

Eudoxus Reals

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Abstract

In this project, we present a peculiar construction of the real numbers, called “Eudoxus reals”, using Isabelle/HOL. Similar to the classical method of Dedekind cuts, our approach starts from first principles. However, unlike Dedekind cuts, Eudoxus reals directly derive real numbers from integers, bypassing the intermediate step of constructing rational numbers.

This construction of the real numbers was first discovered by Stephen Schanuel. Schanuel named his construction after the ancient Greek philosopher Eudoxus, who developed a theory of magnitude and proportion to explain the relations between the discrete and the continuous. Our formalization is based on R.D. Arthan’s paper detailing the construction [1]. For establishing the existence of multiplicative inverses for positive slopes, we used the idea of finding a suitable representative from Sławomir Kołodyński’s construction on IsarMathLib which is based on Zermelo–Fraenkel set theory.

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```
theory Slope
imports HOL.Archimedean-Field
begin
```

1 Slopes

1.1 Bounded Functions

```
definition bounded :: ('a  $\Rightarrow$  int)  $\Rightarrow$  bool where
  bounded f  $\longleftrightarrow$  bdd-above (( $\lambda$ z. |f z|) ‘ UNIV)
```

```
lemma boundedI:
  assumes  $\bigwedge z. |f z| \leq C$ 
  shows bounded f
   $\langle$ proof $\rangle$ 
```

```
lemma boundedE[elim]:
  assumes bounded f  $\exists C. (\forall z. |f z| \leq C) \wedge 0 \leq C \implies P$ 
  shows P
   $\langle$ proof $\rangle$ 
```

```
lemma boundedE-strict:
  assumes bounded f  $\exists C. (\forall z. |f z| < C) \wedge 0 < C \implies P$ 
  shows P
   $\langle$ proof $\rangle$ 
```

```
lemma bounded-alt-def: bounded f  $\longleftrightarrow$  ( $\exists C. \forall z. |f z| \leq C$ )  $\langle$ proof $\rangle$ 
```

```
lemma bounded-iff-finite-range: bounded f  $\longleftrightarrow$  finite (range f)
 $\langle$ proof $\rangle$ 
```

```
lemma bounded-constant:
  shows bounded ( $\lambda$ -. c)
   $\langle$ proof $\rangle$ 
```

```
lemma bounded-add:
  assumes bounded f bounded g
  shows bounded ( $\lambda$ z. f z + g z)
   $\langle$ proof $\rangle$ 
```

```
lemma bounded-mult:
  assumes bounded f bounded g
  shows bounded ( $\lambda$ z. f z * g z)
   $\langle$ proof $\rangle$ 
```

```
lemma bounded-mult-const:
  assumes bounded f
```

shows *bounded* $(\lambda z. c * f z)$
 $\langle proof \rangle$

lemma *bounded-uminus*:
assumes *bounded* f
shows *bounded* $(\lambda x. - f x)$
 $\langle proof \rangle$

lemma *bounded-comp*:
assumes *bounded* f
shows *bounded* $(f o g)$ **and** *bounded* $(g o f)$
 $\langle proof \rangle$

1.2 Properties of Slopes

definition *slope* $:: (int \Rightarrow int) \Rightarrow bool$ **where**
 $slope\ f \iff bounded\ (\lambda(m, n). f\ (m + n) - (f\ m + f\ n))$

lemma *bounded-slopeI*:
assumes *bounded* f
shows *slope* f
 $\langle proof \rangle$

lemma *slopeE[elim]*:
assumes *slope* f
obtains C **where** $\bigwedge m\ n. |f\ (m + n) - (f\ m + f\ n)| \leq C\ 0 \leq C$ $\langle proof \rangle$

lemma *slope-add*:
assumes *slope* f *slope* g
shows *slope* $(\lambda z. f\ z + g\ z)$
 $\langle proof \rangle$

lemma *slope-symmetric-bound*:
assumes *slope* f
obtains C **where** $\bigwedge p\ q. |p * f\ q - q * f\ p| \leq (|p| + |q| + 2) * C\ 0 \leq C$
 $\langle proof \rangle$

lemma *slope-linear-bound*:
assumes *slope* f
obtains $A\ B$ **where** $\forall n. |f\ n| \leq A * |n| + B\ 0 \leq A\ 0 \leq B$
 $\langle proof \rangle$

lemma *slope-comp*:
assumes *slope* f *slope* g
shows *slope* $(f o g)$
 $\langle proof \rangle$

lemma *slope-scale*: *slope* $((*)\ a)$ $\langle proof \rangle$

lemma *slope-zero*: *slope* ($\lambda\cdot. 0$) \langle *proof* \rangle

lemma *slope-one*: *slope* *id* \langle *proof* \rangle

lemma *slope-uminus*: *slope* *uminus* \langle *proof* \rangle

lemma *slope-uminus'*:
 assumes *slope* *f*
 shows *slope* ($\lambda x. - f x$)
 \langle *proof* \rangle

lemma *slope-minus*:
 assumes *slope* *f* *slope* *g*
 shows *slope* ($\lambda x. f x - g x$)
 \langle *proof* \rangle

lemma *slope-comp-commute*:
 assumes *slope* *f* *slope* *g*
 shows *bounded* ($\lambda z. (f \circ g) z - (g \circ f) z$)
 \langle *proof* \rangle

lemma *int-set-infiniteI*:
 assumes $\bigwedge C. C \geq 0 \implies \exists N \geq C. N \in (A :: \text{int set})$
 shows *infinite* *A*
 \langle *proof* \rangle

lemma *int-set-infiniteD*:
 assumes *infinite* (*A* :: *int set*) $C \geq 0$
 obtains *z* **where** $z \in A \ C \leq |z|$
 \langle *proof* \rangle

lemma *bounded-odd*:
 fixes *f* :: *int* \Rightarrow *int*
 assumes $\bigwedge z. z < 0 \implies f z = -f (-z) \ \bigwedge n. n > 0 \implies |f n| \leq C$
 shows *bounded* *f*
 \langle *proof* \rangle

lemma *slope-odd*:
 assumes $\bigwedge z. z < 0 \implies f z = -f (-z)$
 $\bigwedge m n. \llbracket m > 0; n > 0 \rrbracket \implies |f (m + n) - (f m + f n)| \leq C$
 shows *slope* *f*
 \langle *proof* \rangle

lemma *slope-bounded-comp-right-abs*:
 assumes *slope* *f* *bounded* (*f* \circ *abs*)
 shows *bounded* *f*
 \langle *proof* \rangle

corollary *slope-finite-range-iff*:

assumes *slope f*
shows *finite (range f) \longleftrightarrow finite (f ‘ {0..}) (is ?lhs \longleftrightarrow ?rhs)*
 <proof>

lemma *slope-positive-lower-bound:*
assumes *slope f infinite (f ‘ {0..} \cap {0<..}) D > 0*
obtains *M where M > 0 \wedge m. m > 0 \implies (m + 1) * D \leq f (m * M)*
 <proof>

1.3 Set Membership of *Inf* and *Sup* on Integers

lemma *int-Inf-mem:*
fixes *S :: int set*
assumes *S \neq {} bdd-below S*
shows *Inf S \in S*
 <proof>

lemma *int-Sup-mem:*
fixes *S :: int set*
assumes *S \neq {} bdd-above S*
shows *Sup S \in S*
 <proof>

end

theory *Eudoxus*
imports *Slope*
begin

2 Eudoxus Reals

2.1 Type Definition

Two slopes are said to be equivalent if their difference is bounded.

definition *eudoxus-rel :: (int \Rightarrow int) \Rightarrow (int \Rightarrow int) \Rightarrow bool (infix \sim_e 50) where*

$$f \sim_e g \equiv \text{slope } f \wedge \text{slope } g \wedge \text{bounded } (\lambda n. f \ n - g \ n)$$

lemma *eudoxus-rel-equiv:*
part-equivp eudoxus-rel
 <proof>

We define the reals as the set of all equivalence classes of the relation (\sim_e).

quotient-type *real = (int \Rightarrow int) / partial: eudoxus-rel*
 <proof>

lemma *real-quot-type: quot-type (\sim_e) Abs-real Rep-real*

$\langle proof \rangle$
lemma *slope-refl*: $slope\ f = (f \sim_e f)$
 $\langle proof \rangle$
declare *slope-refl*[*THEN iffD2, simp*]
lemmas *slope-refl* = *slope-refl*[*THEN iffD1*]
lemma *slope-induct*[*consumes 0, case-names slope*]:
assumes $\bigwedge f. slope\ f \implies P\ (abs-real\ f)$
shows $P\ x$
 $\langle proof \rangle$
lemma *abs-real-eq-iff*: $f \sim_e g \iff slope\ f \wedge slope\ g \wedge abs-real\ f = abs-real\ g$
 $\langle proof \rangle$
lemma *abs-real-eqI*[*intro*]: $f \sim_e g \implies abs-real\ f = abs-real\ g$ $\langle proof \rangle$
lemmas *eudoxus-rel-sym*[*sym*] = *Quotient-symp*[*OF Quotient-real, THEN sympD*]
lemmas *eudoxus-rel-trans*[*trans*] = *Quotient-transp*[*OF Quotient-real, THEN transpD*]
lemmas *rep-real-abs-real-refl* = *Quotient-rep-abs*[*OF Quotient-real, OF slope-refl*[*THEN iffD1*], *intro!*]
lemmas *rep-real-iff* = *Quotient-rel-rep*[*OF Quotient-real, iff*]
declare *Quotient-abs-rep*[*OF Quotient-real, simp*]
lemma *slope-rep-real*: $slope\ (rep-real\ x)$ $\langle proof \rangle$
lemma *eudoxus-relI*:
assumes $slope\ f\ slope\ g \wedge n. n \geq N \implies |f\ n - g\ n| \leq C$
shows $f \sim_e g$
 $\langle proof \rangle$

2.2 Addition and Subtraction

We define addition, subtraction and the additive identity as follows.

instantiation *real* :: {*zero, plus, minus, uminus*}
begin

quotient-definition

0 :: *real* **is** *abs-real* ($\lambda-. 0$) $\langle proof \rangle$

declare *slope-zero*[*intro!, simp*]

lemma *zero-iff-bounded*: $f \sim_e (\lambda-. 0) \iff bounded\ f$ $\langle proof \rangle$

lemma *zero-iff-bounded'*: $x = 0 \iff bounded\ (rep-real\ x)$ $\langle proof \rangle$

lemma *zero-def*: $0 = \text{abs-real } (\lambda-. 0)$ $\langle \text{proof} \rangle$

definition *eudoxus-plus* :: $(\text{int} \Rightarrow \text{int}) \Rightarrow (\text{int} \Rightarrow \text{int}) \Rightarrow (\text{int} \Rightarrow \text{int})$ (**infixl** $+_e$ 60) **where**

$(f :: \text{int} \Rightarrow \text{int}) +_e g = (\lambda z. f z + g z)$

declare *slope-add*[*intro*, *simp*]

quotient-definition

$(+)$:: $(\text{real} \Rightarrow \text{real} \Rightarrow \text{real})$ **is** $(+_e)$
 $\langle \text{proof} \rangle$

lemmas *eudoxus-plus-cong* = *apply-rsp'*[*OF plus-real.rsp*, *THEN rel-funD*, *intro*]

lemma *abs-real-plus*[*simp*]:

assumes *slope f slope g*

shows $\text{abs-real } f + \text{abs-real } g = \text{abs-real } (f +_e g)$

$\langle \text{proof} \rangle$

definition *eudoxus-uminus* :: $(\text{int} \Rightarrow \text{int}) \Rightarrow (\text{int} \Rightarrow \text{int})$ $(-_e)$ **where**

$-_e (f :: \text{int} \Rightarrow \text{int}) = (\lambda x. - f x)$

declare *slope-uminus'*[*intro*, *simp*]

quotient-definition

(uminus) :: $(\text{real} \Rightarrow \text{real})$ **is** $-_e$
 $\langle \text{proof} \rangle$

lemmas *eudoxus-uminus-cong* = *apply-rsp'*[*OF uminus-real.rsp*, *simplified*, *intro*]

lemma *abs-real-uminus*[*simp*]:

assumes *slope f*

shows $-\text{abs-real } f = \text{abs-real } (-_e f)$

$\langle \text{proof} \rangle$

definition $x - (y :: \text{real}) = x + - y$

declare *slope-minus*[*intro*, *simp*]

lemma *abs-real-minus*[*simp*]:

assumes *slope g slope f*

shows $\text{abs-real } g - \text{abs-real } f = \text{abs-real } (g +_e (-_e f))$

$\langle \text{proof} \rangle$

instance $\langle \text{proof} \rangle$

end

The Eudoxus reals equipped with addition and negation specified as above constitute an Abelian group.

instance *real* :: *ab-group-add*
 ⟨*proof*⟩

2.3 Multiplication

We define multiplication as the composition of two slopes.

instantiation *real* :: {*one, times*}
begin

quotient-definition

1 :: *real* **is** *abs-real id* ⟨*proof*⟩

declare *slope-one*[*intro!*, *simp*]

lemma *one-def*: *1 = abs-real id* ⟨*proof*⟩

definition *eudoxus-times* :: (*int* ⇒ *int*) ⇒ (*int* ⇒ *int*) ⇒ *int* ⇒ *int* (**infixl** $*_e$ 60)

where

f $*_e$ *g* = *f o g*

declare *slope-comp*[*intro*, *simp*]

declare *slope-scale*[*intro*, *simp*]

quotient-definition

($*$) :: *real* ⇒ *real* ⇒ *real* **is** ($*_e$)
 ⟨*proof*⟩

lemmas *eudoxus-times-cong* = *apply-rsp*'[*OF times-real.rsp*, *THEN rel-funD*, *intro*]

lemmas *eudoxus-rel-comp* = *eudoxus-times-cong*[*unfolded eudoxus-times-def*]

lemma *eudoxus-times-commute*:

assumes *slope f slope g*

shows (*f* $*_e$ *g*) \sim_e (*g* $*_e$ *f*)

⟨*proof*⟩

lemma *abs-real-times*[*simp*]:

assumes *slope f slope g*

shows *abs-real f* $*$ *abs-real g* = *abs-real* (*f* $*_e$ *g*)

⟨*proof*⟩

instance ⟨*proof*⟩

end

lemma *neg-one-def*: $- 1 = abs-real (-_e id)$ ⟨*proof*⟩

lemma *slope-neg-one*[*intro*, *simp*]: *slope* ($-_e id$) ⟨*proof*⟩

With the definitions provided above, the Eudoxus reals are a commutative ring with unity.

instance *real* :: *comm-ring-1*
 ⟨*proof*⟩

lemma *real-of-nat*:
of-nat *n* = *abs-real* ((*) (*of-nat* *n*))
 ⟨*proof*⟩

lemma *real-of-int*:
of-int *z* = *abs-real* ((*) *z*)
 ⟨*proof*⟩

The Eudoxus reals are a ring of characteristic $0::'a$.

instance *real* :: *ring-char-0*
 ⟨*proof*⟩

2.4 Ordering

We call a slope positive, if it tends to infinity. Similarly, we call a slope negative if it tends to negative infinity.

instantiation *real* :: {*ord*, *abs*, *sgn*}
begin

definition *pos* :: (*int* ⇒ *int*) ⇒ *bool* **where**
pos *f* = (∀ *C* ≥ 0. ∃ *N*. ∀ *n* ≥ *N*. *f* *n* ≥ *C*)

definition *neg* :: (*int* ⇒ *int*) ⇒ *bool* **where**
neg *f* = (∀ *C* ≥ 0. ∃ *N*. ∀ *n* ≥ *N*. *f* *n* ≤ −*C*)

lemma *pos-neg-exclusive*: ¬ (*pos* *f* ∧ *neg* *f*) ⟨*proof*⟩

lemma *pos-iff-neg-uminus*: *pos* *f* = *neg* (−_{*e*} *f*) ⟨*proof*⟩

lemma *neg-iff-pos-uminus*: *neg* *f* = *pos* (−_{*e*} *f*) ⟨*proof*⟩

lemma *pos-iff*:
assumes *slope* *f*
shows *pos* *f* = *infinite* (*f* ‘ {0..} ∩ {0<..}) (**is** ?*lhs* = ?*rhs*)
 ⟨*proof*⟩

lemma *neg-iff*:
assumes *slope* *f*
shows *neg* *f* = *infinite* (*f* ‘ {0..} ∩ {..*0*}) (**is** ?*lhs* = ?*rhs*)
 ⟨*proof*⟩

lemma *pos-cong*:
assumes *f* ∼_{*e*} *g*
shows *pos* *f* = *pos* *g*
 ⟨*proof*⟩

lemma *neg-cong*:
assumes $f \sim_e g$
shows $\text{neg } f = \text{neg } g$
 $\langle \text{proof} \rangle$

lemma *pos-iff-nonneg-nonzero*:
assumes *slope* f
shows $\text{pos } f \longleftrightarrow (\neg \text{neg } f) \wedge (\neg \text{bounded } f)$ (**is** $?lhs \longleftrightarrow ?rhs$)
 $\langle \text{proof} \rangle$

lemma *neg-iff-nonpos-nonzero*:
assumes *slope* f
shows $\text{neg } f \longleftrightarrow (\neg \text{pos } f) \wedge (\neg \text{bounded } f)$
 $\langle \text{proof} \rangle$

We define the sign of a slope to be *id* if it is positive, $-_e$ *id* if it is negative and λ -. *0* otherwise.

definition *eudoxus-sgn* :: $(\text{int} \Rightarrow \text{int}) \Rightarrow (\text{int} \Rightarrow \text{int})$ **where**
eudoxus-sgn $f = (\text{if } \text{pos } f \text{ then } \text{id} \text{ else if } \text{neg } f \text{ then } -_e \text{ id} \text{ else } (\lambda\text{. } 0))$

lemma *eudoxus-sgn-iff*:
assumes *slope* f
shows $\text{eudoxus-sgn } f = (\lambda\text{. } 0) \longleftrightarrow \text{bounded } f$
 $\text{eudoxus-sgn } f = \text{id} \longleftrightarrow \text{pos } f$
 $\text{eudoxus-sgn } f = (-_e \text{ id}) \longleftrightarrow \text{neg } f$
 $\langle \text{proof} \rangle$

quotient-definition
 $(\text{sgn} :: \text{real} \Rightarrow \text{real})$ **is** *eudoxus-sgn*
 $\langle \text{proof} \rangle$

lemmas *eudoxus-sgn-cong* = *apply-rsp*'[*OF sgn-real.rsp, intro*]

lemma *eudoxus-sgn-cong*'[*cong*]:
assumes $f \sim_e g$
shows $\text{eudoxus-sgn } f = \text{eudoxus-sgn } g$
 $\langle \text{proof} \rangle$

lemma *sgn-range*: $\text{sgn } (x :: \text{real}) \in \{-1, 0, 1\}$ $\langle \text{proof} \rangle$

lemma *sgn-abs-real-zero-iff*:
assumes *slope* f
shows $\text{sgn } (\text{abs-real } f) = 0 \longleftrightarrow (\text{eudoxus-sgn } f = (\lambda\text{. } 0))$ (**is** $?lhs \longleftrightarrow ?rhs$)
 $\langle \text{proof} \rangle$

lemma *sgn-zero-iff*'[*simp*]: $\text{sgn } (x :: \text{real}) = 0 \longleftrightarrow x = 0$
 $\langle \text{proof} \rangle$

lemma *sgn-zero*[*simp*]: $\text{sgn } (0 :: \text{real}) = 0$ *<proof>*

lemma *sgn-abs-real-one-iff*:

assumes *slope f*

shows $\text{sgn } (\text{abs-real } f) = 1 \iff \text{pos } f$

<proof>

lemmas *sgn-pos = sgn-abs-real-one-iff*[*THEN iffD2, simp*]

lemma *sgn-one*[*simp*]: $\text{sgn } (1 :: \text{real}) = 1$ *<proof>*

lemma *sgn-abs-real-neg-one-iff*:

assumes *slope f*

shows $\text{sgn } (\text{abs-real } f) = -1 \iff \text{neg } f$

<proof>

lemmas *sgn-neg = sgn-abs-real-neg-one-iff*[*THEN iffD2, simp*]

lemma *sgn-neg-one*[*simp*]: $\text{sgn } (-1 :: \text{real}) = -1$ *<proof>*

lemma *sgn-plus*:

assumes $\text{sgn } x = (1 :: \text{real})$ $\text{sgn } y = 1$

shows $\text{sgn } (x + y) = 1$

<proof>

lemma *sgn-times*: $\text{sgn } ((x :: \text{real}) * y) = \text{sgn } x * \text{sgn } y$

<proof>

lemma *sgn-uminus*: $\text{sgn } (- (x :: \text{real})) = - \text{sgn } x$ *<proof>*

lemma *sgn-plus'*:

assumes $\text{sgn } x = (-1 :: \text{real})$ $\text{sgn } y = -1$

shows $\text{sgn } (x + y) = -1$

<proof>

lemma *pos-dual-def*:

assumes *slope f*

shows $\text{pos } f = (\forall C \geq 0. \exists N. \forall n \leq N. f n \leq -C)$

<proof>

lemma *neg-dual-def*:

assumes *slope f*

shows $\text{neg } f = (\forall C \geq 0. \exists N. \forall n \leq N. f n \geq C)$

<proof>

lemma *pos-representative*:

assumes *slope f pos f*

obtains *g* **where** $f \sim_e g \wedge n. n \geq N \implies g n \geq C$

<proof>

lemma *pos-representative'*:
assumes *slope f pos f*
obtains *g where $f \sim_e g \wedge n. g n \geq C \implies n \geq N$*
 \langle *proof* \rangle

lemma *neg-representative*:
assumes *slope f neg f*
obtains *g where $f \sim_e g \wedge n. n \geq N \implies g n \leq -C$*
 \langle *proof* \rangle

lemma *neg-representative'*:
assumes *slope f neg f*
obtains *g where $f \sim_e g \wedge n. g n \leq -C \implies n \geq N$*
 \langle *proof* \rangle

We call a real x less than another real y , if their difference is positive.

definition
 $x < (y::real) \equiv \text{sgn } (y - x) = 1$

definition
 $x \leq (y::real) \equiv x < y \vee x = y$

definition
 $\text{abs-real}: |x :: real| = (\text{if } 0 \leq x \text{ then } x \text{ else } -x)$

instance \langle *proof* \rangle
end

instance *real :: linorder*
 \langle *proof* \rangle

lemma *real-leI*:
assumes $\text{sgn } (y - x) \in \{0 :: real, 1\}$
shows $x \leq y$
 \langle *proof* \rangle

lemma *real-lessI*:
assumes $\text{sgn } (y - x) = (1 :: real)$
shows $x < y$
 \langle *proof* \rangle

lemma *abs-real-leI*:
assumes $\text{slope } f \text{ slope } g \wedge z. z \geq N \implies f z \geq g z$
shows $\text{abs-real } f \geq \text{abs-real } g$
 \langle *proof* \rangle

lemma *abs-real-lessI*:

assumes $\text{slope } f \text{ slope } g \wedge z. z \geq N \implies f z \geq g z \wedge C. C \geq 0 \implies \exists z. f z \geq g z + C$
shows $\text{abs-real } f > \text{abs-real } g$
 $\langle \text{proof} \rangle$

lemma *abs-real-lessD*:

assumes $\text{slope } f \text{ slope } g \text{ abs-real } f > \text{abs-real } g$
obtains z **where** $z \geq N f z > g z$
 $\langle \text{proof} \rangle$

2.5 Multiplicative Inverse

We now define the multiplicative inverse. We start by constructing a candidate for positive slopes first and then extend it to the entire domain using the choice function *Eps*.

instantiation $\text{real} :: \{\text{inverse}\}$
begin

definition *eudoxus-pos-inverse* :: $(\text{int} \Rightarrow \text{int}) \Rightarrow (\text{int} \Rightarrow \text{int})$ **where**
 $\text{eudoxus-pos-inverse } f z = \text{sgn } z * \text{Inf } (\{0..\} \cap \{n. f n \geq |z|\})$

lemma *eudoxus-pos-inverse*:

assumes $\text{slope } f \text{ pos } f$
obtains g **where** $f \sim_e g \text{ slope } (\text{eudoxus-pos-inverse } g) \text{ eudoxus-pos-inverse } g *_e f \sim_e \text{id}$
 $\langle \text{proof} \rangle$

definition *eudoxus-inverse* :: $(\text{int} \Rightarrow \text{int}) \Rightarrow (\text{int} \Rightarrow \text{int})$ **where**
 $\text{eudoxus-inverse } f = (\text{if } \neg \text{bounded } f \text{ then } \text{SOME } g. \text{slope } g \wedge (g *_e f) \sim_e \text{id} \text{ else } (\lambda-. 0))$

lemma

assumes $\text{slope } f$
shows $\text{slope-eudoxus-inverse: slope } (\text{eudoxus-inverse } f) \text{ (is ?slope) and}$
 $\text{eudoxus-inverse-id: } \neg \text{bounded } f \implies \text{eudoxus-inverse } f *_e f \sim_e \text{id} \text{ (is } \neg \text{bounded } f \implies ?\text{id})$
 $\langle \text{proof} \rangle$

quotient-definition

$(\text{inverse} :: \text{real} \Rightarrow \text{real})$ **is** *eudoxus-inverse*
 $\langle \text{proof} \rangle$

definition

$x \text{ div } (y::\text{real}) = \text{inverse } y * x$

instance $\langle \text{proof} \rangle$

end

lemmas *eudoxus-inverse-cong* = $\text{apply-rsp}'[\text{OF } \text{inverse-real.rsp}, \text{intro}]$

lemma *eudoxus-inverse-abs*[simp]:
assumes *slope f* \neg *bounded f*
shows *inverse (abs-real f) * abs-real f = 1*
 \langle *proof* \rangle

The Eudoxus reals are a field, with inverses defined as above.

instance *real* :: *field*
 \langle *proof* \rangle

instantiation *real* :: *distrib-lattice*
begin

definition
 $(inf :: real \Rightarrow real \Rightarrow real) = min$

definition
 $(sup :: real \Rightarrow real \Rightarrow real) = max$

instance \langle *proof* \rangle

end

The ordering on the Eudoxus reals is linear.

instance *real* :: *linordered-field*
 \langle *proof* \rangle

The Eudoxus reals fulfill the Archimedean property.

instance *real* :: *archimedean-field*
 \langle *proof* \rangle

2.6 Completeness

To show that the Eudoxus reals are complete, we first introduce the floor function.

instantiation *real* :: *floor-ceiling*
begin

definition
 $(floor :: (real \Rightarrow int)) = (\lambda x. (SOME z. of-int z \leq x \wedge x < of-int z + 1))$

instance
 \langle *proof* \rangle
end

lemma *eudoxus-dense-rational*:
fixes *x y* :: *real*
assumes $x < y$

obtains $m\ n$ **where** $x < (of-int\ m / of-int\ n) (of-int\ m / of-int\ n) < y\ n > 0$
<proof>

The Eudoxus reals are a complete field.

lemma *eudoxus-complete*:

assumes $S \neq \{\}$ *bdd-above* S

obtains $u :: real$ **where** $\bigwedge s. s \in S \implies s \leq u \wedge y. (\bigwedge s. s \in S \implies s \leq y) \implies u$
 $\leq y$
<proof>

end

References

- [1] R. D. Arthan. The Eudoxus real numbers. arXiv:math/0405454, 2004.