# The Falling Factorial of a Sum 

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#### Abstract

This entry shows that the falling factorial of a sum can be computed with an expression using binomial coefficients and the falling factorial of its summands. The entry provides three different proofs: a combinatorial proof, an induction proof and an algebraic proof using the Vandermonde identity.

The three formalizations try to follow their informal presentations from a Mathematics Stack Exchange page [1, 2, 3, 4] as close as possible. The induction and algebraic formalization end up to be very close to their informal presentation, whereas the combinatorial proof first requires the introduction of list interleavings, and significant more detail than its informal presentation.


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1 Proving Falling Factorial of a Sum with Combi- natorics
theory Falling-Factorial-Sum-Combinatorics
imports

Discrete-Summation.Factorials
Card-Partitions.Injectivity-Solver
begin

### 1.1 Preliminaries

### 1.1.1 Addition to Factorials Theory

lemma card-lists-distinct-length-eq: assumes finite $A$
shows card $\{x s$. length $x s=n \wedge$ distinct $x s \wedge$ set $x s \subseteq A\}=$ ffact $n(\operatorname{card} A)$
$\langle p r o o f\rangle$

### 1.2 Interleavings of Two Lists

inductive interleavings $::$ 'a list $\Rightarrow{ }^{\prime}$ 'a list $\Rightarrow{ }^{\prime}$ 'a list $\Rightarrow$ bool
where
interleavings [] ys ys
| interleavings xs [] xs
| interleavings xs ys zs $\Longrightarrow$ interleavings $(x \# x s)$ ys $(x \# z s)$
| interleavings $x s$ ys $z s \Longrightarrow$ interleavings $x s(y \# y s)(y \# z s)$
lemma interleaving-Nil-implies-eq1:
assumes interleavings xs ys zs
assumes $x s=[]$
shows $y s=z s$
$\langle p r o o f\rangle$
lemma interleaving-Nil-iff1:
interleavings [] ys zs $\longleftrightarrow(y s=z s)$
$\langle p r o o f\rangle$
lemma interleaving-Nil-implies-eq2:
assumes interleavings xs ys zs
assumes $y s=[]$
shows $x s=z s$
$\langle p r o o f\rangle$
lemma interleaving-Nil-iff2:
interleavings $x s[] z s \longleftrightarrow(x s=z s)$
$\langle$ proof $\rangle$
lemma interleavings-Cons:
$\{z s$. interleavings $(x \# x s)(y \# y s) z s\}=$
$\{x \# z s \mid z s$. interleavings $x s(y \# y s) z s\} \cup\{y \# z s \mid z s$. interleavings $(x \# x s) y s z s\}$ (is ? $S=$ ? expr $)$
〈proof〉
lemma interleavings-filter:
assumes $X \cap Y=\{ \}$ set $z s \subseteq X \cup Y$

```
    shows interleavings [z\leftarrowzs.z\inX][z\leftarrowzs.z\inY]zs
<proof\rangle
lemma interleavings-filter-eq1:
    assumes interleavings xs ys zs
    assumes ( }\forallx\in\mathrm{ set xs. P x)}\wedge(\forally\in\mathrm{ set ys. ᄀP y)
    shows filter P zs = xs
<proof>
lemma interleavings-filter-eq2:
    assumes interleavings xs ys zs
    assumes ( }\forallx\in\mathrm{ set xs. ᄀ P x)^( }\forally\in\mathrm{ set ys. P y)
    shows filter P zs = ys
<proof>
lemma interleavings-length:
    assumes interleavings xs ys zs
    shows length xs + length ys = length zs
<proof>
lemma interleavings-set:
    assumes interleavings xs ys zs
    shows set xs U set ys = set zs
<proof\rangle
lemma interleavings-distinct:
    assumes interleavings xs ys zs
    shows distinct xs ^ distinct ys }\wedge\mathrm{ set xs }\cap\mathrm{ set ys = {} }\longleftrightarrow\mathrm{ distinct zs
<proof>
lemma two-mutual-lists-induction:
    assumes \ys. P [] ys
    assumes \xs. P xs []
    assumes \x xs y ys.P xs (y#ys)\LongrightarrowP(x#xs) ys \LongrightarrowP(x#xs)(y#ys)
    shows P xs ys
<proof\rangle
lemma finite-interleavings:
    finite {zs. interleavings xs ys zs}
<proof>
lemma card-interleavings:
    assumes set xs \cap set ys = {}
    shows card {zs. interleavings xs ys zs} = (length xs + length ys choose (length
xs))
<proof\rangle
```


### 1.3 Cardinality of Distinct Fixed-Length Lists from a Union of Two Sets

```
lemma lists-distinct-union-by-interleavings:
    assumes \(X \cap Y=\{ \}\)
    shows \(\{z s\). length \(z s=n \wedge\) distinct zs \(\wedge\) set \(z s \subseteq X \cup Y\}=d o\{\)
        \(k \leftarrow\{0 . . n\} ;\)
        \(x s \leftarrow\{x s\). length \(x s=k \wedge\) distinct \(x s \wedge\) set \(x s \subseteq X\}\);
        \(y s \leftarrow\{\) ys. length ys \(=n-k \wedge\) distinct \(y s \wedge\) set \(y s \subseteq Y\} ;\)
        \(\{z s\). interleavings xs ys \(z s\}\)
    \(\}(\) is ? \(S=\) ? \(\operatorname{expr})\)
〈proof〉
lemma interleavings-inject:
    assumes \(\left(\right.\) set \(x s \cup\) set \(\left.x s^{\prime}\right) \cap\left(\right.\) set ys \(\cup\) set \(\left.y s^{\prime}\right)=\{ \}\)
    assumes interleavings \(x s\) ys zs interleavings \(x s^{\prime} y s^{\prime} z s^{\prime}\)
    assumes \(z s=z s^{\prime}\)
    shows \(x s=x s^{\prime}\) and \(y s=y s^{\prime}\)
\(\langle p r o o f\rangle\)
```

lemma injectivity:
assumes $X \cap Y=\{ \}$
assumes $k \in\{0 . . n\} \wedge k^{\prime} \in\{0 . . n\}$
assumes (length $x s=k \wedge$ distinct $x s \wedge$ set $x s \subseteq X) \wedge\left(\right.$ length $x s^{\prime}=k^{\prime} \wedge$ distinct
$x s^{\prime} \wedge$ set $\left.x s^{\prime} \subseteq X\right)$
assumes (length ys $=n-k \wedge$ distinct ys $\wedge$ set ys $\subseteq Y) \wedge\left(\right.$ length $y^{\prime}=n-k^{\prime}$
$\wedge$ distinct $y s^{\prime} \wedge$ set $\left.y s^{\prime} \subseteq Y\right)$
assumes interleavings xs ys zs $\wedge$ interleavings $x s^{\prime} y s^{\prime} z s^{\prime}$
assumes $z s=z s^{\prime}$
shows $k=k^{\prime}$ and $x s=x s^{\prime}$ and $y s=y s^{\prime}$
$\langle p r o o f\rangle$
lemma card-lists-distinct-length-eq-union:
assumes finite $X$ finite $Y X \cap Y=\{ \}$
shows card $\{z s$. length $z s=n \wedge$ distinct $z s \wedge$ set $z s \subseteq X \cup Y\}=$
$\left(\sum k=0 . . n .(n\right.$ choose $k) *$ ffact $k($ card $X) *$ ffact $(n-k)($ card $\left.Y)\right)$
(is card ? $S=-$ )
$\langle p r o o f\rangle$
lemma
ffact $n(x+y)=\left(\sum k=0 . . n .(n\right.$ choose $k) *$ ffact $k x *$ ffact $\left.(n-k) y\right)$
$\langle$ proof $\rangle$
end

## 2 Proving Falling Factorial of a Sum with Induction

theory Falling-Factorial-Sum-Induction

## imports <br> Discrete-Summation.Factorials <br> begin

Note the potentially special copyright license condition of the following proof.
lemma ffact-add-nat:
ffact $n(x+y)=\left(\sum k=0 . . n .(n\right.$ choose $k) *$ ffact $k x *$ ffact $\left.(n-k) y\right)$〈proof〉
lemma ffact-add:
fixes $x y$ :: 'a::\{ab-group-add, comm-semiring-1-cancel, ring-1\}
shows ffact $n(x+y)=\left(\sum k=0 . . n\right.$. of-nat $(n$ choose $k) *$ ffact $k x * f f a c t(n-$ k) $y$ )
$\langle p r o o f\rangle$
end

## 3 Proving Falling Factorial of a Sum with Vandermonde Identity

theory Falling-Factorial-Sum-Vandermonde imports<br>Discrete-Summation.Factorials<br>begin

Note the potentially special copyright license condition of the following proof.
lemma ffact-add-nat:
shows ffact $k(n+m)=\left(\sum i \leq k .(k\right.$ choose $i) *$ ffact $i n *$ ffact $\left.(k-i) m\right)$
$\langle p r o o f\rangle$
end

## 4 Note on Copyright Licensing

The initial material of the informal proof for this formalisation is provided on Mathematics Stack Exchange under the Creative Commons AttributionShareAlike 3.0 Unported license (CC BY-SA 3.0; https://creativecommons. org/licenses/by-sa/3.0/), which is pointed out on the the Mathematics Stack Exchange terms of use at https://stackexchange.com/legal/terms-of-service. The two main proofs, the induction and the algebraic proof in this AFP entry are (even textually) very close to the initial material from Mathematics Stack Exchange.

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## References

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[2] grapher. Combinatorial proof of falling factorial and binomial theorem. Mathematics Stack Exchange. https://math.stackexchange. com/q/1271688 (version: 2016-08-27), author profile: https://math. stackexchange.com/users/199155/grapher.
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