Euler's Polyhedron Formula

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Abstract

Euler stated in 1752 that every convex polyhedron satisfied the formula $V - E + F = 2$ where V, E and F are the numbers of its vertices, edges, and faces. For three dimensions, the well-known proof involves removing one face and then flattening the remainder to form a planar graph, which then is iteratively transformed to leave a single triangle. The history of that proof is extensively discussed and elaborated by Imre Lakatos [1], leaving one finally wondering whether the theorem even holds. The formal proof provided here has been ported from HOL Light, where it is credited to Lawrence [2]. The proof generalises Euler's observation from solid polyhedra to convex polytopes of arbitrary dimension.

Contents

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1 Library Extras

For adding to the repository

theory *Library-Extras* **imports**
  HOL−Analysis.Polytope

begin

2 Preliminaries

lemma *Inter-over-Union*:
  $\bigcap \{\bigcup (\mathcal{F}\ x) \mid x.\ x \in S\} = \bigcup \{\bigcap (G\ `\ S) \mid G.\ \forall x{\in}S.\ G\ x \in \mathcal{F}\ x\}$
$\langle proof \rangle$

lemmas *closure-Int-convex* = *convex-closure-inter-two*

lemmas *span-not-UNIV-orthogonal* = *span-not-univ-orthogonal*

lemma *convex-closure-rel-interior-Int*:
  assumes $\bigwedge S.\ S{\in}\mathcal{F} \Longrightarrow convex\ (S :: 'n{::}euclidean\text{-}space\ set)$
    and $\bigcap(rel\text{-}interior\ `\ \mathcal{F}) \neq \{\}$
  shows $\bigcap(closure\ `\ \mathcal{F}) \subseteq closure\ (\bigcap(rel\text{-}interior\ `\ \mathcal{F}))$
$\langle proof \rangle$

lemma *closure-Inter-convex*:
  fixes $\mathcal{F} :: 'n{::}euclidean\text{-}space\ set\ set$
  assumes $\bigwedge S.\ S \in \mathcal{F} \Longrightarrow convex\ S$ **and** $\bigcap(rel\text{-}interior\ `\ \mathcal{F}) \neq \{\}$
  shows $closure(\bigcap\mathcal{F}) = \bigcap(closure\ `\ \mathcal{F})$
$\langle proof \rangle$

lemma *closure-Inter-convex-open*:
  $(\bigwedge S{::}'n{::}euclidean\text{-}space\ set.\ S \in \mathcal{F} \Longrightarrow convex\ S \wedge open\ S)$
      $\Longrightarrow closure(\bigcap\mathcal{F}) = (if\ \bigcap\mathcal{F} = \{\}\ then\ \{\}\ else\ \bigcap(closure\ `\ \mathcal{F}))$
  $\langle proof \rangle$

lemma *empty-interior-subset-hyperplane-aux*:
  fixes $S :: 'a{::}euclidean\text{-}space\ set$
  assumes $convex\ S\ 0 \in S$ **and** *empty-int*: $interior\ S = \{\}$
  shows $\exists\, a\ b.\ a{\neq}0 \wedge S \subseteq \{x.\ a \cdot x = b\}$
$\langle proof \rangle$

lemma *empty-interior-subset-hyperplane*:
  fixes $S :: 'a{::}euclidean\text{-}space\ set$
  assumes *convex S* **and** *int*: $interior\ S = \{\}$
  obtains *a b* **where** $a \neq 0\ S \subseteq \{x.\ a \cdot x = b\}$
$\langle proof \rangle$

lemma *aff-dim-psubset*:
$(\mathit{affine\ hull}\ S) \subset (\mathit{affine\ hull}\ T) \Longrightarrow \mathit{aff\text{-}dim}\ S < \mathit{aff\text{-}dim}\ T$
⟨*proof*⟩

lemma *aff-dim-eq-full-gen*:
$S \subseteq T \Longrightarrow (\mathit{aff\text{-}dim}\ S = \mathit{aff\text{-}dim}\ T \longleftrightarrow \mathit{affine\ hull}\ S = \mathit{affine\ hull}\ T)$
⟨*proof*⟩

lemma *aff-dim-eq-full*:
fixes $S :: {'n}::\mathit{euclidean\text{-}space\ set}$
shows $\mathit{aff\text{-}dim}\ S = (\mathit{DIM}('n)) \longleftrightarrow \mathit{affine\ hull}\ S = \mathit{UNIV}$
⟨*proof*⟩

3 Conic sets and conic hull

definition $\mathit{conic} :: {'a}::\mathit{real\text{-}vector\ set} \Rightarrow \mathit{bool}$
where $\mathit{conic}\ S \equiv \forall x\ c.\ x \in S \longrightarrow 0 \le c \longrightarrow (c *_R x) \in S$

lemma *conicD*: $[\![\mathit{conic}\ S;\ x \in S;\ 0 \le c]\!] \Longrightarrow (c *_R x) \in S$
⟨*proof*⟩

lemma *subspace-imp-conic*: $\mathit{subspace}\ S \Longrightarrow \mathit{conic}\ S$
⟨*proof*⟩

lemma *conic-empty* [*simp*]: $\mathit{conic}\ \{\}$
⟨*proof*⟩

lemma *conic-UNIV*: $\mathit{conic}\ \mathit{UNIV}$
⟨*proof*⟩

lemma *conic-Inter*: $(\bigwedge S.\ S \in \mathcal{F} \Longrightarrow \mathit{conic}\ S) \Longrightarrow \mathit{conic}(\bigcap \mathcal{F})$
⟨*proof*⟩

lemma *conic-linear-image*:
$[\![\mathit{conic}\ S;\ \mathit{linear}\ f]\!] \Longrightarrow \mathit{conic}(f\ `\ S)$
⟨*proof*⟩

lemma *conic-linear-image-eq*:
$[\![\mathit{linear}\ f;\ \mathit{inj}\ f]\!] \Longrightarrow \mathit{conic}\ (f\ `\ S) \longleftrightarrow \mathit{conic}\ S$
⟨*proof*⟩

lemma *conic-mul*: $[\![\mathit{conic}\ S;\ x \in S;\ 0 \le c]\!] \Longrightarrow (c *_R x) \in S$
⟨*proof*⟩

lemma *conic-conic-hull*: $\mathit{conic}(\mathit{conic\ hull}\ S)$
⟨*proof*⟩

lemma *conic-hull-eq*: $(\mathit{conic\ hull}\ S = S) \longleftrightarrow \mathit{conic}\ S$
⟨*proof*⟩

lemma *conic-hull-UNIV* [*simp*]: *conic hull UNIV* = *UNIV*
⟨*proof*⟩

lemma *conic-negations*: *conic S* $\Longrightarrow$ *conic* (*image uminus S*)
⟨*proof*⟩

lemma *conic-span* [*iff*]: *conic*(*span S*)
⟨*proof*⟩

lemma *conic-hull-explicit*:
conic hull S = $\{c *_R x \mid c\ x.\ 0 \le c \wedge x \in S\}$
⟨*proof*⟩

lemma *conic-hull-as-image*:
conic hull S = ($\lambda z.$ *fst z* $*_R$ *snd z*) ‘ ($\{0..\} \times S$)
⟨*proof*⟩

lemma *conic-hull-linear-image*:
linear f $\Longrightarrow$ *conic hull f* ‘ *S* = *f* ‘ (*conic hull S*)
⟨*proof*⟩

lemma *conic-hull-image-scale*:
assumes $\bigwedge x.\ x \in S \Longrightarrow 0 < c\ x$
shows *conic hull* ($\lambda x.\ c\ x *_R x$) ‘ *S* = *conic hull S*
⟨*proof*⟩

lemma *convex-conic-hull*:
assumes *convex S*
shows *convex* (*conic hull S*)
⟨*proof*⟩

lemma *conic-halfspace-le*: *conic* $\{x.\ a \cdot x \le 0\}$
⟨*proof*⟩

lemma *conic-halfspace-ge*: *conic* $\{x.\ a \cdot x \ge 0\}$
⟨*proof*⟩

lemma *conic-hull-empty* [*simp*]: *conic hull* {} = {}
⟨*proof*⟩

lemma *conic-contains-0*: *conic S* $\Longrightarrow$ ($0 \in S \longleftrightarrow S \neq \{\}$)
⟨*proof*⟩

lemma *conic-hull-eq-empty*: *conic hull S* = {} $\longleftrightarrow$ (*S* = {})
⟨*proof*⟩

lemma *conic-sums*: ⟦*conic S*; *conic T*⟧ $\Longrightarrow$ *conic* ($\bigcup x\in S.\ \bigcup y \in T.\ \{x + y\}$)
⟨*proof*⟩

lemma *conic-Times*: $[\![conic\ S;\ conic\ T]\!] \Longrightarrow conic(S \times T)$
$\langle proof \rangle$

lemma *conic-Times-eq*:
$conic(S \times T) \longleftrightarrow S = \{\} \vee T = \{\} \vee conic\ S \wedge conic\ T$ (**is** *?lhs = ?rhs*)
$\langle proof \rangle$

lemma *conic-hull-0* [*simp*]: $conic\ hull\ \{0\} = \{0\}$
$\langle proof \rangle$

lemma *conic-hull-contains-0* [*simp*]: $0 \in conic\ hull\ S \longleftrightarrow (S \neq \{\})$
$\langle proof \rangle$

lemma *conic-hull-eq-sing*:
$conic\ hull\ S = \{x\} \longleftrightarrow S = \{0\} \wedge x = 0$
$\langle proof \rangle$

lemma *conic-hull-Int-affine-hull*:
assumes $T \subseteq S\ 0 \notin affine\ hull\ S$
shows $(conic\ hull\ T) \cap (affine\ hull\ S) = T$
$\langle proof \rangle$

lemma *open-in-subset-relative-interior*:
fixes S :: *'a::euclidean-space set*
shows *openin* (*top-of-set* (*affine hull T*)) $S \Longrightarrow (S \subseteq rel\text{-}interior\ T) = (S \subseteq T)$
$\langle proof \rangle$

lemma *conic-hull-eq-span-affine-hull*:
fixes S :: *'a::euclidean-space set*
assumes $0 \in rel\text{-}interior\ S$
shows $conic\ hull\ S = span\ S \wedge conic\ hull\ S = affine\ hull\ S$
$\langle proof \rangle$

lemma *conic-hull-eq-span*:
fixes S :: *'a::euclidean-space set*
assumes $0 \in rel\text{-}interior\ S$
shows $conic\ hull\ S = span\ S$
$\langle proof \rangle$

lemma *conic-hull-eq-affine-hull*:
fixes S :: *'a::euclidean-space set*
assumes $0 \in rel\text{-}interior\ S$
shows $conic\ hull\ S = affine\ hull\ S$
$\langle proof \rangle$

lemma *conic-hull-eq-span-eq*:

fixes S :: $'a$::*euclidean-space set*
shows $0 \in$ *rel-interior*(*conic hull* S) $\longleftrightarrow$ *conic hull* S = *span* S (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

4 Closure of conic hulls

proposition *closedin-conic-hull*:
fixes S :: $'a$::*euclidean-space set*
assumes *compact* T $0 \notin T$ $T \subseteq S$
shows *closedin* (*top-of-set* (*conic hull* S)) (*conic hull* T)
⟨*proof*⟩

lemma *closed-conic-hull*:
fixes S :: $'a$::*euclidean-space set*
assumes $0 \in$ *rel-interior* S $\vee$ *compact* S $\wedge$ $0 \notin S$
shows *closed*(*conic hull* S)
⟨*proof*⟩

lemma *conic-closure*:
fixes S :: $'a$::*euclidean-space set*
shows *conic* $S \Longrightarrow$ *conic*(*closure* S)
⟨*proof*⟩

lemma *closure-conic-hull*:
fixes S :: $'a$::*euclidean-space set*
assumes $0 \in$ *rel-interior* S $\vee$ *bounded* S $\wedge$ ~($0 \in$ *closure* S)
shows *closure*(*conic hull* S) = *conic hull* (*closure* S)
⟨*proof*⟩

lemma *faces-of-linear-image*:
⟦*linear f*; *inj f*⟧ $\Longrightarrow$ {T. T *face-of* (f ‘ S)} = (*image f*) ‘ {T. T *face-of* S}
⟨*proof*⟩

lemma *face-of-conic*:
assumes *conic* S f *face-of* S
shows *conic* f
⟨*proof*⟩

lemma *extreme-point-of-conic*:
assumes *conic* S **and** x: x *extreme-point-of* S
shows $x = 0$
⟨*proof*⟩

5 Convex cones and corresponding hulls

definition *convex-cone* :: $'a$::*real-vector set* $\Rightarrow$ *bool*
where *convex-cone* $\equiv \lambda S.$ $S \neq \{\}$ $\wedge$ *convex* S $\wedge$ *conic* S

lemma *convex-cone-iff*:
  convex-cone $S \longleftrightarrow$
    $0 \in S \wedge (\forall x \in S. \forall y \in S. x + y \in S) \wedge (\forall x \in S. \forall c \geq 0. c *_R x \in S)$
  ⟨*proof*⟩

lemma *convex-cone-add*: $[\![$*convex-cone* S; $x \in S$; $y \in S]\!] \Longrightarrow x+y \in S$
  ⟨*proof*⟩

lemma *convex-cone-scaleR*: $[\![$*convex-cone* S; $0 \leq c$; $x \in S]\!] \Longrightarrow c *_R x \in S$
  ⟨*proof*⟩

lemma *convex-cone-nonempty*: *convex-cone* $S \Longrightarrow S \neq \{\}$
  ⟨*proof*⟩

lemma *convex-cone-linear-image*:
  convex-cone $S \wedge$ *linear* $f \Longrightarrow$ *convex-cone*$(f$ ‘ $S)$
  ⟨*proof*⟩

lemma *convex-cone-linear-image-eq*:
  $[\![$*linear* f; *inj* $f]\!] \Longrightarrow$ (*convex-cone*$(f$ ‘ $S) \longleftrightarrow$ *convex-cone* S)
  ⟨*proof*⟩

lemma *convex-cone-halfspace-ge*: *convex-cone* $\{x.\ a \cdot x \geq 0\}$
  ⟨*proof*⟩

lemma *convex-cone-halfspace-le*: *convex-cone* $\{x.\ a \cdot x \leq 0\}$
  ⟨*proof*⟩

lemma *convex-cone-contains-0*: *convex-cone* $S \Longrightarrow 0 \in S$
  ⟨*proof*⟩

lemma *convex-cone-Inter*:
  $(\bigwedge S.\ S \in f \Longrightarrow$ *convex-cone* $S) \Longrightarrow$ *convex-cone*$(\bigcap f)$
  ⟨*proof*⟩

lemma *convex-cone-convex-cone-hull*: *convex-cone*(*convex-cone hull* S)
  ⟨*proof*⟩

lemma *convex-convex-cone-hull*: *convex*(*convex-cone hull* S)
  ⟨*proof*⟩

lemma *conic-convex-cone-hull*: *conic*(*convex-cone hull* S)
  ⟨*proof*⟩

lemma *convex-cone-hull-nonempty*: *convex-cone hull* $S \neq \{\}$
  ⟨*proof*⟩

lemma *convex-cone-hull-contains-0*: $0 \in$ *convex-cone hull* S

$\langle proof \rangle$

lemma *convex-cone-hull-add*:
$[\![x \in \textit{convex-cone hull } S;\ y \in \textit{convex-cone hull } S]\!] \Longrightarrow x + y \in \textit{convex-cone hull } S$
$\langle proof \rangle$

lemma *convex-cone-hull-mul*:
$[\![x \in \textit{convex-cone hull } S;\ 0 \le c]\!] \Longrightarrow (c *_R x) \in \textit{convex-cone hull } S$
$\langle proof \rangle$

lemma *convex-cone-sums*:
$[\![\textit{convex-cone } S;\ \textit{convex-cone } T]\!] \Longrightarrow \textit{convex-cone } (\bigcup x\in S.\ \bigcup y \in T.\ \{x + y\})$
$\langle proof \rangle$

lemma *convex-cone-Times*:
$[\![\textit{convex-cone } S;\ \textit{convex-cone } T]\!] \Longrightarrow \textit{convex-cone}(S \times T)$
$\langle proof \rangle$

lemma *convex-cone-Times-D1*: $\textit{convex-cone } (S \times T) \Longrightarrow \textit{convex-cone } S$
$\langle proof \rangle$

lemma *convex-cone-Times-eq*:
$\textit{convex-cone}(S \times T) \longleftrightarrow \textit{convex-cone } S \wedge \textit{convex-cone } T$
$\langle proof \rangle$

lemma *convex-cone-hull-Un*:
$\textit{convex-cone hull}(S \cup T) = (\bigcup x \in \textit{convex-cone hull } S.\ \bigcup y \in \textit{convex-cone hull } T.\ \{x + y\})$
(**is** *?lhs = ?rhs*)
$\langle proof \rangle$

lemma *convex-cone-singleton* [*iff*]: $\textit{convex-cone } \{0\}$
$\langle proof \rangle$

lemma *convex-hull-subset-convex-cone-hull*:
$\textit{convex hull } S \subseteq \textit{convex-cone hull } S$
$\langle proof \rangle$

lemma *conic-hull-subset-convex-cone-hull*:
$\textit{conic hull } S \subseteq \textit{convex-cone hull } S$
$\langle proof \rangle$

lemma *subspace-imp-convex-cone*: $\textit{subspace } S \Longrightarrow \textit{convex-cone } S$
$\langle proof \rangle$

lemma *convex-cone-span*: $\textit{convex-cone}(\textit{span } S)$

$\langle proof \rangle$

lemma *convex-cone-negations*:
$\textit{convex-cone } S \Longrightarrow \textit{convex-cone } (\textit{image uminus } S)$
$\langle proof \rangle$

lemma *subspace-convex-cone-symmetric*:
$\textit{subspace } S \longleftrightarrow \textit{convex-cone } S \wedge (\forall x \in S.\ -x \in S)$
$\langle proof \rangle$

lemma *convex-cone-hull-separate-nonempty*:
assumes $S \neq \{\}$
shows $\textit{convex-cone hull } S = \textit{conic hull } (\textit{convex hull } S)$ (**is** $?lhs = ?rhs$)
$\langle proof \rangle$

lemma *convex-cone-hull-empty* [*simp*]: $\textit{convex-cone hull } \{\} = \{0\}$
$\langle proof \rangle$

lemma *convex-cone-hull-separate*:
$\textit{convex-cone hull } S = \textit{insert } 0\ (\textit{conic hull } (\textit{convex hull } S))$
$\langle proof \rangle$

lemma *convex-cone-hull-convex-hull-nonempty*:
$S \neq \{\} \Longrightarrow \textit{convex-cone hull } S = (\bigcup x \in \textit{convex hull } S.\ \bigcup c\in\{0..\}.\ \{c *_R x\})$
$\langle proof \rangle$

lemma *convex-cone-hull-convex-hull*:
$\textit{convex-cone hull } S = \textit{insert } 0\ (\bigcup x \in \textit{convex hull } S.\ \bigcup c\in\{0..\}.\ \{c *_R x\})$
$\langle proof \rangle$

lemma *convex-cone-hull-linear-image*:
$\textit{linear } f \Longrightarrow \textit{convex-cone hull } (f \text{ ` } S) = \textit{image } f\ (\textit{convex-cone hull } S)$
$\langle proof \rangle$

5.1 Finitely generated cone is polyhedral, and hence closed

proposition *polyhedron-convex-cone-hull*:
fixes $S :: 'a::\textit{euclidean-space set}$
assumes *finite S*
shows $\textit{polyhedron}(\textit{convex-cone hull } S)$
$\langle proof \rangle$

lemma *closed-convex-cone-hull*:
fixes $S :: 'a::\textit{euclidean-space set}$
shows $\textit{finite } S \Longrightarrow \textit{closed}(\textit{convex-cone hull } S)$

$\langle proof \rangle$

lemma *polyhedron-convex-cone-hull-polytope*:
fixes S :: $'a$::*euclidean-space set*
shows *polytope* $S \Longrightarrow$ *polyhedron*(*convex-cone hull* S)
$\langle proof \rangle$

lemma *polyhedron-conic-hull-polytope*:
fixes S :: $'a$::*euclidean-space set*
shows *polytope* $S \Longrightarrow$ *polyhedron*(*conic hull* S)
$\langle proof \rangle$

lemma *closed-conic-hull-strong*:
fixes S :: $'a$::*euclidean-space set*
shows $0 \in$ *rel-interior* $S \vee$ *polytope* $S \vee$ *compact* $S \wedge$ ~$(0 \in S) \Longrightarrow$ *closed*(*conic hull* S)
$\langle proof \rangle$

end

6 Inclusion-exclusion principle

Inclusion-exclusion principle, the usual and generalized forms.

theory *Inclusion-Exclusion*
imports *Main*
begin

lemma *subset-insert-lemma*:
$\{T.\ T \subseteq (insert\ a\ S) \wedge P\ T\} = \{T.\ T \subseteq S \wedge P\ T\} \cup \{insert\ a\ T\ | T.\ T \subseteq S \wedge P(insert\ a\ T)\}$ (**is** *?L=?R*)
$\langle proof \rangle$

locale *Incl-Excl* =
fixes P :: $'a\ set \Rightarrow bool$ **and** f :: $'a\ set \Rightarrow 'b$::*ring-1*
assumes *disj-add*: $[\![P\ S;\ P\ T;\ disjnt\ S\ T]\!] \Longrightarrow f(S \cup T) = f\ S + f\ T$
and *empty*: $P\{\}$
and *Int*: $[\![P\ S;\ P\ T]\!] \Longrightarrow P(S \cap T)$
and *Un*: $[\![P\ S;\ P\ T]\!] \Longrightarrow P(S \cup T)$
and *Diff*: $[\![P\ S;\ P\ T]\!] \Longrightarrow P(S - T)$

begin

lemma *f-empty* [*simp*]: $f\{\} = 0$
$\langle proof \rangle$

lemma *f-Un-Int*: $[\![P\ S;\ P\ T]\!] \Longrightarrow f(S \cup T) + f(S \cap T) = f\ S + f\ T$
$\langle proof \rangle$

lemma *restricted-indexed*:
assumes *finite* A **and** X: $\bigwedge a.\ a \in A \Longrightarrow P(X\ a)$
shows $f(\bigcup(X\ `\ A)) = (\sum B \mid B \subseteq A \land B \neq \{\}.\ (-\ 1)\ \hat{}\ (card\ B + 1) * f\ (\bigcap (X\ `\ B)))$
⟨*proof*⟩

lemma *restricted*:
assumes *finite* A $\bigwedge a.\ a \in A \Longrightarrow P\ a$
shows $f(\bigcup\ A) = (\sum B \mid B \subseteq A \land B \neq \{\}.\ (-\ 1)\ \hat{}\ (card\ B + 1) * f\ (\bigcap\ B))$
⟨*proof*⟩

end

6.1 Versions for unrestrictedly additive functions

lemma *Incl-Excl-UN*:
fixes $f :: {'a}\ set \Rightarrow {'b}::ring\text{-}1$
assumes $\bigwedge S\ T.\ disjnt\ S\ T \Longrightarrow f(S \cup T) = f\ S + f\ T$ *finite* A
shows $f(\bigcup(G\ `\ A)) = (\sum B \mid B \subseteq A \land B \neq \{\}.\ (-1)\ \hat{}\ (card\ B + 1) * f\ (\bigcap (G\ `\ B)))$
⟨*proof*⟩

lemma *Incl-Excl-Union*:
fixes $f :: {'a}\ set \Rightarrow {'b}::ring\text{-}1$
assumes $\bigwedge S\ T.\ disjnt\ S\ T \Longrightarrow f(S \cup T) = f\ S + f\ T$ *finite* A
shows $f(\bigcup\ A) = (\sum B \mid B \subseteq A \land B \neq \{\}.\ (-\ 1)\ \hat{}\ (card\ B + 1) * f\ (\bigcap\ B))$
⟨*proof*⟩

The famous inclusion-exclusion formula for the cardinality of a union

lemma *int-card-UNION*:
assumes *finite* A $\bigwedge K.\ K \in A \Longrightarrow$ *finite* K
shows $int\ (card\ (\bigcup A)) = (\sum I \mid I \subseteq A \land I \neq \{\}.\ (-\ 1)\ \hat{}\ (card\ I + 1) * int\ (card\ (\bigcap I)))$
⟨*proof*⟩

A more conventional form

lemma *inclusion-exclusion*:
assumes *finite* A $\bigwedge K.\ K \in A \Longrightarrow$ *finite* K
shows $int(card(\bigcup\ A)) =$
$(\sum n{=}1..card\ A.\ (-1)\ \hat{}\ (Suc\ n) * (\sum B \mid B \subseteq A \land card\ B = n.\ int\ (card\ (\bigcap\ B))))$ (**is** $\text{-}{=}?R$)
⟨*proof*⟩

lemma *card-UNION*:
assumes *finite* A **and** $\bigwedge K.\ K \in A \Longrightarrow$ *finite* K
shows $card\ (\bigcup A) = nat\ (\sum I \mid I \subseteq A \land I \neq \{\}.\ (-\ 1)\ \hat{}\ (card\ I + 1) * int\ (card\ (\bigcap I)))$
⟨*proof*⟩

lemma *card-UNION-nonneg*:

assumes *finite* A **and** $\bigwedge K.\ K \in A \Longrightarrow$ *finite* K
shows $(\sum I \mid I \subseteq A \wedge I \neq \{\}.\ (-\ 1)\ \hat{}\ (card\ I + 1) * int\ (card\ (\bigcap I))) \geq 0$
$\langle proof \rangle$

6.2 a general "Moebius inversion" inclusion-exclusion principle. This "symmetric" form is from Ira Gessel: "Symmetric Inclusion-Exclusion"

lemma *sum-Un-eq*:
$[\![S \cap T = \{\};\ S \cup T = U;\ finite\ U]\!]$
$\Longrightarrow (sum\ f\ S + sum\ f\ T = sum\ f\ U)$
$\langle proof \rangle$

lemma *card-adjust-lemma*: $[\![inj\text{-}on\ f\ S;\ x = y + card\ (f\ `\ S)]\!] \Longrightarrow x = y + card\ S$
$\langle proof \rangle$

lemma *card-subsets-step*:
assumes *finite* $S\ x \notin S\ U \subseteq S$
shows $card\ \{T.\ T \subseteq (insert\ x\ S) \wedge U \subseteq T \wedge odd(card\ T)\}$
$= card\ \{T.\ T \subseteq S \wedge U \subseteq T \wedge odd(card\ T)\} + card\ \{T.\ T \subseteq S \wedge U \subseteq T$
$\wedge\ even(card\ T)\} \wedge$
$card\ \{T.\ T \subseteq (insert\ x\ S) \wedge U \subseteq T \wedge even(card\ T)\}$
$= card\ \{T.\ T \subseteq S \wedge U \subseteq T \wedge even(card\ T)\} + card\ \{T.\ T \subseteq S \wedge U \subseteq T$
$\wedge\ odd(card\ T)\}$
$\langle proof \rangle$

lemma *card-subsupersets-even-odd*:
assumes *finite* $S\ U \subset S$
shows $card\ \{T.\ T \subseteq S \wedge U \subseteq T \wedge even(card\ T)\}$
$= card\ \{T.\ T \subseteq S \wedge U \subseteq T \wedge odd(card\ T)\}$
$\langle proof \rangle$

lemma *sum-alternating-cancels*:
assumes *finite* S $card\ \{x.\ x \in S \wedge even(f\ x)\} = card\ \{x.\ x \in S \wedge odd(f\ x)\}$
shows $(\sum x{\in}S.\ (-1)\ \hat{}\ f\ x) = (0{::}'b{::}ring\text{-}1)$
$\langle proof \rangle$

lemma *inclusion-exclusion-symmetric*:
fixes $f :: 'a\ set \Rightarrow 'b{::}ring\text{-}1$
assumes §: $\bigwedge S.\ finite\ S \Longrightarrow g\ S = (\sum T \in Pow\ S.\ (-1)\ \hat{}\ card\ T * f\ T)$
and *finite* S
shows $f\ S = (\sum T \in Pow\ S.\ (-1)\ \hat{}\ card\ T * g\ T)$
$\langle proof \rangle$

The more typical non-symmetric version.

lemma *inclusion-exclusion-mobius*:
fixes $f :: 'a\ set \Rightarrow 'b{::}ring\text{-}1$
assumes §: $\bigwedge S.\ finite\ S \Longrightarrow g\ S = sum\ f\ (Pow\ S)$ **and** *finite* S
shows $f\ S = (\sum T \in Pow\ S.\ (-1)\ \hat{}\ (card\ S - card\ T) * g\ T)$ (**is** - = *?rhs*)

$\langle proof \rangle$

end

7 Euler's Polyhedron Formula

One of the Famous 100 Theorems, ported from HOL Light

Cited source: Lawrence, J. (1997). A Short Proof of Euler's Relation for Convex Polytopes. *Canadian Mathematical Bulletin*, **40**(4), 471–474.

theory *Euler-Formula*
imports
HOL−Analysis.Analysis
Library-Extras
Inclusion-Exclusion
begin

Interpret which "side" of a hyperplane a point is on.

definition *hyperplane-side*
where $hyperplane\text{-}side \equiv \lambda(a,b).\ \lambda x.\ sgn\ (a \cdot x - b)$

Equivalence relation imposed by a hyperplane arrangement.

definition *hyperplane-equiv*
where $hyperplane\text{-}equiv \equiv \lambda A\ x\ y.\ \forall h \in A.\ hyperplane\text{-}side\ h\ x = hyperplane\text{-}side\ h\ y$

lemma *hyperplane-equiv-refl* [*iff*]: $hyperplane\text{-}equiv\ A\ x\ x$
$\langle proof \rangle$

lemma *hyperplane-equiv-sym*:
$hyperplane\text{-}equiv\ A\ x\ y \longleftrightarrow hyperplane\text{-}equiv\ A\ y\ x$
$\langle proof \rangle$

lemma *hyperplane-equiv-trans*:
$[\![hyperplane\text{-}equiv\ A\ x\ y;\ hyperplane\text{-}equiv\ A\ y\ z]\!] \Longrightarrow hyperplane\text{-}equiv\ A\ x\ z$
$\langle proof \rangle$

lemma *hyperplane-equiv-Un*:
$hyperplane\text{-}equiv\ (A \cup B)\ x\ y \longleftrightarrow hyperplane\text{-}equiv\ A\ x\ y \wedge hyperplane\text{-}equiv\ B\ x\ y$
$\langle proof \rangle$

7.1 Cells of a hyperplane arrangement

definition $hyperplane\text{-}cell :: ('a::real\text{-}inner \times real)\ set \Rightarrow 'a\ set \Rightarrow bool$
where $hyperplane\text{-}cell \equiv \lambda A\ C.\ \exists x.\ C = Collect\ (hyperplane\text{-}equiv\ A\ x)$

lemma *hyperplane-cell*: $hyperplane\text{-}cell\ A\ C \longleftrightarrow (\exists x.\ C = \{y.\ hyperplane\text{-}equiv\ A\ x\ y\})$

⟨*proof*⟩

lemma *not-hyperplane-cell-empty* [*simp*]: $\neg$ *hyperplane-cell* A {}
⟨*proof*⟩

lemma *nonempty-hyperplane-cell*: *hyperplane-cell* A $C \Longrightarrow (C \neq \{\})$
⟨*proof*⟩

lemma *Union-hyperplane-cells*: $\bigcup$ {C. *hyperplane-cell* A C} = *UNIV*
⟨*proof*⟩

lemma *disjoint-hyperplane-cells*:
⟦*hyperplane-cell* A *C1*; *hyperplane-cell* A *C2*; *C1* $\neq$ *C2*⟧ $\Longrightarrow$ *disjnt C1 C2*
⟨*proof*⟩

lemma *disjoint-hyperplane-cells-eq*:
⟦*hyperplane-cell* A *C1*; *hyperplane-cell* A *C2*⟧ $\Longrightarrow$ (*disjnt C1 C2* $\longleftrightarrow$ (*C1* $\neq$ *C2*))
⟨*proof*⟩

lemma *hyperplane-cell-empty* [*iff*]: *hyperplane-cell* {} $C \longleftrightarrow C$ = *UNIV*
⟨*proof*⟩

lemma *hyperplane-cell-singleton-cases*:
assumes *hyperplane-cell* $\{(a,b)\}$ C
shows $C = \{x.\ a \cdot x = b\} \vee C = \{x.\ a \cdot x < b\} \vee C = \{x.\ a \cdot x > b\}$
⟨*proof*⟩

lemma *hyperplane-cell-singleton*:
hyperplane-cell $\{(a,b)\}$ $C \longleftrightarrow$
(*if* $a = 0$ *then* C = *UNIV* *else* $C = \{x.\ a \cdot x = b\} \vee C = \{x.\ a \cdot x < b\} \vee C = \{x.\ a \cdot x > b\}$)
⟨*proof*⟩

lemma *hyperplane-cell-Un*:
hyperplane-cell $(A \cup B)$ $C \longleftrightarrow$
$C \neq \{\} \wedge$
($\exists$ *C1 C2*. *hyperplane-cell* A *C1* $\wedge$ *hyperplane-cell* B *C2* $\wedge$ C = *C1* $\cap$ *C2*)
⟨*proof*⟩

lemma *finite-hyperplane-cells*:
finite $A \Longrightarrow$ *finite* {C. *hyperplane-cell* A C}
⟨*proof*⟩

lemma *finite-restrict-hyperplane-cells*:
finite $A \Longrightarrow$ *finite* {C. *hyperplane-cell* A C $\wedge$ P C}
⟨*proof*⟩

lemma *finite-set-of-hyperplane-cells*:

$[\![$*finite* A; $\bigwedge C$. $C \in \mathcal{C} \Longrightarrow$ *hyperplane-cell* A $C]\!] \Longrightarrow$ *finite* $\mathcal{C}$
⟨*proof*⟩

lemma *pairwise-disjoint-hyperplane-cells*:
($\bigwedge C$. $C \in \mathcal{C} \Longrightarrow$ *hyperplane-cell* A C) $\Longrightarrow$ *pairwise disjnt* $\mathcal{C}$
⟨*proof*⟩

lemma *hyperplane-cell-Int-open-affine*:
assumes *finite* A *hyperplane-cell* A C
obtains S T **where** *open* S *affine* T $C = S \cap T$
⟨*proof*⟩

lemma *hyperplane-cell-relatively-open*:
assumes *finite* A *hyperplane-cell* A C
shows *openin* (*subtopology euclidean* (*affine hull* C)) C
⟨*proof*⟩

lemma *hyperplane-cell-relative-interior*:
$[\![$*finite* A; *hyperplane-cell* A $C]\!] \Longrightarrow$ *rel-interior* $C = C$
⟨*proof*⟩

lemma *hyperplane-cell-convex*:
assumes *hyperplane-cell* A C
shows *convex* C
⟨*proof*⟩

lemma *hyperplane-cell-Inter*:
assumes $\bigwedge C$. $C \in \mathcal{C} \Longrightarrow$ *hyperplane-cell* A C
and $\mathcal{C} \neq \{\}$ **and** *INT*: $\bigcap \mathcal{C} \neq \{\}$
shows *hyperplane-cell* A ($\bigcap \mathcal{C}$)
⟨*proof*⟩

lemma *hyperplane-cell-Int*:
$[\![$*hyperplane-cell* A S; *hyperplane-cell* A T; $S \cap T \neq \{\}]\!] \Longrightarrow$ *hyperplane-cell* A ($S \cap T$)
⟨*proof*⟩

7.2 A cell complex is considered to be a union of such cells

definition *hyperplane-cellcomplex*
where *hyperplane-cellcomplex* A $S \equiv$
$\exists \mathcal{T}$. ($\forall C \in \mathcal{T}$. *hyperplane-cell* A C) $\wedge$ $S = \bigcup \mathcal{T}$

lemma *hyperplane-cellcomplex-empty* [*simp*]: *hyperplane-cellcomplex* A $\{\}$
⟨*proof*⟩

lemma *hyperplane-cell-cellcomplex*:
hyperplane-cell A $C \Longrightarrow$ *hyperplane-cellcomplex* A C

⟨*proof*⟩

lemma *hyperplane-cellcomplex-Union*:
assumes $\bigwedge S.\ S \in \mathcal{C} \Longrightarrow$ *hyperplane-cellcomplex A S*
shows *hyperplane-cellcomplex A* $(\bigcup\ \mathcal{C})$
⟨*proof*⟩

lemma *hyperplane-cellcomplex-Un*:
⟦*hyperplane-cellcomplex A S*; *hyperplane-cellcomplex A T*⟧
$\Longrightarrow$ *hyperplane-cellcomplex A* $(S \cup T)$
⟨*proof*⟩

lemma *hyperplane-cellcomplex-UNIV* [*simp*]: *hyperplane-cellcomplex A UNIV*
⟨*proof*⟩

lemma *hyperplane-cellcomplex-Inter*:
assumes $\bigwedge S.\ S \in \mathcal{C} \Longrightarrow$ *hyperplane-cellcomplex A S*
shows *hyperplane-cellcomplex A* $(\bigcap \mathcal{C})$
⟨*proof*⟩

lemma *hyperplane-cellcomplex-Int*:
⟦*hyperplane-cellcomplex A S*; *hyperplane-cellcomplex A T*⟧
$\Longrightarrow$ *hyperplane-cellcomplex A* $(S \cap T)$
⟨*proof*⟩

lemma *hyperplane-cellcomplex-Compl*:
assumes *hyperplane-cellcomplex A S*
shows *hyperplane-cellcomplex A* $(-\ S)$
⟨*proof*⟩

lemma *hyperplane-cellcomplex-diff*:
⟦*hyperplane-cellcomplex A S*; *hyperplane-cellcomplex A T*⟧
$\Longrightarrow$ *hyperplane-cellcomplex A* $(S\ -\ T)$
⟨*proof*⟩

lemma *hyperplane-cellcomplex-mono*:
assumes *hyperplane-cellcomplex A S* $A \subseteq B$
shows *hyperplane-cellcomplex B S*
⟨*proof*⟩

lemma *finite-hyperplane-cellcomplexes*:
assumes *finite A*
shows *finite* $\{C.$ *hyperplane-cellcomplex A C*$\}$
⟨*proof*⟩

lemma *finite-restrict-hyperplane-cellcomplexes*:
finite A $\Longrightarrow$ *finite* $\{C.$ *hyperplane-cellcomplex A C* $\wedge$ *P C*$\}$
⟨*proof*⟩

lemma *finite-set-of-hyperplane-cellcomplex*:
  assumes *finite A* $\bigwedge C.\ C \in \mathcal{C} \Longrightarrow$ *hyperplane-cellcomplex A C*
  shows *finite* $\mathcal{C}$
  $\langle proof \rangle$

lemma *cell-subset-cellcomplex*:
  $[\![$*hyperplane-cell A C*; *hyperplane-cellcomplex A S*$]\!] \Longrightarrow C \subseteq S \longleftrightarrow \sim$ *disjnt C S*
  $\langle proof \rangle$

7.3 Euler characteristic

definition *Euler-characteristic* :: $('a::euclidean\text{-}space \times real)\ set \Rightarrow 'a\ set \Rightarrow int$
  where *Euler-characteristic A S* $\equiv$
    $(\sum C \mid hyperplane\text{-}cell\ A\ C \wedge C \subseteq S.\ (-1)\ \hat{}\ nat\ (aff\text{-}dim\ C))$

lemma *Euler-characteristic-empty* [*simp*]: *Euler-characteristic A* $\{\} = 0$
  $\langle proof \rangle$

lemma *Euler-characteristic-cell-Union*:
  assumes $\bigwedge C.\ C \in \mathcal{C} \Longrightarrow$ *hyperplane-cell A C*
  shows *Euler-characteristic A* $(\bigcup \mathcal{C}) = (\sum C{\in}\mathcal{C}.\ (-\ 1)\ \hat{}\ nat\ (aff\text{-}dim\ C))$
$\langle proof \rangle$

lemma *Euler-characteristic-cell*:
  hyperplane-cell A C $\Longrightarrow$ *Euler-characteristic A C* $= (-1)\ \hat{}(nat(aff\text{-}dim\ C))$
  $\langle proof \rangle$

lemma *Euler-characteristic-cellcomplex-Un*:
  assumes *finite A hyperplane-cellcomplex A S*
    and *AT*: *hyperplane-cellcomplex A T* **and** *disjnt S T*
  shows *Euler-characteristic A* $(S \cup T) =$
    Euler-characteristic A S + *Euler-characteristic A T*
$\langle proof \rangle$

lemma *Euler-characteristic-cellcomplex-Union*:
  assumes *finite A*
    and $\mathcal{C}$: $\bigwedge C.\ C \in \mathcal{C} \Longrightarrow$ *hyperplane-cellcomplex A C pairwise disjnt* $\mathcal{C}$
  shows *Euler-characteristic A* $(\bigcup \mathcal{C}) =$ *sum* (*Euler-characteristic A*) $\mathcal{C}$
$\langle proof \rangle$

lemma *Euler-characteristic*:
  fixes $A :: ('n::euclidean\text{-}space * real)\ set$
  assumes *finite A*
  shows *Euler-characteristic A S* =
    $(\sum d = 0..DIM('n).\ (-1)\ \hat{}\ d * int\ (card\ \{C.\ hyperplane\text{-}cell\ A\ C \wedge C \subseteq S \wedge aff\text{-}dim\ C = int\ d\}))$
    (**is** - = *?rhs*)
$\langle proof \rangle$

7.4 Show that the characteristic is invariant w.r.t. hyperplane arrangement.

lemma *hyperplane-cells-distinct-lemma*:
$\{x.\ a \cdot x = b\} \cap \{x.\ a \cdot x < b\} = \{\} \wedge$
$\{x.\ a \cdot x = b\} \cap \{x.\ a \cdot x > b\} = \{\} \wedge$
$\{x.\ a \cdot x < b\} \cap \{x.\ a \cdot x = b\} = \{\} \wedge$
$\{x.\ a \cdot x < b\} \cap \{x.\ a \cdot x > b\} = \{\} \wedge$
$\{x.\ a \cdot x > b\} \cap \{x.\ a \cdot x = b\} = \{\} \wedge$
$\{x.\ a \cdot x > b\} \cap \{x.\ a \cdot x < b\} = \{\}$
⟨*proof*⟩

proposition *Euler-characterstic-lemma*:
assumes *finite A* **and** *hyperplane-cellcomplex A S*
shows *Euler-characteristic* (*insert h A*) *S* = *Euler-characteristic A S*
⟨*proof*⟩

lemma *Euler-characterstic-invariant-aux*:
assumes *finite B finite A hyperplane-cellcomplex A S*
shows *Euler-characteristic* ($A \cup B$) *S* = *Euler-characteristic A S*
⟨*proof*⟩

lemma *Euler-characterstic-invariant*:
assumes *finite A finite B hyperplane-cellcomplex A S hyperplane-cellcomplex B S*
shows *Euler-characteristic A S* = *Euler-characteristic B S*
⟨*proof*⟩

lemma *Euler-characteristic-inclusion-exclusion*:
assumes *finite A finite* $\mathcal{S}$ $\bigwedge K.\ K \in \mathcal{S} \Longrightarrow$ *hyperplane-cellcomplex A K*
shows *Euler-characteristic A* $(\bigcup \mathcal{S}) = (\sum \mathcal{T} \mid \mathcal{T} \subseteq \mathcal{S} \wedge \mathcal{T} \neq \{\}.\ (-\ 1)\ \hat{}\ (card\ \mathcal{T} + 1) * \textit{Euler-characteristic}\ A\ (\bigcap \mathcal{T}))$
⟨*proof*⟩

7.5 Euler-type relation for full-dimensional proper polyhedral cones

lemma *Euler-polyhedral-cone*:
fixes *S* :: *'n::euclidean-space set*
assumes *polyhedron S conic S* **and** *intS*: *interior* $S \neq \{\}$ **and** $S \neq UNIV$
shows $(\sum d = 0..DIM('n).\ (-\ 1)\ \hat{}\ d * int\ (card\ \{f.\ f\ \textit{face-of}\ S \wedge \textit{aff-dim}\ f = int\ d\})) = 0$ (**is** *?lhs = 0*)
⟨*proof*⟩

7.6 Euler-Poincare relation for special $(n-1)$-dimensional polytope

lemma *Euler-Poincare-lemma*:
fixes *p* :: *'n::euclidean-space set*

assumes $DIM('n) \geq 2$ *polytope p* $i \in Basis$ **and** *affp*: *affine hull p* $= \{x.\ x \cdot i = 1\}$
shows $(\sum d = 0..DIM('n) - 1.\ (-1)\ \hat{}\ d * int\ (card\ \{f.\ f\ face\text{-}of\ p \wedge aff\text{-}dim\ f = int\ d\})) = 1$
$\langle proof \rangle$

corollary *Euler-poincare-special*:
fixes $p :: 'n::euclidean\text{-}space\ set$
assumes $2 \leq DIM('n)$ *polytope p* $i \in Basis$ **and** *affp*: *affine hull p* $= \{x.\ x \cdot i = 0\}$
shows $(\sum d = 0..DIM('n) - 1.\ (-1)\ \hat{}\ d * card\ \{f.\ f\ face\text{-}of\ p \wedge aff\text{-}dim\ f = d\}) = 1$
$\langle proof \rangle$

7.7 Now Euler-Poincare for a general full-dimensional polytope

theorem *Euler-Poincare-full*:
fixes $p :: 'n::euclidean\text{-}space\ set$
assumes *polytope p aff-dim p* $= DIM('n)$
shows $(\sum d = 0..DIM('n).\ (-1)\ \hat{}\ d * (card\ \{f.\ f\ face\text{-}of\ p \wedge aff\text{-}dim\ f = d\})) = 1$
$\langle proof \rangle$

In particular, the Euler relation in 3 dimensions

corollary *Euler-relation*:
fixes $p :: 'n::euclidean\text{-}space\ set$
assumes *polytope p aff-dim p* $= 3$ $DIM('n) = 3$
shows $(card\ \{v.\ v\ face\text{-}of\ p \wedge aff\text{-}dim\ v = 0\} + card\ \{f.\ f\ face\text{-}of\ p \wedge aff\text{-}dim\ f = 2\}) - card\ \{e.\ e\ face\text{-}of\ p \wedge aff\text{-}dim\ e = 1\} = 2$
$\langle proof \rangle$

end

References

[1] I. Lakatos. *Proofs and Refutations: The Logic of Mathematical Discovery*. 1976.

[2] J. Lawrence. A short proof of Euler's relation for convex polytopes. *Canadian Mathematical Bulletin*, 40(4):471–474, 1997.