Farey Sequences and Ford Circles

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18 May 2025

Abstract

The sequence F_n of *Farey fractions* of order n has the form

$$\frac{0}{1}, \frac{1}{n}, \frac{1}{n-1}, \ldots, \frac{n-1}{n}, \frac{1}{1}$$

where the fractions appear in numerical order and have denominators at most n. The transformation from F_n to F_{n+1} can be effected by combining adjacent elements of the sequence F_n, using an operation called the *mediant*. Adjacent (reduced) fractions $(a/b) < (c/d)$ satisfy the *unimodular* relation $bc - ad = 1$ and their mediant is $\frac{a+c}{b+d}$. A *Ford circle* is specified by a rational number, and interesting consequences follow in the case of Ford circles obtained from some Farey sequence F_n. The formalised material is drawn from Apostol's *Modular Functions and Dirichlet Series in Number Theory* [1].

1 Farey Sequences and Ford Circles

theory *Farey-Ford*
imports *HOL−Analysis.Analysis HOL−Number-Theory.Totient HOL−Library.Sublist*

begin

1.1 Library material

lemma *sum-squared-le-sum-of-squares*:
fixes f :: $'a \Rightarrow real$
assumes *finite I*
shows $(\sum i{\in}I.\ f\ i)^2 \le (\sum y{\in}I.\ (f\ y)^2) * card\ I$
⟨*proof*⟩

lemma *sum-squared-le-sum-of-squares-2*:
$(x{+}y)/2 \le sqrt\ ((x^2 + y^2)\ /\ 2)$
⟨*proof*⟩

lemma *sphere-scale*:
assumes $a \neq 0$
shows $(\lambda x.\ a *_R x)$ ‘ $sphere\ c\ r = sphere\ (a *_R c :: 'a :: real\text{-}normed\text{-}vector)$ $(|a| * r)$
⟨*proof*⟩

lemma *sphere-cscale*:
assumes $a \neq 0$
shows $(\lambda x.\ a * x)$ ‘ $sphere\ c\ r = sphere\ (a * c :: complex)\ (cmod\ a * r)$
⟨*proof*⟩

lemma *Complex-divide-complex-of-real*: $Complex\ x\ y\ /\ of\text{-}real\ r = Complex\ (x/r)$ (y/r)
⟨*proof*⟩
lemma *cmod-neg-real*: $cmod\ (Complex\ (-x)\ y) = cmod\ (Complex\ x\ y)$
⟨*proof*⟩

1.2 Farey sequences

lemma *sorted-two-sublist*:
fixes x:: $'a$::*order*
assumes *sorted*: *sorted-wrt* $(<)$ l
shows *sublist* $[x, y]$ $l \longleftrightarrow x < y \land x \in set\ l \land y \in set\ l \land (\forall z \in set\ l.\ z \le x$ $\lor z \ge y)$
⟨*proof*⟩

lemma *quotient-of-rat-of-int* [*simp*]: *quotient-of* (*rat-of-int i*) = (*i*, *1*)
⟨*proof*⟩

lemma *quotient-of-rat-of-nat* [*simp*]: *quotient-of* (*rat-of-nat i*) = (*int i*, *1*)
⟨*proof*⟩

lemma *int-div-le-self*:
‹*x div k* ≤ *x*› **if** ‹*0* < *x*› **for** *x k* :: *int*
⟨*proof*⟩

lemma *transp-add1-int*:
assumes ⋀*n*::*int*. *R* (*f n*) (*f* (*1* + *n*))
and *n* < *n*′
and *transp R*
shows *R* (*f n*) (*f n*′)
⟨*proof*⟩

lemma *refl-transp-add1-int*:
assumes ⋀*n*::*int*. *R* (*f n*) (*f* (*1* + *n*))
and *n* ≤ *n*′
and *reflp R transp R*
shows *R* (*f n*) (*f n*′)
⟨*proof*⟩

lemma *transp-Suc*:
assumes ⋀*n*. *R* (*f n*) (*f* (*Suc n*))
and *n* < *n*′
and *transp R*
shows *R* (*f n*) (*f n*′)
⟨*proof*⟩

lemma *refl-transp-Suc*:
assumes ⋀*n*. *R* (*f n*) (*f* (*Suc n*))
and *n* ≤ *n*′
and *reflp R transp R*
shows *R* (*f n*) (*f n*′)
⟨*proof*⟩

lemma *sorted-subset-imp-subseq*:
fixes *xs* :: ′*a*::*order list*
assumes *set xs* ⊆ *set ys sorted-wrt* (<) *xs sorted-wrt* (≤) *ys*
shows *subseq xs ys*
⟨*proof*⟩

lemma *coprime-unimodular-int*:

fixes $a\ b{::}int$
assumes $coprime\ a\ b\ a{>}1\ b{>}1$
obtains $x\ y$ **where** $a{*}x - b{*}y = 1\ 0 < x\ x < b\ 0 < y\ y < a$
⟨*proof*⟩

1.3 Farey Fractions

type-synonym $farey = rat$

definition $num\text{-}farey :: farey \Rightarrow int$
where $num\text{-}farey \equiv \lambda x.\ fst\ (quotient\text{-}of\ x)$

definition $denom\text{-}farey :: farey \Rightarrow int$
where $denom\text{-}farey \equiv \lambda x.\ snd\ (quotient\text{-}of\ x)$

definition $farey :: [int,int] \Rightarrow farey$
where $farey \equiv \lambda a\ b.\ max\ 0\ (min\ 1\ (Fract\ a\ b))$

lemma *farey01* [*simp*]: $0 \le farey\ a\ b\ farey\ a\ b \le 1$
⟨*proof*⟩

lemma *farey-0* [*simp*]: $farey\ 0\ n = 0$
⟨*proof*⟩

lemma *farey-1* [*simp*]: $farey\ 1\ 1 = 1$
⟨*proof*⟩

lemma *num-farey-nonneg*: $x \in \{0..1\} \Longrightarrow num\text{-}farey\ x \ge 0$
⟨*proof*⟩

lemma *num-farey-le-denom*: $x \in \{0..1\} \Longrightarrow num\text{-}farey\ x \le denom\text{-}farey\ x$
⟨*proof*⟩

lemma *denom-farey-pos*: $denom\text{-}farey\ x > 0$
⟨*proof*⟩

lemma *coprime-num-denom-farey* [*intro*]: $coprime\ (num\text{-}farey\ x)\ (denom\text{-}farey\ x)$
⟨*proof*⟩

lemma *rat-of-farey-conv-num-denom*:
$x = rat\text{-}of\text{-}int\ (num\text{-}farey\ x)\ /\ rat\text{-}of\text{-}int\ (denom\text{-}farey\ x)$
⟨*proof*⟩

lemma *num-denom-farey-eqI*:
assumes $x = of\text{-}int\ a\ /\ of\text{-}int\ b\ b > 0\ coprime\ a\ b$
shows $num\text{-}farey\ x = a\ denom\text{-}farey\ x = b$
⟨*proof*⟩

lemma *farey-cases* [*cases type*, *case-names farey*]:

assumes $x \in \{0..1\}$
obtains a b **where** $0 \le a$ $a \le b$ *coprime a b* $x = \mathit{Fract}\ a\ b$
⟨*proof*⟩

lemma *rat-of-farey*: $[\![x = \mathit{of\text{-}int}\ a\ /\ \mathit{of\text{-}int}\ b;\ x \in \{0..1\}]\!] \Longrightarrow x = \mathit{farey}\ a\ b$
⟨*proof*⟩

lemma *farey-num-denom-eq* [*simp*]: $x \in \{0..1\} \Longrightarrow \mathit{farey}\ (\mathit{num\text{-}farey}\ x)\ (\mathit{denom\text{-}farey}\ x) = x$
⟨*proof*⟩

lemma *farey-eqI*:
assumes $\mathit{num\text{-}farey}\ x = \mathit{num\text{-}farey}\ y$ $\mathit{denom\text{-}farey}\ x = \mathit{denom\text{-}farey}\ y$
shows $x=y$
⟨*proof*⟩

lemma
assumes *coprime a b* $0 \le a$ $a<b$
shows *num-farey-eq* [*simp*]: $\mathit{num\text{-}farey}\ (\mathit{farey}\ a\ b) = a$
and *denom-farey-eq* [*simp*]: $\mathit{denom\text{-}farey}\ (\mathit{farey}\ a\ b) = b$
⟨*proof*⟩

lemma
assumes $0 \le a$ $a \le b$ $0<b$
shows *num-farey*: $\mathit{num\text{-}farey}\ (\mathit{farey}\ a\ b) = a\ \mathit{div}\ (\mathit{gcd}\ a\ b)$
and *denom-farey*: $\mathit{denom\text{-}farey}\ (\mathit{farey}\ a\ b) = b\ \mathit{div}\ (\mathit{gcd}\ a\ b)$
⟨*proof*⟩

lemma
assumes *coprime a b* $0<b$
shows *num-farey-Fract* [*simp*]: $\mathit{num\text{-}farey}\ (\mathit{Fract}\ a\ b) = a$
and *denom-farey-Fract* [*simp*]: $\mathit{denom\text{-}farey}\ (\mathit{Fract}\ a\ b) = b$
⟨*proof*⟩

lemma *num-farey-0* [*simp*]: $\mathit{num\text{-}farey}\ 0 = 0$
and *denom-farey-0* [*simp*]: $\mathit{denom\text{-}farey}\ 0 = 1$
and *num-farey-1* [*simp*]: $\mathit{num\text{-}farey}\ 1 = 1$
and *denom-farey-1* [*simp*]: $\mathit{denom\text{-}farey}\ 1 = 1$
⟨*proof*⟩

lemma *num-farey-neq-denom*: $\mathit{denom\text{-}farey}\ x \ne 1 \Longrightarrow \mathit{num\text{-}farey}\ x \ne \mathit{denom\text{-}farey}\ x$
⟨*proof*⟩

lemma *num-farey-0-iff* [*simp*]: $\mathit{num\text{-}farey}\ x = 0 \longleftrightarrow x = 0$
⟨*proof*⟩

lemma *denom-farey-le1-cases*:
assumes $\mathit{denom\text{-}farey}\ x \le 1$ $x \in \{0..1\}$

shows $x = 0 \lor x = 1$
⟨*proof*⟩

definition *mediant* :: *farey* ⇒ *farey* ⇒ *farey* **where**
mediant ≡ λ*x y*. *Fract* (*fst* (*quotient-of x*) + *fst* (*quotient-of y*))
(*snd* (*quotient-of x*) + *snd* (*quotient-of y*))

lemma *mediant-eq-Fract*:
mediant x y = *Fract* (*num-farey x* + *num-farey y*) (*denom-farey x* + *denom-farey y*)
⟨*proof*⟩

lemma *mediant-eq-farey*:
assumes $x \in \{0..1\}$ $y \in \{0..1\}$
shows *mediant x y* = *farey* (*num-farey x* + *num-farey y*) (*denom-farey x* + *denom-farey y*)
⟨*proof*⟩

definition *farey-unimodular* :: *farey* ⇒ *farey* ⇒ *bool* **where**
farey-unimodular x y ⟷
denom-farey x ∗ *num-farey y* − *num-farey x* ∗ *denom-farey y* = *1*

lemma *farey-unimodular-imp-less*:
assumes *farey-unimodular x y*
shows $x < y$
⟨*proof*⟩

lemma *denom-mediant*: *denom-farey* (*mediant x y*) ≤ *denom-farey x* + *denom-farey y*
⟨*proof*⟩

lemma *unimodular-imp-both-coprime*:
fixes *a*:: *'a*::{*algebraic-semidom*,*comm-ring-1*}
assumes $b*c - a*d = 1$
shows *coprime a b coprime c d*
⟨*proof*⟩

lemma *unimodular-imp-coprime*:
fixes *a*:: *'a*::{*algebraic-semidom*,*comm-ring-1*}
assumes $b*c - a*d = 1$
shows *coprime* (*a*+*c*) (*b*+*d*)
⟨*proof*⟩

definition *fareys* :: *int* ⇒ *rat list*
where *fareys n* ≡ *sorted-list-of-set* {$x \in \{0..1\}$. *denom-farey x* ≤ *n*}

lemma *strict-sorted-fareys*: *sorted-wrt* (<) (*fareys n*)
⟨*proof*⟩

lemma *farey-set-UN-farey*: $\{x \in \{0..1\}.\ \textit{denom-farey}\ x \leq n\} = (\bigcup b \in \{1..n\}.\ \bigcup a \in \{0..b\}.\ \{\textit{farey}\ a\ b\})$
⟨*proof*⟩

lemma *farey-set-UN-farey'*: $\{x \in \{0..1\}.\ \textit{denom-farey}\ x \leq n\} = (\bigcup b \in \{1..n\}.\ \bigcup a \in \{0..b\}.\ \textit{if coprime}\ a\ b\ \textit{then}\ \{\textit{farey}\ a\ b\}\ \textit{else}\ \{\})$
⟨*proof*⟩

lemma *farey-set-UN-Fract*: $\{x \in \{0..1\}.\ \textit{denom-farey}\ x \leq n\} = (\bigcup b \in \{1..n\}.\ \bigcup a \in \{0..b\}.\ \{\textit{Fract}\ a\ b\})$
⟨*proof*⟩

lemma *farey-set-UN-Fract'*: $\{x \in \{0..1\}.\ \textit{denom-farey}\ x \leq n\} = (\bigcup b \in \{1..n\}.\ \bigcup a \in \{0..b\}.\ \textit{if coprime}\ a\ b\ \textit{then}\ \{\textit{Fract}\ a\ b\}\ \textit{else}\ \{\})$
⟨*proof*⟩

lemma *finite-farey-set*: $\textit{finite}\ \{x \in \{0..1\}.\ \textit{denom-farey}\ x \leq n\}$
⟨*proof*⟩

lemma *denom-in-fareys-iff*: $x \in \textit{set}\ (\textit{fareys}\ n) \longleftrightarrow \textit{denom-farey}\ x \leq \textit{int}\ n \land x \in \{0..1\}$
⟨*proof*⟩

lemma *denom-fareys-leI*: $x \in \textit{set}\ (\textit{fareys}\ n) \Longrightarrow \textit{denom-farey}\ x \leq n$
⟨*proof*⟩

lemma *denom-fareys-leD*: $[\![\textit{denom-farey}\ x \leq \textit{int}\ n;\ x \in \{0..1\}]\!] \Longrightarrow x \in \textit{set}\ (\textit{fareys}\ n)$
⟨*proof*⟩

lemma *fareys-increasing-1*: $\textit{set}\ (\textit{fareys}\ n) \subseteq \textit{set}\ (\textit{fareys}\ (1 + n))$
⟨*proof*⟩

definition *fareys-new* :: *int* ⇒ *rat set* **where**
$\textit{fareys-new}\ n \equiv \{\textit{Fract}\ a\ n|\ a.\ \textit{coprime}\ a\ n \land a \in \{0..n\}\}$

lemma *fareys-new-0* [*simp*]: $\textit{fareys-new}\ 0 = \{\}$
⟨*proof*⟩

lemma *fareys-new-1* [*simp*]: $\textit{fareys-new}\ 1 = \{0,1\}$
⟨*proof*⟩

lemma *fareys-new-not01*:
assumes $n>1$
shows $0 \notin (\textit{fareys-new}\ n)\ 1 \notin (\textit{fareys-new}\ n)$
⟨*proof*⟩

lemma *inj-num-farey*: $\textit{inj-on}\ \textit{num-farey}\ (\textit{fareys-new}\ n)$

⟨*proof*⟩

lemma *finite-fareys-new* [*simp*]: *finite* (*fareys-new n*)
⟨*proof*⟩

lemma *card-fareys-new*:
assumes *n* > *1*
shows *card* (*fareys-new* (*int n*)) = *totient n*
⟨*proof*⟩

lemma *disjoint-fareys-plus1*:
assumes *n* > *0*
shows *disjnt* (*set* (*fareys n*)) (*fareys-new* (*1* + *n*))
⟨*proof*⟩

lemma *set-fareys-plus1*: *set* (*fareys* (*1* + *n*)) = *set* (*fareys n*) ∪ *fareys-new* (*1* + *n*)
⟨*proof*⟩

lemma *length-fareys-Suc*:
assumes *n*>*0*
shows *length* (*fareys* (*1* + *int n*)) = *length* (*fareys n*) + *totient* (*Suc n*)
⟨*proof*⟩

lemma *fareys-0* [*simp*]: *fareys 0* = []
⟨*proof*⟩

lemma *fareys-1* [*simp*]: *fareys 1* = [*0*, *1*]
⟨*proof*⟩

lemma *fareys-2* [*simp*]: *fareys 2* = [*0*, *farey 1 2*, *1*]
⟨*proof*⟩

lemma *length-fareys-1*:
shows *length* (*fareys 1*) = *1* + *totient 1*
⟨*proof*⟩

lemma *length-fareys*: *n*>*0* ⟹ *length* (*fareys n*) = *1* + (∑ *k*=*1*..*n*. *totient k*)
⟨*proof*⟩

lemma *subseq-fareys-1*: *subseq* (*fareys n*) (*fareys* (*1* + *n*))
⟨*proof*⟩

lemma *monotone-fareys*: *monotone* (≤) *subseq fareys*
⟨*proof*⟩

lemma *farey-unimodular-0-1* [*simp*, *intro*]: *farey-unimodular 0 1*

⟨*proof*⟩

Apostol's Theorem 5.2 for integers

lemma *mediant-lies-betw-int*:
fixes *a b c d*::*int*
assumes *rat-of-int a / of-int b < of-int c / of-int d b>0 d>0*
shows *rat-of-int a / of-int b < (of-int a + of-int c) / (of-int b + of-int d)*
(rat-of-int a + of-int c) / (of-int b + of-int d) < of-int c / of-int d
⟨*proof*⟩

Apostol's Theorem 5.2

theorem *mediant-inbetween*:
fixes *x y*::*farey*
assumes *x < y*
shows *x < mediant x y mediant x y < y*
⟨*proof*⟩

lemma *sublist-fareysD*:
assumes *sublist [x,y] (fareys n)*
obtains *x ∈ set (fareys n) y ∈ set (fareys n)*
⟨*proof*⟩

Adding the denominators of two consecutive Farey fractions

lemma *sublist-fareys-add-denoms*:
fixes *a b c d*::*int*
defines *x ≡ Fract a b*
defines *y ≡ Fract c d*
assumes *sub*: *sublist [x,y] (fareys (int n))* **and** *b>0 d>0 coprime a b coprime c d*
shows *b + d > n*
⟨*proof*⟩

1.4 Apostol's Theorems 5.3–5.5

theorem *consec-subset-fareys*:
fixes *a b c d*::*int*
assumes *abcd*: *0 ≤ Fract a b Fract a b < Fract c d Fract c d ≤ 1*
and *consec*: *b∗c − a∗d = 1*
and *max*: *max b d ≤ n n < b+d*
and *b>0*
shows *sublist [Fract a b, Fract c d] (fareys n)*
⟨*proof*⟩

lemma *farey-unimodular-mediant*:
assumes *farey-unimodular x y*
shows *farey-unimodular x (mediant x y) farey-unimodular (mediant x y) y*
⟨*proof*⟩

Apostol's Theorem 5.4

theorem *mediant-unimodular*:

fixes *a b c d::int*
assumes *abcd*: *0 ≤ Fract a b Fract a b < Fract c d Fract c d ≤ 1*
and *consec*: *b∗c − a∗d = 1*
and *0*: *b>0 d>0*
defines *h ≡ a+c*
defines *k ≡ b+d*
obtains *Fract a b < Fract h k Fract h k < Fract c d coprime h k*
b∗h − a∗k = 1 c∗k − d∗h = 1
⟨*proof*⟩

Apostol's Theorem 5.5, first part: "Each fraction in *F (n + 1)* which is not in *F n* is the mediant of a pair of consecutive fractions in *F n*

lemma *get-consecutive-parents*:
fixes *m n::int*
assumes *coprime m n 0<m m<n*
obtains *a b c d* **where** *m = a+c n = b+d b∗c − a∗d = 1 a≥0 b>0 c>0 d>0 a<b c≤d*
⟨*proof*⟩

theorem *fareys-new-eq-mediant*:
assumes *x ∈ fareys-new n n>1*
obtains *a b c d* **where**
sublist [Fract a b, Fract c d] (fareys (n−1))
x = mediant (Fract a b) (Fract c d) coprime a b coprime c d a≥0 b>0 c>0 d>0
⟨*proof*⟩

Apostol's Theorem 5.5, second part: "Moreover, if *a / b < c / d* are consecutive in any *F n*, then they satisfy the unimodular relation *bc − ad = 1*.

theorem *consec-imp-unimodular*:
assumes *sublist [Fract a b, Fract c d] (fareys (int n)) b>0 d>0 coprime a b coprime c d*
shows *b∗c − a∗d = 1*
⟨*proof*⟩

1.5 Ford circles

definition *Ford-center :: rat ⇒ complex* **where**
Ford-center r ≡ (λ(h,k). Complex (h/k) (1/(2 ∗ k^2))) (quotient-of r)

definition *Ford-radius :: rat ⇒ real* **where**
Ford-radius r ≡ (λ(h,k). 1/(2 ∗ k^2)) (quotient-of r)

definition *Ford-tan :: [rat,rat] ⇒ bool* **where**
Ford-tan r s ≡ dist (Ford-center r) (Ford-center s) = Ford-radius r + Ford-radius s

definition *Ford-circle :: rat ⇒ complex set* **where**

Ford-circle r ≡ *sphere* (*Ford-center r*) (*Ford-radius r*)

lemma *Im-Ford-center* [*simp*]: *Im* (*Ford-center r*) = *Ford-radius r*
⟨*proof*⟩

lemma *Ford-radius-nonneg*: *Ford-radius r* ≥ *0*
⟨*proof*⟩

lemma *two-Ford-tangent*:
assumes *r*: (*a*,*b*) = *quotient-of r* **and** *s*: (*c*,*d*) = *quotient-of s*
shows (*dist* (*Ford-center r*) (*Ford-center s*))^*2* − (*Ford-radius r* + *Ford-radius s*)^*2*
= ((*a*∗*d* − *b*∗*c*)^*2* − *1*) / (*b*∗*d*)^*2*
⟨*proof*⟩

Apostol's Theorem 5.6

lemma *two-Ford-tangent-iff*:
assumes *r*: (*a*,*b*) = *quotient-of r* **and** *s*: (*c*,*d*) = *quotient-of s*
shows *Ford-tan r s* ⟷ |*b* ∗ *c* − *a* ∗ *d*| = *1*
⟨*proof*⟩

Also Apostol's Theorem 5.6: Distinct Ford circles do not overlap

lemma *Ford-no-overlap*:
assumes *r* ≠ *s*
shows *dist* (*Ford-center r*) (*Ford-center s*) ≥ *Ford-radius r* + *Ford-radius s*
⟨*proof*⟩

lemma *Ford-aux1*:
assumes *a*≠*0*
shows *cmod* (*Complex* (*b* / (*a* ∗ (a^2 + b^2))) (*1* / (*2* ∗ a^2) − *inverse* (a^2 + b^2))) = *1* / (*2* ∗ a^2)
(**is** *cmod ?z* = *?r*)
⟨*proof*⟩

lemma *Ford-aux2*:
assumes *a*≠*0*
shows *cmod* (*Complex* (*a* / (*b* ∗ (b^2 + a^2)) − *1* / (*a* ∗ *b*)) (*1* / (*2* ∗ a^2) − *inverse* (b^2 + a^2))) = *1* / (*2* ∗ a^2)
(**is** *cmod ?z* = *?r*)
⟨*proof*⟩

definition *Radem-trans* :: *rat* ⇒ *complex* ⇒ *complex* **where**
Radem-trans ≡ λ*r* τ. *let* (*h*,*k*) = *quotient-of r in* − i ∗ *of-int k* ^ *2* ∗ (τ − *of-rat r*)

Theorem 5.8 first part

lemma *Radem-trans-image*: *Radem-trans r* ‘ *Ford-circle r* = *sphere* (*of-rat* (*1*/*2*)) (*1*/*2*)
⟨*proof*⟩

locale *three-Ford* =
  fixes *N*::*nat*
  fixes *h1 k1 h k h2 k2*::*int*
  assumes *sub1*: *sublist* [*Fract h1 k1*, *Fract h k*] (*fareys* (*int N*))
  assumes *sub2*: *sublist* [*Fract h k*, *Fract h2 k2*] (*fareys* (*int N*))
  assumes *coprime*: *coprime h1 k1 coprime h k coprime h2 k2*
  assumes *k-pos*: *k1* > *0 k* > *0 k2* > *0*

begin

definition *r1* ≡ *Fract h1 k1*
definition *r* ≡ *Fract h k*
definition *r2* ≡ *Fract h2 k2*

lemma *N-pos*: *N*>*0*
  ⟨*proof*⟩

lemma *r-eq-quotient*:
  (*h1*,*k1*) = *quotient-of r1* (*h*,*k*) = *quotient-of r* (*h2*,*k2*) = *quotient-of r2*
  ⟨*proof*⟩

lemma *r-eq-divide*:
  r1 = *of-int h1* / *of-int k1 r* = *of-int h* / *of-int k r2* = *of-int h2*/ *of-int k2*
  ⟨*proof*⟩

lemma *collapse-r*:
  real-of-int h1 / *of-int k1* = *of-rat r1*
  real-of-int h / *of-int k* = *of-rat r real-of-int h2*/ *of-int k2* = *of-rat r2*
  ⟨*proof*⟩

lemma *unimod1*: *k1*∗*h* − *h1*∗*k* = *1*
  and *unimod2*: *k*∗*h2* − *h*∗*k2* = *1*
  ⟨*proof*⟩

lemma *r-less*: *r1* < *r r* < *r2*
  ⟨*proof*⟩

lemma *r01*:
  obtains *r1* ∈ {*0..1*} *r* ∈ {*0..1*} *r2* ∈ {*0..1*}
  ⟨*proof*⟩

lemma *atMost-N*:
  obtains *k1* ≤ *N k* ≤ *N k2* ≤ *N*
  ⟨*proof*⟩

lemma *greaterN1*: *k1* + *k* > *N*
  ⟨*proof*⟩

lemma *greaterN2*: $k + k2 > N$
⟨*proof*⟩

definition *alpha1* ≡ *Complex* $(h/k - k1 \ / \ of\text{-}int(k * (k^2 + k1^2)))$ $(inverse\ (of\text{-}int\ (k^2 + k1^2)))$
definition *alpha2* ≡ *Complex* $(h/k + k2 \ / \ of\text{-}int(k * (k^2 + k2^2)))$ $(inverse\ (of\text{-}int\ (k^2 + k2^2)))$

definition *zed1* ≡ *Complex* (k^2) $(k*k1) \ / \ ((k^2 + k1^2))$
definition *zed2* ≡ *Complex* (k^2) $(- \ k*k2) \ / \ ((k^2 + k2^2))$

Apostol's Theorem 5.7

lemma *three-Ford-tangent*:
obtains *alpha1* ∈ *Ford-circle r alpha1* ∈ *Ford-circle r1*
alpha2 ∈ *Ford-circle r alpha2* ∈ *Ford-circle r2*
⟨*proof*⟩

Theorem 5.8 second part, for alpha1

lemma *Radem-trans-alpha1*: *Radem-trans r alpha1* = *zed1*
⟨*proof*⟩

Theorem 5.8 second part, for alpha2

lemma *Radem-trans-alpha2*: *Radem-trans r alpha2* = *zed2*
⟨*proof*⟩

Theorem 5.9, for zed1

lemma *cmod-zed1*: *cmod zed1* = $k \ / \ sqrt\ (k^2 + k1^2)$
⟨*proof*⟩

Theorem 5.9, for zed2

lemma *cmod-zed2*: *cmod zed2* = $k \ / \ sqrt\ (k^2 + k2^2)$
⟨*proof*⟩

The last part of theorem 5.9

lemma *RMS-calc*:
assumes $k' > 0$ $k' + k > int\ N$
shows $k \ / \ sqrt\ (k^2 + k'^2) < sqrt\ 2 * k/N$
⟨*proof*⟩

lemma *on-chord-bounded-cmod*:
assumes $z \in closed\text{-}segment\ zed1\ zed2$
shows $cmod\ z < sqrt\ 2 * k/N$
⟨*proof*⟩

end

end

Acknowledgements Manual Eberl set up the initial Farey development.

References

[1] T. M. Apostol. *Modular Functions and Dirichlet Series in Number Theory*. Springer, 1990.