

# The Error Function

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## Abstract

This entry provides the definitions and basic properties of the complex and real error function  $\operatorname{erf}$  and the complementary error function  $\operatorname{erfc}$ . Additionally, it gives their full asymptotic expansions.

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# 1 The complex and real error function

**theory** *Error-Function*

**imports** *HOL-Complex-Analysis.Complex-Analysis HOL-Library.Landau-Symbols*  
**begin**

## 1.1 Auxiliary Facts

**lemma** *tendsto-sandwich-mono*:

**assumes**  $(\lambda n. f \text{ (real } n)) \longrightarrow (c::\text{real})$

**assumes** *eventually*  $(\lambda x. \forall y z. x \leq y \wedge y \leq z \longrightarrow f y \leq f z)$  *at-top*

**shows**  $(f \longrightarrow c)$  *at-top*

*<proof>*

**lemma** *tendsto-sandwich-antimono*:

**assumes**  $(\lambda n. f \text{ (real } n)) \longrightarrow (c::\text{real})$

**assumes** *eventually*  $(\lambda x. \forall y z. x \leq y \wedge y \leq z \longrightarrow f y \geq f z)$  *at-top*

**shows**  $(f \longrightarrow c)$  *at-top*

*<proof>*

**lemma** *has-bochner-integral-completion* [*intro*]:

**fixes**  $f :: 'a \Rightarrow 'b::\{\text{banach, second-countable-topology}\}$

**shows** *has-bochner-integral*  $M f I \Longrightarrow \text{has-bochner-integral (completion } M) f I$

*<proof>*

**lemma** *has-bochner-integral-imp-has-integral*:

*has-bochner-integral lebesgue*  $(\lambda x. \text{indicator } S x *_R f x) I \Longrightarrow$

$(f \text{ has-integral } (I :: 'b :: \text{euclidean-space})) S$

*<proof>*

**lemma** *has-bochner-integral-imp-has-integral'*:

*has-bochner-integral lborel*  $(\lambda x. \text{indicator } S x *_R f x) I \Longrightarrow$

$(f \text{ has-integral } (I :: 'b :: \text{euclidean-space})) S$

*<proof>*

**lemma** *has-bochner-integral-erf-aux*:

*has-bochner-integral lborel*  $(\lambda x. \text{indicator } \{0..\} x *_R \exp (-x^2)) (\text{sqrt pi} / 2)$

*<proof>*

**lemma** *has-integral-erf-aux*:  $((\lambda t::\text{real. } \exp (-t^2)) \text{ has-integral } (\text{sqrt pi} / 2)) \{0..\}$

*<proof>*

**lemma** *contour-integrable-on-linepath-neg-exp-squared* [*simp, intro*]:

$(\lambda t. \exp (-t^2)) \text{ contour-integrable-on linepath } 0 z$

*<proof>*

**lemma** *holomorphic-on-chain*:

$g \text{ holomorphic-on } t \Longrightarrow f \text{ holomorphic-on } s \Longrightarrow f' s \subseteq t \Longrightarrow$

$(\lambda x. g (f x)) \text{ holomorphic-on } s$

*<proof>*

**lemma** *holomorphic-on-chain-UNIV*:

*g holomorphic-on UNIV  $\implies$  f holomorphic-on  $s \implies$   
 $(\lambda x. g (f x))$  holomorphic-on  $s$   
 ⟨proof⟩*

**lemmas** *holomorphic-on-exp' [holomorphic-intros] =  
 holomorphic-on-exp [THEN holomorphic-on-chain-UNIV]*

**lemma** *leibniz-rule-field-derivative-real*:

**fixes** *f::'a::{real-normed-field, banach}  $\Rightarrow$  real  $\Rightarrow$  'a*  
**assumes** *fx:  $\bigwedge x t. x \in U \implies t \in \{a..b\} \implies ((\lambda x. f x t)$  has-field-derivative  $fx$   
 $x t)$  (at  $x$  within  $U$ )*  
**assumes** *integrable-f2:  $\bigwedge x. x \in U \implies (f x)$  integrable-on  $\{a..b\}$*   
**assumes** *cont-fx: continuous-on  $(U \times \{a..b\})$   $(\lambda(x, t). f x t)$*   
**assumes** *U:  $x0 \in U$  convex  $U$*   
**shows**  *$((\lambda x. \text{integral } \{a..b\} (f x))$  has-field-derivative  $\text{integral } \{a..b\} (f x x0))$  (at  
 $x0$  within  $U$ )*  
 ⟨proof⟩

**lemma** *has-vector-derivative-linepath-within [derivative-intros]*:

**assumes** *[derivative-intros]:*  
*(f has-vector-derivative  $f'$ ) (at  $x$  within  $S)$  ( $g$  has-vector-derivative  $g'$ ) (at  $x$   
 within  $S)$   
*(h has-real-derivative  $h'$ ) (at  $x$  within  $S)$*   
**shows**  *$((\lambda x. \text{linepath } (f x) (g x) (h x))$  has-vector-derivative  
 $(1 - h x) *_R f' + h x *_R g' - h' *_R (f x - g x)$ ) (at  $x$  within  $S)$*   
 ⟨proof⟩*

**lemma** *has-field-derivative-linepath-within [derivative-intros]*:

**assumes** *[derivative-intros]:*  
*(f has-field-derivative  $f'$ ) (at  $x$  within  $S)$  ( $g$  has-field-derivative  $g'$ ) (at  $x$  within  
 $S)$   
*(h has-real-derivative  $h'$ ) (at  $x$  within  $S)$*   
**shows**  *$((\lambda x. \text{linepath } (f x) (g x) (h x))$  has-field-derivative  
 $(1 - h x) *_R f' + h x *_R g' - h' *_R (f x - g x)$ ) (at  $x$  within  $S)$*   
 ⟨proof⟩*

**lemma** *continuous-on-linepath' [continuous-intros]*:

**assumes** *[continuous-intros]: continuous-on  $A$   $f$  continuous-on  $A$   $g$  continuous-on  
 $A$   $h$*   
**shows** *continuous-on  $A$   $(\lambda x. \text{linepath } (f x) (g x) (h x))$*   
 ⟨proof⟩

**lemma** *contour-integral-has-field-derivative*:

**assumes** *A: open  $A$  convex  $A$   $a \in A$   $z \in A$*   
**assumes** *integrable:  $\bigwedge z. z \in A \implies f$  contour-integrable-on linepath  $a z$*   
**assumes** *holo:  $f$  holomorphic-on  $A$*   
**shows**  *$((\lambda z. \text{contour-integral } (\text{linepath } a z) f)$  has-field-derivative  $f z)$  (at  $z$*

within B)  
<proof>

## 1.2 Definition of the error function

**definition** *erf-coeffs* :: nat  $\Rightarrow$  real **where**

*erf-coeffs* n =  
(if odd n then 2 / sqrt pi \* (-1)  $^$ (n div 2) / (of-nat n \* fact (n div 2))  
else 0)

**lemma** *summable-erf*:

**fixes** z :: 'a :: {real-normed-div-algebra, banach}  
**shows** *summable* ( $\lambda$ n. of-real (*erf-coeffs* n) \* z  $^$ n)  
<proof>

**definition** *erf* :: ('a :: {real-normed-field, banach})  $\Rightarrow$  'a **where**

*erf* z = ( $\sum$  n. of-real (*erf-coeffs* n) \* z  $^$ n)

**lemma** *erf-converges*: ( $\lambda$ n. of-real (*erf-coeffs* n) \* z  $^$ n) *sums erf* z  
<proof>

**lemma** *erf-0* [*simp*]: *erf* 0 = 0  
<proof>

**lemma** *erf-minus* [*simp*]: *erf* (-z) = - *erf* z  
<proof>

**lemma** *erf-of-real* [*simp*]: *erf* (of-real x) = of-real (*erf* x)  
<proof>

**lemma** *of-real-erf-numeral* [*simp*]: of-real (*erf* (numeral n)) = *erf* (numeral n)  
<proof>

**lemma** *of-real-erf-1* [*simp*]: of-real (*erf* 1) = *erf* 1  
<proof>

**lemma** *erf-has-field-derivative*:

(*erf* has-field-derivative of-real (2 / sqrt pi) \* exp  $-(z^2)$ ) (at z within A)  
<proof>

**lemmas** *erf-has-field-derivative'* [*derivative-intros*] =  
*erf-has-field-derivative* [THEN *DERIV-chain2*]

**lemma** *erf-continuous-on*: continuous-on A *erf*  
<proof>

**lemma** *continuous-on-compose2-UNIV*:

continuous-on UNIV g  $\implies$  continuous-on s f  $\implies$  continuous-on s ( $\lambda$ x. g (f x))

*<proof>*

**lemmas** *erf-continuous-on'* [*continuous-intros*] =  
*erf-continuous-on* [*THEN continuous-on-compose2-UNIV*]

**lemma** *erf-continuous* [*continuous-intros*]: *continuous* (at *x* within *A*) *erf*  
*<proof>*

**lemmas** *erf-continuous'* [*continuous-intros*] =  
*continuous-within-compose2*[*OF - erf-continuous*]

**lemmas** *tendsto-erf* [*tendsto-intros*] = *isCont-tendsto-compose*[*OF erf-continuous*]

**lemma** *erf-cnj* [*simp*]: *erf* (*cnj z*) = *cnj* (*erf z*)  
*<proof>*

**lemma** *integral-exp-minus-squared-real*:

**assumes**  $a \leq b$

**shows**  $((\lambda t. \exp(-(t^2))) \text{ has-integral } (\text{sqrt pi} / 2 * (\text{erf } b - \text{erf } a))) \{a..b\}$   
*<proof>*

**lemma** *erf-real-altdef-nonneg*:

$x \geq 0 \implies \text{erf } (x::\text{real}) = 2 / \text{sqrt pi} * \text{integral } \{0..x\} (\lambda t. \exp(-(t^2)))$

*<proof>*

**lemma** *erf-real-altdef-nonpos*:

$x \leq 0 \implies \text{erf } (x::\text{real}) = -2 / \text{sqrt pi} * \text{integral } \{0..-x\} (\lambda t. \exp(-(t^2)))$

*<proof>*

**lemma** *less-imp-erf-real-less*:

**assumes**  $a < (b::\text{real})$

**shows**  $\text{erf } a < \text{erf } b$

*<proof>*

**lemma** *le-imp-erf-real-le*:  $a \leq (b::\text{real}) \implies \text{erf } a \leq \text{erf } b$

*<proof>*

**lemma** *erf-real-less-cancel* [*simp*]:  $(\text{erf } (a :: \text{real}) < \text{erf } b) \longleftrightarrow a < b$

*<proof>*

**lemma** *erf-real-eq-iff* [*simp*]:  $\text{erf } (a::\text{real}) = \text{erf } b \longleftrightarrow a = b$

*<proof>*

**lemma** *erf-real-le-cancel* [*simp*]:  $(\text{erf } (a :: \text{real}) \leq \text{erf } b) \longleftrightarrow a \leq b$

*<proof>*

**lemma** *inj-on-erf-real* [*intro*]: *inj-on* (*erf* :: *real*  $\Rightarrow$  *real*) *A*

*<proof>*

**lemma** *strict-mono-erf-real* [*intro*]: *strict-mono* (*erf* :: *real*  $\Rightarrow$  *real*)  
*<proof>*

**lemma** *mono-erf-real* [*intro*]: *mono* (*erf* :: *real*  $\Rightarrow$  *real*)  
*<proof>*

**lemma** *erf-real-ge-0-iff* [*simp*]: *erf* (*x*::*real*)  $\geq 0 \iff x \geq 0$   
*<proof>*

**lemma** *erf-real-le-0-iff* [*simp*]: *erf* (*x*::*real*)  $\leq 0 \iff x \leq 0$   
*<proof>*

**lemma** *erf-real-gt-0-iff* [*simp*]: *erf* (*x*::*real*)  $> 0 \iff x > 0$   
*<proof>*

**lemma** *erf-real-less-0-iff* [*simp*]: *erf* (*x*::*real*)  $< 0 \iff x < 0$   
*<proof>*

**lemma** *erf-at-top* [*tendsto-intros*]: ((*erf* :: *real*  $\Rightarrow$  *real*)  $\longrightarrow 1$ ) *at-top*  
*<proof>*

**lemma** *erf-at-bot* [*tendsto-intros*]: ((*erf* :: *real*  $\Rightarrow$  *real*)  $\longrightarrow -1$ ) *at-bot*  
*<proof>*

**lemmas** *tendsto-erf-at-top* [*tendsto-intros*] = *filterlim-compose*[*OF erf-at-top*]

**lemmas** *tendsto-erf-at-bot* [*tendsto-intros*] = *filterlim-compose*[*OF erf-at-bot*]

### 1.3 The complimentary error function

**definition** *erfc* where *erfc* *z* =  $1 - \text{erf } z$

**lemma** *erf-conv-erfc*: *erf* *z* =  $1 - \text{erfc } z$  *<proof>*

**lemma** *erfc-0* [*simp*]: *erfc*  $0 = 1$   
*<proof>*

**lemma** *erfc-minus*: *erfc* ( $-z$ ) =  $2 - \text{erfc } z$   
*<proof>*

**lemma** *erfc-of-real* [*simp*]: *erfc* (*of-real* *x*) = *of-real* (*erfc* *x*)  
*<proof>*

**lemma** *of-real-erfc-numeral* [*simp*]: *of-real* (*erfc* (*numeral* *n*)) = *erfc* (*numeral* *n*)  
*<proof>*

**lemma** *of-real-erfc-1* [*simp*]: *of-real* (*erfc*  $1$ ) = *erfc*  $1$

*<proof>*

**lemma** *less-imp-erfc-real-less*:  $a < (b::real) \implies \text{erfc } a > \text{erfc } b$   
*<proof>*

**lemma** *le-imp-erfc-real-le*:  $a \leq (b::real) \implies \text{erfc } a \geq \text{erfc } b$   
*<proof>*

**lemma** *erfc-real-less-cancel* [*simp*]:  $(\text{erfc } (a :: real) < \text{erfc } b) \longleftrightarrow a > b$   
*<proof>*

**lemma** *erfc-real-eq-iff* [*simp*]:  $\text{erfc } (a::real) = \text{erfc } b \longleftrightarrow a = b$   
*<proof>*

**lemma** *erfc-real-le-cancel* [*simp*]:  $(\text{erfc } (a :: real) \leq \text{erfc } b) \longleftrightarrow a \geq b$   
*<proof>*

**lemma** *inj-on-erfc-real* [*intro*]:  $\text{inj-on } (\text{erfc} :: real \Rightarrow real) A$   
*<proof>*

**lemma** *antimono-erfc-real* [*intro*]:  $\text{antimono } (\text{erfc} :: real \Rightarrow real)$   
*<proof>*

**lemma** *erfc-real-ge-0-iff* [*simp*]:  $\text{erfc } (x::real) \geq 1 \longleftrightarrow x \leq 0$   
*<proof>*

**lemma** *erfc-real-le-0-iff* [*simp*]:  $\text{erfc } (x::real) \leq 1 \longleftrightarrow x \geq 0$   
*<proof>*

**lemma** *erfc-real-gt-0-iff* [*simp*]:  $\text{erfc } (x::real) > 1 \longleftrightarrow x < 0$   
*<proof>*

**lemma** *erfc-real-less-0-iff* [*simp*]:  $\text{erfc } (x::real) < 1 \longleftrightarrow x > 0$   
*<proof>*

**lemma** *erfc-has-field-derivative*:  
 $(\text{erfc has-field-derivative } -\text{of-real } (2 / \text{sqrt } \pi) * \text{exp } (-(z^2)))$  (at  $z$  within  $A$ )  
*<proof>*

**lemmas** *erfc-has-field-derivative'* [*derivative-intros*] =  
*erfc-has-field-derivative* [*THEN DERIV-chain2*]

**lemma** *erfc-continuous-on*:  $\text{continuous-on } A \text{ erfc}$   
*<proof>*

**lemmas** *erfc-continuous-on'* [*continuous-intros*] =  
*erfc-continuous-on* [*THEN continuous-on-compose2-UNIV*]

**lemma** *erfc-continuous* [*continuous-intros*]: *continuous (at x within A) erfc*  
⟨*proof*⟩

**lemmas** *erfc-continuous'* [*continuous-intros*] =  
*continuous-within-compose2*[*OF - erfc-continuous*]

**lemmas** *tendsto-erfc* [*tendsto-intros*] = *isCont-tendsto-compose*[*OF erfc-continuous*]

**lemma** *erfc-at-top* [*tendsto-intros*]:  $((erfc :: real \Rightarrow real) \longrightarrow 0)$  *at-top*  
⟨*proof*⟩

**lemma** *erfc-at-bot* [*tendsto-intros*]:  $((erfc :: real \Rightarrow real) \longrightarrow 2)$  *at-bot*  
⟨*proof*⟩

**lemmas** *tendsto-erfc-at-top* [*tendsto-intros*] = *filterlim-compose*[*OF erfc-at-top*]

**lemmas** *tendsto-erfc-at-bot* [*tendsto-intros*] = *filterlim-compose*[*OF erfc-at-bot*]

**lemma** *integrable-exp-minus-squared*:

**assumes**  $A \subseteq \{0..\}$   $A \in sets\ lborel$

**shows** *set-integrable lborel A*  $(\lambda t::real. exp(-t^2))$  (**is** *?thesis1*)

**and**  $(\lambda t::real. exp(-t^2))$  *integrable-on A* (**is** *?thesis2*)

⟨*proof*⟩

**lemma**

**assumes**  $x \geq 0$

**shows** *erfc-real-altdef-nonneg*:  $erfc\ x = 2 / \sqrt{\pi} * \int \{x..\}$   $(\lambda t. exp(-t^2))$

**and** *has-integral-erfc*:  $((\lambda t. exp(-t^2))\ has-integral(\sqrt{\pi} / 2 * erfc\ x))$   
 $\{x..\}$

⟨*proof*⟩

**lemma** *erfc-real-gt-0* [*simp, intro*]:  $erfc(x::real) > 0$   
⟨*proof*⟩

**lemma** *erfc-real-less-2* [*intro*]:  $erfc(x::real) < 2$   
⟨*proof*⟩

**lemma** *erf-real-gt-neg1* [*intro*]:  $erf(x::real) > -1$   
⟨*proof*⟩

**lemma** *erf-real-less-1* [*intro*]:  $erf(x::real) < 1$   
⟨*proof*⟩

**lemma** *erfc-cnj* [*simp*]:  $erfc(cn\ j\ z) = cn\ j\ (erfc\ z)$   
⟨*proof*⟩



## 1.4 Specific facts about the complex case

**lemma** *erf-complex-altdef*:

$erf\ z = of\text{-real}\ (2 / \text{sqrt}\ \pi) * \text{contour-integral}\ (\text{linepath}\ 0\ z)\ (\lambda t.\ \text{exp}\ (-(t^2)))$   
<proof>

**lemma** *erf-holomorphic-on*: *erf* holomorphic-on *A*

<proof>

**lemmas** *erf-holomorphic-on'* [*holomorphic-intros*] =  
*erf-holomorphic-on* [*THEN* *holomorphic-on-chain-UNIV*]

**lemma** *erf-analytic-on*: *erf* analytic-on *A*

<proof>

**lemma** *erf-analytic-on'* [*analytic-intros*]:

**assumes** *f* analytic-on *A*

**shows**  $(\lambda x.\ erf\ (f\ x))$  analytic-on *A*

<proof>

**lemma** *erfc-holomorphic-on*: *erfc* holomorphic-on *A*

<proof>

**lemmas** *erfc-holomorphic-on'* [*holomorphic-intros*] =  
*erfc-holomorphic-on* [*THEN* *holomorphic-on-chain-UNIV*]

**lemma** *erfc-analytic-on*: *erfc* analytic-on *A*

<proof>

**lemma** *erfc-analytic-on'* [*analytic-intros*]:

**assumes** *f* analytic-on *A*

**shows**  $(\lambda x.\ erfc\ (f\ x))$  analytic-on *A*

<proof>

end

## 1.5 Asymptotics

**theory** *Error-Function-Asymptotics*

**imports** *Error-Function* *Landau-Symbols*.*Landau-More*

**begin**

**lemma** *real-powr-eq-powerI*:

$x > 0 \implies y = \text{real}\ y' \implies x^{\text{powr}\ y} = x^y$

<proof>

**definition** *erf-remainder-integral* **where**

*erf-remainder-integral* *n* *x* =

$\lim\ (\lambda m.\ \text{integral}\ \{x..x + \text{real}\ m\}\ (\lambda t.\ \text{exp}\ (-(t^2)) / t^{(2*n)}))$

The following is the remainder term in the asymptotic expansion of  $\text{erfc}$ .

**definition** *erf-remainder* where

$$\text{erf-remainder } n \ x = \\ ((-1)^n * 2 * \text{fact } (2*n)) / (\text{sqrt } \pi * 4^n * \text{fact } n) * \\ \text{erf-remainder-integral } n \ x$$

**lemma** *erf-remainder-integral-aux-nonneg*:

$$x > 0 \implies \text{integral } \{x..x + \text{real } m\} (\lambda t. \exp(-(t^2)) / t^{(2*n)}) \geq 0 \\ \langle \text{proof} \rangle$$

**lemma** *erf-remainder-integral-aux-bound*:

$$\text{assumes } x > 0 \\ \text{shows } \text{norm } (\text{integral } \{x..x + \text{real } m\} (\lambda t. \exp(-t^2) / t^{(2*n)})) \leq \exp(-x^2) / x^{(2*n+1)} \\ \text{and } \text{integral } \{x..x + \text{real } m\} (\lambda t. \exp(-t^2) / t^{(2*n)}) \leq \exp(-x^2) / x^{(2*n+1)} \\ \langle \text{proof} \rangle$$

**lemma** *convergent-erf-remainder-integral*:

$$\text{assumes } x > 0 \\ \text{shows } \text{convergent } (\lambda m. \text{integral } \{x..x + \text{real } m\} (\lambda t. \exp(-(t^2)) / t^{(2*n)})) \\ \langle \text{proof} \rangle$$

**lemma** *LIMSEQ-erf-remainder-integral*:

$$x > 0 \implies (\lambda m. \text{integral } \{x..x + \text{real } m\} (\lambda t. \exp(-(t^2)) / t^{(2*n)})) \longrightarrow \\ \text{erf-remainder-integral } n \ x \\ \langle \text{proof} \rangle$$

We show some bounds on the remainder term.

**lemma**

$$\text{assumes } x > 0 \\ \text{shows } \text{erf-remainder-integral-nonneg: } \text{erf-remainder-integral } n \ x \geq 0 \\ \text{and } \text{erf-remainder-integral-bound: } \text{erf-remainder-integral } n \ x \leq \exp(-x^2) / x^{(2*n+1)} \\ \langle \text{proof} \rangle$$

**lemma** *erf-remainder-integral-bigo*:

$$\text{erf-remainder-integral } n \in O(\lambda x. \exp(-x^2) / x^{(2*n+1)}) \\ \langle \text{proof} \rangle$$

**theorem** *erf-remainder-bigo*:  $\text{erf-remainder } n \in O(\lambda x. \exp(-x^2) / x^{(2*n+1)})$

$\langle \text{proof} \rangle$

Next, we unroll the remainder term to develop the asymptotic expansion.

**lemma** *erf-remainder-integral-0-conv-erfc*:

$$\text{assumes } (x::\text{real}) > 0 \\ \text{shows } \text{erf-remainder-integral } 0 \ x = \text{sqrt } \pi / 2 * \text{erfc } x \\ \langle \text{proof} \rangle$$

The first remainder is the *erfc* function itself.

**lemma** *erf-remainder-0-conv-erfc*:  $x > 0 \implies \text{erf-remainder } 0 \ x = \text{erfc } x$   
 ⟨proof⟩

Also, the following recurrence allows us to get the next term of the asymptotic expansion.

**lemma** *erf-remainder-integral-conv-Suc*:

**assumes**  $x > 0$

**shows**  $\text{erf-remainder-integral } n \ x = \text{exp } (-x^2) / (2 * x^{2 * n + 1}) -$   
 $\text{real } (2 * n + 1) / 2 * \text{erf-remainder-integral } (\text{Suc } n) \ x$

⟨proof⟩

**lemma** *erf-remainder-conv-Suc*:

**assumes**  $x > 0$

**shows**  $\text{erf-remainder } n \ x = (-1)^n * \text{fact } (2 * n) / (\text{sqrt } \pi * 4^n * \text{fact } n) *$

$\text{exp } (-x^2) / (x^{2 * n + 1}) + \text{erf-remainder } (\text{Suc } n) \ x$

⟨proof⟩

Finally, this gives us the full asymptotic expansion for *erfc*:

**theorem** *erfc-unroll*:

**assumes**  $x > 0$

**shows**  $\text{erfc } x = \text{exp } (-x^2) / \text{sqrt } \pi *$   
 $(\sum_{i < n.} (-1)^i * \text{fact } (2 * i) / (4^i * \text{fact } i) / x^{2 * i + 1}) + \text{erf-remainder}$   
 $n \ x$

⟨proof⟩

For convenience, we define another auxiliary function that is more suitable for use in an automated expansion framework, since it has a simple asymptotic expansion in powers of  $x$ .

**definition** *erfc-aux* **where**  $\text{erfc-aux } x = \text{exp } (x^2) * \text{sqrt } \pi * \text{erfc } x$

**definition** *erf-remainder'* **where**  $\text{erf-remainder}' \ n \ x = \text{exp } (x^2) * \text{sqrt } \pi * \text{erf-remainder}$   
 $n \ x$

**lemma** *erfc-aux-unroll*:

$x > 0 \implies$

$\text{erfc-aux } x = (\sum_{i < n.} (-1)^i * \text{fact } (2 * i) / (4^i * \text{fact } i) / x^{2 * i + 1}) +$   
 $\text{erf-remainder}' \ n \ x$

⟨proof⟩

**lemma** *erf-remainder'-bigo*:  $\text{erf-remainder}' \ n \in O(\lambda x. 1 / x^{2 * n + 1})$

⟨proof⟩

**lemma** *has-field-derivative-erfc-aux*:

$(\text{erfc-aux has-field-derivative } (2 * x * \text{erfc-aux } x - 2)) \ (\text{at } x)$

⟨proof⟩

**end**