wf*: *phis'* *g* (*n*, *v*) = *Some args* $\Longrightarrow$ *length* (*predecessors g n*) = *length args*
⟨*proof*⟩

lemma *simpleDefs-phiDefs-disjoint*: $n \in set\ (\alpha n\ g) \Longrightarrow defs'\ g\ n \cap \{v.\ (n,\ v) \in dom\ (phis'\ g)\} = \{\}$
⟨*proof*⟩

lemma *oldDefs-correct*: $defs\ g\ n = var\ g\ `\ defs'\ g\ n$
⟨*proof*⟩

lemma *oldUses-correct*: $n \in set\ (\alpha n\ g) \Longrightarrow uses\ g\ n = var\ g\ `\ uses'\ g\ n$
⟨*proof*⟩

lemmas *base-SSA-defs* = *CFG-SSA-base.CFG-SSA-defs*

sublocale *braun-ssa*: *CFG-SSA* $\alpha e\ \alpha n$ *invar inEdges' Entry defs' uses' phis'*
⟨*proof*⟩
end

declare (**in** *CFG*) *invar*[*rule del*]
declare (**in** *CFG*) *Entry-no-predecessor*[*simp del*]
context *CFG-Construct*
begin
declare *invar*[*intro*!]
declare *Entry-no-predecessor*[*simp*]

lemma *no-disjoint-cycle*[*simp*]:
assumes $g \vdash n{-}ns{\rightarrow}n$ *distinct ns*
shows $ns = [n]$
⟨*proof*⟩

lemma *lookupDef-path*:
assumes $m \in set\ (\alpha n\ g)$
obtains *ns* **where** $g \vdash lookupDefNode\ g\ m\ v{-}ns{\rightarrow}m\ (\forall x \in set\ (tl\ ns).\ v \notin defs\ g\ x)$
⟨*proof*⟩

lemma *lookupDef-path-conventional*:
assumes $g \vdash n{-}ns{\rightarrow}m\ n = lookupDefNode\ g\ m\ v\ n \notin set\ (tl\ ns)\ x \in set\ (tl\ ns)\ v' \in braun\text{-}ssa.allDefs\ g\ x$
shows $var\ g\ v' \neq v$
⟨*proof*⟩

lemma *allUse-lookupDef*:
assumes $v \in braun\text{-}ssa.allUses\ g\ m\ m \in set\ (\alpha n\ g)$
shows $lookupDef\ g\ m\ (var\ g\ v) = v$
⟨*proof*⟩

lemma *phis'-fst*:
assumes $phis'\ g\ (n,v) = Some\ vs\ v' \in set\ vs$
shows $var\ g\ v' = var\ g\ v$

⟨*proof*⟩

lemma *allUse-simpleUse*:
assumes $v \in$ *braun-ssa.allUses g m* $m \in$ *set* (αn *g*)
obtains *ms m′* **where** $g \vdash m{-}ms{\rightarrow}m'$ *var g v* $\in$ *uses g m′* $\forall x \in$ *set* (*tl ms*). *var g v* $\notin$ *defs g x*
⟨*proof*⟩

lemma *defs′*: $v \in$ *defs′ g n* $\longleftrightarrow$ *var g v* $\in$ *defs g n* $\wedge$ *defKind v* = *SimpleDef* $\wedge$ *defNode v* = *n*
⟨*proof*⟩

lemma *use-implies-allDef*:
assumes *lookupDef g m* (*var g v*) = *v* *m* $\in$ *set* (αn *g*) *var g v* $\in$ *uses g m′ g* $\vdash m{-}ms{\rightarrow}m' \forall x \in$ *set* (*tl ms*). *var g v* $\notin$ *defs g x*
shows *v* $\in$ *braun-ssa.allDefs g* (*defNode v*)
⟨*proof*⟩

lemma *allUse-defNode-in-αn*[*simp*]:
assumes *v* $\in$ *braun-ssa.allUses g m m* $\in$ *set* (αn *g*)
shows *defNode v* $\in$ *set* (αn *g*)
⟨*proof*⟩

lemma *allUse-implies-allDef*:
assumes *v* $\in$ *braun-ssa.allUses g m m* $\in$ *set* (αn *g*)
shows *v* $\in$ *braun-ssa.allDefs g* (*defNode v*)
⟨*proof*⟩

lemma *conventional*:
assumes *g* $\vdash n{-}ns{\rightarrow}m$ *n* $\notin$ *set* (*tl ns*) *v* $\in$ *braun-ssa.allDefs g n v* $\in$ *braun-ssa.allUses g m*
x $\in$ *set* (*tl ns*) *v′* $\in$ *braun-ssa.allDefs g x*
shows *var g v′* $\neq$ *var g v*
⟨*proof*⟩

lemma *allDefs-var-disjoint-aux*: *n* $\in$ *set* (αn *g*) $\Longrightarrow$ *v* $\in$ *defs g n* $\Longrightarrow$ *n* $\notin$ *phiDefNodes g v*
⟨*proof*⟩

lemma *allDefs-var-disjoint*: ⟦*n* $\in$ *set* (αn *g*); *v* $\in$ *braun-ssa.allDefs g n*; *v′* $\in$ *braun-ssa.allDefs g n*; *v* $\neq$ *v′*⟧ $\Longrightarrow$ *var g v′* $\neq$ *var g v*
⟨*proof*⟩

lemma[*simp*]: *n* $\in$ *set* (αn *g*) $\Longrightarrow$ *v* $\in$ *defs g n* $\Longrightarrow$ *lookupDefNode g n v* = *n*
⟨*proof*⟩

lemma[*simp*]: *n* $\in$ *set* (αn *g*) $\Longrightarrow$ *length* (*predecessors g n*) $\neq$ *1* $\Longrightarrow$ *lookupDefNode g n v* = *n*
⟨*proof*⟩

lemma *lookupDef-idem*[*simp*]:
assumes $n \in set\ (\alpha n\ g)$
shows *lookupDef g* (*lookupDefNode g n v*) *v* = *lookupDef g n v*
⟨*proof*⟩
end

locale *CFG-Construct-wf* = *CFG-Construct* $\alpha e\ \alpha n$ *invar inEdges' Entry defs uses* + *CFG-wf* $\alpha e\ \alpha n$ *invar inEdges' Entry defs uses*
for
$\alpha e :: 'g \Rightarrow ('node::linorder \times 'edgeD \times 'node)\ set$ **and**
$\alpha n :: 'g \Rightarrow 'node\ list$ **and**
invar :: $'g \Rightarrow bool$ **and**
inEdges' :: $'g \Rightarrow 'node \Rightarrow ('node \times 'edgeD)\ list$ **and**
Entry::$'g \Rightarrow 'node$ **and**
defs :: $'g \Rightarrow 'node \Rightarrow 'var::linorder\ set$ **and**
uses :: $'g \Rightarrow 'node \Rightarrow 'var\ set$
begin
lemma *def-ass-allUses-aux*:
assumes $g \vdash Entry\ g{-}ns{\rightarrow}n$
shows *lookupDefNode g n* (*var g v*) $\in$ *set ns*
⟨*proof*⟩

lemma *def-ass-allUses*:
assumes $v \in$ *braun-ssa.allUses g n n* $\in set\ (\alpha n\ g)$
shows *braun-ssa.defAss g n v*
⟨*proof*⟩

lemma *Empty-no-phis*:
shows *phis' g* (*Entry g*, *v*) = *None*
⟨*proof*⟩

lemma *braun-ssa-CFG-SSA-wf*:
CFG-SSA-wf $\alpha e\ \alpha n$ *invar inEdges' Entry defs' uses' phis'*
⟨*proof*⟩

sublocale *braun-ssa*: *CFG-SSA-wf* $\alpha e\ \alpha n$ *invar inEdges' Entry defs' uses' phis'*
⟨*proof*⟩

lemma *braun-ssa-CFG-SSA-Transformed*:
CFG-SSA-Transformed $\alpha e\ \alpha n$ *invar inEdges' Entry defs uses defs' uses' phis' var*
⟨*proof*⟩

sublocale *braun-ssa*: *CFG-SSA-Transformed* $\alpha e\ \alpha n$ *invar inEdges' Entry defs uses defs' uses' phis' var*
⟨*proof*⟩

lemma *PhiDef-defNode-eq*:

assumes $n \in set\ (\alpha n\ g)\ n \in phiDefNodes\ g\ v\ v \in vars\ g$
shows *braun-ssa.defNode* $g\ (v,n,PhiDef) = n$
⟨*proof*⟩

lemma *phiDefNodes-aux-pruned-aux*:
assumes $n \in phiDefNodes\text{-}aux\ g\ v\ (\alpha n\ g)\ nUse\ v \in uses\ g\ nUse\ g \vdash n{-}ns{\rightarrow}m\ g \vdash m{-}ms{\rightarrow}nUse\ braun\text{-}ssa.liveVal\ g\ (lookupDef\ g\ m\ v)\ \forall n \in set\ (ns@ms).\ v \notin defs\ g\ n$
shows *braun-ssa.liveVal* $g\ (v,n,PhiDef)$
⟨*proof*⟩

lemma *phiDefNodes-aux-pruned*:
assumes $m \in phiDefNodes\text{-}aux\ g\ v\ (\alpha n\ g)\ n\ n \in set\ (\alpha n\ g)\ v \in uses\ g\ n$
shows *braun-ssa.liveVal* $g\ (v,\ m,\ PhiDef)$
⟨*proof*⟩

theorem *phis'-pruned*: *braun-ssa.pruned* g
⟨*proof*⟩

declare *var-def*[*simp del*]

declare *no-disjoint-cycle* [*simp del*]
declare *lookupDef-looksup* [*simp del*]

declare *lookupDef.simps* [*code*]
declare *phiDefNodes-aux.simps* [*code*]
declare *phiDefNodes-def* [*code*]
declare *defs'-def* [*code*]
declare *uses'-def* [*code*]
declare *phis'-def* [*code*]
declare *predecessors-def* [*code*]
end

end

4.2 Inductive Removal of Trivial Phi Functions

theory *Construct-SSA-notriv*
imports *SSA-CFG Minimality HOL−Library.While-Combinator*
begin

locale *CFG-SSA-Transformed-notriv-base* = *CFG-SSA-Transformed-base* $\alpha e\ \alpha n$ *invar inEdges' Entry oldDefs oldUses defs uses phis var*
for
$\alpha e :: 'g \Rightarrow ('node::linorder \times 'edgeD \times 'node)\ set$ **and**
$\alpha n :: 'g \Rightarrow 'node\ list$ **and**
$invar :: 'g \Rightarrow bool$ **and**
$inEdges' :: 'g \Rightarrow 'node \Rightarrow ('node \times 'edgeD)\ list$ **and**
$Entry::'g \Rightarrow 'node$ **and**

oldDefs :: *'g* ⇒ *'node* ⇒ *'var*::*linorder set* **and**
oldUses :: *'g* ⇒ *'node* ⇒ *'var set* **and**
defs :: *'g* ⇒ *'node* ⇒ *'val*::*linorder set* **and**
uses :: *'g* ⇒ *'node* ⇒ *'val set* **and**
phis :: *'g* ⇒ (*'node*, *'val*) *phis* **and**
var :: *'g* ⇒ *'val* ⇒ *'var* +
fixes *chooseNext-all* :: (*'node* ⇒ *'val set*) ⇒ (*'node*, *'val*) *phis* ⇒ *'g* ⇒ (*'node* × *'val*)
begin
abbreviation *chooseNext g* ≡ *snd* (*chooseNext-all* (*uses g*) (*phis g*) *g*)
abbreviation *chooseNext' g* ≡ *chooseNext-all* (*uses g*) (*phis g*) *g*

definition *substitution g* ≡ *THE v'*. *isTrivialPhi g* (*chooseNext g*) *v'*
definition *substNext g* ≡ λ*v*. *if v* = *chooseNext g then substitution g else v*
definition[*simp*]: *uses' g n* ≡ *substNext g* ‘ *uses g n*
definition[*simp*]: *phis' g x* ≡ *case x of* (*n*,*v*) ⇒ *if v* = *chooseNext g*
then None
else map-option (*map* (*substNext g*)) (*phis g* (*n*,*v*))
end

locale *CFG-SSA-Transformed-notriv* = *CFG-SSA-Transformed* α*e* α*n invar inEdges' Entry oldDefs oldUses defs uses phis var*
+ *CFG-SSA-Transformed-notriv-base* α*e* α*n invar inEdges' Entry oldDefs oldUses defs uses phis var chooseNext-all*
for
α*e* :: *'g* ⇒ (*'node*::*linorder* × *'edgeD* × *'node*) *set* **and**
α*n* :: *'g* ⇒ *'node list* **and**
invar :: *'g* ⇒ *bool* **and**
inEdges' :: *'g* ⇒ *'node* ⇒ (*'node* × *'edgeD*) *list* **and**
Entry::*'g* ⇒ *'node* **and**
oldDefs :: *'g* ⇒ *'node* ⇒ *'var*::*linorder set* **and**
oldUses :: *'g* ⇒ *'node* ⇒ *'var set* **and**
defs :: *'g* ⇒ *'node* ⇒ *'val*::*linorder set* **and**
uses :: *'g* ⇒ *'node* ⇒ *'val set* **and**
phis :: *'g* ⇒ (*'node*, *'val*) *phis* **and**
var :: *'g* ⇒ *'val* ⇒ *'var* **and**
chooseNext-all :: (*'node* ⇒ *'val set*) ⇒ (*'node*, *'val*) *phis* ⇒ *'g* ⇒ (*'node* × *'val*)
+
assumes *chooseNext-all*: *CFG-SSA-Transformed* α*e* α*n invar inEdges' Entry oldDefs oldUses defs u p var* ⟹
CFG-SSA-wf-base.redundant α*n inEdges' defs u p g* ⟹
chooseNext-all (*u g*) (*p g*) *g* ∈ *dom* (*p g*) ∧
CFG-SSA-wf-base.trivial α*n inEdges' defs u p g* (*snd* (*chooseNext-all* (*u g*) (*p g*) *g*))
begin
lemma *chooseNext'*:*redundant g* ⟹ *chooseNext' g* ∈ *dom* (*phis g*) ∧ *trivial g* (*chooseNext g*)
⟨*proof*⟩

lemma *chooseNext*: *redundant g* $\Longrightarrow$ *chooseNext g* $\in$ *allVars g* $\wedge$ *trivial g* (*chooseNext g*)
⟨*proof*⟩

lemmas *chooseNext-in-allVars*[*simp*] = *chooseNext*[*THEN conjunct1*]

lemma *isTrivialPhi-det*: *trivial g v* $\Longrightarrow$ $\exists!v'$. *isTrivialPhi g v v'*
⟨*proof*⟩

lemma *trivialPhi-strict-dom*:
assumes[*simp*]: *v* $\in$ *allVars g* **and** *triv*: *isTrivialPhi g v v'*
shows *strict-def-dom g v' v*
⟨*proof*⟩

lemma *isTrivialPhi-asymmetric*:
assumes *isTrivialPhi g a b*
and *isTrivialPhi g b a*
shows *False*
⟨*proof*⟩

lemma *substitution*[*intro*]: *redundant g* $\Longrightarrow$ *isTrivialPhi g* (*chooseNext g*) (*substitution g*)
⟨*proof*⟩

lemma *trivialPhi-in-allVars*[*simp*]:
assumes *isTrivialPhi g v v'* **and**[*simp*]: *v* $\in$ *allVars g*
shows *v'* $\in$ *allVars g*
⟨*proof*⟩

lemma *substitution-in-allVars*[*simp*]:
assumes *redundant g*
shows *substitution g* $\in$ *allVars g*
⟨*proof*⟩

lemma *defs-uses-disjoint-inv*:
assumes[*simp*]: *n* $\in$ *set* (αn *g*) *redundant g*
shows *defs g n* $\cap$ *uses' g n* = {}
⟨*proof*⟩
end

context *CFG-SSA-wf*
begin
inductive *liveVal'* :: *'g* $\Rightarrow$ *'val list* $\Rightarrow$ *bool*
for *g* :: *'g*
where
liveSimple': ⟦*n* $\in$ *set* (αn *g*); *val* $\in$ *uses g n*⟧ $\Longrightarrow$ *liveVal' g* [*val*]
| *livePhi'*: ⟦*liveVal' g* (*v*#*vs*); *phiArg g v v'*⟧ $\Longrightarrow$ *liveVal' g* (*v'*#*v*#*vs*)

lemma *liveVal'-suffix*:

assumes *liveVal′ g vs suffix vs′ vs vs′* ≠ []
shows *liveVal′ g vs′*
⟨*proof*⟩

lemma *liveVal′I*:
assumes *liveVal g v*
obtains *vs* **where** *liveVal′ g* (*v*#*vs*)
⟨*proof*⟩

lemma *liveVal′D*:
assumes *liveVal′ g vs vs* = *v*#*vs′*
shows *liveVal g v*
⟨*proof*⟩
end

locale *CFG-SSA-step* = *CFG-SSA-Transformed-notriv* α*e* α*n invar inEdges′ Entry oldDefs oldUses defs uses phis var chooseNext-all*
for
α*e* :: *′g* ⇒ (*′node*::*linorder* × *′edgeD* × *′node*) *set* **and**
α*n* :: *′g* ⇒ *′node list* **and**
invar :: *′g* ⇒ *bool* **and**
inEdges′ :: *′g* ⇒ *′node* ⇒ (*′node* × *′edgeD*) *list* **and**
Entry::*′g* ⇒ *′node* **and**
oldDefs :: *′g* ⇒ *′node* ⇒ *′var*::*linorder set* **and**
oldUses :: *′g* ⇒ *′node* ⇒ *′var set* **and**
defs :: *′g* ⇒ *′node* ⇒ *′val*::*linorder set* **and**
uses :: *′g* ⇒ *′node* ⇒ *′val set* **and**
phis :: *′g* ⇒ (*′node*, *′val*) *phis* **and**
var :: *′g* ⇒ *′val* ⇒ *′var* **and**
chooseNext-all :: (*′node* ⇒ *′val set*) ⇒ (*′node*, *′val*) *phis* ⇒ *′g* ⇒ (*′node* × *′val*)
and
g :: *′g* +
assumes *redundant*[*simp*]: *redundant g*
begin
abbreviation *u-g* ≡ *uses*(*g*:=*uses′ g*)
abbreviation *p-g* ≡ *phis*(*g*:=*phis′ g*)

sublocale *step*: *CFG-SSA-Transformed-notriv-base* α*e* α*n invar inEdges′ Entry oldDefs oldUses defs u-g p-g var chooseNext-all* ⟨*proof*⟩

lemma *simpleDefs-phiDefs-disjoint-inv*:
assumes *n* ∈ *set* (α*n g*)
shows *defs g n* ∩ *step.phiDefs g n* = {}
⟨*proof*⟩

lemma *allDefs-disjoint-inv*:
assumes *n* ∈ *set* (α*n g*) *m* ∈ *set* (α*n g*) *n* ≠ *m*
shows *step.allDefs g n* ∩ *step.allDefs g m* = {}
⟨*proof*⟩

lemma *phis-finite-inv*:
shows *finite* (*dom* (*phis′ g*))
⟨*proof*⟩

lemma *phis-wf-inv*:
assumes *phis′ g* (*n*, *v*) = *Some args*
shows *length* (*old.predecessors g n*) = *length args*
⟨*proof*⟩

sublocale *step*: *CFG-SSA αe αn invar inEdges′ Entry defs u-g p-g*
⟨*proof*⟩

lemma *allUses-narrows*:
assumes *n* ∈ *set* (*αn g*)
shows *step.allUses g n* ⊆ *substNext g ‘ allUses g n*
⟨*proof*⟩

lemma *allDefs-narrows*[*simp*]: *v* ∈ *step.allDefs g n* ⟹ *v* ∈ *allDefs g n*
⟨*proof*⟩

lemma *allUses-def-ass-inv*:
assumes *v′* ∈ *step.allUses g n n* ∈ *set* (*αn g*)
shows *step.defAss g n v′*
⟨*proof*⟩

lemma *Entry-no-phis-inv*: *phis′ g* (*Entry g*, *v*) = *None*
⟨*proof*⟩

sublocale *step*: *CFG-SSA-wf αe αn invar inEdges′ Entry defs u-g p-g*
⟨*proof*⟩

lemma *chooseNext-eliminated*: *chooseNext g* ∉ *step.allDefs g* (*defNode g* (*chooseNext g*))
⟨*proof*⟩

lemma *oldUses-inv*:
assumes *n* ∈ *set* (*αn g*)
shows *oldUses g n* = *var g ‘ u-g g n*
⟨*proof*⟩

lemma *conventional-inv*:
assumes *g* ⊢ *n*−*ns*→*m n* ∉ *set* (*tl ns*) *v* ∈ *step.allDefs g n v* ∈ *step.allUses g m x* ∈ *set* (*tl ns*) *v′* ∈ *step.allDefs g x*
shows *var g v′* ≠ *var g v*
⟨*proof*⟩

lemma[*simp*]: *var g* (*substNext g v*) = *var g v*

⟨*proof*⟩

lemma *phis-same-var-inv*:
assumes *phis' g (n,v) = Some vs v' ∈ set vs*
shows *var g v' = var g v*
⟨*proof*⟩

lemma *allDefs-var-disjoint-inv*: ⟦*n ∈ set (αn g)*; *v ∈ step.allDefs g n*; *v' ∈ step.allDefs g n*; *v ≠ v'*⟧ ⟹ *var g v' ≠ var g v*
⟨*proof*⟩

lemma *step-CFG-SSA-Transformed-notriv*: *CFG-SSA-Transformed-notriv αe αn invar inEdges' Entry oldDefs oldUses defs u-g p-g var chooseNext-all*
⟨*proof*⟩

sublocale *step*: *CFG-SSA-Transformed-notriv αe αn invar inEdges' Entry oldDefs oldUses defs u-g p-g var chooseNext-all*
⟨*proof*⟩

lemma *step-defNode*: *v ∈ allVars g* ⟹ *v ≠ chooseNext g* ⟹ *step.defNode g v = defNode g v*
⟨*proof*⟩

lemma *step-phi*: *v ∈ allVars g* ⟹ *v ≠ chooseNext g* ⟹ *step.phi g v = map-option (map (substNext g)) (phi g v)*
⟨*proof*⟩

lemma *liveVal'-inv*:
assumes *liveVal' g (v#vs) v ≠ chooseNext g*
obtains *vs'* **where** *step.liveVal' g (v#vs')*
⟨*proof*⟩

lemma *liveVal-inv*:
assumes *liveVal g v v ≠ chooseNext g*
shows *step.liveVal g v*
⟨*proof*⟩

lemma *pruned-inv*:
assumes *pruned g*
shows *step.pruned g*
⟨*proof*⟩
end

context *CFG-SSA-Transformed-notriv-base*
begin
abbreviation *inst g u p ≡ CFG-SSA-Transformed-notriv αe αn invar inEdges' Entry oldDefs oldUses defs (uses(g:=u)) (phis(g:=p)) var chooseNext-all*
abbreviation *inst' g ≡ λ(u,p). inst g u p*

interpretation *uninst*: *CFG-SSA-Transformed-notriv-base* αe αn *invar inEdges' Entry oldDefs oldUses defs u p var chooseNext-all*
for *u* **and** *p*
⟨*proof*⟩

definition *cond g* ≡ λ(*u*,*p*). *uninst.redundant* (*uses*(*g*:=*u*)) (*phis*(*g*:=*p*)) *g*
definition *step g* ≡ λ(*u*,*p*). (*uninst.uses'* (*uses*(*g*:=*u*)) (*phis*(*g*:=*p*)) *g*,
uninst.phis' (*uses*(*g*:=*u*)) (*phis*(*g*:=*p*)) *g*)
definition[*code*]: *substAll g* ≡ *while* (*cond g*) (*step g*) (*uses g*,*phis g*)

definition[*code*]: *uses'-all g* ≡ *fst* (*substAll g*)
definition[*code*]: *phis'-all g* ≡ *snd* (*substAll g*)

lemma *uninst-allVars-simps* [*simp*]:
uninst.allVars u (λ-. *p g*) *g* = *uninst.allVars u p g*
uninst.allVars (λ-. *u g*) *p g* = *uninst.allVars u p g*
uninst.allVars (*uses*(*g*:=*u g*)) *p g* = *uninst.allVars u p g*
uninst.allVars u (*phis*(*g*:=*p g*)) *g* = *uninst.allVars u p g*
⟨*proof*⟩

lemma *uninst-trivial-simps* [*simp*]:
uninst.trivial u (λ-. *p g*) *g* = *uninst.trivial u p g*
uninst.trivial (λ-. *u g*) *p g* = *uninst.trivial u p g*
uninst.trivial (*uses*(*g*:=*u g*)) *p g* = *uninst.trivial u p g*
uninst.trivial u (*phis*(*g*:=*p g*)) *g* = *uninst.trivial u p g*
⟨*proof*⟩

end

context *CFG-SSA-Transformed-notriv*
begin
declare *fun-upd-apply*[*simp del*] *fun-upd-same*[*simp*]

lemma *substAll-wf*:
assumes[*simp*]: *redundant g*
shows *card* (*dom* (*phis' g*)) < *card* (*dom* (*phis g*))
⟨*proof*⟩

lemma *step-preserves-inst*:
assumes *inst' g* (*u*,*p*)
and *CFG-SSA-wf-base.redundant* αn *inEdges' defs* (*uses*(*g*:=*u*)) (*phis*(*g*:=*p*)) *g*
shows *inst' g* (*step g* (*u*,*p*))
⟨*proof*⟩

lemma *substAll*:
assumes *P* (*uses g*, *phis g*)
assumes ⋀*x*. *P x* ⟹ *inst' g x* ⟹ *cond g x* ⟹ *P* (*step g x*)

assumes $\bigwedge x.\ P\ x \Longrightarrow inst'\ g\ x \Longrightarrow \neg cond\ g\ x \Longrightarrow Q\ (fst\ x)\ (snd\ x)$
shows $inst\ g\ (uses'\text{-}all\ g)\ (phis'\text{-}all\ g)\ Q\ (uses'\text{-}all\ g)\ (phis'\text{-}all\ g)$
⟨*proof*⟩

sublocale *notriv*: *CFG-SSA-Transformed* $\alpha e\ \alpha n$ *invar inEdges' Entry oldDefs oldUses defs uses'-all phis'-all*
⟨*proof*⟩

theorem *not-redundant*: $\neg notriv.redundant\ g$
⟨*proof*⟩

corollary *minimal*: $old.reducible\ g \Longrightarrow notriv.cytronMinimal\ g$
⟨*proof*⟩

theorem *pruned-invariant*:
assumes *pruned g*
shows *notriv.pruned g*
⟨*proof*⟩
end

end

5 Proof of Semantic Equivalence

theory *SSA-Semantics* **imports** *Construct-SSA* **begin**

type-synonym $('node, 'var)\ state = 'var \rightharpoonup 'node$

context *CFG-SSA-Transformed*
begin
declare *invar*[*intro*!]

definition *step* ::
$'g \Rightarrow 'node \Rightarrow ('node, 'var)\ state \Rightarrow ('node, 'var)\ state$
where
$step\ g\ m\ s\ v \equiv if\ v \in oldDefs\ g\ m\ then\ Some\ m\ else\ s\ v$

inductive $bs :: 'g \Rightarrow 'node\ list \Rightarrow ('node, 'var)\ state \Rightarrow bool$ (- ⊢ -⇓- [50, 50, 50] 50)
where
$g \vdash Entry\ g{-}ns{\rightarrow}last\ ns \Longrightarrow g \vdash ns⇓(fold\ (step\ g)\ ns\ Map.empty)$

definition *ssaStep* ::
$'g \Rightarrow 'node \Rightarrow nat \Rightarrow ('node, 'val)\ state \Rightarrow ('node, 'val)\ state$
where
$ssaStep\ g\ m\ i\ s\ v \equiv$
$if\ v \in defs\ g\ m\ then$

Some m
else
case phis g (m,v) of
Some phiParams ⇒ s (phiParams ! i)
| None ⇒ s v

inductive *ssaBS* :: *'g ⇒ 'node list ⇒ ('node, 'val) state ⇒ bool* (- ⊢ -⇓$_s$ - [*50*, *50*, *50*] *50*)
for
g :: *'g*
where
empty: *g ⊢ [Entry g]⇓$_s$(ssaStep g (Entry g) 0 Map.empty)*
| snoc: *⟦g ⊢ ns⇓$_s$s; last ns = old.predecessors g m ! i; m ∈ set (αn g); i < length (old.predecessors g m)⟧ ⟹*
g ⊢ (ns@[m])⇓$_s$(ssaStep g m i s)

lemma *ssaBS-I*:
assumes *g ⊢ Entry g−ns→n*
obtains *s* **where** *g ⊢ ns⇓$_s$s*
⟨*proof*⟩

lemma *ssaBS-nonempty*[*simp*]: *¬ (g ⊢ []⇓$_s$s)*
⟨*proof*⟩

lemma *ssaBS-hd*[*simp*]: *g ⊢ ns⇓$_s$s ⟹ hd ns = Entry g*
⟨*proof*⟩

lemma *equiv-aux*:
assumes *g ⊢ ns⇓s g ⊢ ns⇓$_s$s' g ⊢ last ns−ms→m v ∈ allUses g m ∀ n ∈ set (tl ms). var g v ∉ var g ‘ allDefs g n*
shows *s (var g v) = s' v*
⟨*proof*⟩

theorem *equiv*:
assumes *g ⊢ ns⇓s g ⊢ ns⇓$_s$s' v ∈ uses g (last ns)*
shows *s (var g v) = s' v*
⟨*proof*⟩
end

end

6 Code Generation

6.1 While Combinator Extensions

theory *While-Combinator-Exts* **imports**
HOL−Library.While-Combinator
begin

lemma *while-option-None-invD*:
assumes *while-option b c s* = *None* **and** *wf r*
and *I s* **and** $\bigwedge s.\ [\![\ I\ s;\ b\ s\ ]\!] \Longrightarrow I\ (c\ s)$
and $\bigwedge s.\ [\![\ I\ s;\ b\ s\ ]\!] \Longrightarrow (c\ s,\ s\ ) \in r$
shows *False*
⟨*proof*⟩

lemma *while-option-NoneD*:
assumes *while-option b c s* = *None*
and *wf r* **and** $\bigwedge s.\ b\ s \Longrightarrow (c\ s,\ s) \in r$
shows *False*
⟨*proof*⟩

lemma *while-option-sim*:
assumes *start*: *R* (*Some s1*) (*Some s2*)
and *cond*: $\bigwedge s1\ s2.\ [\![\ R\ (Some\ s1)\ (Some\ s2);\ I\ s1\ ]\!] \Longrightarrow b1\ s1 = b2\ s2$
and *step* : $\bigwedge s1\ s2.\ [\![\ R\ (Some\ s1)\ (Some\ s2);\ I\ s1;\ b1\ s1\ ]\!] \Longrightarrow R\ (Some\ (c1\ s1))\ (Some\ (c2\ s2))$
and *diverge*: *R None None*
and *inv-start*: *I s1*
and *inv-step*: $\bigwedge s1.\ [\![\ I\ s1;\ b1\ s1\ ]\!] \Longrightarrow I\ (c1\ s1)$
shows *R* (*while-option b1 c1 s1*) (*while-option b2 c2 s2*)
⟨*proof*⟩

end

theory *SSA-CFG-code* **imports**
SSA-CFG
Mapping-Exts
HOL−Library.Product-Lexorder
begin

definition *Union-of* :: (*'a* ⇒ *'b set*) ⇒ *'a set* ⇒ *'b set*
where *Union-of f A* ≡ $\bigcup$(*f* ‘ *A*)

lemma *Union-of-alt-def*: *Union-of f A* = ($\bigcup x \in A.\ f\ x$)
⟨*proof*⟩

type-synonym (*'node*, *'val*) *phis-code* = (*'node* × *'val*, *'val list*) *mapping*

context *CFG-base* **begin**
definition *addN* :: *'g* ⇒ *'node* ⇒ (*'var*, *'node set*) *mapping* ⇒ (*'var*, *'node set*) *mapping*
where *addN g n* ≡ *fold* (λ*v*. *Mapping.map-default v* {} (*insert n*)) (*sorted-list-of-set* (*uses g n*))
definition *addN'* *g n* = *fold* (λ*v m*. *m*(*v* ↦ *case-option* {*n*} (*insert n*) (*m v*))) (*sorted-list-of-set* (*uses g n*))

lemma *addN-transfer* [*transfer-rule*]:
rel-fun (=) (*rel-fun* (=) (*rel-fun* (*pcr-mapping* (=) (=)) (*pcr-mapping* (=) (=)))) *addN' addN*
⟨*proof*⟩

definition *useNodes-of g* = *fold* (*addN g*) (*αn g*) *Mapping.empty*
lemmas *useNodes-of-code* = *useNodes-of-def* [*unfolded addN-def* [*abs-def*]]
declare *useNodes-of-code* [*code*]

lemma *lookup-useNodes-of'*:
assumes [*simp*]: ⋀*n*. *finite* (*uses g n*)
shows *Mapping.lookup* (*useNodes-of g*) *v* =
(*if* (∃ *n* ∈ *set* (*αn g*). *v* ∈ *uses g n*) *then Some* {*n* ∈ *set* (*αn g*). *v* ∈ *uses g n*} *else None*)
⟨*proof*⟩
end

context *CFG* **begin**
lift-definition *useNodes-of'* :: *'g* ⇒ (*'var*, *'node set*) *mapping*
is *λg v*. *if* (∃ *n* ∈ *set* (*αn g*). *v* ∈ *uses g n*) *then Some* {*n* ∈ *set* (*αn g*). *v* ∈ *uses g n*} *else None* ⟨*proof*⟩

lemma *useNodes-of'*: *useNodes-of'* = *useNodes-of*
⟨*proof*⟩

declare *useNodes-of'.transfer* [*unfolded useNodes-of'*, *transfer-rule*]

lemma *lookup-useNodes-of*: *Mapping.lookup* (*useNodes-of g*) *v* =
(*if* (∃ *n* ∈ *set* (*αn g*). *v* ∈ *uses g n*) *then Some* {*n* ∈ *set* (*αn g*). *v* ∈ *uses g n*} *else None*)
⟨*proof*⟩

end

context *CFG-SSA-base* **begin**
definition *phis-addN*
where *phis-addN g n* = *fold* (*λv*. *Mapping.map-default v* {} (*insert n*)) (*case-option* [] *id* (*phis g n*))

definition *phidefNodes* **where** [*code*]:
phidefNodes g = *fold* (*λ*(*n*,*v*). *Mapping.update v n*) (*sorted-list-of-set* (*dom* (*phis g*))) *Mapping.empty*

lemma *keys-phidefNodes*:
assumes *finite* (*dom* (*phis g*))
shows *Mapping.keys* (*phidefNodes g*) = *snd* ‘ *dom* (*phis g*)
⟨*proof*⟩

definition *phiNodes-of* :: *'g* ⇒ (*'val*, (*'node* × *'val*) *set*) *mapping*

where *phiNodes-of g = fold (phis-addN g) (sorted-list-of-set (dom (phis g))) Mapping.empty*

lemma *lookup-phiNodes-of*:
assumes [*simp*]: *finite (dom (phis g))*
shows *Mapping.lookup (phiNodes-of g) v =*
(if (∃ n ∈ dom (phis g). v ∈ set (the (phis g n))) then Some {n ∈ dom (phis g). v ∈ set (the (phis g n))} else None)
⟨*proof*⟩

lemmas *phiNodes-of-code = phiNodes-of-def* [*unfolded phis-addN-def* [*abs-def*]]
declare *phiNodes-of-code* [*code*]

lemma *phis-transfer* [*transfer-rule*]:
includes *lifting-syntax*
shows ((=) ===> *pcr-mapping* (=) (=)) *phis* (λ*g*. *Mapping.Mapping (phis g)*)
⟨*proof*⟩

end

context *CFG-SSA* **begin**
declare *lookup-phiNodes-of* [*OF phis-finite*, *simp*]
declare *keys-phidefNodes* [*OF phis-finite*, *simp*]
end

locale *CFG-SSA-ext-base = CFG-SSA-base αe αn invar inEdges' Entry defs uses phis*
for *αe* :: *'g* ⇒ (*'node*::*linorder* × *'edgeD* × *'node*) *set*
and *αn* :: *'g* ⇒ *'node list*
and *invar* :: *'g* ⇒ *bool*
and *inEdges'* :: *'g* ⇒ *'node* ⇒ (*'node* × *'edgeD*) *list*
and *Entry* :: *'g* ⇒ *'node*
and *defs* :: *'g* ⇒ *'node* ⇒ *'val*::*linorder set*
and *uses* :: *'g* ⇒ *'node* ⇒ *'val set*
and *phis* :: *'g* ⇒ (*'node*, *'val*) *phis*
begin
abbreviation *cache g f ≡ Mapping.tabulate (αn g) f*

lemma *lookup-cache*[*simp*]: *n ∈ set (αn g) ⟹ Mapping.lookup (cache g f) n = Some (f n)*
⟨*proof*⟩

lemma *lookup-cacheD* [*dest*]: *Mapping.lookup (cache g f) x = Some y ⟹ y = f x*
⟨*proof*⟩

lemma *lookup-cache-usesD*: *Mapping.lookup (cache g (uses g)) n = Some vs ⟹ vs = uses g n*
⟨*proof*⟩

end

definition[*simp*]: *usesOf m n* ≡ *case-option* {} *id* (*Mapping.lookup m n*)

locale *CFG-SSA-ext* = *CFG-SSA-ext-base* αe αn *invar inEdges' Entry defs uses phis*
+ *CFG-SSA* αe αn *invar inEdges' Entry defs uses phis*
for αe :: $'g \Rightarrow ('node::linorder \times 'edgeD \times 'node)$ *set*
and αn :: $'g \Rightarrow 'node$ *list*
and *invar* :: $'g \Rightarrow$ *bool*
and *inEdges'* :: $'g \Rightarrow 'node \Rightarrow ('node \times 'edgeD)$ *list*
and *Entry* :: $'g \Rightarrow 'node$
and *defs* :: $'g \Rightarrow 'node \Rightarrow 'val::linorder$ *set*
and *uses* :: $'g \Rightarrow 'node \Rightarrow 'val$ *set*
and *phis* :: $'g \Rightarrow ('node, 'val)$ *phis*
begin
lemma *usesOf-cache*[*abs-def*, *simp*]: *usesOf* (*cache g* (*uses g*)) *n* = *uses g n*
⟨*proof*⟩
end

locale *CFG-SSA-base-code* = *CFG-SSA-ext-base* αe αn *invar inEdges' Entry defs usesOf* ∘ *uses* λg. *Mapping.lookup* (*phis g*)
for αe :: $'g \Rightarrow ('node::linorder \times 'edgeD \times 'node)$ *set*
and αn :: $'g \Rightarrow 'node$ *list*
and *invar* :: $'g \Rightarrow$ *bool*
and *inEdges'* :: $'g \Rightarrow 'node \Rightarrow ('node \times 'edgeD)$ *list*
and *Entry* :: $'g \Rightarrow 'node$
and *defs* :: $'g \Rightarrow 'node \Rightarrow 'val::linorder$ *set*
and *uses* :: $'g \Rightarrow ('node, 'val\ set)$ *mapping*
and *phis* :: $'g \Rightarrow ('node, 'val)$ *phis-code*
begin
declare *phis-transfer* [*simplified*, *transfer-rule*]

lemma *phiDefs-code* [*code*]:
phiDefs g n = *snd* ‘ *Set.filter* ($\lambda(n',v)$. $n' = n$) (*Mapping.keys* (*phis g*))
⟨*proof*⟩

lemmas *phiUses-code* [*code*] = *phiUses-def* [*folded Union-of-alt-def*]
declare *allUses-def* [*code*]
lemmas *allVars-code* [*code*] = *allVars-def* [*folded Union-of-alt-def*]
end

locale *CFG-SSA-code* = *CFG-SSA-base-code* αe αn *invar inEdges' Entry defs uses phis*
+ *CFG-SSA-ext* αe αn *invar inEdges' Entry defs usesOf* ∘ *uses* λg. *Mapping.lookup* (*phis g*)
for αe :: $'g \Rightarrow ('node::linorder \times 'edgeD \times 'node)$ *set*
and αn :: $'g \Rightarrow 'node$ *list*
and *invar* :: $'g \Rightarrow$ *bool*

and *inEdges'* :: *'g* ⇒ *'node* ⇒ (*'node* × *'edgeD*) *list*
and *Entry* :: *'g* ⇒ *'node*
and *defs* :: *'g* ⇒ *'node* ⇒ *'val*::*linorder set*
and *uses* :: *'g* ⇒ (*'node*, *'val set*) *mapping*
and *phis* :: *'g* ⇒ (*'node*, *'val*) *phis-code*

definition *the-trivial v vs* = (*case* (*foldl* (λ(*good*,*v'*) *w*. *if w* = *v then* (*good*,*v'*)
else case v' of Some v' ⇒ (*good* ∧ *w* = *v'*, *Some v'*)
| *None* ⇒ (*good*, *Some w*))
(*True*, *None*) *vs*)
of (*False*, -) ⇒ *None* | (*True*,*v*) ⇒ *v*)

lemma *the-trivial-Nil* [*simp*]: *the-trivial x* [] = *None*
⟨*proof*⟩

lemma *the-trivialI*:
assumes *set vs* ⊆ {*v*, *v'*}
and *v'* ≠ *v*
shows *the-trivial v vs* = (*if set vs* ⊆ {*v*} *then None else Some v'*)
⟨*proof*⟩

lemma *the-trivial-conv*:
shows *the-trivial v vs* = (*if* ∃ *v'* ∈ *set vs*. *v'* ≠ *v* ∧ *set vs* − {*v'*} ⊆ {*v*} *then Some* (*THE v'*. *v'* ∈ *set vs* ∧ *v'* ≠ *v* ∧ *set vs* − {*v'*} ⊆ {*v*}) *else None*)
⟨*proof*⟩

lemma *the-trivial-SomeE*:
assumes *the-trivial v vs* = *Some v'*
obtains *v* ≠ *v'* **and** *set vs* = {*v'*} | *v* ≠ *v'* **and** *set vs* = {*v*,*v'*}
⟨*proof*⟩

locale *CFG-SSA-wf-base-code* = *CFG-SSA-base-code* α*e* α*n invar inEdges' Entry defs uses phis*
+ *CFG-SSA-wf-base* α*e* α*n invar inEdges' Entry defs usesOf* ∘ *uses* λ*g*. *Mapping.lookup* (*phis g*)
for α*e* :: *'g* ⇒ (*'node*::*linorder* × *'edgeD* × *'node*) *set*
and α*n* :: *'g* ⇒ *'node list*
and *invar* :: *'g* ⇒ *bool*
and *inEdges'* :: *'g* ⇒ *'node* ⇒ (*'node* × *'edgeD*) *list*
and *Entry* :: *'g* ⇒ *'node*
and *defs* :: *'g* ⇒ *'node* ⇒ *'val*::*linorder set*
and *uses* :: *'g* ⇒ (*'node*, *'val set*) *mapping*
and *phis* :: *'g* ⇒ (*'node*, *'val*) *phis-code*
begin
definition [*code*]:
trivial-code (*v*::*'val*) *vs* = (*the-trivial v vs* ≠ *None*)
definition[*code*]: *trivial-phis g* = *Set.filter* (λ(*n*,*v*). *trivial-code v* (*the* (*Mapping.lookup* (*phis g*) (*n*,*v*)))) (*Mapping.keys* (*phis g*))

definition [*code*]: *redundant-code g* = (*trivial-phis g* ≠ {})
end

locale *CFG-SSA-wf-code* = *CFG-SSA-code αe αn invar inEdges' Entry defs uses phis*
+ *CFG-SSA-wf-base-code αe αn invar inEdges' Entry defs uses phis*
+ *CFG-SSA-wf αe αn invar inEdges' Entry defs usesOf* ∘ *uses* λ*g. Mapping.lookup* (*phis g*)
for *αe* :: *'g* ⇒ (*'node*::*linorder* × *'edgeD* × *'node*) *set*
and *αn* :: *'g* ⇒ *'node list*
and *invar* :: *'g* ⇒ *bool*
and *inEdges'* :: *'g* ⇒ *'node* ⇒ (*'node* × *'edgeD*) *list*
and *Entry* :: *'g* ⇒ *'node*
and *defs* :: *'g* ⇒ *'node* ⇒ *'val*::*linorder set*
and *uses* :: *'g* ⇒ (*'node*, *'val set*) *mapping*
and *phis* :: *'g* ⇒ (*'node*, *'val*) *phis-code*
begin
lemma *trivial-code*:
phi g v = *Some vs* ⟹ *trivial g v* = *trivial-code v vs*
⟨*proof*⟩

lemma *trivial-phis*:
trivial-phis g = {(*n*,*v*). *Mapping.lookup* (*phis g*) (*n*,*v*) ≠ *None* ∧ *trivial g v*}
⟨*proof*⟩

lemma *redundant-code*:
redundant g = *redundant-code g*
⟨*proof*⟩

lemma *trivial-code-mapI*:
⟦ *trivial-code v vs*; *f* ‘ (*set vs* − {*v*}) ≠ {*v*} ; *f v* = *v* ⟧ ⟹ *trivial-code v* (*map f vs*)
⟨*proof*⟩

lemma *trivial-code-map-conv*:
f v = *v* ⟹ *trivial-code v* (*map f vs*) ⟷ (∃ *v'*∈*set vs*. *f v'* ≠ *v* ∧ (*f* ‘ *set vs*) − {*f v'*} ⊆ {*v*})
⟨*proof*⟩

end

locale *CFG-SSA-Transformed-code* = *ssa*: *CFG-SSA-wf-code αe αn invar inEdges' Entry defs uses phis*
+
CFG-SSA-Transformed αe αn invar inEdges' Entry oldDefs oldUses defs usesOf ∘ *uses* λ*g. Mapping.lookup* (*phis g*) *var*
for
αe :: *'g* ⇒ (*'node*::*linorder* × *'edgeD* × *'node*) *set* **and**
αn :: *'g* ⇒ *'node list* **and**

invar :: *'g* ⇒ *bool* **and**
inEdges' :: *'g* ⇒ *'node* ⇒ (*'node* × *'edgeD*) *list* **and**
Entry::*'g* ⇒ *'node* **and**
oldDefs :: *'g* ⇒ *'node* ⇒ *'var*::*linorder set* **and**
oldUses :: *'g* ⇒ *'node* ⇒ *'var set* **and**
defs :: *'g* ⇒ *'node* ⇒ *'val*::*linorder set* **and**
uses :: *'g* ⇒ (*'node*, *'val set*) *mapping* **and**
phis :: *'g* ⇒ (*'node*, *'val*) *phis-code* **and**
var :: *'g* ⇒ *'val* ⇒ *'var*
+
assumes *dom-uses-in-graph*: *Mapping.keys* (*uses g*) ⊆ *set* (*αn g*)

end

6.2 Code Equations for SSA Construction

theory *Construct-SSA-code* **imports**
SSA-CFG-code
Construct-SSA
Mapping-Exts
HOL−Library.Product-Lexorder
begin

definition[*code*]: *lookup-multimap m k* ≡ (*case-option* {} *id* (*Mapping.lookup m k*))

locale *CFG-Construct-linorder* = *CFG-Construct-wf αe αn invar inEdges' Entry defs uses*
for
αe :: *'g* ⇒ (*'node*::*linorder* × *'edgeD* × *'node*) *set* **and**
αn :: *'g* ⇒ *'node list* **and**
invar :: *'g* ⇒ *bool* **and**
inEdges' :: *'g* ⇒ *'node* ⇒ (*'node* × *'edgeD*) *list* **and**
Entry::*'g* ⇒ *'node* **and**
defs :: *'g* ⇒ *'node* ⇒ (*'var*::*linorder*) *set* **and**
uses :: *'g* ⇒ *'node* ⇒ *'var set*
begin
type-synonym (*'n*, *'v*) *sparse-phis* = (*'n* × *'v*, (*'n*, *'v*) *ssaVal list*) *mapping*

function *readVariableRecursive* :: *'g* ⇒ *'var* ⇒ *'node* ⇒ (*'node*, *'var*) *sparse-phis* ⇒ ((*'node*, *'var*) *ssaVal* × (*'node*, *'var*) *sparse-phis*)
and *readArgs* :: *'g* ⇒ *'var* ⇒ *'node* ⇒ (*'node*, *'var*) *sparse-phis* ⇒ *'node list* ⇒ (*'node*, *'var*) *sparse-phis* × (*'node*, *'var*) *ssaVal list*
where[*code*]: *readVariableRecursive g v n phis* = (*if v* ∈ *defs g n then* ((*v*,*n*,*SimpleDef*), *phis*)
else case predecessors g n of
[] ⇒ ((*v*,*n*,*PhiDef*), *Mapping.update* (*n*,*v*) [] *phis*)
| [*m*] ⇒ *readVariableRecursive g v m phis*
| *ms* ⇒ (*case Mapping.lookup phis* (*n*,*v*) *of*

```
      Some - ⇒ ((v,n,PhiDef),phis)
    | None ⇒
      let phis = Mapping.update (n,v) [] phis in
      let (phis,args) = readArgs g v n phis ms in
      ((v,n,PhiDef), Mapping.update (n,v) args phis)
))
| readArgs g v n phis [] = (phis,[])
| readArgs g v n phis (m#ms) = (
    let (phis,args) = readArgs g v n phis ms in
    let (v,phis) = readVariableRecursive g v m phis in
    (phis,v#args))
⟨proof⟩

lemma length-filter-less2:
  assumes x ∈ set xs ¬P x Q x ⋀x. P x ⟹ Q x
  shows length (filter P xs) < length (filter Q xs)
⟨proof⟩

lemma length-filter-le2:
  assumes ⋀x. P x ⟹ Q x
  shows length (filter P xs) ≤ length (filter Q xs)
⟨proof⟩

abbreviation phis-measure g v phis ≡ length [n ← αn g. Mapping.lookup phis
(n,v) = None]

lemma phis-measure-update-le: phis-measure g v (Mapping.update k a p) ≤
phis-measure g v p
⟨proof⟩

lemma phis-measure-update-le': phis-measure g v p ≤ phis-measure g v (Mapping.update
k [] phis) ⟹
    phis-measure g v (Mapping.update k a p) ≤ phis-measure g v phis
⟨proof⟩

lemma readArgs-phis-le:
  readVariableRecursive-readArgs-dom (Inl (g, v, n, phis)) ⟹ (val,p) = read-
VariableRecursive g v n phis ⟹ phis-measure g v p ≤ phis-measure g v phis
  readVariableRecursive-readArgs-dom (Inr (g, v, n, phis, ms)) ⟹ (p,u) =
readArgs g v n phis ms ⟹ phis-measure g v p ≤ phis-measure g v phis
⟨proof⟩

termination
⟨proof⟩

declare readVariableRecursive.simps[simp del] readArgs.simps[simp del]

lemma fst-readVariableRecursive:
  assumes n ∈ set (αn g)
```

shows *fst* (*readVariableRecursive g v n phis*) = *lookupDef g n v*
⟨*proof*⟩

definition *phis'-aux g v ns* (*phis*:: (*'node*,*'var*) *sparse-phis*) ≡ *Mapping.Mapping* (λ(*m*,*v*$_2$).
(*if* *v*$_2$=*v* ∧ *m* ∈ ⋃(*phiDefNodes-aux g v* [*n* ← α*n g*. (*n*,*v*) ∉ *Mapping.keys phis*] ‘ *ns*) ∧ *v* ∈ *vars g then Some* (*map* (λ*m*. *lookupDef g m v*) (*predecessors g m*)) *else* (*Mapping.lookup phis* (*m*,*v*$_2$))))

lemma *phis'-aux-keys-super*: *Mapping.keys* (*phis'-aux g v ns phis*) ⊇ *Mapping.keys phis*
⟨*proof*⟩

lemma *phiDefNodes-aux-in-unvisited*:
shows *phiDefNodes-aux g v un n* ⊆ *set un*
⟨*proof*⟩

lemma *phiDefNodes-aux-unvisited-monotonic*:
assumes *set un* ⊆ *set un'*
shows *phiDefNodes-aux g v un n* ⊆ *phiDefNodes-aux g v un' n*
⟨*proof*⟩

lemma *phiDefNodes-aux-single-pred*:
assumes *predecessors g n* = [*m*]
shows *phiDefNodes-aux g v* (*removeAll n un*) *m* = *phiDefNodes-aux g v un m*
⟨*proof*⟩

lemma *phis'-aux-finite*:
assumes *finite* (*Mapping.keys phis*)
shows *finite* (*Mapping.keys* (*phis'-aux g v ns phis*))
⟨*proof*⟩

lemma *phiDefNodes-aux-redirect*:
assumes *asm*: *g* ⊢ *n*−*ns*→*m* ∀ *n* ∈ *set ns*. *v* ∉ *defs g n length* (*predecessors g n*) ≠ *1 unvisitedPath un ns*
assumes *n'*: *n'* ∈ *set ns n'* ∈ *phiDefNodes-aux g v un m' m'* ∈ *set* (α*n g*)
shows *n* ∈ *phiDefNodes-aux g v un m'*
⟨*proof*⟩

lemma *snd-readVariableRecursive*:
assumes *v* ∈ *vars g n* ∈ *set* (α*n g*) *finite* (*Mapping.keys phis*)
⋀*n*. (*n*,*v*) ∈ *Mapping.keys phis* ⟹ *length* (*predecessors g n*) ≠ *1 Mapping.lookup phis* (*Entry g*,*v*) ∈ {*None*, *Some* []}
shows
phis'-aux g v {*n*} *phis* = *snd* (*readVariableRecursive g v n phis*)
set ms ⊆ *set* (α*n g*) ⟹ (*phis'-aux g v* (*set ms*) *phis*, *map* (λ*m*. *lookupDef g m v*) *ms*) = *readArgs g v n phis ms*
⟨*proof*⟩

definition *aux-1 g n* = (λ*v* (*uses*,*phis*).
let (*use*,*phis′*) = *readVariableRecursive g v n phis in*
(*Mapping.update n* (*insert use* (*lookup-multimap uses n*)) *uses*, *phis′*)
)

definition *aux-2 g n* = *foldr* (*aux-1 g n*) (*sorted-list-of-set* (*uses g n*))

abbreviation *init-state* ≡ (*Mapping.empty*, *Mapping.empty*)
abbreviation *from-sparse* ≡ λ(*n*,*v*). (*n*,(*v*,*n*,*PhiDef*))
definition *uses′-phis′ g* = (
let (*u*,*p*) = *foldr* (*aux-2 g*) (α*n g*) *init-state in*
(*u*, *map-keys from-sparse p*)
)

lemma *from-sparse-inj*: *inj from-sparse*
⟨*proof*⟩

declare *uses′-phis′-def*[*unfolded aux-2-def*[*abs-def*] *aux-1-def*, *code*]

lift-definition *phis′-code* :: *′g* ⇒ (*′node*, (*′node*, *′var*) *ssaVal*) *phis-code* **is** *phis′*
⟨*proof*⟩

lemma *foldr-prod*: *foldr* (λ*x y*. (*f1 x* (*fst y*), *f2 x* (*snd y*))) *xs y* = (*foldr f1 xs* (*fst y*), *foldr f2 xs* (*snd y*))
⟨*proof*⟩

lemma *foldr-aux-1*:
assumes *set us* ⊆ *uses g n Mapping.lookup u n* = *None foldr* (*aux-1 g n*) *us* (*u*,*p*) = (*u′*,*p′*) (**is** *foldr ?f - - = -*)
assumes *finite* (*Mapping.keys p*) ⋀*n v*. (*n*,*v*) ∈ *Mapping.keys p* ⟹ *length* (*predecessors g n*) ≠ *1* ⋀*v*. *Mapping.lookup p* (*Entry g*,*v*) ∈ {*None*, *Some* []}
shows *lookupDef g n ‘ set us* = *lookup-multimap u′ n* ⋀*m*. *m* ≠ *n* ⟹ *Mapping.lookup u′ m* = *Mapping.lookup u m*
⋀*m v*. (*if m* ∈ *phiDefNodes-aux g v* [*n* ← α*n g*. (*n*,*v*) ∉ *Mapping.keys p*] *n* ∧ *v* ∈ *set us then*
Some (*map* (λ*m*. *lookupDef g m v*) (*predecessors g m*)) *else*
(*Mapping.lookup p* (*m*,*v*))) = *Mapping.lookup p′* (*m*,*v*)
⟨*proof*⟩

lemma *foldr-aux-2*:
assumes *set ns* ⊆ *set* (α*n g*) *distinct ns foldr* (*aux-2 g*) *ns init-state* = (*u′*,*p′*)
shows ⋀*n*. *n* ∈ *set ns* ⟹ *uses′ g n* = *lookup-multimap u′ n* ⋀*n*. *n* ∉ *set ns* ⟹ *Mapping.lookup u′ n* = *None*
⋀*m v*. (*if* ∃ *n* ∈ *set ns*. *m* ∈ *phiDefNodes-aux g v* (α*n g*) *n* ∧ *v* ∈ *uses g n then*
Some (*map* (λ*m*. *lookupDef g m v*) (*predecessors g m*)) *else*
None) = *Mapping.lookup p′* (*m*,*v*)
⟨*proof*⟩

lemma *fst-uses'-phis'*: *uses' g* = *lookup-multimap* (*fst* (*uses'-phis' g*))
⟨*proof*⟩

lemma *fst-uses'-phis'-in-αn*: *Mapping.keys* (*fst* (*uses'-phis' g*)) ⊆ *set* (*αn g*)
⟨*proof*⟩

lemma *snd-uses'-phis'*: *phis'-code g* = *snd* (*uses'-phis' g*)
⟨*proof*⟩
end

end

6.3 Locales Transfer Rules

theory *SSA-Transfer-Rules* **imports**
SSA-CFG
Construct-SSA-code
begin

context includes *lifting-syntax*
begin

lemmas *weak-All-transfer1* [*transfer-rule*] = *iffD1* [*OF right-total-alt-def2*]
lemma *weak-All-transfer2* [*transfer-rule*]: *right-total R* ⟹ ((*R* ===> (=)) ===> (⟶)) *All All*
⟨*proof*⟩

lemma *weak-imp-transfer* [*transfer-rule*]:
((=) ===> (=) ===> (⟶)) (⟶) (⟶)
⟨*proof*⟩

lemma *weak-conj-transfer* [*transfer-rule*]:
((⟶) ===> (⟶) ===> (⟶)) (∧) (∧)
⟨*proof*⟩

lemma *graph-path-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *right-total G*
and [*transfer-rule*]: (*G* ===> (=)) *αe αe2*
and [*transfer-rule*]: (*G* ===> (=)) *αn αn2*
and [*transfer-rule*]: (*G* ===> (=)) *invar invar2*
and [*transfer-rule*]: (*G* ===> (=)) *inEdges inEdges2*
shows (⟶) (*graph-path αe αn invar inEdges*) (*graph-path αe2 αn2 invar2 inEdges2*)
⟨*proof*⟩

end

context *graph-path-base* **begin**

context includes *lifting-syntax*
begin

lemma *inEdges-transfer* [*transfer-rule*]:
  assumes [*transfer-rule*]: *right-total A*
    and [*transfer-rule*]: $(A ===> (=))$ αe $\alpha e2$
    and [*transfer-rule*]: $(A ===> (=))$ αn $\alpha n2$
    and [*transfer-rule*]: $(A ===> (=))$ *invar invar2*
    and [*transfer-rule*]: $(A ===> (=))$ *inEdges' inEdges2*
  shows $(A ===> (=))$ *inEdges* (*graph-path-base.inEdges inEdges2*)
⟨*proof*⟩

lemma *predecessors-transfer* [*transfer-rule*]:
  assumes [*transfer-rule*]: *right-total A*
    and [*transfer-rule*]: $(A ===> (=))$ αe $\alpha e2$
    and [*transfer-rule*]: $(A ===> (=))$ αn $\alpha n2$
    and [*transfer-rule*]: $(A ===> (=))$ *invar invar2*
    and [*transfer-rule*]: $(A ===> (=))$ *inEdges' inEdges2*
  shows $(A ===> (=))$ *predecessors* (*graph-path-base.predecessors inEdges2*)
⟨*proof*⟩

lemma *successors-transfer* [*transfer-rule*]:
  assumes [*transfer-rule*]: *right-total A*
    and [*transfer-rule*]: $(A ===> (=))$ αe $\alpha e2$
    and [*transfer-rule*]: $(A ===> (=))$ αn $\alpha n2$
    and [*transfer-rule*]: $(A ===> (=))$ *invar invar2*
    and [*transfer-rule*]: $(A ===> (=))$ *inEdges' inEdges2*
  shows $(A ===> (=))$ *successors* (*graph-path-base.successors* $\alpha n2$ *inEdges2*)
⟨*proof*⟩

lemma *path-transfer* [*transfer-rule*]:
  assumes [*transfer-rule*]: *right-total A*
    and [*transfer-rule*]: $(A ===> (=))$ αe $\alpha e2$
    and [*transfer-rule*]: $(A ===> (=))$ αn $\alpha n2$
    and [*transfer-rule*]: $(A ===> (=))$ *invar invar2*
    and [*transfer-rule*]: $(A ===> (=))$ *inEdges' inEdges2*
  shows $(A ===> (=))$ *path* (*graph-path-base.path* $\alpha n2$ *invar2 inEdges2*)
⟨*proof*⟩

lemma *path2-transfer* [*transfer-rule*]:
  assumes [*transfer-rule*]: *right-total A*
    and [*transfer-rule*]: $(A ===> (=))$ αe $\alpha e2$
    and [*transfer-rule*]: $(A ===> (=))$ αn $\alpha n2$
    and [*transfer-rule*]: $(A ===> (=))$ *invar invar2*
    and [*transfer-rule*]: $(A ===> (=))$ *inEdges' inEdges2*
  shows $(A ===> (=))$ *path2* (*graph-path-base.path2* $\alpha n2$ *invar2 inEdges2*)
⟨*proof*⟩

lemma *weak-Ex-transfer* [*transfer-rule*]: (((=) ===> (⟶)) ===> (⟶)) *Ex Ex*
⟨*proof*⟩

lemmas *transfer-rules* = *inEdges-transfer predecessors-transfer successors-transfer path-transfer path2-transfer*

end

end

lemma *graph-Entry-transfer* [*transfer-rule*]:
includes *lifting-syntax*
assumes [*transfer-rule*]: *right-total G*
and [*transfer-rule*]: (*G* ===> (=)) *αe1 αe2*
and [*transfer-rule*]: (*G* ===> (=)) *αn1 αn2*
and [*transfer-rule*]: (*G* ===> (=)) *invar1 invar2*
and [*transfer-rule*]: (*G* ===> (=)) *inEdges1 inEdges2*
and [*transfer-rule*]: (*G* ===> (=)) *Entry1 Entry2*
shows (⟶) (*graph-Entry αe1 αn1 invar1 inEdges1 Entry1*) (*graph-Entry αe2 αn2 invar2 inEdges2 Entry2*)
⟨*proof*⟩

context *graph-Entry-base* **begin**

lemma *dominates-transfer* [*transfer-rule*]:
includes *lifting-syntax*
assumes [*transfer-rule*]: *right-total G*
and [*transfer-rule*]: (*G* ===> (=)) *αe αe2*
and [*transfer-rule*]: (*G* ===> (=)) *αn αn2*
and [*transfer-rule*]: (*G* ===> (=)) *invar invar2*
and [*transfer-rule*]: (*G* ===> (=)) *inEdges′ inEdges2*
and [*transfer-rule*]: (*G* ===> (=)) *Entry Entry2*
shows (*G* ===> (=)) *dominates* (*graph-Entry-base.dominates αn2 invar2 inEdges2 Entry2*)
⟨*proof*⟩

end

context *graph-Entry* **begin**

context includes *lifting-syntax*
begin

lemma *shortestPath-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *right-total G*
and [*transfer-rule*]: (*G* ===> (=)) *αe αe2*
and [*transfer-rule*]: (*G* ===> (=)) *αn αn2*
and [*transfer-rule*]: (*G* ===> (=)) *invar invar2*
and [*transfer-rule*]: (*G* ===> (=)) *inEdges′ inEdges2*

and [*transfer-rule*]: $(G ===> (=))$ *Entry Entry2*
shows $(G ===> (=))$ *shortestPath* (*graph-Entry.shortestPath* $\alpha n2$ *invar2 inEdges2 Entry2*)
⟨*proof*⟩

lemma *dominators-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *right-total G*
and [*transfer-rule*]: $(G ===> (=))$ αe $\alpha e2$
and [*transfer-rule*]: $(G ===> (=))$ αn $\alpha n2$
and [*transfer-rule*]: $(G ===> (=))$ *invar invar2*
and [*transfer-rule*]: $(G ===> (=))$ *inEdges' inEdges2*
and [*transfer-rule*]: $(G ===> (=))$ *Entry Entry2*
shows $(G ===> (=))$ *dominators* (*graph-Entry.dominators* $\alpha n2$ *invar2 inEdges2 Entry2*)
⟨*proof*⟩

lemma *isIdom-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *right-total G*
and [*transfer-rule*]: $(G ===> (=))$ αe $\alpha e2$
and [*transfer-rule*]: $(G ===> (=))$ αn $\alpha n2$
and [*transfer-rule*]: $(G ===> (=))$ *invar invar2*
and [*transfer-rule*]: $(G ===> (=))$ *inEdges' inEdges2*
and [*transfer-rule*]: $(G ===> (=))$ *Entry Entry2*
shows $(G ===> (=))$ *isIdom* (*graph-Entry.isIdom* $\alpha n2$ *invar2 inEdges2 Entry2*)
⟨*proof*⟩

lemma *idom-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *right-total G*
and [*transfer-rule*]: $(G ===> (=))$ αe $\alpha e2$
and [*transfer-rule*]: $(G ===> (=))$ αn $\alpha n2$
and [*transfer-rule*]: $(G ===> (=))$ *invar invar2*
and [*transfer-rule*]: $(G ===> (=))$ *inEdges' inEdges2*
and [*transfer-rule*]: $(G ===> (=))$ *Entry Entry2*
shows $(G ===> (=))$ *idom* (*graph-Entry.idom* $\alpha n2$ *invar2 inEdges2 Entry2*)
⟨*proof*⟩

lemmas *graph-Entry-transfer* =
dominates-transfer
shortestPath-transfer
dominators-transfer
isIdom-transfer
idom-transfer
end

end

lemma *CFG-transfer* [*transfer-rule*]:
includes *lifting-syntax*
assumes [*transfer-rule*]: *right-total G*

and [*transfer-rule*]: (*G* ===> (=)) *αe1 αe2*
and [*transfer-rule*]: (*G* ===> (=)) *αn1 αn2*
and [*transfer-rule*]: (*G* ===> (=)) *invar1 invar2*
and [*transfer-rule*]: (*G* ===> (=)) *inEdges1 inEdges2*
and [*transfer-rule*]: (*G* ===> (=)) *Entry1 Entry2*
and [*transfer-rule*]: (*G* ===> (=)) *defs1 defs2*
and [*transfer-rule*]: (*G* ===> (=)) *uses1 uses2*
shows *SSA-CFG.CFG αe1 αn1 invar1 inEdges1 Entry1 defs1 uses1*
⟶ *SSA-CFG.CFG αe2 αn2 invar2 inEdges2 Entry2 defs2 uses2*
⟨*proof*⟩

context *CFG-base* **begin**

context includes *lifting-syntax*
begin

lemma *vars-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *right-total G*
and [*transfer-rule*]: (*G* ===> (=)) *αe αe2*
and [*transfer-rule*]: (*G* ===> (=)) *αn αn2*
and [*transfer-rule*]: (*G* ===> (=)) *invar invar2*
and [*transfer-rule*]: (*G* ===> (=)) *inEdges′ inEdges2*
and [*transfer-rule*]: (*G* ===> (=)) *Entry Entry2*
and [*transfer-rule*]: (*G* ===> (=)) *defs defs2*
and [*transfer-rule*]: (*G* ===> (=)) *uses uses2*
shows (*G* ===> (=)) *vars* (*CFG-base.vars αn2 uses2*)
⟨*proof*⟩

lemma *defAss′-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *right-total G*
and [*transfer-rule*]: (*G* ===> (=)) *αe αe2*
and [*transfer-rule*]: (*G* ===> (=)) *αn αn2*
and [*transfer-rule*]: (*G* ===> (=)) *invar invar2*
and [*transfer-rule*]: (*G* ===> (=)) *inEdges′ inEdges2*
and [*transfer-rule*]: (*G* ===> (=)) *Entry Entry2*
and [*transfer-rule*]: (*G* ===> (=)) *defs defs2*
and [*transfer-rule*]: (*G* ===> (=)) *uses uses2*
shows (*G* ===> (=)) *defAss′* (*CFG-base.defAss′ αn2 invar2 inEdges2 Entry2 defs2*)
⟨*proof*⟩

lemma *defAss′Uses-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *right-total G*
and [*transfer-rule*]: (*G* ===> (=)) *αe αe2*
and [*transfer-rule*]: (*G* ===> (=)) *αn αn2*
and [*transfer-rule*]: (*G* ===> (=)) *invar invar2*
and [*transfer-rule*]: (*G* ===> (=)) *inEdges′ inEdges2*
and [*transfer-rule*]: (*G* ===> (=)) *Entry Entry2*
and [*transfer-rule*]: (*G* ===> (=)) *defs defs2*

and [*transfer-rule*]: (*G* ===> (=)) *uses uses2*
shows (*G* ===> (=)) *defAss'Uses* (*CFG-base.defAss'Uses* $\alpha n2$ *invar2 inEdges2 Entry2 defs2 uses2*)
⟨*proof*⟩

lemmas *CFG-transfers* =
vars-transfer
defAss'-transfer
defAss'Uses-transfer

end

end

context includes *lifting-syntax*
begin

lemma *CFG-Construct-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *right-total G*
and [*transfer-rule*]: (*G* ===> (=)) $\alpha e1$ $\alpha e2$
and [*transfer-rule*]: (*G* ===> (=)) $\alpha n1$ $\alpha n2$
and [*transfer-rule*]: (*G* ===> (=)) *invar1 invar2*
and [*transfer-rule*]: (*G* ===> (=)) *inEdges1 inEdges2*
and [*transfer-rule*]: (*G* ===> (=)) *Entry1 Entry2*
and [*transfer-rule*]: (*G* ===> (=)) *defs1 defs2*
and [*transfer-rule*]: (*G* ===> (=)) *uses1 uses2*
shows *CFG-Construct* $\alpha e1$ $\alpha n1$ *invar1 inEdges1 Entry1 defs1 uses1*
⟶ *CFG-Construct* $\alpha e2$ $\alpha n2$ *invar2 inEdges2 Entry2 defs2 uses2*
⟨*proof*⟩

lemma *CFG-Construct-linorder-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *right-total G*
and [*transfer-rule*]: (*G* ===> (=)) $\alpha e1$ $\alpha e2$
and [*transfer-rule*]: (*G* ===> (=)) $\alpha n1$ $\alpha n2$
and [*transfer-rule*]: (*G* ===> (=)) *invar1 invar2*
and [*transfer-rule*]: (*G* ===> (=)) *inEdges1 inEdges2*
and [*transfer-rule*]: (*G* ===> (=)) *Entry1 Entry2*
and [*transfer-rule*]: (*G* ===> (=)) *defs1 defs2*
and [*transfer-rule*]: (*G* ===> (=)) *uses1 uses2*
shows *CFG-Construct-linorder* $\alpha e1$ $\alpha n1$ *invar1 inEdges1 Entry1 defs1 uses1*
⟶ *CFG-Construct-linorder* $\alpha e2$ $\alpha n2$ *invar2 inEdges2 Entry2 defs2 uses2*
⟨*proof*⟩

end

context *CFG-Construct* **begin**

context includes *lifting-syntax*
begin

lemma *phiDefNodes-aux-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *right-total G*
and [*transfer-rule*]: (*G* ===> (=)) *αe αe2*
and [*transfer-rule*]: (*G* ===> (=)) *αn αn2*
and [*transfer-rule*]: (*G* ===> (=)) *invar invar2*
and [*transfer-rule*]: (*G* ===> (=)) *inEdges' inEdges2*
and [*transfer-rule*]: (*G* ===> (=)) *Entry Entry2*
and [*transfer-rule*]: (*G* ===> (=)) *defs defs2*
and [*transfer-rule*]: (*G* ===> (=)) *uses uses2*
shows (*G* ===> (=)) *phiDefNodes-aux* (*CFG-Construct.phiDefNodes-aux inEdges2 defs2*)
⟨*proof*⟩

lemma *phiDefNodes-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *right-total G*
and [*transfer-rule*]: (*G* ===> (=)) *αe αe2*
and [*transfer-rule*]: (*G* ===> (=)) *αn αn2*
and [*transfer-rule*]: (*G* ===> (=)) *invar invar2*
and [*transfer-rule*]: (*G* ===> (=)) *inEdges' inEdges2*
and [*transfer-rule*]: (*G* ===> (=)) *Entry Entry2*
and [*transfer-rule*]: (*G* ===> (=)) *defs defs2*
and [*transfer-rule*]: (*G* ===> (=)) *uses uses2*
shows (*G* ===> (=)) *phiDefNodes* (*CFG-Construct.phiDefNodes αn2 inEdges2 defs2 uses2*)
⟨*proof*⟩

lemma *lookupDef-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *right-total G*
and [*transfer-rule*]: (*G* ===> (=)) *αe αe2*
and [*transfer-rule*]: (*G* ===> (=)) *αn αn2*
and [*transfer-rule*]: (*G* ===> (=)) *invar invar2*
and [*transfer-rule*]: (*G* ===> (=)) *inEdges' inEdges2*
and [*transfer-rule*]: (*G* ===> (=)) *Entry Entry2*
and [*transfer-rule*]: (*G* ===> (=)) *defs defs2*
and [*transfer-rule*]: (*G* ===> (=)) *uses uses2*
shows (*G* ===> (=)) *lookupDef* (*CFG-Construct.lookupDef αn2 inEdges2 defs2*)
⟨*proof*⟩

lemma *defs'-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *right-total G*
and [*transfer-rule*]: (*G* ===> (=)) *αe αe2*
and [*transfer-rule*]: (*G* ===> (=)) *αn αn2*
and [*transfer-rule*]: (*G* ===> (=)) *invar invar2*
and [*transfer-rule*]: (*G* ===> (=)) *inEdges' inEdges2*
and [*transfer-rule*]: (*G* ===> (=)) *Entry Entry2*

and [*transfer-rule*]: $(G \Longrightarrow (=))$ *defs defs2*
and [*transfer-rule*]: $(G \Longrightarrow (=))$ *uses uses2*
shows $(G \Longrightarrow (=))$ *defs'* (*CFG-Construct.defs' defs2*)
⟨*proof*⟩

lemma *uses'-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *right-total G*
and [*transfer-rule*]: $(G \Longrightarrow (=))$ αe $\alpha e2$
and [*transfer-rule*]: $(G \Longrightarrow (=))$ αn $\alpha n2$
and [*transfer-rule*]: $(G \Longrightarrow (=))$ *invar invar2*
and [*transfer-rule*]: $(G \Longrightarrow (=))$ *inEdges' inEdges2*
and [*transfer-rule*]: $(G \Longrightarrow (=))$ *Entry Entry2*
and [*transfer-rule*]: $(G \Longrightarrow (=))$ *defs defs2*
and [*transfer-rule*]: $(G \Longrightarrow (=))$ *uses uses2*
shows $(G \Longrightarrow (=))$ *uses'* (*CFG-Construct.uses'* $\alpha n2$ *inEdges2 defs2 uses2*)
⟨*proof*⟩

lemma *phis'-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *right-total G*
and [*transfer-rule*]: $(G \Longrightarrow (=))$ αe $\alpha e2$
and [*transfer-rule*]: $(G \Longrightarrow (=))$ αn $\alpha n2$
and [*transfer-rule*]: $(G \Longrightarrow (=))$ *invar invar2*
and [*transfer-rule*]: $(G \Longrightarrow (=))$ *inEdges' inEdges2*
and [*transfer-rule*]: $(G \Longrightarrow (=))$ *Entry Entry2*
and [*transfer-rule*]: $(G \Longrightarrow (=))$ *defs defs2*
and [*transfer-rule*]: $(G \Longrightarrow (=))$ *uses uses2*
shows $(G \Longrightarrow (=))$ *phis'* (*CFG-Construct.phis'* $\alpha n2$ *inEdges2 defs2 uses2*)
⟨*proof*⟩

lemmas *CFG-Construct-transfer-rules* =
phiDefNodes-aux-transfer
phiDefNodes-transfer
lookupDef-transfer
defs'-transfer
uses'-transfer
phis'-transfer
end

end

context *CFG-SSA-base* **begin**

context includes *lifting-syntax*
begin

lemma *phiDefs-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *right-total G*
and [*transfer-rule*]: $(G \Longrightarrow (=))$ αe $\alpha e2$
and [*transfer-rule*]: $(G \Longrightarrow (=))$ αn $\alpha n2$

and [*transfer-rule*]: (*G* ===> (=)) *invar invar2*
and [*transfer-rule*]: (*G* ===> (=)) *inEdges' inEdges2*
and [*transfer-rule*]: (*G* ===> (=)) *Entry Entry2*
and [*transfer-rule*]: (*G* ===> (=)) *defs defs2*
and [*transfer-rule*]: (*G* ===> (=)) *uses uses2*
and [*transfer-rule*]: (*G* ===> (=)) *phis phis2*
shows (*G* ===> (=)) *phiDefs* (*CFG-SSA-base.phiDefs phis2*)
⟨*proof*⟩

lemma *allDefs-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *right-total G*
and [*transfer-rule*]: (*G* ===> (=)) *αe αe2*
and [*transfer-rule*]: (*G* ===> (=)) *αn αn2*
and [*transfer-rule*]: (*G* ===> (=)) *invar invar2*
and [*transfer-rule*]: (*G* ===> (=)) *inEdges' inEdges2*
and [*transfer-rule*]: (*G* ===> (=)) *Entry Entry2*
and [*transfer-rule*]: (*G* ===> (=)) *defs* (*defs2*::*'a* ⇒ *'node* ⇒ *'val set*)
and [*transfer-rule*]: (*G* ===> (=)) *uses* (*uses2*::*'a* ⇒ *'node* ⇒ *'val set*)
and [*transfer-rule*]: (*G* ===> (=)) *phis phis2*
shows (*G* ===> (=)) *allDefs* (*CFG-SSA-base.allDefs defs2 phis2*)
⟨*proof*⟩

lemma *phiUses-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *right-total G*
and [*transfer-rule*]: (*G* ===> (=)) *αe αe2*
and [*transfer-rule*]: (*G* ===> (=)) *αn αn2*
and [*transfer-rule*]: (*G* ===> (=)) *invar invar2*
and [*transfer-rule*]: (*G* ===> (=)) *inEdges' inEdges2*
and [*transfer-rule*]: (*G* ===> (=)) *Entry Entry2*
and [*transfer-rule*]: (*G* ===> (=)) *defs defs2*
and [*transfer-rule*]: (*G* ===> (=)) *uses uses2*
and [*transfer-rule*]: (*G* ===> (=)) *phis phis2*
shows (*G* ===> (=)) *phiUses* (*CFG-SSA-base.phiUses αn2 inEdges2 phis2*)
⟨*proof*⟩

lemma *allUses-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *right-total G*
and [*transfer-rule*]: (*G* ===> (=)) *αe αe2*
and [*transfer-rule*]: (*G* ===> (=)) *αn αn2*
and [*transfer-rule*]: (*G* ===> (=)) *invar invar2*
and [*transfer-rule*]: (*G* ===> (=)) *inEdges' inEdges2*
and [*transfer-rule*]: (*G* ===> (=)) *Entry Entry2*
and [*transfer-rule*]: (*G* ===> (=)) *defs defs2*
and [*transfer-rule*]: (*G* ===> (=)) *uses uses2*
and [*transfer-rule*]: (*G* ===> (=)) *phis phis2*
shows (*G* ===> (=)) *allUses* (*CFG-SSA-base.allUses αn2 inEdges2 uses2 phis2*)
⟨*proof*⟩

lemma *allVars-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *right-total G*
and [*transfer-rule*]: (*G* ===> (=)) *αe αe2*
and [*transfer-rule*]: (*G* ===> (=)) *αn αn2*
and [*transfer-rule*]: (*G* ===> (=)) *invar invar2*
and [*transfer-rule*]: (*G* ===> (=)) *inEdges' inEdges2*
and [*transfer-rule*]: (*G* ===> (=)) *Entry Entry2*
and [*transfer-rule*]: (*G* ===> (=)) *defs defs2*
and [*transfer-rule*]: (*G* ===> (=)) *uses uses2*
and [*transfer-rule*]: (*G* ===> (=)) *phis phis2*
shows (*G* ===> (=)) *allVars* (*CFG-SSA-base.allVars αn2 inEdges2 defs2 uses2 phis2*)
⟨*proof*⟩

lemma *defAss-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *right-total G*
and [*transfer-rule*]: (*G* ===> (=)) *αe αe2*
and [*transfer-rule*]: (*G* ===> (=)) *αn αn2*
and [*transfer-rule*]: (*G* ===> (=)) *invar invar2*
and [*transfer-rule*]: (*G* ===> (=)) *inEdges' inEdges2*
and [*transfer-rule*]: (*G* ===> (=)) *Entry Entry2*
and [*transfer-rule*]: (*G* ===> (=)) *defs defs2*
and [*transfer-rule*]: (*G* ===> (=)) *uses uses2*
and [*transfer-rule*]: (*G* ===> (=)) *phis phis2*
shows (*G* ===> (=)) *defAss* (*CFG-SSA-base.defAss αn2 invar2 inEdges2 Entry2 defs2 phis2*)
⟨*proof*⟩

lemmas *CFG-SSA-base-transfer-rules* =
phiDefs-transfer
allDefs-transfer
phiUses-transfer
allUses-transfer
allVars-transfer
defAss-transfer
end

end

context *CFG-SSA-base-code* **begin**

lemma *CFG-SSA-base-code-transfer-rules* [*transfer-rule*]:
includes *lifting-syntax*
assumes [*transfer-rule*]: *right-total G*
and [*transfer-rule*]: (*G* ===> (=)) *αe αe2*
and [*transfer-rule*]: (*G* ===> (=)) *αn αn2*
and [*transfer-rule*]: (*G* ===> (=)) *invar invar2*
and [*transfer-rule*]: (*G* ===> (=)) *inEdges' inEdges2*
and [*transfer-rule*]: (*G* ===> (=)) *Entry Entry2*

and [*transfer-rule*]: (*G* ===> (=)) *defs defs2*
and [*transfer-rule*]: (*G* ===> (=)) *uses uses2*
and [*transfer-rule*]: (*G* ===> (=)) *phis phis2*
shows (*G* ===> (=)) *phiDefs* (*CFG-SSA-base.phiDefs* (λ*g*. *Mapping.lookup* (*phis2 g*)))
(*G* ===> (=)) *allDefs* (*CFG-SSA-base.allDefs defs2* (λ*g*. *Mapping.lookup* (*phis2 g*)))
(*G* ===> (=)) *phiUses* (*CFG-SSA-base.phiUses* α*n2 inEdges2* (λ*g*. *Mapping.lookup* (*phis2 g*)))
(*G* ===> (=)) *allUses* (*CFG-SSA-base.allUses* α*n2 inEdges2* (*usesOf* ∘ *uses2*) (λ*g*. *Mapping.lookup* (*phis2 g*)))
(*G* ===> (=)) *defAss* (*CFG-SSA-base.defAss* α*n2 invar2 inEdges2 Entry2 defs2* (λ*g*. *Mapping.lookup* (*phis2 g*)))
⟨*proof*⟩

end

lemma *CFG-SSA-transfer* [*transfer-rule*]:
includes *lifting-syntax*
assumes [*transfer-rule*]: *right-total G*
and [*transfer-rule*]: (*G* ===> (=)) α*e1* α*e2*
and [*transfer-rule*]: (*G* ===> (=)) α*n1* α*n2*
and [*transfer-rule*]: (*G* ===> (=)) *invar1 invar2*
and [*transfer-rule*]: (*G* ===> (=)) *inEdges1 inEdges2*
and [*transfer-rule*]: (*G* ===> (=)) *Entry1 Entry2*
and [*transfer-rule*]: (*G* ===> (=)) *defs1 defs2*
and [*transfer-rule*]: (*G* ===> (=)) *uses1 uses2*
and [*transfer-rule*]: (*G* ===> (=)) *phis1 phis2*
shows *CFG-SSA* α*e1* α*n1 invar1 inEdges1 Entry1 defs1 uses1 phis1*
⟶ *CFG-SSA* α*e2* α*n2 invar2 inEdges2 Entry2 defs2 uses2 phis2*
⟨*proof*⟩

end

6.4 Code Equations for SSA Minimization

theory *Construct-SSA-notriv-code* **imports**
SSA-CFG-code
Construct-SSA-notriv
While-Combinator-Exts
begin

abbreviation (*input*) *const x* ≡ (λ-. *x*)

context *CFG-SSA-Transformed-notriv-base* **begin**
definition [*code*]: *substitution-code g next* = *the* (*the-trivial* (*snd next*) (*the* (*phis g next*)))
definition [*code*]: *substNext-code g next* ≡ λ*v*. *if v* = *snd next then substitution-code g next else v*

definition [*code*]: *uses'-code g next n* ≡ *substNext-code g next ‘ uses g n*

lemma *substNext-code-alt-def*:
substNext-code g next = *id*(*snd next* := *substitution-code g next*)
⟨*proof*⟩
end

type-synonym (*'g*, *'node*, *'val*) *chooseNext-code* = (*'node* ⇒ *'val set*) ⇒ (*'node*, *'val*) *phis-code* ⇒ *'g* ⇒ (*'node* × *'val*)

locale *CFG-SSA-Transformed-notriv-base-code* =
ssa:*CFG-SSA-wf-base-code* α*e* α*n invar inEdges' Entry defs uses phis* +
CFG-SSA-Transformed-notriv-base α*e* α*n invar inEdges' Entry oldDefs oldUses defs usesOf* ∘ *uses* λ*g*. *Mapping.lookup* (*phis g*) *var* λ*uses phis*. *chooseNext-all uses* (*Mapping.Mapping phis*)
for
α*e* :: *'g* ⇒ (*'node*::*linorder* × *'edgeD* × *'node*) *set* **and**
α*n* :: *'g* ⇒ *'node list* **and**
invar :: *'g* ⇒ *bool* **and**
inEdges' :: *'g* ⇒ *'node* ⇒ (*'node* × *'edgeD*) *list* **and**
Entry::*'g* ⇒ *'node* **and**
oldDefs :: *'g* ⇒ *'node* ⇒ *'var*::*linorder set* **and**
oldUses :: *'g* ⇒ *'node* ⇒ *'var set* **and**
defs :: *'g* ⇒ *'node* ⇒ *'val*::*linorder set* **and**
uses :: *'g* ⇒ (*'node*, *'val set*) *mapping* **and**
phis :: *'g* ⇒ (*'node*, *'val*) *phis-code* **and**
var :: *'g* ⇒ *'val* ⇒ *'var* **and**
chooseNext-all :: (*'g*, *'node*, *'val*) *chooseNext-code*
begin
definition [*code*]: *cond-code g* = *ssa.redundant-code g*

definition *uses'-codem* :: *'g* ⇒ *'node* × *'val* ⇒ *'val* ⇒ (*'val*, *'node set*) *mapping* ⇒ (*'node*, *'val set*) *mapping*
where [*code*]: *uses'-codem g next next' nodes-of-uses* =
fold (λ*n*. *Mapping.update n* (*Set.insert next'* (*Set.remove* (*snd next*) (*the* (*Mapping.lookup* (*uses g*) *n*)))))
(*sorted-list-of-set* (*case-option* {} *id* (*Mapping.lookup nodes-of-uses* (*snd next*))))
(*uses g*)

definition *nodes-of-uses'* :: *'g* ⇒ *'node* × *'val* ⇒ *'val* ⇒ *'val set* ⇒ (*'val*, *'node set*) *mapping* ⇒ (*'val*, *'node set*) *mapping*
where [*code*]: *nodes-of-uses' g next next' phiVals nodes-of-uses* =
(*let users* = *case-option* {} *id* (*Mapping.lookup nodes-of-uses* (*snd next*))
in
if (*next'* ∈ *phiVals*) *then Mapping.map-default next'* {} (λ*ns*. *ns* ∪ *users*) (*Mapping.delete* (*snd next*) *nodes-of-uses*)
else Mapping.delete (*snd next*) *nodes-of-uses*)

definition [*code*]: *phis'-code g next* ≡ *map-values* (λ(*n*,*v*) *vs*. *if v* = *snd next then None else Some* (*map* (*substNext-code g next*) *vs*)) (*phis g*)

definition [*code*]: *phis'-codem g next next' nodes-of-phis* =
fold (λ*n*. *Mapping.update n* (*List.map* (*id*(*snd next* := *next'*)) (*the* (*Mapping.lookup* (*phis g*) *n*)))) (*sorted-list-of-set* (*case-option* {} (*Set.remove next*) (*Mapping.lookup nodes-of-phis* (*snd next*)))) (*Mapping.delete next* (*phis g*))

definition *nodes-of-phis'* :: *'g* ⇒ *'node* × *'val* ⇒ *'val* ⇒ (*'val*, (*'node* × *'val*) *set*) *mapping* ⇒ (*'val*, (*'node* × *'val*) *set*) *mapping*
where [*code*]: *nodes-of-phis' g next next' nodes-of-phis* =
(*let old-phis* = *Set.remove next* (*case-option* {} *id* (*Mapping.lookup nodes-of-phis* (*snd next*)));
nop = *Mapping.delete* (*snd next*) *nodes-of-phis*
in
Mapping.map-default next' {} (λ*ns*. (*Set.remove next ns*) ∪ *old-phis*) *nop*)

definition [*code*]: *triv-phis' g next triv-phis nodes-of-phis*
= (*Set.remove next triv-phis*) ∪ (*Set.filter* (λ*n*. *ssa.trivial-code* (*snd n*) (*the* (*Mapping.lookup* (*phis g*) *n*))) (*case-option* {} (*Set.remove next*) (*Mapping.lookup nodes-of-phis* (*snd next*))))

definition [*code*]: *step-code g* = (*let next* = *chooseNext' g in* (*uses'-code g next*, *phis'-code g next*))
definition [*code*]: *step-codem g next next' nodes-of-uses nodes-of-phis* = (*uses'-codem g next next' nodes-of-uses*, *phis'-codem g next next' nodes-of-phis*)

definition *phi-equiv-mapping* :: *'g* ⇒ (*'val*, *'a set*) *mapping* ⇒ (*'val*, *'a set*) *mapping* ⇒ *bool* (- ⊢ - ≈$_\varphi$ - 50)
where *g* ⊢ *nou*$_1$ ≈$_\varphi$ *nou*$_2$ ≡ ∀ *v* ∈ *Mapping.keys* (*ssa.phidefNodes g*). *case-option* {} *id* (*Mapping.lookup nou*$_1$ *v*) = *case-option* {} *id* (*Mapping.lookup nou*$_2$ *v*)
end

locale *CFG-SSA-Transformed-notriv-linorder* = *CFG-SSA-Transformed-notriv-base αe αn invar inEdges' Entry oldDefs oldUses defs uses phis var chooseNext-all*
+ *CFG-SSA-Transformed-notriv αe αn invar inEdges' Entry oldDefs oldUses defs uses phis var chooseNext-all*
for
αe :: *'g* ⇒ (*'node*::*linorder* × *'edgeD* × *'node*) *set* **and**
αn :: *'g* ⇒ *'node list* **and**
invar :: *'g* ⇒ *bool* **and**
inEdges' :: *'g* ⇒ *'node* ⇒ (*'node* × *'edgeD*) *list* **and**
Entry::*'g* ⇒ *'node* **and**
oldDefs :: *'g* ⇒ *'node* ⇒ *'var*::*linorder set* **and**
oldUses :: *'g* ⇒ *'node* ⇒ *'var set* **and**

defs :: *'g* ⇒ *'node* ⇒ *'val*::*linorder set* **and**
uses :: *'g* ⇒ *'node* ⇒ *'val set* **and**
phis :: *'g* ⇒ (*'node*, *'val*) *phis* **and**
var :: *'g* ⇒ *'val* ⇒ *'var* **and**
chooseNext-all :: (*'node* ⇒ *'val set*) ⇒ (*'node*, *'val*) *phis* ⇒ *'g* ⇒ (*'node* × *'val*)
begin
lemma *isTrivial-the-trivial*: ⟦ *phi g v* = *Some vs*; *isTrivialPhi g v v'* ⟧ ⟹ *the-trivial v vs* = *Some v'*
⟨*proof*⟩

lemma *the-trivial-THE-isTrivial*: ⟦ *phi g v* = *Some vs*; *trivial g v* ⟧ ⟹ *the-trivial v vs* = *Some* (*The* (*isTrivialPhi g v*))
⟨*proof*⟩

lemma *substitution-code-correct*:
assumes *redundant g*
shows *substitution g* = *substitution-code g* (*chooseNext' g*)
⟨*proof*⟩

lemma *substNext-code-correct*:
assumes *redundant g*
shows *substNext g* = *substNext-code g* (*chooseNext' g*)
⟨*proof*⟩

lemma *uses'-code-correct*:
assumes *redundant g*
shows *uses' g* = *uses'-code g* (*chooseNext' g*)
⟨*proof*⟩

end

context *CFG-SSA-Transformed-notriv-linorder*
begin
lemma *substAll-terminates*: *while-option* (*cond g*) (*step g*) (*uses g*, *phis g*) ≠ *None*
⟨*proof*⟩
end

locale *CFG-SSA-Transformed-notriv-linorder-code* =
CFG-SSA-Transformed-code αe αn invar inEdges' Entry oldDefs oldUses defs uses phis var
+ *CFG-SSA-Transformed-notriv-base-code αe αn invar inEdges' Entry oldDefs oldUses defs uses phis var chooseNext-all*
+ *CFG-SSA-Transformed-notriv-linorder αe αn invar inEdges' Entry oldDefs oldUses defs usesOf* ∘ *uses* λ*g*. *Mapping.lookup* (*phis g*) *var*
λ*uses phis*. *chooseNext-all uses* (*Mapping.Mapping phis*)
for
αe :: *'g* ⇒ (*'node*::*linorder* × *'edgeD* × *'node*) *set* **and**
αn :: *'g* ⇒ *'node list* **and**

invar :: *'g* ⇒ *bool* **and**
inEdges' :: *'g* ⇒ *'node* ⇒ (*'node* × *'edgeD*) *list* **and**
Entry::*'g* ⇒ *'node* **and**
oldDefs :: *'g* ⇒ *'node* ⇒ *'var*::*linorder set* **and**
oldUses :: *'g* ⇒ *'node* ⇒ *'var set* **and**
defs :: *'g* ⇒ *'node* ⇒ *'val*::*linorder set* **and**
uses :: *'g* ⇒ (*'node*, *'val set*) *mapping* **and**
phis :: *'g* ⇒ (*'node*, *'val*) *phis-code* **and**
var :: *'g* ⇒ *'val* ⇒ *'var* **and**
chooseNext-all :: (*'g*, *'node*, *'val*) *chooseNext-code*
+
assumes *chooseNext-all-code*:
CFG-SSA-Transformed-code αe αn invar inEdges' Entry oldDefs oldUses defs u p var ⟹
CFG-SSA-wf-base-code.redundant-code p g ⟹
chooseNext-all (*usesOf* (*u g*)) (*p g*) *g* = *Max* (*CFG-SSA-wf-base-code.trivial-phis p g*)

locale *CFG-SSA-step-code* =
step-code: *CFG-SSA-Transformed-notriv-linorder-code αe αn invar inEdges' Entry oldDefs oldUses defs uses phis var chooseNext-all*
+
CFG-SSA-step αe αn invar inEdges' Entry oldDefs oldUses defs usesOf ∘ *uses λg. Mapping.lookup* (*phis g*) *var λuses phis. chooseNext-all uses* (*Mapping.Mapping phis*) *g*
for
αe :: *'g* ⇒ (*'node*::*linorder* × *'edgeD* × *'node*) *set* **and**
αn :: *'g* ⇒ *'node list* **and**
invar :: *'g* ⇒ *bool* **and**
inEdges' :: *'g* ⇒ *'node* ⇒ (*'node* × *'edgeD*) *list* **and**
Entry::*'g* ⇒ *'node* **and**
oldDefs :: *'g* ⇒ *'node* ⇒ *'var*::*linorder set* **and**
oldUses :: *'g* ⇒ *'node* ⇒ *'var set* **and**
defs :: *'g* ⇒ *'node* ⇒ *'val*::*linorder set* **and**
uses :: *'g* ⇒ (*'node*, *'val set*) *mapping* **and**
phis :: *'g* ⇒ (*'node*, *'val*) *phis-code* **and**
var :: *'g* ⇒ *'val* ⇒ *'var* **and**
chooseNext-all :: (*'g*, *'node*, *'val*) *chooseNext-code* **and**
g :: *'g*

context *CFG-SSA-Transformed-notriv-linorder-code*
begin
abbreviation *u-g g u* ≡ *uses*(*g*:=*u*)
abbreviation *p-g g p* ≡ *phis*(*g*:=*p*)
abbreviation *cN* ≡ (*λuses phis. chooseNext-all uses* (*Mapping.Mapping phis*))

interpretation *uninst-code*: *CFG-SSA-Transformed-notriv-base-code αe αn invar inEdges' Entry oldDefs oldUses defs u p var chooseNext-all*
for *u p*

⟨*proof*⟩

interpretation *uninst*: *CFG-SSA-Transformed-notriv-base* αe αn *invar inEdges' Entry oldDefs oldUses defs u p var cN*
for *u p*
⟨*proof*⟩

lemma *phis'-code-correct*:
assumes *ssa.redundant g*
shows *phis' g* = *Mapping.lookup* (*phis'-code g* (*chooseNext' g*))
⟨*proof*⟩

lemma *redundant-ign*[*simp*]: *uninst-code.ssa.redundant-code* (*const p*) *g* = *uninst-code.ssa.redundant-code* (*phis*(*g*:=*p*)) *g*
⟨*proof*⟩

lemma *uses'-ign*[*simp*]: *uninst-code.uses'-codem* (*const u*) *g* = *uninst-code.uses'-codem* (*u-g g u*) *g*
⟨*proof*⟩

lemma *phis'-ign*[*simp*]: *uninst-code.phis'-code* (*const p*) *g* = *uninst-code.phis'-code* (*phis*(*g*:=*p*)) *g*
⟨*proof*⟩

lemma *phis'm-ign*[*simp*]: *uninst-code.phis'-codem* (*const p*) *g* = *uninst-code.phis'-codem* (*phis*(*g*:=*p*)) *g*
⟨*proof*⟩

lemma *set-sorted-list-of-set-phis-dom* [*simp*]:
set (*sorted-list-of-set* {*x* ∈ *dom* (*Mapping.lookup* (*phis g*)). *P x*}) = {*x* ∈ *dom* (*Mapping.lookup* (*phis g*)). *P x*}
⟨*proof*⟩

lemma *phis'-codem-correct*:
assumes *g* ⊢ *nodes-of-phis* $\approx_{\varphi}$ (*ssa.phiNodes-of g*) **and** *next* ∈ *Mapping.keys* (*phis g*)
shows *phis'-codem g next* (*substitution-code g next*) *nodes-of-phis* = *phis'-code g next*
⟨*proof*⟩

lemma *uses-transfer* [*transfer-rule*]: (*rel-fun* (=) (*pcr-mapping* (=) (=))) (λ*g n*. *Mapping.lookup* (*uses g*) *n*) *uses*
⟨*proof*⟩

lemma *uses'-codem-correct*:
assumes *g* ⊢ *nodes-of-uses* $\approx_{\varphi}$ *ssa.useNodes-of g* **and** *next* ∈ *Mapping.keys* (*phis g*)
shows *usesOf* (*uses'-codem g next* (*substitution-code g next*) *nodes-of-uses*) = *uses'-code g next*

⟨*proof*⟩

lemma *step-ign*[*simp*]: *uninst-code.step-codem* (*const u*) (*const p*) *g* = *uninst-code.step-codem* (*u-g g u*) (*phis*(*g*:=*p*)) *g*
⟨*proof*⟩

lemma *cN-transfer* [*transfer-rule*]: (*rel-fun* (=) (*rel-fun* (*pcr-mapping* (=) (=)) (=))) *cN chooseNext-all*
⟨*proof*⟩

lemma *usesOf-transfer* [*transfer-rule*]: (*rel-fun* (*pcr-mapping* (=) (=)) (=)) (λ*m x*. *case-option* {} *id* (*m x*)) *usesOf*
⟨*proof*⟩

lemma *dom-phis'-codem*:
assumes ⋀*ns*. *Mapping.lookup nodes-of-phis* (*snd next*) = *Some ns* ⟹ *finite ns*
shows *dom* (*Mapping.lookup* (*phis'-codem g next next' nodes-of-phis*)) = *dom* (*Mapping.lookup* (*phis g*)) ∪ (*case-option* {} *id* (*Mapping.lookup nodes-of-phis* (*snd next*))) − {*next*}
⟨*proof*⟩

lemma *dom-phis'-code* [*simp*]:
shows *dom* (*Mapping.lookup* (*phis'-code g next*)) = *dom* (*Mapping.lookup* (*phis g*)) − {*v*. *snd v* = *snd next*}
⟨*proof*⟩

lemma *nodes-of-phis-finite* [*simplified*]:
assumes *g* ⊢ *nodes-of-phis* ≈$_\varphi$ *ssa.phiNodes-of g* **and** *Mapping.lookup nodes-of-phis v* = *Some ns* **and** *v* ∈ *Mapping.keys* (*ssa.phidefNodes g*)
shows *finite ns*
⟨*proof*⟩

lemma *lookup-phis'-codem-next*:
assumes ⋀*ns*. *Mapping.lookup nodes-of-phis* (*snd next*) = *Some ns* ⟹ *finite ns*
shows *Mapping.lookup* (*phis'-codem g next next' nodes-of-phis*) *next* = *None*
⟨*proof*⟩

lemma *lookup-phis'-codem-other*:
assumes *g* ⊢ *nodes-of-phis* ≈$_\varphi$ (*ssa.phiNodes-of g*)
and *next* ∈ *Mapping.keys* (*phis g*) **and** *next* ≠ *φ*
shows *Mapping.lookup* (*phis'-codem g next* (*substitution-code g next*) *nodes-of-phis*) *φ* =
map-option (*map* (*substNext-code g next*)) (*Mapping.lookup* (*phis g*) *φ*)
⟨*proof*⟩

lemma *lookup-nodes-of-phis'-subst* [*simp*]:
Mapping.lookup (*nodes-of-phis' g next* (*substitution-code g next*) *nodes-of-phis*) (*substitution-code g next*) =
Some ((*case-option* {} (*Set.remove next*) (*Mapping.lookup nodes-of-phis* (*substitution-code*

g next))) ∪ (*case-option* {} (*Set.remove next*) (*Mapping.lookup nodes-of-phis* (*snd next*))))
⟨*proof*⟩

lemma *lookup-nodes-of-phis'-not-subst*:
v ≠ *substitution-code g next* ⟹
Mapping.lookup (*nodes-of-phis' g next* (*substitution-code g next*) *nodes-of-phis*) *v* = (*if v* = *snd next then None else Mapping.lookup nodes-of-phis v*)
⟨*proof*⟩

lemma *lookup-phis'-code*:
Mapping.lookup (*phis'-code g next*) *v* = (*if snd v* = *snd next then None else map-option* (*map* (*substNext-code g next*)) (*Mapping.lookup* (*phis g*) *v*))
⟨*proof*⟩

lemma *phi-equiv-mappingE'*:
assumes *g* ⊢ *m*$_1$ ≈$_\varphi$ *ssa.phiNodes-of g*
and *Mapping.lookup* (*phis g*) *x* = *Some vs* **and** *b* ∈ *set vs* **and** *b* ∈ *snd* ‘ *Mapping.keys* (*phis g*)
obtains *Mapping.lookup m*$_1$ *b* = *Some* {*n* ∈ *Mapping.keys* (*phis g*). *b* ∈ *set* (*the* (*Mapping.lookup* (*phis g*) *n*))}
⟨*proof*⟩

lemma *phi-equiv-mappingE*:
assumes *g* ⊢ *m*$_1$ ≈$_\varphi$ *ssa.phiNodes-of g* **and** *b* ∈ *Mapping.keys* (*phis g*)
and *Mapping.lookup* (*phis g*) *x* = *Some vs* **and** *snd b* ∈ *set vs*
obtains *ns* **where** *Mapping.lookup m*$_1$ (*snd b*) = *Some* {*n* ∈ *Mapping.keys* (*phis g*). *snd b* ∈ *set* (*the* (*Mapping.lookup* (*phis g*) *n*))}
⟨*proof*⟩

lemma *phi-equiv-mappingE2'*:
assumes *g* ⊢ *m*$_1$ ≈$_\varphi$ *ssa.phiNodes-of g*
and *b* ∈ *snd* ‘ *Mapping.keys* (*phis g*)
and ∀ *φ* ∈ *Mapping.keys* (*phis g*). *b* ∉ *set* (*the* (*Mapping.lookup* (*phis g*) *φ*))
shows *Mapping.lookup m*$_1$ *b* = *None* ∨ *Mapping.lookup m*$_1$ *b* = *Some* {}
⟨*proof*⟩

lemma *keys-phis'-codem* [*simp*]: *Mapping.keys* (*phis'-codem g next next'* (*ssa.phiNodes-of g*)) = *Mapping.keys* (*phis g*) − {*next*}
⟨*proof*⟩

lemma *keys-phis'-codem'*:
assumes *g* ⊢ *nodes-of-phis* ≈$_\varphi$ *ssa.phiNodes-of g* **and** *next* ∈ *Mapping.keys* (*phis g*)
shows *Mapping.keys* (*phis'-codem g next next' nodes-of-phis*) = *Mapping.keys* (*phis g*) − {*next*}
⟨*proof*⟩

lemma *triv-phis'-correct*:

assumes $g \vdash$ *nodes-of-phis* $\approx_\varphi$ *ssa.phiNodes-of g* **and** *next* $\in$ *Mapping.keys* (*phis g*) **and** *ssa.trivial g* (*snd next*)
shows *uninst-code.triv-phis'* (*const* (*phis'-codem g next* (*substitution-code g next*) *nodes-of-phis*)) *g next* (*ssa.trivial-phis g*) *nodes-of-phis* = *uninst-code.ssa.trivial-phis* (*const* (*phis'-codem g next* (*substitution-code g next*) *nodes-of-phis*)) *g*
⟨*proof*⟩

lemma *nodes-of-phis'-correct*:
assumes $g \vdash$ *nodes-of-phis* $\approx_\varphi$ *ssa.phiNodes-of g*
and *next* $\in$ *Mapping.keys* (*phis g*) **and** *ssa.trivial g* (*snd next*)
shows $g \vdash$ (*nodes-of-phis' g next* (*substitution-code g next*) *nodes-of-phis*) $\approx_\varphi$ (*uninst-code.ssa.phiNodes-of* (*const* (*phis'-codem g next* (*substitution-code g next*) *nodes-of-phis*)) *g*)
⟨*proof*⟩

lemma *nodes-of-uses'-correct*:
assumes $g \vdash$ *nodes-of-uses* $\approx_\varphi$ *ssa.useNodes-of g*
and *next* $\in$ *Mapping.keys* (*phis g*) **and** *ssa.trivial g* (*snd next*)
shows $g \vdash$ (*nodes-of-uses' g next* (*substitution-code g next*) (*Mapping.keys* (*ssa.phidefNodes g*)) *nodes-of-uses*) $\approx_\varphi$ (*uninst-code.ssa.useNodes-of* (*const* (*uses'-codem g next* (*substitution-code g next*) *nodes-of-uses*)) *g*)
⟨*proof*⟩

definition[*code*]: *substAll-efficient g* ≡
let phiVals = *Mapping.keys* (*ssa.phidefNodes g*);
u = *uses g*;
p = *phis g*;
tp = *ssa.trivial-phis g*;
nou = *ssa.useNodes-of g*;
nop = *ssa.phiNodes-of g*
in
while
(λ((*u*,*p*),*triv-phis*,*nodes-of-uses*,*nodes-of-phis*). ¬ *Set.is-empty triv-phis*)
(λ((*u*,*p*),*triv-phis*,*nodes-of-uses*,*nodes-of-phis*). *let*
next = *Max triv-phis*;
next' = *uninst-code.substitution-code* (*const p*) *g next*;
(*u'*,*p'*) = *uninst-code.step-codem* (*const u*) (*const p*) *g next next' nodes-of-uses nodes-of-phis*;
tp' = *uninst-code.triv-phis'* (*const p'*) *g next triv-phis nodes-of-phis*;
nou' = *uninst-code.nodes-of-uses' g next next' phiVals nodes-of-uses*;
nop' = *uninst-code.nodes-of-phis' g next next' nodes-of-phis*
in ((*u'*, *p'*), *tp'*, *nou'*, *nop'*))
((*u*, *p*), *tp*, *nou*, *nop*)

abbreviation *u-c x* ≡ *const* (*usesOf* (*fst x*))
abbreviation *p-c x* ≡ *const* (*Mapping.lookup* (*snd x*))
abbreviation *u g x* ≡ *u-g g* (*fst x*)
abbreviation *p g x* ≡ *p-g g* (*snd x*)

lemma *usesOf-upd* [*simp*]: (*usesOf* ∘ *u g s1*)(*g* := *usesOf us*) = *usesOf* ∘ *u-g g us*
⟨*proof*⟩

lemma *keys-uses′-codem* [*simp*]: *Mapping.keys* (*uses′-codem g next* (*substitution-code g next*) (*ssa.useNodes-of g*)) = *Mapping.keys* (*uses g*)
⟨*proof*⟩

lemma *keys-uses′-codem′*: ⟦ *g* ⊢ *nodes-of-uses* ≈$_{\varphi}$ *ssa.useNodes-of g*; *next* ∈ *Mapping.keys* (*phis g*) ⟧
⟹ *Mapping.keys* (*uses′-codem g next* (*substitution-code g next*) *nodes-of-uses*) = *Mapping.keys* (*uses g*)
⟨*proof*⟩

lemma *triv-phis-base* [*simp*]: *uninst-code.ssa.trivial-phis* (*const* (*phis g*)) *g* = *ssa.trivial-phis g*
⟨*proof*⟩
lemma *useNodes-of-base* [*simp*]: *uninst-code.ssa.useNodes-of* (*const* (*uses g*)) *g* = *ssa.useNodes-of g*
⟨*proof*⟩

lemma *phiNodes-of-base* [*simp*]: *uninst-code.ssa.phiNodes-of* (*const* (*phis g*)) *g* = *ssa.phiNodes-of g*
⟨*proof*⟩

lemma *phi-equiv-mapping-refl* [*simp*]: *uninst-code.phi-equiv-mapping ph g m m*
⟨*proof*⟩

lemma *substAll-efficient-code* [*code*]:
substAll g = *map-prod usesOf Mapping.lookup* (*fst* (*substAll-efficient g*))
⟨*proof*⟩

end

end

6.5 Generic Code Extraction Based on typedefs

theory *Generic-Interpretation*
imports
Construct-SSA-code
Construct-SSA-notriv-code
RBT-Mapping-Exts
SSA-Transfer-Rules
HOL−Library.RBT-Set
HOL−Library.Code-Target-Numeral
begin

record (*′node*, *′var*, *′edge*) *gen-cfg* =

gen-αe :: (*'node*, *'edge*) *edge set*
gen-αn :: *'node list*
gen-inEdges :: *'node* ⇒ (*'node*, *'edge*) *edge list*
gen-Entry :: *'node*
gen-defs :: *'node* ⇒ *'var set*
gen-uses :: *'node* ⇒ *'var set*

abbreviation *trivial-gen-cfg ext* ≡ *gen-cfg-ext* {} [*undefined*] (*const* []) *undefined* (*const* {}) (*const* {}) *ext*
abbreviation (*input*) *ign f g* (-::*unit*) ≡ *f g*

lemma *set-iterator-foldri-Nil* [*simp*, *intro*!]: *set-iterator* (*foldri* []) {}
⟨*proof*⟩

lemma *set-iterator-foldri-one* [*simp*, *intro*!]: *set-iterator* (*foldri* [*a*]) {*a*}
⟨*proof*⟩

abbreviation *gen-inEdges' g n* ≡ *map* (λ(*f*,*d*,*t*). (*f*,*d*)) (*gen-inEdges g n*)

lemma *gen-cfg-inhabited*: *let g* = *trivial-gen-cfg ext in CFG-wf* (*ign gen-αe g*) (*ign gen-αn g*) (*const True*) (*ign gen-inEdges' g*) (*ign gen-Entry g*) (*ign gen-defs g*) (*ign gen-uses g*)
⟨*proof*⟩

typedef (*'node*, *'var*, *'edge*) *gen-cfg-wf* = {*g* :: (*'node*::*linorder*, *'var*::*linorder*, *'edge*) *gen-cfg*.
CFG-wf (*ign gen-αe g*) (*ign gen-αn g*) (*const True*) (*ign gen-inEdges' g*) (*ign gen-Entry g*) (*ign gen-defs g*) (*ign gen-uses g*)}
⟨*proof*⟩

setup-lifting *type-definition-gen-cfg-wf*

lift-definition *gen-wf-αn* :: (*'node*::*linorder*, *'var*::*linorder*, *'edge*) *gen-cfg-wf* ⇒ *'node list* **is** *gen-αn* ⟨*proof*⟩
lift-definition *gen-wf-αe* :: (*'node*::*linorder*, *'var*::*linorder*, *'edge*) *gen-cfg-wf* ⇒ (*'node*, *'edge*) *edge set* **is** *gen-αe* ⟨*proof*⟩
lift-definition *gen-wf-inEdges* :: (*'node*::*linorder*, *'var*::*linorder*, *'edge*) *gen-cfg-wf* ⇒ *'node* ⇒ (*'node*, *'edge*) *edge list* **is** *gen-inEdges* ⟨*proof*⟩
lift-definition *gen-wf-Entry* :: (*'node*::*linorder*, *'var*::*linorder*, *'edge*) *gen-cfg-wf* ⇒ *'node* **is** *gen-Entry* ⟨*proof*⟩
lift-definition *gen-wf-defs* :: (*'node*::*linorder*, *'var*::*linorder*, *'edge*) *gen-cfg-wf* ⇒ *'node* ⇒ *'var set* **is** *gen-defs* ⟨*proof*⟩
lift-definition *gen-wf-uses* :: (*'node*::*linorder*, *'var*::*linorder*, *'edge*) *gen-cfg-wf* ⇒ *'node* ⇒ *'var set* **is** *gen-uses* ⟨*proof*⟩

abbreviation *gen-wf-invar* ≡ *const True*
abbreviation *gen-wf-inEdges' g n* ≡ *map* (λ(*f*,*d*,*t*). (*f*,*d*)) (*gen-wf-inEdges g n*)

lemma *gen-wf-inEdges'-transfer* [*transfer-rule*]: *rel-fun cr-gen-cfg-wf* (=) *gen-inEdges'*

gen-wf-inEdges'
⟨*proof*⟩

lemma *gen-wf-invar-trans*: *rel-fun cr-gen-cfg-wf* (=) *gen-wf-invar gen-wf-invar*
⟨*proof*⟩

declare *graph-path-base.transfer-rules*[*OF gen-cfg-wf.right-total gen-wf-αe.transfer gen-wf-αn.transfer gen-wf-invar-trans gen-wf-inEdges'-transfer*, *transfer-rule*]
declare *CFG-base.defAss'-transfer*[*OF gen-cfg-wf.right-total gen-wf-αe.transfer gen-wf-αn.transfer gen-wf-invar-trans gen-wf-inEdges'-transfer*, *transfer-rule*]

global-interpretation *gen-wf*: *CFG-Construct-linorder gen-wf-αe gen-wf-αn gen-wf-invar gen-wf-inEdges' gen-wf-Entry gen-wf-defs gen-wf-uses*
defines
gen-wf-predecessors = *gen-wf.predecessors* **and**
gen-wf-successors = *gen-wf.successors* **and**
gen-wf-defs' = *gen-wf.defs'* **and**
gen-wf-vars = *gen-wf.vars* **and**
gen-wf-var = *gen-wf.var* **and**
gen-wf-readVariableRecursive = *gen-wf.readVariableRecursive* **and**
gen-wf-readArgs = *gen-wf.readArgs* **and**
gen-wf-uses'-phis' = *gen-wf.uses'-phis'*
⟨*proof*⟩

record ('*node*, '*var*, '*edge*, '*val*) *gen-ssa-cfg* = ('*node*, '*var*, '*edge*) *gen-cfg* +
gen-ssa-defs :: '*node* ⇒ '*val set*
gen-ssa-uses :: ('*node*, '*val set*) *mapping*
gen-phis :: ('*node*, '*val*) *phis-code*
gen-var :: '*val* ⇒ '*var*

typedef ('*node*, '*var*, '*edge*, '*val*) *gen-ssa-cfg-wf* = {*g* :: ('*node*::*linorder*, '*var*::*linorder*, '*edge*, '*val*::*linorder*) *gen-ssa-cfg*.
CFG-SSA-Transformed-code (*ign gen-αe g*) (*ign gen-αn g*) (*const True*) (*ign gen-inEdges' g*) (*ign gen-Entry g*) (*ign gen-defs g*) (*ign gen-uses g*) (*ign gen-ssa-defs g*) (*ign gen-ssa-uses g*) (*ign gen-phis g*) (*ign gen-var g*)}
⟨*proof*⟩

setup-lifting *type-definition-gen-ssa-cfg-wf*

lift-definition *gen-ssa-wf-αn* :: ('*node*::*linorder*, '*var*::*linorder*, '*edge*, '*val*::*linorder*) *gen-ssa-cfg-wf* ⇒ '*node list* **is** *gen-αn* ⟨*proof*⟩
lift-definition *gen-ssa-wf-αe* :: ('*node*::*linorder*, '*var*::*linorder*, '*edge*, '*val*::*linorder*) *gen-ssa-cfg-wf* ⇒ ('*node*, '*edge*) *edge set* **is** *gen-αe* ⟨*proof*⟩
lift-definition *gen-ssa-wf-inEdges* :: ('*node*::*linorder*, '*var*::*linorder*, '*edge*, '*val*::*linorder*) *gen-ssa-cfg-wf* ⇒ '*node* ⇒ ('*node*, '*edge*) *edge list* **is** *gen-inEdges* ⟨*proof*⟩
lift-definition *gen-ssa-wf-Entry* :: ('*node*::*linorder*, '*var*::*linorder*, '*edge*, '*val*::*linorder*) *gen-ssa-cfg-wf* ⇒ '*node* **is** *gen-Entry* ⟨*proof*⟩
lift-definition *gen-ssa-wf-defs* :: ('*node*::*linorder*, '*var*::*linorder*, '*edge*, '*val*::*linorder*)

gen-ssa-cfg-wf ⇒ *'node* ⇒ *'var set* **is** *gen-defs* ⟨*proof*⟩
lift-definition *gen-ssa-wf-uses* :: (*'node*::*linorder*, *'var*::*linorder*, *'edge*, *'val*::*linorder*)
gen-ssa-cfg-wf ⇒ *'node* ⇒ *'var set* **is** *gen-uses* ⟨*proof*⟩
lift-definition *gen-ssa-wf-ssa-defs* :: (*'node*::*linorder*, *'var*::*linorder*, *'edge*, *'val*::*linorder*)
gen-ssa-cfg-wf ⇒ *'node* ⇒ *'val set* **is** *gen-ssa-defs* ⟨*proof*⟩
lift-definition *gen-ssa-wf-ssa-uses* :: (*'node*::*linorder*, *'var*::*linorder*, *'edge*, *'val*::*linorder*)
gen-ssa-cfg-wf ⇒ (*'node*, *'val set*) *mapping* **is** *gen-ssa-uses* ⟨*proof*⟩
lift-definition *gen-ssa-wf-phis* :: (*'node*::*linorder*, *'var*::*linorder*, *'edge*, *'val*::*linorder*)
gen-ssa-cfg-wf ⇒ (*'node*, *'val*) *phis-code* **is** *gen-phis* ⟨*proof*⟩
lift-definition *gen-ssa-wf-var* :: (*'node*::*linorder*, *'var*::*linorder*, *'edge*, *'val*::*linorder*)
gen-ssa-cfg-wf ⇒ *'val* ⇒ *'var* **is** *gen-var* ⟨*proof*⟩

abbreviation *gen-ssa-wf-inEdges' g n* ≡ *map* (λ(*f*,*d*,*t*). (*f*,*d*)) (*gen-ssa-wf-inEdges g n*)

lemma *gen-ssa-wf-inEdges'-transfer* [*transfer-rule*]: *rel-fun cr-gen-ssa-cfg-wf* (=) *gen-inEdges' gen-ssa-wf-inEdges'*
⟨*proof*⟩

global-interpretation *uninst*: *CFG-SSA-wf-base-code gen-ssa-wf-*α*e gen-ssa-wf-*α*n gen-wf-invar gen-ssa-wf-inEdges' gen-ssa-wf-Entry gen-ssa-wf-ssa-defs u p*
for *u* **and** *p*
defines
uninst-predecessors = *uninst.predecessors*
and *uninst-successors* = *uninst.successors*
and *uninst-phiDefs* = *uninst.phiDefs*
and *uninst-phiUses* = *uninst.phiUses*
and *uninst-allDefs* = *uninst.allDefs*
and *uninst-allUses* = *uninst.allUses*
and *uninst-allVars* = *uninst.allVars*
and *uninst-isTrivialPhi* = *uninst.isTrivialPhi*
and *uninst-trivial* = *uninst.trivial-code*
and *uninst-redundant* = *uninst.redundant-code*
and *uninst-phi* = *uninst.phi*
and *uninst-defNode* = *uninst.defNode*
and *uninst-trivial-phis* = *uninst.trivial-phis*
and *uninst-phidefNodes* = *uninst.phidefNodes*
and *uninst-useNodes-of* = *uninst.useNodes-of*
and *uninst-phiNodes-of* = *uninst.phiNodes-of*
⟨*proof*⟩

definition *uninst-chooseNext u p g* ≡ *Max* (*uninst-trivial-phis* (*const p*) *g*)

lemma *gen-ssa-wf-invar-trans*: *rel-fun cr-gen-ssa-cfg-wf* (=) *gen-wf-invar gen-wf-invar*
⟨*proof*⟩

declare *graph-path-base.transfer-rules*[*OF gen-ssa-cfg-wf.right-total gen-ssa-wf-*α*e.transfer gen-ssa-wf-*α*n.transfer gen-ssa-wf-invar-trans gen-ssa-wf-inEdges'-transfer*, *transfer-rule*]

declare *CFG-base.defAss'-transfer*[*OF gen-ssa-cfg-wf.right-total gen-ssa-wf-αe.transfer gen-ssa-wf-αn.transfer gen-ssa-wf-invar-trans gen-ssa-wf-inEdges'-transfer*, *transfer-rule*]
declare *CFG-SSA-base-code.CFG-SSA-base-code-transfer-rules*[*OF gen-ssa-cfg-wf.right-total gen-ssa-wf-αe.transfer gen-ssa-wf-αn.transfer gen-ssa-wf-invar-trans gen-ssa-wf-inEdges'-transfer gen-ssa-wf-Entry.transfer gen-ssa-wf-ssa-defs.transfer gen-ssa-wf-ssa-uses.transfer gen-ssa-wf-phis.transfer*, *transfer-rule*]

lemma *path2-ign*[*simp*]: *graph-path-base.path2* (*ign gen-αn g*) *gen-wf-invar* (*ign gen-inEdges' g*) *g' n ns m* ⟷ *graph-path-base.path2 gen-αn gen-wf-invar gen-inEdges' g n ns m*
⟨*proof*⟩
lemma *allDefs-ign*[*simp*]: *CFG-SSA-base.allDefs* (*ign gen-ssa-defs g*) (*ign Mapping.lookup* (*gen-phis g*)) *ga n* = *CFG-SSA-base.allDefs gen-ssa-defs* (λ*g. Mapping.lookup* (*gen-phis g*)) *g n*
⟨*proof*⟩
lemma *successors-ign*[*simp*]: *graph-path-base.successors* (*ign gen-αn g*) (*ign gen-inEdges' g*) *ga n* = *graph-path-base.successors gen-αn gen-inEdges' g n*
⟨*proof*⟩
lemma *predecessors-ign*[*simp*]: *graph-path-base.predecessors* (*ign gen-inEdges' g*) *ga n* = *graph-path-base.predecessors gen-inEdges' g n*
⟨*proof*⟩
lemma *phiDefs-ign*[*simp*]: *CFG-SSA-base.phiDefs* (*ign Mapping.lookup* (*gen-phis g*)) *ga* = *CFG-SSA-base.phiDefs* (λ*g. Mapping.lookup* (*gen-phis g*)) *g*
⟨*proof*⟩
lemma *defAss-ign*[*simp*]: *CFG-SSA-base.defAss* (*ign gen-αn g*) *gen-wf-invar* (*ign gen-inEdges' g*) (*ign gen-Entry g*) (*ign gen-ssa-defs g*) (*ign Mapping.lookup* (*gen-phis g*)) *ga*
= *CFG-SSA-base.defAss gen-αn gen-wf-invar gen-inEdges' gen-Entry gen-ssa-defs* (λ*g. Mapping.lookup* (*gen-phis g*)) *g*
⟨*proof*⟩
lemma *allUses-ign*[*simp*]: *CFG-SSA-base.allUses* (*ign gen-αn g*) (*ign gen-inEdges' g*) (*usesOf* ∘ *ign gen-ssa-uses g*) (*ign Mapping.lookup* (*gen-phis g*)) *ga m*
= *CFG-SSA-base.allUses gen-αn gen-inEdges'* (*usesOf* ∘ *gen-ssa-uses*) (λ*g. Mapping.lookup* (*gen-phis g*)) *g m*
⟨*proof*⟩
lemma *defAss'-ign*[*simp*]: *CFG-base.defAss'* (*ign gen-αn g*) *gen-wf-invar* (*ign gen-inEdges' g*) (*ign gen-Entry g*) (*ign gen-defs g*) *ga*
= *CFG-base.defAss' gen-αn gen-wf-invar gen-inEdges' gen-Entry gen-defs g*
⟨*proof*⟩

global-interpretation *gen-ssa-wf-notriv*: *CFG-SSA-Transformed-notriv-linorder-code gen-ssa-wf-αe gen-ssa-wf-αn gen-wf-invar gen-ssa-wf-inEdges' gen-ssa-wf-Entry gen-ssa-wf-defs gen-ssa-wf-uses gen-ssa-wf-ssa-defs gen-ssa-wf-ssa-uses gen-ssa-wf-phis gen-ssa-wf-var uninst-chooseNext*
defines
gen-ssa-wf-notriv-substAll = *gen-ssa-wf-notriv.substAll* **and**
gen-ssa-wf-notriv-substAll-efficient = *gen-ssa-wf-notriv.substAll-efficient*

⟨*proof*⟩

global-interpretation *uninst-code*: *CFG-SSA-Transformed-notriv-base-code gen-ssa-wf-αe gen-ssa-wf-αn gen-wf-invar gen-ssa-wf-inEdges′ gen-ssa-wf-Entry gen-ssa-wf-defs gen-ssa-wf-uses gen-ssa-wf-ssa-defs u p gen-ssa-wf-var uninst-chooseNext*
for *u* **and** *p*
defines
uninst-code-step-code = *uninst-code.step-codem* **and**
uninst-code-phis′ = *uninst-code.phis′-codem* **and**
uninst-code-uses′ = *uninst-code.uses′-codem* **and**
uninst-code-substNext = *uninst-code.substNext-code* **and**
uninst-code-substitution = *uninst-code.substitution-code* **and**
uninst-code-triv-phis′ = *uninst-code.triv-phis′* **and**
uninst-code-nodes-of-uses′ = *uninst-code.nodes-of-uses′* **and**
uninst-code-nodes-of-phis′ = *uninst-code.nodes-of-phis′*
⟨*proof*⟩

lift-definition *gen-cfg-wf-extend* :: ('*a*::*linorder*, '*b*::*linorder*, '*c*) *gen-cfg-wf* ⇒ '*d* ⇒ ('*a*, '*b*, '*c*, '*d*) *gen-cfg-scheme*
is *gen-cfg.extend* ⟨*proof*⟩

lemma *gen-αe-wf-extend* [*simp*]:
gen-αe (*gen-cfg-wf-extend gen-cfg-wf* (|*gen-ssa-defs* = *d*, *gen-ssa-uses* = *u*, *gen-phis* = *p*, *gen-var* = *v*|))
= *gen-wf-αe gen-cfg-wf*
⟨*proof*⟩

lemma *gen-αn-wf-extend* [*simp*]:
gen-αn (*gen-cfg-wf-extend gen-cfg-wf* (|*gen-ssa-defs* = *d*, *gen-ssa-uses* = *u*, *gen-phis* = *p*, *gen-var* = *v*|))
= *gen-wf-αn gen-cfg-wf*
⟨*proof*⟩

lemma *gen-inEdges-wf-extend* [*simp*]:
gen-inEdges (*gen-cfg-wf-extend gen-cfg-wf* (|*gen-ssa-defs* = *d*, *gen-ssa-uses* = *u*, *gen-phis* = *p*, *gen-var* = *v*|))
= *gen-wf-inEdges gen-cfg-wf*
⟨*proof*⟩

lemma *gen-Entry-wf-extend* [*simp*]:
gen-Entry (*gen-cfg-wf-extend gen-cfg-wf* (|*gen-ssa-defs* = *d*, *gen-ssa-uses* = *u*, *gen-phis* = *p*, *gen-var* = *v*|))
= *gen-wf-Entry gen-cfg-wf*
⟨*proof*⟩

lemma *gen-defs-wf-extend* [*simp*]:
gen-defs (*gen-cfg-wf-extend gen-cfg-wf* (|*gen-ssa-defs* = *d*, *gen-ssa-uses* = *u*, *gen-phis* = *p*, *gen-var* = *v*|))
= *gen-wf-defs gen-cfg-wf*

⟨*proof*⟩

lemma *gen-uses-wf-extend* [*simp*]:
gen-uses (*gen-cfg-wf-extend gen-cfg-wf* (|*gen-ssa-defs* = *d*, *gen-ssa-uses* = *u*, *gen-phis* = *p*, *gen-var* = *v*|))
= *gen-wf-uses gen-cfg-wf*
⟨*proof*⟩

lemma *gen-ssa-defs-wf-extend* [*simp*]:
gen-ssa-defs (*gen-cfg-wf-extend gen-cfg-wf* (|*gen-ssa-defs* = *d*, *gen-ssa-uses* = *u*, *gen-phis* = *p*, *gen-var* = *v*|))
= *d*
⟨*proof*⟩

lemma *gen-ssa-uses-wf-extend* [*simp*]:
gen-ssa-uses (*gen-cfg-wf-extend gen-cfg-wf* (|*gen-ssa-defs* = *d*, *gen-ssa-uses* = *u*, *gen-phis* = *p*, *gen-var* = *v*|))
= *u*
⟨*proof*⟩

lemma *gen-phis-wf-extend* [*simp*]:
gen-phis (*gen-cfg-wf-extend gen-cfg-wf* (|*gen-ssa-defs* = *d*, *gen-ssa-uses* = *u*, *gen-phis* = *p*, *gen-var* = *v*|))
= *p*
⟨*proof*⟩

lemma *gen-var-wf-extend* [*simp*]:
gen-var (*gen-cfg-wf-extend gen-cfg-wf* (|*gen-ssa-defs* = *d*, *gen-ssa-uses* = *u*, *gen-phis* = *p*, *gen-var* = *v*|))
= *v*
⟨*proof*⟩

lemma *CFG-SSA-Transformed-codeI*:
assumes *CFG-SSA-Transformed* αe αn *invar inEdges Entry oldDefs oldUses defs* (λg. *lookup-multimap* (*uses g*)) (λg. *Mapping.lookup* (*phis g*)) *var*
and $\bigwedge g$. *Mapping.keys* (*uses g*) $\subseteq$ *set* (αn *g*)
shows *CFG-SSA-Transformed-code* αe αn *invar inEdges Entry oldDefs oldUses defs uses phis var*
⟨*proof*⟩

lemma *CFG-SSA-Transformed-ign*:
CFG-SSA-Transformed (*ign gen-wf-*αe *gen-cfg-wf*) (*ign gen-wf-*αn *gen-cfg-wf*) *gen-wf-invar*
(*const* (*gen-wf-inEdges'* *gen-cfg-wf*)) (*ign gen-wf-Entry gen-cfg-wf*) (*ign gen-wf-defs gen-cfg-wf*)
(*ign gen-wf-uses gen-cfg-wf*) (*ign gen-wf-defs'* *gen-cfg-wf*) (*ign gen-wf.uses'* *gen-cfg-wf*)
(*ign gen-wf.phis'* *gen-cfg-wf*)
(*ign gen-wf-var gen-cfg-wf*)

⟨*proof*⟩

lift-definition *gen-ssa-cfg-wf* :: ('*node*::*linorder*, '*var*::*linorder*, '*edge*) *gen-cfg-wf* ⇒ ('*node*, '*var*, '*edge* , ('*node*,'*var*) *ssaVal*) *gen-ssa-cfg-wf*
is λ*g*. *let* (*uses*,*phis*) = *gen-wf-uses'-phis'* *g* *in* (*gen-cfg-wf-extend* *g*)(|
gen-ssa-defs = *gen-wf-defs'* *g*,
gen-ssa-uses = *uses*,
gen-phis = *phis*,
gen-var = *gen-wf-var* *g*
|)
⟨*proof*⟩

declare *uninst.defNode-code*[*abs-def*, *code*] *uninst.allVars-code*[*abs-def*, *code*] *uninst.allUses-def*[*abs-def*, *code*] *uninst.allDefs-def*[*abs-def*, *code*]
uninst.phiUses-code[*abs-def*, *code*] *uninst.phi-def*[*abs-def*, *code*] *uninst.redundant-code-def*[*abs-def*, *code*]
declare *uninst-code.uses'-code-def*[*abs-def*, *code*] *uninst-code.substNext-code-def*[*abs-def*, *code*] *uninst-code.substitution-code-def*[*abs-def*, *folded uninst-phi-def*, *code*]
declare *uninst-code.phis'-code-def*[*folded uninst-code-substNext-def*, *code*] *uninst-code.step-code-def*[*folded uninst-code.uses'-code-def uninst-code.phis'-code-def*, *code*]
uninst-code.cond-code-def[*folded uninst-redundant-def*, *code*]
declare *gen-ssa-wf-notriv.substAll-efficient-def*
[*folded uninst-code-nodes-of-phis'-def uninst-code-nodes-of-uses'-def uninst-code-triv-phis'-def*
uninst-code-substitution-def
uninst-code-step-code-def uninst-code-phis'-def uninst-code-uses'-def uninst-trivial-phis-def
uninst-phidefNodes-def uninst-useNodes-of-def uninst-phiNodes-of-def, *code*]
declare *keys-dom-lookup* [*symmetric*, *code-unfold*]

definition *map-keys-from-sparse* ≡ *map-keys gen-wf.from-sparse*

declare *map-keys-code*[*OF gen-wf.from-sparse-inj*, *folded map-keys-from-sparse-def*, *code*]
declare *map-keys-from-sparse-def*[*symmetric*, *code-unfold*]

lemma *fold-Cons-commute*: (⋀*a* *b*. ⟦*a* ∈ *set* (*x* # *xs*); *b* ∈ *set* (*x* # *xs*)⟧ ⟹ *f* *a* ∘ *f* *b* = *f* *b* ∘ *f* *a*)
⟹ *fold* *f* (*x* # *xs*) = *f* *x* ∘ (*fold* *f* *xs*)
⟨*proof*⟩

lemma *Union-of-code* [*code*]: *Union-of* *f* (*RBT-Set.Set* *r*) = *RBT.fold* (λ*a* -. (∪) (*f* *a*)) *r* {}
⟨*proof*⟩

definition[*code*]: *disjoint* *xs* *ys* = (*xs* ∩ *ys* = {})

definition *gen-ssa-wf-notriv-substAll'* = *fst* ∘ *gen-ssa-wf-notriv-substAll-efficient*

definition *fold-set* *f* *A* ≡ *fold* *f* (*sorted-list-of-set* *A*)
declare *fold-set-def* [*symmetric*, *code-unfold*]

declare *fold-set-def*
[**where** *A=RBT-Set.Set r* **for** *r*,
unfolded sorted-list-set fold-keys-def-alt [*symmetric*,*abs-def*] *fold-keys-def* [*abs-def*],
code]

declare *graph-path-base.inEdges-def* [*code*]

end

theory *Generic-Extract* **imports**
Generic-Interpretation
begin

export-code open
set sorted-list-of-set disjoint RBT.fold
gen-ssa-cfg-wf gen-wf-var gen-ssa-wf-notriv-substAll'
in *OCaml* **module-name** *BraunSSA*

end

theory *Disjoin-Transform* **imports**
Slicing.AdditionalLemmas
begin

inductive *subcmd* :: *cmd* ⇒ *cmd* ⇒ *bool* **where**
sub-Skip: *subcmd c Skip*
| *sub-Base*: *subcmd c c*
| *sub-Seq1*: *subcmd c1 c* ⟹ *subcmd* (*c1*;;*c2*) *c*
| *sub-Seq2*: *subcmd c2 c* ⟹ *subcmd* (*c1*;;*c2*) *c*
| *sub-If1*: *subcmd c1 c* ⟹ *subcmd* (*if* (*b*) *c1 else c2*) *c*
| *sub-If2*:*subcmd c2 c* ⟹ *subcmd* (*if* (*b*) *c1 else c2*) *c*
| *sub-While*: *subcmd c' c* ⟹ *subcmd* (*while* (*b*) *c'*) *c*

fun *maxVnameLen-aux* :: *expr* ⇒ *nat* **where**
maxVnameLen-aux (*Val* -) = *0*
| *maxVnameLen-aux* (*Var V*) = *length V*
| *maxVnameLen-aux* (*e1* « - » *e2*) = *max* (*maxVnameLen-aux e1*) (*maxVnameLen-aux e2*)

fun *maxVnameLen* :: *cmd* ⇒ *nat* **where**
maxVnameLen Skip = *0*
| *maxVnameLen* (*V*:=*e*) = *max* (*length V*) (*maxVnameLen-aux e*)
| *maxVnameLen* (c_1;;c_2) = *max* (*maxVnameLen* c_1) (*maxVnameLen* c_2)
| *maxVnameLen* (*if* (*b*) c_1 *else* c_2) = *max* (*maxVnameLen* c_1) (*max* (*maxVnameLen-aux b*) (*maxVnameLen* c_2))
| *maxVnameLen* (*while* (*b*) *c*) = *max* (*maxVnameLen c*) (*maxVnameLen-aux b*)

definition *tempName* :: *cmd* ⇒ *vname* **where** *tempName c* ≡ *replicate* (*Suc* (*maxVnameLen c*)) (*CHR* ''*a*'')

inductive *newname* :: *cmd* ⇒ *vname* ⇒ *bool* **where**
  newname Skip V
| *V* ∉ {*V'*} ∪ *rhs-aux e* ⟹ *newname* (*V'*:=*e*) *V*
| ⟦*newname c1 V*; *newname c2 V*⟧ ⟹ *newname* (*c1*;;*c2*) *V*
| ⟦*newname c1 V*; *newname c2 V*; *V* ∉ *rhs-aux b*⟧ ⟹ *newname* (*if* (*b*) *c1 else c2*) *V*
| ⟦*newname c V*; *V* ∉ *rhs-aux b*⟧ ⟹ *newname* (*while* (*b*) *c*) *V*

lemma *maxVnameLen-aux-newname*: *length V* > *maxVnameLen-aux e* ⟹ *V* ∉ *rhs-aux e*
⟨*proof*⟩

lemma *maxVnameLen-newname*: *length V* > *maxVnameLen c* ⟹ *newname c V*
⟨*proof*⟩

lemma *tempname-newname*[*intro*]: *newname c* (*tempName c*)
  ⟨*proof*⟩

fun *transform-aux* :: *vname* ⇒ *cmd* ⇒ *cmd* **where**
  transform-aux - Skip = *Skip*
| *transform-aux V'* (*V*:=*e*) =
    (*if V* ∈ *rhs* (*V*:=*e*) *then V'*:=*e*;; *V*:=*Var V'*
     else V:=*e*)
| *transform-aux V'* (*c*$_1$;;*c*$_2$) = *transform-aux V' c*$_1$;; *transform-aux V' c*$_2$
| *transform-aux V'* (*if* (*b*) *c*$_1$ *else c*$_2$) =
    (*if* (*b*) *transform-aux V' c*$_1$ *else transform-aux V' c*$_2$)
| *transform-aux V'* (*while* (*b*) *c*) = (*while* (*b*) *transform-aux V' c*)

abbreviation *transform* :: *cmd* ⇒ *cmd* **where**
  transform c ≡ *transform-aux* (*tempName c*) *c*

fun *leftmostCmd* :: *cmd* ⇒ *cmd* **where**
  leftmostCmd (*c1*;;*c2*) = *leftmostCmd c1*
| *leftmostCmd c* = *c*

lemma *leftmost-lhs*[*simp*]: *lhs* (*leftmostCmd c*) = *lhs c*
⟨*proof*⟩

lemma *leftmost-rhs*[*simp*]: *rhs* (*leftmostCmd c*) = *rhs c*
⟨*proof*⟩

lemma *leftmost-subcmd*[*intro*]: *subcmd c* (*leftmostCmd c*)
⟨*proof*⟩

lemma *leftmost-labels*: *labels c n c'* ⟹ *subcmd c* (*leftmostCmd c'*)
⟨*proof*⟩

theorem *transform-disjoint*:
  assumes *subcmd* (*transform-aux temp c*) (*V*:=*e*) *newname c temp*
  shows *V* ∉ *rhs-aux e*
⟨*proof*⟩

lemma *transform-disjoint'*: *subcmd* (*transform c*) (*leftmostCmd c'*) ⟹ *lhs c'* ∩ *rhs c'* = {}
  ⟨*proof*⟩

corollary *Defs-Uses-transform-disjoint* [*simp*]: *Defs* (*transform c*) *n* ∩ *Uses* (*transform c*) *n* = {}
  ⟨*proof*⟩

end

6.5.1 Instantiation for a Simple While Language

theory *WhileGraphSSA* **imports**
Generic-Interpretation
Disjoin-Transform
HOL−Library.List-Lexorder
HOL−Library.Char-ord
begin

instantiation *w-node* :: *ord*
begin

fun *less-eq-w-node* **where**
  (-*Entry*-) ≤ *x* = *True*
| (- *n* -) ≤ *x* = (*case x of*
    (-*Entry*-) ⇒ *False*
  | (- *m* -) ⇒ *n* ≤ *m*
  | (-*Exit*-) ⇒ *True*)
| (-*Exit*-) ≤ *x* = (*x* = (-*Exit*-))

fun *less-w-node* **where**
  (-*Entry*-) < *x* = (*x* ≠ (-*Entry*-))
| (- *n* -) < *x* = (*case x of*
    (-*Entry*-) ⇒ *False*
  | (- *m* -) ⇒ *n* < *m*
  | (-*Exit*-) ⇒ *True*)
| (-*Exit*-) < *x* = *False*

instance ⟨*proof*⟩
end

instance *w-node* :: *linorder* ⟨*proof*⟩

declare *Defs.simps* [*simp del*]
declare *Uses.simps* [*simp del*]
declare *Let-def* [*simp*]

declare *finite-valid-nodes* [*simp*, *intro*!]

lemma *finite-valid-edge* [*simp*, *intro*!]: *finite* (*Collect* (*valid-edge c*))
⟨*proof*⟩

lemma *uses-expr-finite*: *finite* (*rhs-aux e*)
⟨*proof*⟩

lemma *uses-cmd-finite*: *finite* (*rhs c*)
⟨*proof*⟩

lemma *defs-cmd-finite*: *finite* (*lhs c*)
⟨*proof*⟩

lemma *finite-labels'*: *finite* {(*l*,*c*). *labels prog l c*}
⟨*proof*⟩

lemma *finite-Defs* [*simp*, *intro*!]: *finite* (*Defs c n*)
⟨*proof*⟩

lemma *finite-Uses* [*simp*, *intro*!]: *finite* (*Uses c n*)
⟨*proof*⟩

definition *while-cfg-*α*e c* = *Collect* (*valid-edge* (*transform c*))
definition *while-cfg-*α*n c* = *sorted-list-of-set* (*Collect* (*valid-node* (*transform c*)))
definition *while-cfg-invar c* = *True*
definition *while-cfg-inEdges' c t* = (*SOME ls. distinct ls* ∧ *set ls* = {(*sourcenode e*, *kind e*)| *e. valid-edge* (*transform c*) *e* ∧ *targetnode e* = *t*})
definition *while-cfg-Entry c* = (*-Entry-*)
definition *while-cfg-defs c* = (*Defs* (*transform c*))((*-Entry-*) := {*v*. ∃ *n*. *v* ∈ *Uses* (*transform c*) *n*})
definition *while-cfg-uses c* = *Uses* (*transform c*)

abbreviation *while-cfg-inEdges c t* ≡ *map* (λ(*f*,*d*). (*f*,*d*,*t*)) (*while-cfg-inEdges' c t*)

lemmas *while-cfg-defs* = *while-cfg-*α*e-def while-cfg-*α*n-def*
while-cfg-invar-def while-cfg-inEdges'-def
while-cfg-Entry-def while-cfg-defs-def
while-cfg-uses-def

interpretation *while*: *graph-path while-cfg-*α*e while-cfg-*α*n while-cfg-invar while-cfg-inEdges'*
⟨*proof*⟩

lemma *right-total-const*: *right-total* (λ*x y*. *x* = *c*)

⟨*proof*⟩

lemma *const-transfer*: *rel-fun* (λ*x y*. *x* = *c*) (=) *f* (λ-. *f c*)
⟨*proof*⟩

interpretation *while-ign*: *graph-path* λ-. *while-cfg-*α*e cmd* λ-. *while-cfg-*α*n cmd* λ-. *while-cfg-invar cmd* λ-. *while-cfg-inEdges*′ *cmd*
⟨*proof*⟩

definition *gen-while-cfg g* ≡ (|
*gen-*α*e* = *while-cfg-*α*e g*,
*gen-*α*n* = *while-cfg-*α*n g*,
gen-inEdges = *while-cfg-inEdges g*,
gen-Entry = *while-cfg-Entry g*,
gen-defs = *while-cfg-defs g* ,
gen-uses = *while-cfg-uses g*
|)

lemma *while-path-graph-pathD*: *While-CFG.path* (*transform c*) *n es m* ⟹ *while.path2 c n* (*n*#*map targetnode es*) *m*
⟨*proof*⟩

lemma *Uses-Entry* [*simp*]: *Uses c* (*-Entry-*) = {}
⟨*proof*⟩

lemma *in-Uses-valid-node*: *V* ∈ *Uses c n* ⟹ *valid-node c n*
⟨*proof*⟩

lemma *while-cfg-CFG-wf-impl*:
SSA-CFG.CFG-wf (λ-. *gen-*α*e* (*gen-while-cfg cmd*)) (λ-. *gen-*α*n* (*gen-while-cfg cmd*))
(λ-. *while-cfg-invar cmd*) (λ-. *gen-inEdges*′ (*gen-while-cfg cmd*))
(λ-. *gen-Entry* (*gen-while-cfg cmd*)) (λ-. *gen-defs* (*gen-while-cfg cmd*))
(λ-. *gen-uses* (*gen-while-cfg cmd*))
⟨*proof*⟩

lift-definition *gen-while-cfg-wf* :: *cmd* ⇒ (*w-node*, *vname*, *state edge-kind*) *gen-cfg-wf*
is *gen-while-cfg*
⟨*proof*⟩

definition *build-ssa cmd* = *gen-ssa-wf-notriv-substAll* (*gen-ssa-cfg-wf* (*gen-while-cfg-wf cmd*))

end

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