Exponents 3 and 4 of Fermat's Last Theorem and the Parametrisation of Pythagorean Triples

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Abstract

This document gives a formal proof of the cases $n = 3$ and $n = 4$ (and all their multiples) of Fermat's Last Theorem: if $n > 2$ then for all integers x, y, z:

$$x^n + y^n = z^n \Longrightarrow xyz = 0.$$

Both proofs only use facts about the integers and are developed along the lines of the standard proofs (see, for example, sections 1 and 2 of the book by Edwards [Edw77]).

First, the framework of 'infinite descent' is being formalised and in both proofs there is a central role for the lemma

$$\textit{coprime} a b \ \wedge\ ab = c^n \Longrightarrow \exists\, k : |a| = k^n.$$

Furthermore, the proof of the case $n = 4$ uses a parametrisation of the Pythagorean triples. The proof of the case $n = 3$ contains a study of the quadratic form $x^2 + 3y^2$. This study is completed with a result on which prime numbers can be written as $x^2 + 3y^2$.

The case $n = 4$ of FLT, in contrast to the case $n = 3$, has already been formalised (in the proof assistant Coq) [DM05]. The parametrisation of the Pythagorean Triples can be found as number 23 on the list of 'top 100 mathematical theorems' [Wie].

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1 Pythagorean triples and Fermat's last theorem, case $n = 4$

theory *Fermat4*
imports *HOL−Computational-Algebra.Primes*
begin

context
begin

private lemma *nat-relprime-power-divisors*:
assumes *n0*: $0 < n$ **and** *abc*: $(a::nat)*b = c\,\hat{}\,n$ **and** *relprime*: *coprime a b*
shows $\exists\ k.\ a = k\,\hat{}\,n$
⟨*proof*⟩ **lemma** *int-relprime-power-divisors*:
assumes $0 < n$ **and** $0 \le a$ **and** $0 \le b$ **and** $(a::int) * b = c\ \hat{}\ n$ **and** *coprime a b*
shows $\exists k.\ a = k\,\hat{}\,n$
⟨*proof*⟩

Proof of Fermat's last theorem for the case $n = 4$:

$$\forall x, y, z:\ x^4 + y^4 = z^4 \Longrightarrow xyz = 0.$$

private lemma *nat-power2-diff*: $a \ge (b::nat) \Longrightarrow (a-b)\,\hat{}\,2 = a\,\hat{}\,2 + b\,\hat{}\,2 - 2*a*b$
⟨*proof*⟩ **lemma** *nat-power-le-imp-le-base*: $[\![\ n \ne 0;\ a\,\hat{}\,n \le b\,\hat{}\,n\ ]\!] \Longrightarrow (a::nat) \le b$
⟨*proof*⟩ **lemma** *nat-power-inject-base*: $[\![\ n \ne 0;\ a\,\hat{}\,n = b\,\hat{}\,n\ ]\!] \Longrightarrow (a::nat) = b$
⟨*proof*⟩

1.1 Parametrisation of Pythagorean triples (over $\mathbb{N}$ and $\mathbb{Z}$)

private theorem *nat-euclid-pyth-triples*:
assumes *abc*: $(a::nat)\,\hat{}\,2 + b\,\hat{}\,2 = c\,\hat{}\,2$ **and** *ab-relprime*: *coprime a b* **and** *aodd*: *odd a*
shows $\exists\ p\ q.\ a = p\,\hat{}\,2 - q\,\hat{}\,2 \wedge b = 2*p*q \wedge c = p\,\hat{}\,2 + q\,\hat{}\,2 \wedge$ *coprime p q*
⟨*proof*⟩

Now for the case of integers. Based on *nat-euclid-pyth-triples*.

private corollary *int-euclid-pyth-triples*: $[\![$ *coprime* $(a::int)$ b; *odd a*; $a\,\hat{}\,2 + b\,\hat{}\,2 = c\,\hat{}\,2$
$]\!]$
$\Longrightarrow \exists\ p\ q.\ a = p\,\hat{}\,2 - q\,\hat{}\,2 \wedge b = 2*p*q \wedge |c| = p\,\hat{}\,2 + q\,\hat{}\,2 \wedge$ *coprime p q*
⟨*proof*⟩

1.2 Fermat's last theorem, case $n = 4$

Core of the proof. Constructs a smaller solution over $\mathbb{Z}$ of

$$a^4 + b^4 = c^2 \wedge \mathit{coprime}\, a\, b \wedge abc \ne 0 \wedge a \text{ odd}.$$

private lemma *smaller-fermat4*:
assumes *abc*: $(a::int)\,\hat{}\,4+b\,\hat{}\,4=c\,\hat{}\,2$ **and** *abc0*: $a*b*c \ne 0$ **and** *aodd*: *odd a*
and *ab-relprime*: *coprime a b*
shows

$\exists\ p\ q\ r.\ (p\hat{\ }4+q\hat{\ }4=r\hat{\ }2 \land p*q*r \neq 0 \land odd\ p \land coprime\ p\ q \land r\hat{\ }2 < c\hat{\ }2)$
⟨*proof*⟩

Show that no solution exists, by infinite descent of c^2.

private lemma *no-rewritten-fermat4*:
$\neg\ (\exists\ (a{::}int)\ b.\ (a\hat{\ }4 + b\hat{\ }4 = c\hat{\ }2 \land a*b*c \neq 0 \land odd\ a \land coprime\ a\ b))$
⟨*proof*⟩

The theorem. Puts equation in requested shape.

theorem *fermat-4*:
assumes *ass*: $(x{::}int)\hat{\ }4 + y\hat{\ }4 = z\hat{\ }4$
shows $x*y*z=0$
⟨*proof*⟩

corollary *fermat-mult4*:
assumes *xyz*: $(x{::}int)\hat{\ }n + y\hat{\ }n = z\hat{\ }n$ **and** *n*: $4\ dvd\ n$
shows $x*y*z=0$
⟨*proof*⟩

end

end

2 The quadratic form $x^2 + Ny^2$

theory *Quad-Form*
imports
HOL−Number-Theory.Number-Theory
begin

context
begin

Shows some properties of the quadratic form x^2+Ny^2, such as how to multiply and divide them. The second part focuses on the case $N = 3$ and is used in the proof of the case $n = 3$ of Fermat's last theorem. The last part – not used for FLT3 – shows which primes can be written as $x^2 + 3y^2$.

2.1 Definitions and auxiliary results

private lemma *best-division-abs*: $(n{::}int) > 0 \Longrightarrow \exists\ k.\ 2 * |a - k*n| \leq n$
⟨*proof*⟩

lemma *prime-power-dvd-cancel-right*:
$p\ \hat{\ }\ n\ dvd\ a$ **if** $prime\ (p{::}'a{::}semiring\text{-}gcd)\ \neg\ p\ dvd\ b\ p\ \hat{\ }\ n\ dvd\ a * b$
⟨*proof*⟩

definition
is-qfN :: *int* ⇒ *int* ⇒ *bool* **where**
$is\text{-}qfN\ A\ N \longleftrightarrow (\exists\ x\ y.\ A = x\hat{\ }2 + N*y\hat{\ }2)$

definition
is-cube-form :: *int* ⇒ *int* ⇒ *bool* **where**
is-cube-form a b ⟷ (∃ *p q. a* = *p*ˆ*3* − *9*∗*p*∗*q*ˆ*2* ∧ *b* = *3*∗*p*ˆ*2*∗*q* − *3*∗*q*ˆ*3*)

private lemma *abs-eq-impl-unitfactor*: |*a*::*int*| = |*b*| ⟹ ∃ *u. a* = *u*∗*b* ∧ |*u*|=*1*
⟨*proof*⟩ **lemma** *prime-3-nat*: *prime* (*3*::*nat*) ⟨*proof*⟩

2.2 Basic facts if $N \geq 1$

lemma *qfN-pos*: ⟦ *N* ≥ *1*; *is-qfN A N* ⟧ ⟹ *A* ≥ *0*
⟨*proof*⟩

lemma *qfN-zero*: ⟦ (*N*::*int*) ≥ *1*; *a*ˆ*2* + *N*∗*b*ˆ*2* = *0* ⟧ ⟹ (*a* = *0* ∧ *b* = *0*)
⟨*proof*⟩

2.3 Multiplication and division

lemma *qfN-mult1*: ((*a*::*int*)ˆ*2* + *N*∗*b*ˆ*2*)∗(*c*ˆ*2* + *N*∗*d*ˆ*2*)
= (*a*∗*c*+*N*∗*b*∗*d*)ˆ*2* + *N*∗(*a*∗*d*−*b*∗*c*)ˆ*2*
⟨*proof*⟩

lemma *qfN-mult2*: ((*a*::*int*)ˆ*2* + *N*∗*b*ˆ*2*)∗(*c*ˆ*2* + *N*∗*d*ˆ*2*)
= (*a*∗*c*−*N*∗*b*∗*d*)ˆ*2* + *N*∗(*a*∗*d*+*b*∗*c*)ˆ*2*
⟨*proof*⟩

corollary *is-qfN-mult*: *is-qfN A N* ⟹ *is-qfN B N* ⟹ *is-qfN* (*A*∗*B*) *N*
⟨*proof*⟩

corollary *is-qfN-power*: (*n*::*nat*) > *0* ⟹ *is-qfN A N* ⟹ *is-qfN* (*A*ˆ*n*) *N*
⟨*proof*⟩

lemma *qfN-div-prime*:
fixes *p* :: *int*
assumes *ass*: *prime* (*p*ˆ*2*+*N*∗*q*ˆ*2*) ∧ (*p*ˆ*2*+*N*∗*q*ˆ*2*) *dvd* (*a*ˆ*2*+*N*∗*b*ˆ*2*)
shows ∃ *u v. a*ˆ*2*+*N*∗*b*ˆ*2* = (*u*ˆ*2*+*N*∗*v*ˆ*2*)∗(*p*ˆ*2*+*N*∗*q*ˆ*2*)
∧ (∃ *e. a* = *p*∗*u*+*e*∗*N*∗*q*∗*v* ∧ *b* = *p*∗*v* − *e*∗*q*∗*u* ∧ |*e*|=*1*)
⟨*proof*⟩

corollary *qfN-div-prime-weak*:
⟦ *prime* (*p*ˆ*2*+*N*∗*q*ˆ*2*::*int*); (*p*ˆ*2*+*N*∗*q*ˆ*2*) *dvd* (*a*ˆ*2*+*N*∗*b*ˆ*2*) ⟧
⟹ ∃ *u v. a*ˆ*2*+*N*∗*b*ˆ*2* = (*u*ˆ*2*+*N*∗*v*ˆ*2*)∗(*p*ˆ*2*+*N*∗*q*ˆ*2*)
⟨*proof*⟩

corollary *qfN-div-prime-general*: ⟦ *prime P*; *P dvd A*; *is-qfN A N*; *is-qfN P N* ⟧
⟹ ∃ *Q. A* = *Q*∗*P* ∧ *is-qfN Q N*
⟨*proof*⟩

lemma *qfN-power-div-prime*:
fixes *P* :: *int*
assumes *ass*: *prime P* ∧ *odd P* ∧ *P dvd A* ∧ *P*ˆ*n* = *p*ˆ*2*+*N*∗*q*ˆ*2*
∧ *A*ˆ*n* = *a*ˆ*2*+*N*∗*b*ˆ*2* ∧ *coprime a b* ∧ *coprime p* (*N*∗*q*) ∧ *n*>*0*
shows ∃ *u v. a*ˆ*2*+*N*∗*b*ˆ*2* = (*u*ˆ*2* + *N*∗*v*ˆ*2*)∗(*p*ˆ*2*+*N*∗*q*ˆ*2*) ∧ *coprime u v*

$\wedge$ ($\exists$ *e*. *a* = *p*∗*u*+*e*∗*N*∗*q*∗*v* $\wedge$ *b* = *p*∗*v*−*e*∗*q*∗*u* $\wedge$ |*e*| = *1*)
⟨*proof*⟩

lemma *qfN-primedivisor-not*:
assumes *ass*: *prime P* $\wedge$ *Q* > *0* $\wedge$ *is-qfN* (*P*∗*Q*) *N* $\wedge$ ¬ *is-qfN P N*
shows $\exists$ *R*. (*prime R* $\wedge$ *R dvd Q* $\wedge$ ¬ *is-qfN R N*)
⟨*proof*⟩

lemma *prime-factor-int*:
fixes *k* :: *int*
assumes |*k*| ≠ *1*
obtains *p* **where** *prime p p dvd k*
⟨*proof*⟩

lemma *qfN-oddprime-cube*:
⟦ *prime* (*p*ˆ*2*+*N*∗*q*ˆ*2*::*int*); *odd* (*p*ˆ*2*+*N*∗*q*ˆ*2*); *p* ≠ *0*; *N* ≥ *1* ⟧
⟹ $\exists$ *a b*. (*p*ˆ*2*+*N*∗*q*ˆ*2*)ˆ*3* = *a*ˆ*2* + *N*∗*b*ˆ*2* $\wedge$ *coprime a* (*N*∗*b*)
⟨*proof*⟩

2.4 Uniqueness ($N > 1$)

lemma *qfN-prime-unique*:
⟦ *prime* (*a*ˆ*2*+*N*∗*b*ˆ*2*::*int*); *N* > *1*; *a*ˆ*2*+*N*∗*b*ˆ*2* = *c*ˆ*2*+*N*∗*d*ˆ*2* ⟧
⟹ (|*a*| = |*c*| $\wedge$ |*b*| = |*d*|)
⟨*proof*⟩

lemma *qfN-square-prime*:
assumes *ass*:
prime (*p*ˆ*2*+*N*∗*q*ˆ*2*::*int*) $\wedge$ *N*>*1* $\wedge$ (*p*ˆ*2*+*N*∗*q*ˆ*2*)ˆ*2* = *r*ˆ*2*+*N*∗*s*ˆ*2* $\wedge$ *coprime r s*
shows |*r*| = |*p*ˆ*2*−*N*∗*q*ˆ*2*| $\wedge$ |*s*| = |*2*∗*p*∗*q*|
⟨*proof*⟩

lemma *qfN-cube-prime*:
assumes *ass*: *prime* (*p*ˆ*2* + *N*∗*q*ˆ*2*::*int*) $\wedge$ *N* > *1*
$\wedge$ (*p*ˆ*2* + *N*∗*q*ˆ*2*)ˆ*3* = *a*ˆ*2* + *N*∗*b*ˆ*2* $\wedge$ *coprime a b*
shows |*a*| = |*p*ˆ*3*− *3*∗*N*∗*p*∗*q*ˆ*2*| $\wedge$ |*b*| = |*3*∗*p*ˆ*2*∗*q*−*N*∗*q*ˆ*3*|
⟨*proof*⟩

2.5 The case $N = 3$

lemma *qf3-even*: *even* (*a*ˆ*2*+*3*∗*b*ˆ*2*) ⟹ $\exists$ *B*. *a*ˆ*2*+*3*∗*b*ˆ*2* = *4*∗*B* $\wedge$ *is-qfN B 3*
⟨*proof*⟩

lemma *qf3-even-general*: ⟦ *is-qfN A 3*; *even A* ⟧
⟹ $\exists$ *B*. *A* = *4*∗*B* $\wedge$ *is-qfN B 3*
⟨*proof*⟩

lemma *qf3-oddprimedivisor-not*:
assumes *ass*: *prime P* $\wedge$ *odd P* $\wedge$ *Q*>*0* $\wedge$ *is-qfN* (*P*∗*Q*) *3* $\wedge$ ¬ *is-qfN P 3*
shows $\exists$ *R*. *prime R* $\wedge$ *odd R* $\wedge$ *R dvd Q* $\wedge$ ¬ *is-qfN R 3*
⟨*proof*⟩

lemma *qf3-oddprimedivisor*:
  ⟦ *prime* (*P*::*int*); *odd P*; *coprime a b*; *P dvd* (*a*ˆ*2*+*3*∗*b*ˆ*2*) ⟧
  ⟹ *is-qfN P 3*
⟨*proof*⟩

lemma *qf3-cube-prime-impl-cube-form*:
  assumes *ab-relprime*: *coprime a b* **and** *abP*: *P*ˆ*3* = *a*ˆ*2* + *3*∗*b*ˆ*2*
  and *P*: *prime P* ∧ *odd P*
  shows *is-cube-form a b*
⟨*proof*⟩

lemma *cube-form-mult*: ⟦ *is-cube-form a b*; *is-cube-form c d*; |*e*| = *1* ⟧
  ⟹ *is-cube-form* (*a*∗*c*+*e*∗*3*∗*b*∗*d*) (*a*∗*d*−*e*∗*b*∗*c*)
⟨*proof*⟩

lemma *qf3-cube-primelist-impl-cube-form*: ⟦ (∀ *p*∈*set-mset ps*. *prime p*); *odd* (*int* (∏ *i*∈#*ps*. *i*)) ⟧ ⟹
  (!! *a b*. *coprime a b* ⟹ *a*ˆ*2* + *3*∗*b*ˆ*2* = (*int*(∏ *i*∈#*ps*. *i*))ˆ*3* ⟹ *is-cube-form a b*)
⟨*proof*⟩

lemma *qf3-cube-impl-cube-form*:
  assumes *ass*: *coprime a b* ∧ *a*ˆ*2* + *3*∗*b*ˆ*2* = *w*ˆ*3* ∧ *odd w*
  shows *is-cube-form a b*
⟨*proof*⟩

2.6 Existence ($N = 3$)

This part contains the proof that all prime numbers $\equiv 1 \bmod 6$ can be written as $x^2 + 3y^2$.

First show $(\frac{a}{p})(\frac{b}{p}) = (\frac{ab}{p})$, where p is an odd prime.

lemma *Legendre-zmult*: ⟦ *p* > *2*; *prime p* ⟧
  ⟹ (*Legendre* (*a*∗*b*) *p*) = (*Legendre a p*)∗(*Legendre b p*)
⟨*proof*⟩

Now show $(\frac{-3}{p}) = +1$ for primes $p \equiv 1 \bmod 6$.

lemma *Legendre-1mod6*: *prime* (*6*∗*m*+*1*) ⟹ *Legendre* (−*3*) (*6*∗*m*+*1*) = *1*
⟨*proof*⟩

Use this to prove that such primes can be written as $x^2 + 3y^2$.

lemma *qf3-prime-exists*: *prime* (*6*∗*m*+*1*::*int*) ⟹ ∃ *x y*. *6*∗*m*+*1* = *x*ˆ*2* + *3*∗*y*ˆ*2*
⟨*proof*⟩

end

end

3 Fermat's last theorem, case $n = 3$

theory *Fermat3*
imports *Quad-Form*

begin

context
begin

Proof of Fermat's last theorem for the case $n = 3$:

$$\forall x, y, z: \; x^3 + y^3 = z^3 \Longrightarrow xyz = 0.$$

private lemma *nat-relprime-power-divisors*:
  assumes *n0*: $0 < n$ **and** *abc*: $(a::nat)*b = c\hat{~}n$ **and** *relprime*: *coprime a b*
  shows $\exists\ k.\ a = k\hat{~}n$
⟨*proof*⟩ **lemma** *int-relprime-odd-power-divisors*:
  assumes *odd n* **and** $(a::int) * b = c\ \hat{~}\ n$ **and** *coprime a b*
  shows $\exists k.\ a = k\hat{~}n$
⟨*proof*⟩ **lemma** *factor-sum-cubes*: $(x::int)\hat{~}3 + y\hat{~}3 = (x+y)*(x\hat{~}2 - x*y + y\hat{~}2)$
  ⟨*proof*⟩ **lemma** *two-not-abs-cube*: $|x\hat{~}3| = (2::int) \Longrightarrow False$
⟨*proof*⟩

Shows there exists no solution $v^3 + w^3 = x^3$ with $vwx \neq 0$ and *coprimevw* and x even, by constructing a solution with a smaller $|x^3|$.

private lemma *no-rewritten-fermat3*:
  $\neg\ (\exists\ v\ w.\ v\hat{~}3+w\hat{~}3 = x\hat{~}3 \wedge v*w*x \neq 0 \wedge even\ (x::int) \wedge coprime\ v\ w)$
⟨*proof*⟩

The theorem. Puts equation in requested shape.

theorem *fermat-3*:
  assumes *ass*: $(x::int)\hat{~}3 + y\hat{~}3 = z\hat{~}3$
  shows $x*y*z=0$
⟨*proof*⟩

corollary *fermat-mult3*:
  assumes *xyz*: $(x::int)\hat{~}n + y\hat{~}n = z\hat{~}n$ **and** *n*: *3 dvd n*
  shows $x*y*z=0$
⟨*proof*⟩

end

end

References

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