The Falling Factorial of a Sum

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Abstract

This entry shows that the falling factorial of a sum can be computed with an expression using binomial coefficients and the falling factorial of its summands. The entry provides three different proofs: a combinatorial proof, an induction proof and an algebraic proof using the Vandermonde identity.

The three formalizations try to follow their informal presentations from a Mathematics Stack Exchange page [1, 2, 3, 4] as close as possible. The induction and algebraic formalization end up to be very close to their informal presentation, whereas the combinatorial proof first requires the introduction of list interleavings, and significant more detail than its informal presentation.

Contents

1 Proving Falling Factorial of a Sum with Combinatorics

theory *Falling-Factorial-Sum-Combinatorics*
imports

Discrete-Summation.Factorials
Card-Partitions.Injectivity-Solver
begin

1.1 Preliminaries

1.1.1 Addition to Factorials Theory

lemma *card-lists-distinct-length-eq*:
assumes *finite A*
shows $card\ \{xs.\ length\ xs = n \wedge distinct\ xs \wedge set\ xs \subseteq A\} = ffact\ n\ (card\ A)$
$\langle proof \rangle$

1.2 Interleavings of Two Lists

inductive $interleavings :: 'a\ list \Rightarrow 'a\ list \Rightarrow 'a\ list \Rightarrow bool$
where
$interleavings\ []\ ys\ ys$
$|\ interleavings\ xs\ []\ xs$
$|\ interleavings\ xs\ ys\ zs \Longrightarrow interleavings\ (x\#xs)\ ys\ (x\#zs)$
$|\ interleavings\ xs\ ys\ zs \Longrightarrow interleavings\ xs\ (y\#ys)\ (y\#zs)$

lemma *interleaving-Nil-implies-eq1*:
assumes *interleavings xs ys zs*
assumes $xs = []$
shows $ys = zs$
$\langle proof \rangle$

lemma *interleaving-Nil-iff1*:
$interleavings\ []\ ys\ zs \longleftrightarrow (ys = zs)$
$\langle proof \rangle$

lemma *interleaving-Nil-implies-eq2*:
assumes *interleavings xs ys zs*
assumes $ys = []$
shows $xs = zs$
$\langle proof \rangle$

lemma *interleaving-Nil-iff2*:
$interleavings\ xs\ []\ zs \longleftrightarrow (xs = zs)$
$\langle proof \rangle$

lemma *interleavings-Cons*:
$\{zs.\ interleavings\ (x\#xs)\ (y\#ys)\ zs\} =$
$\{x\#zs|zs.\ interleavings\ xs\ (y\#ys)\ zs\} \cup \{y\#zs|zs.\ interleavings\ (x\#xs)\ ys\ zs\}$
(**is** $?S = ?expr$)
$\langle proof \rangle$

lemma *interleavings-filter*:
assumes $X \cap Y = \{\}$ $set\ zs \subseteq X \cup Y$

shows $\mathit{interleavings}\ [z{\leftarrow}zs\ .\ z \in X]\ [z{\leftarrow}zs\ .\ z \in Y]\ zs$
⟨*proof*⟩

lemma *interleavings-filter-eq1*:
assumes $\mathit{interleavings}\ xs\ ys\ zs$
assumes $(\forall x{\in}\mathit{set}\ xs.\ P\ x) \wedge (\forall y{\in}\mathit{set}\ ys.\ \neg\ P\ y)$
shows $\mathit{filter}\ P\ zs = xs$
⟨*proof*⟩

lemma *interleavings-filter-eq2*:
assumes $\mathit{interleavings}\ xs\ ys\ zs$
assumes $(\forall x{\in}\mathit{set}\ xs.\ \neg\ P\ x) \wedge (\forall y{\in}\mathit{set}\ ys.\ P\ y)$
shows $\mathit{filter}\ P\ zs = ys$
⟨*proof*⟩

lemma *interleavings-length*:
assumes $\mathit{interleavings}\ xs\ ys\ zs$
shows $\mathit{length}\ xs + \mathit{length}\ ys = \mathit{length}\ zs$
⟨*proof*⟩

lemma *interleavings-set*:
assumes $\mathit{interleavings}\ xs\ ys\ zs$
shows $\mathit{set}\ xs \cup \mathit{set}\ ys = \mathit{set}\ zs$
⟨*proof*⟩

lemma *interleavings-distinct*:
assumes $\mathit{interleavings}\ xs\ ys\ zs$
shows $\mathit{distinct}\ xs \wedge \mathit{distinct}\ ys \wedge \mathit{set}\ xs \cap \mathit{set}\ ys = \{\} \longleftrightarrow \mathit{distinct}\ zs$
⟨*proof*⟩

lemma *two-mutual-lists-induction*:
assumes $\bigwedge ys.\ P\ []\ ys$
assumes $\bigwedge xs.\ P\ xs\ []$
assumes $\bigwedge x\ xs\ y\ ys.\ P\ xs\ (y\#ys) \Longrightarrow P\ (x\#xs)\ ys \Longrightarrow P\ (x\#xs)\ (y\#ys)$
shows $P\ xs\ ys$
⟨*proof*⟩

lemma *finite-interleavings*:
$\mathit{finite}\ \{zs.\ \mathit{interleavings}\ xs\ ys\ zs\}$
⟨*proof*⟩

lemma *card-interleavings*:
assumes $\mathit{set}\ xs \cap \mathit{set}\ ys = \{\}$
shows $\mathit{card}\ \{zs.\ \mathit{interleavings}\ xs\ ys\ zs\} = (\mathit{length}\ xs + \mathit{length}\ ys\ \mathit{choose}\ (\mathit{length}\ xs))$
⟨*proof*⟩

1.3 Cardinality of Distinct Fixed-Length Lists from a Union of Two Sets

lemma *lists-distinct-union-by-interleavings*:
assumes $X \cap Y = \{\}$
shows $\{zs.\ length\ zs = n \wedge distinct\ zs \wedge set\ zs \subseteq X \cup Y\} = do\ \{$
$k \leftarrow \{0..n\};$
$xs \leftarrow \{xs.\ length\ xs = k \wedge distinct\ xs \wedge set\ xs \subseteq X\};$
$ys \leftarrow \{ys.\ length\ ys = n - k \wedge distinct\ ys \wedge set\ ys \subseteq Y\};$
$\{zs.\ interleavings\ xs\ ys\ zs\}$
$\}$ (**is** $?S = ?expr$)
⟨*proof*⟩

lemma *interleavings-inject*:
assumes $(set\ xs \cup set\ xs') \cap (set\ ys \cup set\ ys') = \{\}$
assumes $interleavings\ xs\ ys\ zs\ interleavings\ xs'\ ys'\ zs'$
assumes $zs = zs'$
shows $xs = xs'$ **and** $ys = ys'$
⟨*proof*⟩

lemma *injectivity*:
assumes $X \cap Y = \{\}$
assumes $k \in \{0..n\} \wedge k' \in \{0..n\}$
assumes $(length\ xs = k \wedge distinct\ xs \wedge set\ xs \subseteq X) \wedge (length\ xs' = k' \wedge distinct\ xs' \wedge set\ xs' \subseteq X)$
assumes $(length\ ys = n - k \wedge distinct\ ys \wedge set\ ys \subseteq Y) \wedge (length\ ys' = n - k' \wedge distinct\ ys' \wedge set\ ys' \subseteq Y)$
assumes $interleavings\ xs\ ys\ zs \wedge interleavings\ xs'\ ys'\ zs'$
assumes $zs = zs'$
shows $k = k'$ **and** $xs = xs'$ **and** $ys = ys'$
⟨*proof*⟩

lemma *finite-length-distinct*: $finite\ X \Longrightarrow finite\ \{xs.\ length\ xs = k \wedge distinct\ xs \wedge set\ xs \subseteq X\}$
⟨*proof*⟩

lemma *card-lists-distinct-length-eq-union*:
assumes $finite\ X\ finite\ Y\ X \cap Y = \{\}$
shows $card\ \{zs.\ length\ zs = n \wedge distinct\ zs \wedge set\ zs \subseteq X \cup Y\} =$
$(\sum k{=}0..n.\ (n\ choose\ k) * ffact\ k\ (card\ X) * ffact\ (n - k)\ (card\ Y))$
(**is** $card\ ?S = -$)
⟨*proof*⟩

lemma
$ffact\ n\ (x + y) = (\sum k{=}0..n.\ (n\ choose\ k) * ffact\ k\ x * ffact\ (n - k)\ y)$
⟨*proof*⟩

end

2 Proving Falling Factorial of a Sum with Induction

theory *Falling-Factorial-Sum-Induction*
imports
Discrete-Summation.Factorials
begin

Note the potentially special copyright license condition of the following proof.

lemma *ffact-add-nat*:
$ffact\ n\ (x + y) = (\sum k{=}0..n.\ (n\ choose\ k) * ffact\ k\ x * ffact\ (n - k)\ y)$
$\langle proof \rangle$

lemma *ffact-add*:
fixes $x\ y :: 'a::\{ab\text{-}group\text{-}add,\ comm\text{-}semiring\text{-}1\text{-}cancel,\ ring\text{-}1\}$
shows $ffact\ n\ (x + y) = (\sum k{=}0..n.\ of\text{-}nat\ (n\ choose\ k) * ffact\ k\ x * ffact\ (n - k)\ y)$
$\langle proof \rangle$

end

3 Proving Falling Factorial of a Sum with Vandermonde Identity

theory *Falling-Factorial-Sum-Vandermonde*
imports
Discrete-Summation.Factorials
begin

Note the potentially special copyright license condition of the following proof.

lemma *ffact-add-nat*:
shows $ffact\ k\ (n + m) = (\sum i{\le}k.\ (k\ choose\ i) * ffact\ i\ n * ffact\ (k - i)\ m)$
$\langle proof \rangle$

end

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