

Exponents 3 and 4 of Fermat's Last Theorem and the Parametrisation of Pythagorean Triples

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Abstract

This document gives a formal proof of the cases $n = 3$ and $n = 4$ (and all their multiples) of Fermat's Last Theorem: if $n > 2$ then for all integers x, y, z :

$$x^n + y^n = z^n \implies xyz = 0.$$

Both proofs only use facts about the integers and are developed along the lines of the standard proofs (see, for example, sections 1 and 2 of the book by Edwards [Edw77]).

First, the framework of 'infinite descent' is being formalised and in both proofs there is a central role for the lemma

$$\text{coprime } ab \wedge ab = c^n \implies \exists k : |a| = k^n.$$

Furthermore, the proof of the case $n = 4$ uses a parametrisation of the Pythagorean triples. The proof of the case $n = 3$ contains a study of the quadratic form $x^2 + 3y^2$. This study is completed with a result on which prime numbers can be written as $x^2 + 3y^2$.

The case $n = 4$ of FLT, in contrast to the case $n = 3$, has already been formalised (in the proof assistant Coq) [DM05]. The parametrisation of the Pythagorean Triples can be found as number 23 on the list of 'top 100 mathematical theorems' [Wie].

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1 Pythagorean triples and Fermat's last theorem, case $n = 4$

theory *Fermat4*

imports *HOL-Computational-Algebra.Primes*

begin

context

begin

private lemma *nat-relprime-power-divisors*:

assumes $n0: 0 < n$ **and** $abc: (a::nat)*b = c^n$ **and** $relprime: coprime\ a\ b$

shows $\exists k. a = k^n$

<proof> **lemma** *int-relprime-power-divisors*:

assumes $0 < n$ **and** $0 \leq a$ **and** $0 \leq b$ **and** $(a::int) * b = c^n$ **and** $coprime\ a\ b$

shows $\exists k. a = k^n$

<proof>

Proof of Fermat's last theorem for the case $n = 4$:

$$\forall x, y, z : x^4 + y^4 = z^4 \implies xyz = 0.$$

private lemma *nat-power2-diff*: $a \geq (b::nat) \implies (a-b)^2 = a^2 + b^2 - 2*a*b$

<proof> **lemma** *nat-power-le-imp-le-base*: $\llbracket n \neq 0; a^n \leq b^n \rrbracket \implies (a::nat) \leq b$

<proof> **lemma** *nat-power-inject-base*: $\llbracket n \neq 0; a^n = b^n \rrbracket \implies (a::nat) = b$

<proof>

1.1 Parametrisation of Pythagorean triples (over \mathbb{N} and \mathbb{Z})

private theorem *nat-euclid-pyth-triples*:

assumes $abc: (a::nat)^2 + b^2 = c^2$ **and** $ab-relprime: coprime\ a\ b$ **and** $aodd: odd\ a$

shows $\exists p\ q. a = p^2 - q^2 \wedge b = 2*p*q \wedge c = p^2 + q^2 \wedge coprime\ p\ q$

<proof>

Now for the case of integers. Based on *nat-euclid-pyth-triples*.

private corollary *int-euclid-pyth-triples*: $\llbracket coprime\ (a::int)\ b; odd\ a; a^2 + b^2 = c^2$

\rrbracket

$\implies \exists p\ q. a = p^2 - q^2 \wedge b = 2*p*q \wedge |c| = p^2 + q^2 \wedge coprime\ p\ q$

<proof>

1.2 Fermat's last theorem, case $n = 4$

Core of the proof. Constructs a smaller solution over \mathbb{Z} of

$$a^4 + b^4 = c^2 \wedge coprime\ a\ b \wedge abc \neq 0 \wedge a\ odd.$$

private lemma *smaller-fermat4*:

assumes $abc: (a::int)^4 + b^4 = c^2$ **and** $abc0: a*b*c \neq 0$ **and** $aodd: odd\ a$

and $ab-relprime: coprime\ a\ b$

shows

$\exists p q r. (p^4 + q^4 = r^2 \wedge p * q * r \neq 0 \wedge \text{odd } p \wedge \text{coprime } p q \wedge r^2 < c^2)$
 <proof>

Show that no solution exists, by infinite descent of c^2 .

private lemma *no-rewritten-fermat4*:

$\neg (\exists (a::\text{int}) b. (a^4 + b^4 = c^2 \wedge a * b * c \neq 0 \wedge \text{odd } a \wedge \text{coprime } a b))$
 <proof>

The theorem. Puts equation in requested shape.

theorem *fermat-4*:

assumes *ass*: $(x::\text{int})^4 + y^4 = z^4$
shows $x * y * z = 0$
 <proof>

corollary *fermat-mult4*:

assumes *xyz*: $(x::\text{int})^n + y^n = z^n$ **and** $n: 4 \text{ dvd } n$
shows $x * y * z = 0$
 <proof>

end

end

2 The quadratic form $x^2 + Ny^2$

theory *Quad-Form*

imports

HOL-Number-Theory.Number-Theory

begin

context

begin

Shows some properties of the quadratic form $x^2 + Ny^2$, such as how to multiply and divide them. The second part focuses on the case $N = 3$ and is used in the proof of the case $n = 3$ of Fermat's last theorem. The last part – not used for FLT3 – shows which primes can be written as $x^2 + 3y^2$.

2.1 Definitions and auxiliary results

private lemma *best-division-abs*: $(n::\text{int}) > 0 \implies \exists k. 2 * |a - k * n| \leq n$
 <proof>

lemma *prime-power-dvd-cancel-right*:

$p \wedge n \text{ dvd } a$ **if** *prime* $(p::'a::\text{semiring-gcd}) \neg p \text{ dvd } b$ $p \wedge n \text{ dvd } a * b$
 <proof>

definition

is-qn $:: \text{int} \Rightarrow \text{int} \Rightarrow \text{bool}$ **where**
is-qn $A N \longleftrightarrow (\exists x y. A = x^2 + N * y^2)$

definition

is-cube-form :: *int* \Rightarrow *int* \Rightarrow *bool* **where**

is-cube-form *a b* $\longleftrightarrow (\exists p q. a = p^3 - 9*p*q^2 \wedge b = 3*p^2*q - 3*q^3)$

private lemma *abs-eq-impl-unitfactor*: $|a::int| = |b| \implies \exists u. a = u*b \wedge |u|=1$
 $\langle proof \rangle$ **lemma** *prime-3-nat*: *prime* ($3::nat$) $\langle proof \rangle$

2.2 Basic facts if $N \geq 1$

lemma *qfN-pos*: $\llbracket N \geq 1; is-qfN A N \rrbracket \implies A \geq 0$
 $\langle proof \rangle$

lemma *qfN-zero*: $\llbracket (N::int) \geq 1; a^2 + N*b^2 = 0 \rrbracket \implies (a = 0 \wedge b = 0)$
 $\langle proof \rangle$

2.3 Multiplication and division

lemma *qfN-mult1*: $((a::int)^2 + N*b^2)*(c^2 + N*d^2)$
 $= (a*c + N*b*d)^2 + N*(a*d - b*c)^2$
 $\langle proof \rangle$

lemma *qfN-mult2*: $((a::int)^2 + N*b^2)*(c^2 + N*d^2)$
 $= (a*c - N*b*d)^2 + N*(a*d + b*c)^2$
 $\langle proof \rangle$

corollary *is-qfN-mult*: $is-qfN A N \implies is-qfN B N \implies is-qfN (A*B) N$
 $\langle proof \rangle$

corollary *is-qfN-power*: $(n::nat) > 0 \implies is-qfN A N \implies is-qfN (A^n) N$
 $\langle proof \rangle$

lemma *qfN-div-prime*:

fixes *p* :: *int*

assumes *ass*: $prime (p^2 + N*q^2) \wedge (p^2 + N*q^2) \text{ dvd } (a^2 + N*b^2)$

shows $\exists u v. a^2 + N*b^2 = (u^2 + N*v^2)*(p^2 + N*q^2)$

$\wedge (\exists e. a = p*u + e*N*q*v \wedge b = p*v - e*q*u \wedge |e|=1)$

$\langle proof \rangle$

corollary *qfN-div-prime-weak*:

$\llbracket prime (p^2 + N*q^2::int); (p^2 + N*q^2) \text{ dvd } (a^2 + N*b^2) \rrbracket$

$\implies \exists u v. a^2 + N*b^2 = (u^2 + N*v^2)*(p^2 + N*q^2)$

$\langle proof \rangle$

corollary *qfN-div-prime-general*: $\llbracket prime P; P \text{ dvd } A; is-qfN A N; is-qfN P N \rrbracket$

$\implies \exists Q. A = Q*P \wedge is-qfN Q N$

$\langle proof \rangle$

lemma *qfN-power-div-prime*:

fixes *P* :: *int*

assumes *ass*: $prime P \wedge odd P \wedge P \text{ dvd } A \wedge P^n = p^2 + N*q^2$

$\wedge A^n = a^2 + N*b^2 \wedge coprime a b \wedge coprime p (N*q) \wedge n > 0$

shows $\exists u v. a^2 + N*b^2 = (u^2 + N*v^2)*(p^2 + N*q^2) \wedge coprime u v$

$\langle \text{proof} \rangle \quad \wedge (\exists e. a = p*u + e*N*q*v \wedge b = p*v - e*q*u \wedge |e| = 1)$

lemma *qfN-primedivisor-not*:

assumes *ass*: $\text{prime } P \wedge Q > 0 \wedge \text{is-qfN } (P*Q) \ N \wedge \neg \text{is-qfN } P \ N$
shows $\exists R. (\text{prime } R \wedge R \text{ dvd } Q \wedge \neg \text{is-qfN } R \ N)$

$\langle \text{proof} \rangle$

lemma *prime-factor-int*:

fixes $k :: \text{int}$
assumes $|k| \neq 1$
obtains p **where** $\text{prime } p \ p \text{ dvd } k$

$\langle \text{proof} \rangle$

lemma *qfN-oddprime-cube*:

$\llbracket \text{prime } (p^2 + N*q^2 :: \text{int}); \text{odd } (p^2 + N*q^2); p \neq 0; N \geq 1 \rrbracket$
 $\implies \exists a \ b. (p^2 + N*q^2)^3 = a^2 + N*b^2 \wedge \text{coprime } a \ (N*b)$

$\langle \text{proof} \rangle$

2.4 Uniqueness ($N > 1$)

lemma *qfN-prime-unique*:

$\llbracket \text{prime } (a^2 + N*b^2 :: \text{int}); N > 1; a^2 + N*b^2 = c^2 + N*d^2 \rrbracket$
 $\implies (|a| = |c| \wedge |b| = |d|)$

$\langle \text{proof} \rangle$

lemma *qfN-square-prime*:

assumes *ass*:
 $\text{prime } (p^2 + N*q^2 :: \text{int}) \wedge N > 1 \wedge (p^2 + N*q^2)^2 = r^2 + N*s^2 \wedge \text{coprime } r \ s$
shows $|r| = |p^2 - N*q^2| \wedge |s| = |2*p*q|$

$\langle \text{proof} \rangle$

lemma *qfN-cube-prime*:

assumes *ass*: $\text{prime } (p^2 + N*q^2 :: \text{int}) \wedge N > 1$
 $\wedge (p^2 + N*q^2)^3 = a^2 + N*b^2 \wedge \text{coprime } a \ b$
shows $|a| = |p^3 - 3*N*p*q^2| \wedge |b| = |3*p^2*q - N*q^3|$

$\langle \text{proof} \rangle$

2.5 The case $N = 3$

lemma *qf3-even*: $\text{even } (a^2 + 3*b^2) \implies \exists B. a^2 + 3*b^2 = 4*B \wedge \text{is-qfN } B \ 3$

$\langle \text{proof} \rangle$

lemma *qf3-even-general*: $\llbracket \text{is-qfN } A \ 3; \text{even } A \rrbracket$

$\implies \exists B. A = 4*B \wedge \text{is-qfN } B \ 3$

$\langle \text{proof} \rangle$

lemma *qf3-oddprimedivisor-not*:

assumes *ass*: $\text{prime } P \wedge \text{odd } P \wedge Q > 0 \wedge \text{is-qfN } (P*Q) \ 3 \wedge \neg \text{is-qfN } P \ 3$
shows $\exists R. \text{prime } R \wedge \text{odd } R \wedge R \text{ dvd } Q \wedge \neg \text{is-qfN } R \ 3$

$\langle \text{proof} \rangle$

lemma *qf3-oddprimedivisor*:

[[*prime* ($P::int$); *odd* P ; *coprime* a b ; P *div* (a^2+3*b^2)]]
 \implies *is-qfN* P 3
 <proof>

lemma *qf3-cube-prime-impl-cube-form*:

assumes *ab-relprime*: *coprime* a b **and** *abP*: $P^3 = a^2 + 3*b^2$
and P : *prime* $P \wedge$ *odd* P
shows *is-cube-form* a b
 <proof>

lemma *cube-form-mult*: [[*is-cube-form* a b ; *is-cube-form* c d ; $|e| = 1$]]

\implies *is-cube-form* ($a*c+e*3*b*d$) ($a*d-e*b*c$)
 <proof>

lemma *qf3-cube-primelist-impl-cube-form*: [[($\forall p \in$ *set-mset* ps . *prime* p); *odd* (int ($\prod_{i \in \#ps}$ i))]]

\implies
 (!! a b . *coprime* a $b \implies a^2 + 3*b^2 = (int(\prod_{i \in \#ps} i))^3 \implies$ *is-cube-form* a b)
 <proof>

lemma *qf3-cube-impl-cube-form*:

assumes *ass*: *coprime* a $b \wedge a^2 + 3*b^2 = w^3 \wedge$ *odd* w
shows *is-cube-form* a b
 <proof>

2.6 Existence ($N = 3$)

This part contains the proof that all prime numbers $\equiv 1 \pmod{6}$ can be written as $x^2 + 3y^2$.

First show $\left(\frac{a}{p}\right)\left(\frac{b}{p}\right) = \left(\frac{ab}{p}\right)$, where p is an odd prime.

lemma *Legendre-zmult*: [[$p > 2$; *prime* p]]

\implies (*Legendre* ($a*b$) p) = (*Legendre* a p)*(*Legendre* b p)
 <proof>

Now show $\left(\frac{-3}{p}\right) = +1$ for primes $p \equiv 1 \pmod{6}$.

lemma *Legendre-1mod6*: *prime* ($6*m+1$) \implies (*Legendre* (-3) ($6*m+1$)) = 1

<proof>

Use this to prove that such primes can be written as $x^2 + 3y^2$.

lemma *qf3-prime-exists*: *prime* ($6*m+1::int$) $\implies \exists x y$. $6*m+1 = x^2 + 3*y^2$

<proof>

end

end

3 Fermat's last theorem, case $n = 3$

theory *Fermat3*

imports *Quad-Form*

begin

context

begin

Proof of Fermat's last theorem for the case $n = 3$:

$$\forall x, y, z : x^3 + y^3 = z^3 \implies xyz = 0.$$

private lemma *nat-relprime-power-divisors*:

assumes $n0: 0 < n$ **and** $abc: (a::nat)*b = c^n$ **and** $relprime: coprime\ a\ b$

shows $\exists k. a = k^n$

<proof> **lemma** *int-relprime-odd-power-divisors*:

assumes $odd\ n$ **and** $(a::int) * b = c^n$ **and** $coprime\ a\ b$

shows $\exists k. a = k^n$

<proof> **lemma** *factor-sum-cubes*: $(x::int)^3 + y^3 = (x+y)*(x^2 - xy + y^2)$

<proof> **lemma** *two-not-abs-cube*: $|x^3| = (2::int) \implies False$

<proof>

Shows there exists no solution $v^3 + w^3 = x^3$ with $vwx \neq 0$ and $coprime\ v\ w$ and x even, by constructing a solution with a smaller $|x^3|$.

private lemma *no-rewritten-fermat3*:

$\neg (\exists v\ w. v^3 + w^3 = x^3 \wedge v*w*x \neq 0 \wedge even\ (x::int) \wedge coprime\ v\ w)$

<proof>

The theorem. Puts equation in requested shape.

theorem *fermat-3*:

assumes $ass: (x::int)^3 + y^3 = z^3$

shows $x*y*z=0$

<proof>

corollary *fermat-mult3*:

assumes $xyz: (x::int)^n + y^n = z^n$ **and** $n: 3\ dvd\ n$

shows $x*y*z=0$

<proof>

end

end

References

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