The Error Function

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Abstract

This entry provides the definitions and basic properties of the complex and real error function erf and the complementary error function erfc. Additionally, it gives their full asymptotic expansions.

Contents

1 The complex and real error function

theory *Error-Function*
imports *HOL−Complex-Analysis.Complex-Analysis HOL−Library.Landau-Symbols*
begin

1.1 Auxiliary Facts

lemma *tendsto-sandwich-mono*:
assumes $(\lambda n.\ f\ (real\ n)) \longlonglongrightarrow (c::real)$
assumes *eventually* $(\lambda x.\ \forall y\ z.\ x \le y \wedge y \le z \longrightarrow f\ y \le f\ z)$ *at-top*
shows $(f \longrightarrow c)$ *at-top*
⟨*proof*⟩

lemma *tendsto-sandwich-antimono*:
assumes $(\lambda n.\ f\ (real\ n)) \longlonglongrightarrow (c::real)$
assumes *eventually* $(\lambda x.\ \forall y\ z.\ x \le y \wedge y \le z \longrightarrow f\ y \ge f\ z)$ *at-top*
shows $(f \longrightarrow c)$ *at-top*
⟨*proof*⟩

lemma *has-bochner-integral-completion* [*intro*]:
fixes $f :: 'a \Rightarrow 'b::\{banach,\ second\text{-}countable\text{-}topology\}$
shows *has-bochner-integral M f I* $\Longrightarrow$ *has-bochner-integral* (*completion M*) *f I*
⟨*proof*⟩

lemma *has-bochner-integral-imp-has-integral*:
has-bochner-integral lebesgue $(\lambda x.\ indicator\ S\ x *_R f\ x)$ $I \Longrightarrow$
$(f$ *has-integral* $(I :: 'b :: euclidean\text{-}space))$ S
⟨*proof*⟩

lemma *has-bochner-integral-imp-has-integral′*:
has-bochner-integral lborel $(\lambda x.\ indicator\ S\ x *_R f\ x)$ $I \Longrightarrow$
$(f$ *has-integral* $(I :: 'b :: euclidean\text{-}space))$ S
⟨*proof*⟩

lemma *has-bochner-integral-erf-aux*:
has-bochner-integral lborel $(\lambda x.\ indicator\ \{0..\}\ x *_R exp\ (-\ x^2))$ $(sqrt\ pi\ /\ 2)$
⟨*proof*⟩

lemma *has-integral-erf-aux*: $((\lambda t::real.\ exp\ (-t^2))$ *has-integral* $(sqrt\ pi\ /\ 2))\ \{0..\}$
⟨*proof*⟩

lemma *contour-integrable-on-linepath-neg-exp-squared* [*simp*, *intro*]:
$(\lambda t.\ exp\ (-(t\hat{\ }2)))$ *contour-integrable-on linepath 0 z*
⟨*proof*⟩

lemma *holomorphic-on-chain*:
g holomorphic-on t $\Longrightarrow$ *f holomorphic-on s* $\Longrightarrow$ f ‘ $s \subseteq t \Longrightarrow$
$(\lambda x.\ g\ (f\ x))$ *holomorphic-on s*
⟨*proof*⟩

lemma *holomorphic-on-chain-UNIV*:
g holomorphic-on UNIV $\Longrightarrow$ *f holomorphic-on s* $\Longrightarrow$
(λx. *g* (*f x*)) *holomorphic-on s*
⟨*proof*⟩

lemmas *holomorphic-on-exp′* [*holomorphic-intros*] =
holomorphic-on-exp [*THEN holomorphic-on-chain-UNIV*]

lemma *leibniz-rule-field-derivative-real*:
fixes *f*::′*a*::{*real-normed-field*, *banach*} ⇒ *real* ⇒ ′*a*
assumes *fx*: $\bigwedge$*x t*. *x* ∈ *U* $\Longrightarrow$ *t* ∈ {*a*..*b*} $\Longrightarrow$ ((λx. *f x t*) *has-field-derivative fx x t*) (*at x within U*)
assumes *integrable-f2*: $\bigwedge$*x*. *x* ∈ *U* $\Longrightarrow$ (*f x*) *integrable-on* {*a*..*b*}
assumes *cont-fx*: *continuous-on* (*U* × {*a*..*b*}) (λ(*x*, *t*). *fx x t*)
assumes *U*: *x0* ∈ *U convex U*
shows ((λx. *integral* {*a*..*b*} (*f x*)) *has-field-derivative integral* {*a*..*b*} (*fx x0*)) (*at x0 within U*)
⟨*proof*⟩

lemma *has-vector-derivative-linepath-within* [*derivative-intros*]:
assumes [*derivative-intros*]:
(*f has-vector-derivative f′*) (*at x within S*) (*g has-vector-derivative g′*) (*at x within S*)
(*h has-real-derivative h′*) (*at x within S*)
shows ((λx. *linepath* (*f x*) (*g x*) (*h x*)) *has-vector-derivative*
(*1* − *h x*) $*_R$ *f′* + *h x* $*_R$ *g′* − *h′* $*_R$ (*f x* − *g x*)) (*at x within S*)
⟨*proof*⟩

lemma *has-field-derivative-linepath-within* [*derivative-intros*]:
assumes [*derivative-intros*]:
(*f has-field-derivative f′*) (*at x within S*) (*g has-field-derivative g′*) (*at x within S*)
(*h has-real-derivative h′*) (*at x within S*)
shows ((λx. *linepath* (*f x*) (*g x*) (*h x*)) *has-field-derivative*
(*1* − *h x*) $*_R$ *f′* + *h x* $*_R$ *g′* − *h′* $*_R$ (*f x* − *g x*)) (*at x within S*)
⟨*proof*⟩

lemma *continuous-on-linepath′* [*continuous-intros*]:
assumes [*continuous-intros*]: *continuous-on A f continuous-on A g continuous-on A h*
shows *continuous-on A* (λx. *linepath* (*f x*) (*g x*) (*h x*))
⟨*proof*⟩

lemma *contour-integral-has-field-derivative*:
assumes *A*: *open A convex A a* ∈ *A z* ∈ *A*
assumes *integrable*: $\bigwedge$*z*. *z* ∈ *A* $\Longrightarrow$ *f contour-integrable-on linepath a z*
assumes *holo*: *f holomorphic-on A*
shows ((λz. *contour-integral* (*linepath a z*) *f*) *has-field-derivative f z*) (*at z*

within B)
⟨*proof*⟩

1.2 Definition of the error function

definition *erf-coeffs* :: *nat* ⇒ *real* **where**
erf-coeffs n =
(*if odd n then 2 / sqrt pi* ∗ (−*1*) ^ (*n div 2*) / (*of-nat n* ∗ *fact* (*n div 2*))
else 0)

lemma *summable-erf*:
fixes *z* :: *'a* :: {*real-normed-div-algebra*, *banach*}
shows *summable* (λ*n*. *of-real* (*erf-coeffs n*) ∗ *z* ^ *n*)
⟨*proof*⟩

definition *erf* :: (*'a* :: {*real-normed-field*, *banach*}) ⇒ *'a* **where**
erf z = (∑ *n*. *of-real* (*erf-coeffs n*) ∗ *z* ^ *n*)

lemma *erf-converges*: (λ*n*. *of-real* (*erf-coeffs n*) ∗ *z* ^ *n*) *sums erf z*
⟨*proof*⟩

lemma *erf-0* [*simp*]: *erf 0* = *0*
⟨*proof*⟩

lemma *erf-minus* [*simp*]: *erf* (−*z*) = − *erf z*
⟨*proof*⟩

lemma *erf-of-real* [*simp*]: *erf* (*of-real x*) = *of-real* (*erf x*)
⟨*proof*⟩

lemma *of-real-erf-numeral* [*simp*]: *of-real* (*erf* (*numeral n*)) = *erf* (*numeral n*)
⟨*proof*⟩

lemma *of-real-erf-1* [*simp*]: *of-real* (*erf 1*) = *erf 1*
⟨*proof*⟩

lemma *erf-has-field-derivative*:
(*erf has-field-derivative of-real* (*2 / sqrt pi*) ∗ *exp* (−(*z*^*2*))) (*at z within A*)
⟨*proof*⟩

lemmas *erf-has-field-derivative'* [*derivative-intros*] =
erf-has-field-derivative [*THEN DERIV-chain2*]

lemma *erf-continuous-on*: *continuous-on A erf*
⟨*proof*⟩

lemma *continuous-on-compose2-UNIV*:
continuous-on UNIV g ⟹ *continuous-on s f* ⟹ *continuous-on s* (λ*x*. *g* (*f x*))

⟨*proof*⟩

lemmas *erf-continuous-on'* [*continuous-intros*] =
erf-continuous-on [*THEN continuous-on-compose2-UNIV*]

lemma *erf-continuous* [*continuous-intros*]: *continuous* (*at x within A*) *erf*
⟨*proof*⟩

lemmas *erf-continuous'* [*continuous-intros*] =
continuous-within-compose2[*OF - erf-continuous*]

lemmas *tendsto-erf* [*tendsto-intros*] = *isCont-tendsto-compose*[*OF erf-continuous*]

lemma *erf-cnj* [*simp*]: *erf* (*cnj z*) = *cnj* (*erf z*)
⟨*proof*⟩

lemma *integral-exp-minus-squared-real*:
assumes $a \leq b$
shows $((\lambda t.\ exp\ (-(t\hat{\ }2)))\ has\text{-}integral\ (sqrt\ pi\ /\ 2 * (erf\ b - erf\ a)))\ \{a..b\}$
⟨*proof*⟩

lemma *erf-real-altdef-nonneg*:
$x \geq 0 \Longrightarrow erf\ (x::real) = 2\ /\ sqrt\ pi * integral\ \{0..x\}\ (\lambda t.\ exp\ (-(t\hat{\ }2)))$
⟨*proof*⟩

lemma *erf-real-altdef-nonpos*:
$x \leq 0 \Longrightarrow erf\ (x::real) = -2\ /\ sqrt\ pi * integral\ \{0..-x\}\ (\lambda t.\ exp\ (-(t\hat{\ }2)))$
⟨*proof*⟩

lemma *less-imp-erf-real-less*:
assumes $a < (b::real)$
shows $erf\ a < erf\ b$
⟨*proof*⟩

lemma *le-imp-erf-real-le*: $a \leq (b::real) \Longrightarrow erf\ a \leq erf\ b$
⟨*proof*⟩

lemma *erf-real-less-cancel* [*simp*]: $(erf\ (a :: real) < erf\ b) \longleftrightarrow a < b$
⟨*proof*⟩

lemma *erf-real-eq-iff* [*simp*]: $erf\ (a::real) = erf\ b \longleftrightarrow a = b$
⟨*proof*⟩

lemma *erf-real-le-cancel* [*simp*]: $(erf\ (a :: real) \leq erf\ b) \longleftrightarrow a \leq b$
⟨*proof*⟩

lemma *inj-on-erf-real* [*intro*]: *inj-on* (*erf* :: *real* ⇒ *real*) *A*
⟨*proof*⟩

lemma *strict-mono-erf-real* [*intro*]: *strict-mono* (*erf* :: *real* ⇒ *real*)
⟨*proof*⟩

lemma *mono-erf-real* [*intro*]: *mono* (*erf* :: *real* ⇒ *real*)
⟨*proof*⟩

lemma *erf-real-ge-0-iff* [*simp*]: *erf* (*x*::*real*) ≥ *0* ⟷ *x* ≥ *0*
⟨*proof*⟩

lemma *erf-real-le-0-iff* [*simp*]: *erf* (*x*::*real*) ≤ *0* ⟷ *x* ≤ *0*
⟨*proof*⟩

lemma *erf-real-gt-0-iff* [*simp*]: *erf* (*x*::*real*) > *0* ⟷ *x* > *0*
⟨*proof*⟩

lemma *erf-real-less-0-iff* [*simp*]: *erf* (*x*::*real*) < *0* ⟷ *x* < *0*
⟨*proof*⟩

lemma *erf-at-top* [*tendsto-intros*]: ((*erf* :: *real* ⇒ *real*) ⟶ *1*) *at-top*
⟨*proof*⟩

lemma *erf-at-bot* [*tendsto-intros*]: ((*erf* :: *real* ⇒ *real*) ⟶ −*1*) *at-bot*
⟨*proof*⟩

lemmas *tendsto-erf-at-top* [*tendsto-intros*] = *filterlim-compose*[*OF erf-at-top*]
lemmas *tendsto-erf-at-bot* [*tendsto-intros*] = *filterlim-compose*[*OF erf-at-bot*]

1.3 The complimentary error function

definition *erfc* **where** *erfc z* = *1* − *erf z*

lemma *erf-conv-erfc*: *erf z* = *1* − *erfc z* ⟨*proof*⟩

lemma *erfc-0* [*simp*]: *erfc 0* = *1*
⟨*proof*⟩

lemma *erfc-minus*: *erfc* (−*z*) = *2* − *erfc z*
⟨*proof*⟩

lemma *erfc-of-real* [*simp*]: *erfc* (*of-real x*) = *of-real* (*erfc x*)
⟨*proof*⟩

lemma *of-real-erfc-numeral* [*simp*]: *of-real* (*erfc* (*numeral n*)) = *erfc* (*numeral n*)
⟨*proof*⟩

lemma *of-real-erfc-1* [*simp*]: *of-real* (*erfc 1*) = *erfc 1*

⟨proof⟩

lemma *less-imp-erfc-real-less*: $a < (b::real) \Longrightarrow erfc\ a > erfc\ b$
⟨proof⟩

lemma *le-imp-erfc-real-le*: $a \leq (b::real) \Longrightarrow erfc\ a \geq erfc\ b$
⟨proof⟩

lemma *erfc-real-less-cancel* [*simp*]: $(erfc\ (a :: real) < erfc\ b) \longleftrightarrow a > b$
⟨proof⟩

lemma *erfc-real-eq-iff* [*simp*]: $erfc\ (a::real) = erfc\ b \longleftrightarrow a = b$
⟨proof⟩

lemma *erfc-real-le-cancel* [*simp*]: $(erfc\ (a :: real) \leq erfc\ b) \longleftrightarrow a \geq b$
⟨proof⟩

lemma *inj-on-erfc-real* [*intro*]: *inj-on* $(erfc :: real \Rightarrow real)\ A$
⟨proof⟩

lemma *antimono-erfc-real* [*intro*]: *antimono* $(erfc :: real \Rightarrow real)$
⟨proof⟩

lemma *erfc-real-ge-0-iff* [*simp*]: $erfc\ (x::real) \geq 1 \longleftrightarrow x \leq 0$
⟨proof⟩

lemma *erfc-real-le-0-iff* [*simp*]: $erfc\ (x::real) \leq 1 \longleftrightarrow x \geq 0$
⟨proof⟩

lemma *erfc-real-gt-0-iff* [*simp*]: $erfc\ (x::real) > 1 \longleftrightarrow x < 0$
⟨proof⟩

lemma *erfc-real-less-0-iff* [*simp*]: $erfc\ (x::real) < 1 \longleftrightarrow x > 0$
⟨proof⟩

lemma *erfc-has-field-derivative*:
(*erfc has-field-derivative* $-$*of-real* $(2\ /\ sqrt\ pi) * exp\ (-(z\hat{\ }2)))$ (*at z within A*)
⟨proof⟩

lemmas *erfc-has-field-derivative′* [*derivative-intros*] =
erfc-has-field-derivative [*THEN DERIV-chain2*]

lemma *erfc-continuous-on*: *continuous-on A erfc*
⟨proof⟩

lemmas *erfc-continuous-on′* [*continuous-intros*] =
erfc-continuous-on [*THEN continuous-on-compose2-UNIV*]

lemma *erfc-continuous* [*continuous-intros*]: *continuous* (*at x within A*) *erfc*
⟨*proof*⟩

lemmas *erfc-continuous'* [*continuous-intros*] =
continuous-within-compose2[*OF - erfc-continuous*]

lemmas *tendsto-erfc* [*tendsto-intros*] = *isCont-tendsto-compose*[*OF erfc-continuous*]

lemma *erfc-at-top* [*tendsto-intros*]: ((*erfc* :: *real* ⇒ *real*) ⟶ *0*) *at-top*
⟨*proof*⟩

lemma *erfc-at-bot* [*tendsto-intros*]: ((*erfc* :: *real* ⇒ *real*) ⟶ *2*) *at-bot*
⟨*proof*⟩

lemmas *tendsto-erfc-at-top* [*tendsto-intros*] = *filterlim-compose*[*OF erfc-at-top*]
lemmas *tendsto-erfc-at-bot* [*tendsto-intros*] = *filterlim-compose*[*OF erfc-at-bot*]

lemma *integrable-exp-minus-squared*:
assumes *A* ⊆ {*0*..} *A* ∈ *sets lborel*
shows *set-integrable lborel A* (λ*t*::*real*. *exp* ($-t^2$)) (**is** *?thesis1*)
and (λ*t*::*real*. *exp* ($-t^2$)) *integrable-on A* (**is** *?thesis2*)
⟨*proof*⟩

lemma
assumes *x* ≥ *0*
shows *erfc-real-altdef-nonneg*: *erfc x* = *2* / *sqrt pi* ∗ *integral* {*x*..} (λ*t*. *exp* ($-t^2$))
and *has-integral-erfc*: ((λ*t*. *exp* ($-t^2$)) *has-integral* (*sqrt pi* / *2* ∗ *erfc x*)) {*x*..}
⟨*proof*⟩

lemma *erfc-real-gt-0* [*simp*, *intro*]: *erfc* (*x*::*real*) > *0*
⟨*proof*⟩

lemma *erfc-real-less-2* [*intro*]: *erfc* (*x*::*real*) < *2*
⟨*proof*⟩

lemma *erf-real-gt-neg1* [*intro*]: *erf* (*x*::*real*) > −*1*
⟨*proof*⟩

lemma *erf-real-less-1* [*intro*]: *erf* (*x*::*real*) < *1*
⟨*proof*⟩

lemma *erfc-cnj* [*simp*]: *erfc* (*cnj z*) = *cnj* (*erfc z*)
⟨*proof*⟩

1.4 Specific facts about the complex case

lemma *erf-complex-altdef*:
  erf z = *of-real* (*2* / *sqrt pi*) ∗ *contour-integral* (*linepath 0 z*) (λ*t*. *exp* (−(*t*^*2*)))
⟨*proof*⟩

lemma *erf-holomorphic-on*: *erf holomorphic-on A*
  ⟨*proof*⟩

lemmas *erf-holomorphic-on′* [*holomorphic-intros*] =
  erf-holomorphic-on [*THEN holomorphic-on-chain-UNIV*]

lemma *erf-analytic-on*: *erf analytic-on A*
  ⟨*proof*⟩

lemma *erf-analytic-on′* [*analytic-intros*]:
  assumes *f analytic-on A*
  shows (λ*x*. *erf* (*f x*)) *analytic-on A*
⟨*proof*⟩

lemma *erfc-holomorphic-on*: *erfc holomorphic-on A*
  ⟨*proof*⟩

lemmas *erfc-holomorphic-on′* [*holomorphic-intros*] =
  erfc-holomorphic-on [*THEN holomorphic-on-chain-UNIV*]

lemma *erfc-analytic-on*: *erfc analytic-on A*
  ⟨*proof*⟩

lemma *erfc-analytic-on′* [*analytic-intros*]:
  assumes *f analytic-on A*
  shows (λ*x*. *erfc* (*f x*)) *analytic-on A*
⟨*proof*⟩

end

1.5 Asymptotics

theory *Error-Function-Asymptotics*
  imports *Error-Function Landau-Symbols.Landau-More*
begin

lemma *real-powr-eq-powerI*:
  x > *0* ⟹ *y* = *real y′* ⟹ *x powr y* = *x* ^ *y′*
  ⟨*proof*⟩

definition *erf-remainder-integral* **where**
  erf-remainder-integral n x =
    lim (λ*m*. *integral* {*x*..*x* + *real m*} (λ*t*. *exp* (−(*t*^*2*)) / *t* ^ (*2*∗*n*)))

The following is the remainder term in the asymptotic expansion of *erfc*.

definition *erf-remainder* **where**
erf-remainder n x =
((−*1*) ^*n* ∗ *2* ∗ *fact* (*2*∗*n*)) / (*sqrt pi* ∗ *4* ^ *n* ∗ *fact n*) ∗
erf-remainder-integral n x

lemma *erf-remainder-integral-aux-nonneg*:
x > *0* ⟹ *integral* {*x*..*x* + *real m*} (λ*t*. *exp* (−(*t*^*2*)) / *t* ^(*2*∗*n*)) ≥ *0*
⟨*proof*⟩

lemma *erf-remainder-integral-aux-bound*:
assumes *x* > *0*
shows *norm* (*integral* {*x*..*x* + *real m*} (λ*t*. *exp* ($-t^2$) / *t* ^(*2*∗*n*))) ≤ *exp* ($-x^2$) / *x* ^(*2*∗*n*+*1*)
and *integral* {*x*..*x* + *real m*} (λ*t*. *exp* ($-t^2$) / *t* ^(*2*∗*n*)) ≤ *exp* ($-x^2$) / *x* ^ (*2*∗*n*+*1*)
⟨*proof*⟩

lemma *convergent-erf-remainder-integral*:
assumes *x* > *0*
shows *convergent* (λ*m*. *integral* {*x*..*x* + *real m*} (λ*t*. *exp* (−(*t*^*2*)) / *t* ^(*2*∗*n*)))
⟨*proof*⟩

lemma *LIMSEQ-erf-remainder-integral*:
x > *0* ⟹ (λ*m*. *integral* {*x*..*x* + *real m*} (λ*t*. *exp* (−(*t*^*2*)) / *t* ^(*2*∗*n*))) ⟶
erf-remainder-integral n x
⟨*proof*⟩

We show some bounds on the remainder term.

lemma
assumes *x* > *0*
shows *erf-remainder-integral-nonneg*: *erf-remainder-integral n x* ≥ *0*
and *erf-remainder-integral-bound*: *erf-remainder-integral n x* ≤ *exp* ($-x^2$) / *x* ^(*2*∗*n*+*1*)
⟨*proof*⟩

lemma *erf-remainder-integral-bigo*:
erf-remainder-integral n ∈ *O*(λ*x*. *exp* ($-x^2$) / *x* ^(*2*∗*n*+*1*))
⟨*proof*⟩

theorem *erf-remainder-bigo*: *erf-remainder n* ∈ *O*(λ*x*. *exp* ($-x^2$) / *x* ^(*2*∗*n*+*1*))
⟨*proof*⟩

Next, we unroll the remainder term to develop the asymptotic expansion.

lemma *erf-remainder-integral-0-conv-erfc*:
assumes (*x*::*real*) > *0*
shows *erf-remainder-integral 0 x* = *sqrt pi* / *2* ∗ *erfc x*
⟨*proof*⟩

The first remainder is the *erfc* function itself.

lemma *erf-remainder-0-conv-erfc*: $x > 0 \Longrightarrow$ *erf-remainder 0 x = erfc x*
⟨*proof*⟩

Also, the following recurrence allows us to get the next term of the asymptotic expansion.

lemma *erf-remainder-integral-conv-Suc*:
assumes $x > 0$
shows *erf-remainder-integral n x = exp* $(-x^2)$ / $(2*x\hat{}(2*n+1))$ −
real $(2*n+1)$ / 2 ∗ *erf-remainder-integral* (*Suc n*) *x*
⟨*proof*⟩

lemma *erf-remainder-conv-Suc*:
assumes $x > 0$
shows *erf-remainder n x* = (− 1) ^ *n* ∗ *fact* (2 ∗ *n*) / (*sqrt pi* ∗ 4 ^ *n* ∗ *fact n*) ∗
exp $(- x^2)$ / (*x* ^ (2 ∗ *n* + 1)) + *erf-remainder* (*Suc n*) *x*
⟨*proof*⟩

Finally, this gives us the full asymptotic expansion for *erfc*:

theorem *erfc-unroll*:
assumes $x > 0$
shows *erfc x = exp* $(-x^2)$ / *sqrt pi* ∗
(∑ *i*<*n*. (−1)^*i* ∗ *fact* (2∗*i*) / (4^*i*∗*fact i*) / *x*^(2∗*i*+1)) + *erf-remainder n x*
⟨*proof*⟩

For convenience, we define another auxiliary function that is more suitable for use in an automated expansion framework, since it has a simple asymptotic expansion in powers of x.

definition *erfc-aux* **where** *erfc-aux x = exp* (x^2) ∗ *sqrt pi* ∗ *erfc x*
definition *erf-remainder′* **where** *erf-remainder′ n x = exp* (x^2) ∗ *sqrt pi* ∗ *erf-remainder n x*

lemma *erfc-aux-unroll*:
$x > 0 \Longrightarrow$
erfc-aux x = (∑ *i*<*n*. (−1)^*i* ∗ *fact* (2∗*i*) / (4^*i*∗*fact i*) / *x*^(2∗*i*+1)) + *erf-remainder′ n x*
⟨*proof*⟩

lemma *erf-remainder′-bigo*: *erf-remainder′ n* ∈ *O*(λ*x*. 1 / *x* ^ (2∗*n*+1))
⟨*proof*⟩

lemma *has-field-derivative-erfc-aux*:
(*erfc-aux has-field-derivative* (2 ∗ *x* ∗ *erfc-aux x* − 2)) (*at x*)
⟨*proof*⟩

end