The Falling Factorial of a Sum

Lukas Bulwahn

May 26, 2024

Abstract

This entry shows that the falling factorial of a sum can be computed with an expression using binomial coefficients and the falling factorial of its summands. The entry provides three different proofs: a combinatorial proof, an induction proof and an algebraic proof using the Vandermonde identity.

The three formalizations try to follow their informal presentations from a Mathematics Stack Exchange page [1, 2, 3, 4] as close as possible. The induction and algebraic formalization end up to be very close to their informal presentation, whereas the combinatorial proof first requires the introduction of list interleavings, and significant more detail than its informal presentation.

Contents

1 Proving Falling Factorial of a Sum with Combinatorics 1

1.1 Preliminaries 2

1.1.1 Addition to Factorials Theory 2

1.2 Interleavings of Two Lists 2

1.3 Cardinality of Distinct Fixed-Length Lists from a Union of Two Sets 4

2 Proving Falling Factorial of a Sum with Induction 5

3 Proving Falling Factorial of a Sum with Vandermonde Identity 5

4 Note on Copyright Licensing 5

1 Proving Falling Factorial of a Sum with Combinatorics

theory *Falling-Factorial-Sum-Combinatorics*
imports

Discrete-Summation.Factorials
Card-Partitions.Injectivity-Solver
begin

1.1 Preliminaries

1.1.1 Addition to Factorials Theory

lemma *card-lists-distinct-length-eq*:
assumes *finite A*
shows $card\ \{xs.\ length\ xs = n \wedge distinct\ xs \wedge set\ xs \subseteq A\} = ffact\ n\ (card\ A)$
$\langle proof \rangle$

1.2 Interleavings of Two Lists

inductive $interleavings :: 'a\ list \Rightarrow 'a\ list \Rightarrow 'a\ list \Rightarrow bool$
where
$interleavings\ []\ ys\ ys$
$|\ interleavings\ xs\ []\ xs$
$|\ interleavings\ xs\ ys\ zs \Longrightarrow interleavings\ (x\#xs)\ ys\ (x\#zs)$
$|\ interleavings\ xs\ ys\ zs \Longrightarrow interleavings\ xs\ (y\#ys)\ (y\#zs)$

lemma *interleaving-Nil-implies-eq1*:
assumes *interleavings xs ys zs*
assumes $xs = []$
shows $ys = zs$
$\langle proof \rangle$

lemma *interleaving-Nil-iff1*:
$interleavings\ []\ ys\ zs \longleftrightarrow (ys = zs)$
$\langle proof \rangle$

lemma *interleaving-Nil-implies-eq2*:
assumes *interleavings xs ys zs*
assumes $ys = []$
shows $xs = zs$
$\langle proof \rangle$

lemma *interleaving-Nil-iff2*:
$interleavings\ xs\ []\ zs \longleftrightarrow (xs = zs)$
$\langle proof \rangle$

lemma *interleavings-Cons*:
$\{zs.\ interleavings\ (x\#xs)\ (y\#ys)\ zs\} =$
$\{x\#zs|zs.\ interleavings\ xs\ (y\#ys)\ zs\} \cup \{y\#zs|zs.\ interleavings\ (x\#xs)\ ys\ zs\}$
(**is** $?S = ?expr$)
$\langle proof \rangle$

lemma *interleavings-filter*:
assumes $X \cap Y = \{\}$ $set\ zs \subseteq X \cup Y$

shows *interleavings* $[z{\leftarrow}zs\ .\ z \in X]$ $[z{\leftarrow}zs\ .\ z \in Y]$ *zs*
⟨*proof*⟩

lemma *interleavings-filter-eq1*:
assumes *interleavings xs ys zs*
assumes $(\forall x{\in}set\ xs.\ P\ x) \wedge (\forall y{\in}set\ ys.\ \neg\ P\ y)$
shows *filter P zs = xs*
⟨*proof*⟩

lemma *interleavings-filter-eq2*:
assumes *interleavings xs ys zs*
assumes $(\forall x{\in}set\ xs.\ \neg\ P\ x) \wedge (\forall y{\in}set\ ys.\ P\ y)$
shows *filter P zs = ys*
⟨*proof*⟩

lemma *interleavings-length*:
assumes *interleavings xs ys zs*
shows *length xs + length ys = length zs*
⟨*proof*⟩

lemma *interleavings-set*:
assumes *interleavings xs ys zs*
shows $set\ xs \cup set\ ys = set\ zs$
⟨*proof*⟩

lemma *interleavings-distinct*:
assumes *interleavings xs ys zs*
shows $distinct\ xs \wedge distinct\ ys \wedge set\ xs \cap set\ ys = \{\} \longleftrightarrow distinct\ zs$
⟨*proof*⟩

lemma *two-mutual-lists-induction*:
assumes $\bigwedge ys.\ P\ []\ ys$
assumes $\bigwedge xs.\ P\ xs\ []$
assumes $\bigwedge x\ xs\ y\ ys.\ P\ xs\ (y\#ys) \Longrightarrow P\ (x\#xs)\ ys \Longrightarrow P\ (x\#xs)\ (y\#ys)$
shows *P xs ys*
⟨*proof*⟩

lemma *finite-interleavings*:
$finite\ \{zs.\ interleavings\ xs\ ys\ zs\}$
⟨*proof*⟩

lemma *card-interleavings*:
assumes $set\ xs \cap set\ ys = \{\}$
shows $card\ \{zs.\ interleavings\ xs\ ys\ zs\} = (length\ xs + length\ ys\ choose\ (length\ xs))$
⟨*proof*⟩

1.3 Cardinality of Distinct Fixed-Length Lists from a Union of Two Sets

lemma *lists-distinct-union-by-interleavings*:
assumes $X \cap Y = \{\}$
shows $\{zs.\ length\ zs = n \wedge distinct\ zs \wedge set\ zs \subseteq X \cup Y\} = do\ \{$
$k \leftarrow \{0..n\};$
$xs \leftarrow \{xs.\ length\ xs = k \wedge distinct\ xs \wedge set\ xs \subseteq X\};$
$ys \leftarrow \{ys.\ length\ ys = n - k \wedge distinct\ ys \wedge set\ ys \subseteq Y\};$
$\{zs.\ interleavings\ xs\ ys\ zs\}$
$\}$ (**is** $?S = ?expr$)
$\langle proof \rangle$

lemma *interleavings-inject*:
assumes $(set\ xs \cup set\ xs') \cap (set\ ys \cup set\ ys') = \{\}$
assumes $interleavings\ xs\ ys\ zs\ interleavings\ xs'\ ys'\ zs'$
assumes $zs = zs'$
shows $xs = xs'$ **and** $ys = ys'$
$\langle proof \rangle$

lemma *injectivity*:
assumes $X \cap Y = \{\}$
assumes $k \in \{0..n\} \wedge k' \in \{0..n\}$
assumes $(length\ xs = k \wedge distinct\ xs \wedge set\ xs \subseteq X) \wedge (length\ xs' = k' \wedge distinct\ xs' \wedge set\ xs' \subseteq X)$
assumes $(length\ ys = n - k \wedge distinct\ ys \wedge set\ ys \subseteq Y) \wedge (length\ ys' = n - k' \wedge distinct\ ys' \wedge set\ ys' \subseteq Y)$
assumes $interleavings\ xs\ ys\ zs \wedge interleavings\ xs'\ ys'\ zs'$
assumes $zs = zs'$
shows $k = k'$ **and** $xs = xs'$ **and** $ys = ys'$
$\langle proof \rangle$

lemma *finite-length-distinct*: $finite\ X \Longrightarrow finite\ \{xs.\ length\ xs = k \wedge distinct\ xs \wedge set\ xs \subseteq X\}$
$\langle proof \rangle$

lemma *card-lists-distinct-length-eq-union*:
assumes $finite\ X\ finite\ Y\ X \cap Y = \{\}$
shows $card\ \{zs.\ length\ zs = n \wedge distinct\ zs \wedge set\ zs \subseteq X \cup Y\} =$
$(\sum k{=}0..n.\ (n\ choose\ k) * ffact\ k\ (card\ X) * ffact\ (n - k)\ (card\ Y))$
(**is** $card\ ?S = -$)
$\langle proof \rangle$

lemma
$ffact\ n\ (x + y) = (\sum k{=}0..n.\ (n\ choose\ k) * ffact\ k\ x * ffact\ (n - k)\ y)$
$\langle proof \rangle$

end

2 Proving Falling Factorial of a Sum with Induction

theory *Falling-Factorial-Sum-Induction*
imports
Discrete-Summation.Factorials
begin

Note the potentially special copyright license condition of the following proof.

lemma *ffact-add-nat*:
$ffact\ n\ (x + y) = (\sum k{=}0..n.\ (n\ choose\ k) * ffact\ k\ x * ffact\ (n - k)\ y)$
$\langle proof \rangle$

lemma *ffact-add*:
fixes $x\ y :: 'a::\{ab\text{-}group\text{-}add,\ comm\text{-}semiring\text{-}1\text{-}cancel,\ ring\text{-}1\}$
shows $ffact\ n\ (x + y) = (\sum k{=}0..n.\ of\text{-}nat\ (n\ choose\ k) * ffact\ k\ x * ffact\ (n - k)\ y)$
$\langle proof \rangle$

end

3 Proving Falling Factorial of a Sum with Vandermonde Identity

theory *Falling-Factorial-Sum-Vandermonde*
imports
Discrete-Summation.Factorials
begin

Note the potentially special copyright license condition of the following proof.

lemma *ffact-add-nat*:
shows $ffact\ k\ (n + m) = (\sum i \le k.\ (k\ choose\ i) * ffact\ i\ n * ffact\ (k - i)\ m)$
$\langle proof \rangle$

end

4 Note on Copyright Licensing

The initial material of the informal proof for this formalisation is provided on Mathematics Stack Exchange under the Creative Commons Attribution-ShareAlike 3.0 Unported license (CC BY-SA 3.0; https://creativecommons.org/licenses/by-sa/3.0/), which is pointed out on the the Mathematics Stack Exchange terms of use at https://stackexchange.com/legal/terms-of-service.

The two main proofs, the induction and the algebraic proof in this AFP entry are (even textually) very close to the initial material from Mathematics Stack Exchange.

In case the two Isabelle proofs are judged to build upon the main proofs from Mathematics Stack Exchange, the CC BY-SA 3.0 license requires that these proofs must be available under the same license, and hence, these proofs are consequently licensed under CC BY-SA 3.0. In case the two Isabelle proofs are not judged to build upon the material from Mathematics Stack Exchange, I as an author provide them under the 3-Clause BSD License (https://opensource.org/licenses/BSD-3-Clause) to allow their seemless integration into the Isabelle repository at any point in time.

All other content that does not build upon the material from Mathematics Stack Exchange is licensed under the 3-clause BSD License, and can be copied, moved or integrated in other work licensed under the 3-clause BSD License without further consideration of the different obligations of the existing copyright licensing.

References

[1] ajotatxe. Combinatorial proof of falling factorial and binomial theorem. Mathematics Stack Exchange. https://math.stackexchange.com/q/1271700 (version: 2015-05-07), author profile: https://math.stackexchange.com/users/132456/ajotatxe.

[2] grapher. Combinatorial proof of falling factorial and binomial theorem. Mathematics Stack Exchange. https://math.stackexchange.com/q/1271688 (version: 2016-08-27), author profile: https://math.stackexchange.com/users/199155/grapher.

[3] F. John. Combinatorial proof of falling factorial and binomial theorem. Mathematics Stack Exchange. https://math.stackexchange.com/q/2161558 (version: 2017-11-14), author profile: https://math.stackexchange.com/users/302692/foobaz-john.

[4] B. M. Scott. Combinatorial proof of falling factorial and binomial theorem. Mathematics Stack Exchange. https://math.stackexchange.com/q/1271983 (version: 2015-05-07), author profile: https://math.stackexchange.com/users/12042/brian-m-scott.