

The Error Function

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Abstract

This entry provides the definitions and basic properties of the complex and real error function erf and the complementary error function erfc . Additionally, it gives their full asymptotic expansions.

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1 The complex and real error function

theory *Error-Function*

imports *HOL-Complex-Analysis.Complex-Analysis HOL-Library.Landau-Symbols*
begin

1.1 Auxiliary Facts

lemma *tendsto-sandwich-mono*:

assumes $(\lambda n. f \text{ (real } n)) \longrightarrow (c::\text{real})$

assumes *eventually* $(\lambda x. \forall y z. x \leq y \wedge y \leq z \longrightarrow f y \leq f z)$ *at-top*

shows $(f \longrightarrow c)$ *at-top*

proof (*rule tendsto-sandwich*)

from *assms*(2) **obtain** *C* **where** $C: \bigwedge x y. C \leq x \implies x \leq y \implies f x \leq f y$

by (*auto simp: eventually-at-top-linorder*)

show *eventually* $(\lambda x. f \text{ (real (nat } \lfloor x \rfloor))} \leq f x)$ *at-top*

using *eventually-gt-at-top*[*of 0::real*] *eventually-gt-at-top*[*of C+1::real*]

by *eventually-elim* (*rule C, linarith+*)

show *eventually* $(\lambda x. f \text{ (real (Suc (nat } \lfloor x \rfloor))} \geq f x)$ *at-top*

using *eventually-gt-at-top*[*of 0::real*] *eventually-gt-at-top*[*of C+1::real*]

by *eventually-elim* (*rule C, linarith+*)

qed (*auto intro!: filterlim-compose*[*OF assms*(1)])

filterlim-compose[*OF filterlim-nat-sequentially*]

filterlim-compose[*OF filterlim-Suc*] *filterlim-floor-sequentially*

simp del: of-nat-Suc)

lemma *tendsto-sandwich-antimono*:

assumes $(\lambda n. f \text{ (real } n)) \longrightarrow (c::\text{real})$

assumes *eventually* $(\lambda x. \forall y z. x \leq y \wedge y \leq z \longrightarrow f y \geq f z)$ *at-top*

shows $(f \longrightarrow c)$ *at-top*

proof (*rule tendsto-sandwich*)

from *assms*(2) **obtain** *C* **where** $C: \bigwedge x y. C \leq x \implies x \leq y \implies f x \geq f y$

by (*auto simp: eventually-at-top-linorder*)

show *eventually* $(\lambda x. f \text{ (real (nat } \lfloor x \rfloor))} \geq f x)$ *at-top*

using *eventually-gt-at-top*[*of 0::real*] *eventually-gt-at-top*[*of C+1::real*]

by *eventually-elim* (*rule C, linarith+*)

show *eventually* $(\lambda x. f \text{ (real (Suc (nat } \lfloor x \rfloor))} \leq f x)$ *at-top*

using *eventually-gt-at-top*[*of 0::real*] *eventually-gt-at-top*[*of C+1::real*]

by *eventually-elim* (*rule C, linarith+*)

qed (*auto intro!: filterlim-compose*[*OF assms*(1)])

filterlim-compose[*OF filterlim-nat-sequentially*]

filterlim-compose[*OF filterlim-Suc*] *filterlim-floor-sequentially*

simp del: of-nat-Suc)

lemma *has-bochner-integral-completion* [*intro*]:

fixes $f :: 'a \Rightarrow 'b::\{\text{banach, second-countable-topology}\}$

shows *has-bochner-integral* $M f I \implies \text{has-bochner-integral (completion } M) f I$

by (*auto simp: has-bochner-integral-iff integrable-completion integral-completion*
borel-measurable-integrable)

lemma *has-bochner-integral-imp-has-integral*:
has-bochner-integral lebesgue ($\lambda x. \text{indicator } S \ x \ *_{\mathbb{R}} \ f \ x$) $I \implies$
($f \text{ has-integral } (I :: 'b :: \text{euclidean-space})) \ S$
using *has-integral-set-lebesgue*[of $S \ f$]
by (*simp add: has-bochner-integral-iff set-integrable-def set-lebesgue-integral-def*)

lemma *has-bochner-integral-imp-has-integral'*:
has-bochner-integral lborel ($\lambda x. \text{indicator } S \ x \ *_{\mathbb{R}} \ f \ x$) $I \implies$
($f \text{ has-integral } (I :: 'b :: \text{euclidean-space})) \ S$
by (*intro has-bochner-integral-imp-has-integral has-bochner-integral-completion*)

lemma *has-bochner-integral-erf-aux*:
has-bochner-integral lborel ($\lambda x. \text{indicator } \{0..\} \ x \ *_{\mathbb{R}} \ \exp \ (- \ x^2)$) ($\text{sqrt } \pi \ / \ 2$)
proof –
let $?pI = \lambda f. (\int^{+s}. f \ (s::\text{real}) \ * \ \text{indicator } \{0..\} \ s \ \partial \text{lborel})$
let $?gauss = \lambda x. \exp \ (- \ x^2)$
let $?I = \text{indicator } \{0<..\} :: \text{real} \implies \text{real}$
let $?ff = \lambda x \ s. x \ * \ \exp \ (- \ x^2 \ * \ (1 + s^2)) :: \text{real}$
have $*$: $?pI \ ?gauss = (\int^{+x}. ?gauss \ x \ * \ ?I \ x \ \partial \text{lborel})$
by (*intro nn-integral-cong-AE AE-I[where $N=\{0\}$]*) (*auto split: split-indicator*)

have $?pI \ ?gauss \ * \ ?pI \ ?gauss = (\int^{+x}. \int^{+s}. ?gauss \ x \ * \ ?gauss \ s \ * \ ?I \ s \ * \ ?I \ x \ \partial \text{lborel} \ \partial \text{lborel})$
by (*auto simp: nn-integral-cmult[symmetric] nn-integral-multc[symmetric] * ennreal-mult[symmetric] intro!: nn-integral-cong split: split-indicator*)

also have $\dots = (\int^{+x}. \int^{+s}. ?ff \ x \ s \ * \ ?I \ s \ * \ ?I \ x \ \partial \text{lborel} \ \partial \text{lborel})$
proof (*rule nn-integral-cong, cases*)
fix $x :: \text{real}$ **assume** $x \neq 0$
then show $(\int^{+s}. ?gauss \ x \ * \ ?gauss \ s \ * \ ?I \ s \ * \ ?I \ x \ \partial \text{lborel}) =$
 $(\int^{+s}. ?ff \ x \ s \ * \ ?I \ s \ * \ ?I \ x \ \partial \text{lborel})$
by (*subst nn-integral-real-affine[where $t=0$ and $c=x$]*)
(*auto simp: mult-exp-exp nn-integral-cmult[symmetric] field-simps zero-less-mult-iff ennreal-mult[symmetric] intro!: nn-integral-cong split: split-indicator*)

qed simp

also have $\dots = \int^{+s}. \int^{+x}. ?ff \ x \ s \ * \ ?I \ s \ * \ ?I \ x \ \partial \text{lborel} \ \partial \text{lborel}$
by (*rule lborel-pair.Fubini'[symmetric]*) *auto*

also have $\dots = ?pI \ (\lambda s. ?pI \ (\lambda x. ?ff \ x \ s))$
by (*rule nn-integral-cong-AE*)
(*auto intro!: nn-integral-cong-AE AE-I[where $N=\{0\}$] split: split-indicator-asm*)

also have $\dots = ?pI \ (\lambda s. \text{ennreal } (1 / (2 * (1 + s^2))))$
proof (*intro nn-integral-cong ennreal-mult-right-cong*)
fix $s :: \text{real}$ **show** $?pI \ (\lambda x. ?ff \ x \ s) = \text{ennreal } (1 / (2 * (1 + s^2)))$
proof (*subst nn-integral-FTC-atLeast*)
have $((\lambda a. - (\exp \ (- \ ((1 + s^2) * a^2)) / (2 + 2 * s^2))) \longrightarrow - (0 / (2 + 2 * s^2)))) \ \text{at-top}$
by (*intro tendsto-intros filterlim-compose[OF exp-at-bot] filterlim-compose[OF filterlim-uminus-at-bot-at-top]*)

```

      filterlim-tendsto-pos-mult-at-top)
    (auto intro!: filterlim-at-top-mult-at-top[OF filterlim-ident filterlim-ident]
      simp: add-pos-nonneg power2-eq-square add-nonneg-eq-0-iff)
  then show (( $\lambda a. - (\exp (- a^2 - s^2 * a^2) / (2 + 2 * s^2))$ )  $\longrightarrow 0$ ) at-top
    by (simp add: field-simps)
  qed (auto intro!: derivative-eq-intros simp: field-simps add-nonneg-eq-0-iff)
qed
also have ... = ennreal (pi / 4)
proof (subst nn-integral-FTC-atLeast)
  show (( $\lambda a. \arctan a / 2$ )  $\longrightarrow (pi / 2) / 2$ ) at-top
    by (intro tendsto-intros) (simp-all add: tendsto-arctan-at-top)
qed (auto intro!: derivative-eq-intros simp: add-nonneg-eq-0-iff field-simps power2-eq-square)
finally have ?pI ?gauss2 = pi / 4
  by (simp add: power2-eq-square)
then have ?pI ?gauss = sqrt (pi / 4)
  using power-eq-iff-eq-base[of 2 enn2real (?pI ?gauss) sqrt (pi / 4)]
  by (cases ?pI ?gauss) (auto simp: power2-eq-square ennreal-mult[symmetric]
ennreal-top-mult)
also have ?pI ?gauss = ( $\int^+ x. \text{indicator } \{0..\} x *_{\mathbb{R}} \exp (- x^2) \partial \text{lborel}$ )
  by (intro nn-integral-cong) (simp split: split-indicator)
also have sqrt (pi / 4) = sqrt pi / 2
  by (simp add: real-sqrt-divide)
finally show ?thesis
  by (rule has-bochner-integral-nn-integral[rotated 3])
    auto
qed

lemma has-integral-erf-aux: (( $\lambda t::\text{real}. \exp (-t^2)$ ) has-integral (sqrt pi / 2)) {0..}
  by (intro has-bochner-integral-imp-has-integral' has-bochner-integral-erf-aux)

lemma contour-integrable-on-linepath-neg-exp-squared [simp, intro]:
  ( $\lambda t. \exp (-t^2)$ ) contour-integrable-on linepath 0 z
  by (auto intro!: contour-integrable-continuous-linepath continuous-intros)

lemma holomorphic-on-chain:
  g holomorphic-on t  $\implies$  f holomorphic-on s  $\implies$  f ' s  $\subseteq$  t  $\implies$ 
  ( $\lambda x. g (f x)$ ) holomorphic-on s
  using holomorphic-on-compose-gen[of f s g t] by (simp add: o-def)

lemma holomorphic-on-chain-UNIV:
  g holomorphic-on UNIV  $\implies$  f holomorphic-on s  $\implies$ 
  ( $\lambda x. g (f x)$ ) holomorphic-on s
  using holomorphic-on-compose-gen[of f s g UNIV] by (simp add: o-def)

lemmas holomorphic-on-exp' [holomorphic-intros] =
  holomorphic-on-exp [THEN holomorphic-on-chain-UNIV]

lemma leibniz-rule-field-derivative-real:
  fixes f::'a::{real-normed-field, banach}  $\Rightarrow$  real  $\Rightarrow$  'a

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assumes fx : $\bigwedge x t. x \in U \implies t \in \{a..b\} \implies ((\lambda x. f x t) \text{ has-field-derivative } fx \ x t) \text{ (at } x \text{ within } U)$
assumes $integrable-f2$: $\bigwedge x. x \in U \implies (f x) \text{ integrable-on } \{a..b\}$
assumes $cont-fx$: $\text{continuous-on } (U \times \{a..b\}) (\lambda(x, t). fx \ x \ t)$
assumes U : $x0 \in U \text{ convex } U$
shows $((\lambda x. \text{integral } \{a..b\} (f x)) \text{ has-field-derivative integral } \{a..b\} (fx \ x0)) \text{ (at } x0 \text{ within } U)$
using $leibniz-rule-field-derivative$ [of $U \ a \ b \ f \ fx \ x0$] **assms by simp**

lemma $has-vector-derivative-linepath-within$ [derivative-intros]:

assumes [derivative-intros]:
 $(f \text{ has-vector-derivative } f') \text{ (at } x \text{ within } S) (g \text{ has-vector-derivative } g') \text{ (at } x \text{ within } S)$
 $(h \text{ has-real-derivative } h') \text{ (at } x \text{ within } S)$
shows $((\lambda x. \text{linepath } (f x) (g x) (h x)) \text{ has-vector-derivative } (1 - h x) *_R f' + h x *_R g' - h' *_R (f x - g x)) \text{ (at } x \text{ within } S)$
unfolding $linepath-def$ [abs-def]
by (auto intro!: derivative-eq-intros simp: field-simps scaleR-diff-right)

lemma $has-field-derivative-linepath-within$ [derivative-intros]:

assumes [derivative-intros]:
 $(f \text{ has-field-derivative } f') \text{ (at } x \text{ within } S) (g \text{ has-field-derivative } g') \text{ (at } x \text{ within } S)$
 $(h \text{ has-real-derivative } h') \text{ (at } x \text{ within } S)$
shows $((\lambda x. \text{linepath } (f x) (g x) (h x)) \text{ has-field-derivative } (1 - h x) *_R f' + h x *_R g' - h' *_R (f x - g x)) \text{ (at } x \text{ within } S)$
unfolding $linepath-def$ [abs-def]
by (auto intro!: derivative-eq-intros simp: field-simps scaleR-diff-right)

lemma $continuous-on-linepath'$ [continuous-intros]:

assumes [continuous-intros]: $\text{continuous-on } A \ f \ \text{continuous-on } A \ g \ \text{continuous-on } A \ h$
shows $\text{continuous-on } A \ (\lambda x. \text{linepath } (f x) (g x) (h x))$
using $assms$ **unfolding** $linepath-def$ **by** (auto intro!: continuous-intros)

lemma $contour-integral-has-field-derivative$:

assumes A : $\text{open } A \ \text{convex } A \ a \in A \ z \in A$
assumes $integrable$: $\bigwedge z. z \in A \implies f \text{ contour-integrable-on linepath } a \ z$
assumes $holo$: $f \text{ holomorphic-on } A$
shows $((\lambda z. \text{contour-integral } (\text{linepath } a \ z) f) \text{ has-field-derivative } f z) \text{ (at } z \text{ within } B)$
proof –
have $(f \text{ has-field-derivative deriv } f z) \text{ (at } z) \text{ if } z \in A \ \text{for } z$
using $that \ assms \ \text{by } (auto \ intro! : \text{holomorphic-derivI})$
note [derivative-intros] = $DERIV-chain2$ [OF this]
note [continuous-intros] =
 $\text{continuous-on-compose2}$ [OF $\text{holomorphic-on-imp-continuous-on}$ [OF $holo$]]
 $\text{continuous-on-compose2}$ [OF $\text{holomorphic-on-imp-continuous-on}$ [OF holomorphic-deriv [OF $holo$]]]

have [*derivative-intros*]:
 (($\lambda x. \text{linepath } a \ x \ t$) *has-field-derivative of-real t*) (at x within A) **for** $t \ x$
by (*auto simp: linepath-def scaleR-conv-of-real intro!: derivative-eq-intros*)

have *: *linepath* $a \ b \ t \in A$ **if** $a \in A \ b \in A \ t \in \{0..1\}$ **for** $a \ b \ t$
using *that linepath-in-convex-hull[of a A b t] <convex A>* **by** (*simp add: hull-same*)

have (($\lambda z. \text{integral } \{0..1\} (\lambda x. f (\text{linepath } a \ z \ x)) * (z - a)$) *has-field-derivative*
integral $\{0..1\} (\lambda t. \text{deriv } f (\text{linepath } a \ z \ t) * \text{of-real } t) * (z - a) +$
integral $\{0..1\} (\lambda x. f (\text{linepath } a \ z \ x))$) (at z within A)
 (**is** (- *has-field-derivative ?I*) -)
by (*rule derivative-eq-intros leibniz-rule-field-derivative-real*) +
 (*insert assms,*
auto intro!: derivative-eq-intros leibniz-rule-field-derivative-real
integrable-continuous-real continuous-intros
*simp: split-beta scaleR-conv-of-real **)

also have ($\lambda z. \text{integral } \{0..1\} (\lambda x. f (\text{linepath } a \ z \ x)) * (z - a)$) =
 ($\lambda z. \text{contour-integral } (\text{linepath } a \ z) \ f$)
by (*simp add: contour-integral-integral*)

also have $?I = \text{integral } \{0..1\} (\lambda x. \text{deriv } f (\text{linepath } a \ z \ x) * \text{of-real } x * (z - a) +$
 $f (\text{linepath } a \ z \ x))$ (**is** - = *integral - ?g*)
by (*subst integral-mult-left [symmetric], subst integral-add [symmetric]*)
 (*insert assms, auto intro!: integrable-continuous-real continuous-intros simp:*
 *)

also have ($?g$ *has-integral of-real 1 * f (linepath a z 1) - of-real 0 * f (linepath*
a z 0)) $\{0..1\}$
using * A
by (*intro fundamental-theorem-of-calculus*)
 (*auto intro!: derivative-eq-intros has-vector-derivative-real-field*
simp: linepath-def scaleR-conv-of-real)

hence *integral* $\{0..1\} ?g = f (\text{linepath } a \ z \ 1)$ **by** (*simp add: has-integral-iff*)

also have *linepath* $a \ z \ 1 = z$ **by** (*simp add: linepath-def*)

also from $\langle z \in A \rangle$ **and** $\langle \text{open } A \rangle$ **have** *at z within A = at z* **by** (*rule at-within-open*)

finally show $?thesis$ **by** (*rule DERIV-subset*) *simp-all*
qed

1.2 Definition of the error function

definition *erf-coeffs* :: $\text{nat} \Rightarrow \text{real}$ **where**

erf-coeffs $n =$
 (*if odd n then* $2 / \text{sqrt } \pi * (-1) ^ (n \text{ div } 2) / (\text{of-nat } n * \text{fact } (n \text{ div } 2))$
else 0)

lemma *summable-erf*:

fixes $z :: 'a :: \{\text{real-normed-div-algebra, banach}\}$
shows *summable* ($\lambda n. \text{of-real } (\text{erf-coeffs } n) * z ^ n$)

proof -

define b **where** $b = (\lambda n. 2 / \text{sqrt } \pi * (\text{if odd } n \text{ then inverse (fact } (n \text{ div } 2)))$

```

else 0))
show ?thesis
proof (rule summable-comparison-test[OF exI[of - 1]], clarify)
  fix n :: nat assume n: n ≥ 1
  hence norm (of-real (erf-coeffs n) * z ^ n) ≤ b n * norm z ^ n
  unfolding norm-mult norm-power erf-coeffs-def b-def
  by (intro mult-right-mono) (auto simp: field-simps norm-divide abs-mult)
  thus norm (of-real (erf-coeffs n) * z ^ n) ≤ b n * norm z ^ n
  by (simp add: mult-ac)
next
  have summable (λn. (norm z * 2 / sqrt pi) * (inverse (fact n) * norm z ^
(2*n)))
  (is summable ?c) unfolding power-mult by (intro summable-mult summable-exp)
  also have ?c = (λn. b (2*n+1) * norm z ^ (2*n+1))
  unfolding b-def by (auto simp: fun-eq-iff b-def)
  also have summable ... ↔ summable (λn. b n * norm z ^ n)
  using summable-mono-reindex [of λn. 2*n+1]
  by (intro summable-mono-reindex [of λn. 2*n+1])
  (auto elim!: oddE simp: strict-mono-def b-def)
  finally show ... .
qed
qed

```

definition *erf* :: ('a :: {real-normed-field, banach}) ⇒ 'a **where**
erf z = (∑ n. of-real (erf-coeffs n) * z ^ n)

lemma *erf-converges*: (λn. of-real (erf-coeffs n) * z ^ n) sums *erf* z
using *summable-erf* **by** (*simp add: sums-iff erf-def*)

lemma *erf-0* [*simp*]: *erf* 0 = 0
unfolding *erf-def* *power-zero* **by** (*simp add: erf-coeffs-def*)

lemma *erf-minus* [*simp*]: *erf* (-z) = - *erf* z
unfolding *erf-def*
by (*subst suminf-minus* [OF *summable-erf*, *symmetric*], *rule suminf-cong*)
(*simp-all add: erf-coeffs-def*)

lemma *erf-of-real* [*simp*]: *erf* (of-real x) = of-real (*erf* x)
unfolding *erf-def* **using** *summable-erf*[of x]
by (*subst suminf-of-real*) (*simp-all add: summable-erf*)

lemma *of-real-erf-numeral* [*simp*]: of-real (*erf* (numeral n)) = *erf* (numeral n)
by (*simp only: erf-of-real* [*symmetric*] *of-real-numeral*)

lemma *of-real-erf-1* [*simp*]: of-real (*erf* 1) = *erf* 1
by (*simp only: erf-of-real* [*symmetric*] *of-real-1*)

lemma *erf-has-field-derivative*:

(*erf has-field-derivative of-real (2 / sqrt pi) * exp (-(z²))*) (at z within A)
proof –
define *a'* **where** $a' = (\lambda n. 2 / \text{sqrt } \pi * (\text{if even } n \text{ then } (-1)^{\wedge (n \text{ div } 2)} / \text{fact } (n \text{ div } 2) \text{ else } 0))$
have (*erf has-field-derivative*
 $(\sum n. \text{diffs } (\lambda n. \text{of-real } (\text{erf-coeffs } n)) n * z^{\wedge n})$) (at z)
using *summable-erf unfolding erf-def* **by** (*rule termdiffs-strong-converges-everywhere*)
also have $\text{diffs } (\lambda n. \text{of-real } (\text{erf-coeffs } n)) = (\lambda n. \text{of-real } (a' n) :: 'a)$
by (*simp add: erf-coeffs-def a'-def diffs-def fun-eq-iff del: of-nat-Suc*)
hence $(\sum n. \text{diffs } (\lambda n. \text{of-real } (\text{erf-coeffs } n)) n * z^{\wedge n}) =$
 $(\sum n. \text{of-real } (a' n) * z^{\wedge n})$ **by** *simp*
also have $\dots = (\sum n. \text{of-real } (a' (2*n)) * z^{\wedge (2*n)})$
by (*intro suminf-mono-reindex [symmetric]*) (*auto simp: strict-mono-def a'-def elim!: evenE*)
also have $(\lambda n. \text{of-real } (a' (2*n)) * z^{\wedge (2*n)}) =$
 $(\lambda n. \text{of-real } (2 / \text{sqrt } \pi) * (\text{inverse } (\text{fact } n) * (-(z^{\wedge 2}))^{\wedge n}))$
by (*simp add: fun-eq-iff power-mult [symmetric] a'-def field-simps power-minus'*)
also have $\text{suminf } \dots = \text{of-real } (2 / \text{sqrt } \pi) * \text{exp } (-(z^{\wedge 2}))$
by (*subst suminf-mult, intro summable-exp*)
(auto simp: field-simps scaleR-conv-of-real exp-def)
finally show *?thesis* **by** (*rule DERIV-subset*) *simp-all*
qed

lemmas *erf-has-field-derivative'* [*derivative-intros*] =
erf-has-field-derivative [THEN DERIV-chain2]

lemma *erf-continuous-on: continuous-on A erf*
by (*rule DERIV-continuous-on erf-has-field-derivative*)+

lemma *continuous-on-compose2-UNIV*:
 $\text{continuous-on UNIV } g \implies \text{continuous-on } s \ f \implies \text{continuous-on } s \ (\lambda x. g \ (f \ x))$
by (*rule continuous-on-compose2[of UNIV g s f]*) *simp-all*

lemmas *erf-continuous-on'* [*continuous-intros*] =
erf-continuous-on [THEN continuous-on-compose2-UNIV]

lemma *erf-continuous* [*continuous-intros*]: *continuous (at x within A) erf*
by (*rule continuous-within-subset[OF - subset-UNIV]*)
(insert erf-continuous-on[of UNIV], auto simp: continuous-on-eq-continuous-at)

lemmas *erf-continuous'* [*continuous-intros*] =
continuous-within-compose2[OF - erf-continuous]

lemmas *tendsto-erf* [*tendsto-intros*] = *isCont-tendsto-compose[OF erf-continuous]*

lemma *erf-cnjl* [*simp*]: $\text{erf } (\text{cnj } z) = \text{cnj } (\text{erf } z)$

proof –
interpret *bounded-linear cnj* **by** (*rule bounded-linear-cnjl*)

from *suminf*[*OF summable-erf*] **show** *?thesis* **by** (*simp add: erf-def erf-coeffs-def*)
qed

lemma *integral-exp-minus-squared-real*:

assumes $a \leq b$
shows $((\lambda t. \exp(-(t^2))) \text{ has-integral } (\sqrt{\pi} / 2 * (\text{erf } b - \text{erf } a))) \{a..b\}$
proof –
have $((\lambda t. \exp(-(t^2))) \text{ has-integral } (\sqrt{\pi} / 2 * \text{erf } b - \sqrt{\pi} / 2 * \text{erf } a))$
 $\{a..b\}$
using *assms*
by (*intro fundamental-theorem-of-calculus*)
(auto intro!: derivative-eq-intros
simp: has-field-derivative-iff-has-vector-derivative [symmetric])
thus *?thesis* **by** (*simp add: field-simps*)
qed

lemma *erf-real-altdef-nonneg*:

$x \geq 0 \implies \text{erf } (x::\text{real}) = 2 / \sqrt{\pi} * \text{integral } \{0..x\} (\lambda t. \exp(-(t^2)))$
using *integral-exp-minus-squared-real*[*of 0 x*]
by (*simp add: has-integral-iff field-simps*)

lemma *erf-real-altdef-nonpos*:

$x \leq 0 \implies \text{erf } (x::\text{real}) = -2 / \sqrt{\pi} * \text{integral } \{0..-x\} (\lambda t. \exp(-(t^2)))$
using *erf-real-altdef-nonneg*[*of -x*] **by** *simp*

lemma *less-imp-erf-real-less*:

assumes $a < (b::\text{real})$
shows $\text{erf } a < \text{erf } b$
proof –
from *assms* **have** $\exists z. z > a \wedge z < b \wedge \text{erf } b - \text{erf } a = (b - a) * (2 / \sqrt{\pi} * \exp(-z^2))$
by (*intro MVT2*) (*auto intro!: derivative-eq-intros simp: field-simps*)
then obtain z **where** $z: a < z < b$
and $\text{erf } b - \text{erf } a = (b - a) * (2 / \sqrt{\pi} * \exp(-z^2))$
by *blast*
note *erf*
also from *assms* **have** $(b - a) * (2 / \sqrt{\pi} * \exp(-z^2)) > 0$
by (*intro mult-pos-pos divide-pos-pos*) *simp-all*
finally show *?thesis* **by** *simp*
qed

lemma *le-imp-erf-real-le*: $a \leq (b::\text{real}) \implies \text{erf } a \leq \text{erf } b$

by (*cases a < b*) (*auto dest: less-imp-erf-real-less*)

lemma *erf-real-less-cancel* [*simp*]: $(\text{erf } (a :: \text{real}) < \text{erf } b) \longleftrightarrow a < b$

using *less-imp-erf-real-less*[*of a b*] *less-imp-erf-real-less*[*of b a*]
by (*cases a b rule: linorder-cases*) *simp-all*

lemma *erf-real-eq-iff* [simp]: $\text{erf } (a::\text{real}) = \text{erf } b \longleftrightarrow a = b$
by (cases a b rule: linorder-cases) (auto dest: less-imp-erf-real-less)

lemma *erf-real-le-cancel* [simp]: $(\text{erf } (a::\text{real}) \leq \text{erf } b) \longleftrightarrow a \leq b$
by (cases a b rule: linorder-cases) (auto simp: less-eq-real-def)

lemma *inj-on-erf-real* [intro]: $\text{inj-on } (\text{erf } :: \text{real} \Rightarrow \text{real}) A$
by (auto simp: inj-on-def)

lemma *strict-mono-erf-real* [intro]: $\text{strict-mono } (\text{erf } :: \text{real} \Rightarrow \text{real})$
by (auto simp: strict-mono-def)

lemma *mono-erf-real* [intro]: $\text{mono } (\text{erf } :: \text{real} \Rightarrow \text{real})$
by (auto simp: mono-def)

lemma *erf-real-ge-0-iff* [simp]: $\text{erf } (x::\text{real}) \geq 0 \longleftrightarrow x \geq 0$
using *erf-real-le-cancel*[of 0 x] **unfolding** *erf-0* .

lemma *erf-real-le-0-iff* [simp]: $\text{erf } (x::\text{real}) \leq 0 \longleftrightarrow x \leq 0$
using *erf-real-le-cancel*[of x 0] **unfolding** *erf-0* .

lemma *erf-real-gt-0-iff* [simp]: $\text{erf } (x::\text{real}) > 0 \longleftrightarrow x > 0$
using *erf-real-less-cancel*[of 0 x] **unfolding** *erf-0* .

lemma *erf-real-less-0-iff* [simp]: $\text{erf } (x::\text{real}) < 0 \longleftrightarrow x < 0$
using *erf-real-less-cancel*[of x 0] **unfolding** *erf-0* .

lemma *erf-at-top* [tendsto-intros]: $((\text{erf } :: \text{real} \Rightarrow \text{real}) \longrightarrow 1)$ *at-top*
proof –
have *: $(\bigcup n. \{0..real\ } n) = \{0.. \}$ **by** (auto intro!: real-nat-ceiling-ge)
let ?f = $\lambda t::\text{real}. \exp(-t^2)$
have $(\lambda n. \text{set-lebesgue-integral } \text{lborel } \{0..real\ } n \ ?f)$
 $\longrightarrow \text{set-lebesgue-integral } \text{lborel } (\bigcup n. \{0..real\ } n) \ ?f$
using *has-bochner-integral-erf-aux*
by (intro *set-integral-cont-up*)
*(insert *, auto simp: incseq-def has-bochner-integral-iff set-integrable-def)*
also note *
also have $(\lambda n. \text{set-lebesgue-integral } \text{lborel } \{0..real\ } n \ ?f) = (\lambda n. \text{integral } \{0..real\ } n \ ?f)$
proof –
have $\bigwedge n. \text{set-integrable } \text{lborel } \{0..real\ } n \ (\lambda x. \exp(-x^2))$
unfolding *set-integrable-def*
by (intro *borel-integrable-compact*) (auto intro!: *continuous-intros*)
then show *?thesis*
by (intro *set-borel-integral-eq-integral ext*)
qed
also have $\dots = (\lambda n. \text{sqrt } \pi / 2 * \text{erf } (\text{real } n))$ **by** (*simp add: erf-real-altdef-nonneg*)

also have *set-lebesgue-integral lborel* $\{0..\}$ $?f = \text{sqrt } \pi / 2$
using *has-bochner-integral-erf-aux* **by** (*simp add: has-bochner-integral-iff set-lebesgue-integral-def*)
finally have $(\lambda n. 2 / \text{sqrt } \pi * (\text{sqrt } \pi / 2 * \text{erf } (\text{real } n))) \longrightarrow$
 $(2 / \text{sqrt } \pi) * (\text{sqrt } \pi / 2)$ **by** (*intro tendsto-intros*)
hence $(\lambda n. \text{erf } (\text{real } n)) \longrightarrow 1$ **by** *simp*
thus *?thesis* **by** (*rule tendsto-sandwich-mono*) *auto*
qed

lemma *erf-at-bot* [*tendsto-intros*]: $((\text{erf } :: \text{real} \Rightarrow \text{real}) \longrightarrow -1)$ *at-bot*
by (*simp add: filterlim-at-bot-mirror tendsto-minus-cancel-left erf-at-top*)

lemmas *tendsto-erf-at-top* [*tendsto-intros*] = *filterlim-compose*[*OF erf-at-top*]
lemmas *tendsto-erf-at-bot* [*tendsto-intros*] = *filterlim-compose*[*OF erf-at-bot*]

1.3 The complimentary error function

definition *erfc* **where** $\text{erfc } z = 1 - \text{erf } z$

lemma *erf-conv-erfc*: $\text{erf } z = 1 - \text{erfc } z$ **by** (*simp add: erfc-def*)

lemma *erfc-0* [*simp*]: $\text{erfc } 0 = 1$
by (*simp add: erfc-def*)

lemma *erfc-minus*: $\text{erfc } (-z) = 2 - \text{erfc } z$
by (*simp add: erfc-def*)

lemma *erfc-of-real* [*simp*]: $\text{erfc } (\text{of-real } x) = \text{of-real } (\text{erfc } x)$
by (*simp add: erfc-def*)

lemma *of-real-erfc-numeral* [*simp*]: $\text{of-real } (\text{erfc } (\text{numeral } n)) = \text{erfc } (\text{numeral } n)$
by (*simp add: erfc-def*)

lemma *of-real-erfc-1* [*simp*]: $\text{of-real } (\text{erfc } 1) = \text{erfc } 1$
by (*simp add: erfc-def*)

lemma *less-imp-erfc-real-less*: $a < (b :: \text{real}) \implies \text{erfc } a > \text{erfc } b$
by (*simp add: erfc-def*)

lemma *le-imp-erfc-real-le*: $a \leq (b :: \text{real}) \implies \text{erfc } a \geq \text{erfc } b$
by (*simp add: erfc-def*)

lemma *erfc-real-less-cancel* [*simp*]: $(\text{erfc } (a :: \text{real}) < \text{erfc } b) \iff a > b$
by (*simp add: erfc-def*)

lemma *erfc-real-eq-iff* [*simp*]: $\text{erfc } (a :: \text{real}) = \text{erfc } b \iff a = b$
by (*simp add: erfc-def*)

lemma *erfc-real-le-cancel* [*simp*]: $(\text{erfc } (a :: \text{real}) \leq \text{erfc } b) \iff a \geq b$

by (*simp add: erfc-def*)

lemma *inj-on-erfc-real* [*intro*]: *inj-on (erfc :: real \Rightarrow real) A*
by (*auto simp: inj-on-def*)

lemma *antimono-erfc-real* [*intro*]: *antimono (erfc :: real \Rightarrow real)*
by (*auto simp: antimono-def*)

lemma *erfc-real-ge-0-iff* [*simp*]: *erfc (x::real) $\geq 1 \iff x \leq 0$*
by (*simp add: erfc-def*)

lemma *erfc-real-le-0-iff* [*simp*]: *erfc (x::real) $\leq 1 \iff x \geq 0$*
by (*simp add: erfc-def*)

lemma *erfc-real-gt-0-iff* [*simp*]: *erfc (x::real) $> 1 \iff x < 0$*
by (*simp add: erfc-def*)

lemma *erfc-real-less-0-iff* [*simp*]: *erfc (x::real) $< 1 \iff x > 0$*
by (*simp add: erfc-def*)

lemma *erfc-has-field-derivative*:
*(erfc has-field-derivative -of-real (2 / sqrt pi) * exp $-(z^2)$) (at z within A)*
unfolding *erfc-def [abs-def]* **by** (*auto intro!: derivative-eq-intros*)

lemmas *erfc-has-field-derivative'* [*derivative-intros*] =
erfc-has-field-derivative [THEN DERIV-chain2]

lemma *erfc-continuous-on*: *continuous-on A erfc*
by (*rule DERIV-continuous-on erfc-has-field-derivative*)⁺

lemmas *erfc-continuous-on'* [*continuous-intros*] =
erfc-continuous-on [THEN continuous-on-compose2-UNIV]

lemma *erfc-continuous* [*continuous-intros*]: *continuous (at x within A) erfc*
by (*rule continuous-within-subset[OF - subset-UNIV]*)
(insert erfc-continuous-on[of UNIV], auto simp: continuous-on-eq-continuous-at)

lemmas *erfc-continuous'* [*continuous-intros*] =
continuous-within-compose2[OF - erfc-continuous]

lemmas *tendsto-erfc* [*tendsto-intros*] = *isCont-tendsto-compose[OF erfc-continuous]*

lemma *erfc-at-top* [*tendsto-intros*]: *((erfc :: real \Rightarrow real) $\longrightarrow 0$) at-top*
unfolding *erfc-def [abs-def]* **by** (*auto intro!: tendsto-eq-intros*)

lemma *erfc-at-bot* [*tendsto-intros*]: *((erfc :: real \Rightarrow real) $\longrightarrow 2$) at-bot*
unfolding *erfc-def [abs-def]* **by** (*auto intro!: tendsto-eq-intros*)

lemmas *tendsto-erfc-at-top* [*tendsto-intros*] = *filterlim-compose*[*OF erfc-at-top*]

lemmas *tendsto-erfc-at-bot* [*tendsto-intros*] = *filterlim-compose*[*OF erfc-at-bot*]

lemma *integrable-exp-minus-squared*:

assumes $A \subseteq \{0..\}$ $A \in \text{sets lborel}$

shows *set-integrable lborel* A $(\lambda t::\text{real. exp } (-t^2))$ (**is** *?thesis1*)

and $(\lambda t::\text{real. exp } (-t^2))$ *integrable-on* A (**is** *?thesis2*)

proof –

show *?thesis1*

by (*rule set-integrable-subset*[*of* - $\{0..\}$])

(*insert assms has-bochner-integral-erf-aux, auto simp: has-bochner-integral-iff set-integrable-def*)

thus *?thesis2* **by** (*rule set-borel-integral-eq-integral*)

qed

lemma

assumes $x \geq 0$

shows *erfc-real-altdef-nonneg*: $\text{erfc } x = 2 / \text{sqrt } \pi * \text{integral } \{x..\}$ $(\lambda t. \text{exp } (-t^2))$

and *has-integral-erfc*: $((\lambda t. \text{exp } (-t^2)) \text{ has-integral } (\text{sqrt } \pi / 2 * \text{erfc } x)) \{x..\}$

proof –

let $?f = \lambda t::\text{real. exp } (-t^2)$

have *int*: *set-integrable lborel* $\{0..\}$ $?f$

using *has-bochner-integral-erf-aux* **by** (*simp add: has-bochner-integral-iff set-integrable-def*)

from *assms* **have** $\{(0::\text{real})..\} = \{0..x\} \cup \{x..\}$ **by** *auto*

have *set-lebesgue-integral lborel* $(\{0..x\} \cup \{x..\})$ $?f =$

set-lebesgue-integral lborel $\{0..x\}$ $?f + \text{set-lebesgue-integral lborel } \{x..\}$

$?f$

by (*subst set-integral-Un-AE; (rule set-integrable-subset*[*OF int*])*?)*

(*insert assms AE-lborel-singleton*[*of x*], *auto elim!: eventually-mono*)

also note $*$ [*symmetric*]

also have *set-lebesgue-integral lborel* $\{0..\}$ $?f = \text{sqrt } \pi / 2$

using *has-bochner-integral-erf-aux* **by** (*simp add: has-bochner-integral-iff set-lebesgue-integral-def*)

also have *set-lebesgue-integral lborel* $\{0..x\}$ $?f = \text{sqrt } \pi / 2 * \text{erf } x$

by (*subst set-borel-integral-eq-integral*(2)[*OF set-integrable-subset*[*OF int*]])

(*insert assms, auto simp: erf-real-altdef-nonneg*)

also have *set-lebesgue-integral lborel* $\{x..\}$ $?f = \text{integral } \{x..\}$ $?f$

by (*subst set-borel-integral-eq-integral*(2)[*OF set-integrable-subset*[*OF int*]])

(*insert assms, auto*)

finally show $\text{erfc } x = 2 / \text{sqrt } \pi * \text{integral } \{x..\}$ $?f$ **by** (*simp add: field-simps erfc-def*)

with *integrable-exp-minus-squared*(2)[*of* $\{x..\}$] *assms*

show $(?f \text{ has-integral } (\text{sqrt } \pi / 2 * \text{erfc } x)) \{x..\}$

by (*simp add: has-integral-iff*)

qed

```

lemma erfc-real-gt-0 [simp, intro]: erfc (x::real) > 0
proof (cases x ≥ 0)
  case True
    have  $0 < \text{integral } \{x..x+1\} (\lambda t. \exp(-(x+1)^2))$  by simp
    also from True have  $\dots \leq \text{integral } \{x..x+1\} (\lambda t. \exp(-t^2))$ 
      by (intro integral-le)
      (auto intro!: integrable-continuous-real continuous-intros power-mono)
    also have  $\dots \leq \text{sqrt } \pi / 2 * \text{erfc } x$ 
      by (rule has-integral-subset-le[OF - integrable-integral has-integral-erfc])
      (auto intro!: integrable-continuous-real continuous-intros True)
    finally have  $\text{sqrt } \pi / 2 * \text{erfc } x > 0$  .
    hence  $\dots * (2 / \text{sqrt } \pi) > 0$  by (rule mult-pos-pos) simp-all
    thus erfc x > 0 by simp
  next
    case False
      have  $0 \leq (1::\text{real})$  by simp
      also from False have  $\dots < \text{erfc } x$  by simp
      finally show ?thesis .
qed

```

```

lemma erfc-real-less-2 [intro]: erfc (x::real) < 2
  using erfc-real-gt-0[of -x] unfolding erfc-minus by simp

```

```

lemma erf-real-gt-neg1 [intro]: erf (x::real) > -1
  using erfc-real-less-2[of x] unfolding erfc-def by simp

```

```

lemma erf-real-less-1 [intro]: erf (x::real) < 1
  using erfc-real-gt-0[of x] unfolding erfc-def by simp

```

```

lemma erfc-cnj [simp]: erfc (cnj z) = cnj (erfc z)
  by (simp add: erfc-def)

```

1.4 Specific facts about the complex case

```

lemma erf-complex-altdef:
  erf z = of-real ( $2 / \text{sqrt } \pi$ ) * contour-integral (linepath 0 z) ( $\lambda t. \exp(-(t^2)$ )
proof -
  define A where A = ( $\lambda z. \text{contour-integral } (\text{linepath } 0 z) (\lambda t. \exp(-(t^2)))$ )
  have [derivative-intros]: (A has-field-derivative  $\exp(-(z^2))$ ) (at z) for z ::
complex
    unfolding A-def
    by (rule contour-integral-has-field-derivative[where A = UNIV])
    (auto intro!: holomorphic-intros)
  have erf z -  $2 / \text{sqrt } \pi * A z$  = erf 0 -  $2 / \text{sqrt } \pi * A 0$ 
    by (rule has-derivative-zero-unique [where f =  $\lambda z. \text{erf } z - 2 / \text{sqrt } \pi * A z$ 
and s = UNIV])
    (auto intro!: has-field-derivative-imp-has-derivative derivative-eq-intros)
  also have A 0 = 0 by (simp only: A-def contour-integral-trivial)
  finally show ?thesis unfolding A-def by (simp add: algebra-simps)

```

qed

lemma *erf-holomorphic-on: erf holomorphic-on A*
by (auto simp: holomorphic-on-def field-differentiable-def intro!: erf-has-field-derivative)

lemmas *erf-holomorphic-on' [holomorphic-intros] =*
erf-holomorphic-on [THEN holomorphic-on-chain-UNIV]

lemma *erf-analytic-on: erf analytic-on A*
by (auto simp: analytic-on-def) (auto intro!: exI[of - 1] holomorphic-intros)

lemma *erf-analytic-on' [analytic-intros]:*

assumes *f analytic-on A*

shows $(\lambda x. \text{erf } (f x))$ *analytic-on A*

proof –

from *assms* and *erf-analytic-on* have *erf* \circ *f* *analytic-on A*

by (rule *analytic-on-compose-gen*) auto

thus ?thesis by (simp add: *o-def*)

qed

lemma *erfc-holomorphic-on: erfc holomorphic-on A*
by (auto simp: holomorphic-on-def field-differentiable-def intro!: erfc-has-field-derivative)

lemmas *erfc-holomorphic-on' [holomorphic-intros] =*
erfc-holomorphic-on [THEN holomorphic-on-chain-UNIV]

lemma *erfc-analytic-on: erfc analytic-on A*
by (auto simp: analytic-on-def) (auto intro!: exI[of - 1] holomorphic-intros)

lemma *erfc-analytic-on' [analytic-intros]:*

assumes *f analytic-on A*

shows $(\lambda x. \text{erfc } (f x))$ *analytic-on A*

proof –

from *assms* and *erfc-analytic-on* have *erfc* \circ *f* *analytic-on A*

by (rule *analytic-on-compose-gen*) auto

thus ?thesis by (simp add: *o-def*)

qed

end

1.5 Asymptotics

theory *Error-Function-Asymptotics*

imports *Error-Function Landau-Symbols.Landau-More*

begin

lemma *real-powr-eq-powerI:*

$x > 0 \implies y = \text{real } y' \implies x \text{ powr } y = x \wedge y'$

by (simp add: *powr-realpow*)

definition *erf-remainder-integral* **where**

$$\text{erf-remainder-integral } n \ x = \\ \lim (\lambda m. \text{integral } \{x..x + \text{real } m\} (\lambda t. \exp(-t^2) / t^{(2*n)}))$$

The following is the remainder term in the asymptotic expansion of *erfc*.

definition *erf-remainder* **where**

$$\text{erf-remainder } n \ x = \\ ((-1)^n * 2 * \text{fact } (2*n)) / (\text{sqrt } \pi * 4^n * \text{fact } n) * \\ \text{erf-remainder-integral } n \ x$$

lemma *erf-remainder-integral-aux-nonneg*:

$$x > 0 \implies \text{integral } \{x..x + \text{real } m\} (\lambda t. \exp(-t^2) / t^{(2*n)}) \geq 0$$

by (*intro integral-nonneg integrable-continuous-real*) (*auto intro!: continuous-intros*)

lemma *erf-remainder-integral-aux-bound*:

assumes $x > 0$

$$\text{shows } \text{norm } (\text{integral } \{x..x + \text{real } m\} (\lambda t. \exp(-t^2) / t^{(2*n)})) \leq \exp(-x^2) / x^{(2*n+1)}$$

$$\text{and } \text{integral } \{x..x + \text{real } m\} (\lambda t. \exp(-t^2) / t^{(2*n)}) \leq \exp(-x^2) / x^{(2*n+1)}$$

proof -

$$\text{have } \text{norm } (\text{integral } \{x..x + \text{real } m\} (\lambda t. \exp(-t^2) / t^{(2*n)})) \leq \\ \text{integral } \{x..x + \text{real } m\} (\lambda t. \exp(-x*t) / x^{(2*n)})$$

proof (*intro integral-norm-bound-integral ballI*)

fix t **assume** $t: t \in \{x..x + \text{real } m\}$

from t **have** $\text{norm } (\exp(-t^2) / t^{(2*n)}) = \exp(-t^2) / t^{(2*n)}$ **by** *simp*

also have $\dots \leq \exp(-x*t) / x^{(2*n)}$ **using** t *assms*

by (*intro frac-le*) (*simp-all add: self-le-power power2-eq-square power-mono*)

finally show $\text{norm } (\exp(-t^2) / t^{(2*n)}) \leq \dots$ **by** *simp*

qed (*insert assms, auto intro!: continuous-intros integrable-continuous-real*)

$$\text{also have } \dots = -\exp(-x*(x + \text{real } m)) / x^{(2*n+1)} - (-\exp(-x*x) / x^{(2*n+1)})$$

using *assms*

by (*intro integral-unique fundamental-theorem-of-calculus*)

(*auto simp: has-field-derivative-iff-has-vector-derivative [symmetric]*)

intro!: derivative-eq-intros)

also have $\dots \leq \exp(-x^2) / x^{(2*n+1)}$ **using** *assms* **by** (*simp add: power2-eq-square*)

finally show $*$: $\text{norm } (\text{integral } \{x..x + \text{real } m\} (\lambda t. \exp(-t^2) / t^{(2*n)})) \leq \exp(-x^2) / x^{(2*n+1)}$.

have $\text{integral } \{x..x + \text{real } m\} (\lambda t. \exp(-t^2) / t^{(2*n)}) \leq$

$$\text{norm } (\text{integral } \{x..x + \text{real } m\} (\lambda t. \exp(-t^2) / t^{(2*n)})) \text{ by } \text{simp}$$

also note $*$

$$\text{finally show } \text{integral } \{x..x + \text{real } m\} (\lambda t. \exp(-t^2) / t^{(2*n)}) \leq \exp(-x^2) / x^{(2*n+1)} .$$

qed

lemma *convergent-erf-remainder-integral*:

assumes $x > 0$
shows $\text{convergent } (\lambda m. \text{integral } \{x..x + \text{real } m\} (\lambda t. \text{exp } (-(t^2)) / t^{(2*n)}))$
proof (*intro Bseq-mono-convergent BseqI'; clarify?*)
fix $m :: \text{nat}$
show $\text{norm } (\text{integral } \{x..x + \text{real } m\} (\lambda t. \text{exp } (-t^2) / t^{(2*n)})) \leq \text{exp } (-x^2) / x^{(2*n+1)}$
using *assms by (rule erf-remainder-integral-aux-bound)*
qed (*insert assms, auto intro!: integral-subset-le integrable-continuous-real continuous-intros*)

lemma *LIMSEQ-erf-remainder-integral:*
 $x > 0 \implies (\lambda m. \text{integral } \{x..x + \text{real } m\} (\lambda t. \text{exp } (-(t^2)) / t^{(2*n)})) \longrightarrow \text{erf-remainder-integral } n \ x$
using *convergent-erf-remainder-integral[of x]*
by (*simp add: convergent-LIMSEQ-iff erf-remainder-integral-def*)

We show some bounds on the remainder term.

lemma
assumes $x > 0$
shows *erf-remainder-integral-nonneg: erf-remainder-integral n x ≥ 0*
and *erf-remainder-integral-bound: erf-remainder-integral n x $\leq \text{exp } (-x^2) / x^{(2*n+1)}$*
proof –
note $*$ = *LIMSEQ-erf-remainder-integral[OF assms]*
show *erf-remainder-integral n x ≥ 0*
by (*intro tendsto-le[OF - * tendsto-const] always-eventually erf-remainder-integral-aux-nonneg allI assms sequentially-bot*)
show *erf-remainder-integral n x $\leq \text{exp } (-x^2) / x^{(2*n+1)}$*
by (*intro tendsto-le[OF - tendsto-const *] always-eventually erf-remainder-integral-aux-bound allI assms sequentially-bot*)
qed

lemma *erf-remainder-integral-bigo:*
 $\text{erf-remainder-integral } n \in O(\lambda x. \text{exp } (-x^2) / x^{(2*n+1)})$
using *erf-remainder-integral-nonneg erf-remainder-integral-bound*
by (*auto intro!: bigoI[of - 1] eventually-mono [OF eventually-gt-at-top[of 0::real]]*)

theorem *erf-remainder-bigo: erf-remainder n $\in O(\lambda x. \text{exp } (-x^2) / x^{(2*n+1)})$*
using *erf-remainder-integral-bigo[of n] by (simp add: erf-remainder-def [abs-def])*

Next, we unroll the remainder term to develop the asymptotic expansion.

lemma *erf-remainder-integral-0-conv-erfc:*
assumes $(x::\text{real}) > 0$
shows $\text{erf-remainder-integral } 0 \ x = \text{sqrt } \text{pi} / 2 * \text{erfc } x$
proof –
have $(\lambda m. \text{sqrt } \text{pi} / 2 * (\text{erf } (x + \text{real } m) - \text{erf } x)) \longrightarrow \text{sqrt } \text{pi} / 2 * \text{erfc } x$
(is filterlim ?f - -) unfolding erfc-def
by (*intro tendsto-intros filterlim-tendsto-add-at-top[OF tendsto-const filterlim-real-sequentially]*)

also have $?f = (\lambda m. \text{integral } \{x..x + \text{real } m\} (\lambda t. \text{exp } (-t^2)))$
by (*auto simp: fun-eq-iff integral-unique*[*OF integral-exp-minus-squared-real*])
finally have $(\lambda m. \text{integral } \{x..x + \text{real } m\} (\lambda t. \text{exp } (-t^2))) \longrightarrow \text{sqrt } \pi / 2$
 $* \text{erfc } x$.
moreover have $(\lambda m. \text{integral } \{x..x + \text{real } m\} (\lambda t. \text{exp } (-t^2))) \longrightarrow \text{erf-remainder-integral}$
 $0\ x$
using *LIMSEQ-erf-remainder-integral*[*of x 0*] **assms** **by** *simp*
ultimately show $\text{erf-remainder-integral } 0\ x = \text{sqrt } \pi / 2 * \text{erfc } x$
by (*intro LIMSEQ-unique*)
qed

The first remainder is the *erfc* function itself.

lemma *erf-remainder-0-conv-erfc*: $x > 0 \implies \text{erf-remainder } 0\ x = \text{erfc } x$
by (*simp add: erf-remainder-def erf-remainder-integral-0-conv-erfc*)

Also, the following recurrence allows us to get the next term of the asymptotic expansion.

lemma *erf-remainder-integral-conv-Suc*:

assumes $x > 0$

shows $\text{erf-remainder-integral } n\ x = \text{exp } (-x^2) / (2 * x^{(2 * n + 1)}) -$
 $\text{real } (2 * n + 1) / 2 * \text{erf-remainder-integral } (\text{Suc } n)\ x$

proof –

let $?A = \lambda m. \{x..x + \text{real } m\}$

let $?J = \lambda m\ n. \text{integral } \{x..x + \text{real } m\} (\lambda t. \text{exp } (-t^2) / t^{(2 * n)})$

define I **where**

$I = (\lambda m. \text{exp } (- (x + \text{real } m)^2) / (- 2 * (x + \text{real } m)^{(2 * n + 1)}) -$
 $\text{exp } (-x^2) * \text{inverse } (- 2 * x^{(2 * n + 1)}) - \text{real } (2 * n + 1) / 2 * ?J$

$m\ (\text{Suc } n))$

have $I\text{-eq}$: $I = (\lambda m. \text{integral } (?A\ m) (\lambda t. \text{exp } (-t^2) / t^{(2 * n)}))$

proof

fix $m :: \text{nat}$

have $((\lambda t. (- 2 * t * \text{exp } (-t^2))) * \text{inverse } (- 2 * t^{(2 * n + 1)})) \text{has-integral } I\ m)$
 $(?A\ m)$

proof (*rule integration-by-parts*[*OF bounded-bilinear-mult*])

fix t **assume** $t \in ?A\ m$

with *assms* **show** $((\lambda t. \text{exp } (-t^2)) \text{has-vector-derivative } - 2 * t * \text{exp } (-t^2))$
 $(\text{at } t)$

by (*auto simp: has-field-derivative-iff-has-vector-derivative* [*symmetric*]
field-simps intro!: derivative-eq-intros)

from *assms* t **have** $((\lambda t. -(1/2) * t \text{powr } (- 2 * n - 1)) \text{has-field-derivative}$
 $(2 * n + 1) / 2 * t \text{powr } (- 2 * n - 2)) (\text{at } t)$

by (*auto intro!: derivative-eq-intros simp: field-simps powr-numeral power2-eq-square*
powr-minus powr-divide [*symmetric*] *powr-add*)

also have $?this \longleftrightarrow ((\lambda t. \text{inverse } (- 2 * t^{(2 * n + 1)})) \text{has-field-derivative}$
 $(2 * n + 1) / 2 / t^{(2 * \text{Suc } n)}) (\text{at } t)$ **using** t

using *eventually-nhds-in-open*[*of* $\{0 < ..\}$] t *assms*

by (*intro DERIV-cong-ev refl*)

(*auto elim!: eventually-mono simp: powr-minus field-simps powr-diff*)

powr-realpow power2-eq-square intro!: real-powr-eq-powerI

finally show $((\lambda t. \text{inverse } (-2 * t^{2 * n + 1}))) \text{ has-vector-derivative } (2 * n + 1) / 2 / t^{(2 * \text{Suc } n)} \text{ (at } t)$

by *(simp add: has-field-derivative-iff-has-vector-derivative)*

next

have $((\lambda t. \text{real } (2 * n + 1) / 2 * (\exp (- t^2) / t^{(2 * \text{Suc } n)})) \text{ has-integral } \text{real } (2 * n + 1) / 2 * ?J m (\text{Suc } n) (?A m) \text{ (is } (?f \text{ has-integral } ?a) -)$

using *assms*

by *(intro has-integral-mult-right integrable-integral integrable-continuous-real (auto intro!: continuous-intros))*

also have $?f = (\lambda t. \exp (- t^2) * (\text{real } (2 * n + 1) / 2 / t^{(2 * \text{Suc } n)}))$

by *(simp add: fun-eq-iff field-simps)*

also have $?a = \exp (- (x + \text{real } m)^2) * \text{inverse } (- 2 * (x + \text{real } m)^{(2 * n + 1)}) -$

$\exp (- x^2) * \text{inverse } (- 2 * x^{(2 * n + 1)}) - I m$ **using**

assms

by *(simp add: I-def algebra-simps inverse-eq-divide)*

finally show $((\lambda t. \exp (- t^2) * (\text{real } (2 * n + 1) / 2 / t^{(2 * \text{Suc } n)})) \text{ has-integral } \dots)$

$\{x..x + \text{real } m\} .$

qed *(insert assms, auto intro!: continuous-intros)*

hence $I m = \text{integral } \{x..x + \text{real } m\} (\lambda t. - 2 * t * \exp (- t^2) * \text{inverse } (- 2 * t^{(2 * n + 1)}))$

by *(simp add: has-integral-iff)*

also have $\dots = \text{integral } \{x..x + \text{real } m\} (\lambda t. \exp (- t^2) / t^{(2 * n)})$

using *assms* **by** *(intro integral-cong) (simp-all add: field-simps)*

finally show $I m = \dots .$

qed

have $\text{filterlim } (\lambda m. (- \exp (- (x + \text{real } m)^2)) / (2 * (x + \text{real } m)^{(2 * n + 1)}))$

(nhds 0) at-top

by *(rule real-tendsto-divide-at-top filterlim-real-sequentially tendsto-minus filterlim-compose[OF exp-at-bot] filterlim-compose[OF filterlim-uminus-at-bot-at-top] filterlim-pow-at-top filterlim-tendsto-add-at-top tendsto-const filterlim-ident filterlim-tendsto-pos-mult-at-top | simp)+*

hence $*: \text{filterlim } (\lambda m. (\exp (- (x + \text{real } m)^2)) / (- 2 * (x + \text{real } m)^{(2 * n + 1)}))$

(nhds 0) at-top **by** *(simp add: add-ac)*

have $I \longrightarrow 0 - \exp (- x^2) * \text{inverse } (- 2 * x^{(2 * n + 1)}) - \text{real } (2 * n + 1) / 2 * \text{erf-remainder-integral } (\text{Suc } n) x$

unfolding *I-def*

by *(intro tendsto-diff * tendsto-const tendsto-mult LIMSEQ-erf-remainder-integral assms)*

moreover from *LIMSEQ-erf-remainder-integral[OF assms, of n] I-eq*

have $I \longrightarrow \text{erf-remainder-integral } n x$ **by** *simp*

ultimately have $0 - \exp (- x^2) * \text{inverse } (- 2 * x^{(2 * n + 1)}) - \text{real } (2 * n + 1) / 2 * \text{erf-remainder-integral } (\text{Suc } n) x = \text{erf-remainder-integral } n x$

by (rule LIMSEQ-unique)
 thus ?thesis by (simp add: field-simps)
 qed

lemma erf-remainder-conv-Suc:

assumes $x > 0$

shows $\text{erf-remainder } n \ x = (-1)^n * \text{fact } (2 * n) / (\text{sqrt } \pi * 4^n * \text{fact } n) * \text{exp } (-x^2) / (x^{2 * n + 1}) + \text{erf-remainder } (\text{Suc } n) \ x$

proof –

have $\text{erf-remainder } n \ x =$

$(-1)^n * 2 * \text{fact } (2 * n) / (\text{sqrt } \pi * 4^n * \text{fact } n) * \text{exp } (-x^2) / (2 * x^{2 * n + 1}) + -$

$(-1)^n * 2 * \text{fact } (2 * n) / (\text{sqrt } \pi * 4^n * \text{fact } n) * \text{real } (2 * n + 1) / 2 * \text{erf-remainder-integral } (\text{Suc } n) \ x$ (is - = ?A +

?B)

unfolding erf-remainder-def using assms

by (subst erf-remainder-integral-conv-Suc)

(auto simp: assms algebra-simps simp del: power-Suc)

also have ?B = erf-remainder (Suc n) x

by (simp add: divide-simps erf-remainder-def)

also have ?A = $(-1)^n * \text{fact } (2 * n) / (\text{sqrt } \pi * 4^n * \text{fact } n) * \text{exp } (-x^2) / (x^{2 * n + 1})$

by (simp add: divide-simps)

finally show ?thesis .

qed

Finally, this gives us the full asymptotic expansion for erfc:

theorem erfc-unroll:

assumes $x > 0$

shows $\text{erfc } x = \text{exp } (-x^2) / \text{sqrt } \pi *$

$(\sum_{i < n}. (-1)^i * \text{fact } (2 * i) / (4^i * \text{fact } i) / x^{2 * i + 1}) + \text{erf-remainder}$

$n \ x$

proof (induction n)

case (Suc n)

note Suc.IH

also note erf-remainder-conv-Suc[OF assms, of n]

also have $\text{exp } (-x^2) / \text{sqrt } \pi *$

$(\sum_{i < n}. (-1)^i * \text{fact } (2 * i) / (4^i * \text{fact } i) / x^{2 * i + 1}) +$

$((-1)^n * \text{fact } (2 * n) / (\text{sqrt } \pi * 4^n * \text{fact } n) * \text{exp } (-x^2) /$

$x^{2 * n + 1}) +$

$\text{erf-remainder } (\text{Suc } n) \ x =$

$\text{exp } (-x^2) / \text{sqrt } \pi *$

$(\sum_{i < \text{Suc } n}. (-1)^i * \text{fact } (2 * i) / (4^i * \text{fact } i) / x^{2 * i$

$+ 1)) +$

$\text{erf-remainder } (\text{Suc } n) \ x$

by (subst sum.lessThan-Suc) (simp add: algebra-simps)

finally show ?case .

qed (auto simp: assms erf-remainder-0-conv-erfc)

For convenience, we define another auxiliary function that is more suitable for use in an automated expansion framework, since it has a simple asymptotic expansion in powers of x .

definition *erfc-aux* **where** *erfc-aux* $x = \exp(x^2) * \text{sqrt pi} * \text{erfc } x$

definition *erf-remainder'* **where** *erf-remainder' n x* $= \exp(x^2) * \text{sqrt pi} * \text{erf-remainder } n x$

lemma *erfc-aux-unroll*:

$x > 0 \implies$
 $\text{erfc-aux } x = (\sum_{i < n}. (-1)^i * \text{fact } (2*i) / (4^i * \text{fact } i) / x^{(2*i+1)}) + \text{erf-remainder' } n x$
using *erfc-unroll*[of x n]
by (*simp add: erfc-aux-def erf-remainder'-def exp-minus field-simps del: of-nat-Suc*)

lemma *erf-remainder'-bigo*: *erf-remainder' n* $\in O(\lambda x. 1 / x^{(2*n+1)})$

proof –

have $(\lambda x. \exp(x^2) * \text{erf-remainder } n x) \in O(\lambda x. \exp(x^2) * (\exp(-x^2) / x^{(2*n+1)}))$

by (*intro landau-o.big.mult erf-remainder-bigo simp-all*)

thus *?thesis* **by** (*simp add: exp-minus erf-remainder'-def [abs-def]*)

qed

lemma *has-field-derivative-erfc-aux*:

$(\text{erfc-aux has-field-derivative } (2 * x * \text{erfc-aux } x - 2)) \text{ (at } x)$

by (*auto simp: erfc-aux-def [abs-def] exp-minus field-simps intro!: derivative-eq-intros*)

end