

The Falling Factorial of a Sum

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Abstract

This entry shows that the falling factorial of a sum can be computed with an expression using binomial coefficients and the falling factorial of its summands. The entry provides three different proofs: a combinatorial proof, an induction proof and an algebraic proof using the Vandermonde identity.

The three formalizations try to follow their informal presentations from a Mathematics Stack Exchange page [1, 2, 3, 4] as close as possible. The induction and algebraic formalization end up to be very close to their informal presentation, whereas the combinatorial proof first requires the introduction of list interleavings, and significant more detail than its informal presentation.

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1 Proving Falling Factorial of a Sum with Combinatorics

```
theory Falling-Factorial-Sum-Combinatorics
imports
```

Discrete-Summation.Factorials
Card-Partitions.Injectivity-Solver

begin

1.1 Preliminaries

1.1.1 Addition to Factorials Theory

lemma *card-lists-distinct-length-eq*:

assumes *finite A*

shows $\text{card } \{xs. \text{length } xs = n \wedge \text{distinct } xs \wedge \text{set } xs \subseteq A\} = \text{ffact } n \text{ (card } A)$

<proof>

1.2 Interleavings of Two Lists

inductive *interleavings* :: 'a list \Rightarrow 'a list \Rightarrow 'a list \Rightarrow bool

where

interleavings [] *ys ys*

| *interleavings xs* [] *xs*

| *interleavings xs ys zs* \Longrightarrow *interleavings (x#xs) ys (x#zs)*

| *interleavings xs ys zs* \Longrightarrow *interleavings xs (y#ys) (y#zs)*

lemma *interleaving-Nil-implies-eq1*:

assumes *interleavings xs ys zs*

assumes *xs = []*

shows *ys = zs*

<proof>

lemma *interleaving-Nil-iff1*:

interleavings [] ys zs \longleftrightarrow (*ys = zs*)

<proof>

lemma *interleaving-Nil-implies-eq2*:

assumes *interleavings xs ys zs*

assumes *ys = []*

shows *xs = zs*

<proof>

lemma *interleaving-Nil-iff2*:

interleavings xs [] zs \longleftrightarrow (*xs = zs*)

<proof>

lemma *interleavings-Cons*:

$\{zs. \text{interleavings } (x\#xs) (y\#ys) zs\} =$

$\{x\#zs | zs. \text{interleavings } xs (y\#ys) zs\} \cup \{y\#zs | zs. \text{interleavings } (x\#xs) ys zs\}$

(**is** ?S = ?expr)

<proof>

lemma *interleavings-filter*:

assumes $X \cap Y = \{\}$ *set zs* $\subseteq X \cup Y$

shows *interleavings* $[z \leftarrow zs . z \in X] [z \leftarrow zs . z \in Y] zs$
(*proof*)

lemma *interleavings-filter-eq1*:
assumes *interleavings* $xs\ ys\ zs$
assumes $(\forall x \in \text{set } xs. P\ x) \wedge (\forall y \in \text{set } ys. \neg P\ y)$
shows *filter* $P\ zs = xs$
(*proof*)

lemma *interleavings-filter-eq2*:
assumes *interleavings* $xs\ ys\ zs$
assumes $(\forall x \in \text{set } xs. \neg P\ x) \wedge (\forall y \in \text{set } ys. P\ y)$
shows *filter* $P\ zs = ys$
(*proof*)

lemma *interleavings-length*:
assumes *interleavings* $xs\ ys\ zs$
shows $\text{length } xs + \text{length } ys = \text{length } zs$
(*proof*)

lemma *interleavings-set*:
assumes *interleavings* $xs\ ys\ zs$
shows $\text{set } xs \cup \text{set } ys = \text{set } zs$
(*proof*)

lemma *interleavings-distinct*:
assumes *interleavings* $xs\ ys\ zs$
shows $\text{distinct } xs \wedge \text{distinct } ys \wedge \text{set } xs \cap \text{set } ys = \{\}$ \longleftrightarrow *distinct* zs
(*proof*)

lemma *two-mutual-lists-induction*:
assumes $\bigwedge ys. P\ []\ ys$
assumes $\bigwedge xs. P\ xs\ []$
assumes $\bigwedge x\ xs\ y\ ys. P\ xs\ (y\#\!ys) \implies P\ (x\#\!xs)\ ys \implies P\ (x\#\!xs)\ (y\#\!ys)$
shows $P\ xs\ ys$
(*proof*)

lemma *finite-interleavings*:
finite $\{zs. \text{interleavings } xs\ ys\ zs\}$
(*proof*)

lemma *card-interleavings*:
assumes $\text{set } xs \cap \text{set } ys = \{\}$
shows $\text{card } \{zs. \text{interleavings } xs\ ys\ zs\} = (\text{length } xs + \text{length } ys\ \text{choose } (\text{length } xs))$
(*proof*)

1.3 Cardinality of Distinct Fixed-Length Lists from a Union of Two Sets

lemma *lists-distinct-union-by-interleavings*:

assumes $X \cap Y = \{\}$
shows $\{zs. \text{length } zs = n \wedge \text{distinct } zs \wedge \text{set } zs \subseteq X \cup Y\} = \text{do } \{$
 $k \leftarrow \{0..n\};$
 $xs \leftarrow \{xs. \text{length } xs = k \wedge \text{distinct } xs \wedge \text{set } xs \subseteq X\};$
 $ys \leftarrow \{ys. \text{length } ys = n - k \wedge \text{distinct } ys \wedge \text{set } ys \subseteq Y\};$
 $\{zs. \text{interleavings } xs \ ys \ zs\}$
 $\} \text{ (is } ?S = ?\text{expr)}$
 $\langle \text{proof} \rangle$

lemma *interleavings-inject*:

assumes $(\text{set } xs \cup \text{set } xs') \cap (\text{set } ys \cup \text{set } ys') = \{\}$
assumes *interleavings* $xs \ ys \ zs \ \text{interleavings } xs' \ ys' \ zs'$
assumes $zs = zs'$
shows $xs = xs' \ \text{and} \ ys = ys'$
 $\langle \text{proof} \rangle$

lemma *injectivity*:

assumes $X \cap Y = \{\}$
assumes $k \in \{0..n\} \wedge k' \in \{0..n\}$
assumes $(\text{length } xs = k \wedge \text{distinct } xs \wedge \text{set } xs \subseteq X) \wedge (\text{length } xs' = k' \wedge \text{distinct } xs' \wedge \text{set } xs' \subseteq X)$
assumes $(\text{length } ys = n - k \wedge \text{distinct } ys \wedge \text{set } ys \subseteq Y) \wedge (\text{length } ys' = n - k' \wedge \text{distinct } ys' \wedge \text{set } ys' \subseteq Y)$
assumes *interleavings* $xs \ ys \ zs \ \wedge \ \text{interleavings } xs' \ ys' \ zs'$
assumes $zs = zs'$
shows $k = k' \ \text{and} \ xs = xs' \ \text{and} \ ys = ys'$
 $\langle \text{proof} \rangle$

lemma *card-lists-distinct-length-eq-union*:

assumes *finite* X *finite* Y $X \cap Y = \{\}$
shows $\text{card } \{zs. \text{length } zs = n \wedge \text{distinct } zs \wedge \text{set } zs \subseteq X \cup Y\} =$
 $(\sum k=0..n. (n \ \text{choose } k) * \text{ffact } k \ (\text{card } X) * \text{ffact } (n - k) \ (\text{card } Y))$
(is $\text{card } ?S = -)$
 $\langle \text{proof} \rangle$

lemma

$\text{ffact } n \ (x + y) = (\sum k=0..n. (n \ \text{choose } k) * \text{ffact } k \ x * \text{ffact } (n - k) \ y)$
 $\langle \text{proof} \rangle$

end

2 Proving Falling Factorial of a Sum with Induction

theory *Falling-Factorial-Sum-Induction*

```

imports
  Discrete-Summation.Factorials
begin

```

Note the potentially special copyright license condition of the following proof.

```

lemma ffact-add-nat:
  ffact  $n$   $(x + y) = (\sum k=0..n. (n \text{ choose } k) * \text{ffact } k \ x * \text{ffact } (n - k) \ y)$ 
  <proof>

```

```

lemma ffact-add:
  fixes  $x \ y :: 'a::\{ab\text{-group-add, comm-semiring-1-cancel, ring-1}\}$ 
  shows ffact  $n$   $(x + y) = (\sum k=0..n. \text{of-nat } (n \text{ choose } k) * \text{ffact } k \ x * \text{ffact } (n - k) \ y)$ 
  <proof>

```

```

end

```

3 Proving Falling Factorial of a Sum with Vandermonde Identity

```

theory Falling-Factorial-Sum-Vandermonde
imports
  Discrete-Summation.Factorials
begin

```

Note the potentially special copyright license condition of the following proof.

```

lemma ffact-add-nat:
  shows ffact  $k$   $(n + m) = (\sum i \leq k. (k \text{ choose } i) * \text{ffact } i \ n * \text{ffact } (k - i) \ m)$ 
  <proof>

```

```

end

```

4 Note on Copyright Licensing

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References

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