The Falling Factorial of a Sum

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Abstract

This entry shows that the falling factorial of a sum can be computed with an expression using binomial coefficients and the falling factorial of its summands. The entry provides three different proofs: a combinatorial proof, an induction proof and an algebraic proof using the Vandermonde identity.

The three formalizations try to follow their informal presentations from a Mathematics Stack Exchange page [1, 2, 3, 4] as close as possible. The induction and algebraic formalization end up to be very close to their informal presentation, whereas the combinatorial proof first requires the introduction of list interleavings, and significant more detail than its informal presentation.

Contents

1 Proving Falling Factorial of a Sum with Combinatorics

theory *Falling-Factorial-Sum-Combinatorics*
imports

Discrete-Summation.Factorials
Card-Partitions.Injectivity-Solver
begin

1.1 Preliminaries

1.1.1 Addition to Factorials Theory

lemma *card-lists-distinct-length-eq*:
assumes *finite A*
shows $card\ \{xs.\ length\ xs = n \wedge distinct\ xs \wedge set\ xs \subseteq A\} = ffact\ n\ (card\ A)$
proof *cases*
assume $n \leq card\ A$
have $card\ \{xs.\ length\ xs = n \wedge distinct\ xs \wedge set\ xs \subseteq A\} = \prod\{card\ A - n + 1..card\ A\}$
using ‹*finite A*› ‹$n \leq card\ A$› **by** (*rule card-lists-distinct-length-eq*)
also have $\ldots = ffact\ n\ (card\ A)$
using ‹$n \leq card\ A$› **by** (*simp add*: *prod-rev-ffact-nat*′[*symmetric*])
finally show *?thesis* **.**
next
assume $\neg\ n \leq card\ A$
from *this* ‹*finite A*› **have** $\forall xs.\ length\ xs = n \wedge distinct\ xs \wedge set\ xs \subseteq A \longrightarrow False$
by (*metis card-mono distinct-card*)
from *this* **have** *eq-empty*: $\{xs.\ length\ xs = n \wedge distinct\ xs \wedge set\ xs \subseteq A\} = \{\}$
using ‹*finite A*› **by** *auto*
from ‹$\neg\ n \leq card\ A$› **show** *?thesis*
by (*simp add*: *ffact-nat-triv eq-empty*)
qed

1.2 Interleavings of Two Lists

inductive *interleavings* :: ′*a list* ⇒ ′*a list* ⇒ ′*a list* ⇒ *bool*
where
interleavings [] *ys ys*
| *interleavings xs* [] *xs*
| *interleavings xs ys zs* ⟹ *interleavings* (*x*#*xs*) *ys* (*x*#*zs*)
| *interleavings xs ys zs* ⟹ *interleavings xs* (*y*#*ys*) (*y*#*zs*)

lemma *interleaving-Nil-implies-eq1*:
assumes *interleavings xs ys zs*
assumes *xs* = []
shows *ys* = *zs*
using *assms* **by** (*induct rule*: *interleavings.induct*) *auto*

lemma *interleaving-Nil-iff1*:
interleavings [] *ys zs* ⟷ (*ys* = *zs*)
using *interleaving-Nil-implies-eq1*
by (*auto simp add*: *interleavings.intros*(*1*))

lemma *interleaving-Nil-implies-eq2*:
  assumes *interleavings xs ys zs*
  assumes $ys = []$
  shows $xs = zs$
using *assms* **by** (*induct rule*: *interleavings.induct*) *auto*

lemma *interleaving-Nil-iff2*:
  interleavings xs [] *zs* $\longleftrightarrow$ ($xs = zs$)
using *interleaving-Nil-implies-eq2*
by (*auto simp add*: *interleavings.intros*(*2*))

lemma *interleavings-Cons*:
  $\{zs.\ interleavings\ (x\#xs)\ (y\#ys)\ zs\} =$
    $\{x\#zs|zs.\ interleavings\ xs\ (y\#ys)\ zs\} \cup \{y\#zs|zs.\ interleavings\ (x\#xs)\ ys\ zs\}$
  (**is** $?S = ?expr$)
proof
  show $?S \subseteq ?expr$
    by (*auto elim*: *interleavings.cases*)
next
  show $?expr \subseteq ?S$
    by (*auto intro*: *interleavings.intros*)
qed

lemma *interleavings-filter*:
  assumes $X \cap Y = \{\}$ $set\ zs \subseteq X \cup Y$
  shows *interleavings* $[z{\leftarrow}zs\ .\ z \in X]$ $[z{\leftarrow}zs\ .\ z \in Y]$ *zs*
using *assms* **by** (*induct zs*) (*auto intro*: *interleavings.intros*)

lemma *interleavings-filter-eq1*:
  assumes *interleavings xs ys zs*
  assumes $(\forall x{\in}set\ xs.\ P\ x) \wedge (\forall y{\in}set\ ys.\ \neg\ P\ y)$
  shows $filter\ P\ zs = xs$
using *assms* **by** (*induct rule*: *interleavings.induct*) *auto*

lemma *interleavings-filter-eq2*:
  assumes *interleavings xs ys zs*
  assumes $(\forall x{\in}set\ xs.\ \neg\ P\ x) \wedge (\forall y{\in}set\ ys.\ P\ y)$
  shows $filter\ P\ zs = ys$
using *assms* **by** (*induct rule*: *interleavings.induct*) *auto*

lemma *interleavings-length*:
  assumes *interleavings xs ys zs*
  shows $length\ xs + length\ ys = length\ zs$
using *assms* **by** (*induct xs ys zs rule*: *interleavings.induct*) *auto*

lemma *interleavings-set*:
  assumes *interleavings xs ys zs*
  shows $set\ xs \cup set\ ys = set\ zs$
using *assms* **by** (*induct xs ys zs rule*: *interleavings.induct*) *auto*

lemma *interleavings-distinct*:
  assumes *interleavings xs ys zs*
  shows $distinct\ xs \wedge distinct\ ys \wedge set\ xs \cap set\ ys = \{\} \longleftrightarrow distinct\ zs$
using *assms interleavings-set* **by** (*induct xs ys zs rule*: *interleavings.induct*) *fast-force+*

lemma *two-mutual-lists-induction*:
  assumes $\bigwedge ys.\ P\ []\ ys$
  assumes $\bigwedge xs.\ P\ xs\ []$
  assumes $\bigwedge x\ xs\ y\ ys.\ P\ xs\ (y\#ys) \Longrightarrow P\ (x\#xs)\ ys \Longrightarrow P\ (x\#xs)\ (y\#ys)$
  shows *P xs ys*
using *assms* **by** (*induction-schema*) (*pat-completeness*, *lexicographic-order*)

lemma *finite-interleavings*:
  finite {*zs. interleavings xs ys zs*}
proof (*induct xs ys rule*: *two-mutual-lists-induction*)
  case (*1 ys*)
  show *?case* **by** (*simp add*: *interleaving-Nil-iff1*)
next
  case (*2 xs*)
  then show *?case* **by** (*simp add*: *interleaving-Nil-iff2*)
next
  case (*3 x xs y ys*)
  then show *?case* **by** (*simp add*: *interleavings-Cons*)
qed

lemma *card-interleavings*:
  assumes $set\ xs \cap set\ ys = \{\}$
  shows $card\ \{zs.\ interleavings\ xs\ ys\ zs\} = (length\ xs + length\ ys\ choose\ (length\ xs))$
using *assms*
proof (*induct xs ys rule*: *two-mutual-lists-induction*)
  case (*1 ys*)
  have $card\ \{zs.\ interleavings\ []\ ys\ zs\} = card\ \{ys\}$
    by (*simp add*: *interleaving-Nil-iff1*)
  also have $\ldots = (length\ [] + length\ ys\ choose\ (length\ []))$ **by** *simp*
  finally show *?case* **.**
next
  case (*2 xs*)
  have $card\ \{zs.\ interleavings\ xs\ []\ zs\} = card\ \{xs\}$
    by (*simp add*: *interleaving-Nil-iff2*)
  also have $\ldots = (length\ xs + length\ []\ choose\ (length\ xs))$ **by** *simp*
  finally show *?case* **.**
next
  case (*3 x xs y ys*)
  have $card\ \{zs.\ interleavings\ (x\ \#\ xs)\ (y\ \#\ ys)\ zs\} =$
    $card\ (\{x\#zs|zs.\ interleavings\ xs\ (y\#ys)\ zs\} \cup \{y\#zs|zs.\ interleavings\ (x\#xs)\ ys\ zs\})$

by (*simp add*: *interleavings-Cons*)
also have ... = *card* $\{x\#zs|zs.$ *interleavings xs* $(y\#ys)$ *zs*$\}$ + *card* $\{y\#zs|zs.$ *interleavings* $(x\#xs)$ *ys zs*$\}$
proof −
have *finite* $\{x \# zs |zs.$ *interleavings xs* $(y \# ys)$ *zs*$\}$
by (*simp add*: *finite-interleavings*)
moreover have *finite* $\{y \# zs |zs.$ *interleavings* $(x \# xs)$ *ys zs*$\}$
by (*simp add*: *finite-interleavings*)
moreover have $\{x \# zs |zs.$ *interleavings xs* $(y \# ys)$ *zs*$\} \cap \{y \# zs |zs.$ *interleavings* $(x \# xs)$ *ys zs*$\} = \{\}$
using ‹*set* $(x \# xs) \cap$ *set* $(y \# ys) = \{\}$› **by** *auto*
ultimately show *?thesis* **by** (*simp add*: *card-Un-disjoint*)
qed
also have ... = *card* $((\lambda zs.\ x \# zs)$ ‘ $\{zs.$ *interleavings xs* $(y \# ys)$ *zs*$\})$ +
card $((\lambda zs.\ y \# zs)$ ‘ $\{zs.$ *interleavings* $(x\#xs)$ *ys zs*$\})$
by (*simp add*: *setcompr-eq-image*)
also have ... = *card* $\{zs.$ *interleavings xs* $(y \# ys)$ *zs*$\}$ + *card* $\{zs.$ *interleavings* $(x\#xs)$ *ys zs*$\}$
by (*simp add*: *card-image*)
also have ... = (*length xs* + *length* $(y \# ys)$ *choose length xs*) + (*length* $(x \# xs)$ + *length ys choose length* $(x \# xs)$)
using *3* **by** *simp*
also have ... = *length* $(x \# xs)$ + *length* $(y \# ys)$ *choose length* $(x \# xs)$ **by** *simp*
finally show *?case* **.**
qed

1.3 Cardinality of Distinct Fixed-Length Lists from a Union of Two Sets

lemma *lists-distinct-union-by-interleavings*:
assumes $X \cap Y = \{\}$
shows $\{zs.$ *length zs* $= n \wedge$ *distinct zs* $\wedge$ *set zs* $\subseteq X \cup Y\}$ = *do* {
$k \leftarrow \{0..n\}$;
$xs \leftarrow \{xs.$ *length xs* $= k \wedge$ *distinct xs* $\wedge$ *set xs* $\subseteq X\}$;
$ys \leftarrow \{ys.$ *length ys* $= n - k \wedge$ *distinct ys* $\wedge$ *set ys* $\subseteq Y\}$;
$\{zs.$ *interleavings xs ys zs*$\}$
} (**is** *?S* = *?expr*)
proof
show *?S* $\subseteq$ *?expr*
proof
fix *zs*
assume *zs* $\in$ *?S*
from *this* **have** *length zs* $= n$ **and** *distinct zs* **and** *set zs* $\subseteq X \cup Y$ **by** *auto*
define *xs* **where** *xs* = *filter* $(\lambda z.\ z \in X)$ *zs*
define *ys* **where** *ys* = *filter* $(\lambda z.\ z \in Y)$ *zs*
have *eq*: $[z\leftarrow zs\ .\ z \in Y] = [z\leftarrow zs\ .\ z \notin X]$
using ‹*set zs* $\subseteq X \cup Y$› ‹$X \cap Y = \{\}$›
by (*auto intro*: *filter-cong*)

have *length xs* $\leq$ *n* $\wedge$ *distinct xs* $\wedge$ *set xs* $\subseteq$ *X*
using ‹*length zs* = *n*› ‹*distinct zs*› **unfolding** *xs-def* **by** *auto*
moreover have *length ys* = *n* − *length xs*
using ‹*set zs* $\subseteq$ *X* $\cup$ *Y*› ‹*length zs* = *n*›
unfolding *xs-def ys-def eq*
by (*metis diff-add-inverse sum-length-filter-compl*)
moreover have *distinct ys* $\wedge$ *set ys* $\subseteq$ *Y*
using ‹*distinct zs*› **unfolding** *ys-def* **by** *auto*
moreover have *interleavings xs ys zs*
using *xs-def ys-def* ‹*X* $\cap$ *Y* = {}› ‹*set zs* $\subseteq$ *X* $\cup$ *Y*›
by (*simp add*: *interleavings-filter*)
ultimately show *zs* $\in$ *?expr* **by** *force*
qed
next
show *?expr* $\subseteq$ *?S*
proof
fix *zs*
assume *zs* $\in$ *?expr*
from *this* **obtain** *xs ys* **where** *length xs* $\leq$ *n distinct xs set xs* $\subseteq$ *X*
and *length ys* = *n* − *length xs distinct ys set ys* $\subseteq$ *Y interleavings xs ys zs* **by** *auto*
have *length zs* = *n*
using ‹*length xs* $\leq$ *n*› ‹*length ys* = *n* − *length xs*› ‹*interleavings xs ys zs*›
using *interleavings-length* **by** *force*
moreover have *distinct zs*
using ‹*distinct xs*› ‹*distinct ys*› ‹*interleavings xs ys zs*› ‹*set xs* $\subseteq$ *X*› ‹*set ys* $\subseteq$ *Y*›
using ‹*X* $\cap$ *Y* = {}› *interleavings-distinct* **by** *fastforce*
moreover have *set zs* $\subseteq$ *X* $\cup$ *Y*
using ‹*interleavings xs ys zs*› ‹*set xs* $\subseteq$ *X*› ‹*set ys* $\subseteq$ *Y*› *interleavings-set* **by** *blast*
ultimately show *zs* $\in$ *?S* **by** *blast*
qed
qed

lemma *interleavings-inject*:
assumes (*set xs* $\cup$ *set xs'*) $\cap$ (*set ys* $\cup$ *set ys'*) = {}
assumes *interleavings xs ys zs interleavings xs' ys' zs'*
assumes *zs* = *zs'*
shows *xs* = *xs'* **and** *ys* = *ys'*
proof −
have *xs* = *filter* (λ*z*. *z* $\in$ *set xs* $\cup$ *set xs'*) *zs*
using ‹(*set xs* $\cup$ *set xs'*) $\cap$ (*set ys* $\cup$ *set ys'*) = {}› ‹*interleavings xs ys zs*›
by (*auto intro*: *interleavings-filter-eq1*[*symmetric*])
also have ... = *filter* (λ*z*. *z* $\in$ *set xs* $\cup$ *set xs'*) *zs'*
using ‹*zs* = *zs'*› **by** *simp*
also have ... = *xs'*
using ‹(*set xs* $\cup$ *set xs'*) $\cap$ (*set ys* $\cup$ *set ys'*) = {}› ‹*interleavings xs' ys' zs'*›
by (*auto intro*: *interleavings-filter-eq1*)

finally show $xs = xs'$ by *simp*
have $ys = filter\ (\lambda z.\ z \in set\ ys \cup set\ ys')\ zs$
using ‹$(set\ xs \cup set\ xs') \cap (set\ ys \cup set\ ys') = \{\}$› ‹$interleavings\ xs\ ys\ zs$›
by (*auto intro*: *interleavings-filter-eq2*[*symmetric*])
also have $\ldots = filter\ (\lambda z.\ z \in set\ ys \cup set\ ys')\ zs'$
using ‹$zs = zs'$› by *simp*
also have $\ldots = ys'$
using ‹$(set\ xs \cup set\ xs') \cap (set\ ys \cup set\ ys') = \{\}$› ‹$interleavings\ xs'\ ys'\ zs'$›
by (*auto intro*: *interleavings-filter-eq2*)
finally show $ys = ys'$.
qed

lemma *injectivity*:
assumes $X \cap Y = \{\}$
assumes $k \in \{0..n\} \wedge k' \in \{0..n\}$
assumes $(length\ xs = k \wedge distinct\ xs \wedge set\ xs \subseteq X) \wedge (length\ xs' = k' \wedge distinct\ xs' \wedge set\ xs' \subseteq X)$
assumes $(length\ ys = n - k \wedge distinct\ ys \wedge set\ ys \subseteq Y) \wedge (length\ ys' = n - k' \wedge distinct\ ys' \wedge set\ ys' \subseteq Y)$
assumes $interleavings\ xs\ ys\ zs \wedge interleavings\ xs'\ ys'\ zs'$
assumes $zs = zs'$
shows $k = k'$ and $xs = xs'$ and $ys = ys'$
proof −
from *assms*(*1*,*3*,*4*) have $(set\ xs \cup set\ xs') \cap (set\ ys \cup set\ ys') = \{\}$ by *blast*
from *this assms*(*5*) ‹$zs = zs'$› show $xs = xs'$ and $ys = ys'$
using *interleavings-inject* by *fastforce*+
from *this assms*(*3*) show $k = k'$ by *auto*
qed

lemma *finite-length-distinct*: $finite\ X \Longrightarrow finite\ \{xs.\ length\ xs = k \wedge distinct\ xs \wedge set\ xs \subseteq X\}$
by(*fast elim*: *rev-finite-subset*[*OF finite-subset-distinct*])

lemma *card-lists-distinct-length-eq-union*:
assumes *finite X finite Y* $X \cap Y = \{\}$
shows $card\ \{zs.\ length\ zs = n \wedge distinct\ zs \wedge set\ zs \subseteq X \cup Y\} =$
$(\sum k{=}0..n.\ (n\ choose\ k) * ffact\ k\ (card\ X) * ffact\ (n - k)\ (card\ Y))$
(is *card ?S* = -)
proof −
let $?expr = do\ \{$
$k \leftarrow \{0..n\};$
$xs \leftarrow \{xs.\ length\ xs = k \wedge distinct\ xs \wedge set\ xs \subseteq X\};$
$ys \leftarrow \{ys.\ length\ ys = n - k \wedge distinct\ ys \wedge set\ ys \subseteq Y\};$
$\{zs.\ interleavings\ xs\ ys\ zs\}$
$\}$
from ‹$X \cap Y = \{\}$› have $card\ ?S = card\ ?expr$
by (*simp add*: *lists-distinct-union-by-interleavings*)
let $?S \gg= ?comp = ?expr$
{

fix *k*
assume *k* ∈ *?S*
let *?expr* = *?comp k*
let *?S* ≫= *?comp* = *?expr*
from ‹*finite X*› **have** *finite ?S* **by**(*rule finite-length-distinct*)
moreover {
fix *xs*
assume *xs*: *xs* ∈ *?S*
let *?expr* = *?comp xs*
let *?S* ≫= *?comp* = *?expr*
from ‹*finite Y*› **have** *finite ?S* **by**(*rule finite-length-distinct*)
moreover {
fix *ys*
assume *ys*: *ys* ∈ *?S*
let *?expr* = *?comp ys*
have *finite ?expr*
by (*simp add*: *finite-interleavings*)
moreover have *card ?expr* = (*n choose k*)
using *xs ys* ‹*X* ∩ *Y* = {}› ‹*k* ∈ -›
by (*subst card-interleavings*) *auto*
ultimately have *finite ?expr* ∧ *card ?expr* = (*n choose k*) **..**
}
moreover have *disjoint-family-on ?comp ?S*
using ‹*k* ∈ {*0*..*n*}› ‹*xs* ∈ {*xs*. *length xs* = *k* ∧ *distinct xs* ∧ *set xs* ⊆ *X*}›
by (*injectivity-solver rule*: *injectivity*(*3*)[*OF* ‹*X* ∩ *Y* = {}›])
moreover have *card ?S* = *ffact* (*n* − *k*) (*card Y*)
using ‹*finite Y*› **by** (*simp add*: *card-lists-distinct-length-eq*)
ultimately have *card ?expr* = (*n choose k*) ∗ *ffact* (*n* − *k*) (*card Y*)
by (*subst card-bind-constant*) *auto*
moreover have *finite ?expr*
using ‹*finite ?S*› **by** (*auto intro*!: *finite-bind finite-interleavings*)
ultimately have *finite ?expr* ∧ *card ?expr* = (*n choose k*) ∗ *ffact* (*n* − *k*) (*card Y*)
by *blast*
}
moreover have *disjoint-family-on ?comp ?S*
using ‹*k* ∈ {*0*..*n*}›
by (*injectivity-solver rule*: *injectivity*(*2*)[*OF* ‹*X* ∩ *Y* = {}›])
moreover have *card ?S* = *ffact k* (*card X*)
using ‹*finite X*› **by** (*simp add*: *card-lists-distinct-length-eq*)
ultimately have *card ?expr* = (*n choose k*) ∗ *ffact k* (*card X*) ∗ *ffact* (*n* − *k*) (*card Y*)
by (*subst card-bind-constant*) *auto*
moreover have *finite ?expr*
using ‹*finite ?S*› ‹*finite Y*› **by** (*auto intro*!: *finite-bind finite-interleavings finite-length-distinct*)
ultimately have *finite ?expr* ∧ *card ?expr* = (*n choose k*) ∗ *ffact k* (*card X*) ∗ *ffact* (*n* − *k*) (*card Y*)
by *blast*

}
moreover have *disjoint-family-on ?comp ?S*
by (*injectivity-solver rule*: *injectivity*(*1*)[*OF* ‹*X* ∩ *Y* = {}›])
ultimately have *card ?expr* = ($\sum$ *k*=*0*..*n*. (*n choose k*) ∗ *ffact k* (*card X*) ∗ *ffact* (*n* − *k*) (*card Y*))
by (*auto simp add*: *card-bind*)
from ‹*card* - = *card ?expr*› *this* **show** *?thesis* **by** *simp*
qed

lemma
ffact n (*x* + *y*) = ($\sum$ *k*=*0*..*n*. (*n choose k*) ∗ *ffact k x* ∗ *ffact* (*n* − *k*) *y*)
proof −
define *X* **where** *X* = {..<*x*}
define *Y* **where** *Y* = {*x*..<*x*+*y*}
have *finite X* **and** *card X* = *x* **unfolding** *X-def* **by** *auto*
have *finite Y* **and** *card Y* = *y* **unfolding** *Y-def* **by** *auto*
have *X* ∩ *Y* = {} **unfolding** *X-def Y-def* **by** *auto*
have *ffact n* (*x* + *y*) = *ffact n* (*card X* + *card Y*)
using ‹*card X* = *x*› ‹*card Y* = *y*› **by** *simp*
also have . . . = *ffact n* (*card* (*X* ∪ *Y*))
using ‹*X* ∩ *Y* = {}› ‹*finite X*› ‹*finite Y*› **by** (*simp add*: *card-Un-disjoint*)
also have . . . = *card* {*xs*. *length xs* = *n* ∧ *distinct xs* ∧ *set xs* ⊆ *X* ∪ *Y*}
using ‹*finite X*› ‹*finite Y*› **by** (*simp add*: *card-lists-distinct-length-eq*)
also have . . . = ($\sum$ *k*=*0*..*n*. (*n choose k*) ∗ *ffact k* (*card X*) ∗ *ffact* (*n* − *k*) (*card Y*))
using ‹*X* ∩ *Y* = {}› ‹*finite X*› ‹*finite Y*› **by** (*simp add*: *card-lists-distinct-length-eq-union*)
also have . . . = ($\sum$ *k*=*0*..*n*. (*n choose k*) ∗ *ffact k x* ∗ *ffact* (*n* − *k*) *y*)
using ‹*card X* = *x*› ‹*card Y* = *y*› **by** *simp*
finally show *?thesis* **.**
qed

end

2 Proving Falling Factorial of a Sum with Induction

theory *Falling-Factorial-Sum-Induction*
imports
Discrete-Summation.Factorials
begin

Note the potentially special copyright license condition of the following proof.

lemma *ffact-add-nat*:
ffact n (*x* + *y*) = ($\sum$ *k*=*0*..*n*. (*n choose k*) ∗ *ffact k x* ∗ *ffact* (*n* − *k*) *y*)
proof (*induct n*)
case *0*
show *?case* **by** *simp*

next
  case (*Suc n*)
  let *?s* = λk. (*n choose k*) $*$ *ffact k x* $*$ *ffact* ($n - k$) *y*
  let *?t* = λk. *ffact k x* $*$ *ffact* (*Suc n* $-$ *k*) *y*
  let *?u* = λk. *ffact* (*Suc k*) *x* $*$ *ffact* ($n - k$) *y*
  have *ffact* (*Suc n*) ($x + y$) = ($x + y - n$) $*$ *ffact n* ($x + y$)
    by (*simp add*: *ffact-Suc-rev-nat*)
  also have ... = ($x + y - n$) $*$ ($\sum$ *k* = *0..n*. (*n choose k*) $*$ *ffact k x* $*$ *ffact* (*n*
$-$ *k*) *y*)
    using *Suc.hyps* **by** *simp*
  also have ... = ($\sum$ *k* = *0..n*. *?s k* $*$ ($x + y - n$))
    by (*simp add*: *mult.commute sum-distrib-left*)
  also have ... = ($\sum$ *k* = *0..n*. *?s k* $*$ (($y + k - n$) + ($x - k$)))
  proof $-$
    have *?s k* $*$ ($x + y - n$) = *?s k* $*$ (($y + k - n$) + ($x - k$)) **for** *k*
      by (*cases* $k \le x \vee n - k \le y$) (*auto simp add*: *ffact-nat-triv*)
    from *this* **show** *?thesis*
      by (*auto intro*: *sum.cong simp only*: *refl*)
  qed
  also have ... = ($\sum$ *k* = *0..n*. (*n choose k*) $*$ (*?t k* + *?u k*))
    by (*auto intro*!: *sum.cong simp add*: *Suc-diff-le ffact-Suc-rev-nat*) *algebra*
  also have ... = ($\sum$ *k* = *0..n*. (*n choose k*) $*$ *?t k*) + ($\sum$ *k* = *0..n*. (*n choose k*)
$*$ *?u k*)
    by (*simp add*: *sum.distrib add-mult-distrib2 mult.commute mult.left-commute*)
  also have ... = *?t 0* + ($\sum$ *k* = *0..n*. ((*n choose k*) + (*n choose Suc k*)) $*$ *?u k*)
  proof $-$
    have ... = (*?t 0* + ($\sum$ *k* = *0..n*. (*n choose Suc k*) $*$ *?u k*)) + ($\sum$ *k* = *0..n*. (*n*
choose k) $*$ *?u k*)
    proof $-$
      have ($\sum$ *k* = *Suc 0..n*. (*n choose k*) $*$ *?t k*) = ($\sum$ *k* = *0..n*. (*n choose Suc k*)
$*$ *?u k*)
      proof $-$
        have ($\sum$ *k* = *Suc 0..n*. (*n choose k*) $*$ *?t k*) = ($\sum$ *k* = *Suc 0..Suc n*. (*n*
choose k) $*$ *?t k*)
          by *simp*
        also have ... = (*sum* ((λk. (*n choose k*) $*$ *?t k*) *o Suc*) {*0..n*})
        by (*simp only*: *sum.reindex*[*symmetric*, *of Suc*] *inj-Suc image-Suc-atLeastAtMost*)
        also have ... = ($\sum$ *k* = *0..n*. (*n choose Suc k*) $*$ *?u k*)
          by *simp*
        finally show *?thesis* **.**
      qed
      from *this* **show** *?thesis*
        by (*simp add*: *sum.atLeast-Suc-atMost*[*of - - λk*. (*n choose k*) $*$ *?t k*])
    qed
    also have ... = *?t 0* + ($\sum$ *k* = *0..n*. ((*n choose k*) + (*n choose Suc k*)) $*$ *?u*
k)
      by (*simp add*: *distrib-right sum.distrib*)
    finally show *?thesis* **.**
  qed

also have . . . = ($\sum$ *k* = *0..Suc n*. (*Suc n choose k*) ∗ *ffact k x* ∗ *ffact* (*Suc n* − *k*) *y*)
proof −
let *?v* = λ*k*. (*Suc n choose k*) ∗ *ffact k x* ∗ *ffact* (*Suc n* − *k*) *y*
have . . . = *?v 0* + ($\sum$ *k* = *0..n*. (*Suc n choose* (*Suc k*)) ∗ *?u k*)
by *simp*
also have . . . = *?v 0* + ($\sum$ *k* = *Suc 0..Suc n*. *?v k*)
by (*simp only*: *sum.shift-bounds-cl-Suc-ivl diff-Suc-Suc mult.assoc*)
also have . . . = ($\sum$ *k* = *0..Suc n*. (*Suc n choose k*) ∗ *ffact k x* ∗ *ffact* (*Suc n* − *k*) *y*)
by (*simp add*: *sum.atLeast-Suc-atMost*)
finally show *?thesis* **.**
qed
finally show *?case* **.**
qed

lemma *ffact-add*:
fixes *x y* :: ′*a*::{*ab-group-add*, *comm-semiring-1-cancel*, *ring-1*}
shows *ffact n* (*x* + *y*) = ($\sum$ *k*=*0..n*. *of-nat* (*n choose k*) ∗ *ffact k x* ∗ *ffact* (*n* − *k*) *y*)
proof (*induct n*)
case *0*
show *?case* **by** *simp*
next
case (*Suc n*)
let *?s* = λ*k*. *of-nat* (*n choose k*) ∗ *ffact k x* ∗ *ffact* (*n* − *k*) *y*
let *?t* = λ*k*. *ffact k x* ∗ *ffact* (*Suc n* − *k*) *y*
let *?u* = λ*k*. *ffact* (*Suc k*) *x* ∗ *ffact* (*n* − *k*) *y*
have *ffact* (*Suc n*) (*x* + *y*) = (*x* + *y* − *of-nat n*) ∗ *ffact n* (*x* + *y*)
by (*simp add*: *ffact-Suc-rev*)
also have . . . = (*x* + *y* − *of-nat n*) ∗ ($\sum$ *k* = *0..n*. *of-nat* (*n choose k*) ∗ *ffact k x* ∗ *ffact* (*n* − *k*) *y*)
using *Suc.hyps* **by** *simp*
also have . . . = ($\sum$ *k* = *0..n*. *?s k* ∗ (*x* + *y* − *of-nat n*))
by (*simp add*: *mult.commute sum-distrib-left*)
also have . . . = ($\sum$ *k* = *0..n*. *?s k* ∗ ((*y* + *of-nat k* − *of-nat n*) + (*x* − *of-nat k*)))
by (*auto intro*: *sum.cong simp add*: *diff-add-eq add-diff-eq add.commute*)
also have . . . = ($\sum$ *k* = *0..n*. *of-nat* (*n choose k*) ∗ (*?t k* + *?u k*))
proof −
{
fix *k*
assume *k* ≤ *n*
have *?u k* = *ffact k x* ∗ *ffact* (*n* − *k*) *y* ∗ (*x* − *of-nat k*)
by (*simp add*: *ffact-Suc-rev Suc-diff-le of-nat-diff mult.commute mult.left-commute*)
moreover from ‹*k* ≤ *n*› **have** *?t k* = *ffact k x* ∗ *ffact* (*n* − *k*) *y* ∗ (*y* + *of-nat k* − *of-nat n*)
by (*simp add*: *ffact-Suc-rev Suc-diff-le of-nat-diff diff-diff-eq2 mult.commute*

mult.left-commute)
ultimately have
?s k ∗ ((*y* + *of-nat k* − *of-nat n*) + (*x* − *of-nat k*)) = *of-nat* (*n choose k*) ∗ (*?t k* + *?u k*)
by (*metis* (*no-types*, *lifting*) *distrib-left mult.assoc*)
}
from *this* **show** *?thesis* **by** (*auto intro*: *sum.cong*)
qed
also have ... = ($\sum$ *k* = *0..n*. *of-nat* (*n choose k*) ∗ *?t k*) + ($\sum$ *k* = *0..n*. *of-nat* (*n choose k*) ∗ *?u k*)
by (*simp add*: *sum.distrib distrib-left mult.commute mult.left-commute*)
also have ... = *?t 0* + ($\sum$ *k* = *0..n*. *of-nat* ((*n choose k*) + (*n choose Suc k*)) ∗ *?u k*)
proof −
have ... = (*?t 0* + ($\sum$ *k* = *0..n*. *of-nat* (*n choose Suc k*) ∗ *?u k*)) + ($\sum$ *k* = *0..n*. *of-nat* (*n choose k*) ∗ *?u k*)
proof −
have ($\sum$ *k* = *Suc 0..n*. *of-nat* (*n choose k*) ∗ *?t k*) = ($\sum$ *k* = *0..n*. *of-nat* (*n choose Suc k*) ∗ *?u k*)
proof −
have ($\sum$ *k* = *Suc 0..n*. *of-nat* (*n choose k*) ∗ *?t k*) = ($\sum$ *k* = *Suc 0..Suc n*. *of-nat* (*n choose k*) ∗ *?t k*)
by (*simp add*: *binomial-eq-0*)
also have ... = (*sum* ((λ*k*. *of-nat* (*n choose k*) ∗ *?t k*) *o Suc*) {*0..n*})
by (*simp only*: *sum.reindex*[*symmetric*, *of Suc*] *inj-Suc image-Suc-atLeastAtMost*)
also have ... = ($\sum$ *k* = *0..n*. *of-nat* (*n choose Suc k*) ∗ *?u k*)
by *simp*
finally show *?thesis* **.**
qed
from *this* **show** *?thesis*
by (*simp add*: *sum.atLeast-Suc-atMost*[*of - - λk. of-nat* (*n choose k*) ∗ *?t k*])
qed
also have ... = *?t 0* + ($\sum$ *k* = *0..n*. *of-nat* ((*n choose k*) + (*n choose Suc k*)) ∗ *?u k*)
by (*simp add*: *distrib-right sum.distrib*)
finally show *?thesis* **.**
qed
also have ... = ($\sum$ *k* = *0..Suc n*. *of-nat* (*Suc n choose k*) ∗ *ffact k x* ∗ *ffact* (*Suc n* − *k*) *y*)
proof −
let *?v* = λ*k*. *of-nat* (*Suc n choose k*) ∗ *ffact k x* ∗ *ffact* (*Suc n* − *k*) *y*
have ... = *?v 0* + ($\sum$ *k* = *0..n*. *of-nat* (*Suc n choose* (*Suc k*)) ∗ *?u k*)
by *simp*
also have ... = *?v 0* + ($\sum$ *k* = *Suc 0..Suc n*. *?v k*)
by (*simp only*: *sum.shift-bounds-cl-Suc-ivl diff-Suc-Suc mult.assoc*)
also have ... = ($\sum$ *k* = *0..Suc n*. *of-nat* (*Suc n choose k*) ∗ *ffact k x* ∗ *ffact* (*Suc n* − *k*) *y*)
by (*simp add*: *sum.atLeast-Suc-atMost*)
finally show *?thesis* **.**

```
  qed
  finally show ?case .
qed

end
```

3 Proving Falling Factorial of a Sum with Vandermonde Identity

```
theory Falling-Factorial-Sum-Vandermonde
imports
  Discrete-Summation.Factorials
begin
```

Note the potentially special copyright license condition of the following proof.

```
lemma ffact-add-nat:
  shows ffact k (n + m) = (∑ i≤k. (k choose i) ∗ ffact i n ∗ ffact (k − i) m)
proof −
  have ffact k (n + m) = fact k ∗ ((n + m) choose k)
    by (simp only: ffact-eq-fact-mult-binomial)
  also have . . . = fact k ∗ (∑ i≤k. (n choose i) ∗ (m choose (k − i)))
    by (simp only: vandermonde)
  also have . . . = (∑ i≤k. fact k ∗ (n choose i) ∗ (m choose (k − i)))
    by (simp add: sum-distrib-left field-simps)
  also have . . . = (∑ i≤k. (fact i ∗ fact (k − i) ∗ (k choose i)) ∗ (n choose i) ∗
(m choose (k − i)))
    by (simp add: binomial-fact-lemma)
  also have . . . = (∑ i≤k. (k choose i) ∗ (fact i ∗ (n choose i)) ∗ (fact (k − i) ∗
(m choose (k − i))))
    by (auto intro: sum.cong)
  also have . . . = (∑ i≤k. (k choose i) ∗ ffact i n ∗ ffact (k − i) m)
    by (simp only: ffact-eq-fact-mult-binomial)
  finally show ?thesis .
qed

end
```

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