

Euler's Partition Theorem

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Abstract

Euler's Partition Theorem states that the number of partitions with only distinct parts is equal to the number of partitions with only odd parts. The combinatorial proof follows John Harrison's pre-existing HOL Light formalization [1]. To understand the rough idea of the proof, I read the lecture notes of the MIT course 18.312 on Algebraic Combinatorics [2] by Gregg Musiker. This theorem is the 45th theorem of the Top 100 Theorems list.

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1 Euler's Partition Theorem

```
theory Euler-Partition
imports
  Main
  Card-Number-Partitions.Number-Partition
begin
```

1.1 Preliminaries

1.1.1 Additions to Divides Theory

```
lemma power-div-nat:
```

assumes $c \leq b$
assumes $a > 0$
shows $(a :: \text{nat}) \wedge b \text{ div } a \wedge c = a \wedge (b - c)$
 <proof>

1.1.2 Additions to Groups-Big Theory

lemma *sum-div*:
assumes *finite* A
assumes $\bigwedge a. a \in A \implies (b :: 'b :: \text{euclidean-semiring}) \text{ dvd } f a$
shows $(\sum a \in A. f a) \text{ div } b = (\sum a \in A. (f a) \text{ div } b)$
 <proof>

lemma *sum-mod*:
assumes *finite* A
assumes $\bigwedge a. a \in A \implies f a \text{ mod } b = (0 :: 'b :: \text{unique-euclidean-semiring})$
shows $(\sum a \in A. f a) \text{ mod } b = 0$
 <proof>

1.1.3 Additions to Finite-Set Theory

lemma *finite-exponents*:
finite $\{i. 2 \wedge i \leq (n :: \text{nat})\}$
 <proof>

1.2 Binary Encoding of Natural Numbers

definition *bitset* $:: \text{nat} \Rightarrow \text{nat set}$
where
 $\text{bitset } n = \{i. \text{odd } (n \text{ div } (2 \wedge i))\}$

lemma *in-bitset-bound*:
 $b \in \text{bitset } n \implies 2 \wedge b \leq n$
 <proof>

lemma *in-bitset-bound-weak*:
 $b \in \text{bitset } n \implies b \leq n$
 <proof>

lemma *finite-bitset*:
finite $(\text{bitset } n)$
 <proof>

lemma *bitset-0*:
 $\text{bitset } 0 = \{\}$
 <proof>

lemma *bitset-2n*: $\text{bitset } (2 * n) = \text{Suc } ' (\text{bitset } n)$
 <proof>

lemma *bitset-Suc*:

assumes *even n*

shows $\text{bitset } (n + 1) = \text{insert } 0 (\text{bitset } n)$

<proof>

lemma *bitset-2n1*:

$\text{bitset } (2 * n + 1) = \text{insert } 0 (\text{Suc } (\text{bitset } n))$

<proof>

lemma *sum-bitset*:

$(\sum_{i \in \text{bitset } n} 2^i) = n$

<proof>

lemma *binarysum-div*:

assumes *finite B*

shows $(\sum_{i \in B} (2^{::\text{nat}})^i \text{ div } 2^j) = (\sum_{i \in B} \text{if } i < j \text{ then } 0 \text{ else } 2^{(i - j)})$

(is - = $(\sum_{i \in \cdot} ?f i)$

<proof>

lemma *odd-iff*:

assumes *finite B*

shows $\text{odd } (\sum_{i \in B} \text{if } i < x \text{ then } (0^{::\text{nat}}) \text{ else } 2^{(i - x)}) = (x \in B) \text{ (is odd } (\sum_{i \in \cdot} ?s i) = -)$

<proof>

lemma *bitset-sum*:

assumes *finite B*

shows $\text{bitset } (\sum_{i \in B} 2^i) = B$

<proof>

1.3 Decomposition of a Number into a Power of Two and an Odd Number

function *(sequential) index* :: $\text{nat} \Rightarrow \text{nat}$

where

$\text{index } 0 = 0$

$| \text{index } n = (\text{if odd } n \text{ then } 0 \text{ else } \text{Suc } (\text{index } (n \text{ div } 2)))$

<proof>

termination

<proof>

function *(sequential) oddpart* :: $\text{nat} \Rightarrow \text{nat}$

where

$\text{oddpart } 0 = 0$

$| \text{oddpart } n = (\text{if odd } n \text{ then } n \text{ else } \text{oddpart } (n \text{ div } 2))$

<proof>

termination

<proof>

lemma *odd-oddpart:*

$odd (oddpart\ n) \longleftrightarrow n \neq 0$

<proof>

lemma *index-oddpart-decomposition:*

$n = 2^{\wedge (index\ n)} * oddpart\ n$

<proof>

lemma *oddpart-leq:*

$oddpart\ n \leq n$

<proof>

lemma *index-oddpart-unique:*

assumes $odd\ (m :: nat)\ odd\ m'$

shows $(2^{\wedge i} * m = 2^{\wedge i'} * m') \longleftrightarrow (i = i' \wedge m = m')$

<proof>

lemma *index-oddpart:*

assumes $odd\ m$

shows $index\ (2^{\wedge i} * m) = i\ oddpart\ (2^{\wedge i} * m) = m$

<proof>

1.4 Partitions With Only Distinct and Only Odd Parts

definition *odd-of-distinct* :: $(nat \Rightarrow nat) \Rightarrow nat \Rightarrow nat$

where

$odd-of-distinct\ p = (\lambda i. \text{if } odd\ i \text{ then } (\sum j \mid p\ (2^{\wedge j} * i) = 1. 2^{\wedge j}) \text{ else } 0)$

definition *distinct-of-odd* :: $(nat \Rightarrow nat) \Rightarrow nat \Rightarrow nat$

where

$distinct-of-odd\ p = (\lambda i. \text{if } index\ i \in \text{bitset}\ (p\ (oddpart\ i)) \text{ then } 1 \text{ else } 0)$

lemma *odd:*

$odd-of-distinct\ p\ i \neq 0 \implies odd\ i$

<proof>

lemma *distinct-distinct-of-odd:*

$distinct-of-odd\ p\ i \leq 1$

<proof>

lemma *odd-of-distinct:*

assumes $odd-of-distinct\ p\ i \neq 0$

assumes $\bigwedge i. p\ i \neq 0 \implies i \leq n$

shows $1 \leq i \wedge i \leq n$

<proof>

lemma *distinct-of-odd*:

assumes $\bigwedge i. p\ i * i \leq n \wedge i. p\ i \neq 0 \implies \text{odd } i$

assumes *distinct-of-odd* $p\ i \neq 0$

shows $1 \leq i \wedge i \leq n$

<proof>

lemma *odd-distinct*:

assumes $\bigwedge i. p\ i \neq 0 \implies \text{odd } i$

shows *odd-of-distinct* (*distinct-of-odd* p) = p

<proof>

lemma *distinct-odd*:

assumes $\bigwedge i. p\ i \neq 0 \implies 1 \leq i \wedge i \leq n \wedge i. p\ i \leq 1$

shows *distinct-of-odd* (*odd-of-distinct* p) = p

<proof>

lemma *sum-distinct-of-odd*:

assumes $\bigwedge i. p\ i \neq 0 \implies 1 \leq i \wedge i \leq n$

assumes $\bigwedge i. p\ i * i \leq n$

assumes $\bigwedge i. p\ i \neq 0 \implies \text{odd } i$

shows $(\sum_{i \leq n}. \text{distinct-of-odd } p\ i * i) = (\sum_{i \leq n}. p\ i * i)$

<proof>

lemma *leq-n*:

assumes $\forall i. 0 < p\ i \longrightarrow 1 \leq i \wedge i \leq (n::\text{nat})$

assumes $(\sum_{i \leq n}. p\ i * i) = n$

shows $p\ i * i \leq n$

<proof>

lemma *distinct-of-odd-in-distinct-partitions*:

assumes $p \in \{p. p\ \text{partitions } n \wedge (\forall i. p\ i \neq 0 \longrightarrow \text{odd } i)\}$

shows *distinct-of-odd* $p \in \{p. p\ \text{partitions } n \wedge (\forall i. p\ i \leq 1)\}$

<proof>

lemma *odd-of-distinct-in-odd-partitions*:

assumes $p \in \{p. p\ \text{partitions } n \wedge (\forall i. p\ i \leq 1)\}$

shows *odd-of-distinct* $p \in \{p. p\ \text{partitions } n \wedge (\forall i. p\ i \neq 0 \longrightarrow \text{odd } i)\}$

<proof>

1.5 Euler's Partition Theorem

theorem *Euler-partition-theorem*:

$\text{card } \{p. p\ \text{partitions } n \wedge (\forall i. p\ i \leq 1)\} = \text{card } \{p. p\ \text{partitions } n \wedge (\forall i. p\ i \neq 0 \longrightarrow \text{odd } i)\}$

(is $\text{card } ?\text{distinct-partitions} = \text{card } ?\text{odd-partitions}$)

<proof>

end

References

- [1] J. Harrison. Euler's partition theorem and other elementary partition theorems. <https://github.com/jrh13/hol-light/blob/master/100/euler.ml>.
- [2] G. Musiker. Course 18.312: Algebraic combinatorics, 2009. http://ocw.mit.edu/courses/mathematics/18-312-algebraic-combinatorics-spring-2009/readings-and-lecture-notes/MIT18_312S09_lec10_Patitio.pdf.