Euler's Polyhedron Formula

Lawrence C. Paulson

March 17, 2025

Abstract

Euler stated in 1752 that every convex polyhedron satisfied the formula $V - E + F = 2$ where V, E and F are the numbers of its vertices, edges, and faces. For three dimensions, the well-known proof involves removing one face and then flattening the remainder to form a planar graph, which then is iteratively transformed to leave a single triangle. The history of that proof is extensively discussed and elaborated by Imre Lakatos [1], leaving one finally wondering whether the theorem even holds. The formal proof provided here has been ported from HOL Light, where it is credited to Lawrence [2]. The proof generalises Euler's observation from solid polyhedra to convex polytopes of arbitrary dimension.

Contents

Acknowledgements The author was supported by the ERC Advanced Grant ALEXANDRIA (Project 742178) funded by the European Research Council.

1 Euler's Polyhedron Formula

One of the Famous 100 Theorems, ported from HOL Light

Cited source: Lawrence, J. (1997). A Short Proof of Euler's Relation for Convex Polytopes. *Canadian Mathematical Bulletin*, **40**(4), 471–474.

theory *Euler-Formula*
  imports
    HOL−Analysis.Analysis
begin

Interpret which "side" of a hyperplane a point is on.

definition *hyperplane-side*
  where *hyperplane-side* ≡ λ(*a*,*b*). λ*x*. *sgn* (*a* · *x* − *b*)

Equivalence relation imposed by a hyperplane arrangement.

definition *hyperplane-equiv*
 where *hyperplane-equiv* ≡ λ*A x y*. ∀ *h* ∈ *A*. *hyperplane-side h x* = *hyperplane-side h y*

lemma *hyperplane-equiv-refl* [*iff*]: *hyperplane-equiv A x x*
  ⟨*proof*⟩

lemma *hyperplane-equiv-sym*:
   hyperplane-equiv A x y ⟷ *hyperplane-equiv A y x*
  ⟨*proof*⟩

lemma *hyperplane-equiv-trans*:
   ⟦*hyperplane-equiv A x y*; *hyperplane-equiv A y z*⟧ ⟹ *hyperplane-equiv A x z*
  ⟨*proof*⟩

lemma *hyperplane-equiv-Un*:
   hyperplane-equiv (*A* ∪ *B*) *x y* ⟷ *hyperplane-equiv A x y* ∧ *hyperplane-equiv B x y*
  ⟨*proof*⟩

1.1 Cells of a hyperplane arrangement

definition *hyperplane-cell* :: (*'a*::*real-inner* × *real*) *set* ⇒ *'a set* ⇒ *bool*
  where *hyperplane-cell* ≡ λ*A C*. ∃ *x*. *C* = *Collect* (*hyperplane-equiv A x*)

lemma *hyperplane-cell*: *hyperplane-cell A C* ⟷ (∃ *x*. *C* = {*y*. *hyperplane-equiv A x y*})
  ⟨*proof*⟩

lemma *not-hyperplane-cell-empty* [*simp*]: ¬ *hyperplane-cell A* {}
  ⟨*proof*⟩

lemma *nonempty-hyperplane-cell*: *hyperplane-cell A C* ⟹ (*C* ≠ {})

⟨*proof*⟩

lemma *Union-hyperplane-cells*: $\bigcup$ {*C*. *hyperplane-cell A C*} = *UNIV*
⟨*proof*⟩

lemma *disjoint-hyperplane-cells*:
⟦*hyperplane-cell A C1*; *hyperplane-cell A C2*; *C1* ≠ *C2*⟧ ⟹ *disjnt C1 C2*
⟨*proof*⟩

lemma *disjoint-hyperplane-cells-eq*:
⟦*hyperplane-cell A C1*; *hyperplane-cell A C2*⟧ ⟹ (*disjnt C1 C2* ⟷ (*C1* ≠ *C2*))
⟨*proof*⟩

lemma *hyperplane-cell-empty* [*iff*]: *hyperplane-cell* {} *C* ⟷ *C* = *UNIV*
⟨*proof*⟩

lemma *hyperplane-cell-singleton-cases*:
assumes *hyperplane-cell* {(*a*,*b*)} *C*
shows *C* = {*x*. *a* · *x* = *b*} ∨ *C* = {*x*. *a* · *x* < *b*} ∨ *C* = {*x*. *a* · *x* > *b*}
⟨*proof*⟩

lemma *hyperplane-cell-singleton*:
hyperplane-cell {(*a*,*b*)} *C* ⟷
(*if a* = *0 then C* = *UNIV else C* = {*x*. *a* · *x* = *b*} ∨ *C* = {*x*. *a* · *x* < *b*} ∨ *C* = {*x*. *a* · *x* > *b*})
⟨*proof*⟩

lemma *hyperplane-cell-Un*:
hyperplane-cell (*A* ∪ *B*) *C* ⟷
C ≠ {} ∧
(∃ *C1 C2*. *hyperplane-cell A C1* ∧ *hyperplane-cell B C2* ∧ *C* = *C1* ∩ *C2*)
⟨*proof*⟩

lemma *finite-hyperplane-cells*:
finite A ⟹ *finite* {*C*. *hyperplane-cell A C*}
⟨*proof*⟩

lemma *finite-restrict-hyperplane-cells*:
finite A ⟹ *finite* {*C*. *hyperplane-cell A C* ∧ *P C*}
⟨*proof*⟩

lemma *finite-set-of-hyperplane-cells*:
⟦*finite A*; ⋀*C*. *C* ∈ 𝒞 ⟹ *hyperplane-cell A C*⟧ ⟹ *finite* 𝒞
⟨*proof*⟩

lemma *pairwise-disjoint-hyperplane-cells*:
(⋀*C*. *C* ∈ 𝒞 ⟹ *hyperplane-cell A C*) ⟹ *pairwise disjnt* 𝒞
⟨*proof*⟩

lemma *hyperplane-cell-Int-open-affine*:
assumes *finite A hyperplane-cell A C*
obtains *S T* **where** *open S affine T C* = *S* ∩ *T*
⟨*proof*⟩

lemma *hyperplane-cell-relatively-open*:
assumes *finite A hyperplane-cell A C*
shows *openin* (*subtopology euclidean* (*affine hull C*)) *C*
⟨*proof*⟩

lemma *hyperplane-cell-relative-interior*:
⟦*finite A*; *hyperplane-cell A C*⟧ ⟹ *rel-interior C* = *C*
⟨*proof*⟩

lemma *hyperplane-cell-convex*:
assumes *hyperplane-cell A C*
shows *convex C*
⟨*proof*⟩

lemma *hyperplane-cell-Inter*:
assumes ⋀*C*. *C* ∈ 𝒞 ⟹ *hyperplane-cell A C*
and 𝒞 ≠ {} **and** *INT*: ⋂𝒞 ≠ {}
shows *hyperplane-cell A* (⋂𝒞)
⟨*proof*⟩

lemma *hyperplane-cell-Int*:
⟦*hyperplane-cell A S*; *hyperplane-cell A T*; *S* ∩ *T* ≠ {}⟧ ⟹ *hyperplane-cell A* (*S* ∩ *T*)
⟨*proof*⟩

1.2 A cell complex is considered to be a union of such cells

definition *hyperplane-cellcomplex*
where *hyperplane-cellcomplex A S* ≡
∃𝒯. (∀ *C* ∈ 𝒯. *hyperplane-cell A C*) ∧ *S* = ⋃𝒯

lemma *hyperplane-cellcomplex-empty* [*simp*]: *hyperplane-cellcomplex A* {}
⟨*proof*⟩

lemma *hyperplane-cell-cellcomplex*:
hyperplane-cell A C ⟹ *hyperplane-cellcomplex A C*
⟨*proof*⟩

lemma *hyperplane-cellcomplex-Union*:
assumes ⋀*S*. *S* ∈ 𝒞 ⟹ *hyperplane-cellcomplex A S*
shows *hyperplane-cellcomplex A* (⋃ 𝒞)
⟨*proof*⟩

lemma *hyperplane-cellcomplex-Un*:
$[\![$*hyperplane-cellcomplex A S*; *hyperplane-cellcomplex A T*$]\!]$
$\Longrightarrow$ *hyperplane-cellcomplex A* $(S \cup T)$
⟨*proof*⟩

lemma *hyperplane-cellcomplex-UNIV* [*simp*]: *hyperplane-cellcomplex A UNIV*
⟨*proof*⟩

lemma *hyperplane-cellcomplex-Inter*:
assumes $\bigwedge S.\ S \in \mathcal{C} \Longrightarrow$ *hyperplane-cellcomplex A S*
shows *hyperplane-cellcomplex A* $(\bigcap \mathcal{C})$
⟨*proof*⟩

lemma *hyperplane-cellcomplex-Int*:
$[\![$*hyperplane-cellcomplex A S*; *hyperplane-cellcomplex A T*$]\!]$
$\Longrightarrow$ *hyperplane-cellcomplex A* $(S \cap T)$
⟨*proof*⟩

lemma *hyperplane-cellcomplex-Compl*:
assumes *hyperplane-cellcomplex A S*
shows *hyperplane-cellcomplex A* $(-\ S)$
⟨*proof*⟩

lemma *hyperplane-cellcomplex-diff*:
$[\![$*hyperplane-cellcomplex A S*; *hyperplane-cellcomplex A T*$]\!]$
$\Longrightarrow$ *hyperplane-cellcomplex A* $(S - T)$
⟨*proof*⟩

lemma *hyperplane-cellcomplex-mono*:
assumes *hyperplane-cellcomplex A S* $A \subseteq B$
shows *hyperplane-cellcomplex B S*
⟨*proof*⟩

lemma *finite-hyperplane-cellcomplexes*:
assumes *finite A*
shows *finite* $\{C.$ *hyperplane-cellcomplex A C*$\}$
⟨*proof*⟩

lemma *finite-restrict-hyperplane-cellcomplexes*:
finite A $\Longrightarrow$ *finite* $\{C.$ *hyperplane-cellcomplex A C* $\wedge$ $P\ C\}$
⟨*proof*⟩

lemma *finite-set-of-hyperplane-cellcomplex*:
assumes *finite A* $\bigwedge C.\ C \in \mathcal{C} \Longrightarrow$ *hyperplane-cellcomplex A C*
shows *finite* $\mathcal{C}$
⟨*proof*⟩

lemma *cell-subset-cellcomplex*:

$[\![hyperplane\text{-}cell\ A\ C;\ hyperplane\text{-}cellcomplex\ A\ S]\!] \Longrightarrow C \subseteq S \longleftrightarrow \sim\ disjnt\ C\ S$
⟨*proof*⟩

1.3 Euler characteristic

definition *Euler-characteristic* :: $('a::euclidean\text{-}space \times real)\ set \Rightarrow 'a\ set \Rightarrow int$
where *Euler-characteristic* $A\ S \equiv$
$(\sum C \mid hyperplane\text{-}cell\ A\ C \land C \subseteq S.\ (-1)\ \hat{}\ nat\ (aff\text{-}dim\ C))$

lemma *Euler-characteristic-empty* [*simp*]: *Euler-characteristic* $A\ \{\} = 0$
⟨*proof*⟩

lemma *Euler-characteristic-cell-Union*:
assumes $\bigwedge C.\ C \in \mathcal{C} \Longrightarrow hyperplane\text{-}cell\ A\ C$
shows *Euler-characteristic* $A\ (\bigcup\ \mathcal{C}) = (\sum C{\in}\mathcal{C}.\ (-\ 1)\ \hat{}\ nat\ (aff\text{-}dim\ C))$
⟨*proof*⟩

lemma *Euler-characteristic-cell*:
$hyperplane\text{-}cell\ A\ C \Longrightarrow Euler\text{-}characteristic\ A\ C = (-1)\ \hat{}(nat(aff\text{-}dim\ C))$
⟨*proof*⟩

lemma *Euler-characteristic-cellcomplex-Un*:
assumes *finite A hyperplane-cellcomplex A S*
and *AT*: *hyperplane-cellcomplex A T* **and** *disjnt S T*
shows *Euler-characteristic* $A\ (S \cup T) =$
Euler-characteristic $A\ S$ + *Euler-characteristic* $A\ T$
⟨*proof*⟩

lemma *Euler-characteristic-cellcomplex-Union*:
assumes *finite A*
and $\mathcal{C}$: $\bigwedge C.\ C \in \mathcal{C} \Longrightarrow hyperplane\text{-}cellcomplex\ A\ C\ pairwise\ disjnt\ \mathcal{C}$
shows *Euler-characteristic* $A\ (\bigcup\ \mathcal{C}) = sum\ (Euler\text{-}characteristic\ A)\ \mathcal{C}$
⟨*proof*⟩

lemma *Euler-characteristic*:
fixes A :: $('n::euclidean\text{-}space * real)\ set$
assumes *finite A*
shows *Euler-characteristic* $A\ S =$
$(\sum d = 0..DIM('n).\ (-1)\ \hat{}\ d * int\ (card\ \{C.\ hyperplane\text{-}cell\ A\ C \land C \subseteq S \land aff\text{-}dim\ C = int\ d\}))$
(**is** - = *?rhs*)
⟨*proof*⟩

1.4 Show that the characteristic is invariant w.r.t. hyperplane arrangement.

lemma *hyperplane-cells-distinct-lemma*:
$\{x.\ a \cdot x = b\} \cap \{x.\ a \cdot x < b\} = \{\} \land$
$\{x.\ a \cdot x = b\} \cap \{x.\ a \cdot x > b\} = \{\} \land$

$\{x.\ a \cdot x < b\} \cap \{x.\ a \cdot x = b\} = \{\} \wedge$
$\{x.\ a \cdot x < b\} \cap \{x.\ a \cdot x > b\} = \{\} \wedge$
$\{x.\ a \cdot x > b\} \cap \{x.\ a \cdot x = b\} = \{\} \wedge$
$\{x.\ a \cdot x > b\} \cap \{x.\ a \cdot x < b\} = \{\}$
⟨*proof*⟩

proposition *Euler-characterstic-lemma*:
assumes *finite A* **and** *hyperplane-cellcomplex A S*
shows *Euler-characteristic* (*insert h A*) *S* = *Euler-characteristic A S*
⟨*proof*⟩

lemma *Euler-characterstic-invariant-aux*:
assumes *finite B finite A hyperplane-cellcomplex A S*
shows *Euler-characteristic* ($A \cup B$) *S* = *Euler-characteristic A S*
⟨*proof*⟩

lemma *Euler-characterstic-invariant*:
assumes *finite A finite B hyperplane-cellcomplex A S hyperplane-cellcomplex B S*
shows *Euler-characteristic A S* = *Euler-characteristic B S*
⟨*proof*⟩

lemma *Euler-characteristic-inclusion-exclusion*:
assumes *finite A finite* $\mathcal{S}$ $\bigwedge K.\ K \in \mathcal{S} \Longrightarrow$ *hyperplane-cellcomplex A K*
shows *Euler-characteristic A* ($\bigcup \mathcal{S}$) = ($\sum \mathcal{T} \mid \mathcal{T} \subseteq \mathcal{S} \wedge \mathcal{T} \neq \{\}.\ (-\ 1)\ \hat{}\ ($*card* $\mathcal{T} + 1) *$ *Euler-characteristic A* ($\bigcap \mathcal{T}$))
⟨*proof*⟩

1.5 Euler-type relation for full-dimensional proper polyhedral cones

lemma *Euler-polyhedral-cone*:
fixes *S* :: *'n::euclidean-space set*
assumes *polyhedron S conic S* **and** *intS*: *interior* $S \neq \{\}$ **and** $S \neq$ *UNIV*
shows ($\sum d = 0..DIM('n).\ (-\ 1)\ \hat{}\ d *$ *int* (*card* {*f*. *f face-of S* ∧ *aff-dim f* = *int d*})) = *0* (**is** *?lhs* = *0*)
⟨*proof*⟩

1.6 Euler-Poincare relation for special $(n - 1)$-dimensional polytope

lemma *Euler-Poincare-lemma*:
fixes *p* :: *'n::euclidean-space set*
assumes $DIM('n) \geq 2$ *polytope p* $i \in$ *Basis* **and** *affp*: *affine hull p* = $\{x.\ x \cdot i = 1\}$
shows ($\sum d = 0..DIM('n) - 1.\ (-1)\ \hat{}\ d *$ *int* (*card* {*f*. *f face-of p* ∧ *aff-dim f* = *int d*})) = *1*
⟨*proof*⟩

corollary *Euler-poincare-special*:
fixes p :: $'n$::*euclidean-space set*
assumes $2 \leq DIM('n)$ *polytope p* $i \in$ *Basis* **and** *affp*: *affine hull p* $= \{x.\ x \cdot i = 0\}$
shows $(\sum d = 0..DIM('n) - 1.\ (-1)\ \widehat{}\ d * card\ \{f.\ f\ face\text{-}of\ p \wedge aff\text{-}dim\ f = d\}) = 1$
⟨*proof*⟩

1.7 Now Euler-Poincare for a general full-dimensional polytope

theorem *Euler-Poincare-full*:
fixes p :: $'n$::*euclidean-space set*
assumes *polytope p aff-dim p* $= DIM('n)$
shows $(\sum d = 0..DIM('n).\ (-1)\ \widehat{}\ d * (card\ \{f.\ f\ face\text{-}of\ p \wedge aff\text{-}dim\ f = d\})) = 1$
⟨*proof*⟩

In particular, the Euler relation in 3 dimensions

corollary *Euler-relation*:
fixes p :: $'n$::*euclidean-space set*
assumes *polytope p aff-dim p* $= 3$ $DIM('n) = 3$
shows $(card\ \{v.\ v\ face\text{-}of\ p \wedge aff\text{-}dim\ v = 0\} + card\ \{f.\ f\ face\text{-}of\ p \wedge aff\text{-}dim\ f = 2\}) - card\ \{e.\ e\ face\text{-}of\ p \wedge aff\text{-}dim\ e = 1\} = 2$
⟨*proof*⟩

end

References

[1] I. Lakatos. *Proofs and Refutations: The Logic of Mathematical Discovery.* 1976.

[2] J. Lawrence. A short proof of Euler's relation for convex polytopes. *Canadian Mathematical Bulletin*, 40(4):471–474, 1997.